Kobayashi-Hitchin correspondence for tame harmonic bundles II

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Abstract

Let X be a smooth projective complex variety with an ample line bundle L, and let D be a simple normal crossing divisor. We establish the Kobayashi-Hitchin correspondence between tame harmonic bundles on $X - D$ and μ_L-stable parabolic λ-flat bundles with trivial characteristic numbers on (X, D). Especially, we obtain the quasiprojective version of the Corlette-Simpson correspondence between flat bundles and Higgs bundles.

Contents

1 Introduction

1.1 Main Results

We explain the main results in this paper. We do not recall history or background about the study of Kobayashi-Hitchin correspondence and harmonic bundles, for which we refer the introductions of [36], [22] or [30], for example. The notion of regular filtered λ-flat bundles and parabolic λ-flat bundles are explained in the subsection 2.1. (See also the subsections 3.1–3.2 of [30]. But, we also use a slightly different notation and terminology, as is explained in the subsection 2.1.6.) They are equivalent, and we will not care about the distinction of them. The notion of filtered local systems is explained in the section 6.

1.1.1 Kobayashi-Hitchin Correspondence

Let X be a smooth complex projective variety with an ample line bundle L. Let D be a normal crossing divisor of X. Our main purpose is to show the following theorem.

Theorem 1.1 (Theorem 5.16, Proposition 2.26, Proposition 2.27) *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat bundle on (X, D). We put $E := \boldsymbol{E}_{|X-D}$. Then the following conditions are equivalent.*

- *It is μ_L-polystable with the trivial characteristic numbers* $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$.
- *There exists a pluri-harmonic metric h of $(E, \mathbb{D}^\lambda)$ adapted to the parabolic structure.*

Such a metric is unique up to obvious ambiguity. ▮

Remark 1.2 *The claims of Theorem* 1.1 *in the case $\lambda = 0$ has already been proved in our previous paper* [30]. *Thus we restrict ourselves to the case $\lambda \neq 0$ in this paper.* ▮

Corollary 1.3 (Corollary 5.18) *Let $\mathcal{C}_\lambda^{poly}$ denote the category of μ_L-polystable λ-flat regular filtered bundles on (X, D) with trivial characteristic numbers. Then we have the natural equivalence of the categories $\mathcal{C}_{\lambda_1}^{poly} \simeq \mathcal{C}_{\lambda_2}^{poly}$ for any $\lambda_i \in \boldsymbol{C}$ $(i = 1, 2)$. The equivalence preserves the tensor products, direct sums and duals.* ▮

Remark 1.4 *Let λ_i $(i = 1, 2)$ be two complex numbers. A λ_2-connection $\mathbb{D}^{\lambda_2} = d'' + (\lambda_2/\lambda_1) \cdot d'$ is induced from a λ_1-connection $\mathbb{D}^{\lambda_1} = d'' + d'$. Hence we have the obvious functor* $\mathrm{Obv} : \mathcal{C}_{\lambda_1}^{poly} \longrightarrow \mathcal{C}_{\lambda_2}^{poly}$. *But this is not same as the above functor $\Xi_{\lambda_1,\lambda_2}$.* ▮

Especially, we obtain a generalization of the Corlette-Simpson correspondence between flat bundles and Higgs bundles in the so-called non-abelian Hodge theory.

Corollary 1.5 *We have the equivalences of the following two categories:*

- *The category of μ_L-polystable regular filtered Higgs bundles on (X, D) with trivial characteristic numbers.*
- *The category of μ_L-polystable regular filtered flat bundles on (X, D) with trivial characteristic numbers.* ▮

1.1.2 Bogomolov-Gieseker inequality and some formula for the characteristic numbers

Let X, L and D be as above.

Theorem 1.6 (Corollary 3.22) *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a μ_L-stable regular filtered λ-flat bundle on (X, D) in codimension two. Then we have the following inequality holds for the parabolic characteristic numbers for $\boldsymbol{E}_*$:*

$$\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) \leq \frac{\int_X \text{par-c}^2_{1,L}(\boldsymbol{E}_*)}{2 \operatorname{rank} E}. \tag{1}$$

It is a generalization of the so-called Bogomolov-Gieseker inequality. ▮

In the case $\lambda \neq 0$, we also have some formulas about the parabolic Chern characteristic numbers, which are valid for any parabolic λ-flat bundles in codimension two. One of the formulas can be stated simply, after we see the correspondence of regular filtered λ-flat sheaves and filtered local systems. Let$(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat sheaf on (X, D). As is explained in Remark 1.4, we have the obvious correspondence of flat λ-connection $\mathbb{D}^\lambda = d'' + d'$ $(\lambda \neq 0)$ and flat connection $\mathbb{D}^{\lambda\, f} = d'' + \lambda^{-1} d'$. In particular, we obtain the local system $\mathcal{L}$ on $X - D$ from the flat bundle $(\boldsymbol{E}_*, \mathbb{D}^{\lambda, f})_{|X-D}$. Moreover, the parabolic structure of $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ induces the filtered structure of $\mathcal{L}$, and we have the more refined claims as in the following proposition.

Proposition 1.7 (Corollary 6.4 and Corollary 6.6) *Let $\widetilde{\mathcal{C}}(X, D)$ denote the category of filtered local system on (X, D), and let $\mathcal{C}^{sat}_\lambda(X, D)$ denote the category of saturated regular filtered λ-flat sheaves on (X, D) for $\lambda \neq 0$. Then we have the equivalent functor $\Phi_\lambda : \widetilde{\mathcal{C}}(X, D) \longrightarrow \mathcal{C}^{sat}_\lambda(X, D)$ such that $\text{par-c}_1(\mathcal{L}_*) = \text{par-c}_1\bigl(\Phi_\lambda(\mathcal{L}_*)\bigr)$ and $\int_X \text{par-ch}_{2,L}(\mathcal{L}_*) = \int_X \text{par-ch}_{2,L}\bigl(\Phi_\lambda(\mathcal{L}_*)\bigr)$. The functor Φ_λ preserves the μ_L-stability.* ∎

Remark 1.8 *From Theorem 1.6 and Proposition 1.7, we obtain the Bogomolov-Gieseker inequality for μ_L-stable filtered local systems (Corollary 6.7). Such a kind of the inequality is discussed in [39].* ∎

Remark 1.9 *Let us describe the formula $\int_X \text{par-ch}_{2,L}(\mathcal{L}_*) = \int_X \text{par-ch}_{2,L}(\Phi(\mathcal{L}_*))$ in terms of the $\boldsymbol{c}$-truncation $({}_{\boldsymbol{c}}E_*, \mathbb{D}^\lambda)$ of saturated regular filtered λ-flat bundle $\Phi_\lambda(\mathcal{L}_*)$. For simplicity, we assume $\dim X = 2$.*

$$\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E_*) = \frac{1}{2} \sum_{i \in S} \sum_{u \in \mathcal{KMS}({}_{\boldsymbol{c}}E_*, i)} \bigl(\text{Re}(\lambda^{-1}\alpha) + a\bigr)^2 \cdot r(i, u) \cdot (D_i, D_i)$$
$$+ \frac{1}{2} \sum_{i \in S} \sum_{\substack{j \neq i \\ P \in D_i \cap D_j}} \sum_{(u_i, u_j) \in \mathcal{KMS}({}_{\boldsymbol{c}}E_*, P)} \bigl(\text{Re}\,\lambda^{-1}\alpha_i + a_i\bigr)\bigl(\text{Re}\,\lambda^{-1}\alpha_j + a_j\bigr) \cdot r(P, u_i, u_j). \quad (2)$$

Here, $u = (a, \alpha)$, $u_i = (a_i, \alpha_i)$ and $u_j = (a_j, \alpha_j)$ denote the KMS-spectra, which are elements of $\boldsymbol{R} \times \boldsymbol{C}$. We put $r(i, u) := \text{rank}^{\,i}\, \text{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E_{|D_i})$ for $(a, \alpha) \in \mathcal{KMS}({}_{\boldsymbol{c}}E_*, i)$, and $r(P, u_i, u_j) := \text{rank}^{\,P}\, \text{Gr}^{F,\mathbb{E}}_{(u_i, u_j)}({}_{\boldsymbol{c}}E_{|P})$ for $(u_i, u_j) \in \mathcal{KMS}({}_{\boldsymbol{c}}E, P)$ and $P \in D_i \cap D_j$. And (D_i, D_j) and $\bigl(D_i, c_1(L)\bigr)$ denote the intersection numbers.

We also have some other formulas for $\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E)$ (Proposition 3.24) or some vanishings for the data of $({}_{\boldsymbol{c}}E, \mathbb{D}^\lambda)$ at D (Proposition 3.26 and Proposition 3.27). ∎

1.1.3 Vanishing of the characteristic numbers and existence of the Corlette-Jost-Zuo metric

Due to Proposition 1.7, we obtain the vanishings $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$, when $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ corresponds to the filtered local system whose parabolic structure is trivial. In other words, $\text{Re}\,\alpha + a = 0$ is satisfied for any KMS-spectrum $u = (a, \alpha) \in \mathcal{KMS}(i)$ and for any $i \in S$. We can apply such a consideration to the canonical prolongation of a flat bundle due to P. Deligne [3]. Let (E, ∇) be a flat bundle on $X - D$. Then it is shown that there exists the holomorphic vector bundle $\widetilde{E}$ on X satisfying (i) $\widetilde{E}_{|X-D} = E$ (ii) $\nabla \widetilde{E} \subset \widetilde{E} \otimes \Omega^{1,0}(\log D)$ (iii) the real parts of the eigenvalues of $\text{Res}_i(\nabla)$ are contained in $[0, 1[$. In that case, we have the naturally defined parabolic structure $\boldsymbol{F}$ for which $\text{Re}\,\alpha + a = 0$ are satisfied for any KMS-spectrum (a, α). Hence we obtain the vanishing $\text{par-deg}_L(\widetilde{E}, \boldsymbol{F}) = \int_X \text{par-ch}_{2,L}(\widetilde{E}, \boldsymbol{F}) = 0$.

This vanishing is significant to understand the existence theorem of the Corlette-Jost-Zuo metric from the view point of Kobayashi-Hitchin correspondence. When (E, ∇) is semisimple, we know the existence of a tame pure imaginary pluri-harmonic metric, which we call the Corlette-Jost-Zuo metric. (See [2] for the case $D = \emptyset$ and [14] for the general case. See also [29].) Since semisimplicity obviously implies the μ_L-polystability of $(\widetilde{E}, \boldsymbol{F}, \nabla)$ ([33], for example), we can derive the existence of the Corlette-Jost-Zuo metric from Theorem 1.1 due to the vanishing of the characteristic numbers.

