Variation of toric hyperKähler manifolds

Hiroshi Konno

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Dedicated to Professor Takushiro Ochiai for his 60th birthday

Abstract. A toric hyperKähler manifold is defined to be a smooth hyperKähler quotient of the quaternionic vector space $\mathbb{H}^N$ by a subtorus of T^N. It has two parameters corresponding to the de Rham cohomology classes represented by the Kähler form and the complex symplectic form respectively. We study the variation of its complex structure according to these parameters. After the detailed analysis of the stability condition depending on the first parameter, we show that toric hyperKähler manifolds with the same second parameter are related by a sequence of Mukai's elementary transformations. We also give a complete description of its Kähler cone and discuss when certain rational curves exist.

1 Introduction

A hyperKähler manifold is a $4n$-dimensional Riemannian manifold whose holonomy group is contained in $Sp(n)$. That is, the Riemannian metric g is Kähler with respect to three complex structures I_1, I_2, I_3, which satisfies the quaternion relations. It is a fundamental building block in Riemannian geometry, due to Berger's classification of holonomy groups [2]. Fujiki and Beauville constructed examples of compact irreducible hyperKähler manifolds of higher dimensions [6] [1], which can be considered as higher dimensional analogues of K3 surfaces. On the other hand, many non-compact examples are constructed by the hyperKähler quotient method due to [10].

One of the important problems in hyperKähler geometry is to understand the global structure of the deformation space of hyperKähler manifolds. For a compact irreducible hyperKähler manifold, Beauville introduced the period map and established the local Torelli theorem [1]. Recently Huybrechts proved the surjectivity of the period map, and proved that non-separating points in the moduli space correspond to birational hyperKähler manifolds [11] [12]. In his study the Kähler cone plays an important role to investigate birational transformations between hyperKähler manifolds.

A toric hyperKähler manifold is not a toric manifold in the usual sense, but a corresponding object in hyperKähler geometry. It was introduced by Bielawski and Dancer [4] and further studied in [14] and [9]. It is defined to be a smooth hyperKähler quotient of the quaternionic vector space $\mathbb{H}^N$ by a subtorus of T^N. Therefore it has certain parameters. In this paper, we study the variation of its complex structure according to these parameters and give a global description of this family of hyperKähler manifolds. Bielawski intrinsically characterize a toric hyperKähler manifold as a complete hyperKähler T^n-manifold of dimension $4n$ with

Euclidean volume growth [5]. So our family of hyperKähler manifolds can be considered as a component of the moduli space of such manifolds.

Let us explain our results in more detail. Let K be a subtorus of T^N acting on $\mathbb{H}^N$ with the hyperKähler moment map $\mu = (\mu_1, \mu_{\mathbb{C}})\colon \mathbb{H}^N \to k^* \otimes \mathbb{R}^3 \cong k^* \times k^*_{\mathbb{C}}$. If $(\alpha, \beta) \in k^* \times k^*_{\mathbb{C}}$ is a regular value of μ, then we have a toric hyperKähler manifold $X(\alpha, \beta) = \mu^{-1}(\alpha, \beta)/K$. First we show that, under mild conditions, the parameters $\alpha \in k^*, \beta \in k^*_{\mathbb{C}}$ can be interpreted as its periods. That is, by the natural isomorphism $\kappa^*\colon k^* \to H^2(X(\alpha, \beta); \mathbb{R})$, the parameters α, β are identified with the de Rham cohomology classes represented by the Kähler form and the complex symplectic form respectively.

Next we show that the parameter β determines the birational class of the complex manifold $(X(\alpha, \beta), I_1)$. More precisely, if we fix the parameter $\beta \in k^*_{\mathbb{C}}$, we have the chamber structure in k^*. If $\alpha' \in k^*$ belongs to the same chamber as α, $(X(\alpha', \beta), I_1)$ is canonically isomorphic to $(X(\alpha, \beta), I_1)$ as a complex manifold. If $\alpha, \alpha' \in k^*$ belong to the next chamber to each other, two complex manifolds $(X(\alpha, \beta), I_1)$ and $(X(\alpha', \beta), I_1)$ are related by an elementary transformation introduced by Mukai [16], which is a special birational transformation between complex symplectic manifolds.

In addition, we discuss when $(X(\alpha, \beta), I_1)$ contains a smooth rational curve, whose volume goes to zero as the 'Kähler class' α approaches to the wall. This arrows us to give a complete description of the Kähler cone of every toric hyperKähler manifold. We also discuss when $(X(\alpha, \beta), I_1)$ is affine. These results give a good global description of our family of toric hyperKähler manifolds.

The proof of the above results involves the detailed analysis of the stability. The complexification of the subtorus $K_{\mathbb{C}}$ acts on $\mu_{\mathbb{C}}^{-1}(\beta)$. The stability condition for this action depends on the polarization, which corresponds to $\alpha \in k^*$. The change of the quotient varieties with respect to different polarizations has been studied by many authors [8] [20]. In this case, this change is proved to be expressed by a more specific birational transformation because of their complex symplectic structures. Thaddeus showed similar phenomena in the case of the moduli space of parabolic Higgs bundles [21].

The content of this paper is as follows. In Section 2, we review the definition and basic properties of toric hyperKähler manifolds. The period map is discussed in Section 3. In Section 4, we review an elementary transformation and explain how it appears naturally in our context. Next we study the stability condition and give useful criterions of stable points in Section 5. In Section 6, we study the variation of toric hyperKähler manifolds and prove the main results mentioned above. In the final section, we give an example.

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2 Toric hyperKähler manifolds

The standard hyperKähler structure on the quaternionic vector space $\mathbb{H}^N$ is the standard flat Riemannian metric g together with three complex structures I_1, I_2, I_3, which are defined by the left multiplication of i, j, k respectively. Throughout this paper we identify $\mathbb{H}^N$ with $\mathbb{C}^N \times \mathbb{C}^N$ with respect to I_2. Then these complex structures are given by

$$I_1(z,w) = (\sqrt{-1}z, \sqrt{-1}w), I_2(z,w) = (-\overline{w}, \overline{z}), I_3(z,w) = (-\sqrt{-1}\overline{w}, \sqrt{-1}\overline{z}).$$

The real torus $T^N = \{\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N \mid |\zeta_i| = 1 \text{ for } i = 1, \ldots, N\}$ acts on $\mathbb{H}^N$ by $(z,w)\zeta = (z\zeta, w\zeta^{-1})$, preserving the hyperKähler structure $(g; I_1, I_2, I_3)$ on $\mathbb{H}^N$. Let t^N be the Lie algebra of T^N and $\{X_1, \ldots, X_N\}$ its basis such that $\mathrm{Exp}(\sum_{i=1}^N s_i X_i) = (e^{2\pi\sqrt{-1}s_1}, \ldots, e^{2\pi\sqrt{-1}s_N}) \in T^N$. The dual basis is denoted by $\{u_1, \ldots, u_N\} \subset (t^N)^*$.

Let K be a subtorus of T^N and $T^n = T^N/K$ the quotient torus with the Lie algebras k and t^n respectively. So we have the following exact sequences;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \xrightarrow{\iota} & t^N & \xrightarrow{\pi} & t^n & \longrightarrow & 0, \\ 0 & \longleftarrow & k^* & \xleftarrow{\iota^*} & (t^N)^* & \xleftarrow{\pi^*} & (t^n)^* & \longleftarrow & 0. \end{array}$$

The hyperKähler moment map for the action of K on $\mathbb{H}^N$

$$\mu = (\mu_1, \mu_2, \mu_3) \colon \mathbb{H}^N \to k^* \otimes \mathbb{R}^3 \cong k^* \times k^*_{\mathbb{C}}$$

is given by

$$\mu_1(z,w) = \pi \sum_{i=1}^N (|z_i|^2 - |w_i|^2)\iota^* u_i \in k^*,$$

$$\mu_{\mathbb{C}}(z,w) = -2\pi\sqrt{-1} \sum_{i=1}^N z_i w_i \iota^* u_i \in k^*_{\mathbb{C}},$$

where $\mu_{\mathbb{C}} = \mu_2 + \sqrt{-1}\mu_3$ and $k^*_{\mathbb{C}} = k^* \otimes \mathbb{C}$.

Definition 2.1. If the subtorus K acts on $\mu^{-1}(\alpha, \beta)$ freely for $(\alpha, \beta) \in k^* \times k^*_{\mathbb{C}}$, the hyperKähler quotient $X(\alpha,\beta) = \mu^{-1}(\alpha,\beta)/K$ is called a *toric hyperKähler manifold.*

Remark 2.2. A toric hyperKähler manifold $X(\alpha, \beta)$ is a smooth non-compact complete connected hyperKähler manifold of dimension $4n$. The quotient torus $T^n = T^N/K$ acts on $X(\alpha, \beta)$, preserving the hyperKähler structure. Note that a toric hyperKähler manifold is not a toric manifold in the usual sense, but a corresponding object in hyperKähler geometry.

Throughout this paper we also denote the hyperKähler structure on $X(\alpha, \beta)$ by $(g; I_1, I_2, I_3)$. The corresponding Kähler forms are denoted by $\omega_1, \omega_2, \omega_3$ respectively. If we consider $X(\alpha, \beta)$ as a complex manifold with respect to I_1, we denote it by $(X(\alpha, \beta), I_1)$. Since $\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\omega_3$ is a holomorphic non-degenerate 2-form on $(X(\alpha, \beta), I_1)$, it is called the *complex symplectic form* of $X(\alpha, \beta)$.

In [14] we studied when the hyperKähler quotient $\mu^{-1}(\alpha, \beta)/K$ is a toric hyperKähler manifold.

Proposition 2.3. [14] *Let $\{P_1, \dots, P_l\}$ be the set of all subspaces of codimension one in k^* which are spanned by subsets of $\{\iota^* u_i\}_{i=1}^N$. If we denote the set of regular values of the hyperKähler moment map μ by $(k^* \times k^*_{\mathbb{C}})_{reg}$, then we have*

$$(k^* \times k^*_{\mathbb{C}})_{reg} = (k^* \times k^*_{\mathbb{C}}) \setminus \bigcup_{s=1}^{l} (P_s \times P_{s\mathbb{C}}),$$

where $P_{s\mathbb{C}} = P_s \otimes \mathbb{C} \subset k^*_{\mathbb{C}}$.

Proposition 2.4. [14] *The subtorus K acts freely on $\mu^{-1}(\alpha, \beta)$ if and only if the following conditions* (R) *and* (C1) *hold:*
(R) $(\alpha, \beta) \in (k^* \times k^*_{\mathbb{C}})_{reg}$.
(C1) *If $\{\iota^* u_i\}_{i \in J}$, where $J \subset \{1, \dots, N\}$, is a basis of k^*, then the following holds;*

$$t^N_{\mathbb{Z}} = k_{\mathbb{Z}} + \sum_{i \in \{1,\dots,N\} \setminus J} \mathbb{Z} X_i \quad \textit{as a } \mathbb{Z}\textit{-module},$$

where $t^N_{\mathbb{Z}} = \sum_{i=1}^N \mathbb{Z} X_i$ and $k_{\mathbb{Z}} = k \cap t^N_{\mathbb{Z}}$.

Remark 2.5. The condition (R) ensures that the hyperKähler quotient $\mu^{-1}(\alpha, \beta)/K$ is an orbifold of the expected dimension. The condition (C1) ensures that the orbifold is a manifold. Note that the condition (C1) depends only on the subtorus K, not on (α, β).

Later we will impose the following condition (C2) on the subtorus K:

(C2) $X_i \notin k$ and $\iota^* u_i \neq 0$ for $i = 1, \dots, N$.

The meaning of the condition (C2) is as follows. Suppose that $X_{i_0} \in k$. Then the toric hyperKähler manifold $X(\alpha, \beta)$ can be also obtained as a hyperKähler quotient of $\mathbb{H}^{N-1}$ by $K \cap T^{N-1}$, where $\mathbb{H}^{N-1} = \{(z, w) \in \mathbb{H}^N \mid z_{i_0} = w_{i_0} = 0\}$ and $T^{N-1} = \{\zeta \in T^N \mid \zeta_{i_0} = 1\}$. On the other hand, suppose that $\iota^* u_{i_0} = 0$. Then the subtorus K is a subgroup of T^{N-1}, and $X(\alpha, \beta)$ is decomposed into the product of a hyperKähler quotient of $\mathbb{H}^{N-1}$ by K with $\mathbb{H}$. These cases will not be essential for our argument. The condition (C2) excludes such cases.

