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ASSOCIATIVE, LIE, AND LEFT-SYMMETRIC ALGEBRAS OF DERIVATIONS

Ualbai Umirbaev[1]

Abstract. Let $P_n = k[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over a field k of characteristic zero in the variables $x_1, x_2, \ldots, x_n$ and $\mathscr{L}_n$ be the left-symmetric algebra of all derivations of P_n [4, 18]. Using the language of $\mathscr{L}_n$, for every derivation $D \in \mathscr{L}_n$ we define the associative algebra A_D, the Lie algebra L_D, and the left-symmetric algebra $\mathscr{L}_D$ related to the study of the Jacobian Conjecture. For every derivation $D \in \mathscr{L}_n$ there is a unique n-tuple $F = (f_1, f_2, \ldots, f_n)$ of elements of P_n such that $D = D_F = f_1\partial_1 + f_2\partial_2 + \ldots + f_n\partial_n$. In this case, using an action of the Hopf algebra of noncommutative symmetric functions NSymm on P_n, we show that these algebras are closely related to the description of coefficients of the formal inverse to the polynomial endomorphism $X + tF$, where $X = (x_1, x_2, \ldots, x_n)$ and t is an independent parameter.

We prove that the Jacobian matrix $J(F)$ is nilpotent if and only if all right powers $D_F^{[r]}$ of D_F in $\mathscr{L}_n$ have zero divergence. In particular, if $J(F)$ is nilpotent then D_F is right nilpotent.

We discuss some advantages and shortcomings of these algebras and formulate some open questions.

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1. Introduction

Let k be an arbitrary field of characteristic zero and $P_n = k[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over k in the variables $x_1, x_2, \ldots, x_n$. There are two well known algebras related to the study of derivations of P_n. They are the Witt algebra W_n and the Weyl algebra A_n. Recall that W_n is the Lie algebra of all derivations of P_n and A_n is the associative algebra of all linear differential operators on P_n.

The set of elements $u\partial_i$, where $u = x_1^{s_1} \ldots x_n^{s_n}$ is an arbitrary monomial, $\partial_i = \frac{\partial}{\partial x_i}$, and $1 \leq i \leq n$, forms a linear basis for W_n. For any $u = a\partial_i, v = b\partial_j$, where $a, b \in P_n$ are monomials, put

$$u \cdot v = ((a\partial_i)(b))\partial_j. \tag{1}$$

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Extending this operation by distributivity, we get a well defined bilinear operation $\cdot$ on W_n. Denote this algebra by $\mathscr{L}_n$. It is easy to check (see Section 2) that $\mathscr{L}_n$ is a left-symmetric algebra [4, 18] and its commutator algebra is the Witt algebra W_n. We say that $\mathscr{L}_n$ is *the left-symmetric algebra of derivations* of P_n.

The language of the left-symmetric algebras of derivations is very convenient to describe some important notions of affine algebraic geometry in purely algebraic terms [18]. For example, an element of $\mathscr{L}_n$ is left nilpotent if and only if it is a locally nilpotent derivation of P_n. One of the greatest algebraic advantages of $\mathscr{L}_n$ is that $\mathscr{L}_n$ satisfies an exact analogue of the Cayley-Hamilton trace identity. Recall that W_n and A_n do not have an analogue of this identity.

Let $D \in \mathscr{L}_n$ be an arbitrary derivation of P_n. Denote by $\mathscr{L}_D$ the subalgebra of the left-symmetric algebra $\mathscr{L}_n$ generated by D. Denote by L_D the Lie subalgebra of the Witt algebra W_n generated by all right powers $D^{[p]}$ of D. Obviously, $L_D \subseteq \mathscr{L}_D$. Denote by A_D the subalgebra (with identity) of the Weyl algebra A_n generated by all right powers $D^{[p]}$ of D. So, A_D is an associative enveloping algebra of the Lie algebra L_D. The Lie algebra L_D is a nontrivial Lie algebra ever related to one derivation.

Every n-tuple $F = (f_1, f_2, \ldots, f_n)$ of elements of P_n represents a polynomial endomorphism of the vector space k^n. We denote by F^* the endomorphism of P_n defined by $F^*(x_i) = f_i$ for all i. Also denote by

$$D_F = f_1\partial_1 + f_2\partial_2 + \ldots + f_n\partial_n$$

the derivation of P_n defined by $D_F(x_i) = f_i$ for all i. Note that every derivation D can be uniquely represented as $D = D_F$ for some polynomial n-tuple F. Using this correspondence we often use parallel notations $A_D = A_F$, $L_D = L_F$, and $\mathscr{L}_D = \mathscr{L}_F$ if $D = D_F$

We show that if the Jacobian matrix $J(F)$ is nilpotent then D_F is a right nilpotent element of $\mathscr{L}_n$. We also show that the Jacobian matrix $J(F)$ is nilpotent if and only if all right powers $D_F^{[p]}$ of D_F have zero divergence. Moreover, if $J(F)$ is nilpotent then every element of L_F has zero divergence.

Let t be an independent parameter and

$$(X + tF)^{-1} = X + tF_1 + t^2F_2 + \ldots + t^nF_n + \ldots$$

be the formal (or analytic) inverse to the endomorphism $X + tF$ of $k[t]^n$. There are many interesting papers devoted to the description of F_i [1, 8, 19]. We show that A_F and L_F are also generated by all D_{F_i} where $i \geq 1$. For this reason we can say that A_F and L_F are, respectively, the associative and the Lie algebras of coefficients of the formal inverse to $X + tF$. Notice that $\mathscr{L}_F$ is also the smallest left-symmetric algebra containing all D_{F_i} where $i \geq 1$.

Recall that the Hopf algebra of noncommutative symmetric functions NSymm [9] regarded as an algebra is the free associative algebra

$$\text{NSymm} = k\langle Z_1, Z_2, \ldots, Z_n, \ldots\rangle$$

over k in the variables $Z_1, Z_2, \ldots, Z_n, \ldots$.

We define an action of NSymm on P_n by

$$(X + tF)^*(a) = a + tZ_1(a) + t^2Z_2(a) + \ldots + t^nZ_n(a) + \ldots$$

for any $a \in P_n$. This action represents a natural linearization of the action of $(X + tF)^*$ on P_n. We show that A_F is the image of NSymm under this representation and L_F is the image of the Lie algebra Prim of all primitive elements of NSymm. In this way, A_F and L_F may be considered as linearization algebras of the action of $(X + tF)^*$ on P_n. The left-symmetric algebra $\mathscr{L}_F$ also can be related to further linearizations.

The Hopf algebra of noncommutative symmetric functions NSymm was introduced in [9] as a noncommutative generalization of the Hopf algebra of symmetric functions Symm. Several systems of free and primitive generators of NSymm and relations between them were given in [9]. Some more relations between the generators of NSymm are given in [21].

