Topological Entropy
Versus
Geodesic Entropy

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Abstract

We show that for a compact Riemannian manifold M, the geodesic entropy -defined as the exponential growth rate of the average number of geodesic segments between two points- is $\leq$ than the topological entropy of the geodesic flow of M. We also show that if M is simply connected and $N \subset M$ is a compact simply connected submanifold, then the exponential growth rate of the sequence given by the Betti numbers of the space of paths starting in N and ending in a fixed point of M, is bounded above by the topological entropy of the geodesic flow on the normal sphere bundle of N.

1 *Introduction*

Let M be a compact oriented C^∞ Riemannian manifold and let $N \subset M$ be a compact oriented submanifold. Fix $p \in M$ and $\lambda > 0$. Denote by $n_N(p,\lambda)$ the number of geodesic segments leaving orthogonally from N and terminating at p with length $\leq \lambda$. If p is not a focal point of N one can see that $n_N(p,\lambda)$ is finite. Define

$$I_N(\lambda) = \int_M n_N(p,\lambda)d\mu(p),$$

where μ is the measure induced by the Riemannian structure. Integrals of this sort were already considered in [1] when N is a point and in [4] when N is a totally-geodesic hypersurface. We define σ_N to be the number:

$$\sigma_N = limsup_{\lambda\to+\infty}\frac{1}{\lambda}\, log\ I_N(\lambda). \tag{1}$$

All these numbers are Riemannian invariants and somehow measure the complexity of the geometry of geodesics leaving orthogonally from a submanifold. In Section 2 we will prove using Yomdin's Theorem [8] that all these invariants are bounded above by the topological entropy h_{top} of the geodesic flow. Such a bound was already obtained

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in [7] and implicitly used in [2, Section 2.7] when N is a point. As an application of this generalization we will show the following: let $n(p, q, \lambda)$ denote the number of geodesic segments between p and q with length $\leq \lambda$, then

$$limsup_{\lambda \to +\infty} \frac{1}{\lambda} \; log \int_{M \times M} n(p, q, \lambda) d\mu(p) d\mu(q) \stackrel{\text{def}}{=} h_{geod} \leq h_{top}.$$

The number h_{geod} can be regarded as the *geodesic entropy* of the Riemannian metric. Recently Mañé [6] proved that also one has $h_{geod} \geq h_{top}$.

In Section 3, we will show that if M and N are simply connected there exists a constant $C > 0$ depending only on the geometry of M and N such that if $\Omega(p, N)$ denotes the Hilbert manifold of paths from N to p then for $k \geq 1$

$$\sum_{i=1}^{k} b_i(\Omega(p, N)) \leq \frac{1}{Vol(M)} I_N(Ck),$$

where $b_i(\Omega(p, N))$ are the Betti numbers over a fixed field. This was proved in [3] when N reduces to a point. Finally, as a corollary, we relate the exponential growth rate of the sequence $b_i(\Omega(p, N))$ with the topological entropy of the geodesic flow on the normal sphere bundle of N.

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2 *An upper bound for the geodesic entropy*

Let M be a compact oriented Riemannian manifold and let $N \subset M$ be a compact oriented submanifold. We will denote by $TN^{\perp}$ the normal bundle of N and by $SN^{\perp}$ the (unit) normal sphere bundle. Set $DN^{\perp}_{\lambda} = \{v \in TN^{\perp} : \| v \| \leq \lambda\}$ and $SN^{\perp}_{t} = \{v \in TN^{\perp} : \| v \| = t\}$.

If TM denotes the tangent bundle of M and $\pi : TM \to M$ denotes the canonical projection, then the exponential map is $exp = \pi \circ \phi_1$ where $\phi_t : TM \to TM$ is the geodesic flow. Finally $exp^{\perp} : TN^{\perp} \to M$ will denote the restriction of exp to $TN^{\perp}$.

Let us recall that a point p is called a *focal point* of N if it is a singular value of $exp^{\perp}$. By Sard's Theorem the set of focal points $F(N)$ has measure zero in M. Recall the definition of $I_N(\lambda)$ from the Introduction. Now we will prove:

Proposition 2.1 *$I_N(\lambda) \leq \int_0^{\lambda} Vol(\phi_t(SN^{\perp}))dt$, where $Vol(\phi_t(SN^{\perp}))$ stands for the Riemannian volume of $\phi_t(SN^{\perp})$ with respect of the canonical metric of TM.*