Since the hyperKähler moment map $\mu \colon \mathbb{H}^N \to k^* \otimes \mathbb{R}^3$ is surjective, $X(\alpha, \beta)$ is not empty for all $(\alpha, \beta) \in (k^* \times k^*_{\mathbb{C}})_{reg}$. Moreover we have the following.

Theorem 2.6. *The diffeomorphism type of the toric hyperKähler manifold $X(\alpha, \beta)$ is independent of $(\alpha, \beta) \in (k^* \times k^*_{\mathbb{C}})_{reg}$.*

Remark 2.7. The above result is essentially due to Bielawski and Dancer [4]. They showed that the homeomorphism type is independent by a similar argument due to [15] and [18]. By a slight modification of their argument, the above results can be proved. See Remark 6.2 below for the proof.

Thus we have a family of hyperKähler manifolds parameterized by $(\alpha, \beta) \in (k^* \times k^*_{\mathbb{C}})_{reg}$. In [5] Bielawski gave an intrinsic characterization of toric hyperKähler manifolds as follows.

Theorem 2.8. [5] *Let M be a complete connected hyperKähler manifold of dimension $4n$. Suppose that the torus T^n acts on M effectively, preserving the hyperKähler structure, and that M has Euclidean volume growth. Then M is isomorphic to one of toric hyperKähler manifolds in the sense of Definition 2.1 as a hyperKähler T^n-manifold.*

Thus, roughly speaking, our family of hyperKähler manifolds can be considered as a component of the moduli space of such manifolds. From now on we study the structure of this family of hyperKähler manifolds.

3 The period map

Let $X(\alpha,\beta)=\mu^{-1}(\alpha,\beta)/K$ be a toric hyperKähler manifold. Define $k^*_{\mathbb{Z}}\subset k^*$ by

$$k^*_{\mathbb{Z}}=\{v\in k^*\mid \langle v,Y\rangle\in\mathbb{Z}\quad \text{for all } Y\in k_{\mathbb{Z}}\},$$

where $\langle\ ,\ \rangle\colon k^*\times k\to\mathbb{R}$ is the natural pairing. Due to the condition (C1), it is easy to see that $k^*_{\mathbb{Z}}=\iota^*((t^N_{\mathbb{Z}})^*)$, where $(t^N_{\mathbb{Z}})^*=\sum_{i=1}^N\mathbb{Z}u_i$. Define a K-equivariant line bundle $\widetilde{L}_\rho=\mathbb{H}^N\times\mathbb{C}$ for $\rho\in k^*_{\mathbb{Z}}$ by $((z,w),v)\zeta=((z\zeta,w\zeta^{-1}),ve^{2\pi\sqrt{-1}\langle\rho,Y\rangle})$, where $\zeta=\mathrm{Exp}Y\in K$.

Let $H^*_K(\mathbb{H}^N;\mathbb{Z})$ be the K-equivariant cohomology of $\mathbb{H}^N$. Define the map $c\colon k^*_{\mathbb{Z}}\to H^2_K(\mathbb{H}^N;\mathbb{Z})$ by $c(\rho)=c_{1K}(\widetilde{L}_\rho)$, where $c_{1K}(\widetilde{L}_\rho)\in H^2_K(\mathbb{H}^N;\mathbb{Z})$ is the K-equivariant first Chern class of $\widetilde{L}_\rho$. This induces the isomorphism $\widetilde{c}\colon S^*(k^*_{\mathbb{Z}})\to H^*_K(\mathbb{H}^N;\mathbb{Z})$ as a graded ring, where $S^*(k^*_{\mathbb{Z}})$ is the symmetric product of $k^*_{\mathbb{Z}}$ and each element of $k^*_{\mathbb{Z}}$ is of degree 2. Let $r\colon H^*_K(\mathbb{H}^N;\mathbb{Z})\to H^*_K(\mu^{-1}(\alpha,\beta);\mathbb{Z})$ be the restriction map. Since K acts freely on $\mu^{-1}(\alpha,\beta)$, we have the isomorphism $j\colon H^*_K(\mu^{-1}(\alpha,\beta);\mathbb{Z})\to H^*(X(\alpha,\beta);\mathbb{Z})$. The composition of the above maps

$$\kappa_{\mathbb{Z}}=j\circ r\circ\widetilde{c}\colon S^*(k^*_{\mathbb{Z}})\to H^*(X(\alpha,\beta);\mathbb{Z})$$

is called the *Kirwan map*. Note that $(\widetilde{L}_\rho|_{\mu^{-1}(\alpha,\beta)})/K$ is a line bundle on $X(\alpha,\beta)$. Then we have the following. Since the proof is easy, we omit the proof.

Lemma 3.1. *The Kirwan map $\kappa_{\mathbb{Z}}$ is given by*

$$\kappa_{\mathbb{Z}}(\rho)=c_1((\widetilde{L}_\rho|_{\mu^{-1}(\alpha,\beta)})/K)\in H^2(X(\alpha,\beta);\mathbb{Z})\quad \text{for an arbitrary } \rho\in k^*_{\mathbb{Z}}.$$

In [14] we computed the integral cohomology ring of a toric hyperKähler manifold $X(\alpha,\beta)$ as follows.

Theorem 3.2. [14] *Let K be a subtorus of T^N satisfying the condition* (C1). *Let $\Phi\colon\mathbb{Z}[u_1,\dots,u_N]\to H^*(X(\alpha,\beta);\mathbb{Z})$ be a ring homomorphism defined by $\Phi(u_i)=\kappa_{\mathbb{Z}}(\iota^*u_i)$. Then the map Φ is surjective, and the kernel $\ker\Phi\subset\mathbb{Z}[u_1,\dots,u_N]$ is the ideal generated by all*
(i) $\sum_{i=1}^N a_iu_i\in\ker\iota^*\cap(t^N_{\mathbb{Z}})^*$,
(ii) $\prod_{b_i\neq0}u_i$ *for* $\sum_{i=1}^N b_iX_i\in k\setminus\{0\}$.
That is, $H^*(X(\alpha,\beta);\mathbb{Z})$ is isomorphic to $\mathbb{Z}[u_1,\dots,u_N]/\ker\Phi$ as a ring.

As a corollary, we have the following.

Proposition 3.3. *Let K be a subtorus of T^N satisfying the conditions* (C1) *and* (C2). *Then the restriction of the Kirwan map $\kappa_{\mathbb{Z}}$ to $k_{\mathbb{Z}}^*$ induces the following isomorphism for all $(\alpha, \beta) \in (k^* \times k_{\mathbb{C}}^*)_{reg}$;*

$$\kappa^* = (\kappa_{\mathbb{Z}} \otimes \mathbb{R})|_{k^*} : k^* \to H^2(X(\alpha, \beta); \mathbb{R}).$$

Definition 3.4. Let K be a subtorus of T^N satisfying the conditions (C1) and (C2). Then we define the *period map* $\mathcal{P}\colon (k^* \times k_{\mathbb{C}}^*)_{reg} \to k^* \times k_{\mathbb{C}}^*$ by

$$\mathcal{P}(\alpha, \beta) = ((\kappa^*)^{-1}([\omega_1]), (\kappa^* \otimes \mathbb{C})^{-1}([\omega_{\mathbb{C}}])),$$

where $[\omega_1] \in H^2(X(\alpha, \beta); \mathbb{R})$ and $[\omega_{\mathbb{C}}] \in H^2(X(\alpha, \beta); \mathbb{C})$ denote the de Rham cohomology classes represented by the Kähler form ω_1 and the complex symplectic form $\omega_{\mathbb{C}}$ of $X(\alpha, \beta)$ respectively.

The following theorem implies that, if K be satisfies the conditions (C1) and (C2), the parameters α, β can be interpreted as the periods of $X(\alpha, \beta)$.

Theorem 3.5. *Let K be a subtorus of T^N satisfying the conditions* (C1) *and* (C2). *Then the period map $\mathcal{P}\colon (k^* \times k_{\mathbb{C}}^*)_{reg} \to k^* \times k_{\mathbb{C}}^*$ is given by $\mathcal{P}(\alpha, \beta) = (\alpha, \beta)$.*

Proof. Let $\Omega^p(\mathbb{H}^N)^K$ be the space of smooth K-invariant differential p-forms on $\mathbb{H}^N$. Recall that the complex for the Cartan model of the K-equivariant cohomology is defined by $\Omega_K^*(\mathbb{H}^N) = S^*(k^*) \otimes \Omega^*(\mathbb{H}^N)^K$, where the degree of $\xi \otimes \psi \in \Omega_K^*(\mathbb{H}^N)$ is given by $\deg(\xi \otimes \psi) = \deg \xi + \deg \psi$. Let $\{Y_1, \ldots, Y_{N-n}\}$ be a basis of the Lie algebra k and $\{v_1, \ldots, v_{N-n}\} \subset k^*$ the dual basis. The exterior differential operator $d_c\colon \Omega_K^p(\mathbb{H}^N) \to \Omega_K^{p+1}(\mathbb{H}^N)$ is defined by

$$d_c(\xi \otimes \psi) = \xi \otimes d\psi - \sum_{i=1}^{N-n} (\xi v_i) \otimes i(Y_i^*)\psi,$$

where Y_i^* is the vector fields on $\mathbb{H}^N$ generated by $Y_i \in k$ and $i(Y_i^*)$ denotes the interior product. (See [3] for detail.)

Let $\widetilde{\omega}_j$ be the Kähler form on $\mathbb{H}^N$ with respect to the complex structure I_j for $j = 1, 2, 3$ and $\widetilde{\omega}_{\mathbb{C}} = \widetilde{\omega}_2 + \sqrt{-1}\widetilde{\omega}_3$. Since $\widetilde{\omega}_1 - (\mu_1 - \alpha) \in \Omega_K^2(\mathbb{H}^N)$ and $\widetilde{\omega}_{\mathbb{C}} - (\mu_{\mathbb{C}} - \beta) \in \Omega_K^2(\mathbb{H}^N) \otimes \mathbb{C}$ are d_c-closed and their restrictions to $\mu^{-1}(\alpha, \beta)$ are $\widetilde{\omega}_1|\mu^{-1}(\alpha, \beta)$ and $\widetilde{\omega}_{\mathbb{C}}|\mu^{-1}(\alpha, \beta)$ respectively, we have

$$\kappa^* \circ (\widetilde{c} \otimes \mathbb{R})^{-1}([\widetilde{\omega}_1 - (\mu_1 - \alpha)]) = [\omega_1] \in H^2(X(\alpha, \beta); \mathbb{R}),$$
$$(\kappa^* \otimes \mathbb{C}) \circ (\widetilde{c} \otimes \mathbb{C})^{-1}([\widetilde{\omega}_{\mathbb{C}} - (\mu_{\mathbb{C}} - \beta)]) = [\omega_{\mathbb{C}}] \in H^2(X(\alpha, \beta); \mathbb{C}).$$

On the other hand, if we define 1-forms on $\mathbb{H}^N$ by

$$\eta_1 = \frac{\sqrt{-1}}{4} \sum_{i=1}^{N} (z_i d\overline{z}_i - \overline{z}_i dz_i) + \frac{\sqrt{-1}}{4} \sum_{i=1}^{N} (w_i d\overline{w}_i - \overline{w}_i dw_i),$$
$$\eta_{\mathbb{C}} = \frac{1}{2} \sum_{i=1}^{N} (z_i dw_i - w_i dz_i),$$

then we have $d_c\eta_1 = \widetilde{\omega}_1 - \mu_1$ and $d_c\eta_{\mathbb{C}} = \widetilde{\omega}_{\mathbb{C}} - \mu_{\mathbb{C}}$. This completed the proof. $\square$

4 Elementary transformation

First let us recall a special birational map between complex symplectic manifolds, which was discovered by Mukai [16]. Let X be a complex symplectic manifold of complex dimension $2n$, containing a complex submanifold S isomorphic to $\mathbb{P}^n$. Since S is a complex Lagrangian submanifold, the normal bundle $N_{S/X}$ is isomorphic to the cotangent bundle T^*S. Let $\pi\colon \widehat{X} \to X$ be the blowing-up along S. Then the exceptional divisor $E = \pi^{-1}(S)$ is isomorphic to the projective cotangent bundle $\mathbb{P}(T^*S)$. This can be viewed as the variety of pairs (p, h) such that $p \in S, h \in S^*$ with $p \in h$, where S^* is the space of hyperplanes in S. Therefore we can blow-down $\mathbb{P}(T^*S)$ to S^* and get a new complex symplectic manifold X' and the birational map $elm_S\colon X \dashrightarrow X'$, which is called the *elementary transformation* along S. This construction can be generalized directly to the case where S is a $\mathbb{P}^r$-bundle.

