Covariant symplectic geometry of binary forms and singularities of systems of rays

by

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1. Introduction

Let $(T^*\mathbb{R}^n, \omega_{\mathbb{R}^n})$ be the phase space for the free particle (cf. [1]) with the Hamiltonian function $H(x,p) = \frac{1}{2}(|p|^2-1)$. Let the smooth hypersurface $\tilde{Z} \subset \mathbb{R}^n$, representing an obstacle in the Euclidean space, be described by the smooth function $\tilde{F}: \mathbb{R}^n \to \mathbb{R}$. The corresponding two hypersurfaces of $(T^*\mathbb{R}^n, \omega_{\mathbb{R}^n})$: $Y = \{(x,p);\ H(x,p) = 0\}$ and $Z = \{(x,p);\ F(x,p)=\tilde{F}(x)=0\}$, define the sphere bundle $W = Y \cap Z$ over $\tilde{Z}$. Let us denote by M the symplectic space of all integral curves (oriented lines) of the Hamiltonian system defined by H. Let $\pi: Y \to M$ be its canonical characteristic projection. The critical points of $\pi|_W$ are given by $\Omega = \{(x,p) \in W;\ \{H,F\} = 0\}$. Ω forms the set of all versors in $\mathbb{R}^n$ tangent to $\tilde{Z}$ and $\Xi = \pi(\Omega) \subset M$ is the hypersurface of all lines tangent to $\tilde{Z}$.

In the generic case there exist only three normal forms for W, which in the appropriate Darboux coordinates are described as follows (see [3], Theorem 3)

$$W_1 = \{(q,p) \in T^*\mathbb{R}^n;\ p_o^2 + p_1 = 0,\ q_o = 0\},$$

$$W_2 = \{(q,p) \in T^*\mathbb{R}^n;\ p_o^3 + p_1 p_o + q_1 = 0,\ q_o = 0\},$$

$$W_3 = \{(q,p) \in T^*\mathbb{R}^n;\ p_o^4 + p_1 p_o^2 + q_2 p_o + p_2 = 0,\ q_o = 0\}.$$

In the variational obstacle problem (cf. [6], [14], see also Remark 5.9 and Figure 2) the geodesic flow γ on $\tilde{Z}$ is defined by the mutual position of the source of radiation and the obstacle itself. The flow γ corresponds to the Hamiltonian flow on the hypersurface Ξ of M. The generic geodesic flows on the obstacle are classified by the corresponding Hamiltonian flows for the hypersurfaces $\Xi_i = \pi(W_i) \subset M$, i=1,2,3. The most singular case, i=3, was precisely described by Arnold 3 . Arnold wrote down explicitely the corresponding geodesic trajectories on Ξ_3 and showed that the typical lagrangian variety in $\Xi_3 \subset \mathbb{R}^4$, corresponding to the generic system of gliding rays in the biasymptotic point of the obstacle, is symplectomorphic to that one given by the following local model

$$\{(q_1,q_2,p_1,p_2) \in M;\ \tfrac{1}{5}\lambda^5 + \tfrac{1}{3}p_1\lambda^3 + \tfrac{1}{2}q_2\lambda^2 + p_2\lambda + \tfrac{1}{2}q_1 \text{ has a root of multiplicity } \geq 3\},$$

(cf. Remark 5.9).

It appeared (see [5], [6], [14]) that the above singularities of wave fronts in the presence of an obstacle are also classified by the reflection groups as in the standard A, D, E case (see [6], [13]). The analogous system of rays on the plane gliding along the curve having an inflection point is determined by the reflection group of symmetries of the icosahedron (cf. [5] p. 28).

The aim of this paper is to investigate the combinatorial aspects of the singularities considered above and to give the precise description of the analytical structure of these singularities in the general setting of symplectic geometry and invariant theory of binary forms. We prove that where the obstacle singularities appear, the natural symplectic structures of the spaces are coming, by the symplectic reduction process, (cf. [18]) from the unique $Sl_2(\mathbb{K})$-invariant symplectic structure which appears as a unique tensor invariant of degree two on the space of binary forms of odd degree. In order to prove that fact, in Section 2 we formulate the corresponding umbral approach to the general investigations of the spaces of invariants of binary forms (cf. [12], [15], [16]) and prove the useful polynomial identities appearing in the stabilized hierarchy of polynomial spaces with the fixed root comultiplicity. In Section 3 we construct an umbral approach in order to classify explicitely the tensor invariants of binary forms of degree n (cf. [16]). Following [12] we prove the fundamental theorems concerning the umbral, bracket representation of tensor invariants. In Section 4 we show the existence and uniqueness of the $Sl_2(\mathbb{K})$-invariant symplectic structure on the space of binary forms of odd degree and we write down its explicit normal form. The momentum mapping for the symplectic action of $Sl_2(\mathbb{K})$ in the symplectic space of binary forms is derived and the classical theorems of the theory of polynomial invariants of binary forms (Cayley, Sylvester) is reformulated using the canonical Poisson brackets. In an analogous way the $Sl_2(\mathbb{R})$ -invariant contact structure of the projective space $\mathbb{R}P^n$ of all zero-dimensional submanifolds of degree n in the projective line is indicated. Section 5 is devoted to the generalization of the notion of open swallowtail in all dimensions and to extending the notion of Hamiltonian system generated by translations to the general swquence of coisotropic submanifolds defined by the so--called apolar subspaces of binary forms. The generating one--parameter families for open swallowtails are written down explicitely and their usefulness in the investigation of oscilating integrals in the presence of an obstacle is shown.

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2. Polynomial invariants of binary forms

Let M^{n+1} be the space of binary forms of degree n. A binary form f(x,y) of degree n in the variables x and y is a homogeneous polynomial of degree n in x and y (cf. [16]). Thus we denote

$$(2.1) \qquad M^{n+1} \ni f(x,y) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k y^{n-k} .$$

Let us consider the standard action ν of $Gl_2(\mathbb{K})$ on M^{n+1}. A nonconstant polynomial $I \in \mathbb{K}[a_o,\ldots a_n,x,y]$ is said to be a covariant of index g of binary forms of degree n if for all $h \in Gl_2(\mathbb{K})$ we have

$$(2.2) \qquad \bar{\nu}^*_h I = (\det h)^g I,$$

where $\bar{\nu}$ is the canonical extension of ν to $M^{n+1} \times \mathbb{K}^2$. A polynomial function I defined only on M^{n+1} and invariant with respect to ν , according to the formula (2.2), is said to be an invariant of binary forms. We assume that the coefficients of f belong to a field K of characteristic zero and the action ν of $Gl_2(\mathbb{K})$ is induced by the following transformations of variables x and y:

$$(2.3) \qquad \begin{aligned} x &= c_{11}\bar{x} + c_{12}\bar{y} \\ y &= c_{21}\bar{x} + c_{22}\bar{y} . \end{aligned}$$

A very effective method for indicating the polynomial covariants of binary forms comes from the so-called combinatorial umbral calculus (see [12] [15]). We recall now some of the basic properties of the umbral calculus applied in the invariant theory of binary forms. Using the umbral methods we can reduce computations with binary forms to the special case of binary forms of type $f(x,y)=(\alpha_1 x+\alpha_2 y)^n$ and obtain the uniform theory of covariants. The aim of this paper is to enrich this theory with respect to the invariant (covariant) differential forms defined on the appropriate spaces of binary forms.

Let $P = \{\alpha, \beta, \ldots, \omega, u\}$ be an alphabet consisting of an infinite (finite for some special constructions) supply of Greek letters followed by the single Roman letter u. To each Greek letter, say , and the Roman letter u, we associate two variables α_1, α_2 and u_1, u_2 resp. The ring of polynomials in these variables (possibly infinite dimensional) is a vector space called the standard umbral space $\mathcal{U}$. With every space of binary forms we associate a linear operator, say U(f) (generally also denoted by U), defined from the umbral space $\mathcal{U}$ to the space of polynomials $\mathbb{K}[a_0, \ldots, a_n, x, y]$ in the following way. We define the action of U(f) on the corresponding monomials of $\mathcal{U} = \mathbb{K}[\alpha_1, \alpha_2, \ldots, u_1, u_2]$:

(2.4) $\langle U(f) | \alpha_1^k \alpha_2^{n-k} \rangle = a_k$, $\langle U(f) | \alpha_1^j \alpha_2^k \rangle = 0$ if $j+k \neq n$

for any Greek umbral letter α,

$$\langle U(f) | u_1^k \rangle = (-y)^k, \quad \langle U(f) | u_2^k \rangle = x^k$$

and the multiplicative rule:

$$(2.5) \quad \langle U(f) | \alpha_1^i \alpha_2^j \beta_1^k \beta_2^\ell \ldots u_1^p u_2^q \rangle = \langle U(f) | \alpha_1^i \alpha_2^j \rangle \langle U(f) | \beta_1^k \beta_2^\ell \rangle \ldots$$
$$\langle U(f) | u_1^p \rangle \langle U(f) | u_2^q \rangle \quad .$$

These rules uniquely define, by linearity, the umbral operator U(f) on the umbral space $\mathcal{U}$.

Every polynomial $I(a_0, \ldots, a_n, x, y)$ can be written as $\langle U(f) | Q(\alpha_1, \alpha_2, \ldots, u_1, u_2) \rangle$ for some polynomial $Q \in \mathcal{U}$. The polynomial Q is called an umbral representation for the polynomial I and I is called the umbral evaluation of Q. It is easy to see that, for the monomial $I = a_0^{d_0} a_1^{d_1} \ldots a_n^{d_n} x^{e_1} y^{e_2}$ we have

$$I = \langle U(f) \mid \underbrace{\alpha_1^o\alpha_2^n \ldots \gamma_1^o\gamma_2^n}_{d_o \text{ times}},\ \underbrace{\delta_1^1\delta_2^{n-1} \ldots \varepsilon_1^1\varepsilon_2^{n-1}}_{d_1 \text{ times}}, \ldots, (-u_1^{e_1})u_2^{e_2} \rangle$$

Usually the umbral representation of a polynomial I is far from unique.