1.2 Methods and Difficulty

1.2.1 Perturbation of parabolic structure

Let X be a smooth projective surface, and let D be a simple normal crossing divisor of X. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a parabolic λ-flat bundle on (X, D). For any small $\epsilon > 0$, we take an ϵ-perturbation $\boldsymbol{F}^{(\epsilon)}$ of the parabolic structure, and then $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$ is *graded semisimple* (the subsection 2.1.5). It can be shown that the pseudo curvature of ordinary metrics for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$ $(\epsilon > 0)$ satisfy the appropriate finiteness (the section 3). By using the theorem of Simpson, we can take a Hermitian-Einstein metric $h^{(\epsilon)}_{HE}$ of $(E_{|X-D}, \mathbb{D}^\lambda)$ which is adapted to $\boldsymbol{F}^{(\epsilon)}$ $(\epsilon > 0)$. Then we can easily derive the Bogomolov-Gieseker inequality (Theorem 1.6). We also obtain the formulas by calculating the integrals of the characteristic numbers for pseudo curvatures, for example (2).

Let us consider the existence of a pluri-harmonic metric (Theorem 1.1). Ideally, the limit $\lim_{\epsilon\to 0} h^{(\epsilon)}_{HE}$ should give the desired pluri-harmonic metric for the given flat parabolic bundle $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$. However, it is not easy to show such a convergence. It is the main problem which we have to overcome in this paper.

1.2.2 Difficulty

In [30], we gave an argument to deal with such a convergence problem for the case $\lambda = 0$. The argument doesn't work in the case $\lambda \neq 0$. Let us explain what is the difference heuristically and imprecisely in the case $\lambda = 1$. Since we have $\text{par-deg}_L(E, \boldsymbol{F}^{(\epsilon)}) = 0$, the metrics $h^{(\epsilon)}_{HE}$ give the harmonic metrics in this case. Recall that a harmonic metric can be regarded as a harmonic map, at least locally, and that we know a well established argument for the convergence of a sequence of harmonic maps when the energies are dominated ([7]). In our case, the energies of $h^{(\epsilon)}_{HE}$ over $X - D$ are not finite, in general. Even if we consider the energies over a compact subset $Z \subset X - D$, it is not clear how to derive a uniform estimate which is independent of ϵ. On the other hand, the Higgs field is fixed for such a convergence problem in the case $\lambda = 0$. In particular, the eigenvalues of the Higgs fields are fixed. Then we can derive the estimate of the local L^2-norm of the Higgs fields independently of ϵ. Since such L^2-norms play the role of the energies, the local convergence can be easily shown in the Higgs case, although we need some technical argument for global convergence. On the contrary, even the local convergence is not easy to show in the case $\lambda \neq 0$.

1.2.3 Convergences

To attack the problem, we discuss similar convergence problems in the curve case where the Kobayashi-Hitchin correspondence was established and well understood by C. Simpson ([35]). Let C be a smooth projective curve, and let D be a divisor of C. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a λ-flat stable parabolic bundle on (C, D), and let $\boldsymbol{F}^{(\epsilon)}$ be ϵ-perturbations. Note we have $\det(E, \boldsymbol{F}, \mathbb{D}^\lambda) = \det(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$. We can take a sequence of harmonic metrics $h^{(\epsilon)}$ for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$ $(\epsilon \geq 0)$ such that $\det h^{(\epsilon)} = \det h^{(0)}$, due to the result of Simpson.

First, we will show that the sequence $\{h^{(\epsilon)} \,|\, \epsilon > 0\}$ converges to $h^{(0)}$. Namely, let $h^{(\epsilon)}_{in}$ $(\epsilon > 0)$ be initial metrics for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$, and let $s^{(\epsilon)}$ be the endomorphism determined by $h^{(\epsilon)} = h^{(\epsilon)}_{in} \cdot s^{(\epsilon)}$. Then we can show the following relations:

$$M(h^{(\epsilon)}_{in}, h^{(\epsilon)}) \leq 0, \quad \big|\log s^{(\epsilon)}\big|_{h^{(\epsilon)}_{in}} \leq C_{1,\epsilon} + C_{2,\epsilon} \cdot M(h^{(\epsilon)}_{in}, h^{(\epsilon)}), \quad \big\|\mathbb{D}^\lambda s^{(\epsilon)}\big\|^2_{L^2, h^{(\epsilon)}_{in}, \omega_\epsilon} \leq \int \big|\mathrm{tr}\big(s^{(\epsilon)} \cdot G(h^{(\epsilon)}_{in})\big)\big| \,\mathrm{dvol}_{\omega_\epsilon}. \tag{3}$$

Here $M(h^{(\epsilon)}_{in}, h^{(\epsilon)})$ denote the Donaldson functionals, and ω_ϵ denote appropriate metrics of $C - D$. Hence, if we show that $C_{i,\epsilon}$ can be taken independently of ϵ for some ω_ϵ, and if we can construct appropriate family of initial metrics $h^{(\epsilon)}_{in}$ such that $G(h^{(\epsilon)}_{in})$ are uniformly bounded with respect to ω_ϵ and $h^{(\epsilon)}_{in}$, we obtain the L^2_1-boundedness of the family $\{s^{(\epsilon)}\}$. Then, by using a standard bootstrapping argument, we can show that the sequence $\{s^{(\epsilon)}\}$ is convergent to the identity in the C^∞-sense (the section 4).

Next, suppose that we are given hermitian metrics $\widetilde{h}^{(\epsilon)} := h^{(\epsilon)} \cdot \widetilde{s}^{(\epsilon)}$ for $\epsilon > 0$, with the following properties:

- $\det \widetilde{h}^{(\epsilon)} = \det h^{(\epsilon)}$.
- $\int |G(\widetilde{h}^{(\epsilon)})|^2 \longrightarrow 0$.
- $\big\|\mathbb{D}^\lambda s^{(\epsilon)}\big\|^2 < \infty$. (We do not need uniform bound.)

Then we can show that $\{\widetilde{h}^{(\epsilon)}\}$ is convergent to $h^{(0)}$ (the subsection 5.1).

We apply the above results to our convergence problem explained in the subsection 1.2.1. Due to the standard Mehta-Ramanathan type theorem (Proposition 2.8), the restriction $(E, \boldsymbol{F}, \mathbb{D}^{\lambda})_{|C}$ is also stable for almost every ample $C \subset X$. Let h_C be a harmonic bundle of $(E, \boldsymbol{F}, \mathbb{D}^{\lambda})_{|C}$. Then we can show that $\{h^{(\epsilon)}_{HE\,|\,C}\}$ is convergent to h_C almost everywhere on C for almost every $C \subset X$, by using the above result. Therefore, we obtain a metric $h_{\mathcal{V}}$ defined almost everywhere on $X - D$ such that $h_{\mathcal{V}\,|\,C} = h_C$ on almost everywhere on C for almost every curve $C \subset X$. With some more additional argument, we can show that $h_{\mathcal{V}}$ gives the desired pluri-harmonic metric, indeed (the subsection 5.2).

Remark 1.10 *Perhaps, the argument of this paper can be used in the Higgs case, to show the existence of a pluri-harmonic metric. However, we remark that the argument for a convergence given in* [30] *can be applied in a wider range. In fact, we used it to discuss the convergence of a family of harmonic bundles induced by the constant multiplication of Higgs fields.* ∎

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2 Preliminary

2.1 Generality of Regular Filtered λ-Flat Sheaf in Complex Geometry

The notion of a parabolic bundle, filtered bundle and their characteristic numbers are explained in the sections 3.1–3.2 of [30]. We use the notation there.

2.1.1 λ-connection

Let Y be a complex manifold, and let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Recall that a λ-connection of $\mathcal{E}$ is defined to be a linear map $\mathbb{D}^{\lambda} : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^{1,0}_Y$ satisfying the twisted Leibniz rule $\mathbb{D}^{\lambda}(f \cdot s) = f \cdot \mathbb{D}^{\lambda}(s) + \lambda \cdot d_Y(f) \cdot s$, where f and s denote holomorphic sections of $\mathcal{O}_Y$ and $\mathcal{E}$ respectively. The linear maps $\mathbb{D}^{\lambda} : \mathcal{E} \otimes \Omega^{p,0} \longrightarrow \mathcal{E} \otimes \Omega^{p+1,0}$ are induced. When $\mathbb{D}^{\lambda} \circ \mathbb{D}^{\lambda}$ is satisfied, it is called flat.

Let X be a complex manifold, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. Let $\mathcal{E}_* = \big(\mathcal{E}, \{{}^i\mathcal{F} \,\big|\, i \in S\}\big)$ be a $\boldsymbol{c}$-parabolic sheaf on (X, D) for some $\boldsymbol{c} \in \boldsymbol{R}^S$. A flat (logarithmic) λ-connection $\mathbb{D}^{\lambda}$ of $\mathcal{E}_*$ is defined to be a linear map $\mathbb{D}^{\lambda} : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^{1,0}(\log D)$ satisfying the same twisted Leibniz rule as above, the flatness $\mathbb{D}^{\lambda} \circ \mathbb{D}^{\lambda} = 0$ and $\mathbb{D}^{\lambda}({}^i\mathcal{F}_a) \subset {}^i\mathcal{F}_a \otimes \Omega^{1,0}(\log D)$. Such a tuple $(\mathcal{E}_*, \mathbb{D}^{\lambda})$ will be called a parabolic λ-flat sheaf. When the underlying $\boldsymbol{c}$-parabolic sheaf $\mathcal{E}_*$ is a $\boldsymbol{c}$-parabolic bundle in codimension k, it is called a λ-flat $\boldsymbol{c}$-parabolic bundle in codimension k.

Let $\boldsymbol{E}_* = \big(\boldsymbol{E}, \{{}_{\boldsymbol{c}}E\} \,\big|\, \boldsymbol{c} \in \boldsymbol{R}^S\big)$ be a filtered sheaf on (X, D). A regular λ-connection of $\boldsymbol{E}_*$ is a λ-connection $\mathbb{D}^{\lambda}$ of $\boldsymbol{E}$ satisfying $\mathbb{D}^{\lambda}\big({}_{\boldsymbol{c}}E\big) \subset {}_{\boldsymbol{c}}E \otimes \Omega^{1,0}_X(\log D)$. A tuple $(\boldsymbol{E}_*, \mathbb{D}^{\lambda})$ is called a regular filtered λ-flat sheaf. When the underlying filtered sheaf is a filtered bundle in codimension k, it is called a regular filtered λ-flat bundle in codimension k.

Lemma 2.1 *A regular filtered sheaf on (X, D) is a regular filtered λ-flat bundle in codimension one.*

Proof We have only to check that there exists a subset $W \subset D$ with $\operatorname{codim}_X(W) \geq 2$, such that ${}_{\boldsymbol{c}}E_{*\,|\,X \setminus W}$ is a $\boldsymbol{c}$-parabolic bundle on $(X \setminus W, D \setminus W)$ for some $\boldsymbol{c}$. We can take W as $\bigcup_{i \neq j} D_i \cap D_j \subset W$, and hence we may assume D is smooth. Since $E = \boldsymbol{E}_{|X-D}$ is locally free and ${}_{\boldsymbol{c}}E$ is torsion-free, we can take $W' \subset D$ with $\operatorname{codim}_X(W') \geq 2$ such that ${}_{\boldsymbol{c}}E_{|X-W'}$ is locally free. We may also take a subset $W'' \subset D \setminus W'$ with $\operatorname{codim}_X(W'') \geq 2$ such that the parabolic filtration of ${}_{\boldsymbol{c}}E_{|D \setminus (W' \cup W'')}$ is filtration in the category of vector bundles. Then $W = W' \cup W''$ gives the desired subset. ∎

When X is an n-dimensional projective variety with an ample line bundle L, we can define the μ-stability, μ-semistability, and μ-polystability of regular filtered λ-flat sheaves with respect to L, in the standard manner. "μ-stability with respect to L" will be called μ_L-stability, in this paper.

