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EVERYWHERE NON REDUCED MODULI SPACES

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Dedicated to the memory of Aldo Andreotti

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Introduction

The purpose of this article is to show how often the moduli spaces of surfaces of general type can be everywhere non reduced in the case when the canonical bundle K_S is not ample. On the other hand, by giving a simple criterion which implies that this must happen, we are in fact able to subsume almost all the previously known examples of obstructed deformations in dimension 2 as particular issues of a very general situation; we produce also infinite series of examples, showing in particular that all non Cartier divisors of rigid 3-dimensional weighted projective spaces give rise to this pathology.

To be more precise, let V be an algebraic variety with a finite group Aut(V) of automorphisms (e.g., cf. [Ma], if V is of general type); then, if V admits a space of moduli $\mathfrak{M}(V)$ (cf. [Mu 2]), locally (i.e. in an analytic neighbourhood of the point of $\mathfrak{M}(V)$ correspoding to V) $\mathfrak{M}(V)$ is the quotient of the base Def(V) of the Kuranishi family of deformations of V by the group Aut(V) (cf. [Wa]). It is clear that in this case $\mathfrak{M}(V)$ is (locally) everywhere non reduced, e.n.r. for short (i.e., everywhere singular) if and only if Def((V) is e.n.r. .

We recall the classical terminology: V is said to have obstructed deformations if the germ $\mathrm{Def}(V)$ is singular. The stronger condition that $\mathrm{Def}(V)$ be e.n.r., i.e. everywhere singular, can thus be referred to as V having "everywhere obstructed deformations", and has been regarded up to now as a very pathological phenomenon.

The first example of algebraic varieties V with $\mathrm{Def}(V)$ e.n.r. is due to Kodaira and Mumford ([Ko], [Mu 1]) : here, though, V and its deformations are blow-ups of $\mathbb{P}^3$, hence there are no birational moduli.

After several examples of obstructed deformations were exhibited, e.g. by Kas ([Ka 1], [Ka 2]), by Burns and Wahl ([B-W]), by Horikawa ([Hor 1]), then Horikawa [Hor 2] and, later, Miranda ([Mi]) gave examples of surfaces of general type S (respectively with $p_g = 4,7$, $K^2 = 6$, 14) for which $\mathrm{Def}(S)$ is e.n.r. .

Their approach was through the classification of all the surfaces with those invariants (Miranda uses Castelnuovo's classification [Cas] of surfaces with $K^2 = 3\,p_g - 7$ and $|K_S|$ birational). In both cases the outcome is that the canonical bundle K_S is not ample for all the surfaces corresponding to the points of a component of the moduli space. This research started when I tried to find a direct proof that $\mathrm{Def}(S)$ was e.n.r., and I noticed that for both examples the singular canonical models were hypersurfaces in a 3 dimensional projective space (respectively $X_9 \subset \mathbb{P}(1,1,2,3)$, $X_7 \subset \mathbb{P}(1,1,1,2)$) admitting a double cover ramified on the singular point and a (disjoint) canonical divisor.

It became clear that the essential point had nothing to do with the fact that these two surfaces lie on extremal lines of surface geography, it was instead the common feature of X being the quotient of a smooth surface Z by $\mathbb{Z}/2$ in such a way that all deformations of X remain singular.

In fact, already Burns and Wahl ([B-W]) had already introduced the philosophy that one may have obstructed deformations if K_S is not ample, i.e. the canonical model X is singular. I would like parenthetically to point out that in their examples of obstructed deformations X is a surface with many nodes so that, by Beauville's beautiful remark in [Be], X has many double covers ramified only at the nodes, and thus one falls again in our general picture, which is as follows.

Main Theorem *Let* Z *be a smooth surface with* Def(Z) *smooth, and let* G *be a finite group acting on* Z *in such a way that* $X = Z/G$ *has has only R.D.P.'s (Rational Double Points) as singularities and is indeed singular. Let* $p : Z \longrightarrow X$ *be the quotient map, and assume that the following inclusion of sheaves*

$$(0.1.) \qquad \Omega^1_X \otimes \omega_X \hookrightarrow (p_* \Omega^1_Z)^G \otimes \omega_X$$

induces an injective map between the first cohomology groups.

Assume also that $H^1(\theta_Z)^G$ *surjects onto* $H^1(\theta_X)$: *then, if* S *is the minimal resolution of singularities of* X *, then*

(0.2.) Def(S) *is a product* $\mathrm{Def}(Z)^G \times R$ *, where*

$\mathrm{Def}(Z)^G$ *is a smooth scheme and* R *is a non reduced, connected, 0-dimensional scheme whose length* n *can in fact be arbitrarily large.* □

It is important to observe that the two cohomological conditions are quite reasonable, since the first is verified if ω_X is sufficiently ample, and the second one is automatically verified if p is unramified in codimension 1.

Roughly speaking, the first condition guarantees that all the deformations of X are equisingular, the second one that all the equisingular deformations of X come from deformations of Z which preserve the action of G .

This is the simple idea, and the technique uses, beyond other tools, a useful criterion by Pinkham ([Pi]), which clarifies and extends previous results of Kas[Ka 2] and Burns-Wahl [B-W].

As we already mentioned, our criterion implies that K_S is not ample for all the surfaces corresponding to the points of a component of the moduli space, and so we regard our criterion as a sort of explanation of this pathology.

In fact it makes us still confident about the validity of the following conjecture [Cat] : if S is a surface of general type with $q = 0$ and K_S ample, then the moduli space $\mathfrak{M}(S)$ is smooth on an open dense set.

As to the further contents of the paper, § 1 contains the proof and a more general version of the basic criterion above. § 2 is devoted to examples where $p : Z \longrightarrow X$ is unramified in codimension 1, and we show in particular that if G acts on a 3-fold W, then for all the hypersurfaces Z of large degree we obtain e.n.r. moduli spaces. We also use a slight generalization of Kas' surfaces [Ka 2] , to show that the ratio between tangent dimension and dimension of Def(S) can be arbitrarily large. Finally, most of our effort is spent in § 3 to analyze the somehow "most simple" example, the one of hypersurfaces X in a weighted projective space $\mathbb{P}$.

We have in fact that the examples of Horikawa and Miranda belong to the huge class of non-Cartier hypersurfaces X in a weighted projective space $\mathbb{P} = \mathbb{P}(1,1,p,q)$ with p,q relatively prime.

We first classify all the X's as above with R.D.P.'s, then we apply our criterion.

The striking result is that in this case the deformation space of S is always e.n.r. The underlying philosophy is the following: if $X \subset \mathbb{P}^3$, then all small deformations of X are still surfaces in $\mathbb{P}^3$ (cf. [K-S]); whereas (cf. § 2, $X_8 \subset \mathbb{P}(1,1,2,2)$) this is no longer true in general for all weighted $\mathbb{P}$'s , it is true if $\mathbb{P}$ has isolated singularities and is thus rigid, and then one gets an e.n.r. deformation space because of the infinitesimal deformations coming from the singularities of $\mathbb{P}$.

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The paper grew out of a talk I gave at the problems seminar of Columbia University, during the special year on Algebraic Geometry; there H. Pinkham pointed out that the infinitesimal computation of the obstruction map that I needed was already available in his earlier paper ([Pi]), and I thank him heartily for this precious help.

Thanks for the warm hospitality go to Columbia University and to the Max-Planck-Institut, where I could finish the fastidious computations of § 3 and write the final version.

Finally, the results of this paper were announced at two Conferences, where my participation was supported by G.N.S.A.G.A. of C.N.R. and the Italian M.P.I.

Notation

S a complete smooth minimal surface of general type over $\mathbb{C}$

For any variety X ,

Ω_X^1 is the sheaf of Kähler differentials,

$\Theta_X = Hom_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ is the sheaf of vector fields, and, if B is any subscheme of X with ideal sheaf I_B , $\Theta_X(-\log B)$ is the subsheaf of vector fields carrying $d(I_B)$ into I_B .

If B is reduced, $N'_{B|X}$, the equisingular normal sheaf of B in X , is the quotient $\Theta_X/\Theta_X(-\log B)$ (cf. [Cat], 9.16, 9.17).

ω_X is the dualizing sheaf (here an invertible sheaf, since X shall be Gorenstein)

$Def(X)$, if X is complete, is the basis of the semi-universal family of deformations of X (cf. [Gr]) .

For a vector space V , V^* denotes its dual space.

If G is a finite group acting on a vector space V , V^G denotes the subspace of invariant vectors.

If G is a finite group acting on a variety Z , G_z denotes the stabilizer of a point z .

If $p : Z \longrightarrow X = Z/G$ is the quotient map, and F is a G-linearized sheaf, $p_*(F)^G$ denotes the G-invariant part of the direct image.

A 1-point nilpotent scheme R shall be the spectrum of a local Artin $\mathbb{C}$-algebra of length > 1 .

§ 1 The basic criterion

Let a finite group G act on a smooth (complete) variety Z .

If $z \in Z$ has non trivial stabilizer G_z , then $z \in W = \bigcup_{\substack{g \in G \\ g \neq id}} Fix(g)$, where $Fix(g) = \{z' | g(z') = z'\}$.

Moreover, let $W' \subset W$ be the locus in W-$Sing_*W$ which has codimension 1: then for $z \in W'$ G_z is cyclic and generated by a pseudo-reflection g , where we recall (cf. [Ch]) that

Def. 1.1. g is a pseudo reflection if g is an automorphism of Z of finite order n s.t. for each point z of Z

either a) $g^i(z) \neq z$ for $i < n$,

or b) there exist local coordinates $(t_1, \dots t_m)$ around z such that

(1.2.) $g(t_1, \dots t_m) = (t_1, \dots ,t_{m-1},\varepsilon t_m)$
(where ε is a primitive n^{th} root of unity)

Lemma 1.3. If g is a pseudo reflection, let $X' = Z/g$, let $p' : Z \longrightarrow X'$ be the quotient map and let B' be the (smooth) branch locus of p' .

Then there are natural isomorphisms

$$\Omega^1_{X'} \xrightarrow{j^\vee} p'_*(\Omega^1_Z)^g \qquad \text{and}$$

$$(p_*\theta_Z)^G \xrightarrow{j} \theta_{X'}(-\log B')$$

Proof. Both $j^\vee, j$ are naturally defined and are clearly isomorphisms on the open sets where p' is unramified.

If $p'(z) = x$, $g(z) = \cdot z$, we take local coordinates as in (1.2.) and then the proof follows from a direct computation.

□

We set now $Z'' = Z-(W-W') = \{z \mid G_z$ is generated by a pseudo-reflection$\}$, so that $\operatorname{codim}(Z-Z'') \geq 2$, $X = Z/G$, let $p : Z \longrightarrow X$ be the quotient map, and finally set $X'' = p(Z'')$.

Lemma 1.4. Let B be the (reduced) branch locus of $p : Z \longrightarrow X = Z/G$, $X^0 = X - \operatorname{Sing} X$, $i : X^0 \hookrightarrow X$ the inclusion map. Then the natural homomorphism $(p_*\theta_Z) \xrightarrow{j} \theta_X(-\log B)$ is an isomorphism, whereas $\Omega^1_X \xrightarrow{j^\vee} p_*(\Omega^1_Z)^G$ is injective, an isomorphism on X^0 , and $p_*(\Omega^1_Z)^G$ equals $i_*(\Omega^1_{X^0})$.

Proof. By (1.3.) the assertion is true on the open set $X'' \subset X^0$. Since $\operatorname{codim}(Z-Z'') \geq 2$ and θ_Z , Ω^1_Z are locally free, if $i'' : X'' \longrightarrow X$ is the inclusion, $p_*(\theta_Z)^G = (i'')_*(\theta_{X''}(-\log B)$ and $p_*(\Omega^1_Z)^G = (i''_*)(\Omega^1_{X''}) = i_*(\Omega^1_{X^0})$.

We have now to show that $\theta_X(-\log B) \longrightarrow (i'')_*(\theta_{X''}(-\log B))$ is an isomorphism. This follows since $\theta_X(-\log B)$ is torsion free (if ν is a vector field, f is a function and $\nu(df)|_{X''} \in I_{B\cap X''}$, then $\nu(df) \in I_B$), hence we have an injective homomorphism of which j gives an inverse.

Finally, the inclusion $\mathcal{O}_X \subset p_*\mathcal{O}_Z$ shows that $j^\vee$ is injective

Q.E.D.

We let now $\pi : S \longrightarrow X$ be a resolution of singularities of X, and Y be the normalization of the fibre product $Z \times_X S$.

We then have a diagram

(1.5.)
$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & S \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{p} & X \end{array}$$

and we set $Y^0 = Y - \mathrm{Sing}(Y)$, $S^0 = S - \varphi(\mathrm{Sing}\ Y)$, $Z^0 = Z - \varepsilon(\mathrm{Sing}(Y))$.

Proposition 1.6. For any invertible sheaf $\mathcal{L}$ on X, we have

$$H^0(\Omega^1_S \otimes \pi^*\mathcal{L}) \cong H^0(\Omega^1_Z \otimes p^*\mathcal{L})^G .$$

Proof. By abuse of notation we shall identify $\mathcal{L}$ with its pull-backs, which we shall consider endowed of their natural linearization (notice that G acts on Y, with quotient S).

Since $S - S^0$ has codim ≥ 2, $H^0(\Omega^1_S \otimes \mathcal{L}) = H^0(\Omega^1_{S^0} \otimes \mathcal{L})$; by (1.4.) $(\varphi_*\Omega^1_{Y_0})^G = \Omega^1_{S^0}$, hence $H^0(\Omega^1_{S^0} \otimes \mathcal{L}) = H^0(\Omega^1_{Y_0} \otimes \mathcal{L})^G$ and this last clearly equals $H^0(\Omega^1_Z \otimes \mathcal{L})^G$ since $\varepsilon|_{\varepsilon^{-1}(Z^0)}$ is a modification and Z is smooth.

Q.E.D.

Corollary 1.7. Let $Z, G, \ldots$ be as above and assume X Gorenstein, $\dim Z = 2$. Let T^* be the cokernel of the sequence

$$0 \longrightarrow \Omega^1_X \otimes \omega_X \longrightarrow (p_*\Omega^1_Z)^G \otimes \omega_X \xrightarrow{r} T^* \longrightarrow 0 .$$

Then $H^0(T^*) \cong H^0(\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X))^\vee$ and the image of $H^0(r)$ is isomorphic to the cokernel of

$$H^0(\Omega^1_X \otimes \omega_X) \longrightarrow H^0(\Omega^1_S \otimes \Omega^2_S) .$$

Proof. First of all $\Omega^2_S \cong \omega_S \cong \pi^*(\omega_X)$ since the singularities of X are Gorenstein quotient singularities, hence R.D.P.'s; so that the second statement follows directly from (1.6.) letting

$\mathcal{L}$ be ω_X .