Next we give a fundamental example, where an elementary transformation appears naturally. Consider the subtorus K of T^N whose Lie algebra k is spanned by $X_1 + \cdots + X_N$. Let $\{v\} \subset k^*$ be its dual basis. Then we have $\iota^* u_1 = \cdots = \iota^* u_N = v$. If we identify k^* with $\mathbb{R}$ with respect to the basis $\{v\}$, the hyperKähler moment map for the action of K on $\mathbb{H}^N$

$$\mu = (\mu_1, \mu_{\mathbb{C}})\colon \mathbb{H}^N \to k^* \times k^*_{\mathbb{C}} \cong \mathbb{R} \times \mathbb{C}$$

is given by

$$\begin{aligned}\mu_1(z, w) =& \pi \sum_{i=1}^{N} (|z_i|^2 - |w_i|^2) \in \mathbb{R},\\ \mu_{\mathbb{C}}(z, w) =& -2\pi\sqrt{-1} \sum_{i=1}^{N} z_i w_i \in \mathbb{C}.\end{aligned}$$

Moreover $(\alpha, \beta) \in \mathbb{R} \times \mathbb{C}$ is a regular value of μ if and only if $(\alpha, \beta) \neq (0, 0)$. Since the condition (C1) is also satisfied, we have a toric hyperKähler manifold

$$X(\alpha, \beta) = \{(z, w) \in \mathbb{H}^N \mid \mu_1(z, w) = \alpha,\ \mu_{\mathbb{C}}(z, w) = \beta\}/K$$

for $(\alpha, \beta) \neq (0, 0)$, which is diffeomorphic to the cotangent bundle of the complex projective space $\mathbb{P}^{N-1}$.

Suppose $\beta \neq 0$. If $\mu_{\mathbb{C}}(z, w) = \beta$, there exists $i \in \{1, \ldots, N\}$ such that $z_i \neq 0$ and $w_i \neq 0$. Then we have

$$(X(\alpha, \beta), I_1) = \mu_{\mathbb{C}}^{-1}(\beta)/K_{\mathbb{C}},$$

where $K_{\mathbb{C}}$ is the complexification of K. Since the right hand side is independent of $\alpha \in \mathbb{R}$, $(X(\alpha', \beta), I_1)$ is canonically isomorphic to $(X(\alpha, \beta), I_1)$ as a complex manifold for an arbitrary $\alpha' \in \mathbb{R}$.

Next suppose $\alpha_+ > 0$ and $\beta = 0$. If $\mu_1(z, w) = \alpha_+$, then we have $z \neq 0$. Therefore we have

$$(X(\alpha_+, 0), I_1) = \{(z, w) \in \mu_{\mathbb{C}}^{-1}(0) \mid z \neq 0\}/K_{\mathbb{C}}.$$

Thus $(X(\alpha'_+,0),I_1)$ is canonically isomorphic to $(X(\alpha_+,0),I_1)$ as a complex manifold for an arbitrary $\alpha'_+ > 0$.

Similarly, in the case that $\alpha_- < 0$ and $\beta = 0$, we have

$$(X(\alpha_-,0),I_1) = \{(z,w) \in \mu_{\mathbb{C}}^{-1}(0) \mid w \neq 0\}/K_{\mathbb{C}}.$$

Therefore $(X(\alpha'_-,0),I_1)$ is canonically isomorphic to $(X(\alpha_-,0),I_1)$ as a complex manifold for an arbitrary $\alpha'_- < 0$.

Moreover, if we set $U = \{(z,w) \in \mu_{\mathbb{C}}^{-1}(0) \mid z \neq 0,\ w \neq 0\}/K_{\mathbb{C}}$, then we have

$$\begin{aligned} X(\alpha_+,0) \setminus U &= \{(z,0) \in \mathbb{H}^N \mid z \neq 0\}/K_{\mathbb{C}} \cong \mathbb{P}^{N-1}, \\ X(\alpha_-,0) \setminus U &= \{(0,w) \in \mathbb{H}^N \mid w \neq 0\}/K_{\mathbb{C}} \cong \mathbb{P}^{N-1}. \end{aligned}$$

Thus we have a birational map $f\colon X(\alpha_+,0) \dashrightarrow X(\alpha_-,0)$ such that $f|_U$ is the identity map of U. Since this is the same situation as the one in Section 1 in [21], the birational map f is the elementary transformation along $X(\alpha_+,0) \setminus U$.

5 Stability

Let K be a subtorus of T^N satisfying the condition (C1). We denote its complexification by $K_{\mathbb{C}}$. Let $\mu = (\mu_1, \mu_{\mathbb{C}})\colon \mathbb{H}^N \to k^* \otimes \mathbb{R}^3$ be the hyperKähler moment map for the action of K on $\mathbb{H}^N$. Since a hyperKähler quotient $\mu^{-1}(\alpha,\beta)/K$ is a symplectic quotient of $\mu_{\mathbb{C}}^{-1}(\beta)$ by K, it should be identified with a quotient space in geometric invariant theory (GIT for short). The key notion in GIT is stability, which depends on $\alpha \in k^*$ in our case. Here we give an ad hoc definition of α-stability.

Definition 5.1. Fix $(\alpha,\beta) \in (k^* \times k^*_{\mathbb{C}})_{reg}$. A point $(z,w) \in \mu_{\mathbb{C}}^{-1}(\beta)$ is *α-stable* if and only if

$$\alpha \in \sum_{i \in J^1_{(z,w)}} \mathbb{R}\iota^* u_i + \sum_{i \in J^2_{(z,w)}} \mathbb{R}_{>0}\,\iota^* u_i + \sum_{i \in J^3_{(z,w)}} \mathbb{R}_{<0}\,\iota^* u_i,$$

where $\mathbb{R}_{>0}, \mathbb{R}_{<0}$ are the sets of positive and negative numbers respectively, and $J^1_{(z,w)}$, $J^2_{(z,w)}$, $J^3_{(z,w)}$ are the subsets of $\{1,\ldots,N\}$ defined as follows:

$$\begin{aligned} J^1_{(z,w)} =& \{i \mid z_i \neq 0 \text{ and } w_i \neq 0\}, \\ J^2_{(z,w)} =& \{i \mid z_i \neq 0 \text{ and } w_i = 0\}, \\ J^3_{(z,w)} =& \{i \mid z_i = 0 \text{ and } w_i \neq 0\}. \end{aligned} \tag{5.1}$$

Moreover, we denote the set of α-stable points in $\mu_{\mathbb{C}}^{-1}(\beta)$ by $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$.

It is easy to see that $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$ is preserved by the action of $K_{\mathbb{C}}$ and that $\mu^{-1}(\alpha,\beta) \subset \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$. Therefore we have the following natural map

$$i\colon \mu^{-1}(\alpha,\beta)/K \to \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}/K_{\mathbb{C}}.$$

Theorem 5.2. *Let K be a subtorus of T^N satisfying the condition* (C1)*. Suppose $(\alpha,\beta)\in(k^*\times k^*_{\mathbb{C}})_{reg}$. Then the following holds.*
(1) *$K_{\mathbb{C}}$ acts freely on $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$.*
(2) *The map i is bijective.*

Remark 5.3. Here we discuss the relation between the definitions of α-stability in Definition 5.1 and in GIT. To define a GIT quotient, we fix a linearization of the action, which is a lift of the action of $K_{\mathbb{C}}$ on $\mu_{\mathbb{C}}^{-1}(\beta)$ to a line bundle. This corresponds to $\alpha\in k^*_{\mathbb{Z}}$, because $\widetilde{L}_\alpha|_{\mu_{\mathbb{C}}^{-1}(\beta)}$ is a $K_{\mathbb{C}}$-equivariant line bundle on $\mu_{\mathbb{C}}^{-1}(\beta)$. Then we have the notion of α-stable and α-semi-stable points (in the sense of GIT) in $\mu_{\mathbb{C}}^{-1}(\beta)$ and also the GIT quotient $\mu_{\mathbb{C}}^{-1}(\beta)//_\alpha K_{\mathbb{C}}$ for $\alpha\in k^*_{\mathbb{Z}}$. By the theorem due to [13], we have a bijection $i'\colon \mu^{-1}(\alpha,\beta)/K\to\mu_{\mathbb{C}}^{-1}(\beta)//_\alpha K_{\mathbb{C}}$. See Chapter 8 in [17] for the proof. For non-compact cases, see also Chapter 3 in [19]. In the proof of the theorem due to [13], a similar functional to the one given in (5.3) below plays an important role. The key point of the proof of Theorem 5.2 is to show that the same procedure also works even if we define α-stability as in Definition 5.1. By Theorem 5.2 together with the theorem due to [13], we have the following conclusion. Under the condition $(\alpha,\beta)\in(k^*\times k^*_{\mathbb{C}})_{reg}$, all α-semi-stable points (in the sense of GIT) are α-stable (in the sense of GIT), and α-stability in Definition 5.1 is equivalent to the one in GIT for $\alpha\in k^*_{\mathbb{Z}}$. Or rather, it is better to say that Definition 5.1 is a criterion of α-stable points (in the sense of GIT) for $\alpha\in k^*_{\mathbb{Z}}$. Thus our quotient $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}/K_{\mathbb{C}}$ is identified with the GIT quotient $\mu_{\mathbb{C}}^{-1}(\beta)//_\alpha K_{\mathbb{C}}$ for $\alpha\in k^*_{\mathbb{Z}}$. Moreover, Definition 5.1 can be considered as a natural generalization of the notion of α-stability to all $\alpha\in k^*$.

To prove this theorem, we need two lemmas.

Lemma 5.4. *Suppose $(\alpha,\beta)\in(k^*\times k^*_{\mathbb{C}})_{reg}$ and $(z,w)\in\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$. Then there exists a basis $\{\iota^*u_i\}_{i\in I}$ of k^* satisfying the following conditions:*
(i) $I\subset J_{(z,w)}=J^1_{(z,w)}\cup J^2_{(z,w)}\cup J^3_{(z,w)}$,
(ii) $\alpha\in\sum_{i\in I^1}\mathbb{R}\iota^*u_i+\sum_{i\in I^2}\mathbb{R}_{>0}\,\iota^*u_i+\sum_{i\in I^3}\mathbb{R}_{<0}\,\iota^*u_i$,
(iii) $\beta\in\sum_{i\in I^1}\mathbb{C}\iota^*u_i$,
where $I^j=I\cap J^j_{(z,w)}$ for $j=1,2,3$.

Proof. Let $\mathcal{I}$ be the set of subsets of $\{1,\dots,N\}$ satisfying the conditions (i), (ii) and (iii). Since $(z,w)\in\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$, we have $J_{(z,w)}\in\mathcal{I}$. Moreover, we have the following.

Claim 5.5. *If $\{\iota^*u_i\}_{i\in I}$ are linearly dependent for $I\in\mathcal{I}$, then there exists a subset $I'\subsetneqq I$ such that $I'\in\mathcal{I}$.*

Proof. Suppose that $\{\iota^*u_i\}_{i\in I^1}$ are linearly dependent. Then there exists a subset $I^{1\prime}\subsetneqq I^1$ such that $\{\iota^*u_i\}_{i\in I^{1\prime}}$ is a basis of $\mathrm{span}_{\mathbb{R}}\{\iota^*u_i\}_{i\in I^1}$. Then we have $I'=I^{1\prime}\cup I^2\cup I^3\in\mathcal{I}$. So Claim 5.5 holds in this case.

