The Number of Rational Points on Some Kummer Surfaces

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INTRODUCTION.

In this article I am going to consider a curve of ginus 2 given by $y^2=(x-q_1)...(x-q_6)$, where $q_i \in \mathbf{Q}$. Let J(C) be its jacobian and K(C) is the Kummer surface, i.e. K(C) is the minimal non-singular model of $J(C)/\pm id$. It is a well known fact that $K(C) \subset \mathbb{CP}^5$. So this embedding defines in the usual way the Weil function H, i.e. a height function.

Namely if $x=(x_0,...,x_n) \in \mathbf{CP}^n(\mathbf{Q})$, $x_i \in \mathbb{Z}$ and $g.c.d.(x_0,...,x_n)=1$, then

$H(x):=\max_i |x_i|$. For the definition and the elementary properties of the Weil function see [L]. It is a well know fact that $\#\left\{ x \in \mathbf{CP}^n(\mathbf{Q}) |\ H(x) \leq h \right\} < \infty$.(See[L].) The aim of this note is to prove the following theorem:

THEOREM.

We have with respect to the Weil function on K(C) defined from the embedding $K(C)(\mathbf{Q}) \subset \mathbf{CP}^5(\mathbf{Q})$ the following estimate for big enouph h:

$$\#\left\{ x \in K(C) \subset \mathbf{CP}^n(\mathbf{Q}) |\ H(x) \leq h \right\} \leq c(\log h)^{\frac{1}{2}} h^{\frac{1}{2}}$$

where c is a positive constant.

The proof of this theorem is based on the observation that K(C) has an elliptic fibration structure, i.e. there exists a map π: $K(C) \to \mathbb{CP}^1$ such that the "generic" fibres are non-singular elliptic curves. More over we prove that there exist two sections σ_0 and σ_1 of this elliptic fibrations defined over $\mathbf{Q}$ which are "independent", i.e. the rank of the "Weil-Mordell" group has rank 1 over $\mathbf{C}(t)$)($\mathbf{C}(t)$ is the field of the rational functions over $\mathbf{C}$). From these facts our estimates follows almost directly.

We know that $J(C)/\pm id$ can be embedded in $\mathbb{CP}^3$ and moreover the equation:

$$(*) \qquad A(x^4+y^4+z^4+t^4)+B(x^2y^2+z^2t^2)+C(x^2z^2+y^2t^2)+D(x^2t^2+y^2z^2)+Fxyzt=0$$

with the condition $R(A,B,C,D,F) \dot{=} 0$, where R is the resultant, defines $J(C)/\pm id$. So we can state the following result: If the 16 double points on the surface defined in $\mathbb{CP}^3$ by (*) has rational coordinates then (*) has an infinite number of rational solutions.

Here one should mentioned that Elkies and D. Zagier have proved that the equation $x^4+y^4+z^4+t^4=0$ has infinite number of rational solutions. See [E] and [Z].

This note was inspired from conversations with Yu. Tschinkel at MPI Bonn, who explain me the lectures of Yu. I. Manin given at MIT and the joint work of Tschinkel, Franke and Manin. In his MIT lectures Manin stated some very interesting conjectures and problems, relating the asymptotic behaviour of the counting function with respect to a fixed height function with the geometry of the variety. Lang was the frst who pointed out the importance of studing the following problem; Let H be a height fnction on some algebraic manifold X defined over a number field K try to find the asymptotic behaviour of the $\#\left\{ x\in X(K) |\ H(x)\leq h\right\}$. (See [Sch].) This is an analogue to the Weyl problem "can one hear the shape of the drum" in the formulation of L. Bers. Another very beautiful book, part of which is devoted to the above problem is Serre's book "Lectures on the Mordell-Weil Theorem".

Let me mentioned that in very few cases we have detailed knowledge of the asymptotic behaviour of the counting function with respect to some height. Schanuel gave an answer to that problem for $\mathbf{P}^n$. (See [S].) Tschinkel, Franke and Manin gave very presice and beautiful answer in the case of G/P, where G is a semisimple algebraic group and P is a parabolic group. (See [FMT].) Very interesting and beautiful results are stated in the article [BM].