By (1.4.) the above sequence is obtained by tensoring with ω_X the exact sequence of local cohomology

$$0 \longrightarrow \Omega^1_X \longrightarrow i_*(\Omega^1_{X^0}) \longrightarrow H^1_{\mathrm{Sing}\, X}(\Omega^1_X) \longrightarrow 0$$

and, as in Pinkham's article ([Pi] page 174, (4) of theorem 1), we notice that by local duality the last term is isomorphic to

$$\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)^{\vee} \quad .$$

<u>Q.E.D.</u>

<u>Remark</u>. Notice that sometimes Y is necessarily singular, no matter of which resolution S of X one takes.

I.e., there is no blow up of Z on which G acts as a group generated by pseudo reflections. For instance, take the action of $\mathbb{Z}/4$ such that $(u,v) \longrightarrow (iu,-iv)$: on the first blow-up there is a point where the action has eigenvalues $(i,-1)$, hence this point has to be blown up, and over it lies a point where the eigenvalues are $(-i,-1)$, ... hence the procedure can never end.

From now on we shall assume $\dim Z = 2$, X Gorenstein (i.e., with R.D.P.'s only), and let S be a minimal resolution of singularities of X .

Also we shall use freely the results of [B-W] and [Pi], in particular we shall use the theorem of Burns and Wahl, asserting that there is a fibre product

$$(1.8.)\qquad \begin{array}{ccc} \mathrm{Def}(S) & \longrightarrow & L_S \\ \downarrow & & \downarrow \beta \\ \mathrm{Def}(X) & \xrightarrow{\psi} & L_X \end{array}$$

where L_X , L_S are the bases of the versal family of (local) deformations of the respective germs $(X, \mathrm{Sing}\ X)$, $(S, \pi^{-1}(\mathrm{Sing}\ X))$.

We have that $L_X \cong H^0(\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)) \overset{\mathrm{def}}{=} T_X$, L_S is also smooth and β is a finite cover with zero differential at the origin (β is a direct product, over the singular points of X , of the quotient maps

$$(1.9.)\qquad H^1_{E_x}(\theta_S) \longrightarrow \mathcal{E}xt^1(\Omega^1_{X,x}, \mathcal{O}_{X,x}) \cong H^1_{E_x}(\theta_S)/W_x ,$$

where $E_x = \pi^{-1}(x)$, and W_x is the Weyl group of the singularity acting on the vector space $H^1_{E_x}(\theta_S)$.

Let us now look at the maps of tangent spaces at the base points induced by diagram (1.8.): it fits into a larger diagram (where $E = \bigcup_{x \text{ singular}} E_x$)

$$(1.10)\qquad \begin{array}{ccccccccc} & & H^1(\theta_S) & \rightleftarrows & H^1_E(\theta_S) & \leftrightarrows 0 & & & \\ & \nearrow & \downarrow & & \downarrow & & & & H^2(\theta_S) \\ & & & & & & & & \| \\ 0 \longrightarrow H^1(\theta_X) & \longrightarrow & \mathrm{Ext}^1(\Omega^1_X,\mathcal{O}_X) & \longrightarrow & H^0(\mathcal{E}xt^1(\Omega^1_X,\mathcal{O}_X)) & & \longrightarrow & & H^2(\theta_X) \\ & & & & \| & & & & \\ & & & & T_X & & & & \end{array}$$

Here the inclusion of $H^1_E(\theta_S)$ is induced by the sequence of local cohomology ([B-W] 1.3), on the other hand Burns and Wahl prove that $\pi_*\theta_S = \theta_X$, so that the Leray spectral sequence for the map π (cf. [Pi] page 176) gives the sequence

$$\begin{array}{c} 0 \longrightarrow H^1(\theta_X) \longrightarrow H^1(\theta_S) \longrightarrow H^0(R^1\pi_*\theta_S) \longrightarrow 0 \\ \wr\| \\ H^1_E(\theta_S) \end{array}$$

plus the isomorphism $H^2(\theta_X) \cong H^2(\theta_S)$; thus we obtain the splitting

$$(1.11)\qquad H^1(\theta_S) \cong H^1(\theta_X) \oplus H^1_E(\theta_S) .$$

On the other hand, (cf. [Pi]), the lower exact row of (1.10) is given by the Ext spectral sequence, and Pinkham proves that

(1.12) ob is dual, via Grothendieck duality, to the map

$$\alpha : H^0(\Omega_S^1 \otimes \Omega_S^2) \longrightarrow T_X^*$$

corresponding, using the isomorphisms of corollary 1.7, to the map $H^0(r)$ ibidem.

Definition 1.13. We let ESDef(X), the space of equisingular deformations of X, be (cf. 1.8.) the germ $\psi^{-1}(0)$. Clearly its tangent space (1.10) is $H^1(\theta_X)$.

We can already derive some consequences (the first statement being already in [Pi]):

Proposition 1.14. Def(S) is singular unless ob is the zero map. If ESDef(X) is smooth and ob is injective, then Def(X) = ESDef(X) and Def(S) = ESDef(X) $\times$ R, where R is a non reduced connected 0-dimensional scheme.

Proof. Since β is finite, dim Def(S) = dim Def(X), hence Def(S) is smooth iff Def(X) is smooth and ψ has surjective differential (recall that β has zero differential); by (1.10) the last condition means that ob be the zero map.

For the other assertion, if ob is injective, then the inclusion ESDef(X) $\subset$ Def(X) induces an isomorphism of tangent spaces: if thus ESDef(X) is smooth, by Dini's inversion theorem this inclusion is an isomorphism. This means that the morphism ψ is constant, thus Def(S) = ESDef(X) $\times\ \beta^{-1}(0)$, and our claim follows since β is finite and with 0 differential, hence $R = \beta^{-1}(0)$ is a non reduced 0-dimensional scheme consisting of one point.

Q.E.D.

Remark 1.15. By 1.12., 1.7., the conditions "ob injective" (resp.; non zero) are equivalent to "$H^0(r)$ surjective" (resp. : non zero).

Theorem 1.16. Let Z be a smooth algebraic surface, and let G be a finite group acting on Z, in such a way that the quotient $X = Z/G$ has only R.D.P.'s as singularities. Consider the exact sequence (cf. (1.7.)):

$$0 \longrightarrow H^0(\Omega^1_X \otimes \omega_X) \longrightarrow H^0(\Omega^1_Z \otimes p^*\omega_X)^G \xrightarrow{H^0(r)} T^*_X \longrightarrow$$

then, if S is a minimal resolution of singularities of X,

i) Def(S) is singular if the map $H^0(r)$ is non zero

ii) Def(S) is e.n.r. if the following hypotheses are verified:

a) Def(Z) is smooth

b) $H^1(\theta_Z)^G$ surjects onto $H^1(\theta_X)$

c) $H^0(r)$ is surjective.

Proof. i) is a restatement of 1.14., in view of remark 1.15. For the same reason ii) is proved if we show that ESDef(X) is smooth provided a), b) hold.

Let $Def(Z)^G$ be the subspace of $Def(Z)$ consisting of the deformations preserving the action of G : it is well known that, under the natural inclusion $Def(Z) \subset H^1(\theta_Z)$, $Def(Z)^G =$ $= Def(Z) \cap H^1(\theta_Z)^G$ (cf. e.g. [Cat], Lecture 3). Hence, if $Def(Z)$ is smooth, then $Def(Z)^G$ is smooth with tangent space equal to $H^1(\theta_Z)^G$. We have a natural morphism $Def(Z)^G \longrightarrow ESDef(X)$: by b) it induces a surjective map on tangent spaces, hence $ESDef(X)$ is also smooth, by Dini's theorem on implicit functions.

Q.E.D.

Recall that, by lemma 1.4., we have an exact sequence

$$(1.18) \qquad 0 \longrightarrow p_*(\theta_Z)^G \longrightarrow \theta_X \longrightarrow N'_{B|X} \longrightarrow 0 \ .$$

The map $H^1(\theta_Z)^G \longrightarrow H^1(\theta_X)$ fits thus into the cohomology exact sequence associated to (1.18), hence hypothesis b) is equivalent to the injectivity of $H^1(N'_{B|X}) \longrightarrow H^2(\theta_Z)^G$.

We record for later use the following

Lemma 1.19. Hypothesis b), that $H^1(\theta_Z)^G$ maps onto $H^1(\theta_X)$, is implied by the vanishing of $H^1(N'_{B|X})$ and is equivalent to the following numerical relation

$$h^1(N'_{B|X}) = h^0(\Omega^1_Z \otimes \Omega^2_Z)^G - h^0(\Omega^1_Z \otimes p^*\omega_X)^G \ .$$

Proof. The first assertion is trivial, the second follows from Serre duality. In fact $h^2(\theta_Z)^G = h^0(\Omega^1_Z \otimes \Omega^2_Z)^G$, whereas $h^2(\theta_X) = h^2(\theta_S)$ (cf. 1.10), which equals $h^0(\Omega^1_S \otimes \Omega^2_S) =$ $= h^0(\Omega^1_Z \otimes \omega_X)^G$ by 1.6.

Q.E.D.

Corollary 1.20. Let Z be a smooth algebraic surface with Def(Z) smooth, let G be a finite group acting on Z in such a way that the quotient map $p : Z \longrightarrow X = Z/G$ is unramified in codimension 1, and X has only R.D.P.'s as singularities (and is indeed singular!).

Then, letting as usual S be a minimal resolution of singularities of X , observe that $p^*\omega_X = \Omega^2_Z$; if $H^0(\Omega^1_Z \otimes \Omega^2_Z)^G$ surjects onto T^*_X , we have $\mathrm{Def}(S) \cong \mathrm{Def}(Z)^G \times R$, where R is a 1-point nilpotent scheme of length = dim T^*_X .

Proof. $N'_{B|X} = 0$.

Q.E.D.

We can in fact restate theorem 1.16. with the weaker assumptions we have in fact used

Theorem 1.21. Notation being as in 1.16., $\mathrm{Def}(S) = \mathrm{Def}(Z)^G \times R'$, with R' a 1-point nilpotent scheme of length ≥ length T^*_X , provided:

a) $\mathrm{Def}(Z)^G$ is smooth

b) $H^1(\theta_Z)^G \longrightarrow H^1(\theta_X)$ is surjective

c) there is a singular point x s.t. $H^0(r)$ surjects onto T_x^* .

§ 2 Examples of e.n.r. moduli spaces.

In this paragraph we shall consider basically two types of examples for which cor. 1.20 applies.

As mentioned in the introduction, the first example concerns surfaces in 3-folds, while the second is a slight generalization of Kas' surfaces [Ka 2] , and deals with quotients of products of curves.

In the first example we shall consider the following situation

(2.1.) W is a smooth 3-fold, and G is a finite group acting on W with only a finite number of points $w_1, \dots w_s$ having a non trivial stabilizer. There is also a very ample divisor H such that $\mathcal{O}_W(H)$ has a G-linearization (e.g. if H is a G-invariant effective divisor).

Assume further

(2.2.) There exists an integer $r > 0$ and $Z \in |rH|$ s.t.

i) Z is G-invariant and with R.D.P.'s

ii) Z contains some w_i

iii) if $Z \ni w_i$, then Z is smooth at w_i , and G_{w_i} acts on the tangent plane T_{Z,w_i} with determinant = 1.

Theorem 2.3. If (2.1.), (2.2.) hold and S is a minimal resolution of $X = Z/G$, then for $r >> 0$ S has everywhere obstructed deformations.

Proof. Since, by (2.1.), $p : Z \longrightarrow X$ is unramified in codimension 1, cor. 1.20 applies: in fact, by (2.2.) X has only R.D.P.'s and is singular. We have to verify that $H^0(\Omega^1_Z \otimes \omega_Z)^G$ maps onto $T^*_X = H^0(T^*) = H^0((p_*\Omega^1_Z)^G/\Omega^1_X)$ (cf. 1.7.). Notice that the restriction map $\Omega^1_W \otimes \omega_W(rH) \longrightarrow \Omega^1_Z \otimes \omega_W(rH)$ is a homomorphism of G-modules, while ω_Z and $\omega_W(rH)$ differ just by a character of G.

Now T^* is a quotient of $(p_*\Omega^1_W)^G/\Omega^1_{W/G}$, which has finite length, hence there exists an integer $k > 0$ s.t. T^* is a quotient of $\bar{T} = \bigoplus_{i=1}^{s} \Omega^1_{W,w_i}/\mathfrak{m}^k_{W,w_i} \cdot \Omega^1_{W,w_1}$ where $\mathfrak{m}_{W,w_i}$ is the maximal ideal of the point w_i.

It suffices now to choose $r >> 0$ s.t. $H^0(\Omega^1_W \otimes \omega_W(rH))$ maps onto $\bar{T}$.

Q.E.D.

We show now at least that (2.1.)(2.2.) occur easily.

Example 2.4. We let $W = A$, where A is an abelian 3-fold and let H be any polarization.

$G = \{\pm 1\}$ acts in the standard way, so $w_1, \dots w_{64}$ are the

2-torsion points.

For every r , $H^0(\mathcal{O}_A(rH))$ splits into even and odd part, and we take Z inside $|rH|^-$.

By Lefschetz's theorem $|rH|$ embeds A for $r \geq 3$, therefore by Bertini's theorem, we can choose Z to be smooth and to contain all the 64 2-torsion points.

2.2 iii) holds since $(-1)^2 = 1$, so X has 64 ordinary quadratic points.

It is important to remark that S is a regular surface.

In fact if X^0 is the smooth part of X , we have
$H^0(\Omega^1_S) \longrightarrow H^0(\Omega^1_{X^0}) \longrightarrow H^0(\Omega^1_Z)^+$.

But, by standard exact sequences, since $H^1(\mathcal{O}_A(-rH)) = 0$, we have $H^0(\Omega^1_Z) = H^0(\Omega^1_A)$, so there are no invariant forms.

The next examples we consider are a slight generalization of an example due to Kas, [Ka 2] , and we shall call the corresponding surfaces "generalized Kas surfaces".