There are two well known systems of free primitive generators [12] of NSymm which are dual to each other with respect to the standard involution of the free associative algebra $k\langle Z_1, Z_2, \ldots, Z_n, \ldots\rangle$. It is interesting that one of them corresponds to the right powers $D_F^{[r]}$ of D_F and the other one corresponds to the D_{F_i} for all $i \geq 1$. These observations make the Lie algebra L_F very important in studying the Jacobian Conjecture. The right powers $D_F^{[r]}$ are very convenient to express that $J(F)$ is nilpotent. In order to solve the Jacobian Conjecture it is necessary to prove that there exists a positive integer m such that $D_{F_i} = 0$ for all $i \geq m$.

The action of NSymm on P_n, defined above, corresponds to one of a series of homomorphisms constructed in [21] and the images of primitive generators were calculated in [21].

It is rewarding to initiate a systematic study of the associative algebra A_F, the Lie algebra L_F and the left-symmetric algebra $\mathscr{L}_F$. Using an example of an automorphism studied earlier by A. van den Essen [7] and G. Gorni and G. Zampieri [11], we give an example of F with nilpotent Jacobian matrix $J(F)$ such that L_F is not nilpotent nor solvable.

The paper is organized as follows. Section 2 is devoted to the study of the left-symmetric algebra $\mathscr{L}_n$. In particular, we describe the right and the left multiplication algebras of $\mathscr{L}_n$ and describe an analogue of the Cayley-Hamilton identity. In Section 3 we develop technics for calculation of divergence of elements in $\mathscr{L}_n$. The definition of the Hopf algebra of noncommutative symmetric functions NSymm is given in Section 4. We give also some primitive systems of generators of NSymm and relations between from [9]. The action of NSymm and the images of primitive elements are given in Section 5. In Section 6 we discuss some properties of these algebras towards the Jacobian Conjecture and formulate some open problems.

2. Algebra $\mathscr{L}_n$

If A is an arbitrary linear algebra over a field k then the set $\mathrm{Der}_k A$ of all k-linear derivations of A forms a Lie algebra. If A is a free algebra then it is possible to define a multiplication $\cdot$ on $\mathrm{Der}_k A$ such that it becomes a left-symmetric algebra and its commutator algebra becomes the Lie algebra of derivations $\mathrm{Der}_k A$ of A [18].

Recall that an algebra $\mathscr{L}$ over k is called left-symmetric [3] if $\mathscr{L}$ satisfies the identity

$$(xy)z - x(yz) = (yx)z - y(xz). \tag{2}$$

This means that the associator $(x, y, z) := (xy)z - x(yz)$ is symmetric with respect to two left arguments, i.e.,

$$(x, y, z) = (y, x, z).$$

The variety of left-symmetric algebras is Lie-admissible, i.e., each left-symmetric algebra $\mathscr{L}$ with the operation $[x, y] := xy - yx$ is a Lie algebra.

Recall that the space of the algebra $\mathscr{L}_n$ is W_n and the product is defined by (1).

Lemma 1. [4, 18] *Algebra $\mathscr{L}_n$ is left-symmetric and its commutator algebra is the Witt algebra W_n.*

Proof. Let $x, y \in \mathscr{L}_n$. Denote by $[x, y] = x \cdot y - y \cdot x$ the commutator of x and y in $\mathscr{L}_n$ and denote by $\{x, y\}$ the product of x and y in W_n. We first prove that the commutator algebra of $\mathscr{L}_n$ is W_n, i.e.,

$$[x, y](a) = \{x, y\}(a)$$

for all $a \in P_n$. Note that

$$\{x, y\}(a) = x(y(a)) - y(x(a))$$

by the definition. Taking into account that $[x, y]$ and $\{x, y\}$ are both derivations, we can assume that $a = x_t$. Consequently, it is sufficient to check that

$$(x \cdot y - y \cdot x)(x_t) = x(y(x_t)) - y(x(x_t)).$$

We may also assume that $x = u\partial_i$ and $y = v\partial_j$. If $t \neq i, j$, then all components of the last equality are zeroes. If $t = i \neq j$ or $t = i = j$, then it is also true. Consequently, the commutator algebra of $\mathscr{L}_n$ is W_n.

Assume that $x, y \in \mathscr{L}_n$ and $z = a\partial_t$. Then

$$(x, y, z) = (xy)z - x(yz) = [(xy)(a) - x(y(a))]\partial_t,$$
$$(y, x, z) = (yx)z - y(xz) = [(yx)(a) - y(x(a))]\partial_t.$$

To prove (2) it is sufficient to check that

$$[x, y](a) = x(y(a)) - y(x(a)) = \{x, y\}(a),$$

which is already proved. □

A natural P_n-module structure on $\mathscr{L}_n$ can be defined by $p \cdot u\partial_i = (pu)\partial_i$ for all i and $p, u \in P_n$. Then

$$\mathscr{L}_n = P_n\partial_1 \oplus P_n\partial_2 \oplus \ldots \oplus P_n\partial_n$$

is a free P_n-module.

Consider the grading

$$P_n = A_0 \oplus A_1 \oplus A_2 \oplus \ldots \oplus A_s \oplus \ldots,$$

where A_i the space of homogeneous elements of degree $i \geq 0$. The left-symmetric algebra $\mathscr{L}_n$ has a natural grading

$$\mathscr{L}_n = L_{-1} \oplus L_0 \oplus L_1 \oplus \ldots \oplus L_s \oplus \ldots,$$

where L_i the space of elements of the form $a\partial_j$ with $a \in A_{i+1}$ and $1 \leq j \leq n$. Elements of L_s are called homogeneous derivations of P_n of degree s.

We have $L_{-1} = k\partial_1 + \ldots + k\partial_n$ and L_0 is a subalgebra of $\mathscr{L}_n$ isomorphic to the matrix algebra $M_n(k)$. The element

$$D_X = x_1\partial_1 + x_2\partial_2 + \ldots + x_n\partial_n$$

is the identity element of the matrix algebra L_0 and is the right identity element of $\mathscr{L}_n$. The left-symmetric algebra $\mathscr{L}_n$ has no identity element.

We establish some properties of $\mathscr{L}_n$ related to the Jacobian Conjecture.

For every n-tuple $F = (f_1, f_2, \ldots, f_n)$ of elements of P_n denote by $J(F) = (\partial_j(f_i))_{1\leq i,j\leq n}$ the Jacobian matrix of F. Notice that every derivation D of P_n has the form $D = D_F$ for a unique endomorphism F. Put $J(D) = J(F)$. So, the Jacobian matrix of every derivation D of A is defined.

Lemma 2. [18] *Let F and G be two arbitrary n-tuples of elements of A. Then*

$$D_F D_G = D_{D_F(G)} = D_{J(G)F} = D_{J(D_G)F}.$$

Proof. The definition of the left symmetric product $\cdot$ directly implies that $D_F D_G = D_{D_F(G)}$. Notice that for any $h \in A$ we have

$$D_F(h) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} y_i|_{y_i:=f_i} = (\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n})F.$$

Consequently, $D_{D_F(G)} = D_{J(G)F}$. $\square$

For any $a \in \mathscr{L}_n$ put $a^0 = a^{[0]} = a$, $a^{r+1} = a(a^r)$, and $a^{[r+1]} = (a^{[r]})a$ for any $r \geq 0$. It is natural to say that a is left nilpotent if $a^m = 0$ for some $m \geq 2$. Similarly, a is right nilpotent if $a^{[m]} = 0$ for some $m \geq 2$.