Proof: Let ω denote the Riemannian volume element and let $(exp^{\perp})^{*}\omega$ denote the pull-back of ω under the map $exp^{\perp}$. The same arguments as in [1, 4] show that if $p \notin F(N)$ then $n_N(p, \lambda)$ is finite, $I_N(\lambda)$ is well defined and

$$I_N(\lambda) \leq \int_{DN^{\perp}_{\lambda}} | (exp^{\perp})^{*}\omega |,$$

or in other words

$$I_N(\lambda) \leq \int_{DN_\lambda^\perp} \mid det\ d(exp^\perp)_v) \mid dv.$$

Using Fubini's Theorem and the Gauss Lemma we obtain:

$$\int_{DN_\lambda^\perp} \mid det\ d(exp^\perp)_v \mid dv = \int_0^\lambda dt \int_{SN_t^\perp} \mid det\ d(exp^\perp)_v \mid_{T_v(SN_t^\perp)} \mid dv =$$

$$\int_0^\lambda dt \int_{SN^\perp} \mid det\ d\pi_{\phi_t(v)} \mid \mid det\ (d\phi_t)_v \mid_{T_v(SN^\perp)} \mid dv \leq$$

$$\int_0^\lambda dt \int_{SN^\perp} \mid det\ (d\phi_t)_v \mid_{T_v(SN^\perp)} \mid dv,$$

which yields the desired inequality.

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Let $h_{top}(Y)$ denote the topological entropy of the geodesic flow with respect to the set $Y \subset SM$; h_{top} will always stand for $h_{top}(SM)$. Recall the definition of σ_N in (1).

Corollary 2.2 $\sigma_N \leq h_{top}(SN^\perp)$.

Proof: Yomdin's Theorem gives (cf. [2, 8]):

$$limsup_{t\to+\infty} \frac{1}{t}\ log\ Vol(\phi_t(SN^\perp)) \leq h_{top}(SN^\perp),$$

so the corollary follows directly from the last proposition. ◇

Corollary 2.3 $h_{geod} \leq h_{top}$.

Proof: Let Δ denote the diagonal in $M \times M$ and consider in $M \times M$ the product metric. Then if $x = (p, q) \in M \times M$ one has that $n(p, q, \lambda) = n_\Delta(x, \frac{\lambda}{\sqrt{2}})$ since a geodesic segment between p and q with length L corresponds to a geodesic in $M \times M$ leaving orthogonally from Δ and terminating at x with length $\frac{1}{\sqrt{2}}L$. Thus from the definitions it follows that

$$h_{geod} = \frac{1}{\sqrt{2}}\sigma_\Delta. \tag{2}$$

Now consider the map $f : S(M \times M) \to S^1$ given by $f(v_1, v_2) = (\| v_1 \|, \| v_2 \|)$. Since f is a first integral of the geodesic flow on $S(M \times M)$ it follows that $h_{top}(S(M \times M)) = sup_{c\in S^1} h_{top}(f^{-1}(c))$. But if we write $c = (x, y)$ then $h_{top}(f^{-1}(c)) = (x + y)h_{top}$ and thus the topological entropy of the geodesic flow of $M \times M$ equals $\sqrt{2}\ h_{top}$. Now the corollary follows by combining the equation (2) with Corollary 2.2.

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3 *The path space* $\Omega(p, N)$

Suppose now that M is a compact simply connected manifold and $N \subset M$ a compact simply connected submanifold. Choose a triangulation of M addapted to N and let $\Delta_k(M)$ and $\Delta_k(N)$ denote the k-skeleton of M and N respectively.

Take a point $p \in M$ and consider $\Omega(p, N)$, the Hilbert manifold pf paths starting from a point in N and ending in p. Denote by $\tilde{\Omega}(p, N)$ the subspace of $\Omega(p, N)$ given by all piece-wise linear paths respect to the chosen triangulation. Following Gromov ([3]) we observe that $\tilde{\Omega}(p, N)$ has a natural cell-decomposition as follows. A path $\gamma \in \tilde{\Omega}(p, N)$ can be identified with the sequence of simplices $\sigma_1, ..., \sigma_r$ that it touches on its way from N to p; in this fashion a cell in $\tilde{\Omega}(p, N)$ can be thought as the cartesian product $\sigma_1 \times ... \times \sigma_r$ where two consecutive simplices σ_i, σ_{i+1} are faces of one simplex. Note that $dim\ \sigma_i$ can be any value between 0 and $dim\ M$.

We will now show that the arguments in [3] extend to the case of the path space $\Omega(p, N)$. Since M and N are simply connected the inclusion map

$$(\Delta_1(M), \Delta_1(N)) \hookrightarrow (M, N)$$

is homotopic to a constant map $(q, q) \in (M, N)$ with homotopy

$$g_t : (\Delta_1(M), \Delta_1(N)) \to (M, N).$$

By the homotopy extension lemma (relative version, cf. [5, Corollary 4-10]) g_t can be extended to a homotopy

$$G_t : (M, N) \to (M, N)$$

so that G_0 is the identity map and $\alpha \stackrel{\text{def}}{=} G_1$ contracts the 1-skeleton of M to a point $q \in N$. Moreover, G_t can be chosen smooth.