Consider, for $i = 1,2$, $f_i : C_i \longrightarrow \Gamma_i$ a simple cyclic cover of degree n between complete smooth curves, i.e.

(2.5.) There is an invertible sheaf $\mathcal{L}_i$ on Γ_i and a divisor B_i consisting of distinct points s.t. $\mathcal{O}_{\Gamma_i}(B_i) \cong \mathcal{L}_i^n$; C_i is the subvariety of L_i , the line bundle whose sheaf of sections is $\mathcal{L}_i$, obtained by taking the n^{th} - root of the section defining B_i .

Clearly the group $\mu_n \cong \mathbb{Z}/n$ of n^{th} roots of unity acts on C_i , and $f_i : C_i \longrightarrow \Gamma_i$ is the quotient map.

Notice further that

$$(2.6.) \quad f_*\mathcal{O}_{C_i} = \mathcal{O}_{\Gamma_i} \oplus \mathcal{L}_i^{-1} \oplus \ldots \oplus \mathcal{L}_i^{-(n-1)} .$$

To adhere to our standard notation, we let $Z = C_1 \times C_2$ and we let μ_n act on Z by the (twisted) action

$$(2.7.) \quad \zeta(x,y) = (\zeta x, \zeta^{-1}y) .$$

It follows immediately that if $X = Z/\mu_n$, then the singularities of X are exactly R.D.P.'s of type A_{n-1} , and $p : Z \longrightarrow X$ is unramified in codimension 1.

As a preliminary computation, we notice that $(\Omega = \Omega^1$, for short)

$$(2.8.) \quad \Omega_{C_i} = f_i^*(\Omega_{\Gamma_i} \otimes \mathcal{L}_i^{n-1}) ,$$

$$(f_i)_* \Omega_{C_i} = \Omega_{\Gamma_i} \oplus (\Omega_{\Gamma_i} \otimes \mathcal{L}_i) \oplus \dots \oplus (\Omega_{\Gamma_i} \otimes \mathcal{L}_i^{n-1})$$

(with $\Omega_{\Gamma_i} \otimes \mathcal{L}_i^j$ being the eigensheaf corresponding to the character of μ_n, $\zeta \longrightarrow \zeta^{-j}$) .

$$(f_i)_* \Omega_{C_i}^{\otimes 2} = \Omega_{\Gamma_i}^{\otimes 2}(B_i) \otimes \mathcal{L}_i^{-1} \oplus \Omega_{\Gamma_i}^{\otimes 2}(B_i) \oplus \dots .$$

$$\dots \oplus \Omega_{\Gamma_i}^{\otimes 2}(B_i) \otimes \mathcal{L}_i^{n-2} ,$$

where $\Omega_{\Gamma_i}^{\otimes 2}(B_i) \otimes \mathcal{L}_i^j$ is the $(\zeta \longrightarrow \zeta^{-j})$ eigensheaf.

We apply now the Künneth formula to $Z = C_1 \times C_2$, keeping in mind that the action is twisted on the second factor, to compute the G-invariant sections of $\Omega_Z^1 \otimes \Omega_Z^2$ (where we denote by the same symbol a sheaf on C_i and its pull-back to Z)

$$(2.9.) \quad H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G = H^0((\Omega_{C_1}^{\otimes 2} \otimes \Omega_{C_2}) \oplus (\Omega_{C_1} \otimes \Omega_{C_2}^{\otimes 2}))^G =$$

$$= (H^0(\Omega_{\Gamma_1}^{\otimes 2}(B_1) \otimes \mathcal{L}_1^{-1}) \otimes H^0(\Omega_{\Gamma_2} \otimes \mathcal{L}_2^{n-1})) \oplus \left(\bigoplus_{i=0}^{n-2} \left(H^0\left(\Omega_{\Gamma_1}^{\otimes 2}(B_1) \otimes \mathcal{L}_1^j\right) \otimes\right.\right.$$

$\otimes H^0(\Omega_{\Gamma_2} \otimes \mathcal{L}_2^j))) \oplus \dots$ (symmetrical expression interchanging the roles of Γ_1, Γ_2) .

It is well known (e.g. since the Kodaira-Spencer map is onto for the family of deformations of Z of the form $C_1' \times C_2'$) that Def(Z) is smooth.

In order to check that $H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G \twoheadrightarrow T_X^*$, we set

$$B_1 = b_1' + \dots b_{r_1}'$$

$$B_2 = b_1'' + \dots b_{r_2}''$$

and we choose local coordinates (x,y) on Z vanishing at (b_i', b_j'') . Then the first summand of (2.9.), locally at (b_i', b_j'') contributes expressions of type

$$(2.10.) \quad (a_{-1}x^{n-1}dx^2 \cdot \alpha_{n-1}dy) + (a_0 x^{n-2}dx^2 \alpha_0 y^{n-1}dy) + \\ + \dots (a_j x^{n-2-j}dx^2 \alpha_j y^{n-1-j}dy) + \dots$$

where $a_h = a_h(x^n)$, $\alpha_k = \alpha_k(y^n)$, i.e. the a_h's are pull-backs of local functions on Γ_1 , and similarly for the α_k's .

We have the following lemma, whose proof is straight-forward.

Lemma 2.11. By the quotient map $\begin{cases} x \longmapsto \zeta x \\ y \longmapsto \zeta^{-1} y \end{cases}$, with ζ a generator of μ_n , the quotient $p_*\Omega_Z^1/\Omega_X^1$ is locally generated by the expressions $x^i y^{i+1}dx - y^i x^{i+1}dy$ (for $i = 0, \dots. n-2$) .

Theorem 2.12. Let S be a generalized Kas surface as above of degree n . Then $\mathrm{Def}(S) = \mathrm{Def}(\Gamma_1) \times \mathrm{Def}(\Gamma_2) \times R$ where R is a

1-point nilpotent scheme of length $(n-1)\times(r_1\times r_2)$ if $r_i=\deg(B_i)\le g_i=\text{genus}(\Gamma_i)$ and the branch points are general.

Corollary 2.13. If $\nu = h^1(\theta_S)/\dim \mathrm{Def}(S)$ is the ratio between tangent dimension and dimension of $\mathrm{Def}(S)$ there do exist generalized Kas-surfaces with ν arbitrarily large.

Proof. If S has degree n, $\deg(B_i)=g_i-1$, then $\nu(S) =$ $= (n-1)(g_1-1)(g_2-1)/3[(g_1-1)+(g_2-1)]$. □

Proof of 2.12. By lemma 2.11. it suffices to verify that, for each $h = 0, \ldots, n-2$, we have 2 surjective maps,

$$(2.14.)\qquad H^0(\Omega_{\Gamma_1}^{\otimes 2}(B_1)\otimes \mathcal{L}_1^h)\otimes(\Omega_{\Gamma_2}\otimes\mathcal{L}_2^h) \twoheadrightarrow \bigoplus_{\substack{i=1,..r_1\\ j=1,..r_2}} \mathbb{C}_{ij}$$

(given by $\bigoplus_{i,j}(\mathrm{val}_{b_i'}\otimes \mathrm{val}_{b_j''})$)

and its symmetrical.

Since $H^1(\Omega_{\Gamma_i}^{\otimes 2}\otimes\mathcal{L}_i^h)=0$, we have a surjection $H^0(\Omega_{\Gamma_1}^{\otimes 2}(B_1)\otimes\mathcal{L}_1^h)\twoheadrightarrow \bigoplus_{i=1,..r_1}\mathbb{C}_{b_i'}$, while $H^0(\Omega_{\Gamma_2}\otimes\mathcal{L}_2^h)\twoheadrightarrow$ $\twoheadrightarrow \bigoplus_{j=1,.\Omega_2}\mathbb{C}_{b_j''}$ if and only if $H^1(\Omega_{\Gamma_2}(-B_2)\otimes\mathcal{L}_2^h)$ injects into $H^1(\Omega_{\Gamma_2}\otimes\mathcal{L}_2^h)$.

By duality there must be a surjection $H^0(\mathcal{L}_2^{-h})\twoheadrightarrow H^0(\mathcal{L}_2^{n-h})$,

and this holds $\Longleftrightarrow H^0(\mathcal{L}_2^i) = \begin{cases}0 & \text{for } i<n\\ 1 & \text{for } i=n\end{cases}$.

This condition and the symmetrical one hold if $\deg(B_i) \leq$ $\leq \text{genus}(\Gamma_i)$ and the branch points are general, by an easy argument on the Jacobian of Γ_i .

Q.E.D.

The following is, instead, an example where the moduli space is only singular. Its purpose is to illustrate the following feature: though small deformations of hypersurfaces in $\mathbb{P}^3$ are still hypersurfaces in $\mathbb{P}^3$ (cf. [K-S], [Se]), the same is not true for hypersurfaces of a weighted projective space $\mathbb{P}$ if $\mathbb{P}$ has not isolated singularities.

In fact (as it happens in the examples by Horikawa and Miranda cf. § 3) when all the deformations are surfaces in $\mathbb{P}$, and the surfaces have to pass through the (rigid) singular points of the weighted projective space, then the deformation space becomes everywhere non reduced.

Example 2.15. Let X be a general member of the family of hypersurfaces of degree 8 in $\mathbb{P} = \mathbb{P}(1,1,2,2)$. X has 4 quadratic ordinary singular points, hence the tangent codimension of this family in the Deformation space is at least 4.

But $\omega_X = \mathcal{O}_X(2)$, so that the canonical map of X embeds it as the complete intersection $G_4 \cap Q_2'$ in $\mathbb{P}^4$ where G_4 is a quartic hypersurface in $\mathbb{P}^4$ and Q_2' is a quadric of rank 3. It is easy to see that the complete intersections $G_4 \cap Q_2$ are all the deformations of X , and that the previous family has thus codimension equal to 3.

§ 3 Weighted hypersurfaces with everywhere non reduced moduli spaces.

This paragraph shows somehow that our situation is not "artificial", in fact the examples we shall consider in this section will be hypersurfaces X *of degree* d *in a weighted 3-dimensional projective space* $\mathbb{P}(1,1,p,q)$: *in these examples* S *shall be simply connected, we shall get all the* A_n *singularities and* Def(S) *will be* e.n.r. *for all* d *satisfying certain congruences which guarantee that* X *is generically singular; the examples of Horikawa and Miranda will be two issues of a double infinite series of examples.*

Here $\mathbb{P}^3 = \mathrm{Proj}(\mathbb{C}[y_0,y_1,y_2,y_3])$, $p < q$ are relatively prime integers, $\mathbb{P} = \mathbb{P}(1,1,p,q) = \mathrm{Proj}(\mathbb{C}[x_0,x_1,x_2x_3])$ where $\deg x_0 = \deg x_1 = 1$, $\deg x_2 = p$, $\deg x_3 = q$.

We let $G = \mathbb{Z}/pq\,\mathbb{Z}$ and let $\zeta \in \mathbb{C}^*$ be a primitive $(pq)^{th}$ root of unity; G acts on $\mathbb{P}^3$ by

(3.1.) $(y_0,y_1,y_2,y_3) \longmapsto (y_0,y_1,\zeta^q y_2,\zeta^p y_3)$.

(3.2.) $\psi : \mathbb{P}^3 \longrightarrow \mathbb{P}$, given by $\psi(y_0,y_1,y_2,y_3) = (y_0,y_1,y_2^p,y_3^q)$ is such that $\mathbb{P}^3/G \overset{\psi}{\cong} \mathbb{P}$.

(3.3.) Let $f \in \mathbb{C}[x_0,x_1,x_2,x_3]_d$ define a hypersurface $X = \{f = 0\}$ of degree d in $\mathbb{P}$, and let $Z = \psi^{-1}(X)$

(thus Z is the locus of zeros of
$F(y) \overset{\text{def.}}{=} f(y_0, y_1, y_2^p, y_3^q))$.

Proposition 2.4. Assume $d \geq (p-1)(q-1)$, so that we can write d as $d = bp + aq$ with $a, b \geq 0$. Then, for a general choice of f , Z is smooth, X is singular but has only R.D.P.'s as singularities iff p,q,d are according to the following table, where $P_2 = (0,0,1,0)$, $P_3 = (0,0,0,1)$ are the singular points of $\mathbb{P}$.

Table 2.5.

$\mathbb{P}$	d	Singularity at P_2	Singularity at P_3
$\mathbb{P}(1,1,2,3)$	$d = 1 + 6k$	A_1	A_2
$\mathbb{P}(1,1,p,p+1)$	$d = p(k(p+1)-1)$	/	A_p
$\mathbb{P}(1,1,p,rp-1)$	$d = (kp-1)(rp-1)$	A_{p-1}	/

Proof. First of all let $Q_2 = \psi^{-1}(P_2)$, $Q_3 = \psi^{-1}(P_3)$. It is easily verified that the stabilizer of a point $y \in \mathbb{P}^3$ equals

i) G if $y = Q_2, Q_3$ or y belongs to the line $L = \{y_2 = y_3 = 0\}$
ii) $\mathbb{Z}/p$ if $y_2 = 0$ and y is not as in i)
iii) $\mathbb{Z}/q$ if $y_3 = 0$ and y is not as in i).

Moreover, G_y is generated by pseudo reflections if $y \neq Q_2, Q_3$, so that in particular P_2, P_3 are the only singular points of $\mathbb{P}$. Since a pseudoreflection is characterized by the property of having a fixed locus of codimension 1, we see that if Z is smooth (and G-invariant), then X is smooth outside P_2, P_3.

Since $d = aq + bp$, by Bertini's theorem Z is smooth outside the locus $\{y_0^d = y_1^d = y_2^{bp} y_3^{aq} = 0\}$, i.e. outside Q_2 and Q_3, hence X too is smooth away of P_2, P_3.

Now, Z is smooth at Q_2 iff $f_d(y_0, y_1, 1, y_3^q)$ contains either a constant term, or a monomial of degree 1, i.e. iff either $d \equiv 0$ or $d \equiv 1 \pmod p$. Similarly Z is smooth at $Q_3 \Leftrightarrow d \equiv 0$ or $d \equiv 1 \pmod q$.