Next suppose that $\{\iota^*u_i\}_{i\in I^1}$ are linearly independent. In this case there exists $i_0\in I^2\cup I^3$ such that $\iota^*u_{i_0}=\sum_{i\in I\setminus\{i_0\}}a_i\iota^*u_i$. By the condition (ii) we have $\alpha=$

$\sum_{i\in I} b_i\iota^* u_i$, where $b_i > 0$ for $i \in I^2$ and $b_i < 0$ for $i \in I^3$. So we have

$$\alpha = (1-t)b_{i_0}\iota^* u_{i_0} + \sum_{i\in I\setminus\{i_0\}} (b_i + tb_{i_0}a_i)\iota^* u_i. \tag{5.2}$$

Since $b_{i_0} \neq 0$, the coefficients of $\iota^* u_i$ vary as the parameter t increases from 0. Define a subset $S(t) \subset I^2 \cup I^3$ for $t \geq 0$ by

$$S(t) = \{i \in I^2 \cup I^3 \mid \text{The coefficient of } \iota^* u_i \text{ in (5.2) is zero}\}.$$

Let $t_0 \geq 0$ be the smallest non-negative number such that $S(t_0) \neq \emptyset$. Since $S(1) \neq \emptyset$, we have $t_0 \leq 1$. Thus we have $I' = I \setminus S(t_0) \in \mathcal{I}$. Therefore Claim 5.5 also holds in this case. $\square$

By Claim 5.5 there exists $I \in \mathcal{I}$ such that $\{\iota^* u_i\}_{i\in I}$ is linearly independent. Since $(\alpha, \beta) \in (k^* \times k^*_{\mathbb{C}})_{reg}$, $\{\iota^* u_i\}_{i\in I}$ is a basis of k^*. Thus we finished the proof of Lemma 5.4. $\square$

Define a functional $l_{\alpha,(z,w)}\colon k \to \mathbb{R}$ for $\alpha \in k^*$ and $(z, w) \in \mathbb{H}^N$ by

$$l_{\alpha,(z,w)}(X) = \langle \alpha, X\rangle + \frac{1}{4}\sum_{i=1}^{N} |z_i|^2 e^{-4\pi\langle \iota^* u_i, X\rangle} + \frac{1}{4}\sum_{i=1}^{N} |w_i|^2 e^{4\pi\langle \iota^* u_i, X\rangle}, \tag{5.3}$$

where $\langle\ ,\ \rangle\colon k^* \times k \to \mathbb{R}$ is the natural pairing.

Remark 5.6. This functional is a slight generalization of the one in Appendix 1 in [7]. It was also used in [4]. See Remark 6.13 below.

Lemma 5.7. *Suppose $(\alpha, \beta) \in (k^* \times k^*_{\mathbb{C}})_{reg}$ and $(z, w) \in \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$. Then the functional $l_{\alpha,(z,w)}$ has the unique critical point, where it attains its minimum.*

Proof. First we show the following.

Claim 5.8. *The functional $l_{\alpha,(z,w)}$ is strictly convex.*

Proof. It is enough to show that $\frac{d^2}{dt^2} l_{\alpha,(z,w)}(X + tY)|_{t=0} > 0$ for all $X \in k$ and all $Y \in k \setminus \{0\}$. Take a basis $\{\iota^* u_i\}_{i\in I}$ of k^* in Lemma 5.4. Note that $z_i \neq 0$ or $w_i \neq 0$ for all $i \in I$. Since $Y \neq 0$, there exists $i \in I$ such that $\langle \iota^* u_i, Y\rangle \neq 0$. Therefore we have

$$\frac{d^2}{dt^2} l_{\alpha,(z,w)}(X + tY)|_{t=0} = 4\pi^2 \sum_{i=1}^{N} \langle \iota^* u_i, Y\rangle^2 (|z_i|^2 e^{-4\pi\langle \iota^* u_i, X\rangle} + |w_i|^2 e^{4\pi\langle \iota^* u_i, X\rangle}) > 0.$$

Thus we proved Claim 5.8. $\square$

The next claim is the key to prove Lemma 5.7.

Claim 5.9. $\lim_{t\to\infty} l_{\alpha,(z,w)}(X + tY) = +\infty$ *for all $X \in k$ and all $Y \in k \setminus \{0\}$.*

Proof. Take a basis $\{\iota^* u_i\}_{i\in I}$ of k^* in Lemma 5.4 again. Since $\{\iota^* u_i\}_{i\in I}$ is a basis of k^*, we can express $\alpha = \sum_{i\in I} a_i \iota^* u_i$ uniquely. Due to the condition (ii), we have $z_i \neq 0$ if $a_i > 0$; $w_i \neq 0$ if $a_i < 0$; $z_i \neq 0$, $w_i \neq 0$ if $a_i = 0$ respectively.

Since

$$l_{\alpha,(z,w)}(X+tY) = \langle \alpha, X+tY\rangle + \frac{1}{4}\sum_{i=1}^{N} |z_i|^2 e^{-4\pi\langle \iota^* u_i, X+tY\rangle} + \frac{1}{4}\sum_{i=1}^{N} |w_i|^2 e^{4\pi\langle \iota^* u_i, X+tY\rangle},$$

the claim holds if $\langle \alpha, Y\rangle > 0$.

So we have to prove Claim 5.9 in the case that $0 \geq \langle \alpha, Y\rangle = \sum_{i\in I} a_i \langle \iota^* u_i, Y\rangle$.

Suppose, in addition, that there exists $i \in I$ such that $a_i\langle \iota^* u_i, Y\rangle < 0$. If $a_i > 0$, then $z_i \neq 0$ and $\langle \iota^* u_i, Y\rangle < 0$. If $a_i < 0$, then $w_i \neq 0$ and $\langle \iota^* u_i, Y\rangle > 0$. So Claim 5.9 holds in this case.

Next, suppose that $a_i\langle \iota^* u_i, Y\rangle = 0$ for all $i \in I$. Since $Y \neq 0$, there exists $i_0 \in I$ such that $\langle \iota^* u_{i_0}, Y\rangle \neq 0$. So we have $a_{i_0} = 0$, which implies $z_{i_0} \neq 0$ and $w_{i_0} \neq 0$. Therefore Claim 5.9 holds in this case.

Thus Claim 5.9 holds in all cases. □

By Claim 5.9, there exists $X \in k$ where the functional $l_{\alpha,(z,w)}$ attains its minimum. Then, by Claim 5.8, $X \in k$ is the unique critical point of the functional $l_{\alpha,(z,w)}$. Thus we finished the proof of Lemma 5.7. □

Proof of Theorem 5.2. First we prove (2). Fix $(z,w) \in \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$. Then, by Lemma 5.7, there exists a critical point $X \in k$ of the functional $l_{\alpha,(z,w)}$. Since

$$\frac{d}{dt} l_{\alpha,(z,w)}(X+tY)|_{t=0} = \langle \alpha - \mu_1((z,w)\mathrm{Exp}(\sqrt{-1}X)), Y\rangle,$$

we have $(z,w)\mathrm{Exp}(\sqrt{-1}X) \in \mu_1^{-1}(\alpha)$. This implies the surjectivity of the map i. The uniqueness of the critical point implies the injectivity of the map i.

Next we prove (1). Suppose that $(z,w) \in \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$ is fixed by $\zeta = \mathrm{Exp}(X + \sqrt{-1}Y) \in K_{\mathbb{C}}$, where $X, Y \in k$. By (2), we may assume $(z,w) \in \mu^{-1}(\alpha,\beta)$. Since $(z,w), (z,w)\mathrm{Exp}\sqrt{-1}Y \in \mu_1^{-1}(\alpha)$, 0 and Y are the critical points of the functional $l_{\alpha,(z,w)}$. Since the critical point is unique, we have $Y = 0 \in k$. That is, $\zeta = \mathrm{Exp}X \in K$ fixes $(z,w) \in \mu^{-1}(\alpha,\beta)$. Since K acts freely on $\mu^{-1}(\alpha,\beta)$, we have $\zeta = 1 \in K$. □

Next we give another useful criterion of α-stable points. Let $\{P_1,\dots,P_l\}$ be the set of subspaces of k^* introduced in Proposition 2.3. Fix $Y_s \in k \setminus \{0\}$ such that $P_s = \{v \in k^* \mid \langle v, Y_s\rangle = 0\}$ for $s = 1,\dots,l$. Moreover, define a subset of $\{1,\dots,N\}$ for $s = 1,\dots,l$ by

$$J_s^0 = \{i \in \{1,\dots,N\} \mid \langle \iota^* u_i, Y_s\rangle = 0\}. \tag{5.4}$$

Then we have the following theorem.

Theorem 5.10. *Let K be a subtorus of T^N satisfying the condition* (C1). *Suppose $(\alpha,\beta) \in (k^* \times k_{\mathbb{C}}^*)_{reg}$ and $(z,w) \in \mu_{\mathbb{C}}^{-1}(\beta)$. Define a subset of $\{1,\dots l\}$ by*

$$\Lambda_\beta = \{s \in \{1,\dots l\} \mid \beta \in P_s \otimes \mathbb{C}\}. \tag{5.5}$$

Moreover, define subsets of $\{1,\ldots,N\}$ *for* $s\in\Lambda_\beta$ *by*

$$\begin{aligned} J_s^+ =&\{i\in\{1,\ldots,N\}\mid \langle\iota^*u_i,Y_s\rangle\langle\alpha,Y_s\rangle>0\},\\ J_s^- =&\{i\in\{1,\ldots,N\}\mid \langle\iota^*u_i,Y_s\rangle\langle\alpha,Y_s\rangle<0\}. \end{aligned} \tag{5.6}$$

Then the following (i) *and* (ii) *are equivalent.*
(i) $(z,w)\in\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$.
(ii) *For each* $s\in\Lambda_\beta$, *there exists* $i\in J_s^+\cup J_s^-$ *such that* $z_i\neq 0$ *if* $i\in J_s^+$ *or* $w_i\neq 0$ *if* $i\in J_s^-$.

Remark 5.11. If $(\alpha,\beta)\in(k^*\times k^*_{\mathbb{C}})_{reg}$, then $\alpha\notin P_s$, therefore $\langle\alpha,Y_s\rangle\neq 0$ for all $s\in\Lambda_\beta$. Therefore we have $\{1,\ldots,N\}=J_s^0\cup J_s^+\cup J_s^-$, where the right hand side is disjoint union. Moreover, the subsets J_s^+,J_s^- depends on α, but not on the sign of Y_s.

Remark 5.12. If $\beta\in k^*_{\mathbb{C}}\setminus\bigcup_{s=1}^l(P_s\otimes\mathbb{C})$, then we have $\Lambda_\beta=\emptyset$. By Theorem 5.10 we have $\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}=\mu_{\mathbb{C}}^{-1}(\beta)$ in this case. This is also proved in [4]. See Remark 6.13.

Proof. First we show that (i) $\Longrightarrow$ (ii). Suppose the contrary, that is, there exists $(z,w)\in\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$ and $s\in\Lambda_\beta$ such that $z_i=0$ for all $i\in J_s^+$ and $w_i=0$ for all $i\in J_s^-$. Since (z,w) is α-stable, we have

$$\alpha\in\sum_{i\in J_s^0}\mathbb{R}\iota^*u_i+\sum_{i\in J_s^+}\mathbb{R}_{\leq 0}\,\iota^*u_i+\sum_{i\in J_s^-}\mathbb{R}_{\geq 0}\,\iota^*u_i,$$

where $\mathbb{R}_{\leq 0},\mathbb{R}_{\geq 0}$ are the sets of non-positive and non-negative numbers respectively. Since $\langle\alpha,Y_s\rangle\neq 0$, we have

$$0<\langle\alpha,Y_s\rangle^2\in\sum_{i\in J_s^+}\mathbb{R}_{\leq 0}\,\langle\iota^*u_i,Y_s\rangle\langle\alpha,Y_s\rangle+\sum_{i\in J_s^-}\mathbb{R}_{\geq 0}\,\langle\iota^*u_i,Y_s\rangle\langle\alpha,Y_s\rangle\subset\mathbb{R}_{\leq 0}.$$

This is a contradiction.

Next we show that (ii) $\Longrightarrow$ (i). Suppose that $(z,w)\in\mu_{\mathbb{C}}^{-1}(\beta)$ satisfies the condition (ii). For each $s\in\Lambda_\beta$, fix $i_s\in J_s^+\cup J_s^-$ such that $z_{i_s}\neq 0$ if $i_s\in J_s^+$ or $w_{i_s}\neq 0$ if $i_s\in J_s^-$. Set $\Lambda_\beta^+=\{s\in\Lambda_\beta\mid i_s\in J_s^+\}$ and $\Lambda_\beta^-=\{s\in\Lambda_\beta\mid i_s\in J_s^-\}$. Moreover, note that, if $s\in\Lambda_\beta^c=\{1,\ldots,l\}\setminus\Lambda_\beta$, then there exists $i_s\in\{1,\ldots,N\}\setminus J_s^0$ such that $z_{i_s}\neq 0$ and $w_{i_s}\neq 0$. Fix such i_s for each $s\in\Lambda_\beta^c$. Thus we have fixed $i_s\in\{1,\ldots,N\}\setminus J_s^0$ for each $s=1,\ldots,l$. Note that k^* is spanned by $\iota^*u_{i_1},\ldots,\iota^*u_{i_l}$. In fact, if not, there exists $t\in\{1,\ldots,l\}$ such that $\iota^*u_{i_1},\ldots,\iota^*u_{i_l}\in P_t$. However, $\iota^*u_{i_t}\notin P_t$. This is a contradiction.