2.1.2 KMS-structure

Let X be a complex manifold, and let D be a simple normal crossing divisor with the irreducible decomposition $D=\bigcup_{i\in S}D_i$. Let $(\boldsymbol{E}_*,\mathbb{D}^\lambda)$ be a regular filtered λ-flat bundle in codimension one over (X,D). For simplicity, we consider only the case $\lambda\neq 0$. Let us take any element $\boldsymbol{c}\in\boldsymbol{R}^S$, and the $\boldsymbol{c}$-truncation ${}_{\boldsymbol{c}}E_*$ of $\boldsymbol{E}_*$. We would like to recall the KMS-structure at D_i, or more precisely, at the generic point of D_i. We may assume that $({}_{\boldsymbol{c}}E_*,\mathbb{D}^\lambda)$ is a $\boldsymbol{c}$-parabolic bundle. We have the induced filtration ${}^iF$ on ${}_{\boldsymbol{c}}E_{|D_i}$, which induces the associated graded bundle:

$$ {}^i\mathrm{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})=\bigoplus_{c_i-1<a\leq c_i}{}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E_{|D_i}). $$

Recall that we use the notation $\mathcal{P}ar({}_{\boldsymbol{c}}E_*,i):=\bigl\{a\,\big|\,c_i-1<a\leq c_i,\ {}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E_{|D_i})\neq 0\bigr\}$ and $\mathcal{P}ar(\boldsymbol{E}_*,i):=\bigcup_{\boldsymbol{c}\in\boldsymbol{R}^S}\mathcal{P}ar({}_{\boldsymbol{c}}E_*,i)$. Due to the regularity, we have the residue endomorphism $\mathrm{Res}_i(\mathbb{D}^\lambda)$ on ${}_{\boldsymbol{c}}E_{|D_i}$, which preserves the filtration ${}^iF$, and hence we have the induced endomorphism $\mathrm{Gr}^F\mathrm{Res}_i(\mathbb{D}^\lambda)$ of ${}^i\mathrm{Gr}^F\bigl({}_{\boldsymbol{c}}E_{|D_i}\bigr)$. We remark that the eigenvalues of $\mathrm{Res}_i(\mathbb{D}^\lambda)$ are constant on D_i. In particular, we obtain the generalized eigen decomposition:

$$ {}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E_{|D_i})=\bigoplus_{\alpha\in\boldsymbol{C}}{}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E_{|D_i}). $$

We put $\mathcal{KMS}\bigl({}_{\boldsymbol{c}}E_*,i\bigr):=\bigl\{(a,\alpha)\in]c_i-1,c_i]\times\boldsymbol{R}\,\big|\,{}^i\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E_{|D_i})\neq 0\bigr\}$. Any elements of $\mathcal{KMS}\bigl({}_{\boldsymbol{c}}E_*,i\bigr)$ or $\mathcal{KMS}\bigl(\boldsymbol{E}_*,i\bigr):=\bigcup_{\boldsymbol{c}\in\boldsymbol{R}^S}\mathcal{KMS}\bigl({}_{\boldsymbol{c}}E_*,i\bigr)$ are called a KMS-spectrum.

2.1.3 Prolongment of flat subbundle and Mehta-Ramanathan type theorem

To begin with, we recall a well known fact about regular singularity of a connection.

Lemma 2.2 *Let E be a holomorphic bundle on a disc Δ, and let ∇ be a logarithmic connection of E on (Δ,O), i.e., $\nabla(E)\subset E\otimes\Omega^{1,0}(\log O)$. Let f be a flat section of $E_{|\Delta^*}$. Then f naturally gives a meromorphic section of E.* ∎

Corollary 2.3 *We put $X=\Delta_z\times\Delta_w^n$ and $D=\{0\}\times\Delta_w^n$. Let E be a holomorphic vector bundle on X and ∇ be the logarithmic connection of E on (X,D). Let e be a flat section of $E_{|X-D}$.*

- *e gives a meromorphic section of E.*
- *Assume that e is holomorphic on E and that $e_{|Q}\neq 0$ for some $Q\in D$. Then $e_{|Q'}\neq 0$ for any $Q'\in D$.*

Proof We may assume that we have a holomorphic frame $\boldsymbol{v}$ of E. We have the expression $e=\sum f_i(z,w)\cdot v_i$. When we fix w, then $f_i(z,w)$ are meromorphic with respect to z. Thus we have the least integer $j(w)$ such that the orders of the poles of $f_i(z,w)$ are less than $j(w)$. We put $\mathcal{S}_j:=\{w\,|\,j(w)\leq j\}$. We have $D=\bigcup_j\mathcal{S}_j$. If $\mathcal{S}_j\neq D$, the measure of $\mathcal{S}_j$ is 0. Hence we obtain $\mathcal{S}_j=D$ for some j, which means e is meromorphic. Thus we obtain the first claim.

Assume that e is holomorphic and that $e_{|Q}\neq 0$ for some $Q\in D$. Recall that we have the induced connection ${}^D\nabla$ of $E_{|D}$. Namely, for any holomorphic section $f\in E_{|D}$, take a holomorphic $F\in E$ such that $F_{|D}=f$, and then ${}^D\nabla(f):=\nabla(F)_{|D}$ is well defined. Since we have ${}^D\nabla(e_{|D})=0$, we obtain the second claim. ∎

Corollary 2.4 *We put $X=\Delta^n$, $D_i=\{z_i=0\}$ and $D=\bigcup_{i=1}^n D_i$. Let (E,∇) be a logarithmic connection on (X,D), and let e be a flat section on $X-D$.*

- *e gives a meromorphic section of E.*
- *Assume that e is holomorphic. We put $D_i^\circ:=D_i\setminus\bigcup_{j\neq i}D_j$. If $e_{|Q}\neq 0$ for some $Q\in D_i^\circ$, we have $e_{|Q'}\neq 0$ for any $Q'\in D_i^\circ$.* ∎

Let X be a complex manifold, and let D be a normal crossing divisor of X. Let (E,∇) be a flat bundle on $X-D$. Recall that P. Deligne gave the extension $\widetilde{E}$ of E with the properties: (i) $\widetilde{E}_{|X-D}=E$, (ii) $\nabla(\widetilde{E})\subset\widetilde{E}\otimes\Omega^{1,0}(\log D)$ (iii) the real parts of the eigenvalues of $\mathrm{Res}_i(\nabla)$ are contained in $\{0\leq t<1\}$ ([3]). Such an extension is unique, or in other words, it is unique as the subsheaf of ι_*E, where ι denotes the inclusion $X-D\longrightarrow X$. The prolongment can also be done for λ-flat bundle $(E,\mathbb{D}^{\lambda})$ on $X-D$, or more precisely, for the associated flat bundle $(E,\mathbb{D}^{\lambda\,f})$.

Lemma 2.5 *Let $(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ be a regular filtered λ-flat bundle on (X,D), and we put $(E,\mathbb{D}^{\lambda}):=(\boldsymbol{E}_*,\mathbb{D}^{\lambda})_{|X-D}$. Let $(\widetilde{E},\mathbb{D}^{\lambda})$ be the Deligne extension of $(E,\mathbb{D}^{\lambda})$. Then we have $\boldsymbol{E}=\widetilde{E}\otimes\mathcal{O}_X(*D)$, where $\mathcal{O}_X(*D)$ denotes the sheaf of meromorphic functions on X whose poles are contained in D.*

Proof We have the naturally defined flat section s on $Hom({}_{\boldsymbol{c}}E,\widetilde{E})_{|X-D}$. Due to Corollary 2.4, s is a meromorphic section, and hence we obtain the flat inclusion ${}_{\boldsymbol{c}}E\longrightarrow\widetilde{E}\otimes\mathcal{O}(N\cdot D)$ for some large integer N, which induce the morphism $\boldsymbol{E}=\bigcup{}_{\boldsymbol{c}}E={}_{\boldsymbol{c}}E\otimes\mathcal{O}(*D)\longrightarrow\widetilde{E}\otimes\mathcal{O}(*D)$. Similarly, we obtain the inclusion $\widetilde{E}\longrightarrow{}_{\boldsymbol{c}}E\otimes\mathcal{O}(N\cdot D)$, and $\widetilde{E}\otimes\mathcal{O}(*D)\longrightarrow\boldsymbol{E}$. They are clearly mutually inverse. ∎

Lemma 2.6 *Let $(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ be a regular filtered λ-flat sheaf on (X,D), and let $(\widetilde{E},\mathbb{D}^{\lambda})$ be in the previous lemma. Then we have $\boldsymbol{E}\simeq\widetilde{E}\otimes\mathcal{O}(*D)$ naturally.*

Proof Due to Lemma 2.1 and Lemma 2.5, there exists a subset $W\subset D$ with $\mathrm{codim}_X(W)\geq 2$ such that $\boldsymbol{E}_{|X-W}\simeq\widetilde{E}\otimes\mathcal{O}(*D)_{|X-W}$. Let us fix $\boldsymbol{c}$. There exists a large integer N such that we have ${}_{\boldsymbol{c}}E_{|X-W}\subset\widetilde{E}\otimes\mathcal{O}(N\cdot D)_{|X-W}$. Since $\widetilde{E}$ is locally free, we obtain ${}_{\boldsymbol{c}}E\subset\widetilde{E}\otimes\mathcal{O}(N\cdot D)$, and thus $\boldsymbol{E}\subset\widetilde{E}\otimes\mathcal{O}(*D)$. On the other hand, there exists a large integer N' such that $\widetilde{E}_{|X-W}\subset{}_{\boldsymbol{c}}E\otimes\mathcal{O}(N'\cdot D)_{|X-W}$. Hence $\widetilde{E}\subset{}_{\boldsymbol{c}}E^{\vee\vee}\otimes\mathcal{O}(N'\cdot D)$, where ${}_{\boldsymbol{c}}E^{\vee\vee}$ denotes the double dual of ${}_{\boldsymbol{c}}E$. Hence we obtain $\widetilde{E}\otimes\mathcal{O}(*D)\subset{}_{\boldsymbol{c}}E^{\vee\vee}\otimes\mathcal{O}(*D)$. It is easy to see ${}_{\boldsymbol{c}}E^{\vee\vee}\otimes\mathcal{O}(*D)\simeq{}_{\boldsymbol{c}}E\otimes\mathcal{O}(*D)$. Thus we are done. ∎