Lemma 3. [18] *A derivation D of A is locally nilpotent if and only if D is a left nilpotent element of $\mathscr{L}_n$.*

Proof. Suppose that $D = D_F$ and put

$$H_i = \underbrace{D(D\ldots(D(D}_{i} X))\ldots)$$

for all $i \geq 1$. Note that $H_1 = F$ and $H_2 = D_F(F)$. Consequently, $D^2 = D_{H_2}$ by Lemma 2. Continuing the same calculations, it is easy to show that $D^i = D_{H_i}$ for all i. Consequently, $D^m = 0$ if and only if $H_m = 0$. Note that $H_m = 0$ means that D applied m times to x_i gives 0 for all i. $\square$

Example 1. Consider a well known [2] locally nilpotent derivation

$$D = (x^2 - yz)(z\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y})$$

of $k[x, y, z]$. It is easy to check that D is not right nilpotent. So, the left nilpotency of derivations does not imply their right nilpotency.

Let $\mathscr{L}$ be an arbitrary left-symmetric algebra. Denote by $\mathrm{Hom}_k(\mathscr{L}, \mathscr{L})$ the associative algebra of all k-linear transformations of the vector space $\mathscr{L}$. For any $x \in \mathscr{L}$ denote by $L_x : \mathscr{L} \to \mathscr{L} (a \mapsto xa)$ and $R_x : \mathscr{L} \to \mathscr{L} (a \mapsto ax)$ the operators of left and right multiplication by x, respectively. It follows from (2) that

(3) $$L_{[x,y]} = [L_x, L_y], \quad R_{xy} = R_y R_x + [L_x, R_y].$$

Denote by $M(\mathscr{L})$ the subalgebra of $\mathrm{Hom}_k(\mathscr{L},\mathscr{L})$ (with identity) generated by all R_x, L_x, where $x \in \mathscr{L}$. Algebra $M(\mathscr{L})$ is called the *multiplication* algebra of $\mathscr{L}$. The subalgebra $R(\mathscr{L})$ of $M(\mathscr{L})$ (with identity) generated by all R_x, where $x \in \mathscr{L}$, is called the *right multiplication* algebra of $\mathscr{L}$. Similarly, the subalgebra $L(\mathscr{L})$ of $M(\mathscr{L})$ (with identity) generated by all L_x, where $x \in \mathscr{L}$, is called the *left multiplication* algebra of $\mathscr{L}$.

Lemma 4. *The right multiplication algebra $R(\mathscr{L}_n)$ of $\mathscr{L}_n$ is isomorphic to the matrix algebra $M_n(P_n)$ and there exists a unique isomorphism $\theta : R(\mathscr{L}_n) \to M_n(P_n)$ such that $\theta(R_D) = J(D)$ for all $D \in \mathscr{L}_n$.*

Proof. Let $D \in \mathscr{L}$. Notice that $R_D = 0$ if and only $D \in k\partial_1 + \ldots + k\partial_n = L_0$. In fact, suppose that $R_D = 0$. Then $\partial_i \cdot D = 0$ for all i. This means that if $D = D_F$ then F does not contain x_i for all i and $D \in L_0$. Consequently, $R_D = 0$ if and only if $J(D) = 0$.

Thus the correspondence $R_D \mapsto J(D)$ is well defined. Notice that for any $D = D_F, D_1, \ldots, D_m$ we have

$$R_{D_1} \ldots R_{D_m}(D) = (\ldots(D \cdot D_m)\ldots D_1) = D_{J(D_1)\ldots J(D_m)F}$$

by Lemma 2. This implies that the equality $f(R_{D_1}, \ldots, R_{D_m}) = 0$, where f is an associative polynomial, holds if and only if $f(J(D_1), \ldots, J(D_m)) = 0$. Consequently, there exists a unique monomorphism $\theta : R(\mathscr{L}) \to M_n(P_n)$ such that $\theta(R_D) = J(D)$ for all $D \in \mathscr{L}_n$. The uniqueness of θ is obvious since $R(\mathscr{L}_n)$ is generated by all R_D.

Denote by B the subalgebra of $M_n(A)$ generated by all Jacobian matrices. Denote by e_{ij}, where $1 \leq i, j \leq n$, the matrix with 1 in the (i, j) place and with zeroes everywhere else, i.e., the matrix identities. Consider $F = (f_1, \ldots, f_n)$. If $f_i = x_j$ and $f_s = 0$ for all $s \neq i$ then $J(F) = e_{ij}$ and $e_{ij} \in B$ for all i, j. Let $u = x_1^{s_1} \ldots x_n^{s_n}$ be an arbitrary monomial of P_n. Put $f_1 = 1/(s_1 + 1)x_1^{s_1} x_2^{s_2} \ldots x_n^{s_n}$ and $f_i = 0$ for all $i \geq 2$. Then u becomes the element of $J(F)$ in the place $(1, 1)$. This implies that $e_{i1}J(F)e_{1j} = ue_{ij}$. Consequently, $B = M_n(P_n)$ and θ is a surjection. □

Identities of $\mathscr{L}_n$ are studied by A.S. Dzhumadildaev [4, 5, 6]. If $n = 1$ then $\mathscr{L}_1$ becomes a Novikov algebra and identities of $\mathscr{L}_1$ are studied in [13].

Corollary 1. *The identities of the right multiplication algebra $R(\mathscr{L}_n)$ coinside with the identities of the matrix algebra $M_n(k)$.*

Corollary 2. [18] *Let $D \in \mathscr{L}_n$. Then the Jacobian matrix $J(D)$ of D is nilpotent if and only if R_D is a nilpotent element of $M(\mathscr{L}_n)$.*

Proof. By Lemma 4, $J(D)^s = 0$ if and only if $R_D^s = 0$. □

Consequently, if $J(D)$ is nilpotent then D is right nilpotent. Is the converse true? This question is still open.

Every element $p \in P_n$ can be considered as an element of $\mathrm{Hom}(\mathscr{L}_n, \mathscr{L}_n)$ since $\mathscr{L}_n$. Then $P_nR(\mathscr{L}_n)$ becomes a left P_n-module. Notice that $M_n(P_n)$ is also a P_n-module.