Define an open convex cone $\mathcal{C}\subset k^*$ by

$$\mathcal{C}=\sum_{s\in\Lambda_\beta^c}\mathbb{R}\iota^*u_{i_s}+\sum_{s\in\Lambda_\beta^+}\mathbb{R}_{>0}\,\iota^*u_{i_s}+\sum_{s\in\Lambda_\beta^-}\mathbb{R}_{<0}\,\iota^*u_{i_s}. \tag{5.7}$$

Claim 5.13. $\alpha\in\mathcal{C}$.

Proof. If $\mathcal{C} = k^*$, the claim holds. Suppose $\mathcal{C} \subsetneqq k^*$. Note that $\mathcal{C}$ is an open convex cone in k^* and that each codimension one boundary of $\mathcal{C}$ is an open subset of P_t for some $t \in \{1, \ldots, l\}$. Define a subset $\Lambda'_\beta \subset \{1, \ldots, l\}$ such that $t \in \Lambda'_\beta$ if and only if there exists a codimension one boundary of $\mathcal{C}$ which is an open subset of P_t. Since the range of the coefficient $\iota^* u_{i_s}$ in (5.7) is $\mathbb{R}$ for $s \in \Lambda^c_\beta$, any open subset of the hyperplane P_s cannot be a boundary of $\mathcal{C}$ for $s \in \Lambda^c_\beta$. Therefore we have $\Lambda'_\beta \subset \Lambda_\beta$.

Fix an arbitrary $t \in \Lambda'_\beta$. By the same reason as above, we have $\iota^* u_{i_s} \in P_t$ for all $s \in \Lambda^c_\beta$. Since $\langle \iota^* u_{i_t}, Y_t \rangle \langle \alpha, Y_t \rangle > 0$ for $t \in \Lambda^+_\beta$ and $\langle \iota^* u_{i_t}, Y_t \rangle \langle \alpha, Y_t \rangle < 0$ for $t \in \Lambda^-_\beta$, there exists a point $v \in \mathcal{C}$ such that $\langle v, Y_t \rangle \langle \alpha, Y_t \rangle > 0$. Since an open subset of the hyperplane P_t is the boundary of $\mathcal{C}$, we also have $\langle \iota^* u_{i_s}, Y_t \rangle \langle \alpha, Y_t \rangle \geq 0$ for all $s \in \Lambda^+_\beta$, and $\langle \iota^* u_{i_s}, Y_t \rangle \langle \alpha, Y_t \rangle \leq 0$ for all $s \in \Lambda^-_\beta$. Thus, if we set

$$\mathcal{C}' = \{ v \in k^* \mid \langle v, Y_t \rangle \langle \alpha, Y_t \rangle > 0 \text{ for all } t \in \Lambda'_\beta \},$$

we have $\mathcal{C} \subset \mathcal{C}'$. Since $\mathcal{C}$ is an open convex cone in k^*, by the definition of Λ'_β, we have $\mathcal{C} = \mathcal{C}'$. Therefore we have $\alpha \in \mathcal{C}$. Thus we proved the claim. □

Since $\mathcal{C}$ is open in k^*, it is easy to see that (z, w) is α-stable. Thus we finished the proof of Theorem 5.10. □

6 Variation of complex structures

Let K be a subtorus of T^N satisfying the condition (C1). Let $\{P_1, \ldots, P_l\}$ be the set of subspaces of k^* introduced in Proposition 2.3. Fix an arbitrary $\beta \in k^*_{\mathbb{C}}$. Define a subset $\Lambda_\beta \subset \{1, \ldots l\}$ by (5.5). Each connected component of $k^* \setminus \bigcup_{s \in \Lambda_\beta} P_s$ is called a *chamber*. In this case the hyperplane P_s for $s \in \Lambda_\beta$ is called a *wall*. Fix $Y_s \in k$ such that $P_s = \{ v \in k^* \mid \langle v, Y_s \rangle = 0 \}$ for $s = 1, \ldots, l$. Recall that a vector $Y \in k_{\mathbb{Z}}$ is *primitive* if and only if $Y = aZ$ for $a \in \mathbb{Z}, Z \in k_{\mathbb{Z}}$ implies $a = \pm 1$. We may assume that Y_s is contained in $k_{\mathbb{Z}}$ and primitive. Then Y_s is determined up to sign.

Due to Theorems 5.2 and 5.10, we have

$$\begin{aligned}(X(\alpha, \beta), I_1) =& \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha - st} / K_{\mathbb{C}} \\ =& \{ (z, w) \in \mu_{\mathbb{C}}^{-1}(\beta) \mid \text{ the condition (ii) in Theorem 5.10} \} / K_{\mathbb{C}}.\end{aligned}$$

Since the condition (ii) in Theorem 5.10 depends only on the chamber, to which α belongs, we have the following.

Theorem 6.1. *Let K be a subtorus of T^N satisfying the condition* (C1). *Suppose that α' belong to the same chamber as α given by $\{P_s\}_{s \in \Lambda_\beta}$ for a fixed $\beta \in k^*_{\mathbb{C}}$. Then $(X(\alpha', \beta), I_1)$ is canonically isomorphic to $(X(\alpha, \beta), I_1)$ as a complex manifold.*

Remark 6.2. Here we give a proof of Theorem 2.6. Fix an arbitrary $(\alpha, \beta) \in (k^* \times k^*_{\mathbb{C}})_{reg}$. Since $(k^* \times k^*_{\mathbb{C}})_{reg}$ is connected, we have only to show that there exists an open neighborhood V of (α, β) such that $X(\alpha', \beta')$ is diffeomorphic to $X(\alpha, \beta)$ for all $(\alpha', \beta') \in V$. Here we use the notation $(\alpha_1, \alpha_2, \alpha_3)$ instead of (α, β) if $\alpha = \alpha_1, \beta = \alpha_2 + \sqrt{-1}\alpha_3$ for $\alpha_1, \alpha_2, \alpha_3 \in k^*$. Since $(k^* \times k^*_{\mathbb{C}})_{reg}$ is open, there exists an open neighborhood U_i of α_i in k^* for $i = 1,2,3$ such that $V = U_1 \times U_2 \times U_3 \subset (k^* \times k^*_{\mathbb{C}})_{reg}$. Fix an arbitrary $(\alpha'_1, \alpha'_2, \alpha'_3) \in V$. By Theorem 6.1, $X(\alpha'_1, \alpha'_2, \alpha'_3)$ is diffeomorphic to $X(\alpha_1, \alpha'_2, \alpha'_3)$. By Theorem 6.1 with respect to the complex structure I_2, $X(\alpha_1, \alpha'_2, \alpha'_3)$ is diffeomorphic to $X(\alpha_1, \alpha_2, \alpha'_3)$. By Theorem 6.1 with respect to the complex structure I_3, $X(\alpha_1, \alpha_2, \alpha'_3)$ is diffeomorphic to $X(\alpha_1, \alpha_2, \alpha_3)$. Thus we proved that $X(\alpha'_1, \alpha'_2, \alpha'_3)$ is diffeomorphic to $X(\alpha_1, \alpha_2, \alpha_3)$. This completed the proof of Theorem 2.6.

Thus we have seen that, if we fix $\beta \in k^*_{\mathbb{C}}$, the complex structure I_1 depends only on the chamber. Next we will discuss the relation between two complex structures corresponding to different chambers. Namely, we study the so called wall crossing phenomena.

Theorem 6.3. *Let K be a subtorus of T^N satisfying the conditions* (C1) *and* (C2)*. Suppose that $\alpha_+, \alpha_- \in k^*$ belong to the next chamber to each other. Then two complex manifolds $(X(\alpha_+, \beta), I_1)$ and $(X(\alpha_-, \beta), I_1)$ are related by Mukai's elementary transformation.*

Proof. Suppose that $\alpha_+, \alpha_- \in k^*$ belong to the chambers $\mathcal{C}_+, \mathcal{C}_-$ respectively and that the two chambers $\mathcal{C}_+, \mathcal{C}_-$ have the common wall, say P_{s_0}, as their codimension one boundary. So we assume $s_0 \in \Lambda_\beta$. Define $J^0_s \subset \{1, \ldots, N\}$ by (5.4) and $J^+_s, J^-_s \subset \{1, \ldots, N\}$ by (5.6) with respect to α_+ for $s \in \Lambda_\beta$ respectively. Note that J^+_s, J^-_s is independent of the sign of Y_s.

We set $\widetilde{M}_+ = \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha_+ - st}$, $\widetilde{M}_- = \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha_- - st}$, and $\widetilde{U} = \widetilde{M}_+ \cap \widetilde{M}_-$. To describe these sets explicitly, we introduce the four conditions as follows;

(s) For each $s \in \Lambda_\beta \setminus \{s_0\}$, there exists $i \in J^+_s \cup J^-_s$ such that $z_i \neq 0$ if $i \in J^+_s$ or $w_i \neq 0$ if $i \in J^-_s$;

(s_0+) There exists $i \in J^+_{s_0} \cup J^-_{s_0}$ such that $z_i \neq 0$ if $i \in J^+_{s_0}$ or $w_i \neq 0$ if $i \in J^-_{s_0}$;

(s_0-) There exists $i \in J^+_{s_0} \cup J^-_{s_0}$ such that $w_i \neq 0$ if $i \in J^+_{s_0}$ or $z_i \neq 0$ if $i \in J^-_{s_0}$;

(not s_0-) $w_i = 0$ for all $i \in J^+_{s_0}$ and $z_i = 0$ for all $i \in J^-_{s_0}$.

Due to Theorem 5.10, we have

$$\begin{aligned} \widetilde{M}_+ =& \{(z,w) \in \mu_{\mathbb{C}}^{-1}(\beta) \mid \text{the conditions (s) and } (s_0+)\}, \\ \widetilde{M}_- =& \{(z,w) \in \mu_{\mathbb{C}}^{-1}(\beta) \mid \text{the conditions (s) and } (s_0-)\}, \\ \widetilde{U} =& \{(z,w) \in \mu_{\mathbb{C}}^{-1}(\beta) \mid \text{the conditions (s), } (s_0+) \text{ and } (s_0-)\}. \end{aligned}$$

Due to Theorem 5.2, we have $(X(\alpha_+, \beta), I_1) = \widetilde{M}_+/K_{\mathbb{C}}$, $\quad (X(\alpha_-, \beta), I_1) = \widetilde{M}_-/K_{\mathbb{C}}$. We will show that the subset $X(\alpha_+, \beta) \setminus U$, where $U = \widetilde{U}/K_{\mathbb{C}}$, is a $\mathbb{P}^r$-bundle over a certain toric hyperKähler manifold. To do this, we need several lemmas.

Set $t^{\overline{N}} = \mathrm{span}\{X_i\}_{i \in J^0_{s_0}}$, where $\overline{N}$ is the number of elements of $J^0_{s_0}$. Define the projection $p\colon t^N \to t^{\overline{N}}$ by $p(\sum_{i=1}^N a_i X_i) = \sum_{i \in J^0_{s_0}} a_i X_i$. Let $\overline{k}$ be the Lie subalgebra of $t^{\overline{N}}$ defined to be the image of k by p. Then we have the dual map $(p|_k)^*\colon \overline{k}^* \to k^*$. Let $\overline{\iota}\colon \overline{k} \to t^{\overline{N}}$ be the inclusion. Since $(p|_k)^*(\overline{\iota}^* u_i) = \iota^* u_i$ for $i \in J^0_{s_0}$, we can identify $\overline{k}^*$ with P_{s_0} canonically. So we can consider $\iota^* u_i \in \overline{k}^*$ for $i \in J^0_{s_0}$ and also $\beta \in \overline{k}^*_{\mathbb{C}}$. Moreover, we have $\dim \overline{k} = \dim k - 1 = N - n - 1$ and $\ker p|_k = \mathbb{R} Y_{s_0}$.

Let $\overline{K}$ be the subtorus of $T^{\overline{N}}$ corresponding to the Lie algebra $\overline{k}$. Then we have the following.

Lemma 6.4. *The subtorus $\overline{K}$ of $T^{\overline{N}}$ satisfies the condition* (C1).