Lemma 2.7 *Let $(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ be a regular filtered λ-flat sheaf on (X,D), and we put $(E,\mathbb{D}^{\lambda}):=(\boldsymbol{E}_*,\mathbb{D}^{\lambda})_{|X-D}$. Let E' be a λ-flat subbundle of E. Then we have the corresponding regular filtered λ-flat subsheaf $\boldsymbol{E}'_*\subset\boldsymbol{E}_*$ such that ${}_{\boldsymbol{c}}E'$ are saturated in ${}_{\boldsymbol{c}}E$.*

Proof Let $\widetilde{E}$ denote the Deligne extension of $(E,\mathbb{D}^{\lambda})$. We have the corresponding subbundle $\widetilde{E}'\subset\widetilde{E}$. Therefore, we obtain $\widetilde{\boldsymbol{E}}':=\widetilde{E}'\otimes\mathcal{O}(*D)\subset\widetilde{E}\otimes\mathcal{O}(*D)=\boldsymbol{E}$. For each $\boldsymbol{c}$, the $\boldsymbol{c}$-truncation ${}_{\boldsymbol{c}}E'$ is given by the intersection of ${}_{\boldsymbol{c}}E$ and $\boldsymbol{E}'$ in $\boldsymbol{E}$. Or equivalently, ${}_{\boldsymbol{c}}E'$ can be given by the intersection of ${}_{\boldsymbol{c}}E$ and $\widetilde{E'}(N\cdot D)$ in $\widetilde{E}(N\cdot D)$ for sufficiently large N. Thus we obtain $\boldsymbol{E}'_*\subset\boldsymbol{E}_*$. ∎

Let us show the Mehta-Ramanathan type theorem for regular filtered λ-flat sheaves. Let X be a smooth projective variety with an ample bundle L and a simple normal crossing divisor D. Let $(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ be a regular filtered λ-flat sheaf on (X,D). Let N be a sufficiently large number. We can take a generic hyper-plane section Y of $L^{\otimes N}$ satisfying the properties: (i) $Y\cap D$ is normal crossing, (ii) $\pi_1(Y\setminus D)\longrightarrow\pi_1(X\setminus D)$ is surjective.

Proposition 2.8 *Assume $\dim X\geq 2$. $(\boldsymbol{E}_*,\mathbb{D}^{\lambda})$ is μ_L-stable, if and only if $(\boldsymbol{E}_*,\mathbb{D}^{\lambda})_{|Y}$ is μ_L-stable.*

Proof Let us fix $\boldsymbol{c}$. If $W\subset{}_{\boldsymbol{c}}E$ destabilizes, the restriction $W_{|Y}$ clearly destabilizes. Hence the μ_L-stability of $({}_{\boldsymbol{c}}E_*,\mathbb{D}^{\lambda})_{|Y}$ implies the μ_L-stability of $({}_{\boldsymbol{c}}E_*,\mathbb{D}^{\lambda})$. Assume that $({}_{\boldsymbol{c}}E_*,\mathbb{D}^{\lambda})$ is not μ_L-stable, and let W be a subsheaf of ${}_{\boldsymbol{c}}E$ such that $\mathbb{D}^{\lambda}(W)\subset W\otimes\Omega^{1,0}(\log D)$ and that $\text{par-deg}(W_*)/\operatorname{rank}(W)\geq\text{par-deg}({}_{\boldsymbol{c}}E_*)/\operatorname{rank}E$. Let Q be any point of $X-D$. Take a path γ connecting Q and a point P of $Y\setminus D$. By the parallel transport along the path, we obtain the vector subspace $W'_Q\subset E_{|Q}$. It is independent of choices of P and γ, and we obtain the flat subbundle $W'\subset{}_{\boldsymbol{c}}E_{|X-D}$. Due to Lemma 2.7, we obtain the saturated subsheaf $\widetilde{W}'\subset{}_{\boldsymbol{c}}E$. By a general argument, it can be shown that there exists a subset $Z\subset D$ with $\mathrm{codim}_X(Z)\geq 2$ such that $\widetilde{W}'_{*|X-Z}$ is a parabolic subbundle of ${}_{\boldsymbol{c}}E_{|X-Z}$. Then it is easy to check $\widetilde{W}'$ destabilizes. ∎

2.1.4 Saturated regular filtered λ-flat sheaf

Let X and D be as above. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat sheaf $(\lambda \neq 0)$.

Definition 2.9 $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ *is called saturated, if the following conditions are satisfied:*

- *There exists a subset $Z \subset D$ with $\operatorname{codim}_X(Z) \geq 2$, and each ${}_{\boldsymbol{a}}E$ are determined on ${}_{\boldsymbol{a}}E_{|X-Z}$. Namely, for any open subset $U \subset X$, we have the following:*

$$ {}_{\boldsymbol{a}}E(U) = {}_{\boldsymbol{a}}E(U \setminus Z) \cap \boldsymbol{E}(U). \tag{4} $$

∎

It is easy to see that a regular filtered λ-flat *bundle* is saturated.

Lemma 2.10 *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a saturated regular filtered λ-sheaf on (X, D). Then each $\boldsymbol{c}$-truncation ${}_{\boldsymbol{c}}E$ is reflexive.*

Proof Recall we have already known that ${}_{\boldsymbol{c}}\boldsymbol{E}_*$ is a filtered bundle in codimension one (Lemma 2.1). Let ${}_{\boldsymbol{c}}E^{\vee\vee}$ denote the double dual of ${}_{\boldsymbol{c}}E$. We have the naturally defined injective map ${}_{\boldsymbol{c}}E \longrightarrow {}_{\boldsymbol{c}}E^{\vee\vee}$. Due to the saturatedness, any sections of ${}_{\boldsymbol{c}}E^{\vee\vee}$ naturally gives sections of ${}_{\boldsymbol{c}}E$, i.e., ${}_{\boldsymbol{c}}E$ is isomorphic to ${}_{\boldsymbol{c}}E^{\vee\vee}$. ∎

Lemma 2.11 *A saturated regular filtered λ-flat sheaf $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ on (X, D) is a regular filtered λ-flat bundle in codimension two.*

Proof We have only to show that there exists a subset $Z \subset D$ with $\operatorname{codim}_X(Z) \geq 3$ such that ${}_{\boldsymbol{c}}E_{*\,|\,X-Z}$ is a $\boldsymbol{c}$-parabolic bundle on $(X - Z, D - Z)$ for any $\boldsymbol{c}$. Due to ${}_{\boldsymbol{c}+\boldsymbol{b}}E = {}_{\boldsymbol{c}}E \otimes \mathcal{O}(\boldsymbol{b} \cdot D)$, where $\boldsymbol{b} \cdot D = \sum_{i \in S} b_i \cdot D_i$, we have only to show such a claim for finite number of tuples $\boldsymbol{c}$. Due to Lemma 2.10, there exists a subset $Z' \subset D$ with $\operatorname{codim}_X(Z') \geq 3$ such that ${}_{\boldsymbol{c}}E_{|X-Z'}$ is locally free. Hence we can assume that ${}_{\boldsymbol{c}}E$ is locally free from the beginning.

We have the parabolic filtration ${}^iF = \{{}^iF_a \,|\, c_i - 1 < a \leq c_i\}$ of ${}_{\boldsymbol{c}}E_{|D_i}$. We can take the saturation ${}^i\widetilde{F}_a$ of ${}^iF_a$. Namely, we put $G_a := {}_{\boldsymbol{c}}E_{|D_i} / {}^iF_a$, and let $G_{a\,tor}$ denote the torsion-part of G_a. Let $\pi_a : {}_{\boldsymbol{c}}E_{|D_i} \longrightarrow G_a$ denote the projection, and we put ${}^i\widetilde{F}_a := \pi_a^{-1}(G_{a\,tor})$.

Lemma 2.12 ${}^i\widetilde{F}_a = {}^iF_a$.

Proof By our construction, we have ${}^iF_a \subset {}^i\widetilde{F}_a$, and we also know that there exists a subset $W \subset D_i$ with $\operatorname{codim}_{D_i}(W) \geq 1$ such that ${}^iF_{a\,|\,D_i - W} = {}^i\widetilde{F}_{a\,|\,D_i - W}$.

Let P be any point of D_i. Let g be a germ of a section of ${}^i\widetilde{F}_a$ at P, and let G be a local section of ${}_{\boldsymbol{c}}E$ on an open subset $U$of P in X such that the germ of the restriction of G to D_i gives g. Then $G_{|U \setminus W}$ gives a section of ${}_{\boldsymbol{c}'}E$ on $U \setminus W$, where $\boldsymbol{c}' = (c'_j)$ is determined by $c'_j = c_j$ $(j \neq i)$ and $c_i = a$. Due to the saturatedness, G is a section of ${}_{\boldsymbol{c}'}E$ on U. Thus g is the germ of a section of ${}^iF_a$, and ${}^iF_a = {}^i\widetilde{F}_a$. Hence we obtain Lemma 2.12. ∎

Let us return to the proof of Lemma 2.11. Due to Lemma 2.12, the associated graded vector bundle ${}^i\operatorname{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})$ is torsion free. Hence there exists a subset $Z''_i \subset D_i$ with $\operatorname{codim}_{D_i} Z''_i \geq 2$ such that ${}^iF_{|D_i \setminus Z''_i}$ is a filtration in the category of vector bundles on $D''_i \setminus Z''_i$. Then ${}_{\boldsymbol{c}}E_{*\,|\,X - Z''}$ is a $\boldsymbol{c}$-parabolic locally free sheaf on $(X - Z'', D - Z'')$. Thus we are done. ∎

Remark 2.13 *By the correspondence of saturated regular filtered flat bundles and filtered local systems, we can obtain more concrete picture of the saturated regular filtered flat sheaves. We will see it in the section* 6. ∎

2.1.5 Perturbation of parabolic structure

Let X be a smooth projective *surface* with an ample line bundle L, and D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. Let $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a $\boldsymbol{c}$-parabolic λ-flat bundle over (X, D) for some $\boldsymbol{c} \in \boldsymbol{R}^S$. Assume $\lambda \neq 0$. We also assume $c_i \notin \mathcal{P}ar({}_{\boldsymbol{c}}E, \boldsymbol{F}, i)$ for each $i \in S$, for simplicity.

In the subsection 3.4 of [30], we explained how to perturb the parabolic structure $\boldsymbol{F}$ in the Higgs case. The argument clearly works even in the case $\lambda \neq 0$ (Proposition 2.16). However, we need more concrete way of perturbation, which we will explain in the following.

Remark 2.14 *The construction given in this subsection is valid when the base manifold X is a curve.* ∎

Let $\mathcal{N}_i$ denote the nilpotent part of the induced endomorphism $\mathrm{Gr}^F \mathrm{Res}_i(\mathbb{D}^\lambda)$ on ${}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E_{|D_i})$. Before proceeding, we give a definition of graded semisimplicity, as in the Higgs case.

Definition 2.15 *The λ-flat $\boldsymbol{c}$-parabolic bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \mathbb{D}^\lambda)$ is called graded semisimple, if the nilpotent parts $\mathcal{N}_i$ are 0 for any $i \in S$.* ∎

By the argument given in the subsection 3.4 of [30], we can show the following proposition.