The $\bar{\nu}$-action of $Gl_2(\mathbb{K})$ on the space of binary forms implies the corresponding action of $Gl_2(\mathbb{K})$ on the umbral space $\mathcal{U}$ Let (c_{ij}) be defined as in (2.3). Then the corresponding change of umbral variables (see [12]), say for some Greek letter α, is defined as follows

(2.6) $$\alpha_1 = [\bar{\alpha}\ c]\ ,\quad \alpha_2 = [\bar{\alpha}\ d]\ ,$$

where $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$, $c = (-c_{21}, c_{11})$, $d = (-c_{22}, c_{12})$ and the bracket $[v\ w] = \det\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$ for two vectors $v=(v_1,v_2)$, $w=(w_1,w_2)$.

By that notation we can easily express the umbral representation of any polynomial $I(\bar{a}_o,\ldots,\bar{a}_n,\bar{x},\bar{y})$ in terms of the umbral representation of $I(a_o,\ldots,a_n,x,y)$, namely (see e.g. [12], p.33), let $I \in \mathbb{K}[a_o,\ldots,a_n,x,y]$, $I = \langle U(f) \mid P(\alpha_1,\alpha_2,\ldots,u_1,u_2)\rangle$: Then we have

(2.7) $$I(\bar{a}_o,\ldots,\bar{a}_n,\bar{x},\bar{y}) = \langle U(\bar{f}) \mid P(\alpha_1,\alpha_2,\ldots,u_1,u_2)\rangle = \langle U(f) \mid P([\alpha\ c], [\alpha\ d], \ldots, \frac{[u\ c]}{[c\ d]}, \frac{[u\ d]}{[c\ d]})\rangle\ ,$$

where $\bar{f}(\bar{x},\bar{y})$ is the binary form obtained from $f(x,y)$ by the change of variables (2.3).

By (2.7) we easily have the following explicit expression for the representation ν (cf. [12]),

$$\bar{a}_k = \langle U(f) \mid [\alpha\ c]^k [\alpha\ d]^{n-k}\ \rangle =$$

$$= \sum_{m=0}^{n} \Big(\sum_{i=m-n+k}^{\min(m,k)} \binom{k}{i}\binom{n-k}{m-i} c_{11}^{i} c_{12}^{m-i} c_{21}^{k-i} c_{22}^{n-k-m+i} \Big) a_m\ .$$

We see that the bracket monomials, say $[\alpha\ \beta]$, $[\alpha\ u]$, etc.

are covariants of index 1, i.e. $[\alpha\ \beta] = [c\ d][\bar\alpha\ \bar\beta]$, $[\alpha\ u] = [c\ d][\bar\alpha\ \bar u]$. Thus we can consider, in the umbral space $\mathcal{U}$, the subspace of bracket polynomials defined as the linear combinations of bracket monomials ; i.e. the nonconstant polynomials in $\mathcal{U}$ which can be written as a product of brackets, say $[\alpha\ \beta]\ [\alpha\ \delta] \ldots [w\ u]$.

The fundamental theorem of invariant theory of binary forms in the umbral approach can be formulated as follows

Theorem 2.1. (cf. [12]). A). The umbral evaluation $\langle U|P\rangle$ of a bracket polynomial P, for which in every bracket monomial M the number of brackets in M containing only Greek letters is constant and equal to $g \in \mathbb{N}$, is a covariant of index g.

B). Let I be a covariant of index g of binary forms of degree n. Then there exists a bracket polynomial P of index g (i.e. with the same number of brackets containing only Greek letters in each monomials involved in P) such that $I = \langle U|P\rangle$.

For the proof of this theorem as well as for the exhaustive account of its applications one can see [12] .

Example 2.2. Using the bracket representations we can easily calculate, by the appropriate algorithm (cf.[12]), the respective basic invariants for binary quadric, binary cubic and binary quartic, namely

1. Binary quadric:

$$D = \langle U(f)\,|\,[\alpha\ \beta]^2\rangle$$

2. Binary cubic:

$$\Delta = \langle U(f)\,|\,[\alpha\ \beta]^2[\alpha\ \gamma]\ [\beta\ \delta]\ [\gamma\ \delta]^2\rangle$$

3. Binary quartic:

$$I = \langle U(f)\,|\,[\alpha\ \beta]^4\rangle\ ,$$

$$J = \langle U(f)\,|\,[\alpha\ \beta]^2[\alpha\ \gamma]^2[\beta\ \gamma]^2\rangle\ .$$

In order to express, more precisely, the evaluations of covariants on the space of binary forms with multiple linear factors we have to introduce the so-called apolar covariant (cf. [12],[16]). We show its convenience in the next sections. Let us consider two binary forms, say $f(x,y)$, $g(x,y)$, where f is of degree n, g is of degree m, $m \leq n$. They can be written umbrally $f = \langle U|[\alpha\ u]^n\rangle$, $g = \langle U|[\beta\ u]^m\rangle$. Their apolar covariant $\langle f|g\rangle$ is the binary form of degree $n-m$ defined umbrally by

$$\langle f|g\rangle = \langle U|[\alpha\ \beta]^m[\alpha\ u]^{n-m}\rangle . \tag{2.9}$$

By the straightforward calculations using the explicit formula for (2.9) we obtain an another expression of polynomial identities mentioned in [9] (Theorem 2), (cf. also [5]).

<u>Proposition 2.3.</u> a) Let n be even. Then the apolar covariant $\langle f|f\rangle$ is an invariant for the binary forms of degree n. It can be expressed by

$$\langle f|f\rangle = \frac{1}{n!}(-1)^n \sum_{k=0}^{n} (-1)^k \frac{\partial^k f}{\partial x^k}\frac{\partial^{n-k} f}{\partial x^{n-k}} = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} a_k a_{n-k} . \tag{2.10}$$

b) Let n be odd, say $n=2j+1$. Then the second apolar covariant $\langle f|\frac{\partial f}{\partial x}\rangle$ can be expressed by

$$\langle f|\frac{\partial f}{\partial x}\rangle = \frac{1}{n!}\sum_{k=0}^{j}(-1)^k(n-2k)\frac{\partial^k f}{\partial x^k}\frac{\partial^{n-k} f}{\partial x^{n-k}} = \tag{2.11}$$

$$= \sum_{k=0}^{n-1}(-1)^k n\binom{n-1}{n-k-1} a_{n-k}(xa_{k+1}+ya_k)$$

<u>Corollary 2.4.</u> Let n be even and f have a linear factor of degree greather than $\frac{n}{2}$. Then $\langle f|f\rangle = 0$.

Let $n=2j+1$ and f have a linear factor greather than $j+1$. Then

$$\langle f|\frac{\partial f}{\partial x}\rangle = 0.$$

Proof. The first part follows from the fact that f, having a multiple linear factor of degree greather than or equal to $[\frac{n}{2}]+1$, is the so-called Hilbert zero-form (cf. [19]). We see that on one side $\langle f|\frac{\partial f}{\partial x}\rangle$ has a multiple linear factor of degree at least two (because of $k \leqslant j$ in differential factors $\frac{\partial^k f}{\partial x^k}$). However from the right hand side of (2.11) $\langle f|\frac{\partial f}{\partial x}\rangle$ is a linear form, which gives us a contradiction unless $\langle f|\frac{\partial f}{\partial x}\rangle$ is zero.

Remark 2.5. Instead of binary forms we can consider polynomials $f(x,1)$. Let us denote by $\Sigma_k(n)$ the set of all polynomials of degree n having at least one root of comultiplicity (= n - multiplicity) less than or equal to k. We see that differentiation of polynomials preserves the comultiplicity of roots and defines the mapping $\phi_k(n): \Sigma_k(n+1) \longrightarrow \Sigma_k(n)$. Using Proposition 2.3 and Corollary 2.4 it can be proved (see e.g. [9], p.14) that if $k \leqslant \frac{n-1}{2}$, then ϕ_k is a diffeomorphism of affine algebraic varieties. In this paper we use the umbral methods to prove the fundamental results concerning the symplectic geometry of polynomials, some of them already mentioned in [4],[6], and to show the effective directions for further generalizations.

3. Umbral derivation of tensor invariants of binary forms.

Let U be the elementary umbral space with one umbral letter . $U_n(\alpha)$ denotes the subspace of U of all homogeneous polynomials of degree n. By $D_n(\alpha)$ we denote the vector space of all differential symbols $a_1(\alpha)d\alpha_1 + a_2(\alpha)d\alpha_2$ with coefficients a_1, a_2 belonging to $U_{n-1}(\alpha)$. The corresponding action, say μ, of $Gl_2(\mathbb{K})$ on $D_n(\alpha)$ is canonically induced from the standard $Gl_2(\mathbb{K})$-action on $\mathbb{K}^2$, namely

$$(3.1)\quad Gl_2(\mathbb{K})\times\mathbb{K}^2 \ni \left(\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}, \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right) \longrightarrow \begin{pmatrix}[\alpha\ c]\\ [\alpha\ d]\end{pmatrix} := \begin{pmatrix}\alpha_1 c_{11} - \alpha_2 c_{21}\\ \alpha_1 c_{12} - \alpha_2 c_{22}\end{pmatrix} \in \mathbb{K}^2,$$

(cf. (2.6)).

Let $E_n(\alpha) \subset D_n(\alpha)$ be the subspace of exact differential 1-forms. $E_n(\alpha)$ is generated by the following linearly independent elements: $\{d(\alpha_1^k\alpha_2^{n-k})\}_{k=0}^{n}$. Let $K_n(\alpha)$ denote the subspace of $D_n(\alpha)$ generated by $\{\alpha_1^r\alpha_2^{n-r-2}[\alpha\ d\alpha]\}_{r=0}^{n-2}$, where $\alpha\ d\alpha$ is an ordinary bracket with $d\alpha = (d\alpha_1, d\alpha_2)$.