Proof. Suppose that $\{\overline{\iota}^* u_i\}_{i\in\overline{J}}$, where $\overline{J} \subset J^0_{s_0}$, be a basis of $\overline{k}^*$. Fix an arbitrary $i_0 \in J^+_{s_0} \cup J^-_{s_0}$. If we set $J = \overline{J} \cup \{i_0\}$, then $\{\iota^* u_i\}_{i\in J}$ is a basis of k^*. By the condition (C1) for K, we have

$$t^N_{\mathbb{Z}} = k_{\mathbb{Z}} + \sum_{i\in\{1,\ldots,N\}\setminus J} \mathbb{Z}X_i \quad \text{as a } \mathbb{Z}\text{-module.}$$

Applying the projection $p\colon t^N \to t^{\overline{N}}$ to the both hand sides, we have

$$t^{\overline{N}}_{\mathbb{Z}} = \overline{k}_{\mathbb{Z}} + \sum_{i\in J^0_{s_0}\setminus \overline{J}} \mathbb{Z}X_i \quad \text{as a } \mathbb{Z}\text{-module,}$$

where $t^{\overline{N}}_{\mathbb{Z}} = \sum_{i\in J^0_{s_0}} \mathbb{Z}X_i$ and $\overline{k}_{\mathbb{Z}} = \overline{k} \cap t^{\overline{N}}_{\mathbb{Z}}$. This is the condition (C1) for $\overline{K}$. $\square$

The action of $\overline{K}$ on $\mathbb{H}^{\overline{N}}$ induces the hyperKähler moment map

$$\overline{\mu} = (\overline{\mu}_1, \overline{\mu}_{\mathbb{C}})\colon \mathbb{H}^{\overline{N}} \to \overline{k}^* \otimes \mathbb{R}^3 \cong \overline{k}^* \times \overline{k}^*_{\mathbb{C}}.$$

Denote the set of regular values of $\overline{\mu}$ by $(\overline{k}^* \times \overline{k}^*_{\mathbb{C}})_{reg}$.

Fix an arbitrary $\overline{\alpha}$ in the interior (with respect to the topology of P_{s_0}) of $\partial\mathcal{C}_+ \cap P_{s_0}$, where $\partial\mathcal{C}_+$ is the boundary of $\mathcal{C}_+$ in k^*. Then we have the following.

Lemma 6.5. $(\overline{\alpha}, \beta) \in (\overline{k}^* \times \overline{k}^*_{\mathbb{C}})_{reg}$.

Proof. Let $\{\overline{P}_1, \ldots, \overline{P}_{\overline{l}}\}$ be the set of all subspaces of codimension one in $\overline{k}^*$ which are spanned by subsets of $\{\iota^* u_i\}_{i\in J^0_{s_0}}$. Define a subset of $\{1, \ldots, \overline{l}\}$ by $\overline{\Lambda}_\beta = \{s \mid \beta \in \overline{P}_s \otimes \mathbb{C}\}$. To prove the lemma, we have to show that $\overline{\alpha} \notin \overline{P}_s$ for all $s \in \overline{\Lambda}_\beta$.

Suppose the contrary, that is, there exists $s \in \overline{\Lambda}_\beta$ such that $\overline{\alpha} \in \overline{P}_s$. By the definition of $\overline{P}_s$, there exists a subset $\{i_1, \ldots, i_{N-n-2}\} \subset J^0_{s_0}$ such that $\overline{P}_s$ is spanned by $\iota^* u_{i_1}, \ldots, \iota^* u_{i_{N-n-2}}$. Fix an arbitrary $i_{N-n-1} \in J^+_{s_0}$ (resp. $J^-_{s_0}$). For small $\epsilon > 0$ we have $\overline{\alpha} + \epsilon\iota^* u_{i_{N-n-1}} \in \mathcal{C}_+$ (resp. $\mathcal{C}_-$). Therefore we have $(\overline{\alpha} + \epsilon\iota^* u_{i_{N-n-1}}, \beta) \in (k^* \times k^*_{\mathbb{C}})_{reg}$.

On the other hand, there exists $t \in \{1, \ldots, l\}$ such that P_t is spanned by $\iota^* u_{i_1}, \ldots, \iota^* u_{i_{N-n-1}}$. So we have $(\overline{\alpha} + \epsilon\iota^* u_{i_{N-n-1}}, \beta) \in P_t \times (P_t \otimes \mathbb{C})$. This is a contradiction. Thus we proved Lemma 6.5. $\square$

Due to Lemmas 6.4 and 6.5, we have a toric hyperKähler manifold $\overline{X}(\overline{\alpha}, \beta) = \overline{\mu}^{-1}(\overline{\alpha}, \beta)/\overline{K}$. By Lemma 6.5, the complex structure I_1 on $\overline{X}(\overline{\alpha}, \beta)$ is independent of $\overline{\alpha}$.

When we write $J^0_{s_0} = \{i_1, \ldots, i_{\overline{N}}\}$, define a map $\widetilde{\theta}\colon \mathbb{H}^N \to \mathbb{H}^{\overline{N}}$ by $\widetilde{\theta}(z, w) = (\overline{z}, \overline{w})$, where $\overline{z} = (z_{i_1}, \ldots, z_{i_{\overline{N}}})$ and $\overline{z} = (w_{i_1}, \ldots, w_{i_{\overline{N}}})$. Then we have the following.

Lemma 6.6. $\widetilde{\theta}(z, w) \in \overline{\mu}_{\mathbb{C}}^{-1}(\beta)^{\overline{\alpha}-st}$ *for all* $(z, w) \in \widetilde{M}_+ \setminus \widetilde{U}$.

Proof. Fix an arbitrary $(z,w) \in \widetilde{M_+} \setminus \widetilde{U}$. Since (z,w) satisfies the condition (not s_0-), we have $\widetilde{\theta}(z,w) \in \overline{\mu}_{\mathbb{C}}^{-1}(\beta)$. So we have to show that $\widetilde{\theta}(z,w)$ is $\overline{\alpha}$-stable.

Let $\{\alpha_n\}_{n=1}^{\infty} \subset \mathcal{C}_+$ be a sequence which converges to $\overline{\alpha}$. Since (z,w) is α_+-stable, we have (z,w) is α_n-stable for all n.

If we define the subsets $J^1_{(z,w)}, J^2_{(z,w)}, J^3_{(z,w)} \subset \{1,\dots,N\}$ by (5.1), then we can express $\alpha_n = \sum_{i \in J_{(z,w)}} a_i^{(n)} \iota^* u_i$ (not uniquely), where $J_{(z,w)} = J^1_{(z,w)} \cup J^2_{(z,w)} \cup J^3_{(z,w)}$ and $a_i^{(n)} > 0$ for $i \in J^2_{(z,w)}$, $a_i^{(n)} < 0$ for $i \in J^3_{(z,w)}$ respectively.

Claim 6.7. $\lim_{n\to\infty} a_i^{(n)} = 0$ *for* $i \in J_{(z,w)} \cap (J^+_{s_0} \cup J^-_{s_0})$.

Proof. Since (z,w) satisfies the condition (not s_0-), we have $J_{(z,w)} \cap J^+_{s_0} \subset J^2_{(z,w)}$ and $J_{(z,w)} \cap J^-_{s_0} \subset J^3_{(z,w)}$. Therefore we have $\langle a_i^{(n)} \iota^* u_i, Y_{s_0}\rangle \langle \alpha_+, Y_{s_0}\rangle > 0$ for all $i \in J_{(z,w)} \cap (J^+_{s_0} \cup J^-_{s_0})$. On the other hand, we have

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i \in J_{(z,w)} \cap (J^+_{s_0} \cup J^-_{s_0})} \langle a_i^{(n)} \iota^* u_i, Y_{s_0}\rangle \langle \alpha_+, Y_{s_0}\rangle = \lim_{n\to\infty} \sum_{i \in J_{(z,w)}} \langle a_i^{(n)} \iota^* u_i, Y_{s_0}\rangle \langle \alpha_+, Y_{s_0}\rangle \\
&= \lim_{n\to\infty} \langle \alpha_n, Y_{s_0}\rangle \langle \alpha_+, Y_{s_0}\rangle = \langle \overline{\alpha}, Y_{s_0}\rangle \langle \alpha_+, Y_{s_0}\rangle = 0.
\end{aligned}$$

Therefore the claim follows. □

By Claim 6.7, if we set $\overline{\alpha}_n = \sum_{i \in J_{(z,w)} \cap J^0_{s_0}} a_i^{(n)} \iota^* u_i \in P_{s_0}$, we have $\lim_{n\to\infty} \overline{\alpha}_n = \overline{\alpha}$. By Lemma 6.5, $\overline{\alpha}$ is in a chamber given by the hyperplanes $\{\overline{P}_s\}_{s \in \overline{\Lambda}_\beta}$. Therefore $\overline{\alpha}_n$ belongs to the same chamber as $\overline{\alpha}$ for sufficiently large $n > 0$. Since $\widetilde{\theta}(z,w)$ is $\overline{\alpha}_n$-stable for all n, it is also $\overline{\alpha}$-stable. Thus we proved Lemma 6.6. □

Due to Lemma 6.6, the map $\widetilde{\theta}|_{\widetilde{M_+} \setminus \widetilde{U}} \colon \widetilde{M_+} \setminus \widetilde{U} \to \mathbb{H}^N$ induces a map θ_+ from $X(\alpha_+,\beta) \setminus U = (\widetilde{M_+} \setminus \widetilde{U})/K_{\mathbb{C}}$ to $(\overline{X}(\overline{\alpha},\beta), I_1) = \overline{\mu}_{\mathbb{C}}^{-1}(\beta)^{\overline{\alpha}-st}/\overline{K}_{\mathbb{C}}$. Then we have the following.

Lemma 6.8. (1) $\theta_+ \colon X(\alpha_+,\beta) \setminus U \to \overline{X}(\overline{\alpha},\beta)$ *is a* $\mathbb{P}^{N-\overline{N}-1}$*-bundle.*
(2) *The primitive vector* $Y_{s_0} \in k_{\mathbb{Z}}$*, which satisfies the condition* $\langle \alpha_+, Y_{s_0}\rangle > 0$*, is given by* $Y_{s_0} = \sum_{i \in J^+_{s_0}} X_i - \sum_{i \in J^-_{s_0}} X_i$.
(3) $N - \overline{N} - 1 \geq 1$.

Proof. Fix an arbitrary $(\overline{z},\overline{w}) \in \overline{\mu}_{\mathbb{C}}^{-1}(\beta)^{\overline{\alpha}-st}$. Define a subset $\widetilde{S}^+_{(\overline{z},\overline{w})} \subset \mathbb{H}^N$ by

$$\widetilde{S}^+_{(\overline{z},\overline{w})} = \{(z,w) \in \widetilde{\theta}^{-1}(\overline{z},\overline{w}) \mid \text{the conditions } (s_0+) \text{ and (not } s_0-) \}.$$

Then we have the following.

Claim 6.9. $\widetilde{\theta}^{-1}(\overline{z},\overline{w}) \cap (\widetilde{M_+} \setminus \widetilde{U}) = \widetilde{S}^+_{(\overline{z},\overline{w})}$.

Proof. Since $\subset$ is trivial, it is enough to show $\supset$. Fix an arbitrary $(z,w) \in \widetilde{S}^+_{(\overline{z},\overline{w})}$. Since $(z,w) \notin \widetilde{U}$ by the condition (not s_0-), it is enough to show that $(z,w) \in$

$\widetilde{M}_+ = \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha_+-st}$. Since $(\overline{z}, \overline{w}) \in \overline{\mu}_{\mathbb{C}}^{-1}(\beta)$ and (z, w) satisfies the condition (not s_0-), we have $(z, w) \in \mu_{\mathbb{C}}^{-1}(\beta)$.