Proposition 2.16 *Let ϵ be any sufficiently small positive number. There exists a tuple of the parabolic structure $\boldsymbol{F}^{(\epsilon)} = \bigl({}^iF^{(\epsilon)} \bigm| i \in S\bigr)$ such that the following holds:*

- *$({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)})$ is a graded semisimple $\boldsymbol{c}$-parabolic λ-flat bundle.*
- *We have* $\text{par-deg}_L({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) = \text{par-deg}_L({}_{\boldsymbol{c}}E, \boldsymbol{F})$.
- *There is a constant C, which is independent of ϵ, such that the following holds:*

$$\Bigl|\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) - \int_X \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F})\Bigr| \leq C \cdot \epsilon,$$

$$\Bigl|\int_X \text{par-c}_1^2({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) = \int_X \text{par-c}_1^2({}_{\boldsymbol{c}}E, \boldsymbol{F})\Bigr| \leq C \cdot \epsilon.$$

- $\mathrm{gap}({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) \geq \epsilon / r$. ∎

For later use, we need to take such a perturbation in a more concrete way. Hence, we recall the construction in the following. Let η be a generic point of D_i. We have the weight filtration W_η of the nilpotent map $\mathcal{N}_{i,\eta}$ on ${}^i\mathrm{Gr}^F\bigl({}_{\boldsymbol{c}}E_{|D_i}\bigr)_\eta$, which is indexed by $\mathbb{Z}$. Then we can extend it to the filtration W of ${}^i\mathrm{Gr}^F\bigl({}_{\boldsymbol{c}}E_{|D_i}\bigr)$ in the category of vector bundles on D_i due to $\dim D_i = 1$. By our construction, $\mathcal{N}_i(W_k) \subset W_{k-2}$. The endomorphism $\mathrm{Res}_i(\mathbb{D}^\lambda)$ preserves the filtration W on ${}^i\mathrm{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})$, and the nilpotent part of the induced endomorphisms on $\mathrm{Gr}^W\,{}^i\mathrm{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})$ are trivial. Recall that the flat λ-connection $\mathbb{D}^\lambda$ *locally* induces the λ-connection ${}^i\mathbb{D}^\lambda$ of the vector bundle ${}_{\boldsymbol{c}}E_{|D_i}$ on D_i. Since ${}^i\mathrm{Gr}^F({}^i\mathbb{D}^\lambda)$ commutes with $\mathrm{Res}_i\,\mathbb{D}^\lambda$, it preserves the filtration W.

Let us take the refinement of the filtration ${}^iF$. For any $a \in]c_i - 1, c_i]$, we have the surjection $\pi_a : {}^iF_a({}_{\boldsymbol{c}}E_{|D_i}) \longrightarrow {}^i\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E_{|D_i})$. We put ${}^i\widetilde{F}_{a,k} := \pi_a^{-1}(W_k)$. We use the lexicographic order on $]c_i - 1, c_i] \times \mathbb{Z}$. Thus we obtain the increasing filtration ${}^i\widetilde{F}$ indexed by $]c_i - 1, c_i] \times \mathbb{Z}$. Obviously, the set $\widetilde{S}_i := \bigl\{(a,k) \in]c_i - 1, c_i] \times \mathbb{Z} \bigm| {}^i\mathrm{Gr}^{\widetilde{F}}_{(a,k)} \neq 0\bigr\}$ is finite.

Next, we explain the perturbation of the weight for the parabolic structure. Let ϵ be a small positive number such that $0 < \mathrm{rank}\, E \cdot \epsilon < \mathrm{gap}({}_{\boldsymbol{c}}E, \boldsymbol{F})$. Let us take an increasing map $\varphi_i : \widetilde{S}_i \longrightarrow]c_i - 1, c_i]$ given by $\varphi_i(a,k) = a + \epsilon \cdot k$. Then ${}^i\widetilde{F}$ and φ_i give the $\boldsymbol{c}$-parabolic filtration $\boldsymbol{F}^{(\epsilon)} = \bigl({}^iF^{(\epsilon)} \bigm| i \in S\bigr)$. Thus we obtain the $\boldsymbol{c}$-parabolic λ-flat bundle $\bigl({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda\bigr)$, which are called the ϵ-perturbation of $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \mathbb{D}^\lambda)$.

The following proposition is standard. (See Proposition 3.3 of [30], for example.)

Proposition 2.17 *Assume that $\bigl({}_{\boldsymbol{c}}E, \boldsymbol{F}, \mathbb{D}^\lambda\bigr)$ is μ_L-stable. If ϵ is sufficiently small, then the ϵ-perturbation $\bigl({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda\bigr)$ is also μ_L-stable.* ∎

2.1.6 Remarks about the terminology and the notation

We give some remarks about the terminology "parabolic structure". Let X be a complex manifold, and let D be a simple normal crossing divisor of X with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. We often discuss a $\boldsymbol{c}$-parabolic λ-flat bundle on (X, D) for some $\boldsymbol{c} \in \boldsymbol{R}^S$. In our most arguments, a choice of $\boldsymbol{c}$ are not relevant. In fact, $\boldsymbol{c}$ is fixed to be $(0, \ldots, 0)$ in many references where the parabolic structure is discussed. But, it is sometimes convenient to avoid the case $c_i \in \mathcal{P}ar({}_{\boldsymbol{c}}E_*, i)$, for example, when we consider a perturbation of the parabolic structure. That is the main reason why we consider general $\boldsymbol{c}$-parabolic structure.

In the following argument, we implicitly assume $c_i \notin \mathcal{P}ar({}_{\boldsymbol{c}}E_*, i)$, and we often omit to distinguish $\boldsymbol{c}$, and use the terminology "parabolic structure" instead of "$\boldsymbol{c}$-parabolic structure", when we do not have to care about a choice of $\boldsymbol{c}$. The author hopes that there will be no confusion and that it will reduce unnecessary complexity of the description.

Relatedly we have the remark about the notation to denote parabolic bundles. We often use the notation $({}_{\boldsymbol{c}}E, \boldsymbol{F})$ or ${}_{\boldsymbol{c}}E_*$ to denote a $\boldsymbol{c}$-parabolic bundle, when we would like to distinguish $\boldsymbol{c}$. The notation "${}_{\boldsymbol{c}}E$" is also appropriate and useful, when we regard it as a prolongment of the locally free sheaf E on $X - D$. But, in most part of this paper, a vector bundle is given not only on $X - D$ but also on X from the beginning. And, as is said above, we will not care about a choice of $\boldsymbol{c}$. Therefore, we often prefer to use the notation $(E, \boldsymbol{F})$ or E_* for simplicity.

2.2 Generality for λ-connection in the C^∞-category

We will give some generality for λ-connections. They are straightforward generalization of the argument for Higgs bundles or flat bundles given in Simpson's papers (for example [34] and [36]), and hence we will often omit to give a detailed proof. For simplicity, we will assume $\lambda \neq 0$.

2.2.1 The induced operators

Let X be a complex manifold, and $(E, \mathbb{D}^\lambda)$ be a flat λ-connection on X. We have the decomposition of $\mathbb{D}^\lambda$ into the $(0,1)$-part d''_E and the $(1,0)$-part d'_E. The holomorphic structure of E is given by d''_E. Recall that the twisted Leibniz rule $d'_E(f \cdot v) = \lambda \cdot \partial_X(f) v + f \cdot d'_E v$ holds for $f \in C^\infty(X)$ and $v \in C^\infty(X, E)$. Let h be a hermitian metric of E. From d''_E and h, we obtain the $(1,0)$-operator $\delta'_{E,h}$ determined by $\overline{\partial} h(u, v) = h(d''_E u, v) + h(u, \delta'_{E,h} v)$. From d'_E and h, we obtain the $(0,1)$-operator $\delta''_{E,h}$ determined by $\lambda \partial h(u, v) = h(d'_E u, v) + h(u, \delta''_{E,h} v)$. We remark $\delta''_{E,h}(f \cdot v) = \overline{\lambda} \cdot \overline{\partial}_X f \cdot v + f \cdot \delta''_{E,h}(v)$. We obtain the following operators:

$$\begin{aligned} \overline{\partial}_{E,h} &:= \frac{1}{1+|\lambda|^2}(d''_E + \lambda \delta''_{E,h}), \quad \partial_{E,h} := \frac{1}{1+|\lambda|^2}(\overline{\lambda} d'_E + \delta'_{E,h}), \\ \theta^\dagger_{E,h} &:= \frac{1}{1+|\lambda|^2}(\overline{\lambda} d''_E - \delta''_{E,h}), \quad \theta_{E,h} := \frac{1}{1+|\lambda|^2}(d'_E - \lambda \delta'_{E,h}). \end{aligned} \tag{5}$$

It is easy to see that the following Leibniz rule holds:

$$\overline{\partial}_{E,h}(fs) = \overline{\partial}_X f \cdot s + f \cdot \overline{\partial}_{E,h} s, \quad \partial_{E,h}(fs) = \partial_X f \cdot s + f \cdot \partial_{E,h} s.$$

On the other hand, θ and $\theta^\dagger$ give the sections of $\mathrm{End}(E) \otimes \Omega^{1,0}$ and $\mathrm{End}(E) \otimes \Omega^{0,1}$ respectively. We also have the formulas:

$$d''_E = \overline{\partial}_{E,h} + \lambda \theta^\dagger_{E,h}, \quad d'_E = \lambda \partial_{E,h} + \theta_{E,h}, \quad \delta'_{E,h} = \partial_{E,h} - \overline{\lambda} \theta_{E,h}, \quad \delta''_{E,h} = \lambda \overline{\partial}_{E,h} - \theta^\dagger_{E,h}.$$

Remark 2.18 *The index "E, h" is attached to emphasize the bundle E and the metric h. We will often omit them if there are no confusion.* ▮

We put $\mathbb{D}^{\lambda \star}_h := \delta'_h - \delta''_h = \partial_h + \theta^\dagger_h - \overline{\lambda}(\overline{\partial}_h + \theta_h)$. We have the following formula:

$$\overline{\partial}_h + \theta_h = \frac{\mathbb{D}^\lambda - \lambda \mathbb{D}^{\lambda \star}_h}{1+|\lambda|^2}, \quad \partial_h + \theta^\dagger_h = \frac{\mathbb{D}^{\lambda \star}_h + \overline{\lambda} \mathbb{D}^\lambda}{1+|\lambda|^2}.$$

We recall that h is called a pluri-harmonic metric if $(\overline{\partial}_h+\theta_h)^2=0$ holds, i.e., $(E,\overline{\partial}_h,\theta_h)$ is a Higgs bundle. The condition is equivalent to $\big[\mathbb{D}^{\lambda},\mathbb{D}_h^{\lambda\star}\big]=0$.