Assume now that Z contains Q_2 ($d \equiv 1 \pmod p$); there are then local coordinates (u,v) on Z at Q_2 with G acting by $(u,v) \longmapsto (\zeta^{-q} u, \zeta^{p-q} v)$. We have $\mathbb{C}\{u,v\} \supset \mathbb{C}\{u,v\}^{<\zeta^p>} = \mathbb{C}\{u,v^q\}$: hence, if we set $w = v^q$, $\varepsilon = \zeta^{-q}$, $\mathbb{C}\{u,v\}^G = \mathbb{C}\{u,w\}^{\mathbb{Z}/p}$, where $\mathbb{Z}/p$ acts by $u \longmapsto \varepsilon u$, $w \longmapsto \varepsilon^q w$.

It is well-known that the quotient is a R.D.P. iff ω_X is invertible, i.e. the determinant of the transformation is 1, i.e. $q + 1 \equiv 0 \pmod p$; then one has a singularity of type A_{p-1}.

We proceed in an entirely similar fashion for Q_3, P_3

finding that Z is smooth (resp. empty) at Q_3 , X has a R.D.P. at P_3 (in fact, a singular point of type A_{q-1}) iff $d \equiv 1 \pmod q$ and $p+1 \equiv 0 \pmod q$ (resp.: $d \equiv 0(q)$) .

We have thus four cases: but, if $d \equiv 0 \pmod q$, $d \equiv 0 \pmod p$, then X is smooth, and this case has to be excluded.

If $p+1 \equiv 0 (\mathrm{mod}\ q)$, since $p < q$, then necessarily $q = p+1$.

Clearly then $q+1 \equiv 0 (\mathrm{mod}\ p) \leftrightarrow 2 \equiv 0 (\mathrm{mod}\ p)$, i.e. $p = 2$, $q = 3$ or $p = 1$, $q = 2$.

We fall then into the cases 1), resp. 2) of the table. On the other hand, $q = p+1$, $d \equiv 0 (\mathrm{mod}\ p)$ $d \equiv 1 (\mathrm{mod}\ q)$ iff there exists m such that $d = mp$, and $mp \equiv 1 (\mathrm{mod}\ p+1)$; but $mp \equiv -m (\mathrm{mod}\ p+1)$, hence it must be $m+1 \equiv 0(p+1)$, i.e. we are in case 2) of the table.

Finally, if $q = rp-1$, $d \equiv 0 (\mathrm{mod}\ q)$, $d \equiv 1 (\mathrm{mod}\ p)$ means that $d = m(rp-1)$, and $m(rp-1) \equiv 1 (\mathrm{mod}\ p)$, i.e. $m+1 \equiv 0 (\mathrm{mod}\ p)$, i.e. $m = kp-1$, as in case 3) of the table.

Q.E.D.

Remark 3.6. These varieties X are Weil subcanonical, i.e. $\mathcal{O}(1)$ is a divisorial sheaf (cf. [Re]) associated to a Weil divisor of which a multiple t is a canonical divisor; the subindex i of X is the minimal i such that $\mathcal{O}(i)$ is invertible $(i|t)$.

In the 3 cases of table 2.5. $t = 6(k-1)$, $(p+1)(kp-3)$, $p[r(kp-1)-k-r-1]$, $i = 6$, $p+1$, p respectively.

Theorem 3.7. *Let* $X \subset \mathbb{P}(1,1,p,q) = \mathbb{P}$ *be a general hypersurface as in table 2.5. of degree* d *s.t.* X *is of general type, i.e.* $d > 2+p+q$ *and* $r > p-2$. *If* S *is a minimal resolution of* X , *then every small deformation of* S *is the resolution of a hypersurface of degree* d *in* $\mathbb{P}$, *and* $\mathrm{Def}(S) = \mathrm{ESDef}(X) \times R$, *where* $\mathrm{ESDef}(X)$ *is smooth and* R *is a 1-point nilpotent scheme of length equal to the sum of the Milnor numbers of the singularities.*

Proof. Of course we are going to use theorem 1.16, part ii). It is well known that $\mathrm{Def}(Z)$ is smooth (cf. [K-S] [Se]), and we have to verify the surjections

$$H^1(\theta_Z)^G \longrightarrow\!\!\!\!> H^1(\theta_X)$$

$$H^0(\Omega^1_Z \otimes p^*\omega_X)^G \longrightarrow\!\!\!\!> H^0((p_*\Omega^1_Z)^G/\Omega^1_X \otimes \omega_X) = T^*_X .$$

To verify the first, we consider the exact sequence 1.18. Here $B = B_2 \cup B_3$, where B_i is defined by the equation $x_i = 0$; B_2 , B_3 are smooth and intersect transversally in d points. (The only exception being the II case with $p = 1$, where $B = B_3$) .

Lemma 3.8. $N'_B = N'_{B_2} \oplus N'_{B_3}$.

Proof. At a point p where B_2 , B_3 meet, (x_2,x_3) are local parameters. The vector fields in $\theta_X(-\log B)$ are of the form $a(x_2,x_3)\cdot x_2\partial/\partial x_2 + b(x_2,x_3)x_3\frac{\partial}{\partial x_3}$, hence the cokernel N_B' has local sections of the form $\alpha(x_3)\partial/\partial x_2 + \beta(x_2)\partial/\partial x_3$.

Q.E.D.

Lemma 3.9. If B_3 does not contain P_2, then $N_{B_3}' \cong \mathcal{O}_{B_3}(q)$, otherwise $N_{B_3}' \cong \mathcal{O}_{B_3}(q) \otimes \mathcal{O}_{B_3}(-(p-1)P_2)$.

Proof. The first assertion is trivial. To prove the second statement, cf. the proof of 3.4., we may assume (after a linear change of coordinates) that $u = y_0/y_2$, $v = y_3/y_2$ are local coordinates for Z at Q_2. We let M be the intermediate quotient $Z/(\mathbb{Z}/q)$: hence $u = y_0/y_2$, $w = v^q = x_3/y_2^q$ are local coordinates on M. M is (locally) smooth and the quotient map $M \to X$ is unramified in codimension 1, so that

$$(3.10)\qquad \begin{cases} \theta_X = \theta_M^{\mathbb{Z}/p} \\ \theta_Z^G = \theta_M(-\log \widetilde{B})^{\mathbb{Z}/p} \end{cases}$$

where $\widetilde{B} = \{w = 0\}$.

We claim first of all that N_{B_3}' is invertible, and we are going to exhibit a local generator around P_2. We recall that the local ring $\mathcal{O}_{X,P_2}$ is generated by $x = u^p$, $y = w^p$, $z = uw$, subject to the relation $xy = z^p$.

The sections of θ_X, by 3.10., are of the form (notice that $\mathbb{Z}/p$ acts by $(u,w) \longmapsto (\varepsilon u, \varepsilon^{-1} w)$ where ε is a primitive p^{th} root of unity)

$$(a(x,y,z)w + \alpha(x,y,z)u^{p-1})\,\partial/\partial w + (b(x,y,z)u + \beta(x,y,z)w^{p-1})\,\partial/\partial u$$

whereas the sections of θ_Z^G are of the form

$$a(x,y,z)w\ \partial/\partial w + (b(x,y,z)u + \beta(x,y,z)w^{p-1})\,\partial/\partial u\ .$$

Hence the elements in the quotient N'_{B_3} can locally be written uniquely in the form

$$(3.11)\qquad u^{p-1}\partial/\partial w \cdot \tilde{\alpha}(x)\ .$$

We can also rewrite

$$u^{p-1}\partial/\partial w = x \cdot u^{-1}\partial/\partial w = xu^{-1}(\partial/\partial y\ pw^{p-1} + \partial/\partial z \cdot u) = x\ \partial/\partial z + {} + pz^{p-1}\partial/\partial y$$

so a local generator of N'_{B_3} is

$$(3.12)\qquad u^{p-1}\partial/\partial w = x\ \partial/\partial z + pz^{p-1}\partial/\partial y$$

in a neighbourhood V_2 of P_2.

We let V_i be the open set where $x_i \neq 0$, for $i = 0,1$. Then V_0 , V_1 , V_2 is an open cover of B_3 . On V_0 we have coordinates (for $\mathbb{P}$)

$$(3.13) \qquad \eta = x_1/x_0 \quad \xi = x_2/x_0^p , \quad \zeta = x_3/x_0^q$$

whereas

$$(3.14) \qquad x = x_o^p/x_2 , \quad y = x_3^p/x_2^q , \quad z = \frac{x_0x_3}{x_2^{(q+1/p)}} = \frac{x_0x_3}{x_2^r}$$

(where, cf. table 2.5., $r = 2$ in the first case).

Clearly $\partial/\partial\zeta$ is a local generator for N'_{B_3} on V_0 and since $x = \frac{1}{\xi}$, $y = \zeta^p/\xi^q$, $z = \zeta/\xi^r$, we have $\frac{\partial}{\partial\zeta} = p\,\frac{\zeta^{p-1}}{\xi^q}\,\frac{\partial}{\partial y} + \xi^{-r}\,\frac{\partial}{\partial z} = pz^{p-1}\xi^{1-r}\,\frac{\partial}{\partial y} + x\xi^{1-r}\,\frac{\partial}{\partial z} = \xi^{1-r}\Big(x\,\frac{\partial}{\partial z} + pz^{p-1}\,\frac{\partial}{\partial y}\Big)$. On V_1 we have coordinates $a = x_0/x_1$, $b = x_2/x_1^p$, $c = x_3/x_1^q$, such that $a = 1/\eta$, $b = \xi\eta^{-p}$, $c = \zeta\eta^{-q}$.

A local generator for N'_{B_3} is $\partial/\partial c$, and $\partial/\partial\zeta = \eta^{-q}\partial/\partial c$.

Hence a section of N'_{B_3} is given by functions A_0, A_1, A_2 on $B_3 \cap V_i$ s.t.

$$A_0\ \partial/\partial\zeta = A_2(x\ \partial/\partial z + pz^{p-1}\partial/\partial y) = A_1\partial/\partial c .$$

Hence $A_0 \cdot \xi^{1-r} = A_2$, $A_0 \eta^{-q} = A_1$, i.e.

$$A_2 = A_0 \cdot \frac{x_2^{1-r}}{x_0^{p(1-r)}} \longleftrightarrow A_0 x_0^{q+1-p} = A_2 x_2^{r-1} \quad \text{and} \quad A_0 x_0^q = A_1 x_1^q .$$

Setting $\tilde{A}_0 = A_0$, $\tilde{A}_2 = A_2$, $\tilde{A}_1 = A_1 \cdot \left(\frac{x_1}{x_0}\right)^{p-1}$, we obtain a rational section of $\mathcal{O}_{B_3}(q-(p-1))$. Therefore $N'_{B_3} \cong \mathcal{O}_{B_3}(q-(p-1)) \otimes \mathcal{O}_{B_3}((p-1)D)$ where D is the divisor of zeros of $\left(\frac{x_0}{x_1}\right)$ in V_1 . On the other hand the section x_0 vanishes on $V_1 \cap B_3$ giving D as divisor, and on P_2 with multiplicity 1, hence $\mathcal{O}_{B_3}(D) = \mathcal{O}_{B_3}(1) \otimes \mathcal{O}_{B_3}(-P_2)$.

Q.E.D.

Analogously we have (assuming $p \geq 2$)

(3.15) $$N'_{B_2} \text{ equals } \begin{cases} \mathcal{O}_{B_2}(p) & \text{if } B_2 \not\ni P_3 \\ \mathcal{O}_{B_2}(p) \otimes \mathcal{O}_{B_2}(-pP_3) & \\ \text{if } B_2 \ni P_3 (\text{here } q-1=p) . & \end{cases}$$

To finish the proof, we shall apply lemma 1.19, and we shall show by explicit computations that

(3.16) $$h^0(\Omega^1_Z \otimes \Omega^2_Z)^G - h^0(\Omega^1_Z \otimes p^*\omega_X)^G = h^1(N'_{B_2}) + h^1(N'_{B_3}) \quad (h^1(N'_{B_3}) \text{ if } p = 1).$$

We are first going to compute the left hand side, and in fact we shall show that the following holds true:

(3.17.) Claim: The left hand side of (3.16) equals $h^0(\mathcal{O}_{\mathbb{P}(1,1,p)}(d-p-q-2)) + h^0(\mathcal{O}_{\mathbb{P}(1,1,q)}(d-p-q-2))$ if $p \geq 2$, and equals $h^0(\mathcal{O}_{\mathbb{P}^2}(d-3-q))$ if $p = 1$.

Proof of the claim: The exact sequence

$$(3.18) \qquad 0 \longrightarrow \mathcal{O}_Z(-d+i) \longrightarrow \Omega^1_{\mathbb{P}^3|Z}(i) \longrightarrow \Omega^1_Z(i) \longrightarrow 0$$

shows that for $i < d$, $H^0(\Omega^1_Z(i)) \cong H^0(\Omega^1_{\mathbb{P}^3|Z}(i))$.

By the exact sequence

$$(3.19) \qquad 0 \longrightarrow \Omega^1_{\mathbb{P}^3}(i) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(i-1)^4 \xrightarrow{(y_j)} \mathcal{O}_{\mathbb{P}^3}(i) \longrightarrow 0$$

we get

$$(3.20) \qquad H^0(\Omega^1_Z(i)) \cong [\ker H^0(\mathcal{O}_Z(i-1))^4 \xrightarrow{(y_j)} H^0(\mathcal{O}_Z(i))] =$$

$$= \ker(A_{i-1})^4 \xrightarrow{(y_j)} A_i \ , \text{ where } A_j = H^0(\mathcal{O}_{\mathbb{P}^3}(j)) \ .$$

Then, by the Koszul exact sequence for the regular sequence $(y_0, \dots y_3)$ in $A = \bigoplus_{i \geq 0} A_j$, we get $H^0(\Omega^1_Z(h)) =$

$= \{\Sigma\, u_{ij}(y)\eta_{ij} \mid \eta_{ij} = y_i dy_j - y_j dy_i \ , \ u_{ij} \in A_{h-2}\}$.

Now, the terms in the exact sequence (3.20) are G-modules, therefore we have to check whether the induced G-module structure on $H^0(\Omega^1_Z(i))$ coincides for $i = d - \lambda - p - q$, resp. $i = d - 4$

with the natural ones on $H^0(\Omega_Z^1 \otimes p^*\omega_X)$, resp. $H^0(\Omega_Z^1 \otimes \Omega_Z^2)$.