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#1. Elliptic fibrations of K(C).

Proposition 1.

The linear system $|2C|$ on J(C) gives a two to one map $\phi_{|2C|}:J(C)\to \mathbb{CP}^3$. $\phi_{|2C|}(J(C))$ is a hypersurface of degree 4 which has exactly 16 double rational points.

PROOF: See [G&H] chapter 6.

Q.E.D.

Cor. 1.1. If we blow up the 16 double points on $\phi_{|2C|}(J(C))$ we will get K(C).

PROOF: See [G&H] chapter 6.

Q.E.D.

Proposition 2.

Let A_1 and A_2 are two of the double points on $\phi_{|2C|}(J(C))$. Let $\{H_t\}$ be the pencil of hyperplanes in $\mathbf{CP}^3$ passing through A_1 and A_2. Let $\{\tilde{E}_t\}$ be the pencil of curves, where $\tilde{E}_t:=H_t\cap\phi_{|2C|}(J(C))$. Let $\{E_t\}$ be the pencil of crves on the K(C), where K(C) is obtained from $\phi_{|2C|}(J(C))$ by blowing up the sixteen double rational points C_t is the proper praimage of $\tilde{E}_t$. Then

a) There exists a map π: $K(C)\to\mathbf{CP}^1$ such that $\pi^{-1}(t)=E_t$ and for generic t E_t is a non-singular elliptic curve, i.e. π: $K(C)\to\mathbf{CP}^1$ defines an elliptic fibration on K(C).

b) The singular fibres of the elliptic fibration are of the following types; 6 singular fibres of type I_2 and two singular fibres of type I_0^*.

c) There exists two sections σ_0 and σ_1 of the above defined elliptic fibrartion of K(C) which are independent over $\mathbf{CP}^1$, i.e. they generate the Mordell-Weil group of K(C), if we consider K(C) as an elliptic curve over $\mathbf{CP}^1$. In other words the rank Mordell-Weil group of K(C) over $\mathbf{C}(t)$.

Proof of Proposition 2.a.:

Let $\tilde{E}_t:=\phi_{|2C|}(J(C))\cap H_t$, where H_t is a generic plane in $\mathbb{CP}^3$. Then Bertinni theorem implies that generic $\tilde{E}_t$ is plane curve of degree 4 with two ordinary singular point, so its normalization will be an elliptic curve. This follows directly from the fact that the genus of a non-singular curve of degree 4 is 3. From here it is trivial to see that we get a regular map π: $K(C)\to\mathbf{CP}^1$. So we get an elliptic fibration.

Proof of Proposition 2.b.:

The best way to study the degenerate fibres is to project $\phi_{|2C|}(J(C))$ from the point say A_1 onto $\mathbf{CP}^2$. It is not difficult to prove that the degree of this projection is two. The projection just described we will denote by p_{A_1}. From here one can obtain that the remification divisor is a union of six lines which we will denote by L_i, i=1,..,6. All these lines intersect in 15 points. For the proof of these facts see [G&H] chapter 6. Let us denote again by A_2 the image of $p_{A_1}(A_2)$. It is clear that if we blow up the singular points A_1 and A_2 on $\phi_{|2C|}(J(C))$ and denote the -2 curves which we obtain after those two blows up by P_1 and P_2, then P_1 and P_2 will be two sections of the elliptic fibration obtained in Proposition 2.a.. It is clear that the praimages of the two lines in $\mathbf{CP}^2$ that pass through A_2 will be a conic on $\phi_{|2C|}(J(C))$ which contain exactly 6 double points. After the blows up we get on the aboved described elliptic fibration of K(C) two degenerate fibres of type I_0^*. The other degenerate fibres we will obtain from the hyperplanes that passes through A_1, A_2 and any of the rest of the double points on $\phi_{|2C|}(J(C))$, i.e. those double points that do not lie niether on P_1 nor on P_2. It is easy to see that we have exactly 6 double points that do not lie niether on P_1 nor on P_2. So from here we obtain 6 singular fibres of type I_2. From the formula of the Euler characteristics of K(C), i.e.

$$24=\chi(K(C))=\sum\chi(\text{singular fibres})$$

we get that these are all singular fibres of the elliptic fibration described above. For the proof of the above mentioned formula see [G&H] chapter 4.