<u>Proposition 3.1.</u> There exists a uniquely defined $Gl_2(\mathbb{K})$-equivariant projection $\mathbb{P}\colon D_n(\alpha) \to E_n(\alpha)$, $\mathrm{Ker}\mathbb{P} = K_n(\alpha)$.

Proof. $E_n(\alpha)$, $K_n(\alpha)$ are $Gl_2(\mathbb{K})$-invariant subspaces of $D_n(\alpha)$, $\dim E_n(\alpha) = n+1$, $\dim K_n(\alpha) = n-1$, $\dim D_n(\alpha) = 2n$. Let us choose the following basis in $D_n(\alpha)$,

$$e_{p,i} = \alpha_1^p\alpha_2^{n-p-1}d\alpha_i, \quad p = 0,\dots,n-1, \quad i=1,2.$$

Then the corresponding generators for $E_n(\alpha)$ and $K_n(\alpha)$ can be expressed as

$$d(\alpha_1^k\alpha_2^{n-k}) = ke_{k-1,1} + (n-k)e_{k,2}\ ; \quad k=0,\dots,n$$

and

$$\alpha_1^r\alpha_2^{n-r-2}[\alpha\ d\alpha] = e_{r+1,2} - e_{r,1}; \quad r = 0,\dots,n-2$$

respectively.

The associated $2(n-1)\times 2(n-1)$ matrix

$$\left(\begin{array}{cc|cc} 1 & 0 & n-1 & 0 \\ & \ddots & & \ddots \\ 0 & 1 & 0 & 1 \\ \hline -1 & 0 & 1 & 0 \\ & \ddots & & \ddots \\ 0 & -1 & 0 & 1 \end{array}\right)$$

has the same rank as the following matrix:

$$\left(\begin{array}{cc|cc} 1 & 0 & n-1 & 0 \\ & \ddots & & \ddots \\ 0 & n-1 & 0 & 1 \\ \hline & & n & \\ & 0 & & \ddots \\ & & & \frac{n}{n-1} \end{array}\right)$$

Thus $D_n(\alpha) = E_n(\alpha) \oplus K_n(\alpha)$. The action μ restricted to $E_n(\alpha)$ and $K_n(\alpha)$ is irreducible, which implies the uniqueness of the projection $\mathbb{P}$.

Now we can define the umbral operator U_α^* into the space of differential 1-forms over the space of binary forms M^{n+1}. Let U be the standard umbral operator (see Section 2) defined on $\mathcal{U} = \mathbb{K}[\alpha_1,\alpha_2]$. Let U' be the restriction of U to the space $\mathcal{U}_n(\alpha)$. Now we introduce an operator $\bar{U}$ defined only on $E_n(\alpha)$ $D_n(\alpha)$ and satisfying the following commutation relation

(3.2) $$d \circ U' = \bar{U} \circ d.$$

Thus for the elements of the basis of $E_n(\alpha)$ we have

(3.3) $$<\bar{U}|d(\alpha_1^k\alpha_2^{n-k})> = da_k, \quad k=0,\ldots,n.$$

Definition 3.2. The linear operator

(3.4) $$U_\alpha^* := \bar{U}\circ\mathbb{P}\colon D_n(\alpha) \longrightarrow V^{n+1} = (M^{n+1})^* ,$$

defined from the umbral space $D_n(\alpha)$ to the space V^{n+1} of differential 1-forms on M^{n+1}, is called the elementary umbral operator for the space of binary forms of degree n.

Proposition 3.3. U_α^* is a $Gl_2(\mathbb{K})$-equivariant linear operator.

Proof. On the basis of Proposition 3.1, $\mathbb{P}$ is $Gl_2(\mathbb{K})$-equivariant so we have to prove that $\bar{U}$ is also $Gl_2(\mathbb{K})$-equivariant. In fact, using the bracket notation (3.1) we have (cf.[12]) that for any polynomial $I \in \mathbb{K}[a_0,\ldots,a_n]$ and its umbral representation $P \in \mathbb{K}[\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2]$ the change of variables (c,d) is expressed (by the formula (2.7)) as follows:

$$I(\bar{a}_0,\ldots,\bar{a}_n) = <U(\bar{f})|P(\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2)> =$$
$$= <U(f)|P([\alpha\ c],[\alpha\ d],\ldots,[\beta\ c],[\beta\ d])> .$$

Thus for the operator $\bar{U}$ we have (see (3.2)),

$\langle\bar{U}(f)|d([\alpha\ c]^k[\alpha\ d]^{n-k})\rangle = d\langle U'(f)|[\alpha\ c]^k[\alpha\ d]^{n-k}\rangle =$

$= d\langle U'(\bar{f})|\alpha_1^k\alpha_2^{n-k}\rangle = d\bar{a}_k\ ,\quad k=0,\dots,n\ ;$

where by $\bar{f}(\bar{x},\bar{y}) = \sum_{k=0}^{n}\binom{n}{k}\bar{a}_k\bar{x}^k\bar{y}^{n-k}$ we denote the transformed binary forms in the new variables, and $U(f)$, $U(\bar{f})$ denote the corresponding umbral operator U written using these two types of variables. This completes the proof.

By the extension of U_α^* to the tensor product of p factors, say $W_{n,p} = D_n(\alpha)\otimes\dots\otimes D_n(\beta)$, we obtain the partial umbral operator for representing the corresponding tensor invariants of degree p:

$$U^*_{(\alpha,\dots,\beta)}: D_n(\alpha)\otimes\dots\otimes D_n(\beta) \longrightarrow \overset{p}{\otimes}V^{n+1},$$

(3.5)

$$\langle U^*_{(\alpha,\dots,\beta)}|w_1(\alpha)\otimes\dots\otimes w_p(\beta)\rangle = \langle U_\alpha^*|w_1(\alpha)\rangle\otimes\dots\otimes\langle U_\beta^*|w_p(\beta)\rangle\ .$$

The formula computing the effect of a change of variables in the standard umbral representations of polynomial invariants (cf.[12],[16]) as well as Proposition 3.3, suggest a subspace of the umbral space $W_{n,p}$ whose umbral evaluations are obviously invariants. Let us define the bracket monomials, say for two umbral letters:

$$[\alpha\ \beta] = \alpha_1\beta_2 - \alpha_2\beta_1$$

(3.6)

$$[\alpha\ d\beta] = \alpha_1 d\beta_2 - \alpha_2 d\beta_1$$

$$[d\alpha\otimes d\beta] = d\alpha_1\otimes d\beta_2 - d\alpha_2\otimes d\beta_1 .$$

<u>Definition 3.4.</u> A bracket monomial $q \in W_{n,p}$ is a nonconstant polynomial in $W_{n,p}$ which can be written as an appropriate product of brackets (3.6). A bracket polynomial is a linear combination of bracket monomials. The linear subspace of $W_{n,p}$ formed by the bracket polynomials is denoted by $B_{n,p}$.

The total umbral space is defined as $W_n = \bigoplus_{p\in\mathbb{N}} W_{n,p}$ and the corresponding umbral operator U^* is defined as the respective direct sum of the partial umbral operators. Let $g \in \mathbb{N}$. A nonconstant polynomial $Q \in \bigoplus_{p\in\mathbb{N}} \overset{p}{\otimes} V^{n+1}$ is said to be an invariant of index g if for all binary forms f(x,y) of degree n and for all linear changes of variables (c,d) the following identity holds:

$$\tilde{Q} = [c\ d]^g Q.$$

The aim of this section is to provide the methods for determining as explicitely as possible all the tensor invariants of binary forms.

The index of a bracket monomial $q \in W_n$ is the number of brackets in q. The bracket polynomials in W_n which are linear combinations of bracket monomials all of the same index g, are called bracket polynomials of index g.

<u>Theorem 3.5.</u> Let $U^*_{(\alpha,\dots,\beta)}$ be the umbral operator into the tensor space $\overset{p}{\otimes} V^{n+1}$. Let $\phi \in B_{n,p}$ be the bracket polynomial of index g. Then the umbral evaluation of ϕ , $\langle U^*_{(\alpha,\dots,\beta)}|\phi\rangle$, is an invariant of index g.

Proof. Let q be a bracket monomial of index g. Let us change the umbral variables, i.e.

$$[\tilde{\alpha}\ \tilde{\beta}] = \det\begin{pmatrix}[\alpha\ c] & [\beta\ c]\\ [\alpha\ d] & [\beta\ d]\end{pmatrix} = [c\ d]\,[\alpha\ \beta]$$

$$[\alpha\ d\beta] = \det\begin{pmatrix}[\alpha\ c] & [d\beta\ c]\\ [\alpha\ d] & [d\beta\ d]\end{pmatrix} = [c\ d]\ \ [\alpha\ d\beta]$$

$$[d\alpha\otimes d\beta] = \det\begin{pmatrix}[d\alpha\ c] & [d\beta\ c]\\ [d\alpha\ d] & [d\beta\ d]\end{pmatrix} = [c\ d]\ \ [d\alpha\otimes d\beta].$$

Thus, for any binary form f(x,y) and for any change of variables (c,d), we have on the basis of Proposition 3.3,

$$\langle U^*_{(\alpha,\dots,\beta)}(\bar{f})|q\rangle = \langle U^*_{(\alpha,\dots,\beta)}(f)|[c\ d]^g q\rangle = [c\ d]^g \langle U^*_{(\alpha,\dots,\beta)}(f)|q\rangle,$$

which implies that $\langle U^*_{(\alpha,\ldots,\beta)} | \phi \rangle$ is a tensor invariant of index g and degree p.

The converse of this theorem is also true.