Next we show (z, w) is α_+-stable. By the condition (s_0+), there exists $i_0 \in J_{s_0}^+$ such that $z_{i_0} \neq 0$ or there exists $i_0 \in J_{s_0}^-$ such that $w_{i_0} \neq 0$. Suppose the former. Then we have $\overline{\alpha} + \epsilon\iota^* u_{i_0} \in \mathcal{C}_+$ for sufficiently small $\epsilon > 0$. Since $(\overline{z}, \overline{w})$ is $\overline{\alpha}$-stable and $\mathcal{C}_+$ is open in k^*, it is easy to see that (z, w) is $\overline{\alpha} + \epsilon\iota^* u_{i_0}$-stable. So it is α_+-stable. If the latter holds, we can also show that (z, w) is α_+-stable by a similar argument. Thus we proved Claim 6.9. □

Fix an arbitrary point $[\overline{z}, \overline{w}] \in \overline{X}(\overline{\alpha}, \beta)$, which is a point represented by $(\overline{z}, \overline{w}) \in \overline{\mu}_{\mathbb{C}}^{-1}(\beta)^{\overline{\alpha}-st}$. By Theorem 5.2, $\overline{K}$ acts freely on $\overline{\mu}_{\mathbb{C}}^{-1}(\beta)^{\overline{\alpha}-st}$. Therefore we have $\theta_+^{-1}([\overline{z}, \overline{w}]) = \widetilde{S}^+_{(\overline{z},\overline{w})}/\mathrm{Exp}\mathbb{C}Y_{s_0}$, where $\mathrm{Exp}\mathbb{C}Y_{s_0} = \{\mathrm{Exp}tY_{s_0} \mid t \in \mathbb{C}\} \cong \mathbb{C}^*$. If we write $Y_{s_0} = \sum_{i=1}^N a_i X_i$, then we have

$$J_{s_0}^0 = \{i \mid a_i = 0\}, \quad J_{s_0}^+ = \{i \mid a_i > 0\}, \quad J_{s_0}^- = \{i \mid a_i < 0\}.$$

We have assumed that $a_i \in \mathbb{Z}$ for all $i = 1, \ldots, N$ and that Y_{s_0} is primitive. Since, by Theorem 5.2 again, $\mathrm{Exp}\mathbb{C}Y_{s_0}$ acts freely on $\widetilde{S}^+_{(\overline{z},\overline{w})} \cong \mathbb{C}^{N-\overline{N}} \setminus \{0\}$, we have $Y_{s_0} = \sum_{i \in J_{s_0}^+} X_i - \sum_{i \in J_{s_0}^-} X_i$. Moreover the action of $\mathrm{Exp}\mathbb{C}Y_{s_0}$ on $\widetilde{S}^+_{(\overline{z},\overline{w})}$ is isomorphic to the standard action of $\mathbb{C}^*$ on $\mathbb{C}^{N-\overline{N}-1} \setminus \{0\}$. Therefore we have $\widetilde{S}^+_{(\overline{z},\overline{w})}/\mathrm{Exp}\mathbb{C}Y_{s_0} \cong \mathbb{P}^{N-\overline{N}-1}$.

Finally we show (3). Denote the number of elements of $J_{s_0}^+ \cup J_{s_0}^-$ by $\#(J_{s_0}^+ \cup J_{s_0}^-)$. First we have $\#(J_{s_0}^+ \cup J_{s_0}^-) = N - \overline{N}$. On the other hand, due to the condition (C2), $Y_{s_0} \neq \pm X_i$ for any $i = 1, \ldots, N$. Therefore, by (2), we have $\#(J_{s_0}^+ \cup J_{s_0}^-) \geq 2$. Thus we finish the proof of Lemma 6.8. □

Now we are in the final stage of the proof of Theorem 6.3. Since the codimension of the complex submanifold $X(\alpha_+, \beta) \setminus U$ is $N - \overline{N} - 1 \geq 1$, we have a birational map $F\colon X(\alpha_+, \beta) \dashrightarrow X(\alpha_-, \beta)$ such that $F|_U$ is the identity map of U.

Next we describe a neighborhood of $X(\alpha_+, \beta) \setminus U$. Fix an arbitrary $(\overline{z}, \overline{w}) \in \overline{\mu}_{\mathbb{C}}^{-1}(\beta)^{\overline{\alpha}-st}$. Define a subset $\widetilde{W}^+_{(\overline{z},\overline{w})} \subset \mathbb{H}^N$ by

$$\widetilde{W}^+_{(\overline{z},\overline{w})} = \{(z, w) \in \widetilde{\theta}^{-1}(\overline{z}, \overline{w}) \mid \text{the conditions } (s_0+) \text{ and } \sum_{i \in J_{s_0}^+ \cup J_{s_0}^-} z_i w_i = 0 \ \}.$$

Then, by a similar argument in the proof of Claim 6.9, we have $\widetilde{W}^+_{(\overline{z},\overline{w})} \subset \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha_+-st}$. If we set $\widetilde{W}^+ = \bigcup_{(\overline{z},\overline{w}) \in \overline{\mu}_{\mathbb{C}}^{-1}(\beta)^{\overline{\alpha}-st}} \widetilde{W}^+_{(\overline{z},\overline{w})}$, then we have a neighborhood $W^+ = \widetilde{W}^+/K_{\mathbb{C}}$ of $X(\alpha_+, \beta) \setminus U$. It is easy to see that the neighborhood W^+ has the structure of a fiber bundle $\Theta_+\colon W^+ \to \overline{X}(\overline{\alpha}, \beta)$, whose fiber $\Theta_+^{-1}([\overline{z}, \overline{w}])$ is isomorphic to $\widetilde{W}^+_{(\overline{z},\overline{w})}/\mathrm{Exp}\mathbb{C}Y_{s_0} \cong T^*\mathbb{P}^{N-\overline{N}-1}$.

Similarly, we have the neighborhood W^- of $X(\alpha_-, \beta) \setminus U$ which has the structure of a fiber bundle over the same toric hyperKähler manifold $\overline{X}(\overline{\alpha}, \beta)$. Then the restriction of the birational map $F\colon X(\alpha_+, \beta) \dashrightarrow X(\alpha_-, \beta)$ to W^+ is a family of the birational map f in Section 4 parameterized $\overline{X}(\overline{\alpha}, \beta)$. Therefore the birational map $F\colon X(\alpha_+,\beta) \dashrightarrow X(\alpha_-,\beta)$ is the elementary transformation along $X(\alpha_+,\beta)\backslash U$. Thus we finished the proof of Theorem 6.3. $\square$

In Proposition 3.3 we have an isomorphism $\kappa^*\colon k^* \to H^2(X(\alpha,\beta);\mathbb{R})$ for a subtorus K satisfying the conditions (C1) and (C2). Now we define the dual map $\kappa_*\colon H_2(X(\alpha,\beta);\mathbb{R}) \to k$ by

$$\langle \gamma, \kappa_*([C])\rangle = \langle \kappa^*(\gamma), [C]\rangle' \quad \text{for } \gamma \in k^* \text{ and } [C] \in H_2(X(\alpha,\beta);\mathbb{R}),$$

where $\langle\ ,\ \rangle\colon k^*\times k \to \mathbb{R}$ and $\langle\ ,\ \rangle'\colon H^2(X(\alpha,\beta);\mathbb{R}) \times H_2(X(\alpha,\beta);\mathbb{R}) \to \mathbb{R}$ are the natural pairings.

Recall that the *Kähler cone* of a complex manifold M is the set of de Rham cohomology classes represented by its Kähler classes. It is a convex cone in $H^2(M;\mathbb{R})$. The Kähler cone of a toric hyperKähler manifold is described as follows.

Theorem 6.10. *Let K be a subtorus of T^N satisfying the conditions* (C1) *and* (C2). *Fix an arbitrary $\beta \in k^*_{\mathbb{C}}$ and an arbitrary chamber $\mathcal{C}$ in k^* given by the hyperplanes $\{P_s\}_{s\in\Lambda_\beta}$. Fix also an arbitrary $\alpha \in \mathcal{C}$. Then the following holds.*
(1) *Suppose, in addition, that $P_{s_0}\cap\partial\mathcal{C}$ is a codimension one boundary of $\mathcal{C}$. Then the complex manifold $(X(\alpha,\beta),I_1)$ contains a smooth rational curve C_{s_0} such that $\kappa_*([C_{s_0}]) = Y_{s_0}$, where $Y_{s_0} \in k_{\mathbb{Z}}$ is the primitive vector satisfying $P_{s_0} = \{v \in k^* \mid \langle v, Y_{s_0}\rangle = 0\}$ and $\langle \alpha, Y_{s_0}\rangle > 0$.*
(2) *The Kähler cone $\mathcal{KC}$ of the complex manifold $(X(\alpha,\beta),I_1)$ is given by $\mathcal{KC} = \kappa^*(\mathcal{C}) \subset H^2(X(\alpha,\beta);\mathbb{R})$.*

Remark 6.11. The volume of the rational curve C_{s_0} in $X(\alpha,\beta)$ is given by

$$vol(C_{s_0}) = \int_{C_{s_0}} \omega_1 = \langle \kappa^*(\alpha), [C_{s_0}]\rangle' = \langle \alpha, \kappa_*([C_{s_0}])\rangle = \langle \alpha, Y_{s_0}\rangle.$$

Therefore $vol(C_{s_0})$ goes to zero as the 'Kähler class' α approaches to the wall P_{s_0}.

Proof. (1) We continue to use the notation in the proof of Lemma 6.8 except that $\alpha_+, \mathcal{C}_+, \widetilde{S}^+_{(\overline{z},\overline{w})}$ in the proof of Lemma 6.8 should be replaced by $\alpha, \mathcal{C}, \widetilde{S}_{(\overline{z},\overline{w})}$ respectively. In the proof of Lemma 6.8, we showed that there exists a subset $\widetilde{S}_{(\overline{z},\overline{w})} \subset \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$ such that $S_{(\overline{z},\overline{w})} = \widetilde{S}_{(\overline{z},\overline{w})}/\mathrm{Exp}\mathbb{C}Y_{s_0} \cong \mathbb{P}^{N-\overline{N}-1}$ is a complex submanifold in $(X(\alpha,\beta),I_1)$. So there exists a smooth rational curve C_{s_0} in $X(\alpha,\beta)$, which is the intersection of $N-\overline{N}-2$ generic hyperplanes in $S_{(\overline{z},\overline{w})} \cong \mathbb{P}^{N-\overline{N}-1}$.

We have to show $\kappa_*([C_{s_0}]) = Y_{s_0}$. Recall that we have defined the K-equivariant line bundle $\widetilde{L}_\rho$ on $\mathbb{H}^N$ in Section 3. Let $i\colon S_{(\overline{z},\overline{w})} \to X(\alpha,\beta)$ be the natural inclusion. Then we have

$$i^*((\widetilde{L}_\rho|_{\mu^{-1}(\alpha,\beta)})/K) = (\widetilde{L}_\rho|_{\widetilde{S}_{(\overline{z},\overline{w})}})/\mathrm{Exp}\mathbb{C}Y_{s_0} = H^{\otimes\langle\rho,Y_{s_0}\rangle},$$

where H is the hyperplane bundle on $S_{(\overline{z},\overline{w})} \cong \mathbb{P}^{N-\overline{N}-1}$. By Lemma 3.1 we have $i^*\circ\kappa_{\mathbb{Z}}(\rho) = c_1(H^{\otimes\langle\rho,Y_{s_0}\rangle})$. Therefore we have

$$\langle \rho, \kappa_*([C_{s_0}])\rangle = \langle \kappa^*(\rho), [C_{s_0}]\rangle' = \langle \rho, Y_{s_0}\rangle \quad \text{for an arbitrary } \rho \in k^*_{\mathbb{Z}}.$$

Thus we finished the proof of (1).
(2) First we prove $\mathcal{KC} \supset \kappa^*(\mathcal{C})$. By Theorem 6.1 $(X(\alpha', \beta), I_1)$ is isomorphic to $(X(\alpha, \beta), I_1)$ as a complex manifold for an arbitrary $\alpha' \in \mathcal{C}$. Since $\kappa^*(\alpha')$ is the Kähler class of $(X(\alpha', \beta), I_1) \cong (X(\alpha, \beta), I_1)$, we have $\mathcal{KC} \supset \kappa^*(\mathcal{C})$.