Let us consider the case where X is provided with a Kahler form ω. For a differential operator A of $E\otimes\Omega^{\cdot}$ of degree one, i.e., $A:C^{\infty}(X,E\otimes\Omega^i)\longrightarrow C^{\infty}(X,E\otimes\Omega^{i+1})$, let A^* denote a formal adjoint with respect to ω and h, i.e., $\int_X(Au,v)_{h,\omega}\,\mathrm{dvol}_{\omega}=\int_X(u,A^*v)_{h,\omega}\,\mathrm{dvol}_{\omega}$ hold for any C^{∞}-sections u and v with compact supports. Here $(\cdot,\cdot)_{h,\omega}$ denotes the Hermitiann inner product of appropriate vector bundles induced by h and ω.

Lemma 2.19 $\big(\mathbb{D}^{\lambda\star}\big)^*=\sqrt{-1}\big[\Lambda_{\omega},\mathbb{D}^{\lambda}\big]$ *and* $\big(\mathbb{D}^{\lambda}\big)^*=-\sqrt{-1}\big[\Lambda_{\omega},\mathbb{D}^{\lambda\star}\big]$.

Proof It follows from the relations $\partial^*=\sqrt{-1}[\Lambda_{\omega},\overline{\partial}_E]$, $\overline{\partial}^*=-\sqrt{-1}[\Lambda_{\omega},\partial_E]$, $\theta^*=-\sqrt{-1}[\Lambda_{\omega},\theta^{\dagger}]$ and $(\theta^{\dagger})^*=\sqrt{-1}[\Lambda_{\omega},\theta]$. ∎

The Laplacian $\Delta^{\lambda}_{h,\omega}:C^{\infty}(X,E)\longrightarrow C^{\infty}(X,E)$ is defined by $\Delta^{\lambda}_{h,\omega}:=\sqrt{-1}\Lambda_{\omega}\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}$.

Remark 2.20 *For the differential operators of functions,* $\Delta^{\lambda}_{\omega}:=\sqrt{-1}\Lambda(\overline{\partial}+\lambda\partial)\circ(\partial-\overline{\lambda\partial})=(1+|\lambda|^2)\sqrt{-1}\Lambda\overline{\partial}\partial=(1+|\lambda|^2)\Delta''_{\omega}$, *where* Δ''_{ω} *denotes the usual Laplacian* $\sqrt{-1}\Lambda_{\omega}\overline{\partial}\partial$. ∎

Lemma 2.21 *When* $\lambda\neq 0$, *we have* $\overline{\lambda}^{-1}\partial_h^2+\lambda^{-1}\theta_h^2=0$ *and* $\lambda^{-1}\overline{\partial}_h^2+\overline{\lambda}^{-1}(\theta_h^{\dagger})^2=0$.

Proof From the flatness $(\mathbb{D}^{\lambda})^2=0$, we obtain the following formulas:

$$(\overline{\partial}_h+\lambda\theta_h^{\dagger})^2=\overline{\partial}_h^2+\lambda\overline{\partial}_h\theta_h^{\dagger}+\lambda^2(\theta_h^{\dagger})^2=0,\tag{6}$$

$$(\lambda\partial_h+\theta_h)^2=\lambda^2\partial_h^2+\lambda\partial_h\theta_h+\theta_h^2=0,\tag{7}$$

$$\big[\overline{\partial}_h+\lambda\theta_h^{\dagger},\,\lambda\partial_h+\theta_h\big]=\lambda\Big(\big[\overline{\partial}_h\,,\partial_h\big]+\big[\theta_h^{\dagger},\theta_h\big]\Big)+\overline{\partial}_h\theta_h+\lambda^2\partial_h\theta_h^{\dagger}=0.\tag{8}$$

For a section A of $\mathrm{End}(E)\otimes\Omega^{p,q}$, let $A_h^{\dagger}$ denote the section of $\mathrm{End}(E)\otimes\Omega^{q,p}$ which is the adjoint of A with respect to h in the sense $\big(A\cdot u,v\big)_h=\big(u,A_h^{\dagger}v\big)_h$. Here $(\cdot,\cdot)_h$ denotes the hermitian product $\big(E\otimes\Omega^{\cdot}\big)\otimes\big(E\otimes\Omega^{\cdot}\big)\longrightarrow\Omega^{\cdot}$ induced by h. Then it is easy to see $(\overline{\partial}_h^2)_h^{\dagger}=-\partial_h^2$, $(\overline{\partial}_h\theta_h^{\dagger})^{\dagger}=\partial_h\theta_h$ and $(\theta_h^{\dagger})^2=-(\theta_h^2)^{\dagger}$. Therefore we obtain the following equality from (6):

$$-\partial_h^2+\overline{\lambda}\big(\partial_h\theta_h\big)-\overline{\lambda}^2\theta_h^2=0.\tag{9}$$

From (7) and (9), we obtain $\big(\lambda+\overline{\lambda}^{-1}\big)\partial_h^2+\big(\lambda^{-1}+\overline{\lambda}\big)\theta_h^2=(1+|\lambda|^2)\big(\overline{\lambda}^{-1}\partial_h^2+\lambda^{-1}\theta_h^2\big)=0$, which gives the first formula in the lemma. The second formula can be obtained by taking the adjoint. ∎

Lemma 2.22 *When* $\lambda\neq 0$, *we have* $\overline{\lambda}^{-1}\cdot\partial_h\theta_h^{\dagger}+\lambda^{-1}\cdot\overline{\partial}_h\theta_h=0$ *and* $\big[\partial_h\,,\overline{\partial}_h\big]+\big[\theta_h\,,\theta_h^{\dagger}\big]=0$.

Proof It is easy to check $[\partial_h,\overline{\partial}_h]_h^{\dagger}=-[\partial_h,\overline{\partial}_h]$, $[\theta_h,\theta_h^{\dagger}]_h^{\dagger}=-[\theta_h,\theta_h^{\dagger}]$ and $(\overline{\partial}_h\theta_h)_h^{\dagger}=\partial_h\theta_h^{\dagger}$. Hence we obtain the following equality from (8):

$$-[\overline{\partial}_h,\partial_h]-[\theta_h^{\dagger},\theta_h]+\overline{\lambda}^{-1}\cdot\partial_h\theta_h^{\dagger}+\overline{\lambda}\cdot\overline{\partial}_h\theta_h=0.\tag{10}$$

The claim of the lemma immediately follows from (8) and (10). ∎

Corollary 2.23 *When* $\lambda\neq 0$, *the pluri-harmonicity of the metric* h *is equivalent to the vanishings* $\theta_h^2=0$ *and* $\overline{\partial}_h\theta_h=0$. ∎

2.2.2 Local expression

Let $(E,\mathbb{D}^\lambda)$ be a flat λ-connection, and let h be a C^∞-metric. Let $\boldsymbol{v}=(v_1,\ldots,v_r)$ be a holomorphic frame of E. Let $H=H(h,\boldsymbol{v})$ denote the hermitian matrix valued function of h with respect to $\boldsymbol{v}$, i.e., $H_{i,j}=h(v_i,v_j)$. Let us see the local expression of the induced operators.

Let A denote the $M(r)$-valued $(1,0)$-form of $\mathbb{D}^\lambda$ with respect to $\boldsymbol{v}$, i.e., $\mathbb{D}^\lambda\boldsymbol{v}=\boldsymbol{v}\cdot A$, in other words, $\mathbb{D}^\lambda v_i=\sum A_{j\,i}\cdot v_j$. Let B denote the $(1,0)$-form of δ'_h with respect to $\boldsymbol{v}$, i.e., $\delta'_h\boldsymbol{v}=\boldsymbol{v}\cdot B$, and then we have $\overline{\partial}h(v_i,v_j)=h\bigl(v_i,\delta'_hv_j\bigr)=\sum h\bigl(v_i,B_{k,j}v_k\bigr)$. Hence $\overline{\partial}H=H\cdot\overline{B}$, i.e., we obtain $B=\overline{H}^{-1}\partial\overline{H}$. Let C denote the $(0,1)$-form of δ''_h with respect to $\boldsymbol{v}$, i.e., $\delta''_h\boldsymbol{v}=\boldsymbol{v}\cdot C$, and then we have $\lambda\cdot\partial h(v_i,v_j)=h(d'v_i,v_j)+h(v_i,\delta''_hv_j)=\sum_k h(A_{k,i}v_k,\,v_j)+\sum_k h(v_i,\,C_{k,j}v_k)$. Hence $\lambda\partial H={}^tAH+H\overline{C}$, i.e., we obtain $C=\overline{\lambda}\cdot\overline{H}^{-1}\overline{\partial H}-\overline{H}^{-1}\overline{{}^tAH}$. Thus we obtain the following:

$$\theta_h\boldsymbol{v}=\boldsymbol{v}\cdot\frac{1}{1+|\lambda|^2}\bigl(A-\overline{H}^{-1}\partial\overline{H}\bigr),\quad \overline{\partial}_h\boldsymbol{v}=\boldsymbol{v}\cdot\frac{\lambda}{1+|\lambda|^2}\bigl(\overline{\lambda}\cdot\overline{H}^{-1}\overline{\partial H}-A_h^\dagger\bigr).$$

Here $A^\dagger$ denote the adjoint of A with respect to h, i.e., $A_h^\dagger=\overline{H}^{-1}\cdot{}^t\overline{A}\cdot\overline{H}$.

2.2.3 Pseudo curvature and the Hermitian-Einstein condition

Assume $\lambda\neq 0$. For a flat λ-connection $(E,\mathbb{D}^\lambda)$ with a hermitian metric h, the pseudo curvature $G(h,\mathbb{D}^\lambda)$ is defined as follows:

$$G(h,\mathbb{D}^\lambda):=\bigl[\mathbb{D}^\lambda,\mathbb{D}_h^{\lambda\star}\bigr]=-\frac{(1+|\lambda|^2)^2}{\lambda}(\overline{\partial}_h+\theta_h)^2.$$

Then a hermitian metric h is a pluri-harmonic metric for $(E,\mathbb{D}^\lambda)$, if and only if $G(h,\mathbb{D}^\lambda)=0$ holds. We will often use the notation $G(h)$ or G_h instead of $G(h,\mathbb{D}^\lambda)$ if there are no confusion.

When X is provided with a Kahler form ω, a Hermitian-Einstein condition for h is $\Lambda_\omega G(h,\mathbb{D}^\lambda)^\perp=0$, where "$\perp$" means the trace free part.