Theorem 3.6. Let Q be a tensor invariant of index g and degree p for binary forms of degree n. Then there exists a bracket polynomial $\phi \in B_{n,p}$ of index g such that

$$Q = \langle U^*_{(\alpha,\ldots,\beta)} | \phi \rangle \quad .$$

Proof. In order to prove this theorem we must first study the combinatorics of bracket polynomials (cf. [12]). Let us order the umbral alphabet in such a way that $\alpha < \beta < \ldots < d\alpha < d\beta < \ldots$. Let q be a bracket monomial. Thus q is a product, say $[\alpha\ \beta][\alpha\ \gamma]\ldots[\delta\ d\delta]\otimes[d\varepsilon\otimes d\rho]$ of s brackets. Using the notation introduced in [12] we rewrite q as a tableau of height s:

$$\begin{bmatrix} \alpha & \beta \\ \alpha & \gamma \\ & \vdots \\ \delta & d\delta \\ d\varepsilon & d\rho \end{bmatrix} = [\alpha\ \beta][\alpha\ \gamma]\ldots[\delta\ d\delta]\otimes[d\varepsilon\otimes d\rho]$$

We will call such a tableau ordered if the letters in each row are increasing from left to right, and the letters in each column are nondecreasing from top down.

We easily see, using the syzygy relations

$$[\alpha\ \delta][\beta\ \gamma] = [\alpha\ \gamma][\beta\ \delta] - [\alpha\ \beta][\gamma\ \delta]$$

$$[\gamma\ \delta][d\alpha\otimes d\beta] = [d\alpha\ \gamma]\otimes[d\beta\ \delta] - [d\alpha\ \delta]\otimes[d\beta\ \gamma]$$

$$[\gamma\ \delta][d\alpha\ \beta] = [d\alpha\ \gamma][\beta\ \delta] - [d\alpha\ \delta][\beta\ \gamma]$$

and antisymmetry

$$[\alpha\ \beta] = -[\beta\ \alpha]$$

$$[d\alpha\otimes d\beta] = -[d\beta\otimes d\alpha] = d\alpha_1\otimes d\beta_2 - d\alpha_2\otimes d\beta_1$$

that the ordered bracket monomials form a basis for the vector space of all bracket polynomials. In fact, treating the differentials combinatorially as ordered symbols and assuming the existence of the nontrivial linear dependence relation between ordered bracket monomials with smallest number of distinct symbols and smallest height of bracket monomials, we come easily to a contradiction setting two highest symbols to be equal (cf.[12], p. 37).

Let Q be a tensor invariant of index g and degree p, let $P \in W_{n,p}$ be its umbral representation. As Q is an invariant, we have for any change of variables (c,d)

(i) $\bar{Q}=[c\ d]^g Q = \langle U^*_{(\alpha,\dots,\beta)} | P([\alpha\ c],[\alpha\ d],\dots,[\beta\ c],[\beta\ d])\rangle$.

This identity is true as a polynomial identity in the variables c_1, c_2, d_1, d_2. Using this fact we can prove that $P([\alpha\ c],$ $[\alpha\ d],\dots,[\beta\ c],[\beta\ d]) = [c\ d]^g R(\alpha_1,\alpha_2,\dots,\beta_1,\beta_2)$, where $R \in B_{n,p}$. We can easily see that the polynomial P may be so chosen that the letter c, as well as the letter d, occurs exactly g times in each of the ordered monomials q_k contained in the expansion of $P([\alpha\ c],[\alpha\ d],\dots)$ as a linear combination of ordered bracket monomials. Here the new alphabet $\{c, d, \alpha, \beta,\dots,d\alpha, d\beta,\dots, d\varepsilon\}$ is ordered as follows: $c < d < \alpha < \beta \dots < d\varepsilon$. In fact, replacing c_1 and c_2 by rc_1 and rc_2, we obtain the polynomial identity

$$r^g[c\ d]^g Q = \langle U^*_{(\alpha,\dots,\beta)} | \sum_k b_k r^{c(k)} q_k \rangle,$$

where c(k) is the number of occurences of c in the bracket monomial q_k. Equating coefficients of r^g we see that the bracket monomials with $c(k) \neq g$ can be omitted in P. Let q_k be a bracket monomial in this improved expansion of P as a linear combination of ordered bracket monomials. Let s(k) be the number of brackets

$[c\ d]$ occuring in q_k. Let s be the minimum of these integers $s(k)$. We see that $s \leqslant g$. If $s = g$ we can simply cancel $[c\ d]^g$ from both sides of (i). Let us suppose $s < g$, writting $q_k = [c\ d]^s q'_k$ we can cancel $[c\ d]^s$ from both sides of (i) to obtain

(ii) $$[c\ d]^{g-s} Q = \langle U^*_{(\alpha,\dots,\beta)} | \sum_k b_k q'_k \rangle .$$

Treating (ii) as a polynomial identity in the variables c_1, c_2, d_1, d_2, we can therefore set $c_1 = d_1$, $c_2 = d_2$. This yields the identity:

(iii) $$\langle U^*_{(\alpha,\dots,\beta)} | \sum_k b_k \hat{q}_k \rangle = 0,$$

where $\hat{q}_k$ is obtained from q'_k by setting $c = d$.
As we know the ordered bracket monomials q'_k as well as $\hat{q}_k$ are linearly independent. Because $\sum_k b_k \hat{q}_k \in \operatorname{Ker} U^*_{(\alpha,\dots,\beta)}$ and $\hat{q}_k$ are linearly independent, each monomial $\hat{q}_k$ can be written as a tableau

(iv) $$\begin{bmatrix} c & * \\ \vdots & \\ c & * \\ * & * \\ \vdots & \\ \varepsilon & d\varepsilon \\ \vdots & \\ * & * \end{bmatrix} ,$$

where c occurs in the first $2(g-s)$ rows and an asterisk stands for the rest of letters. By inspection of (iv) we can deduce the corresponding elements q'_k . They can be written in the form

$$\begin{bmatrix} c & * \\ \vdots & \\ c & * \\ d & * \\ \vdots & \\ d & * \\ * & * \\ \vdots & \\ \varepsilon & d\varepsilon \\ \vdots & \\ * & * \end{bmatrix}$$

where c occurs as the first letter in the first g-s rows and d occurs as the first letter in the next g-s rows with the additional bracket $[\varepsilon\ d\varepsilon]$ standing also in (iv). Thus we have that

$$\sum_k b_k q_k' \in \operatorname{Ker} U^*_{(\alpha,\dots,\beta)}$$

and we can cancel in the original polynomial P all terms giving the smallest number s of brackets $[c\ d]$ occuring in q_k. Repeating this procedure we obtain the result of Theorem 3.6.

Corollary 3.7. On the space of binary forms of odd degree the odd degree tensor invariants do not exist.

Proof. On the basis of Theorem 3.6, for the tensor invariant Q of degree p and index g we can write

$$\bar{Q} = \langle U^*_{(\alpha,\dots,\beta)} | R([\alpha\ c],[\alpha\ d],\dots)\rangle = [c\ d]^g Q,$$

where R is the corresponding bracket polynomial. On the other hand, taking the new parameters $c \longrightarrow tc$, $d \longrightarrow td$ we obtain the following equality (polynomial in t):

$$t^{np} \langle U^*_{(\alpha,\dots,\beta)} | R([\alpha\ c],[\alpha\ d],\dots)\rangle = t^{2g}[c\ d]Q.$$

Thus we see that for odd numbers n only for even number p the integer g can exists.

4. Invariant symplectic structure on the space of binary forms

Let us give now the complete classification of the tensor invariants of degree two, i.e. we assume p=2.

Theorem 4.1. There exists only one (up to constant multiples) tensor invariant of degree two on the space of binary forms.

Proof. Let n be the degree of binary forms. $B_{n,2}$ is generated by the following basis of ordered bracket monomials: $v_1=$

$= [\alpha\ \beta]^{n-1}[d\alpha \otimes d\beta]$, $v_2 = [\alpha\ \beta]^{n-2}[\alpha\ d\alpha]\otimes[\beta\ d\beta]$. We see that the third admissible bracket monomial $w = [\alpha\ \beta]^{n-2}[\beta\ d\alpha]\otimes[\alpha\ d\beta]$, by the appropriate syzygy, can be written as follows
$w = [\alpha\ \beta]^{n-2}([\alpha\ d\alpha]\otimes[\beta\ d\beta]-[\alpha\ \beta][d\alpha\otimes d\beta]) = v_2 - v_1$. We see also that $v_2 \in \mathrm{Ker}\ U^*_{(\alpha,\beta)}$. Thus

$$\dim(\ J_{n,2} = \mathrm{Im}\ U^*_{(\alpha,\beta)}|_{B_{n,2}}) = 1,$$

which completes the proof.

Now we are asking for the normal forms of the corresponding tensor invariants.

Proposition 4.2. All tensor invariants of degree two on the space of binary forms of degree n are proportional to the following basic invariant:

$$(4.1)\qquad Q = \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j+1}((-1)^n da_{j+1}\otimes da_{n-j-1} + da_{n-j-1}\otimes da_{j+1}),$$

i.e. $J_{n,2} = \{Q\}$.