Lemma 5. *$P_nR(\mathscr{L}_n) = R(\mathscr{L}_n)$ and the isomorphism $\theta : R(\mathscr{L}_n) \to M_n(P_n)$, constructed in Lemma 4, is an isomorphism of P_n-modules.*

Proof. As in the proof of Lemma 4, for any $p \in P_n$ and $D = D_F, D_1, \ldots, D_m$ we have

$$pR_{D_1} \ldots R_{D_m}(D) = (\ldots(D \cdot D_m)\ldots D_1) = D_{pJ(D_1)\ldots J(D_m)F}$$

by Lemma 2. This implies that the equality $f(R_{D_1},\dots,R_{D_m})=0$, where f is an associative polynomial over P_n, holds if and only if $f(J(D_1),\dots,J(D_m))=0$. Consequently, there exists a unique monomorphism $\overline{\theta}: P_nR(\mathscr{L}_n)\to M_n(P_n)$ of P_n-modules such that $\overline{\theta}(T)=\theta(T)$ for all $T\in R(\mathscr{L}_n)$. Then $\overline{\theta}$ is an isomorphism since θ is an isomorphism. This implies that $P_nR(\mathscr{L}_n)=R(\mathscr{L}_n)$ and $\overline{\theta}=\theta$. □

The isomorphism $\theta: R(\mathscr{L}_n)\to M_n(P_n)$ from Lemma 4 gives us the matrix $\theta(T)$ for any $T\in R(\mathscr{L})$. Notice that R_{D_X} is the identity element of $R(\mathscr{L}_n)$ and will be denoted by E.

Let T be an arbitrary element of $R(\mathscr{L})$. Then the matrix $\Theta=\theta(T)$ satisfies the well-known Cayley-Hamilton identity

$$\Theta^n+a_1\Theta^{n-1}+\dots+a_{n-1}\Theta+a_nI=0,$$

where I is the identity matrix of order n and $a_i\in P_n$. Recall that $a_1,a_2,\dots,a_n$ can be expressed by traces of powers of J. It follows that

$$T^n+a_1T^{n-1}+\dots+a_{n-1}T+a_nE=0 \tag{4}$$

since θ is an isomorphism. This identity is an analogue of the Cayley-Hamilton trace identity for $\mathscr{L}$. Notice that if $T=f(R_{D_1},\dots,R_{D_m})$ then $\Theta=f(J(D_1),\dots,J(D_m))$. So, all coefficients of (4) can be expressed by traces of products of Jacobian matrices.

Yu. Razmyslov proved [15] that all trace identities (in particular, all identities) of the matrix algebra $M_n(k)$ are corollaries of the Cayley-Hamilton trace identity. Consequently, all identities of $R(\mathscr{L}_n)$ are corollaries of (4). Of course, every identity of $R(\mathscr{L}_n)$ gives a right identity of $\mathscr{L}_n$, i.e., an identity of $\mathscr{L}_n$ which can be expressed by right multiplication operators. But it does not mean that every right multiplication operator identity of $\mathscr{L}_n$ is an identity of $R(\mathscr{L}_n)$. For this reason, we cannot say that every right identity of $\mathscr{L}_n$ is a corollary of (4).

Lemma 6. *The left multiplication algebra $L(\mathscr{L}_n)$ of $\mathscr{L}_n$ is isomorphic to the Weyl algebra A_n.*

Proof. Notice that for any $D=D_F,D_1,D_2,\dots,D_m$ we have

$$L_{D_1}L_{D_2}\dots L_{D_m}(D)=(D_1\dots(D_m\cdot D)\dots)=D_{D_1(D_2(\dots D_m(F)\dots))}.$$

This implies that the equality $f(L_{D_1},L_{D_2},\dots,L_{D_m})=0$, where f is an associative polynomial, holds in $L(\mathscr{L}_n)$ if and only if $f(D_1,D_2,\dots,D_m)=0$ holds in A_n. Consequently, there exists a unique monomorphism $\psi: L(\mathscr{L}_n)\to A_n$ such that $\psi(L_D)=D$ for all $D\in\mathscr{L}_n$. Then ψ is an epimorphism since A_n is generated by all derivations. □

So, Lemmas 4 and 6 describe the structure of the right and left multiplicative algebras of $\mathscr{L}_n$, respectively. But at the moment I do not know the structure of the multiplication algebra $M(\mathscr{L}_n)$. Recall that the Weyl algebra A_n does not satisfy any nontrivial identity. The left operator identities of $\mathscr{L}_n$ are very important in studying the locally nilpotent derivations and the Jacobian Conjecture.

Lemma 7. *Let $f=f(z_1,z_2,\dots,z_t)$ be a Lie polynomial. Then $f(z_1,z_2,\dots,z_t)=0$ is an identity of the Witt algebra W_n if and only if $f(L_{z_1},L_{z_2},\dots,L_{z_t})=0$ is a left operator identity of $\mathscr{L}_n$.*

Proof. Let $w_1, w_2, \ldots, w_t \in W_n = \mathscr{L}_n$. Notice that $f(w_1, w_2, \ldots, w_t) = 0$ in W_n if and only if $L_{f(w_1,w_2,\ldots,w_t)} = 0$ in $L(\mathscr{L}_n)$ since the left annihilator of $\mathscr{L}_n$ is trivial. By (3), we get

$$L_{f(w_1,w_2,\ldots,w_t)} = f(L_{w_1}, L_{w_2}, \ldots, L_{w_t}) = 0.$$

This means that the associative polynomial

$$L_f = f(L_{z_1}, L_{z_2}, \ldots, L_{z_t})$$

in $L_{z_1}, L_{z_2}, \ldots, L_{z_t}$ is a left operator identity of $\mathscr{L}_n$ if and only if $f(z_1, z_2, \ldots, z_t)$ is an identity of W_n.

Identities of W_n are studied in [16] and left operator identities of $\mathscr{L}_n$ are studied in [6].

3. Divergence calculations

If D is an arbitrary element of $\mathscr{L}_n$, then there exists a unique n-tuple $F = (f_1, f_2, \ldots, f_n)$ of elements of P_n such that $D = D_F \in \mathscr{L}_n$. Put

$$\mathrm{div}(D) = \mathrm{div}(D_F) = \partial_1(f_1) + \partial_2(f_2) + \ldots + \partial_n(f_n).$$

Consequently, $\mathrm{div}(D) = \mathrm{Tr}(J(D)) = \mathrm{Tr}(J(F))$.

Recall that every n-tuple $F = (f_1, f_2, \ldots, f_n)$ of P_n represents a polynomial mapping of the vector space k^n. Denote by F^* the endomorphism of P_n such that $F^*(x_i) = f_i$ for all i. If F and G are polynomial endomorphisms of k^n then $(F \circ G)^* = G^* \circ F^*$. By definition, $J(F) = J(F^*)$. The chain rule gives that

$$J(G \circ F) = J(F^* \circ G^*) = F^*(J(G^*))J(F^*) = F^*(J(G))J(F). \tag{5}$$

Lemma 8. *Let $T, S \in \mathscr{L}_n$. Then the following statements are true:*
(i) $J(T \cdot S) = T(J(S)) + J(S)J(T)$;
(ii) $J([T, S]) = T(J(S)) - S(J(T))$;
(iii) $\mathrm{div}([T, S]) = T(\mathrm{div}(S)) - S(\mathrm{div}(T))$.