Proof of Proposition 2.c.:

Construction of sections of the elliptic fibration.

Let us denote by $C_1,\dots,C_{16}$ the translates of C in J(C) by the points of order 2, which we will denote by $A_1,\dots,A_{16}$. It is easy to prove that

FACT1. Each C_i contains exactly 6 two torsion points

FACT2. a. Any two different C_i and C_j intersects in two different points of order two on J(C)

b. Through two different A_i and A_j pass exactly two different C_k and C_l.

c. On the union of two different C_i and C_j there are exactly 10 points of order 2.

For the proves of these two facts see [GH] chapter 6.

FACT3. $\phi_{|2C|}(C_i)$ for i=1,...,16 is a plane nonsingular conic on $J(C)/\pm id$ which contains exactly 6 double rational points.

PROOF OF FACT3:

From the way we defined C_i we see that each of the curve C_i is invarian under the involution $x\to -x$, whose fixed points are exctly the two-torsion points on J(C). This is the canonical involution of the hyper-elliptic curves C_i. From here and the fact that the linear system $|2C|$ define a map $J(C) \rightarrow J(C)/\pm id$ FACT 3 follows directly.

Q.E.D.

<u>FACT4.</u> From the way we defined the elliptic pencil on $\phi_{|2C|}(J(C))$ and FACT2 it follows that the two singular fibres of the type I_0^* are obtain from the images of the two say C_1 and C_2 that intersect each other in A_1 and A_2 and the rest of the 8 blown up points that lie on C_1 and C_2.

<u>PROOF OF FACT 4:</u>

This follows directly from the description of the fibres of type I_0^* we gave above and the fact that $|2C|$ gives the caonical map $J(C) \rightarrow J(C)/\pm id \subset \mathbf{CP}^3$.

Q.E.D.

<u>DEFINITION.</u>

Let us denote the points of order two that lie on C_1 and C_2 by $A_1, A_2, \ldots, A_{10}$. Let C_1 contains $A_1, A_2, \ldots, A_6$.

Now we are ready to construct 8 sections from $C_1, \ldots, C_{16}$. Namely an easy calculation show that there are exactly 8 curves among $C_1, \ldots, C_{16}$ say $C_{i_1}, \ldots, C_{i_8}$ with the following property:

(**) Each of the curve C_{i_j} passes through either A_1 or A_2 and contains A_k, where k=3,4,5, 6.

<u>SUBLEMMA 1.</u> Each of the curve C_{i_j} defined by (**) is a section of our elliptic fibration.

<u>PROOF OF THE SUBLEMMA 1:</u>

From FACT4 it follows that the proper image of each of C_{i_k} for k=1,..,8 intersect one of the

singular fibre of type I_0^* in one point, this point is obtained in the following way: Let $C_1 \cap C_{i_k} = A_1 \cup A_3$. Clearly that after we blow up the image of the point A_3 in $\phi_{|2C|}(J(C))$ then C_{i_k} will intersect transversally the -2 curve that is obtain after the blow up exactly in one point. Since blow up of A_1 is a two section of our elliptic fibration we get that each of the C_{i_k}

is a section of our fibration.

Q.E.D.

REMARK. From all this discussion it follows that we can consider K(C) as an elliptic curve over the fileld of rational functions over **C** with one variable **C**(T) since we can fix one of the section as the zero point on K(C).

SUBLEMMA 2. The rank of the Mordell-Weil group of K(C) is equal to one.

PROOF OF THE SUBLEMMA 2:

In order to prove Sublemma we need to use the following formula of Shioda-Tate for elliptic surfaces:

$$\rho = r + 2 + \sum_{v \in S} (m_v - 1)$$

where ρ is the Picard number of the elliptic surface, r is the rank of the Mordell-Weil group and m_v is the number of the irreducible components of the singular fibres. (See [Sh].)

Notice that ρ of K(C) is 17 and simple calculations show that $\sum_{v \in S} (m_v - 1) = 14$. So plugging these numbers in the Shioda-Tate formula we ge that r=1.