To do this we notice that the G-module structure on A_i linearizes the $\mathcal{O}_{\mathbb{P}^3}(i)$'s and we rewrite (3.19) as

$$0 \longrightarrow \Omega^1_{\mathbb{P}^3}(i) \longrightarrow (A_1)^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(i-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(i) \longrightarrow 0 .$$

By which we get $(i = 0)$ $\Omega^3_{\mathbb{P}^3} \cong (\det A_1)^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4)$; then 3.18 gives $\Omega_Z^2 \cong \Omega^3_{\mathbb{P}^3} \otimes \mathcal{O}_Z(d) \cong (\det A_1)^\vee \otimes \mathcal{O}_Z(d-4)$. On the other hand, we have an inclusion of sheaves, induced by multiplication by a constant times $y_2^{p-1}\ y_3^{q-1}$,

(3.21)
$$\begin{array}{ccccc} 0 \longrightarrow & \Omega_Z^1 \otimes p^*\omega_X & \longrightarrow & \Omega_Z^1 \otimes \Omega_Z^2 \\ & \wr\| & & \wr\| \\ & \Omega_Z^1(d-p-q-2) & & \Omega_Z^1(d-4) \end{array}$$

since $(\det A_1)^\vee$ corresponds to the character $\zeta \longmapsto \zeta^{-p-q}$, for which $y_2^{p-1}y_3^{q-1}$ is an eigenvector, it follows that

(3.22)
$$\Omega_Z^1 \otimes p^*(\omega_X) \cong \Omega_Z^1 \cdot (d-p-q-2)$$
$$\Omega_Z^1 \otimes \Omega_Z^2 \cong (\det A_1)^\vee\ \Omega_Z^1(d-4) .$$

It follows that, if we define the weight w of a monomial $y_0^{j_0} y_1^{j_1} y_2^{j_2} y_3^{j_3}$ as the element w of $\frac{1}{pq}\mathbb{Z}/\mathbb{Z}$ given by

$\frac{j_2}{p} + \frac{j_3}{q}$, and we let A_i^w be the subspace of monomials of degree i , weight w , set $A^w = \bigoplus_i A_i^w$, then

$$H^0(\Omega_Z^1 \otimes p^*\omega_X)^G = \{\Sigma\, u_{ij}(y)\eta_{ij} \mid u_{ij} \in A_{d-4-p-q} \text{ and } w(u_{ij}) + w(\eta_{ij}) = 0\}$$

$$H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G = \{\Sigma\, v_{ij}(y)\eta_{ij} \mid v_{ij} \in A_{d-6},\ w(v_{ij}) + w(\eta_{ij}) = -\frac{1}{q} - \frac{1}{p}\}$$

We let μ' be the linear map induced by (3.21), which is clearly injective; then the left hand side of 3.16 is just the dimension of $\operatorname{coker}(\mu')$.

Since μ' is induced by a map $\mu : (A_{d-4-p-q})^6 \longrightarrow (A_{d-6})^6$, (multiplication by $y_2^{p-1}y_3^{q-1}$) where these spaces are viewed as G-modules according to (3.22), it will suffice to identify $W = H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G / H^0(\Omega_Z^1 \otimes p^*\omega_X)^G$ with a quotient of a supplementary space V to $\operatorname{Im}\mu$ inside $((A_{d-6})^6)^G$.

Recall that $H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G = \{\sum_{i<j} v_{ij}\eta_{ij} \mid v_{01} \in A^{-\frac{1}{p}-\frac{1}{q}}$ $v_{02}, v_{12} \in A^{-\frac{2}{p}-\frac{1}{q}}$, $v_{03}, v_{13} \in A^{-\frac{1}{p}-\frac{2}{q}}$, $v_{23} \in A^{-\frac{2}{p}-\frac{2}{q}}\}$.

The above properties imply, since $(p,q) = 1$

$$(3.23)\quad \begin{cases} v_{01} \text{ is divisible by } y_2^{p-1}y_3^{q-1} \\ v_{02}, v_{12} \text{ are divisible by } y_2^{p-2}y_3^{q-1} \text{ if } p \geq 2 \\ \qquad\qquad (\text{by } y_3^{q-1} \text{ if } p = 1) \\ v_{03}, v_{13} \text{ are divisible by } y_2^{p-1}y_3^{q-2} \\ v_{23} \text{ by } y_2^{p-2}y_3^{q-2} \text{ if } p \geq 2 \ (\text{by } y_3^{q-2} \text{ if } p = 1) . \end{cases}$$

Therefore we can choose the space V to consist of the elements such that

$$(3.24)\quad \Sigma\, v_{ij}\eta_{ij} \in V \Leftrightarrow v_{01} = 0 \text{ , and } (\text{for } i = 0,1)$$

$$v_{i2} = w_{i2}y_2^{p-2}y_3^{q-1} \text{ if } p \geq 2 \ (v_{i2} = 0$$

$$\text{if } p = 1) \text{ with } w_{i2} = w_{i2}(y_0, y_1, y_3)$$

$$v_{i3} = w_{i3}y_2^{p-1}y_3^{q-2} \text{ with } w_{i3} = w_{i3}(y_0, y_1, y_2)$$

$$v_{23} = w_{23}y_2^{p-2}y_3^{q-2} \text{ if } p \geq 2 \text{ with } w_{23}$$

not containing monomials divisible by y_2y_3

$$(\text{if } p = 1 \ v_{23} = w_{23}y_3^{q-2} ,$$

$$w_{23} = w_{23}(y_0, y_1, y_2) .$$

Now, a given element $\Sigma\, v_{ij}\eta_{ij}$ in V maps to 0 in W (defined above) iff there do exist u_{ij} as in (3.22) s.t.

$$\Sigma\, v_{ij}\eta_{ij} = y_2^{p-1} y_3^{q-1}\, \Sigma\, u_{ij}\eta_{ij} \ .$$

We can write the above equation as 4 equations, corresponding to the respective coefficients of $dy_0, \ldots, dy_3$.

We must have $0 = \sum_{i<j} (v_{ij} - y_2^{p-1} y_3^{q-1} u_{ij})(y_i dy_j - y_j dy_i) =$

$= \sum_{i\neq j} (v_{ij} - y_2^{p-1} y_3^{q-1} u_{ij}) y_i dy_j$ (if we agree that $v_{ji} = - v_{ij}$, $u_{ji} = - u_{ij}$ for $j > i$) .

Thus, for $j = 0, \ldots 3$ $\sum_{i\neq j} (y_i v_{ij} - y_2^{p-1} y_3^{q-1} y_i u_{ij}) = 0$.
We can write the above, if $p \geq 2$, as

$$(3.25)\quad \begin{cases} y_2^{p-1} y_3^{q-1} (w_{02}(y_0,y_1,y_3) - y_2 u_{02} + w_{03}(y_0,y_1,y_2) - y_3 u_{03} \\ \qquad\qquad - y_1 u_{01}) = 0 \\ y_2^{p-1} y_3^{q-1} (w_{12} - y_2 u_{12} + w_{13} - y_3 u_{13} + y_0 u_{01}) = 0 \\ y_2^{p-2} y_3^{q-2} (y_0 w_{02} + y_1 w_{12} - y_2 y_0 u_{02} - y_2 y_1 u_{12} \\ \qquad\qquad + w_{32} - y_2 y_3 u_{32}) = 0 \\ y_2^{p-1} y_3^{q-2} (y_0 w_{03} + y_1 w_{13} + w_{23} - y_3 y_0 u_{03} - y_3 y_1 u_{13} - y_2 y_3 u_{23}) = 0 \end{cases}$$

The above system is equivalent to the one obtained by equating to zero the terms in the brackets: but then one can observe that (I being the first new equation and so on ...) $y_0 I + y_1 II = III + IV$, hence the IV equation can be discarded.

We write the w_{ij} as Taylor series in y_2 , y_3 , thus we set

$$w_{i2} = \bar{w}_{i2} + y_3 w''_{i2}$$
$$w_{i3} = \bar{w}_{i3} + y_2 w'_{i3}$$
$$w_{23} = \bar{w}_{23} + y_2 w''_{23} + y_3 w'_{23}$$

where $\bar{w}_{ij} \in \mathbb{C}[y_0, y_1]$, $w''_{ij} \in \mathbb{C}[y_0, y_1, y_3]$, $w'_{ij} \in \mathbb{C}[y_0, y_1, y_2]$.

Assume that our element in V is mapped to 0 inside W : then we have

$$(3.26)\quad \begin{cases} \bar{w}_{02} + \bar{w}_{03} = y_1 \bar{u}_{01} \\ \bar{w}_{12} + \bar{w}_{13} = -y_0 \bar{u}_{01} \\ w''_{02} \equiv u_{03} \pmod{y_2} \quad \text{and} \quad w_{23} = y_0 w_{02} + y_1 w_{12} - y_2 y_0 w'_{03} - \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad - y_2 y_1 w'_{13} \\ w'_{03} \equiv u_{02} \pmod{y_3} \\ w'_{13} \equiv u_{12} \pmod{y_3} \\ w''_{12} \equiv u_{13} \pmod{y_2} \end{cases}$$

Conversely, if (3.26) holds, it suffices to choose $u_{01} = \bar{u}_{01}$ $w''_{02} = u_{03}$, $w'_{03} = u_{02}$, $w'_{13} = u_{12}$, $w''_{12} = u_{13}$, $u_{23} = 0$.

Therefore the kernel of $V \twoheadrightarrow W$ is the same as the kernel of the mapping associating to an element of V the pair

$$y_0(\overline{w_{02}} + \overline{w_{03}}) + y_1(\overline{w_{12}} + \overline{w_{13}}) \ ,$$
$$w_{23} - y_0 w_{02} - y_1 w_{12} + y_0 y_2 w'_{03} + y_1 y_2 w'_{13} \ .$$

We conclude that $\dim W = \dim\{w_{23} \in A^G_{d-2-p-q} \mid w_{23}$ does not contain monomials divisible by $y_2 y_3\} + \dim\{\overline{w_{ij}} \in A^G_{d-p-q-2}\} =$
$= \dim\{w' \in A^G_{d-2-p-q} \cap \mathbb{C}[y_0, y_1, y_2]\} +$
$+ \dim\{w'' \in A^G_{d-p-q-2} \cap \mathbb{C}[y_0, y_1, y_3]\} = h^0 \mathcal{O}_{\mathbb{P}(1,1,p)}(d-p-q-2) +$

$+ h^0 \mathcal{O}_{\mathbb{P}(1,1,q)}(d-p-q-2)$.

Whereas, for $p = 1$, only v_{j3} is $\neq 0$, and $v_{j3} = y_3^{q-2} w_{j3}$ with $w_{j3} = w_{j3}(y_0, y_1, y_2)$.

We find then that the element in V maps to 0 into W if

$$(w_{j3} - y_3 u_{j3}) = \sum_{h \neq j} y_h u_{hj}$$

$$\sum_{j \neq 3} y_j (w_{j3} - y_3 u_{j3}) = 0 \quad .$$

We claim then that $\Sigma\, y_j w_{j3} = 0$ in $\mathbb{C}[y_0, y_1, y_2]$ is a necessary and sufficient condition since then by exactness of the Koszul sequence there exist $u_h\ (y_0, y_1, y_2)$ s.t. $w_{j3} = \sum_{h \neq j} y_h u_{hj}$, and then we choose $u_{j3} = 0$.

Thus here $\dim W = h^0(\mathcal{O}_{\mathbb{P}^2}(d-3-q)) = \binom{d-1-q}{2}$.

<u>Q.E.D. for the claim</u>

To deal with the right hand side of (3.16), we use Serre duality and the fact that (cf. [Do], 3.5.2.)

$$\omega_{B_2} = \mathcal{O}_{B_2}(d-q-2) \ , \quad \omega_{B_3} = \mathcal{O}_{B_3}(d-p-2) \ .$$

Then, setting $\delta_2 = 0$ if $B_2 \not\ni P_3$, $\delta_2 = p = q-1$ if $P_3 \in B_2$, $\delta_3 = 0$ if $B_3 \not\ni P_2$, $\delta_3 = p-1$ otherwise, we have

(3.27) $h^1(N'_{B_2}) = h^0(\mathrm{Hom}(\mathcal{O}_{B_2}(p)(-\delta_2 P_3)\ ,\ \mathcal{O}_{B_2}(d-q-2))) =$
$= h^0(\mathcal{O}_{B_2}(d-p-q-2)(\delta_2 P_3))$, and similarly $h^1(N'_{B_3}) =$
$= h^0(\mathcal{O}_{B_3}(d-p-q-2)(\delta_3 P_2))$.

To end the proof, in view of 3.17 and 3.27, it suffices to prove that for our choices of d , p , q ,

(3.28) $h^0(\mathcal{O}_{B_2}(d-p-q-2)(\delta_2 P_3)) = h^0(\mathcal{O}_{\mathbb{P}(1,1,q)}(d-p-q-2))$
if $p > 1$ and
$h^0(\mathcal{O}_{B_3}(d-p-q-2)(\delta_3 P_2)) = h^0(\mathcal{O}_{\mathbb{P}(1,1,p)}(d-p-q-2))$.

Since $B_3 \subset \mathbb{P}(1,1,q)$ and symmetrically $B_2 \subset \mathbb{P}(1,1,p)$, we are considering the following problem:

(3.29) $B \subset \mathbb{P}(1,1,p) = \mathbb{P}$ has degree d , P is the singular point of $\mathbb{P}$, $\delta = 0$ if $B \not\ni P$, $\delta = (p-1)$ otherwise, i is an integer > 0 : is there an equality

$$h^0(\mathcal{O}_{\mathbb{P}}(d-i)) = h^0(\mathcal{O}_B(d-i)(\delta P) \ ?$$

(3.30) The answer is positive if $\delta = 0$, since we have the long exact cohomology sequence

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}}(d-i)) \longrightarrow H^0(\mathcal{O}_B(d-i)) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}}(-i)) = 0 \ .$$

The following lemma follows from a more general result proven in the Appendix.

<u>Lemma 3.31</u>. Let $\Pi : \mathbb{F} = \mathbb{F}_p \longrightarrow \mathbb{P} = \mathbb{P}(1,1,p)$ be the blow up of the singular point P of $\mathbb{P}$, and let E_∞ be the exceptional divisor. Let B be a smooth curve of $\mathbb{P}$, and identify B to its proper transform in the Segre-Hirzebruch surface $\mathbb{F} = \mathbb{F}_p$. Then if E_0 is the curve defined by $\Pi^{-1}(x_2 = 0)$, and F is a fiber of $\mathbb{F}$ (proper transform of $x_0 = 0$), we have: if r is a positive integer and $r = r' + r''p$, with $r' < p$, then $\mathcal{O}_B(r) = \mathcal{O}_B(r'F + r''E_0)$, and $H^0(\mathcal{O}_{\mathbb{P}}(r)) = H^0(\mathcal{O}_{\mathbb{F}}(r'F + r''E_0))$.