Proof. Suppose that $T = D_F$ and $S = D_G$. Then $T \cdot S = D_{D_F(G)}$. Consider the endomorphism $(X + tF)^*$ where t is an independent parameter. Obviously,

$$(X + tF)^*(G) = G + tD_F(G) + t^2 G_2 + \ldots .$$

Consequently, $D_F(G) = \frac{\partial}{\partial t}((X + tF)^* G)|_{t=0}$. By (5), we get

$$\begin{aligned} J((X + tF)^*(G)) = J((X + tF)^* \circ G^*) &= (X + tF)^*(J(G))J(X + tF) \\ &= (J(G) + tD_F(J(G)) + t^2 T_2 + \ldots)(I + tJ(F)) \\ &= J(G) + t(D_F(J(G)) + J(G)J(F)) + t^2 M_2 + \ldots . \end{aligned}$$

Hence

$$J(D_F(G)) = \frac{\partial}{\partial t} J((X + tF)^*(G))|_{t=0} = D_F(J(G)) + J(G)J(F),$$

which proves (i). Notice that (i) directly implies (ii). Besides, Tr is a linear function and for any $D \in \mathscr{L}_n$ and $B \in M_n(A)$ we have $\mathrm{Tr}(D(B)) = D(\mathrm{Tr}(B))$. Consequently, (ii) implies (iii). □

Lemma 9. *Let $D \in \mathscr{L}_n$. Then $J(D)$ is nilpotent if and only if* $\mathrm{div}(D^{[q]}) = 0$ *for all* $q \geq 1$.

Proof. By Lemma 8, we get $J(D^{[2]}) = D(J(D)) + J(D)^2$ and

$$J(D^{[i+1]}) = J(D^{[i]} \cdot D) = D^{[i]}(J(D)) + J(D)J(D^{[i]}).$$

This allows us to prove, by induction on i, that

(6) $J(D^{[i]}) = D^{[i-1]}(J(D)) + J(D)D^{[i-2]}(J(D)) + \ldots + J(D)^{i-2}D(J(D)) + J(D)^i$

for all $i \geq 1$.

Suppose that $J(D)$ is nilpotent. It is well known that $J(D)$ is nilpotent if and only if $\mathrm{Tr}(J(D)^q) = 0$ for all $q \geq 1$. Recall that $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$ for any $T, S \in M_n(A)$. Consequently, for any $D \in \mathscr{L}_n$, $T \in M_n(A)$, and integer $s \geq 1$ we have

$$\mathrm{Tr}(D(T^s)) = \mathrm{Tr}(D(T)T^{s-1} + TD(T^2)T^{s-2} + \ldots + T^{s-2}D(T)T + T^{s-1}D(T)) = s\mathrm{Tr}(T^{s-1}D(T))$$

and consequently,

$$D(\mathrm{Tr}(T^s)) = \mathrm{Tr}(D(T^s)) = s\mathrm{Tr}(T^{s-1}D(T)) \tag{7}$$

Hence $\mathrm{Tr}(T^{s-1}D(T)) = 0$ and (6) implies that $\mathrm{div}(D^{[i]}) = \mathrm{Tr}(J(D^{[i]})) = 0$.

Suppose that $\mathrm{div}(D^{[q]}) = 0$ for all $q \geq 1$. We prove by induction on s that $\mathrm{Tr}(J(D)^s) = 0$ for all $s \geq 1$. Suppose that it is true for all s such that $1 \leq s < i$. Then, (7) gives that $\mathrm{Tr}(J(D)^{s-1}D^{[p]}(J(D))) = 0$. Consequently, (7) implies that $\mathrm{Tr}((J(D)^i) = 0$. $\square$

Let D be an arbitrary element of $\mathscr{L}_n$. Recall that L_D is the Lie algebra generated by all right powers $D^{[i]}$ $(i \geq 1)$ of D.

Theorem 1. *Let $D \in \mathscr{L}_n$. Then the Jacobian matrix $J(D)$ of D is nilpotent if and only if the divergence of every element of L_D is zero.*

Proof. This is a direct corollary of Lemmas 8 and 9. $\square$

Denote by $I(D)$ the L_D-closed subalgebra of A generated by all $\mathrm{Tr}(J(D)^i) = 0, i \geq 1$.

Corollary 3. *Let $D \in \mathscr{L}_n$. Then the divergence of every element of L_D belongs to $I(D)$.*

Proof. The proof of Lemma 9 can be easily adjusted to prove that $\mathrm{div}(D^{[i]}) \in I(D)$. Then Lemma 8 finishes the proof of the corollary. $\square$

The Lie algebra L_D is a small part of the left-symmetric algebra $\mathscr{L}_D$ generated by D. Probably L_D is the maximal subspace of $\mathscr{L}_D$ whose divergence belong to $I(D)$. In other words, I think that if $J(D)$ is nilpotent then L_D is the maximal subspace of elements of $\mathscr{L}_D$ whose divergence are zeroes.

Recall that a derivation D is called *triangular* if $D(x_i) \in k[x_1, \ldots, x_i]$ for all i and *strongly triangular* if $D(x_i) \in k[x_1, \ldots, x_{i-1}]$ for all i. If D is a triangular derivation with a nilpotent Jacobian matrix $J(D)$, then it is easy to check that D is strongly triangular. If D is strongly triangular then $J(D)$ is nilpotent and both algebras L_D and $\mathscr{L}_D$ are nilpotent.

Example 2. Now we give an example of derivation D with a nilpotent Jacobian matrix $J(D)$ such that L_D is not nilpotent nor solvable. Consider the automorphism

$$(x + s(xt - ys), y + t(xt - ys), s + t^3, t)$$

of the polynomial algebra $k[x, y, s, t]$ studied A. van den Essen [7] and G. Gorni and G. Zampieri [11]. Put

$$F = (s(xt - ys), t(xt - ys), t^3, 0).$$

Obviously, $J(F)$ is nilpotent. Consider

$$D = D_F = s(xt - ys)\partial_x + t(xt - ys)\partial_y + t^3\partial_s.$$

Corollary 2 gives that D is a right nilpotent element of $\mathscr{L}_n$. Put $w = xt - ys$. Then,

$$D(w) = -yt^3, \quad D(D(w)) = -t^4 w.$$

Consequently, D is not a locally nilpotent derivation and is not a left nilpotent element of $\mathscr{L}_n$ by Lemma 3. Direct calculations give

$$D^2 = D^{[2]} = t^3(xt - 2ys)\partial_x - yt^4\partial_y, \quad D^{[2]}(w) = wt^4,$$
$$D^{[3]} = st^4 w\partial_x + t^5 w\partial_y, \quad D^{[3]}(w) = 0, \quad D^{[4]} = 0.$$

Consequently, the Lie algebra L_D is generated by two elements $a = D$, $b = D^{[2]}$, and $c = D^{[3]}$. Moreover, we have

$$[a, b] = -2c - 2A, \quad A = t^6 y\partial_x, \quad [b, c] = 2t^4 c, \quad [a, c] = t^4 b.$$

These relations show that L_D is not nilpotent. We also have

$$[a, A] = t^4 b, [A, c] = -t^7 b, [A, b] = 2t^4 A.$$

Let M be the subalgebra of L_D generated by A, b, c. Note that t is a constant for all elements of L_D. The homomorphic image of M under $t \mapsto 1$ becomes a Lie algebra with a linear basis A, b, c and and satisfies the relations

$$[b, c] = 2c, [A, c] = -b, [A, b] = 2A.$$

Consequently, M is not solvable and so is L_D.