Q.E.D.

So Proposition 2. is proved.

Q.E.D.

#2. THE PROOF OF THE THEOREM.

From [G&H] chapter 6 we know that the linear system

$$|4C-\sum_{i=1}^{16} E_i|$$

on $\hat{J}(C)$, where $\hat{J}(C)$ is obtained from $J(C)$ by blowing up the 16 2-torsion points on $J(C)$ gives a regular two to one map of $\hat{J}(C)\rightarrow K(C)\subset \mathbf{CP}^5$.

THEOREM.

Let C be a curve of genus 2 given by $y^2=(x-q_1)...(x-q_6)$, where $q_i\in\mathbf{Q}$. Let K(C) be the Kummer surface defined in the introduction. We know that we have an embedding $K(C)(\mathbf{Q})\subset\mathbf{CP}^5(\mathbf{Q})$. We have with respect to the Weil function on K(C) defined from the embedding $K(C)(\mathbf{Q})\subset\mathbf{CP}^5(\mathbf{Q})$ the following inequality:

$$\#\left\{ x\in K(C)\subset \mathbf{CP}^n(\mathbf{Q})|\ H(x)\leq h\right\}\leq c(\log h)^{\frac{1}{2}}h^{\frac{1}{2}}$$

where c is a positive constant.

PROOF OF THE THEOREM:

The proof of this theorem is based on the following Lemma.

LEMMA. Let C be defined as follows:

$$y^2=(x-q_1)(x-q_2)...(x-q_6),\ \text{where } q_i\in\mathbf{Q}.$$

Let K(C) be the Kummer surface associated with C. We know from proposition 2 that K(C) is an elliptic curve over $\mathbf{C}(T)$ and its Mordell -Weil group has rank 1. Let σ_o be the zero element and σ_1 be the generator of the Mordell-Weil group. Then for any k let $\sigma_k=k\sigma_1$. We know that σ_k are sections of the elliptic fibration $\pi:K(C)\rightarrow\mathbf{CP}^1$. Then all σ_k are defined over Q and more over each σ_k contains an infinite number of rational points.

PROOF OF THE LEMMA:

From Proposition 2 we know that K(C) is an elliptic fibration over $\mathbf{CP}^1$ which have sections constructed in Proposition 2. Here we will make a very important remark, namely since C is defined as follows:

$$y^2=(x-q_1)(x-q_2)\dots(x-q_6), \text{ where } q_i\in \mathbf{Q}$$

and since the images of C_i on K(C) are plane curves of dgree 2 whic contain six rational points, so all these conics are defined over **Q** and contain infinite number of rational points. From here we get that the sections σ_k are defined over **Q** and contain an infinite number of rational points. So any section which we obtain using the group law of the elliptic curve K(C) over **C**(T) are rational curves defined over **Q** and contain infinite number of rational points.

Q.E.D.

<u>REMARK.</u> The arguments, which will be used later are based on this lemma.

Before starting the proof we will remind the reader the following facts:

<u>FACT 1.</u>

Let A be an abelian variety defined over global field F. From the Mordell-Weil Theorem and a refined Neron-Tate height H one deduce that:

$$\#\left\{x\in A(F)|\ H(x)\le h\right\}\sim c(\log h)^{\frac{r}{2}}$$

where r is the rank of A(F) and c is a canstant that can be expressed throughcard$(A(F))_{tor}$ and the volume of the ellipsoid $\hat{H}\le 1$. See [FMT].

<u>FACT 2.</u>

For $\mathbf{P}^n$ Schanuel proved:

$$\#\left\{x\in \mathbb{P}^n(\mathbf{Q})|\ H(x)\le h\right\}\sim ch^{n+1}$$

For the proof of this fact see [Sc].