Next we show $\mathcal{KC} \subset \kappa^*(\mathcal{C})$. Fix an arbitrary $\alpha' \in k^* \setminus \mathcal{C}$. Then there exists $s \in \Lambda_\beta$ such that $P_s \cap \partial\mathcal{C}$ is a codimension one boundary of $\mathcal{C}$ and $\langle \alpha', Y_s \rangle \leq 0$, where $Y_s \in k_{\mathbb{Z}}$ is the primitive vector satisfying $P_s = \{v \in k^* \mid \langle v, Y_s \rangle = 0\}$ and $\langle \alpha, Y_s \rangle > 0$. By (1) there exists a smooth rational curve C_s such that $\kappa_*([C_s]) = Y_s$. Then we have

$$\langle \kappa^*(\alpha'), [C_s] \rangle' = \langle \alpha', \kappa_*([C_s]) \rangle = \langle \alpha', Y_s \rangle \leq 0.$$

Therefore $\kappa^*(\alpha') \in H^2(X(\alpha, \beta); \mathbb{R})$ cannot be a Kähler class of $(X(\alpha, \beta), I_1)$. So we have $\mathcal{KC} \subset \kappa^*(\mathcal{C})$. Thus we proved (2). □

Corollary 6.12. *Let K be a subtorus of T^N satisfying the conditions* (C1) *and* (C2). *Then the following* (i) *and* (ii) *are equivalent.*
(i) $(X(\alpha, \beta), I_1)$ *is affine.*
(ii) $\beta \in k_{\mathbb{C}}^* \setminus \bigcup_{s=1}^{l} (P_s \otimes \mathbb{C})$.

Remark 6.13. (ii) $\Longrightarrow$ (i) is proved in [4]. The outline of their argument is as follows. Using the functional given in (5.3), they showed that $\mu_{\mathbb{C}}^{-1}(\beta) = \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha-st}$ for any $\alpha \in k^*$ under the condition (ii). So they have $(X(\alpha, \beta), I_1) = \mu_{\mathbb{C}}^{-1}(\beta)/K_{\mathbb{C}}$, which implies (i).

Proof. We have only to show (i) $\Longrightarrow$ (ii). Suppose the contrary. That is, there exists $s \in \{1, \ldots, l\}$ such that $\beta \in P_s \otimes \mathbb{C}$. Then, due to Theorem 6.10, there exists a smooth rational curve in $(X(\alpha, \beta), I_1)$. This contradicts to (i). □

7 Example

Consider the subtorus K of T^5 whose Lie algebra k is spanned by $X_1 + X_2 + X_4$ and $X_1 + X_3 + X_5$. Let $\{v_1, v_2\}$ be the dual basis. Then we have $\iota^* u_1 = v_1 + v_2$, $\iota^* u_2 = \iota^* u_4 = v_1$ and $\iota^* u_3 = \iota^* u_5 = v_2$. So we have subspaces $P_1 = \mathbb{R}v_1$, $P_2 = \mathbb{R}(v_1 + v_2)$ and $P_3 = \mathbb{R}v_2$ in k^*. If we set $Y_1 = X_1 + X_3 + X_5$, $Y_2 = X_2 - X_3 + X_4 - X_5$ and $Y_3 = X_1 + X_2 + X_4$, then $P_s = \{v \in k^* \mid \langle v, Y_s \rangle = 0\}$ and $Y_s \in k_{\mathbb{Z}}$ is primitive for $s = 1, 2, 3$.

The hyperKähler moment map for the action of K on $\mathbb{H}^5$

$$\mu = (\mu_1, \mu_{\mathbb{C}}) \colon \mathbb{H}^5 \to k^* \otimes \mathbb{R}^3 \cong k^* \times k_{\mathbb{C}}^*$$

is given by

$$\begin{aligned}
\mu_1(z, w) =& \pi \sum_{i=1,2,4} (|z_i|^2 - |w_i|^2) v_1 + \pi \sum_{i=1,3,5} (|z_i|^2 - |w_i|^2) v_2 \in k^*, \\
\mu_{\mathbb{C}}(z, w) =& -2\pi\sqrt{-1}\Big(\sum_{i=1,2,4} z_i w_i\Big) v_1 - 2\pi\sqrt{-1}\Big(\sum_{i=1,3,5} z_i w_i\Big) v_2 \in k_{\mathbb{C}}^*.
\end{aligned}$$

If $\beta \in k^*_{\mathbb{C}} \setminus \bigcup_{s=1,2,3} P_s$, then we have $\Lambda_\beta = \emptyset$. Therefore we have no wall in k^* and only one chamber in this case.

If $\beta \in P_s \setminus \{O\}$ for some $s = 1,2,3$, then we have $\Lambda_\beta = \{s\}$. Therefore we have only one wall P_s and two chambers in k^* in this case.

If $\beta = 0$, then $\Lambda_\beta = \Lambda_0 = \{1,2,3\}$. So we have chambers $\mathcal{C}_1, \ldots, \mathcal{C}_6$ in k^* as in Figure 1.

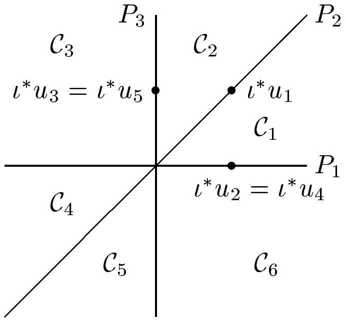

Figure 1

Let us describe wall crossing phenomena in detail. Take an arbitrary $\alpha_i \in \mathcal{C}_i$ for $i = 1, \ldots, 6$. Set $M_i = (X(\alpha_i, 0), I_1)$ and $\widetilde{M}_i = \mu_{\mathbb{C}}^{-1}(0)^{\alpha_i - st}$. We also set $\widetilde{U}_{ij} = \widetilde{M}_i \cap \widetilde{M}_j$ and $U_{ij} = \widetilde{U}_{ij}/K_{\mathbb{C}}$ for $i \neq j$. By Theorem 5.2 we have $M_i = \widetilde{M}_i/K_{\mathbb{C}}$. By Theorem 5.10 we have the following;

$$(z,w) \in \widetilde{M}_1 \Longleftrightarrow \begin{cases} z_1w_1 + z_2w_2 + z_4w_4 = 0, z_1w_1 + z_3w_3 + z_5w_5 = 0, \\ (z_1, z_2, z_4) \neq (0,0,0), (z_1, z_3, z_5) \neq (0,0,0), \\ (z_2, z_4, w_3, w_5) \neq (0,0,0,0), \end{cases} \tag{7.1}$$

$$(z,w) \in \widetilde{M}_2 \Longleftrightarrow \begin{cases} z_1w_1 + z_2w_2 + z_4w_4 = 0, z_1w_1 + z_3w_3 + z_5w_5 = 0, \\ (z_1, z_2, z_4) \neq (0,0,0), (z_1, z_3, z_5) \neq (0,0,0), \\ (w_2, w_4, z_3, z_5) \neq (0,0,0,0). \end{cases} \tag{7.2}$$

$$(z,w) \in \widetilde{M}_6 \Longleftrightarrow \begin{cases} z_1w_1 + z_2w_2 + z_4w_4 = 0, z_1w_1 + z_3w_3 + z_5w_5 = 0, \\ (z_1, z_2, z_4) \neq (0,0,0), (w_1, w_3, w_5) \neq (0,0,0), \\ (z_2, z_4, w_3, w_5) \neq (0,0,0,0). \end{cases} \tag{7.3}$$

First we describe the relation between M_1 and M_2. By (7.1) and (7.2) we have

$$\begin{aligned} &(z,w) \in \widetilde{M}_1 \setminus \widetilde{U}_{12} \\ &\quad \Longleftrightarrow \begin{cases} z_1w_1 + z_2w_2 + z_4w_4 = 0, z_1w_1 + z_3w_3 + z_5w_5 = 0, \\ (z_1, z_2, z_4) \neq (0,0,0), (z_1, z_3, z_5) \neq (0,0,0), \\ (z_2, z_4, w_3, w_5) \neq (0,0,0,0), (w_2, w_4, z_3, z_5) = (0,0,0,0), \end{cases} \\ &\quad \Longleftrightarrow (z_2, z_4, w_3, w_5) \neq (0,0,0,0), (w_1, w_2, w_4, z_3, z_5) = (0,0,0,0,0), z_1 \neq 0. \end{aligned}$$

Since we can normalize z_1 to be 1 by the action of $\mathrm{Exp}\mathbb{C}Y_1$, we have

$$\begin{aligned} &M_1 \setminus U_{12} \\ &= \{(z_2, z_4, w_3, w_5) \neq (0,0,0,0), (z_1, w_1, w_2, w_4, z_3, z_5) = (1,0,0,0,0,0)\}/\mathrm{Exp}\mathbb{C}Y_2, \end{aligned}$$

which is isomorphic to $\mathbb{P}^3$. Therefore the birational transformation $f_{12}\colon M_1 \dashrightarrow M_2$ is the elementary transformation along $M_1 \setminus U_{12} \cong \mathbb{P}^3$.

Next we describe the relation between M_1 and M_6. By (7.1) and (7.3) we have

$$
\begin{aligned}
&(z,w) \in \widetilde{M}_1 \setminus \widetilde{U}_{16} \\
&\Longleftrightarrow \begin{cases} z_1w_1 + z_2w_2 + z_4w_4 = 0, z_1w_1 + z_3w_3 + z_5w_5 = 0, \\ (z_1, z_2, z_4) \neq (0,0,0), (z_1, z_3, z_5) \neq (0,0,0), \\ (z_2, z_4, w_3, w_5) \neq (0,0,0,0), (w_1, w_3, w_5) = (0,0,0), \end{cases} \\
&\Longleftrightarrow \begin{cases} z_2w_2 + z_4w_4 = 0, (z_2, z_4) \neq (0,0), \\ (z_1, z_3, z_5) \neq (0,0,0), (w_1, w_3, w_5) = (0,0,0). \end{cases}
\end{aligned}
$$

Thus we can define a map $\widetilde{\theta}\colon \mathbb{H}^5 \to \mathbb{H}^2$ by $\widetilde{\theta}(z,w) = (\overline{z}, \overline{w})$, where $\overline{z} = (z_2, z_4)$, $\overline{w} = (w_2, w_4)$. Let $\overline{K}$ be the subtorus of T^2 whose Lie algebra $\overline{k}$ is spanned by $X_2 + X_4$. Then the hyperKähler moment map for the action of $\overline{K}$ on $\mathbb{H}^2$

$$\overline{\mu} = (\overline{\mu}_1, \overline{\mu}_{\mathbb{C}})\colon \mathbb{H}^2 \to \overline{k}^* \otimes \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$$

is given by

$$
\begin{aligned}
\overline{\mu}_1(\overline{z}, \overline{w}) =& \pi(|z_2|^2 + |z_4|^2 - |w_2|^2 - |w_4|^2) \in \mathbb{R}, \\
\overline{\mu}_{\mathbb{C}}(\overline{z}, \overline{w}) =& -2\pi\sqrt{-1}(z_2w_2 + z_4w_4) \in \mathbb{C}.
\end{aligned}
$$

Then we have a hyperKähler quotient $\overline{X}(\overline{\alpha}, 0) = \overline{\mu}^{-1}(\overline{\alpha}, 0)/\overline{K}$ for $\overline{\alpha} > 0$. We also have $(\overline{X}(\overline{\alpha}, 0), I_1) = \{(\overline{z}, \overline{w}) \in \overline{\mu}_{\mathbb{C}}^{-1}(0) \mid \overline{z} \neq 0\}/\overline{K}_{\mathbb{C}}$, which we denote by $\overline{M}$. Since this is the same as the example discussed in Section 4, $\overline{M}$ is isomorphic to the cotangent bundle of $\mathbb{P}^1$. Since the image of $\widetilde{M}_1 \setminus \widetilde{U}_{16}$ by the map $\widetilde{\theta}$ is $\{(\overline{z}, \overline{w}) \in \overline{\mu}_{\mathbb{C}}^{-1}(0) \mid \overline{z} \neq 0\}$, we have a surjective map $\theta\colon M_1 \setminus U_{16} \to \overline{M}$. Then we have

$$
\begin{aligned}
&\theta^{-1}([\overline{z}, \overline{w}]) \\
&= \{(z,w) \in \widetilde{\theta}^{-1}(\overline{z}, \overline{w}) \mid (z_1, z_3, z_5) \neq (0,0,0), (w_1, w_3, w_5) = (0,0,0)\}/\mathrm{Exp}\mathbb{C}Y_1,
\end{aligned}
$$

which is isomorphic to $\mathbb{P}^2$. Therefore $M_1 \setminus U_{16}$ is a $\mathbb{P}^2$-bundle over $\overline{M} \cong T^*\mathbb{P}^1$. Thus the birational map $f_{16}\colon M_1 \dashrightarrow M_6$ is the elementary transformation along $M_1 \setminus U_{16}$.

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Hiroshi Konno

Graduate School of Mathematical Sciences, University of Tokyo,

3-8-1, Komaba, Meguro-ku, Tokyo, 153-8914, Japan

e-mail: konno@ms.u-tokyo.ac.jp