Proof. On the basis of Theorem 4.1, as a generator of $J_{n,2}$ we can take the following invariant:

$$(i)\qquad Q = \langle U^*_{(\alpha,\beta)}|n^2[\alpha\ \beta]^{n-1}[d\alpha\otimes d\beta]\rangle = \langle U^*_{(\alpha,\beta)}|n^2\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{n-j-1}($$

$$e_{j,1}\otimes\tilde{e}_{n-j-1,2} - e_{j,2}\otimes\tilde{e}_{n-j-1,1})\rangle ,$$

where we denoted

$$e_{p,i} = \alpha_1^p\alpha_2^{n-p-1}d\alpha_i,\ \tilde{e}_{p,i} = \beta_1^p\beta_2^{n-p-1}d\beta_i,\quad p=0,\dots,n-1;\ i=1,2.$$

Recalling (3.5) we obtain the action of $U^*_{(\alpha,\beta)}$ on the basic elements:

$$\langle U^*_{(\alpha,\beta)}|e_{k,1}\otimes\tilde{e}_{p,1}\rangle = (1/_{n}2)da_{k+1}\otimes da_{p+1},$$

$$\langle U^*_{(\alpha,\beta)}|e_{k,1}\otimes\tilde{e}_{p,2}\rangle = (1/_{n}2)da_{k+1}\otimes da_{p},$$

$$\langle U^*_{(\alpha,\beta)}|e_{k,2}\otimes\tilde{e}_{p,1}\rangle = (1/_{n}2)da_{k}\otimes da_{p+1},$$

$$\langle U^*_{(\alpha,\beta)} | e_{k,2} \otimes \tilde{e}_{p,2} \rangle = (1/_{n}2) da_k \otimes da_p \ , \quad 0 \leqslant p,k \leqslant n-1.$$

Applying these formulae to (i) we obtain the desired invariant Q of the proposition.

From the formula (4.1) we see that Q is a symmetric tensor on the space of binary forms of even degree and antisymmetric if n is odd. Thus we have

<u>Corollary 4.3.</u> On the space of binary forms of odd degree there exists only one (up to constant multiples) $Sl_2(\mathbb{K})$-invariant symplectic structure.

Proof. Taking $n=2k+3$ in (4.1), after straightforward calculations we obtain

$$Q = \sum_{r=0}^{k+1} \binom{n}{r} (-1)^{r+1} da_r \wedge da_{n-r}, \tag{4.2}$$

which is a closed, nondegenerate, $Sl_2(\mathbb{K})$-invariant two-form.

<u>Remark 4.4.</u> On the basis of (4.2) we can choose the symplectic form on M^{n+1} as follows

$$\omega = n!(-1)^{k-1} \sum_{r=0}^{k+1} \binom{n}{r} (-1)^{r+1} da_r \wedge da_{n-r} \ . \tag{4.3}$$

Choosing the new coordinates

$$q_r = \frac{n!}{r!} a_{n-r}, \quad p_r = (-1)^{k-r} \frac{n!}{(n-r)!} a_r, \quad r = 0, \dots, k+1. \tag{4.4}$$

on M^{n+1}, we can write (4.3) in the following Darboux form (cf. [1], [18])

$$\omega = \sum_{j=0}^{k+1} dp_j \wedge dq_j \ ,$$

where the elements of M^{n+1} can be written as follows

$$\begin{aligned} M^{n+1} \ni f(x,y) = {} & q_o \frac{x^{2k+3}}{(2k+3)!} + \dots + q_{k+1} \frac{x^{k+2} y^{k+1}}{(k+2)!} - p_{k+1} \frac{x^{k+1} y^{k+2}}{(k+1)!} + \\ & \dots + (-1)^{k+2} p_o y^{2k+3}. \end{aligned} \tag{4.5}$$

This is exactely the $Sl_2(\mathbb{K})$-invariant symplectic structure mentioned only in [9] in the context of the generalized Newton equation as well as in the obstacle problem [5].

$Sl_2(\mathbb{K})$ acts symplectically on (M^{n+1},ω). Thus we have

Proposition 4.5. The momentum mapping corresponding to the standard $Sl_2(\mathbb{K})$-action on (M^{n+1},ω) is the Ad^*-equivariant quadratic momentum mapping (cf. [2]). In the coordinates of (4.5) it can be written as follows:

$J: M^{n+1} \longrightarrow sl_2(\mathbb{K})^*$; $J(\bar{p}) = (H_+, H_-, H_d)(\bar{p})$, where

$$H_+(\bar{p}) = \sum_{r=1}^{k+1} p_r q_{r-1} + \frac{1}{2} q_{k+1}^2,$$

$$H_-(\bar{p}) = \sum_{r=0}^{k} (2k+3-r)(r+1) p_r q_{r+1} - \frac{1}{2}(k+1)^2 p_{k+1}^2, \tag{4.6}$$

$$H_d(\bar{p}) = \sum_{r=0}^{k+1} (2r-2k-3) q_r p_r,$$

and $\{H_+, H_-\} = H_d$, $n = 2k+3$.

Proof. Taking the standard decomposition of $Sl_2(\mathbb{K})$ onto the three one-parameter subgroups (cf [19])

$$A_+: \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \; ; \; A_-: \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} \; ; \; D: \begin{bmatrix} d^{-1} & 0 \\ 0 & d \end{bmatrix} , \quad \alpha, \beta, d \in \mathbb{K} , \tag{4.7}$$

we obtain the three corresponding Hamiltonian vector fields, say $\bar{A}_+$, $\bar{A}_-$, $\bar{D}$, with the corresponding Hamiltonians H_+, H_-, H_d. Thus, after straightforward calculations, the momentum mapping follows immediately (cf. [1],[2]).

Remark 4.6. Taking into account the relation $\{H_+, H_-\} = H_d$ we can reformulate the fundamental theorem (Cayley, Sylvester [16]) of the theory of invariants of binary forms of odd degree, i.e. we have: A polynomial $\phi(q,p)$ on the space of binary forms M^{n+1} is $Sl_2(\mathbb{K})$-invariant if and only if the following identities are fulfilled: $\{H_+,\phi\} = 0$, $\{H_d,\phi\} = 0$. Thus we easily see that the algebra of polynomial $Sl_2(\mathbb{K})$ invariants of binary forms is endowed with the canonical Poisson structure (cf. [18]).

In the obvious way, by the multiplicative rule, we can extend the umbral operator U^* (see §3) to have an umbral representation of tensor invariants with polynomial coefficients. In fact we define

$$\tilde{W}_n = \bigoplus_{p,s\in\mathbb{N}} W_{n,p}\otimes_{\mathbb{K}}\mathbb{K}_s[\gamma,\ldots,\delta],$$

where "s" is a number of umbral letters $\gamma,\ldots,\delta$. Thus the respective extension of U^*, say $\tilde{U}^*$, on the homogeneous elements of $\tilde{W}_n$ is defined as follows

$$\langle\tilde{U}^*|\phi\otimes\gamma_1^i\gamma_2^j\ldots\delta_1^k\delta_2^l\rangle = \langle U^*|\phi\rangle\langle U|\gamma_1^i\gamma_2^j\rangle \ldots \langle U|\delta_1^k\delta_2^l\rangle \quad \text{(cf. (2.5),§2).}$$

After straightforward reformulations we immediately obtain the fundamental classification theorems, analogous to Theorem 3.5 and Theorem 3.6 of §3. As a simple example, we apply these theorems to classify all tensor invariants of degree one with coefficients being the linear polynomials in variables $a_o,\ldots,a_n$.

Proposition 4.7. All tensor invariants of degree one with linear polynomial coefficients on the space of binary forms of degree n are proportional to the following basic invariant:

$$I = \sum_{j=0}^{n}(-1)^{n-j-1}\binom{n}{j}a_{n-j}da_j \tag{4.8}$$

Proof. Let us take the relations between elementary umbral monomials (cf. Proposition 3.1):

$$e_{k,1} = \frac{1}{n}(w_{k+1} - (n-k-1)g_k)$$

$$e_{k,2} = \frac{1}{n}(w_k + kg_{k-1}), \quad k = 0,\ldots, n-1,$$

where

$$e_{k,i} = \alpha_1^k\alpha_2^{n-k-1}d\alpha_i, \quad w_k = d(\alpha_1^k\alpha_2^{n-k}), \quad g_k \in \mathrm{Ker}U_\alpha^* .$$

It is easy to see that the space of irredundant bracket polynomials in $D_n(\alpha)\otimes\mathcal{U}_n(\beta)$ is spanned by the bracket monomial $n[\alpha\ \beta]^{n-1}[\beta\ d\alpha]$. Thus we have:

$$<\tilde{U}^*_{(\alpha;\beta)}|n[\alpha\ \beta]^{n-1}[\beta\ d\alpha]>= n\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j}<U^*_\alpha|e_{j,2}><U|\beta_1^{n-j}\beta_2^{j}>$$

$$-\ n\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j}<U^*_\alpha|e_{j,1}><U|\beta_1^{n-j-1}\beta_2^{j+1}> \ = I.$$

Corollary 4.8. Let $n=2k+3$. Then the corresponding $Sl_2(\mathbb{K})$ invariant one-form on the space of binary forms of degree n, in Darboux coordinates has the following form

$$\theta = \sum_{j=0}^{k+1}(p_j dq_j - dp_j q_j). \tag{4.9}$$

Remark 4.9. We know (cf. [4],p.306) that the projective space $\mathbb{R}P^n$ of all zero-dimensional submanifolds of degree n in the projective line is endowed with the natural $Sl_2(\mathbb{R})$-invariant contact structure (cf.[1],[6]). Indeed we see that the appropriate $Sl_2(\mathbb{R})$-invariant field of hyperplanes in $\mathbb{R}P^n$ is defined by the $Sl_2(\mathbb{R})$-invariant 1-form $V\lrcorner\omega$, where $V=\sum_{i=0}^{n}a_i\frac{\partial}{\partial a_i}$ is $Sl_2(\mathbb{R})$-invariant and ω is given in Corollary 4.3. In the affine part of $\mathbb{R}P^n$ formed by the zero-dimensional submanifolds of $\mathbb{R}P^1$, which do not contain the point in infinity y=0, this contact structure is given by (see Corollary 4.8)

$$\theta|_{q_0=1} = \sum_{j=1}^{k+1}(p_j dq_j - q_j dp_j) - dp_0.$$

This is the canonical contact structure on the space of polynomials

$$\left\{\frac{x^{2k+3}}{(2k+3)!}+q_1\frac{x^{2k+2}}{(2k+2)!}+\dots+q_{k+1}\frac{x^{k+2}}{(k+2)!}-p_{k+1}\frac{x^{k+1}}{(k+1)!}+\dots+(-1)^{k+2}p_0\right\}.$$

Thus all the results concerning the symplectic geometry of polynomial spaces have a direct reformulation in terms of the above introduced contact geometry (cf. [4]). The more precise analysis of this case we leave to a forthcoming paper.