Moreover $\mathcal{O}_B(\delta P) = \mathcal{O}_B(\delta E_\infty)$ (recall that $E_0 \equiv E_\infty + pF$).

Assume now we are in the case when $B \ni P$, and $(d - i)$ (cf. 3.29) equals $t'p$ (as can be checked), while $d = tp + 1$ (so $t > t'$).

It is then easy to see that $B \equiv tE_0 + F$, and, in view of 3.31 and since $H^0(\mathcal{O}_{\mathbb{P}}(t'p)) = H^0(\mathcal{O}_{\mathbb{F}}(t'E_0)) = H^0(\mathcal{O}_{\mathbb{F}}(t'E_0 + (p-1)E_\infty)$, we are looking at the surjectivity of the map (res) in the following exact sequence on $\mathbb{F}$:

$$(3.32)\qquad H^0(\mathcal{O}_{\mathbb{F}}(t'E_0 + (p-1)E_\infty)) \xrightarrow{\text{res}} H^0(\mathcal{O}_B(t'E_0 + (p-1)E_\infty) \longrightarrow$$

$$\longrightarrow H^1(\mathcal{O}_{\mathbb{F}}(((p-1) + t' - t)E_0 - (p(p-1)+1)F) \xrightarrow{j}$$

$$\xrightarrow{j} H^1(\mathcal{O}_{\mathbb{F}}((t' + p - 1)E_0 - p(p-1)F) .$$

This is equivalent to requiring the injectivity of j.

Set $b = (p-1) - (t-t')$: if $b \leq -1$ the domain of j is a vanishing cohomology group and we are done (cf., for the vanishing, the table at page 612 of [Kon]).

Assume instead $b \geq 0$, and set $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(p)$. Then the map j is the map j'

$$(3.32) \qquad H^1(\mathrm{Sym}^b V \otimes \mathcal{O}(-p(p-1)-1)) \xrightarrow{j'} H^1(\mathrm{Sym}^{b+t} V \otimes \mathcal{O}(-p(p-1))$$

given by the equation of B .

I.e., if $f = \sum_{k=0}^{t} c_{ijk} y_0^i y_1^j y_2^{p(t-k)} = \sum_{\substack{k=0 \\ \deg f_k = kp+1}}^{t} f_k(y_0, y_1) y_2^{p(t-k)}$ is the equation of B , then, since $\mathrm{Sym}^b V = \bigoplus_{i=0}^{b} \mathcal{O}(ip)$ ($\mathcal{O}$ stands for $\mathcal{O}_{\mathbb{P}^1}$!) , $\psi_i \in H^1(\mathcal{O}(ip - p(p-1) - 1)$ is mapped under j' to $\sum_{k=0}^{t} f_k \psi_i$, with $f_k \psi_i \in H^1(\mathcal{O}((i+k)p - p(p-1))$.

On the other hand this last group is zero unless $i + k = h' \leq p-2$, and it is always better to deal with H^0's rather than H^1's , hence we dualize, and ask for the surjectivity of

$$(3.33) \qquad \begin{array}{ccc} \bigoplus_{h'=0}^{p-2} H^0(\mathcal{O}((p-1-h')p-2)) & \longrightarrow\!\!\!\!\to & \bigoplus_{i'=0}^{b} H^0(\mathcal{O}(p(p-1-i')-1)) \\ \| & & \| \\ \bigoplus_{h=1}^{p-1} H^0(\mathcal{O}(hp-2)) & \longrightarrow\!\!\!\!\to & \bigoplus_{i=t-t'}^{p-1} H^0(\mathcal{O}(ip-1)) \end{array}$$

(where $\varphi_h \longmapsto \sum_{\substack{k=0 \\ t-t'\leq k+h\leq p-1}}^{t} f_k\varphi_h)$.

This map is not in general surjective, hence we have to check the three cases of table 2.5.

Case I : for $p = 2$, $b = -2$; for $p = 3$ $b = 0$ and $H^0(\mathbb{O}(1) \oplus \mathbb{O}(4)) \longrightarrow\!\!\!\!\rightarrow H^0(\mathbb{O}(5))$.

Case II : for $p = 1$ there's nothing to check, for $p \geq 2$, there remains to check the case for $\mathbb{F}(1,1,q)$, $q = p+1$, which follows from lemma 3.34, since $b = q-3$

Case III: $b = p-1-r-1$, O.K. if $r > p-2$.

<u>Lemma 3.34</u>. If f has $f_0 \neq 0$, f_0 , f_1 rel. prime, then (3.33) is surjective for $t-t' = 2$.

<u>Proof</u>. Since $\sum_{h=1}^{p-1} h^0(\mathbb{O}(hp-2)) = \sum_{i=2}^{p-1} h^0(\mathbb{O}(ip-1)) + 1$, it suffices to show that the kernel has dimension 1.

The map is given in matrix equation by

$$\begin{pmatrix} \psi_{p-1} \\ \vdots \\ \vdots \\ \psi_2 \end{pmatrix} = \begin{pmatrix} f_0 f_1 & \cdots\cdots & f_{p-2} \\ & f_0 & f_{p-3} \\ & \ddots & \cdots\cdots\cdots \\ & & f_0 f_1 f_2 \\ & & f_0 f_1 \end{pmatrix} \begin{pmatrix} \varphi_{p-1} \\ \vdots \\ \vdots \\ \varphi_1 \end{pmatrix}$$

and it suffices to show, by induction on $i = 1, .. p-2$, that if φ is in the kernel, then $f_0^i \mid \varphi_1, \varphi_2, ... \varphi_{i+1}$ are determined by φ_1, and if $\varphi_1 = f_0^i \cdot g_i$, then $g_i \cdot f_0^{i-(j-1)} \mid \varphi_j$ and $\varphi_{i+1} \equiv \pm f_1^i g_i (\mathrm{mod}\ f_0)$.

In fact, if φ is in the kernel, then $0 = \psi_{i+2} = f_0\varphi_{i+2} + {} + f_1\varphi_{i+1} + . \quad + f_{i+1}\varphi_1$, thus $f_0 \mid \varphi_{i+1}$, hence $f_0 \mid g_i$, $g_i = f_0 g_{i+1}$ and the other inductive assertions are clear since e.g. $\varphi_{i+2} \equiv \pm f_1^{i+1} g_{i+1} (\mathrm{mod}\ f_0)$ is gotten by dividing by f_0 the above equation.

Q.E.D. for the Lemma

The verification of the surjectivity of $H^0(\Omega_Z^1 \otimes p^*\omega_X)^G \twoheadrightarrow T_X^*$ follows easily from the explicit description we gave of $H^0(\Omega_Z^1 \otimes p^*\omega_X)^G = \{ \Sigma\, u_{ij}(y)\eta_{ij} \mid u_{ij} \in A_{d-4-p-q}\ ,\ w(u_{ij}) + {} + w(\eta_{ij}) = 0\}$ and from lemma 2.11 (applied with n equal to p, resp. q).

Q.E.D.

Appendix: formulas for almost simple cyclic covers and cones $\mathbb{P}(1,1,p)$.

To justify the definition of almost simple cyclic covers, let's consider $\mathbb{P} = \mathbb{P}(1,1,n)$, the projective cone over the rational normal curve of degree n, and $\Pi : \mathbb{F} \longrightarrow \mathbb{P}$ where $\mathbb{F}$, the blow up of the vertex of the cone, is the Segre-Hirzebruch surface $\mathbb{F}_n$. We have a commutative diagram

(A1)
$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\psi} & \mathbb{P} \\ \uparrow{\scriptstyle \varepsilon} & & \uparrow{\scriptstyle \Pi} \\ \mathbb{F}_1 & \xrightarrow{\varphi} & \mathbb{F}_n \end{array}$$

where $\psi(y_0,y_1,y_2) = (y_0,y_1,y_2^n)$, ε is the blow-up of the point $y_0 = y_1 = 0$, and

ψ, φ are quotient morphisms by $G \cong \mu_n \cong \mathbb{Z}/n$.

Let E_0 be $\Pi^{-1}(x_2 = 0)$, $E_\infty = \Pi^{-1}(0,0,1)$, $F = \Pi^{-1}(x_0 = 0)$, and set E_0', E_∞', F' their set theoretical inverse images under φ.

An easy computation gives

(A2) $\varphi^*(E_0) = nE_0'$, $\varphi^*(E_\infty) = nE_\infty'$, $\varphi^*(F) = F'$, and E_0, E_∞ are the branch locus of φ, with $E_0 - E_\infty \equiv nF$.

We notice that, by definition (cf. [Do])

(A3) $\mathcal{O}_{\mathbb{P}}(r) = \psi_*(\mathcal{O}_{\mathbb{P}^2}(r))^G$, moreover $\mathcal{O}_{\mathbb{P}^2}(r) = \varepsilon_*\mathcal{O}_{\mathbb{F}_1}(rE_0')$,

hence $\mathcal{O}_{\mathbb{P}}(r) = \Pi_*(\varphi_*(\mathcal{O}_{\mathbb{F}_1}(rE_0'))^G$.

Writing $r = q + r''n$, with $q < n$ we have $(\varphi_*\mathcal{O}_{\mathbb{F}_1}(rE_0'))^G =$ $= \varphi_*\mathcal{O}_{\mathbb{F}_1}(qE_0')^G \otimes \mathcal{O}_{\mathbb{F}_n}(r''E_0)$, hence, in order to demonstrate lemma 3.31, it suffices to show that

(A4) $\varphi_*\mathcal{O}_{\mathbb{F}_1}(qE_0')^G = \mathcal{O}_{\mathbb{F}_n}(qF)$.

This will done in greater generality.

<u>Almost simple cyclic covers.</u>

Let an algebraic variety Y and a line bundle L of rank 1 be given on Y , such that the sheaf of sections of L be isomorphic to $\mathcal{O}_Y(F) = \mathcal{L}$ for some divisor F . Assume that there are given reduced effective divisors E_0 , E_∞ on Y , which are disjoint, and are such that $E_0 \equiv E_\infty + nF$.

<u>Definition A5</u> The almost simple cyclic cover associated to (Y,L,E_0,E_∞) is the subvariety $X \hookrightarrow \mathbb{P}(L \oplus \mathbb{C}_Y)$ defined by the equation $z_1^n e_\infty = e_0 z_0^n$ where e_0 , e_∞ are sections defining E_0 , resp. E_∞ , and z_1 , z_0 are respective linear coordinates on the fibres of L , resp. the trivial bundle $\mathbb{C}_Y$.

The group $G = \mu_n$ acts, if ζ is a primitive n^{th} root of 1 , by $z_1 \longrightarrow \zeta z_1$, $z_0 \longrightarrow z_0$.

Take , as customary, $z = z_1/z_0$ as a coordinate on $V_0 = (X - E_\infty)$, and $z' = z_0/z_1$ on $V_\infty = X - E_0$.

We have

<u>Proposition A6</u> $\varphi_* \mathcal{O}_X \cong \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}(-E_\infty)$, where $\mathcal{L}^{-i}(-E_\infty) \cong \mathcal{L}^{n-i}(-E_0)$ is the eigensheaf corresponding to $\zeta \longmapsto \zeta^i$.

<u>Proof</u>. Write a function on V_0 as $f_0 + f_1 z + \dots f_{n-1} z^{n-1}$ with f_i a section of $\mathcal{L}^{-i}$ on V_0 , and as $g_0 + \quad + g_{n-1}(z')^{n-1}$ on V_∞ , with g_j a section of $\mathcal{L}^j$ on V_∞ ; then notice that $f_i z^i = f_i e_0/e_\infty z'^{n-i}$, hence $f_i e_0/e_\infty = g_{n-i}$.

<u>Q.E.D.</u>

Similarly one proves:

<u>Theorem A7</u> If $\varphi^*(E_0) = nE_0'$, and $q < n$, then

$$\varphi_* \mathcal{O}_X(qE_0') \cong \left(\bigoplus_{i=0}^{q} \mathcal{L}^{q-i}\right) \oplus \left(\bigoplus_{i=q+1}^{n-1} \mathcal{L}^{n+q-i}(-E_0)\right)$$

where the index i gives the $\zeta \longmapsto \zeta^i$ eigensheaf.

<u>Cor. A8</u>. $\varphi_* \mathcal{O}_X(qE_0')^G \cong \mathcal{L}^q$.

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EVERYWHERE NON REDUCED MODULI SPACES

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Dedicated to the memory of Aldo Andreotti

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<u>Added-in-proof</u>: While giving a talk at the Max-Planck-Institut on the above results, I learnt from Masa-Hiko Saito that he has also been studying a situation similar to the one considered in § 1.

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(3.27) $h^1(N'_{B_2}) = h^0(\mathcal{H}om(\mathcal{O}_{B_2}(p)(-\delta_2 P_3)\ ,\ \mathcal{O}_{B_2}(d-q-2))) =$
$= h^0(\mathcal{O}_{B_2}(d-p-q-2)(\delta_2 P_3))$, and similarly $h^1(N'_{B_3}) =$
$= h^0(\mathcal{O}_{B_3}(d-p-q-2)(\delta_3 P_2))$.

To end the proof, in view of 3.17 and 3.27, it suffices to prove that for our choices of d , p , q ,

(3.28) $h^0(\mathcal{O}_{B_2}(d-p-q-2)(\delta_2 P_3)) = h^0(\mathcal{O}_{\mathbb{P}(1,1,q)}(d-p-q-2))$
if $p > 1$ and
$h^0(\mathcal{O}_{B_3}(d-p-q-2)(\delta_3 P_2)) = h^0(\mathcal{O}_{\mathbb{P}(1,1,p)}(d-p-q-2))$.

Since $B_3 \subset \mathbb{P}(1,1,q)$ and symmetrically $B_2 \subset \mathbb{P}(1,1,p)$, we are considering the following problem:

(3.29) $B \subset \mathbb{P}(1,1,p) = \mathbb{P}$ has degree d , P is the singular point of $\mathbb{P}$, $\delta = 0$ if $B \not\ni P$, $\delta = (p-1)$ otherwise, i is an integer > 0 : is there an equality

$$h^0(\mathcal{O}_{\mathbb{P}}(d-i)) = h^0(\mathcal{O}_B(d-i)(\delta P)) \ ?$$

(3.30) The answer is positive if $\delta = 0$, since we have the long exact cohomology sequence

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}}(d-i)) \longrightarrow H^0(\mathcal{O}_B(d-i)) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}}(-i)) = 0 \ .$$

The following lemma follows from a more general result proven in the Appendix.