2.2.4 Some relations between curvature and pseudo curvature

By the construction of δ'_h, the operator $d''+\delta'_h$ is a unitary connection of (E,h). The curvature of $d''+\delta'_h$ is denoted by $R(d'',h)$. We have the following expression of $R(d'',h)$ due to $[d'',d']=0$:

$$R(d'',h)=\bigl[d'',\delta'_h\bigr]=\bigl[d'',\lambda^{-1}d'\bigr]-\frac{1+|\lambda|^2}{\lambda}\bigl[d'',\theta_h\bigr]=-\frac{1+|\lambda|^2}{\lambda}\bigl(\overline{\partial}_h\theta_h+\lambda[\theta_h^\dagger,\theta_h]\bigr). \tag{11}$$

Lemma 2.24 *The following equality holds:*

$$\operatorname{tr}R(d'',h)=\frac{1}{1+|\lambda|^2}\operatorname{tr}G(\mathbb{D}^\lambda,h)=-\frac{1+|\lambda|^2}{\lambda}\overline{\partial}\operatorname{tr}\theta_h. \tag{12}$$

Proof From (11), we obtain $\operatorname{tr}R(d'',h)=-(1+|\lambda|^2)\lambda^{-1}\cdot\overline{\partial}\operatorname{tr}\theta_h$. On the other hand, we have the following:

$$\operatorname{tr}G(h,\mathbb{D}^\lambda)=-\frac{\bigl(1+|\lambda|^2\bigr)^2}{\lambda}\operatorname{tr}\bigl(\overline{\partial}_h^2+\overline{\partial}_h\theta_h+\theta_h^2\bigr)=-\frac{\bigl(1+|\lambda|^2\bigr)^2}{\lambda}\overline{\partial}\operatorname{tr}\theta_h.$$

Here we have used $\operatorname{tr}(\theta_h^2)=0$, which implies $\operatorname{tr}(\overline{\partial}_h^2)=0$ due to Lemma 2.21. Thus we are done. ∎

Lemma 2.25 *In the case* $\dim X=2$*, we have the following formula:*

$$\operatorname{tr}R(h,d'')^2=\frac{1}{(1+|\lambda|^2)^2}\operatorname{tr}G(h,\mathbb{D}^\lambda)^2-\frac{(1+|\lambda|^2)^2}{\lambda}\overline{\partial}\operatorname{tr}(\theta_h^2\cdot\theta_h^\dagger).$$

Proof We have the following:

$$\operatorname{tr} G(h,\mathbb{D}^{\lambda})^2=\frac{(1+|\lambda|^2)^4}{\lambda^2}\Big(\operatorname{tr}\big((\overline{\partial}_h\theta_h)^2\big)+2\operatorname{tr}\big(\overline{\partial}_h^2\cdot\theta_h^2\big)\Big)$$

$$\operatorname{tr} R(h,d'')^2=\frac{(1+|\lambda|^2)^2}{\lambda^2}\Big(\operatorname{tr}\big((\overline{\partial}_h\theta_h)^2\big)+2\lambda\operatorname{tr}\big(\overline{\partial}_h\theta_h\cdot[\theta_h,\theta_h^{\dagger}]\big)+\lambda^2\operatorname{tr}\big([\theta_h,\theta_h^{\dagger}]^2\big)\Big).$$

Since we have $\operatorname{tr}\big([\theta_h,\theta_h^{\dagger}]^2\big)=-2\operatorname{tr}\big(\theta_h^2\theta_h^{\dagger\,2}\big)$ and $(\overline{\partial}_h+\lambda\theta_h^{\dagger})^2=\overline{\partial}_h^2+\lambda\overline{\partial}_h\theta_h^{\dagger}+\lambda^2\theta_h^{\dagger\,2}=0$, we obtain the following:

$$\lambda^2\operatorname{tr}\big([\theta_h,\theta_h^{\dagger}]^2\big)=-2\operatorname{tr}\big(\lambda^2\cdot\theta_h^2\cdot\theta_h^{\dagger\,2}\big)=2\operatorname{tr}\Big(\overline{\partial}_h^2\cdot\theta_h^2+\lambda\cdot\overline{\partial}_h\theta_h^{\dagger}\cdot\theta_h^2\Big).$$

Hence we have the following equality:

$$\operatorname{tr} R(h,d'')^2=\left(\frac{1+|\lambda|^2}{\lambda}\right)^2\Big(\operatorname{tr}\big((\overline{\partial}_h\theta_h)^2\big)+2\lambda\operatorname{tr}\big(\overline{\partial}_h\theta_h\cdot[\theta_h,\,\theta_h^{\dagger}]\big)+2\operatorname{tr}\big(\overline{\partial}_h^2\cdot\theta_h^2\big)+2\lambda\operatorname{tr}\big(\overline{\partial}_h\theta_h^{\dagger}\cdot\theta_h^2\big)\Big).$$

We also remark the following:

$$\begin{aligned}\operatorname{tr}\big(\overline{\partial}_h\theta_h\cdot[\theta_h,\theta_h^{\dagger}]\big)+\operatorname{tr}(\theta_h^2\cdot\overline{\partial}_h\theta_h^{\dagger})&=\operatorname{tr}\big((\overline{\partial}_h\theta_h)\cdot\theta_h\cdot\theta_h^{\dagger}\big)+\operatorname{tr}\big(\overline{\partial}_h\theta_h\cdot\theta_h^{\dagger}\cdot\theta_h\big)-\operatorname{tr}\big(\theta_h\cdot\overline{\partial}_h\theta_h^{\dagger}\cdot\theta_h\big)\\&=\overline{\partial}\operatorname{tr}\big(\theta_h\cdot\theta_h^{\dagger}\cdot\theta_h\big)=-\overline{\partial}\operatorname{tr}(\theta_h^2\cdot\theta_h^{\dagger}).\end{aligned}\quad(13)$$

Then the claim of the lemma immediately follows. ∎

2.2.5 Change of hermitian metrics

Let h_i $(i=1,2)$ be hermitian metrics of E. The endomorphism s is determined by $h_2=h_1\cdot s$, i.e., $h_2(u,v)=h_1\big(s\cdot u,v\big)=h_1\big(u,s\cdot v\big)$, which is self adjoint with respect to both of h_i. Then we have the relations $\delta'_{h_2}=\delta'_{h_1}+s^{-1}\delta'_{h_1}s$ and $\delta''_{h_2}=\delta''_{h_1}+s^{-1}\delta''_{h_1}s$. Therefore we have the following relations from (5):

$$\overline{\partial}_{h_2}=\overline{\partial}_{h_1}+\frac{\lambda}{1+|\lambda|^2}s^{-1}\delta''_{h_1}s,\quad \partial_{h_2}=\partial_{h_1}+\frac{1}{1+|\lambda|^2}s^{-1}\delta'_{h_1}s,$$

$$\theta_{h_2}^{\dagger}=\theta_{h_1}^{\dagger}-\frac{1}{1+|\lambda|^2}s^{-1}\delta''_{h_2}s,\quad \theta_{h_2}=\theta_{h_1}-\frac{\lambda}{1+|\lambda|^2}s^{-1}\delta'_{h_1}s.$$

We also have $\mathbb{D}^{\lambda\star}_{h_2}=\mathbb{D}^{\lambda\star}_{h_1}+s^{-1}\mathbb{D}^{\lambda\star}_{h_1}s$, and thus $\big[\mathbb{D}^{\lambda},\mathbb{D}^{\lambda\star}_{h_2}\big]=\big[\mathbb{D}^{\lambda},\mathbb{D}^{\lambda\star}_{h_1}\big]+\mathbb{D}^{\lambda}(s^{-1})\cdot\mathbb{D}^{\lambda\star}_{h_1}s+s^{-1}\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}_{h_1}s$. Then we obtain the following formula:

$$\Delta^{\lambda}_{h_1,\omega}s=s\sqrt{-1}\big(\Lambda_\omega G(h_2)-\Lambda_\omega G(h_1)\big)+\sqrt{-1}\Lambda_\omega\mathbb{D}^{\lambda}s\cdot s^{-1}\mathbb{D}^{\lambda\,\star}s.\quad(14)$$

In particular, we obtain the following formula by taking the trace:

$$\Delta^{\lambda}_{\omega}\operatorname{tr}(s)=\operatorname{tr}\Big(s\sqrt{-1}\big(\Lambda_\omega G(h_2)-\Lambda_\omega G(h_1)\big)\Big)-\big|\mathbb{D}^{\lambda}(s)s^{-1/2}\big|^2_{h_1,\omega}.\quad(15)$$

As in Lemma 3.1 of [34], we can derive the following inequality for some positive constant C_λ which depends only on λ:

$$\Delta^{\lambda}_{\omega}\log\operatorname{tr}(s)\leq C_\lambda\big(\big|\Lambda_\omega G(h_1)\big|_{h_1}+\big|\Lambda_\omega G(h_2)\big|_{h_2}\big)\quad(16)$$

2.3 Parabolic λ-flat Bundles Associated to Tame Harmonic Bundles

2.3.1 Tame pluri-harmonic metric

Let X be a complex manifold with a simple normal crossing divisor D. Let $(E,\mathbb{D}^{\lambda})$ be a λ-flat bundle on $X-D$. Let h be a pluri-harmonic metric of $(E,\mathbb{D}^{\lambda})$. Then we have the induced Higgs bundle $(E,\overline{\partial}_h,\theta_h)$. Recall the tameness of pluri-harmonic metric. Let P be any point of X, and let $(U_P, z_1, \ldots, z_n)$ be a holomorphic coordinate around P such that $D \cap U_P = \bigcup_{i=1}^{l} \{z_i = 0\}$. Then we have the expression:

$$\theta = \sum_{i=1}^{l} f_i \cdot \frac{dz_i}{z_i} + \sum_{j=l+1}^{n} g_j \cdot dz_j.$$

The pluri-harmonic metric h is called tame, if the coefficients of the characteristic polynomials $\det(t - f_i)$ and $\det(t - g_j)$ are holomorphic on U_P.

Recall also that the curve test for tameness is valid ([28]), namely, a pluri-harmonic h for $(E, \mathbb{D}^\lambda)$ is tame if and only if $h_{|C}$ is tame for any closed curve $C \subset X$ transversal with D.

2.3.2 Prolongation of tame harmonic bundles and uniqueness of pluri-harmonic metrics

Let X be a smooth projective variety with an ample line bundle L, and let D be a simple normal crossing divisor of X with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. Let $(E, \mathbb{D}^\lambda, h)$ be a tame harmonic bundle on $X - D$. Recall that E is prolonged to the filtered bundle $\boldsymbol{E}_* = \bigl({}_{\boldsymbol{c}}E \bigm| \boldsymbol{c} \in \boldsymbol{R}^S\bigr)$ such that $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is a regular filtered λ-flat bundle ([28]). And the metric h is adapted to the parabolic structure. (See the section 3.3 of [30] for the adaptedness, for example.)