This example also shows some limits of divergence calculations. The divergence of every element of L_D is zero, but L_D is not nilpotent nor solvable.

4. Primitives of the Hopf algebra NSymm

As an algebra NSymm [9] is the free associative algebra

$$\mathrm{NSymm} = k\langle Z_1, Z_2, \ldots, Z_n, \ldots\rangle$$

over k in the variables $Z_1, Z_2, \ldots, Z_n, \ldots$. The comultiplication $\triangle$ and the counit ϵ are algebra maps determined by

$$\triangle(Z_n) = \sum_{i+j=n} Z_i \otimes Z_j \ (Z_0 = 1), \quad \varepsilon(Z_n) = 0,$$

for all $n \geq 1$, respectively. The antipod S is an antiisomorphism determined by

$$S(Z_n) = \sum_{i_1+\ldots+i_p=n} (-1)^p Z_{i_1} Z_{i_2} \ldots Z_{i_p}$$

for all $n \geq 1$.

The Hopf algebra of noncommutative symmetric functions was introduced in [9] and many systems of free generators and relations between them were described. It was also proved [9] that NSymm is canonically isomorphic to the Solomon descent algebra [17]. It is also known [9, 14] that the graded dual of NSymm is the Hopf algebra of quasisymmetric functions QSymm [10].

Denote by Prim the set of all primitive elements of NSymm, i.e.,

$$\mathrm{Prim} = \{p \in \mathrm{NSymm} | \bigtriangleup (p) = p \otimes 1 + 1 \otimes p\}.$$

Define the system of elements $U_1, U_2, \ldots, U_i, \ldots$ by

$$\sum_{i=1}^{\infty} t^i U_i = \log(\sum_{i=0}^{\infty} t^i Z_i).$$

Direct calculations give

$$U_m = \sum_{i_1+\ldots+i_k=m} \frac{(-1)^{k-1}}{k} Z_{i_1} \ldots Z_{i_k}$$

and

$$Z_m = \sum_{i_1+\ldots+i_k=m} \frac{1}{k!} U_{i_1} \ldots U_{i_k}$$

for all $m \geq 1$.

It is well known [9, 14] the Lie algebra Prim is a free Lie algebra freely generated by $U_1, U_2, \ldots, U_m \ldots$ and NSymm is the universal enveloping algebra of NSymm.

Consider the following two systems of elements of NSymm :

(8) $$\Theta_n(Z) = \sum_{r_1+\ldots+r_k=n} (-1)^{k-1} r_1 Z_{r_1} Z_{r_2} \ldots Z_{r_k},$$

and

(9) $$\Psi_n(Z) = \sum_{r_1+\ldots+r_k=n} (-1)^{k-1} r_k Z_{r_1} Z_{r_2} \ldots Z_{r_k},$$

where $r_i \in \mathbb{N} = \{1, 2, \ldots\}$ and $n \geq 1$.

Notice that in our notations, Z_i correspond to complete symmetric functions S_i, Ψ_i are the power sums symmetric functions, and U_i correspond to power sums of the second kind Φ_i/i in [9]. The functions corresponding to Θ_i were not considered in [9] since Θ_i can be obtained from Ψ_i by the natural involution of NSymm preserving all Z_i. But in needs of the Jacobian Conjecture it is necessary to study the relations between Θ_i and Ψ_i more deeply.

The systems of elements (8) and (9) are primitive systems of free generators of the free associative algebra NSymm [9] and can be defined recursively by

$$nZ_n = \Theta_n(Z) + \Theta_{n-1}Z_1 + \Theta_{n-2}Z_2 + \ldots + \Theta_1 Z_{n-1}$$

and

$$nZ_n = \Psi_n(Z) + Z_1\Psi_{n-1} + Z_2\Psi_{n-2} + \ldots + Z_{n-1}\Psi_1$$

for all $n \geq 1$.

Recall that a composition is a vector $I = (i_1, \ldots, i_m)$ of nonnegative integers, called the parts of I. The length $l(I)$ of the composition I is the number k of its parts and the weigt of I is the sum $|I| = \Sigma i_j$ of its parts. We use notations

$$Z^I = Z_{i_1} \ldots Z_{i_m}, \quad \Theta^I = \Theta_{i_1} \ldots \Theta_{i_m}, \quad \Psi^I = \Psi_{i_1} \ldots \Psi_{i_m}.$$

Put also

$$\pi_u(I) = i_1(i_1 + i_2) \ldots (i_1 + i_2 + \ldots + i_m)$$

and $\mathrm{lp}(I) = i_m$ (the last part of I). Let J be another composition. We say that $I \preceq J$ if $J = (J_1, \ldots, J_m)$ and $|J_j| = i_j$ for all j. For example, $(3, 2, 6) \preceq (2, 1, 2, 3, 1, 2)$. If $I \preceq J$ then put

$$\pi_u(J, I) = \prod_{i=1}^{m} \pi_u(J_i), \quad \mathrm{lp}(J, I) = \prod_{i=1}^{m} \mathrm{lp}(J_i).$$

The following formulas are proved in [9].

$$Z^I = \sum_{J \succeq I} \frac{1}{\pi_u(J, I)} \Psi^J, \quad \Psi^I = \sum_{J \succeq I} (-1)^{l(J)-l(I)} \mathrm{lp}(J, I) Z^J. \tag{10}$$

Denote by w the natural involution of the free associative algebra NSymm preserving all Z_i. Obviously, $w(\Theta_i) = \Psi_i$ and $w(\Psi_i) = \Theta_i$ for all i. Applying w, from (10) we get

$$Z^{\overline{I}} = \sum_{J \succeq I} \frac{1}{\pi_u(J, I)} \Theta^{\overline{J}}, \quad \Theta^{\overline{I}} = \sum_{J \succeq I} (-1)^{l(J)-l(I)} \mathrm{lp}(J, I) Z^{\overline{I}},$$

where $\overline{I}$ is the mirror image of the composition I, i.e. the new composition obtained by reading I from right to left.

Consequently,

$$Z^I = \sum_{J \succeq I} \frac{1}{\pi_u(\overline{J}, \overline{I})} \Theta^J, \quad \Theta^I = \sum_{J \succeq I} (-1)^{l(J)-l(I)} \mathrm{lp}(\overline{J}, \overline{I}) Z^J. \tag{11}$$

Using (9) and (11), we get

$$\Psi_n = \sum_{|I|=n} (-1)^{l(I)-1} \mathrm{lp}(I) Z^I = \sum_{|I|=n} (-1)^{l(I)-1} \mathrm{lp}(I) \sum_{J \succeq I} \frac{1}{\pi_u(\overline{J}, \overline{I})} \Theta^J,$$

i.e.,

$$\Psi_n = \sum_{J \succeq I, |I|=n} (-1)^{l(I)-1} \frac{\mathrm{lp}(I)}{\pi_u(\overline{J}, \overline{I})} \Theta^J. \tag{12}$$

In fact, Ψ_n can be expressed as a Lie polynomial of $\Theta_1, \ldots, \Theta_n$. We have

$$\Psi_1 = \Theta_1, \ \Psi_2 = \Theta_2, \ \Psi_3 = \Theta_3 + 1/2[\Theta_2, \Theta_1],$$
$$\Psi_4 = \Theta_4 + 2/3[\Theta_3, \Theta_1] + 1/6[[\Theta_2, \Theta_1], \Theta_1].$$

It will be interesting to find the Lie expression of Ψ_n in $\Theta_1, \ldots, \Theta_n$.