This condition and the symmetrical one hold if $\deg(B_i) \leq$ $\leq \text{genus}(\Gamma_i)$ and the branch points are general, by an easy argument on the Jacobian of Γ_i .

Q.E.D.

The following is, instead, an example where the moduli space is only singular. Its purpose is to illustrate the following feature: though small deformations of hypersurfaces in $\mathbb{P}^3$ are still hypersurfaces in $\mathbb{P}^3$ (cf. [K-S], [Se]), the same is not true for hypersurfaces of a weighted projective space $\mathbb{P}$ if $\mathbb{P}$ has not isolated singularities.

In fact (as it happens in the examples by Horikawa and Miranda cf. § 3) when all the deformations are surfaces in $\mathbb{P}$, and the surfaces have to pass through the (rigid) singular points of the weighted projective space, then the deformation space becomes everywhere non reduced.

Example 2.15. Let X be a general member of the family of hypersurfaces of degree 8 in $\mathbb{P} = \mathbb{P}(1,1,2,2)$. X has 4 quadratic ordinary singular points, hence the tangent codimension of this family in the Deformation space is at least 4.

But $\omega_X = \mathcal{O}_X(2)$, so that the canonical map of X embeds it as the complete intersection $G_4 \cap Q_2'$ in $\mathbb{P}^4$ where G_4 is a quartic hypersurface in $\mathbb{P}^4$ and Q_2' is a quadric of rank 3. It is easy to see that the complete intersections $G_4 \cap Q_2$ are all the

Set $b = (p-1)-(t-t')$: if $b \leq -1$ the domain of j is a vanishing cohomology group and we are done (cf., for the vanishing, the table at page 612 of [Kon]).

Assume instead $b \geq 0$, and set $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(p)$. Then the map j is the map j'

$$(3.32) \qquad H^1(\mathrm{Sym}^b V \otimes \mathcal{O}(-p(p-1)-1)) \xrightarrow{j'} H^1(\mathrm{Sym}^{b+t} V \otimes \mathcal{O}(-p(p-1)))$$

given by the equation of B.

I.e., if $f = \sum_{k=0}^{t} c_{ijk} y_0^i y_1^j y_2^{p(t-k)} = \sum_{\substack{k=0 \\ \deg f_k = kp+1}}^{t} f_k(y_0,y_1) y_2^{p(t-k)}$

is the equation of B, then, since $\mathrm{Sym}^b V = \bigoplus_{i=0}^{b} \mathcal{O}(ip)$ ($\mathcal{O}$ stands for $\mathcal{O}_{\mathbb{P}^1}$!), $\psi_i \in H^1(\mathcal{O}(ip - p(p-1)-1)$ is mapped under j' to $\sum_{k=0}^{t} f_k \psi_i$, with $f_k \psi_i \in H^1(\mathcal{O}((i+k)p - p(p-1)))$.

On the other hand this last group is zero unless $i+k = h' \leq p-2$, and it is always better to deal with H^0's rather than H^1's, hence we dualize, and ask for the surjectivity of

$$(3.33) \qquad \begin{array}{ccc} \bigoplus_{h'=0}^{p-2} H^0(\mathcal{O}((p-1-h')p-2)) & \longrightarrow\!\!\!\!\!\rightarrow & \bigoplus_{i'=0}^{b} H^0(\mathcal{O}(p(p-1-i')-1)) \\ \| & & \| \\ \bigoplus_{h=1}^{p-1} H^0(\mathcal{O}(hp-2)) & \longrightarrow\!\!\!\!\!\rightarrow & \bigoplus_{i=t-t'}^{p-1} H^0(\mathcal{O}(ip-1)) \end{array}$$

Lemma 3.31. Let $\Pi : \mathbb{F} = \mathbb{F}_p \longrightarrow \mathbb{P} = \mathbb{P}(1,1,p)$ be the blow up of the singular point P of $\mathbb{P}$, and let E_∞ be the exceptional divisor. Let B be a smooth curve of $\mathbb{P}$, and identify B to its proper transform in the Segre-Hirzebruch surface $\mathbb{F} = \mathbb{F}_p$. Then if E_0 is the curve defined by $\Pi^{-1}(x_2 = 0)$, and F is a fiber of $\mathbb{F}$ (proper transform of $x_0 = 0$), we have: if r is a positive integer and $r = r' + r''p$, with $r' < p$, then $\mathcal{O}_B(r) = \mathcal{O}_B(r'F + r''E_0)$, and $H^0(\mathcal{O}_{\mathbb{P}}(r)) = H^0(\mathcal{O}_{\mathbb{F}}(r'F + r''E_0))$.

Moreover $\mathcal{O}_B(\delta P) = \mathcal{O}_B(\delta E_\infty)$ (recall that $E_0 \equiv E_\infty + pF$).

Assume now we are in the case when $B \ni P$, and $(d-i)$ (cf. 3.29) equals $t'p$ (as can be checked), while $d = tp+1$ (so $t > t'$).

It is then easy to see that $B \equiv tE_0 + F$, and, in view of 3.31 and since $H^0(\mathcal{O}_{\mathbb{P}}(t'p)) = H^0(\mathcal{O}_{\mathbb{F}}(t'E_0)) = H^0(\mathcal{O}_{\mathbb{F}}(t'E_0 + (p-1)E_\infty)$, we are looking at the surjectivity of the map (res) in the following exact sequence on $\mathbb{F}$:

$$(3.32) \qquad H^0(\mathcal{O}_{\mathbb{F}}(t'E_0 + (p-1)E_\infty)) \xrightarrow{res} H^0(\mathcal{O}_B(t'E_0 + (p-1)E_\infty) \longrightarrow$$

$$\longrightarrow H^1(\mathcal{O}_{\mathbb{F}}(((p-1) + t' - t)E_0 - (p(p-1)+1)F) \xrightarrow{j}$$

$$\xrightarrow{j} H^1(\mathcal{O}_{\mathbb{F}}((t'+p-1)E_0 - p(p-1)F) .$$

This is equivalent to requiring the injectivity of j.

$\frac{j_2}{p} + \frac{j_3}{q}$, and we let A_i^w be the subspace of monomials of degree i , weight w , set $A^w = \bigoplus_i A_i^w$, then

$$H^0(\Omega_Z^1 \otimes p^*\omega_X)^G = \{\Sigma\, u_{ij}(y)\eta_{ij} \mid u_{ij} \in A_{d-4-p-q} \text{ and } w(u_{ij}) + w(\eta_{ij}) = 0\}$$

$$H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G = \{\Sigma\, v_{ij}(y)\eta_{ij} \mid v_{ij} \in A_{d-6}\ \ w(v_{ij}) + w(\eta_{ij}) = -\frac{1}{q} - \frac{1}{p}\}$$

We let μ' be the linear map induced by (3.21), which is clearly injective; then the left hand side of 3.16 is just the dimension of $\operatorname{coker}(\mu')$.

Since μ' is induced by a map $\mu : (A_{d-4-p-q})^6 \longrightarrow (A_{d-6})^6$, (multiplication by $y_2^{p-1}y_3^{q-1}$) where these spaces are viewed as G-modules according to (3.22), it will suffice to identify $W = H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G / H^0(\Omega_Z^1 \otimes p^*\omega_X)^G$ with a quotient of a supplementary space V to $\operatorname{Im}\mu$ inside $((A_{d-6})^6)^G$.

Recall that $H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G = \{ \sum_{i<j} v_{ij}\eta_{ij} \mid v_{01} \in A^{-\frac{1}{p}-\frac{1}{q}}$ $v_{02}, v_{12} \in A^{-\frac{2}{p}-\frac{1}{q}}$, $v_{03}, v_{13} \in A^{-\frac{1}{p}-\frac{2}{q}}$, $v_{23} \in A^{-\frac{2}{p}-\frac{2}{q}}\}$.

The above properties imply, since $(p,q) = 1$

with the natural ones on $H^0(\Omega^1_Z \otimes p^*\omega_X)$, resp. $H^0(\Omega^1_Z \otimes \Omega^2_Z)$.

To do this we notice that the G-module structure on A_i linearizes the $\mathcal{O}_{\mathbb{P}^3}(i)$'s and we rewrite (3.19) as

$$0 \longrightarrow \Omega^1_{\mathbb{P}^3}(i) \longrightarrow (A_1)^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(i-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(i) \longrightarrow 0 .$$

By which we get $(i = 0)$ $\Omega^3_{\mathbb{P}^3} \cong (\det A_1)^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4)$; then 3.18 gives $\Omega^2_Z \cong \Omega^3_{\mathbb{P}^3} \otimes \mathcal{O}_Z(d) \cong (\det A_1)^\vee \otimes \mathcal{O}_Z(d-4)$. On the other hand, we have an inclusion of sheaves, induced by multiplication by a constant times $y_2^{p-1}\ y_3^{q-1}$,

$$(3.21)\qquad \begin{array}{ccccc} 0 \longrightarrow & \Omega^1_Z \otimes p^*\omega_X & \longrightarrow & \Omega^1_Z \otimes \Omega^2_Z \\ & \wr\| & & \wr\| \\ & \Omega^1_Z(d-p-q-2) & & \Omega^1_Z(d-4) \end{array}$$

since $(\det A_1)^\vee$ corresponds to the character $\zeta \longmapsto \zeta^{-p-q}$, for which $y_2^{p-1} y_3^{q-1}$ is an eigenvector, it follows that

$$(3.22)\qquad \begin{array}{l} \Omega^1_Z \otimes p^*(\omega_X) \cong \Omega^1_Z \cdot (d-p-q-2) \\ \Omega^1_Z \otimes \Omega^2_Z \cong (\det A_1)^\vee\ \Omega^1_Z(d-4) . \end{array}$$

It follows that, if we define the weight w of a monomial $y_0^{j_0} y_1^{j_1} y_2^{j_2} y_3^{j_3}$ as the element w of $\frac{1}{pq}\mathbb{Z}/\mathbb{Z}$ given by

We let V_i be the open set where $x_i \neq 0$, for $i = 0,1$. Then V_0 , V_1 , V_2 is an open cover of B_3 . On V_0 we have coordinates (for $\mathbb{P}$)

$$(3.13) \qquad \eta = x_1/x_0 \quad \xi = x_2/x_0^p , \quad \zeta = x_3/x_0^q$$

whereas

$$(3.14) \qquad x = x_o^p/x_2 , \quad y = x_3^p/x_2^q , \quad z = \frac{x_0x_3}{x_2{}^{(q+1/p)}} = \frac{x_0x_3}{x_2^r}$$

(where, cf. table 2.5., $r = 2$ in the first case).

Clearly $\partial/\partial\zeta$ is a local generator for N'_{B_3} on V_0 and since $x = \frac{1}{\xi}$, $y = \zeta^p/\xi^q$, $z = \zeta/\xi^r$, we have $\frac{\partial}{\partial\zeta} = p\,\frac{\zeta^{p-1}}{\xi^q}\,\frac{\partial}{\partial y} + \xi^{-r}\,\frac{\partial}{\partial z} = pz^{p-1}\xi^{1-r}\,\frac{\partial}{\partial y} + x\xi^{1-r}\,\frac{\partial}{\partial z} = \xi^{1-r}\Big(x\,\frac{\partial}{\partial z} + pz^{p-1}\,\frac{\partial}{\partial y}\Big)$. On V_1 we have coordinates $a = x_0/x_1$, $b = x_2/x_1^p$, $c = x_3/x_1^q$, such that $a = 1/\eta$, $b = \xi\eta^{-p}$, $c = \zeta\eta^{-q}$.

A local generator for N'_{B_3} is $\partial/\partial c$, and $\partial/\partial\zeta = \eta^{-q}\partial/\partial c$.

Hence a section of N'_{B_3} is given by functions A_0, A_1, A_2 on $B_3 \cap V_i$ s.t.

$$A_0\,\partial/\partial\zeta = A_2(x\,\partial/\partial z + pz^{p-1}\partial/\partial y) = A_1\partial/\partial c .$$

1-point nilpotent scheme of length $(n-1) \times (r_1 \times r_2)$ if $r_i = \deg(B_i) \leq g_i = \text{genus}(\Gamma_i)$ and the branch points are general.

Corollary 2.13. If $\nu = h^1(\theta_S)/\dim \mathrm{Def}(S)$ is the ratio between tangent dimension and dimension of Def(S) there do exist generalized Kas-surfaces with ν arbitrarily large.

Proof. If S has degree n , $\deg(B_i) = g_i - 1$, then $\nu(S) = (n-1)(g_1-1)(g_2-1)/3[(g_1-1)+(g_2-1)]$. □

Proof of 2.12. By lemma 2.11. it suffices to verify that, for each $h = 0, \ldots, n-2$, we have 2 surjective maps,

(2.14.) $$H^0(\Omega_{\Gamma_1}^{\otimes 2}(B_1) \otimes \mathcal{L}_1^h) \otimes (\Omega_{\Gamma_2} \otimes \mathcal{L}_2^h) \twoheadrightarrow \bigoplus_{\substack{i=1,..r_1 \\ j=1,..r_2}} \mathbb{C}_{ij}$$

(given by $\bigoplus_{i,j}(\mathrm{val}_{b_i'} \otimes \mathrm{val}_{b_j''})$)

and its symmetrical.

Since $H^1(\Omega_{\Gamma_i}^{\otimes 2} \otimes \mathcal{L}_i^h) = 0$, we have a surjection $H^0(\Omega_{\Gamma_1}^{\otimes 2}(B_1) \otimes \mathcal{L}_1^h) \twoheadrightarrow \bigoplus_{i=1,..r_1} \mathbb{C}_{b_i'}$, while $H^0(\Omega_{\Gamma_2} \otimes \mathcal{L}_2^h) \twoheadrightarrow \bigoplus_{j=1,..\Omega_2} \mathbb{C}_{b_j''}$ if and only if $H^1(\Omega_{\Gamma_2}(-B_2) \otimes \mathcal{L}_2^h)$ injects into $H^1(\Omega_{\Gamma_2} \otimes \mathcal{L}_2^h)$.