5. The hierarchy of apolar coisotropic manifolds and generalized open swallowtails

Two binary forms $f(x,y)$ and $g(x,y)$ are said to be apolar if their apolar covariant $\langle f|g\rangle$ (cf. §2) is the identically zero form. The apolarity notion is very convenient in describing the varieties of binary forms with multiple linear factors. The main classical result which allows us to associate the apolar covariants with the basic symplectic geometry of binary forms (Sylvester's theorem, see e.g. [12], p.64) is following: Let $\langle f|g\rangle = 0$, where f is a binary form of degree $n=2j+1$ and g is a nonzero form of degree $m=j+1$. Then, $f(x,y)= \sum_{i=1}^{p} h_i(x,y)(\mu_i x - \nu_i y)^{n-m_i+1}$, if $g(x,y) = (\mu_1 x-\nu_1 y)^{m_1}\dots(\mu_p x-\nu_p y)^{m_p}$ is a factorization of g into p distinct linear forms.

Let (M,ω) be a symplectic manifold. The new symplectic structures associated to (M,ω) are provided by the so-called coisotropic submanifolds in M (cf. [1], [18]). We recall that a submanifold $C \subseteq M$ is coisotropic if at each $x \in C$ we have $(T_xC)^{\S}=\{v\in T_xM;\ \langle v \wedge u,\omega\rangle=0,\text{ for every } u \in T_xC\} \subsetneq T_xC$.

The distribution $\Gamma= \bigsqcup_{x\in C} (T_xC)^{\S}$ is the characteristic distribution of $\omega|_C$. Let B be the space of characteristics of it and $\rho: C \to B$ be its canonical projection. It is known (cf. [18]) that if B admits a differentiable structure and ρ is a submersion, then there is a unique symplectic structure β on B such that $\rho^{*}\beta=\omega|_{C}$. The symplectic manifold (B,β) associated in this way with the triplet (M,ω,C) is called the reduced symplectic manifold (cf. [1]).

Let (M^{n+1},ω) be the symplectic space of binary forms (see §4). The canonical subspaces in M^{n+1}, say $C^{(l)}$, $0 \leqslant l \leqslant \frac{n-1}{2}$ of all binary forms apolar to its l-derivatives with respect to x are called the canonical apolar subspaces.

Proposition 5.1. The canonical apolar subspaces $C^{(l)}$, $(0 \leqslant l \leqslant \frac{n-1}{2})$ form the coisotropic varieties of (M^{n+1},ω).

Proof. We see that $C^{(l)}$ is described by the following system of l+1 equations:

$$P_0^{(l)}=\binom{n-l}{0} a_n a_0 - \binom{n-l}{1} a_{n-1}a_1 + \ldots \pm \binom{n-l}{n-l} a_l a_{n-l} = 0$$

$$P_1^{(l)}=\binom{n-l}{0} a_n a_1 - \binom{n-l}{1} a_{n-1}a_2 + \ldots \pm \binom{n-l}{n-l} a_l a_{n-l+1} = 0$$

$$P_l^{(l)}=\binom{n-l}{0} a_n a_l - \binom{n-l}{1} a_{n-1}a_{l+1} + \ldots \pm \binom{n-l}{n-l} a_l a_n = 0 \quad .$$

After straightforward calculations we find that

$$\{P_i^{(l)},\, P_j^{(l)}\}\Big|_{C^{(l)}} = 0,$$

where $\{\phi,\psi\}$ is the appropriate Poisson bracket (cf.[1]);

$$\{\phi,\psi\}=\frac{1}{n!}\sum_{i=0}^{k+1} (-1)^{k-i} \binom{n}{i}^{-1} \left(\frac{\partial\phi}{\partial a_i}\frac{\partial\psi}{\partial a_{n-i}} - \frac{\partial\phi}{\partial a_{n-i}}\frac{\partial\psi}{\partial a_i}\right), \quad n = 2k+3.$$

Thus the system of equations $P_i^{(l)}(a_0,\ldots,a_n)=0$, $i = 0,\ldots,l$ $(\langle f|f_x^{(l)}\rangle = 0)$ form the coisotropic subvariety of (M^{n+1},ω).

Now we pay more attention to the particular case $l = 1$.

Corollary 5.2. Let $l=1$, then the second apolar coisotropic variety $C^{(1)}$ can be expressed as follows:

$$C^{(1)} = \{f \in M^{n+1};\ \langle f|f'_x\rangle = n(P_o^{(1)}y + P_1^{(1)}x) = (-1)^{k+1}\frac{1}{n!}(yH_d + 2H_+x) \underset{x,y}{\equiv} 0 \},$$

where

$$\{P_1^{(1)},\ P_o^{(1)}\} = \frac{1}{nn!}(-1)^{k+1}P_1^{(1)} \quad \text{and } \{H_+,\ H_d\} = H_+.$$

Proof. Immediate, on the basis of (4.6), Proposition 5.1 and simple but tedious calculations (see Proposition 2.3. See also Example 2 in [5]p. 45).

To the space of binary forms of degree n one can easily associate the corresponding spaces of polynomials of one variable putting $y=1$ in (2.1). In order to have the polynomial symplectic spaces adapted to the investigations of singularities in the variational obstacle problem (see [5],[6],[14],[3]) we associate to every symplectic space (M^{n+1},ω) the canonically reduced symplectic space Q^{n-1} of polynomials of degree n-1 where leading term has constant coefficient $\frac{1}{(n-1)!}$. Q^{n-1} $C_o/\sim$, where "$\sim$" is given by the coisotropic submanifold $C_o := \{f \in M^{n+1};\ n!a_n = 1\}$. Q^{n-1} is identified canonically with the space of derivatives $\frac{d}{dx}(f(x,1))$, $f \in M^{n+1}$ belonging to C_o, namely

$$Q^{n-1} \ni \phi(x) = \frac{x^{2k+2}}{(2k+2)!} + q_1\frac{x^{2k+1}}{(2k+1)!} + \dots + q_{k+1}\frac{x^{k+1}}{(k+1)!} - p_{k+1}\frac{x^k}{k!} + \dots$$
$$\dots + (-1)^{k+1}p_1$$

endowed with the reduced symplectic structure

$$\omega' = \sum_{j=1}^{k+1} dp_j \wedge dq_j.$$

<u>Proposition 5.3.</u> The apolar subspaces $C^{(l)}$ $(l = 1,\dots,\frac{n-1}{2})$ of (M^{n+1},ω) induce the corresponding coisotropic subspaces of

(Q^{n-1},ω'), say $\tilde{C}^{(l)}$ $(l=1,\ldots,\frac{n-1}{2})$, described by

$$\tilde{C}^{(l)}=\{\phi\in Q^{n-1};\quad \tilde{P}_s^{(l)}(q,p)=0,\ s=1,\ldots,l\}\ ,\quad l=1,\ldots,k+1,$$

where

$$\tilde{P}_s^{(l)}=\frac{(-1)^k}{n!}\sum_{i=1}^{k-s+1}\binom{n-1}{i}\binom{n}{i}^{-1}q_ip_i+\frac{1}{n!^2}\sum_{i=k-s+2}^{k+1}(-1)^i\binom{n-1}{i}i!(n-s-i)!q_iq_{n-s-i}+\frac{(-1)^k}{n!^2}\sum_{i=k+2}^{n-1}\binom{n-1}{i}i!(n-s-i)!p_{n-i}q_{n-s-i}+a_s\ .$$

Proof. Let us observe that $(P_o^{(l)})^{-1}(0)\cap C_o$ is transversal to the characteristic distribution Γ of C_o. Thus $P_o^{(l)}$ does not give any constraint on the reduced space Q^{n-1}. It is easy to see also that the functions $P_s^{(l)}|_{C_o}$ $(l \geqslant s \geqslant 1)$ are constant along the integral manifolds of distribution Γ . Thus we obtain the new coisotropic constraints defined by these functions on Q^{n-1}, which after straightforward calculations are expressed in the form (5.1).

Now we investigate the properties of the symplectic space induced by the coisotropic submanifold $\tilde{C}^{(1)}$ in (Q^{n-1},ω'), $n=2k+3$.

<u>Proposition 5.4.</u> The reduced symplectic space corresponding to the triplet $(Q^{n-1},\omega',\tilde{C}^{(1)})$ is identified with the following space of polynomials

$$(5.2)\quad Z=\{\frac{x^{2k+1}}{(2k+1)!}+q_1\frac{x^{2k-1}}{(2k-1)!}+\ldots+q_k\frac{x^k}{k!}-p_k\frac{x^{k-1}}{(k-1)!}+\ldots+(-1)^kp_1\}$$

endowed with the reduced symplectic form $\bar{\omega}=\sum_{i=1}^{k}dp_i\wedge dq_i$.

Proof. The function $\tilde{P}_1^{(1)}$ as well as the Hamiltonian H_+ (see Corollary 5.2) corresponding to the one-parameter subgroup A_+ (cf. (4.7)) generates translations along variable x. Thus the space of characteristics of the coisotropic submanifold $\tilde{C}^{(1)}$ can be immediately identified with the derivatives of polynomials;

$$(5.3) \qquad \frac{x^{2k+2}}{(2k+2)!}+\bar{q}_1\frac{x^{2k+1}}{(2k+1)!}+\ldots+\bar{q}_{k+1}\frac{x^{k+1}}{(k+1)!}-\bar{p}_{k+1}\frac{x^k}{k!}+\ldots+(-1)^{k+1}\bar{p}_1$$

with an additional condition that the sum of all roots is equal to zero (cf. [6],[11]). This completes the proof.