5. An action of the Hopf algebra NSymm

We define an action

$$\mathrm{NSymm} \times P_n \longrightarrow P_n \quad ((T, a) \mapsto T \circ a)$$

of NSymm on the polynomial algebra P_n related to an n-tuple F. Since NSymm is a free associative algebra, it is sufficient to define $Z_i \circ a$ for all $i \geq 1$ and $a \in P_n$. For any $a \in P_n$ there exists a unique system of elements $g_i \in P_n, i \geq 1$ such that

$$(X + tF)^*(a) = a + tg_1 + t^2 g_2 + \ldots + t^n g_n + \ldots,$$

where t is an independent variable. Put $Z_i \circ a = g_i$ for all $i \geq 1$. Then

$$(X + tF)^*(a) = a + tZ_1(a) + t^2 Z_2(a) + \ldots + t^n Z_n(a) + \ldots.$$

This formula can be considered as a linearization of the action of $(X+tF)^*$ on P_n. Denote by

$$\lambda : \mathrm{NSymm} \longrightarrow \mathrm{Hom}_k(P_n, P_n)$$

the homomorphism corresponding to this representation, where $\mathrm{Hom}_k(P_n, P_n)$ is the set of all k-linear maps from P_n to P_n. First of all we show that $\lambda(\mathrm{NSymm}) \subseteq A_n$.

Denote by $p : P_n \otimes_k P_n \to P_n$ the product in the polynomial algebra P_n.

Lemma 10. *Let* $T \in \mathrm{NSymm}$. *Then* $\lambda(T)p = p\lambda(\triangle(T))$.

Proof. It is easy to check that the set of elements $T \in \mathrm{NSymm}$ satisfying the statement of the lemma forms a subalgebra. Consequently, we may assume that $T = Z_n$. If $a, b \in A$ then

$$\begin{aligned} \sum_{i=0} t^i \lambda(Z_i)(ab) &= (X + tF)^*(ab) \\ &= ((X + tF)^*(a))((X + tF)^*(b)) \\ &= (\sum_{i=0} t^i \lambda(Z_i)(a))(\sum_{i=0} t^i \lambda(Z_i)(b)). \end{aligned}$$

Comparing coefficients in the degrees of t we get $Z_i(ab) = \sum_{r+s=i} Z_r(a) Z_s(b)$. This means $Z_i p = p \triangle (Z_i)$. □

Lemma 11. $\lambda(\mathrm{Prim}) \subseteq W_n$ *and* $\lambda(\mathrm{NSymm}) \subseteq A_n$.

Proof. If $T \in \mathrm{Prim}$ then, by Lemma 10, we get

$$\begin{aligned} \lambda(T)(ab) = \lambda(T)p(a \otimes b) &= p\lambda(\triangle(T))(a \otimes b) = p\lambda(T \otimes 1 + 1 \otimes T)(a \otimes b) \\ &= p(\lambda(T)(a) \otimes b + a \otimes \lambda(T)(b)) = \lambda(T)(a)b + a\lambda(T)(b), \end{aligned}$$

i.e., $\lambda(T) \in W_n$.

Notice that NSymm is a free associative algebra and any action of NSymm is well defined by the action of any free system of generators. For example, $\Theta_1, \Theta_2, \ldots, \Theta_n, \ldots \in \mathrm{Prim}$ is a free system of generators of NSymm and $\lambda(\Theta_1), \lambda(\Theta_2), \ldots, \lambda(\Theta_n), \ldots \in W_n$. Consequently, for any $T \in \mathrm{NSymm}$ element $\lambda(T)$ is a differential operator on P_n, i.e., $\lambda(T) \in A_n$. □

By this lemma, we have a homomorphism

$$\lambda : \mathrm{NSymm} \longrightarrow A_n. \tag{13}$$

Lemma 12. *Let* $a \in P_n$ *and* $\deg a \leq k$. *Then* $\lambda(Z_i)(a) = 0$ *for all* $i \geq k+1$.

Proof. Obviously, the degree of $(X+tF)(a) = a((x_1+tf_1), \ldots, (x_n+tf_n))$ with respect to t is less than or equal to k. Consequently, $\lambda(Z_i)(a) = 0$ for all $i \geq k+1$. □

Proposition 1. *Let*

$$(X+tF)^{-1} = X + tF_1 + t^2F_2 + \ldots + t^mF_m + \ldots$$

be the formal inverse to the endomorphism $X + tF$ *of* $k[t]^n$. *Then*

$$-\lambda(\Psi_m)(X) = F_m$$

for all $m \geq 1$.

Proof. Consider the endomorphism $(X+tF)^* : k[t]\otimes_k P_n \to k[t]\otimes_k P_n$ of the $k[t]$-algebra. Notice that

$$(X+tF)^* = 1 + t\lambda(Z_1) + t^2\lambda(Z_2) + \ldots + t^n\lambda(Z_n) + \ldots \tag{14}$$

by the definition of $\lambda(Z_i)$. Then,

$$(X+tF)^* = 1 - T, \quad T = -(t\lambda(Z_1) + t^2\lambda(Z_2) + \ldots + t^n\lambda(Z_n) + \ldots),$$

and

$$((X+tF)^*)^{-1} = 1 + T + T^2 + \ldots + T^n + \ldots.$$

Direct calculation gives

$$((X+tF)^*)^{-1} = 1 + tT_1 + t^2T_2 + \ldots + t^nT_n + \ldots,$$

where

$$T_m = \sum_{r_1+\ldots+r_k=m} (-1)^k \lambda(Z_{r_1})\lambda(Z_{r_2})\ldots\lambda(Z_{r_k}), \quad n \geq 1.$$

Notice that $(X+tF)^{-1} = ((X+tF)^*)^{-1}(X)$ and $F_m = T_m(X)$. Then,

$$\begin{aligned} F_n &= \sum_{r_1+\ldots+r_k=m} (-1)^k \lambda(Z_{r_1})\lambda(Z_{r_2})\ldots\lambda(Z_{r_k})(X) \\ &= \sum_{r_1+\ldots+r_k=m} (-1)^k r_k \lambda(Z_{r_1})\lambda(Z_{r_2})\ldots\lambda(Z_{r_k})(X) \end{aligned}$$

by lemma 12. Consequently, $F_m = -\lambda(\Psi_m)(X)$. □

The homomorphism (13) coincides with one of a series of homomorphisms constructed in [21] and the images of primitive generators were calculated in [21].

Lemma 13. *(i)* $\lambda(\Theta_m) = (-1)^{m-1}D_F^{[m]}$ *for all* $m \geq 1$.