Suppose now M has a Riemannian metric and let $L : \Omega(p, N) \to \mathbf{R}$ be the length functional. Note that a smooth map $f : (M, N) \to (M, N)$ induces a map $\hat{f} : \tilde{\Omega}(p, N) \to \Omega(f(p), N)$. We will prove:

Lemma 3.1 *There exists a constant $C > 0$, depending only on M and N such that the natural map*

$$H_i(L^{-1}[0, Ck]) \to H_i(\Omega(p, N)),$$

is surjective for $i \leq k$ and all $k \geq 1$ and any $p \in M$.

Proof: First observe that $\tilde{\Omega}(p, N)$ has the same homotopy type as $\Omega(p, N)$ and it is independent of the point $p \in M$. Hence the lemma is a consequence of the following claim: there exists a constant $C > 0$ depending only on M and N so that

$$\hat{\alpha}(k - skeleton) \subset L^{-1}[0, Ck]$$

for all $k \geq 0$, where $\hat{\alpha}$ is the induced map by α.

Consider a cell $\sigma_1 \times \ldots \times \sigma_r$ in $\tilde{\Omega}(p,N)$ with $dim\ (\sigma_1 \times \ldots \times \sigma_r) \leq k$. Take a path γ in this cell. Since α sends the 1-skeleton to a point we observe that

$$L(\hat{\alpha}(\gamma)) \leq K(L(\gamma) - L(\gamma \mid_{1-skeleton})),$$

for some constant $K > 0$. Call d the maximum of the diameter of all positive dimensional simplices in the triangulation. Then clearly

$$L(\gamma) - L(\gamma \mid_{1-skeleton}) \leq d \ \#\{\sigma_i : \ dim\ \sigma_i > 0\} \leq dk.$$

Thus $L(\hat{\alpha}(\gamma)) \leq Kdk$. If we set $C = Kd$ we obtain the claim.

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Corollary 3.2 *If $p \notin F(N)$ then $\sum_{i=1}^{k} b_i(\Omega(p,N)) \leq n_N(p,Ck)$ for all $k \geq 1$, where $C > 0$ is the constant from the previous lemma and the Betti numbers $b_i(\Omega(p,N))$ are taken over a fixed field.*

Proof: If $p \notin F(N)$, then the length functional L is a Morse function on $\Omega(p,N)$. The critical points of L are precisely the geodesics leaving orthogonally from N and ending in p. Hence $n_N(p,\lambda)$ is nothing but the number of critical points of L with length $\leq \lambda$. If we set, as it is usual, $\Omega^{\lambda}(p,N) = L^{-1}[0,\lambda]$, the Morse inequalities imply:

$$\sum_{i=1}^{r} b_i(\Omega^{\lambda}(p,N)) \leq n_N(p,\lambda),$$

where r is such that $b_i(\Omega^{\lambda}(p,N)) = 0$ for $i > r$.

But from Lemma 3.1 we know that $b_i(\Omega(p,N)) \leq b_i(\Omega^{Ck}(p,N))$ for $i \leq k$ and the corollary follows.

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Define now r_N as:

$$r_N = limsup_{k\to+\infty} \frac{1}{k}\ log\ \sum_{i=1}^{k} b_i(\Omega(p,N)).$$

Corollary 3.3 *For all $k \geq 1$ we have that*

$$\sum_{i=1}^{k} b_i(\Omega(p,N)) \leq \frac{1}{Vol(M)} I_N(Ck) \leq \frac{1}{Vol(M)} \int_0^{Ck} Vol(\phi_t(SN^{\perp}))dt,$$

where $C > 0$ is the constant from Lemma 3.1. Moreover,

$$r_N \leq C\ h_{top}(SN^{\perp}).$$

Proof: It follows directly by integration from the previous corollary, Proposition 2.1 and Corollary 2.2.

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Remark 3.4 The number r_N can thought as a measure of the "complexity" of the embedding $N \hookrightarrow M$. If this embedding is complicated, i.e. if $r_N > 0$, then Corollary 3.3 says that the topological entropy of the geodesic flow over the set $SN^{\perp}$ is positive. In this way Corollary 3.3 could be regarded as follows: the "complexity" of the embedding is bounded by the dymamics of the geodesic flow on the normal sphere bundle.

Finally note that Corollary 3.3 has the following interesting application. *Let M be a simply connected compact manifold and $N \subset M$ a simply connected compact submanifold. Then if every geodesic leaving orthogonally from N returns orthogonally to N at constant time, then $\sum_{i=1}^{k} b_i(\Omega(p, N))$ grows at most like k.* Indeed since there exists $T > 0$ so that $\phi_T(SN^{\perp}) = SN^{\perp}$, it follows that $Vol(\phi_t(SN^{\perp}))$ is bounded which implies the claim via Corollary 3.3.

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