As a polynomial parametrisation of characteristics of $\tilde{C}^{(1)}$, described in Proposition 5.4, we can write the following identification (cf. [11],[10], and (5.3) above),

$$\frac{(x-t)^{2k+1}}{(2k+1)!}+\bar{q}_1\frac{(x-t)^{2k}}{(2k)!}+\ldots+\bar{q}_{k+1}\frac{(x-t)^k}{k!}-\bar{p}_{k+1}\frac{(x-t)^{k-1}}{(k-1)!}+\ldots+(-1)^k\bar{p}_2 =$$
$$\frac{x^{2k+1}}{(2k+1)!}+q_1\frac{x^{2k-1}}{(2k-1)!}+\ldots+q_k\frac{x^k}{k!}-p_k\frac{x^{k-1}}{(k-1)!}+\ldots+(-1)^k p_1.$$

Thus we immediately have

<u>Corollary 5.5.</u> Let $m \geqslant \left[\frac{n}{2}\right]$. Then the sets of polynomials of Z having a root of multiplicity m, say $L^{(n)}_{m-1}$, form the isotropic (see [18]) varieties in $(Z,\tilde{\omega})$. The maximal isotropic variety, i.e. $m = \left[\frac{n}{2}\right]$, is a lagrangian variety (cf. [11]) symplectomorphic, in the case of $n = 7$, to the system of rays on the obstacle, with the highest generic singularity, so-called open swallowtail singularity (cf. [5], [6] and Figure 1, below)

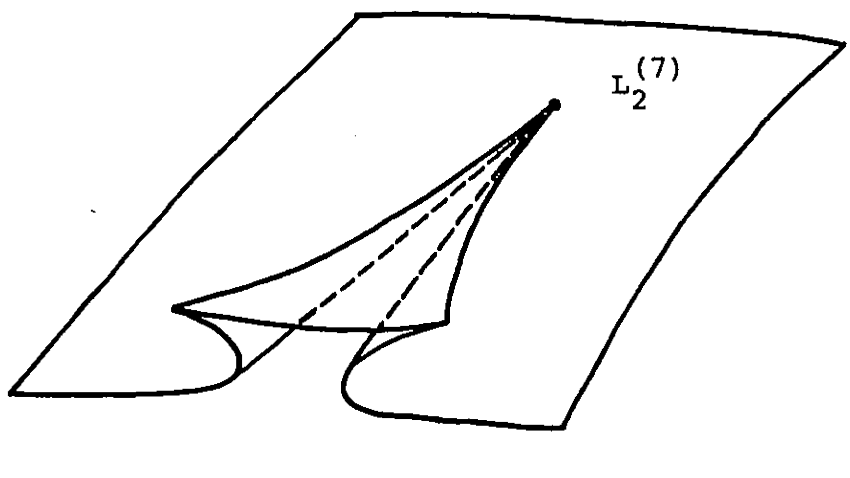

Fig. 1.

<u>Remark 5.6.</u> Let us notice that the open swallowtail singularities in $(Z,\bar{\omega})$ are connected with the structure of the space of the Hilbert's zero-forms (cf. [16],[19]), and are quite exceptional. We can easily see that the variety V of polynomials in $(Z,\bar{\omega})$ with maximal possible number of double roots is not lagrangian (cf. [5],p.37, where it was claimed that it is lagrangian). One can easily check this for k=2. In fact we have

$$\frac{1}{5!}x^5+q_1\frac{1}{3!}x^3+q_2\frac{1}{2}x^2-p_2x+p_1=\frac{1}{5!}(x-\alpha)^2(x-\beta)^2(x-\gamma)$$

and the corresponding immersion of the smooth strata of V is following

$$q_1=\frac{1}{20}(2w-3z^2),\qquad q_2=\frac{1}{30}(wz+z^3),$$
$$p_1=\frac{1}{60}w^2z\ ,\quad p_2=\frac{1}{120}(4wz^2-w^2)\ ,$$

where $z=\alpha+\beta$, $w=\alpha\beta$, $2\alpha+2\beta+\gamma=0$.
By straightforward calculations we obtain

$$\bar{\omega}|_V=dp_1\wedge dq_1+dp_2\wedge dq_2|_V=\left(\frac{1}{450}w^2+\frac{23}{1800}wz^2-\frac{1}{300}z^4\right)dz\wedge dw\neq 0.$$

Following the theory of generating families for the germs of lagrangian varieties presented in[11] one can describe the analytical structure of open swallowtails, i.e. $L_k^{(n)}$, using the polynomial functions. Let us recall that the function $F: Q\times\mathbb{R}^s\to\mathbb{R}$ is a generating family (with s-parameters) for the germ of lagrangian variety $L\subseteq(T^*Q,\omega_Q)$ if L can be locally written in the following way (cf.[18])

$$(5.4)\qquad L=\{(q,p)\in T^*Q;\ \exists_{\lambda\in\mathbb{R}^s},\ \frac{\partial F}{\partial q}(q,\lambda)=p,\ \frac{\partial F}{\partial\lambda}(q,\lambda)=0\}\ .$$

We see that $(Z,\bar{\omega})$ has a canonical cotangent bundle structure, $(Z,\bar{\omega})\simeq(T^*Q,\omega_Q)$. Thus we are able to calculate the global generating families for the general open swallowtails $L_k^{(n)}$.

Proposition 5.7. An open k-dimensional swallowtail $L_k^{(n)} \subseteq (Z,\bar{\omega})$ is represented, in the form (5.4), by the following one-parameter generating family $G_k : Q \times \mathbb{R} \longrightarrow \mathbb{R}$;

$$G_k(q,\lambda)= \sum_{i=-1}^{k-2}\ \sum_{s=2}^{k-i-1}\ \sum_{u=2}^{s}\ \sum_{r=2}^{k-i} D^{(k)}_{k-i,s}A_{s-u}A_{k-i-r}(q_{u-1}+(-1)^u \frac{u-1}{u!}\lambda^u)(q_{r-1}+(-1)^r \frac{r-1}{r!}\lambda^r)\lambda^{n-u-r}+ \frac{1}{2}\sum_{i=0}^{k-2}\ \sum_{u=2}^{k-i}\ \sum_{r=2}^{k-i} D^{(k)}_{k-i,k-i}A_{k-i-u}A_{k-i-r}(q_{u-1}+(-1)^u \frac{u-1}{u!}\lambda^u)(q_{r-1}+(-1)^r \frac{r-1}{r!}\lambda^r)\lambda^{n-u-r}+ \sum_{i=0}^{k-2}\ \sum_{r=2}^{k-i} E^{(k)}_{k-i}A_{k-i-r}(q_{r-1}+(-1)^r \frac{r-1}{r!}\lambda^r)\lambda^{n-r}+ \frac{1}{2}D^{(k)}_{k+1,k+1}\sum_{i=2}^{k+1}\ \sum_{r=2}^{k+1} A_{k+1-i}A_{k+1-r}(q_{i-1}+(-1)^i \frac{i-1}{i!}\lambda^i)(q_{r-1}+(-1)^r \frac{r-1}{r!}\lambda^r)\lambda^{n-i-r}+ E^{(k)}_{k+1}\sum_{i=2}^{k+1} A_{k+1-i}(q_{i-1}+(-1)^i \frac{i-1}{i!}\lambda^i)\lambda^{n-i} - \frac{E_2^{(k)}}{2k+3}\lambda^{2k+3},$$

where

$$D^{(k)}_{r,s}= (-1)^{k-r}\sum_{j=s}^{k+1} \frac{(-1)^{j-s}}{(j-s)!(n-j-r)!},$$

$$E^{(k)}_{r}= (-1)^{k-r}\left(\frac{1}{(n-r)!} - \sum_{j=2}^{k+1} \frac{(-1)^j(j-1)}{j!(n-j-r)!}\right), \quad 1 \leq r,s \leq k+1$$

and the numbers A_k are given by the following recurrential formulae

$$A_0= 1, \qquad A_k = \sum_{i=1}^{k}\frac{1}{i!}(-1)^{i+1}A_{k-i}.$$

Proof. On the basis of Proposition 4.2 in [11] and the formulae for the characteristic curves of $\tilde{C}^{(1)}$. After straightforward calculations we obtain the corresponding generating one-parameter families for the open swallowtails in all dimensions.

Example 5.8. Let $k=1,2$, then the corresponding generating families for the cusp singularity (one-dimensional open swallowtail) and the standard (two-dimensional [3]) open swallowtail singularity of lagrangian varieties can be written directly, by Proposition 5.7, in the following way

cusp:

$$G_1(q,\lambda) = -\frac{1}{40}\lambda^5 - \frac{1}{6}\lambda^3 q - \frac{1}{2}\lambda q^2, \tag{5.5}$$

open swallowtail:

$$G_2(q_1,q_2,\lambda) = -\frac{1}{576}\lambda^7 - \frac{1}{30}\lambda^5 q_1 - \frac{1}{24}\lambda^4 q_2 - \frac{1}{6}\lambda^3 q_1^2 - \frac{1}{2}\lambda^2 q_1 q_2 - \frac{1}{2}\lambda q_2^2. \tag{5.6}$$

Remark 5.9. (singularities in the obstacle geometry [6]). Let Q be a hypersurface in $\mathbb{R}^3$. $T^*\mathbb{R}^3$ is the phase space of free particle. We take the hypersurface Y; $Y = \{(x,p) \in T^*\mathbb{R}^3;\ H(x,p) = \frac{1}{2}(|p|^2-1) = 0\}$. Let M denote the symplectic manifold of integral curves of the characteristic distribution of H. $\pi: Y \to M$ is the canonical projection along the integral curves. M is a symplectic manifold of oriented lines in $\mathbb{R}^3$, $M \cong T^*S^2$ (cf. [6]). Let γ be a geodesic flow on Q (determined by the point source of light in the space, [14]). Let $\tilde{L} \subseteq Y$ be the submanifold formed by versors tangent to the geodesics of γ along the surface Q.