By duality there must be a surjection $H^0(\mathcal{L}_2^{-h}) \twoheadrightarrow H^0(\mathcal{L}_2^{n-h})$, and this holds $\Leftrightarrow H^0(\mathcal{L}_2^i) = \begin{cases} 0 & \text{for } i < n \\ 1 & \text{for } i = n \end{cases}$.

(2.5.) There is an invertible sheaf $\mathcal{L}_i$ on Γ_i and a divisor B_i consisting of distinct points s.t. $\mathcal{O}_{\Gamma_i}(B_i) \cong \mathcal{L}_i^n$; C_i is the subvariety of L_i , the line bundle whose sheaf of sections is $\mathcal{L}_i$, obtained by taking the n^{th} - root of the section defining B_i .

Clearly the group $\mu_n \cong \mathbb{Z}/n$ of n^{th} roots of unity acts on C_i , and $f_i : C_i \longrightarrow \Gamma_i$ is the quotient map.

Notice further that

(2.6.) $$f_*\mathcal{O}_{C_i} = \mathcal{O}_{\Gamma_i} \oplus \mathcal{L}_i^{-1} \oplus \ldots \oplus \mathcal{L}_i^{-(n-1)} .$$

To adhere to our standard notation, we let $Z = C_1 \times C_2$ and we let μ_n act on Z by the (twisted) action

(2.7.) $$\zeta(x,y) = (\zeta x, \zeta^{-1} y) .$$

It follows immediately that if $X = Z/\mu_n$, then the singularities of X are exactly R.D.P.'s of type A_{n-1} , and $p : Z \longrightarrow X$ is unramified in codimension 1.

As a preliminary computation, we notice that $(\Omega = \Omega^1$, for short)

(2.8.) $$\Omega_{C_i} = f_i^*(\Omega_{\Gamma_i} \otimes \mathcal{L}_i^{n-1}) ,$$

Theorem 2.3. If (2.1.), (2.2.) hold and S is a minimal resolution of $X = Z/G$, then for $r >> 0$ S has everywhere obstructed deformations.

Proof. Since, by (2.1.), $p : Z \longrightarrow X$ is unramified in codimension 1, cor. 1.20 applies: in fact, by (2.2.) X has only R.D.P.'s and is singular. We have to verify that $H^0(\Omega^1_Z \otimes \omega_Z)^G$ maps onto $T^*_X = H^0(T^*) = H^0((p_*\Omega^1_Z)^G/\Omega^1_X)$ (cf. 1.7.). Notice that the restriction map $\Omega^1_W \otimes \omega_W(rH) \longrightarrow \Omega^1_Z \otimes \omega_W(rH)$ is a homomorphism of G-modules, while ω_Z and $\omega_W(rH)$ differ just by a character of G.

Now T^* is a quotient of $(p_*\Omega^1_W)^G/\Omega^1_{W/G}$, which has finite length, hence there exists an integer $k > 0$ s.t. T^* is a quotient of $\bar{T} = \bigoplus_{i=1}^{s} \Omega^1_{W,w_i}/\mathfrak{m}^k_{W,w_i} \cdot \Omega^1_{W,w_1}$ where $\mathfrak{m}_{W,w_i}$ is the maximal ideal of the point w_i.

It suffices now to choose $r >> 0$ s.t. $H^0(\Omega^1_W \otimes \omega_W(rH))$ maps onto $\bar{T}$.

Q.E.D.

We show now at least that (2.1.)(2.2.) occur easily.

Example 2.4. We let $W = A$, where A is an abelian 3-fold and let H be any polarization.

$G = \{\pm 1\}$ acts in the standard way, so $w_1, \dots w_{64}$ are the

$\mathcal{L}$ be ω_X .

By (1.4.) the above sequence is obtained by tensoring with ω_X the exact sequence of local cohomology

$$0 \longrightarrow \Omega^1_X \longrightarrow i_*(\Omega^1_{X^0}) \longrightarrow H^1_{\text{Sing } X}(\Omega^1_X) \longrightarrow 0$$

and, as in Pinkham's article ([Pi] page 174, (4) of theorem 1), we notice that by local duality the last term is isomorphic to

$$\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)^\vee \quad .$$

<u>Q.E.D.</u>

<u>Remark</u>. Notice that sometimes Y is necessarily singular, no matter of which resolution S of X one takes.

I.e., there is no blow up of Z on which G acts as a group generated by pseudo reflections. For instance, take the action of $\mathbb{Z}/4$ such that $(u,v) \longrightarrow (iu,-iv)$: on the first blow-up there is a point where the action has eigenvalues $(i,-1)$, hence this point has to be blown up, and over it lies a point where the eigenvalues are $(-i,-1)$, ... hence the procedure can never end.

From now on we shall assume $\dim Z = 2$, X Gorenstein (i.e., with R.D.P.'s only), and let S be a minimal resolution of singularities of X .

$$H^0(\Omega^1_S \otimes \pi^*\mathcal{L}) \cong H^0(\Omega^1_Z \otimes p^*\mathcal{L})^G .$$

Proof. By abuse of notation we shall identify $\mathcal{L}$ with its pull-backs, which we shall consider endowed of their natural linearization (notice that G acts on Y, with quotient S).

Since $S - S^0$ has codim ≥ 2, $H^0(\Omega^1_S \otimes \mathcal{L}) = H^0(\Omega^1_{S^0} \otimes \mathcal{L})$; by (1.4.) $(\varphi_* \Omega^1_{Y_0})^G = \Omega^1_{S^0}$, hence $H^0(\Omega^1_{S^0} \otimes \mathcal{L}) = H^0(\Omega^1_{Y_0} \otimes \mathcal{L})^G$ and this last clearly equals $H^0(\Omega^1_Z \otimes \mathcal{L})^G$ since $\varepsilon|_{\varepsilon^{-1}(Z^0)}$ is a modification and Z is smooth.

Q.E.D.

Corollary 1.7. Let $Z, G, \ldots$ be as above and assume X Gorenstein, $\dim Z = 2$. Let T^* be the cokernel of the sequence

$$0 \longrightarrow \Omega^1_X \otimes \omega_X \longrightarrow (p_*\Omega^1_Z)^G \otimes \omega_X \xrightarrow{r} T^* \longrightarrow 0 .$$

Then $H^0(T^*) \cong H^0(\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X))^\vee$ and the image of $H^0(r)$ is isomorphic to the cokernel of

$$H^0(\Omega^1_X \otimes \omega_X) \longrightarrow H^0(\Omega^1_S \otimes \Omega^2_S) .$$

Proof. First of all $\Omega^2_S \cong \omega_S \cong \pi^*(\omega_X)$ since the singularities of X are Gorenstein quotient singularities, hence R.D.P.'s; so that the second statement follows directly from (1.6.) letting

We have now to show that $\theta_X(-\log B) \longrightarrow (i'')_*(\theta_{X''}(-\log B))$ is an isomorphism. This follows since $\theta_X(-\log B)$ is torsion free (if ν is a vector field, f is a function and $\nu(df)|_{X''} \in \mathcal{I}_{B\cap X''}$, then $\nu(df) \in \mathcal{I}_B$), hence we have an injective homomorphism of which j gives an inverse.

Finally, the inclusion $\mathcal{O}_X \subset p_*\mathcal{O}_Z$ shows that $j^\vee$ is injective

Q.E.D.

We let now $\pi : S \longrightarrow X$ be a resolution of singularities of X, and Y be the normalization of the fibre product $Z \times_X S$.

We then have a diagram

(1.5.)

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & S \\ \downarrow \varepsilon & & \downarrow \pi \\ Z & \xrightarrow{p} & X \end{array}$$

and we set $Y^0 = Y - \mathrm{Sing}(Y)$, $S^0 = S - \varphi(\mathrm{Sing}\ Y)$, $Z^0 = Z - \varepsilon(\mathrm{Sing}(Y))$.

Proposition 1.6. For any invertible sheaf $\mathcal{L}$ on X, we have

Then there are natural isomorphisms

$$\Omega^1_{X'} \xhookrightarrow{j^\vee} p'_*(\Omega^1_Z)^g \quad \text{and}$$

$$(p_*\theta_Z)^G \xrightarrow{j} \theta_{X'}(-\log B')$$

<u>Proof</u>. Both $j^\vee, j$ are naturally defined and are clearly isomorphisms on the open sets where p' is unramified.

If $p'(z) = x$, $g(z) = z$, we take local coordinates as in (1.2.) and then the proof follows from a direct computation.

□

We set now $Z'' = Z-(W-W') = \{z \mid G_z$ is generated by a pseudo-reflection$\}$, so that $\operatorname{codim}(Z - Z'') \geq 2$, $X = Z/G$, let $p : Z \longrightarrow X$ be the quotient map, and finally set $X'' = p(Z'')$.

<u>Lemma 1.4.</u> Let B be the (reduced) branch locus of $p : Z \longrightarrow X = Z/G$, $X^0 = X - \operatorname{Sing} X$, $i : X^0 \hookrightarrow X$ the inclusion map. Then the natural homomorphism $(p_*\theta_Z) \xrightarrow{j} \theta_X(-\log B)$ is an isomorphism, whereas $\Omega^1_X \xhookrightarrow{j^\vee} p_*(\Omega^1_Z)^G$ is injective, an isomorphism on X^0 , and $p_*(\Omega^1_Z)^G$ equals $i_*(\Omega^1_{X^0})$.

<u>Proof</u>. By (1.3.) the assertion is true on the open set $X'' \subset X^0$. Since $\operatorname{codim}(Z - Z'') \geq 2$ and θ_Z , Ω^1_Z are locally free, if $i'' : X'' \longrightarrow X$ is the inclusion, $p_*(\theta_Z)^G = (i'')_*(\theta_{X''}(-\log B)$ and $p_*(\Omega^1_Z)^G = (i''_*)(\Omega^1_{X''}) = i_*(\Omega^1_{X^0})$.

(1.10)

$$\begin{array}{ccccccc}
 & & H^1(\theta_S) & \rightleftarrows & H^1_E(\theta_S) & \leftrightarrows 0 & \\
 & \nearrow & \downarrow & & \downarrow & & H^2(\theta_S) \\
 & & & & & & \| \\
0 \longrightarrow H^1(\theta_X) & \longrightarrow & Ext^1(\Omega^1_X, \mathcal{O}_X) & \longrightarrow & H^0(\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)) & \longrightarrow & H^2(\theta_X) \\
 & & & & \| & & \\
 & & & & T_X & &
\end{array}$$

Here the inclusion of $H^1_E(\theta_S)$ is induced by the sequence of local cohomology ([B-W] 1.3), on the other hand Burns and Wahl prove that $\pi_* \theta_S = \theta_X$, so that the Leray spectral sequence for the map π (cf. [Pi] page 176) gives the sequence

$$\begin{array}{c}
0 \longrightarrow H^1(\theta_X) \longrightarrow H^1(\theta_S) \longrightarrow H^0(R^1 \pi_* \theta_S) \longrightarrow 0 \\
\wr\| \\
H^1_E(\theta_S)
\end{array}$$

plus the isomorphism $H^2(\theta_X) \cong H^2(\theta_S)$; thus we obtain the splitting

(1.11) $\quad H^1(\theta_S) \cong H^1(\theta_X) \oplus H^1_E(\theta_S)$.

On the other hand, (cf. [Pi]), the lower exact row of (1.10) is given by the Ext spectral sequence, and Pinkham proves that

(1.12) ob is dual, via Grothendieck duality, to the map

EVERYWHERE NON REDUCED MODULI SPACES

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Introduction

The purpose of this article is to show how often the moduli spaces of surfaces of general type can be everywhere non reduced in the case when the canonical bundle K_S is not ample. On the other hand, by giving a simple criterion which implies that this must happen, we are in fact able to subsume almost all the previously known examples of obstructed deformations in dimension 2 as particular issues of a very general situation; we produce also infinite series of examples, showing in particular that all non Cartier divisors of rigid 3-dimensional weighted projective spaces give rise to this pathology.

To be more precise, let V be an algebraic variety with a finite group Aut(V) of automorphisms (e.g., cf. [Ma], if V is of general type); then, if V admits a space of moduli $\mathfrak{M}(V)$ (cf. [Mu 2]), locally (i.e. in an analytic neighbourhood of the point of $\mathfrak{M}(V)$ correspoding to V) $\mathfrak{M}(V)$ is the quotient of the base Def(V) of the Kuranishi family of deformations of V by the group Aut(V) (cf. [Wa]). It is clear that in this case $\mathfrak{M}(V)$ is (locally) everywhere non reduced, e.n.r. for short (i.e., everywhere singular) if and only if Def((V) is e.n.r. .

We recall the classical terminology: V is said to have obstructed deformations if the germ $Def(V)$ is singular. The stronger condition that $Def(V)$ be e.n.r., i.e. everywhere singular, can thus be referred to as V having "everywhere obstructed deformations", and has been regarded up to now as a very pathological phenomenon.

The first example of algebraic varieties V with $Def(V)$ e.n.r. is due to Kodaira and Mumford ([Ko], [Mu 1]) : here, though, V and its deformations are blow-ups of $\mathbb{P}^3$, hence there are no birational moduli.

After several examples of obstructed deformations were exhibited, e.g. by Kas ([Ka 1], [Ka 2]), by Burns and Wahl ([B-W]), by Horikawa ([Hor 1]), then Horikawa [Hor 2] and, later, Miranda ([Mi]) gave examples of surfaces of general type S (respectively with $p_g = 4,7$, $K^2 = 6$, 14) for which $Def(S)$ is e.n.r. .

Their approach was through the classification of all the surfaces with those invariants (Miranda uses Castelnuovo's classification [Cas] of surfaces with $K^2 = 3\,p_g - 7$ and $|K_S|$ birational). In both cases the outcome is that the canonical bundle K_S is not ample for all the surfaces corresponding to the points of a component of the moduli space. This research started when I tried to find a direct proof that $Def(S)$ was e.n.r., and I noticed that for both examples the singular canonical models were hypersurfaces in a 3 dimensional projective space (respectively $X_9 \subset \mathbb{P}(1,1,2,3)$, $X_7 \subset \mathbb{P}(1,1,1,2)$) admitting a double cover

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Added-in-proof: While giving a talk at the Max-Planck-Institut on the above results, I learnt from Masa-Hiko Saito that he has also been studying a situation similar to the one considered in § 1.