We proved that the rank of Mordell-Weil group of K(C) over **C**(T) is one. Since Pic $\mathbf{CP}^1=\mathbf{Z}$, from a theorem proved by Silvermann we know that for all points of $\mathbf{CP}^1$ over which the elliptic curves E_t are defined over **Q** except finite number of them the rank of Mordell-Weil group of these elliptic curves will be the same as of $\pi:K(C)\to\mathbf{CP}^1$, i.e. 1. For the proof of this

Theorem of Silvermann see chapter IX of [L]. Let me fix a generic elliptic curve $E(\mathbf{Q})$ of this fibration with Mordell-Weil rank 1. Let $x_1,\ldots,x_n$ be all points on $E(\mathbf{Q})/\mathrm{Tor}E(\mathbf{Q})$ with the property that $H(x_i)<h$. From the LEMMA proved above it follows that all of them are generated by the two sections σ_0 and σ_1 of $K(C)\to\mathbb{CP}^1$. On the other hand we know that

(a) $$n\sim c_1(\log h)^{\frac{1}{2}}$$

(See [FMT].)

We know that each $x_k=\sigma_k\cap E(\mathbf{Q})$, where σ_k is a section equal to $k_l\sigma_1$ for some $k_l\in\mathbf{Z}$ in the sence of group law of the elliptic curve $K(C)$ over $\mathbf{C}(T)$. On each σ_k we have by the theorem of Schanuel:

(**) $\#\left\{x\in\sigma_k \mid H(x)<h\right\}\sim c_2 h^{\frac{2}{\deg\sigma_k}}$, where $\deg\sigma_k=(\sigma_k,L)$ L is the hyperplane in $\mathbb{CP}^5$(See [L]).

So from (*) and (**) we get:

(***) $\#\left\{x\in K(C) \mid H(x)\leq h\right\}\sim\#(\text{torsion points of } K(C)/\mathbf{C}(T))\sum_{k=1}^{n} c_2 h^{\frac{1}{\deg\sigma_k}}$, where $n\sim c_1\log h$.

So from (***) we get the following estimate:

(***) $\#\left\{x\in K(C) \mid H(x)\leq h\right\}\leq\#(\text{torsion points of } K(C)/\mathbf{C}(T)c_1c_2(\log h)^{\frac{1}{2}}h^{\frac{1}{\min_k\deg\sigma_k}}$

From Proposition 2 it follows immediately that

$$\min_k \deg\sigma_k=2$$

i.e. it is achieved on the images of C_i under the map of the linear system $|4C-\sum E_i|$. So our theorem is proved.

Q.E.D.

SOME REMARKS.

a. Mukai and Mori proved that if X is any algebraic K3 surface, then after blowing up certain amount of points on X, then we have an elliptic fibration $\pi:\tilde{X}\rightarrow \mathbf{CP}^1$, even more they constructed a section σ of π. So we have bascally the same situation as in the case of the Kummer surface, that was treated in this article. But it is impossible for me to see if this section is defined over **Q**. So one should expect the same estimates as we obtained for the Kummer surface.

b. It will be interesring to check the Bogomolov's conjecure, namely that any rational point on a K3 surface lie on a rational curve for K(C).

BIBLIOGRAPHY.

[BM] V. BATYREV, Yu. MANIN, "SUR LE NOMBRE DES POINTS RATIONNELS DE HAUTER BORNE DES VARIETES ALGEBRIQUES", Math. Ann. 286, 27-43 (1990).

[E] ELKIES, "On $A^4+B^4+C^4=D^4$", Math. of Computation 51(1988) 825-837

[FMT] J. FRANKE, Yu. MANIN, Yu. Tschinkel "RATIONAL POINTS OF BOUNDED HEIGHT ON FANO VARIETIES", Invent. Math., vol. 95, 421-435(1989)

[GH] Ph. GRIFFITHS and J. HARRIS "PRINCIPLES OF ALGEBRAIC GEOMETRY" .

[L] S. LANG, "DIOPHANT GEOMETRY", Russian edition, "Mup" 1984.

[M] Yu. I. MANIN, MIT LECTURES, 1990.

[S] S. H. SCHANUEL, "HEIGHTS IN NUMBER FIELDS", Bull. Soc. Math. France 107, 433-449(1979).

[SH] T. SHIODA, "ON ELLIPTIC MODULAR CURVES.", J. Math. Soc., Japan, vol. 24, No1, 1972.

[Z] D. ZAGIER, UNPUBLISHED MANUSCRIPT.