Proposition. (cf. [3],[14]). A). $L=\pi(\tilde{L})$ is a lagrangian subvariety of $(M,\tilde{\omega})$. L is singular in the asymptotic points of γ (i.e. the corresponding line of L is also an asymptotic direction on Q) in a hyperbolic region of Q. Typically the asymptotic points of γ form a curve, say $\ell \subseteq Q$.

B). Let $p_o \in \ell$ be such that the corresponding geodesic of γ going through p_o is tangent to ℓ in p_o. Then the corresponding germ of lagrangian variety $(\pi(\tilde{L}),w_o)$ is the open swallowtail singularity (cf. [3]) symplectomorphic to $L_2^{(7)}$ described in Corollary 5.5. (the corresponding variety of rays gliding along the obstacle on the plane with the inflection point, is illustrated in Figure 2, below).

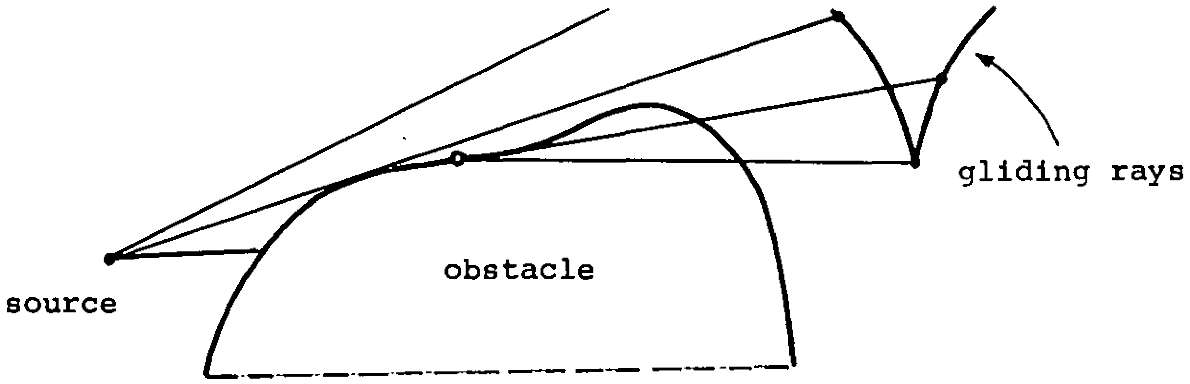

Figure 2.

Using the Huyghens principle (cf. [7],[8]) one can express the asymptotic intensity of radiation in the presence of an obstacle by the appropriate rapidly oscilating integrals with singular stationary varieties represented by the corresponding phase functions (optical distances), say

$$(5.7) \qquad \int_{\mathbb{R}^n} e^{i\tau\phi(x,\lambda)} a(x,\lambda,\tau)d\lambda \ , \ \tau \longrightarrow \infty.$$

For the open swallowtail singularities the phase functions (families) are indicated, by Proposition 5.7, in the following way:

Let us take the product symplectic manifold

$$\Xi = (T^*\mathbb{R}^3 \times_{-} M, \tilde{\omega}\Theta\omega_{\mathbb{R}^3}), \quad \text{(see [11])}.$$

We know that graph$\pi \subseteq \Xi$ is a lagrangian submanifold of Ξ. Then there exists its local Morse family (cf. [18]), say $K : \mathbb{R}^3 \times X \times \mathbb{R}^{\mu} \longrightarrow \mathbb{R}$

$(x,q,\mu) \longrightarrow K(x,q,\mu)$, where T^*X is an appropriate local cotangent bundle structure on M (see[1]). Let $G_k(q,\lambda)$ be the generating family for $L_k^{(n)}$ given in Proposition 5.7. Then the corresponding phase family in (5.7) is a generating family for the pullback (cf.[11])

$$(\text{graph } \pi)^t(L_k^{(n)}).$$

Thus the corresponding optical distance (time), say $\psi_k(x)$, is described by the following equations:

$$\psi_k(x) = \mathrm{Stat}_{q,\mu,\lambda}(G_k(q,\lambda) - K(x,q,\mu)). \tag{5.8}$$

Example 5.10. Now we exactely calculate the planar case $k=1$. In this case the local Morse family for graph π is following

$$K(x_1,x_2,q) = x_2 q - x_1\sqrt{1-q^2}, \quad q \neq 1.$$

Thus taking the generating family (5.5) for $L_1^{(7)}$ we obtain the corresponding family of optical distance functions (cf. (5.7))

$$\phi(x,\lambda_1,\lambda_2) = -\frac{1}{40}\lambda_2^5 - \frac{1}{6}\lambda_2^3\lambda_1 - \frac{1}{2}\lambda_2\lambda_1^2 - x_2\lambda_1 - x_1\sqrt{1-\lambda_1^2}$$

and the graph of phase function $\psi_1(x)$, $\Sigma_1 = \{\ \psi_1(x) - t = 0\}$ (see Figure 3, below),

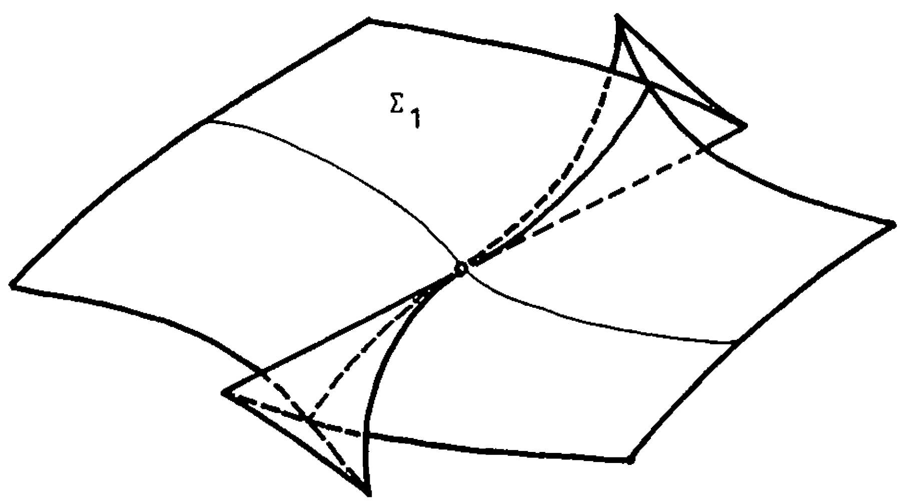

Figure 3.

By the straightforward calculations, using this family, we obtain the corresponding family of wave fronts parametrized by the optical time t;

$$x_1 = \left(\frac{1}{10}\mu^5 - t\right)\sqrt{1 - \frac{1}{4}\mu^4}\ ,$$

$$x_2 = \frac{1}{3}\mu^3 - \frac{1}{2}\mu^2\left(\frac{1}{10}\mu^5 - t\right), \qquad \text{(see Figure 4, below)}$$

which are exactely the level-sets of the phase function $\psi_1(x)$ in the planar obstacle problem (see Figure 2) with inflection point [5].

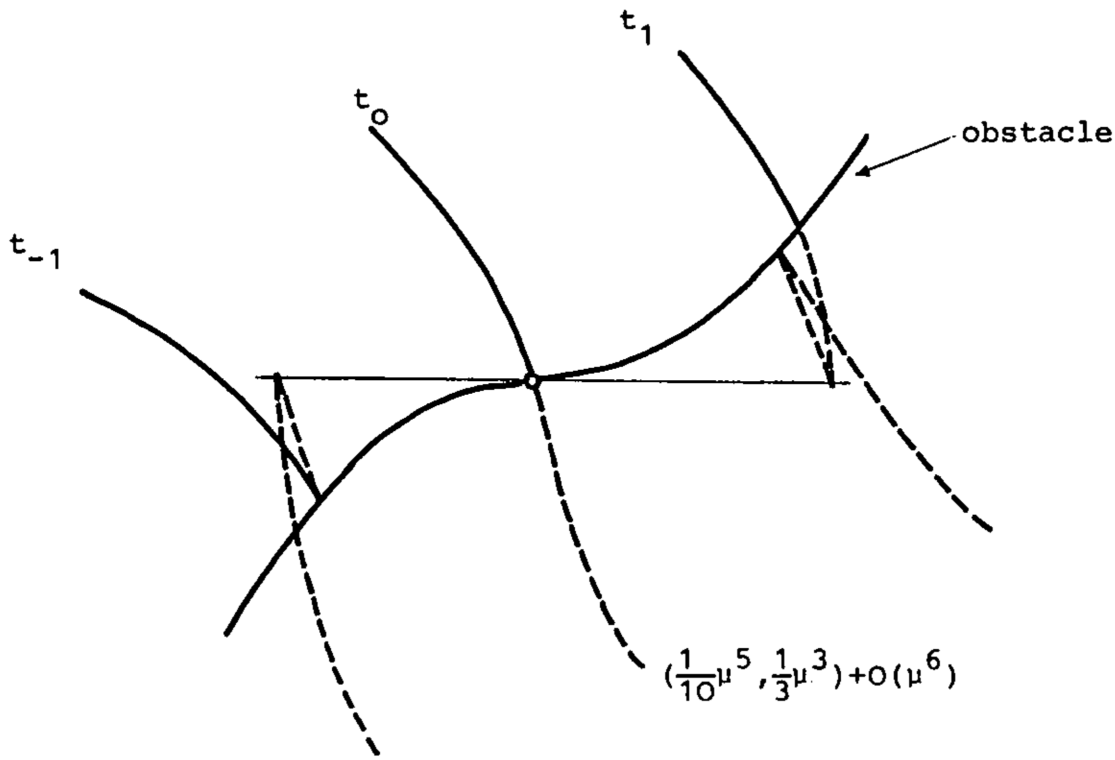

Figure 4.

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