(ii) $\lambda(\Psi_m) = -D_{F_m}$ *for all* $m \geq 1$.

Proof. By Lemma 11, $\lambda(\Theta_m)$ and $\lambda(\Psi_m)$ are derivations of P_n. Consequently, it is sufficient to prove that $\lambda(\Theta_m)(X) = (-1)^{m-1}D_F^{[m]}(X)$ and $\lambda(\Psi_m)(X) = -D_{F_m}(X) = -F_m$. Proposition 1 implies (ii). We have $\lambda(\Theta_1)(X) = F = D_F(X)$ since $\Theta_1 = Z_1$. Then, $\lambda(\Theta_1) = D_F$. Leading an induction on m, by (8) and Lemma 12, we get

$$\lambda(\Theta_m)(X) = -\lambda(\Theta_{m-1})\lambda(Z_1)(X) = -\lambda(\Theta_{m-1})\lambda(Z_1)(X)$$
$$= (-1)^{m-1}D_F^{[m-1]}(F) = (-1)^{m-1}D_F^{[m]}(X). \quad \square$$

Put $D = D_F$. Recall that L_D is the subalgebra of W_n generated by all right powers $D^{[m]}$ $(m \geq 1)$ of D and A_D is the subalgebra of A_n generated by the same elements.

Corollary 4. *Let $D = D_F$. Then $\lambda(\mathrm{Prim}) = L_D$ and $\lambda(\mathrm{NSymm}) = A_D$.*

Proof. This is an immediate corollary of Lemmas 11 and 13. $\square$

Theorem 2. *Let $F = (f_1, \ldots, f_n)$ be an arbitrary n-tuple of the polynomial algebra $P_n = k[x_1, \ldots, x_n]$, $L_D = L_F$ be the Lie algebra generated by all right powers $D^{[m]}$ $(m \geq 1)$ of $D = D_F$, and*

$$(X + tF)^{-1} = X + tF_1 + t^2F_2 + \ldots + t^mF_m + \ldots$$

be the formal inverse to the endomorphism $X + tF$ of $k[t]^n$. Then the Lie algebra L_D is generated by all D_{F_m} where $m \geq 1$.

Proof. By Corollary 4, L_D is the image of the Lie algebra Prim of all primitive elements of NSymm under λ. The set of elements Ψ_m, where $m \geq 1$, is also generates Prim. Consequently, Lemma 13 implies the statement (i). $\square$

One more interesting system of generators $\lambda(U_1), \ldots, \lambda(U_m), \ldots$ of the Lie algebra L_D corresponds to the coefficients of $D - \log$ of $X + tF$ considered in [20, 21].

6. Comments and some open questions

So, we introduced three algebras A_F, L_F, and $\mathscr{L}_F$ related to the study of the Jacobian Conjecture, i.e., to the study of the polynomial endomorphism $X + tF$ with a nilpotent Jacobian matrix $J(F)$. If $J(F)$ is nilpotent then $D = D_F$ is right nilpotent by Corollary 2. Let p be a positive integer such that $D^{[p]} = 0$. In order to solve the Jacobian Conjecture, it is necessary to prove that there exists $m = m(F)$ such that $F_i = 0$ for all $i \geq m$ in notations of Theorem 2. Using Lemma 13 and (12), we get

$$D_{F_i} = \sum_{p \geq J \succeq I, |I| = n} (-1)^{l(I)+|J|-l(J)} \frac{\mathrm{lp}(I)}{\pi_u(\overline{J}, \overline{I})} D^J, \tag{15}$$

where $D^J = D^{[j_1]} \ldots D^{[j_s]}$ for any $J = (j_1, \ldots, j_s)$ and $p \geq J$ means that $p \geq j_i$ for all i. Moreover, the right hand side of this equation is a Lie polynomial in $D^{[s]}$ where $s \geq 1$ and the Jacobian Conjecture can be considered as a problem of the algebra L_F. But I cannot see how to use the degree of F in this formula. We cannot prove that $D_{F_i} = 0$ without this.

Let's come back to formula (10) and Lemma 12. Suppose that the degree of F is m. A composition $I = (i_k, \ldots, i_1)$ of length k is called m-reduced if $i_1 = 1$, $i_2 \leq m$, and $i_j \leq (i_1 + \ldots + i_{j-1})(m-1) + 1$ for all $3 \leq j \leq k$. Let T_n be the set of all m-reduced compositions I with $|I| = n$. Notice that $\mathrm{lp}(I) = 1$ if I is m-reduced. If I is not m-reduced then $\lambda(Z^I)(X) = 0$ by Lemma 12. For this reason we can consider only m-reduced compositions in (10). Then we get

$$D_{F_i} = \sum_{J \succeq I, I \in T_n} (-1)^{l(I)+|J|-l(J)} \frac{1}{\pi_u(\overline{J}, \overline{I})} D^J \tag{16}$$

in A_F but not in L_F. So, we did not get $D_{F_i} = 0$ yet. In fact, to derive (16) we used only the nilpotency of D and the degree of F. In connection with this, the following question is very interesting.

Problem 1. *Is the Jacobian matrix $J(D)$ of D nilpotent if D is a right nilpotent element of $\mathscr{L}_n$?*

If the answer to this question is negative, then we probably cannot prove that $D_{F_i} = 0$ in A_F.

The formula (16) can be considered as a formula in the left-symmetric algebra $\mathscr{L}_D$ where the associative product D^J is changed by the left normed product. For this reason left operator identities of $\mathscr{L}_n$ are very important. Notice that $J(F)$ is nilpotent if and only if R_D is nilpotent by Corollary 2. So, this condition is expressed in the language of right multiplication operators but (16) is expressed in the language of left operators.

The following problem is interesting in connection with Lemmas 4 and 6.

Problem 2. *Describe the structure of the multiplication algebra $M(\mathscr{L}_n)$ of the left-symmetric algebra $\mathscr{L}_n$.*

It is well known that all trace identities of matrix algebras are corollaries of the Cayley-Hamilton trace identities [15].

Problem 3. *Is every trace identity (or identity) of $\mathscr{L}_n$ a corollary of the Cayley-Hamilton trace identities (4).*

By Lemma 7, a positive answer to this question implies that every identity of W_n is a corollary of the Cayley-Hamilton trace identities.

In order to solve the Jacobian Conjecture we need more information about left operator identities of $\mathscr{L}_n$.

Problem 4. *Describe all left operator identities of $\mathscr{L}_n$.*

It is interesting to know that what types of properties can be better described in the language of A_D.

Problem 5. *Describe all $D \in \mathscr{L}_n$ such that A_D is a simple algebra.*

Problem 6. *Is there any derivation D with nilpotent Jacobian matrix $J(D)$ such that A_D is a simple algebra?*

Example 1 shows that the nilpotency of $J(F)$ does not imply neither nilpotency nor solvability of L_F.

Problem 7. *Describe necessary and sufficient conditions of the nilpotency (and solvability) of the Lie algebra L_D.*

At the moment I know that $\mathscr{L}_D$ is nilpotent if and only if $\mathrm{div}(\mathscr{L}_D) = 0$.

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