Proposition 2.26 *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be as above.*

- *$(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is μ_L-polystable with* $\text{par-deg}_L(\boldsymbol{E}_*) = 0$.
- *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda) = \bigoplus_j (\boldsymbol{E}_{j\,*}, \mathbb{D}^\lambda_j) \otimes \boldsymbol{C}^{p(j)}$ be the canonical decomposition of μ_L-polystable regular filtered λ-flat bundle. Then we have the corresponding decomposition of the metric $h = \bigoplus h_i \otimes g_i$, where h_i denote pluri-harmonic metrics of $(E_i, \mathbb{D}^\lambda_i)$ adapted to the parabolic structure, and g_i denote metrics of $\boldsymbol{C}^{p(i)}$.*
- *We have the vanishings of characteristic numbers:*

$$\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = \int_X \text{par-c}_{1,L}^2(\boldsymbol{E}_*) = 0.$$

Proof The first two claims can be shown by the same argument as the proof of Proposition 5.1 of [30]. The third claim can be shown by an argument similar to the proof of Proposition 5.3 of [30], which we explain briefly. We have only to consider the case $\dim X = 2$. Since h is pluri-harmonic, we have the equalities $\operatorname{tr} R(d'', h) = (1 + |\lambda|^2)^{-1} \operatorname{tr} G(h, \mathbb{D}^\lambda) = 0$ and $\operatorname{tr}\bigl(R(d'', h)^2\bigr) = (1 + |\lambda|^2)^{-2} \cdot \operatorname{tr}\bigl(G(h, \mathbb{D}^\lambda)^2\bigr) = 0$, due to Lemma 2.24 and Lemma 2.25 on $X - D$. Hence we have only to show the following:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \bigl(\operatorname{tr} R(d'', h)\bigr)^2 = \int_X \text{par-c}_1^2(\boldsymbol{E}_*), \quad \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\bigl(R(d'', h)^2\bigr) = \int_X 2\,\text{par-ch}_2(\boldsymbol{E}).$$

It can be shown by the same argument as the proof of Proposition 5.3 of [30]. ∎

Proposition 2.27 *Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a regular filtered λ-flat bundle. We put $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-D}$. Let h_a $(a = 1, 2)$ be pluri-harmonic metrics of $(E, \mathbb{D}^\lambda)$ on $X - D$ which is adapted to the parabolic structure. Then we have the decomposition $(E, \mathbb{D}^\lambda) = \bigoplus (E_i, \mathbb{D}^\lambda)$ with the following properties:*

- *The decomposition is orthogonal with respect to both of h_a $(a = 1, 2)$. Hence we have the decomposition $h_a = \bigoplus_i h_{a,i}$.*
- *There exist positive numbers b_i such that $h_{1,i} = b_i \cdot h_{2,i}$.*

The decomposition on $X - D$ is prolonged to the decomposition $(\boldsymbol{E}_*, \mathbb{D}^\lambda) = \bigoplus (\boldsymbol{E}_{i\,*}, \mathbb{D}^\lambda)$ on X.

Proof Similar to Proposition 5.2 of [30]. ∎

2.4 Review of Existence Result of a Hermitian-Einstein Metric due to Simpson

2.4.1 Analytic stability of flat λ-bundle

Let X be a complex manifold with a Kahler form ω. In this subsection, we impose the following condition as in [34].

Condition 2.28

1. *The volume of X with respect to ω is finite.*
2. *There exists a C^∞-function $\phi : X \longrightarrow \boldsymbol{R}_{\geq 0}$ with the following properties:*
   - *$\{x \in X \mid \phi(x) \leq a\}$ is compact for any a.*
   - *$0 \leq \sqrt{-1}\partial\overline{\partial}\phi \leq C \cdot \omega$, and $\overline{\partial}\phi$ is bounded with respect to ω.*
3. *There exists a continuous increasing function $a : [0,\infty[\longrightarrow [0,\infty[$ with the following properties:*
   - *$a(0) = 0$ and $a(t) = t$ for $t \geq 1$.*
   - *Let f be a positive bounded function on X such that $\Delta_\omega f \leq B$ for some $B \in \boldsymbol{R}$. Then there exists a constant $C(B)$, depending only on B, such that $\sup_X |f| \leq C(B) \cdot a\left(\int_X |f| \cdot \mathrm{dvol}_\omega\right)$. Moreover, $\Delta_\omega(f) \leq 0$ implies $\Delta_\omega(f) = 0$.* ∎

Let $(E, \mathbb{D}^\lambda)$ be a flat λ-connection on X. There are two conditions on the finiteness of the pseudo curvature of $(E, \mathbb{D}^\lambda, h)$. The stronger one is as follows:

$$\sup |G(h, \mathbb{D}^\lambda)|_{h,\omega} < \infty. \tag{17}$$

The finiteness (17) implies the weaker one:

$$\sup |\Lambda_\omega G(h, \mathbb{D}^\lambda)|_{h,\omega} < \infty. \tag{18}$$

When a hermitian metric h of E is given with the finiteness (18), the degree $\deg_\omega(E, h)$ is defined as follows:

$$\deg_\omega(E, h) := \frac{\sqrt{-1}}{2\pi} \int_X \frac{\operatorname{tr} G(h, \mathbb{D}^\lambda)}{1 + |\lambda|^2} \cdot \omega^{n-1} = \frac{\sqrt{-1}}{2\pi} \int_X \operatorname{tr} R(h, d'') \cdot \omega^{n-1}.$$

Here we have used (12). For any λ-flat bundle $(V, \mathbb{D}^\lambda_V) \subset (E, \mathbb{D}^\lambda)$, the restriction $h_V := h_{|V}$ induces $\deg_\omega(V, h_V)$. As in Lemma 3.2 of [34], we have the Chern-Weil formula. The proof is same.

Lemma 2.29 *Let π_V denote the orthogonal projection of E onto V. Then the following holds, for some positive constant C:*

$$\deg_\omega(V, h_V) = \frac{1}{2\pi} \frac{1}{1 + |\lambda|^2} \left(\sqrt{-1} \int_X \operatorname{tr}\bigl(\pi_V \circ G(h, \mathbb{D}^\lambda)\bigr) \cdot \omega^{n-1} - \int_X \bigl|\mathbb{D}^\lambda \pi_V\bigr|^2_{h,\omega} \right)$$

The value is finite or $-\infty$, when (18) is satisfied. ∎

Definition 2.30 *$(E, \mathbb{D}^\lambda, h)$ is defined to be analytically stable with respect to ω, if the inequality*

$$\frac{\deg_\omega(V, h_V)}{\operatorname{rank} V} < \frac{\deg_\omega(E, h)}{\operatorname{rank} E}$$

holds for any $(V, \mathbb{D}^\lambda_V) \subset (E, \mathbb{D}^\lambda)$. ∎

2.4.2 Existence theorem of Simpson and some consequence

Proposition 2.31 *Let (X, ω) be a Kahler manifold satisfying Condition 2.28, and let $(E, \mathbb{D}^\lambda, h_0)$ be a metrized flat λ-connection satisfying (17). Assume that $(E, \mathbb{D}^\lambda, h_0)$ is analytically stable with respect to ω. Then there exists a hermitian metric $h = h_0 \cdot s$ satisfying the following conditions:*

- *h and h_0 are mutually bounded.*
- $\det(h) = \det(h_0)$
- *$\mathbb{D}^\lambda(s)$ is L^2 with respect to h_0 and ω.*
- *It satisfies the Hermitian Einstein condition $\Lambda_\omega G(h)^\perp = 0$, where $G(h)^\perp$ denotes the trace free part of $G(h)$.*
- *The following equalities hold:*

$$\int_Y \mathrm{tr}\Big(G(h)^2\Big) \cdot \omega^{n-2} = \int_Y \mathrm{tr}\Big(G(h_0)^2\Big) \cdot \omega^{n-2}, \qquad \int_Y \mathrm{tr}\Big(G(h)^{\perp\, 2}\Big) \cdot \omega^{n-2} = \int_Y \mathrm{tr}\Big(G(h_0)^{\perp\, 2}\Big) \cdot \omega^{n-2}.$$

We do not give a proof of this proposition, because we need only minor modification of the proof of Theorem 1, Proposition 3.5 and Lemma 7.4 of [34]. Indeed, we have only to replace D'', D' and $F(h)$ with $\mathbb{D}^\lambda$, $\mathbb{D}^{\lambda\star}$ and $G(h)$, and to make some obvious modification of positive constant multiplications, as was suggested by Simpson himself. (See the page 754 of [35], for example. Remark that "D^c" corresponds to our $-\mathbb{D}^{\lambda\star}$, and hence our $G(h)$ is slightly different from his.) The author recommends the reader to read a quite excellent discussion in [34]. However, we will use some results related with the Donaldson functional, which are obtained from the proof. Hence we recall a brief outline of the proof of Proposition 2.31. We will use the notation in the subsection 2.5.

Let h_0 be a metric for $(E, \mathbb{D}^\lambda)$ satisfying the finiteness (18). Let us consider the heat equation for the self adjoint endomorphisms s_t with respect to h_0:

$$s_t^{-1} \frac{ds_t}{dt} = -\sqrt{-1}\Lambda_\omega G(h_t)^\perp. \tag{19}$$

A detailed argument to solve (19) is given in the section 6 of [34]. Moreover, $\Lambda_\omega G(h_t)$ is shown to be uniformly bounded. We do not reproduce them here.

Then we would like to show the existence of an appropriate subsequence $t_i \to \infty$ such that $\{s_{t_i}\}$ converges to s_∞ weakly in L_2^p locally on X, and we would like to show that $h_\infty = h_0 \cdot s_\infty$ gives the desired Hermitian-Einstein metric. For that purpose, Simpson used the Donaldson functional $M\big(h_0, h_0 \cdot s_{t_i}\big)$. (We recall the definition and some fundamental property in the subsection 2.5, below.) He showed that there exist positive constants C_i $(i = 1, 2)$ such that the following holds: (Proposition 5.3 of [34]. We review it in Proposition 2.38. We will use the notation there in the following.)

$$\sup |s_t| \leq C_1 + C_2 \cdot M(h_0, h_0 s_t). \tag{20}$$

He also showed (Lemma 7.1 of [34]) that $M(h_0, h_0 \cdot s_t)$ is C^1 with respect to t, and that the following formula holds:

$$\frac{d}{dt} M\big(h_0, h_0 \cdot s_t\big) = -\int_X \big|\Lambda_\omega G(h_t)^\perp\big|^2_{h_t,\omega} \leq 0. \tag{21}$$

Since we have $M(h_0, h_0) = 0$ by definition, we obtain $M(h_0, h_0 \cdot s_t) \leq 0$ from (21). Then we obtain the boundedness of s_t from (20). For the solution of (19), we have $\det(s_t) = 1$. Hence we also obtain the boundedness of s_t^{-1}. We also obtain the existence of a subsequence $\{t'_i\}$ such that $\big|\Lambda_\omega G(h'_{t_i})\big|_{L^2} \longrightarrow 0$.

From the uniform boundedness of s_t and $\Lambda_\omega G(h_t)$, we obtain the lower bound of $M\big(h_0, h_0 s_t\big)$. (See Corollary 2.37 in this paper, for example.) Moreover, we obtain the uniform bound of $\int_X \big|\mathbb{D}^\lambda u_t\big|^2_{h_0}$ due to the positivity of Ψ given in (26), where $s_t = \exp(u_t)$. Due to the boundedness of s_t and s_t^{-1}, we also obtain the boundedness of $\int_X \big|\mathbb{D}^\lambda s_t\big|^2_{h_0}$. Then we obtain the L_1^2 boundedness. Hence we can take a subsequence $\{t''_i\}$ such that $s_{t''_i}$

converges to some s_∞ weakly in L_1^2 locally on $X-D$. Due to some more excellent additional argument given in the page 895 of [34], it can be shown that the convergence is weakly L_2^p locally on $X-D$, for any p. As a result, we obtain the Hermitian-Einstein metric.

By the above argument, we can derive the following lemma, which we would like to use in later discussion.

Lemma 2.32 *Let h_0 be the hermitian metric satisfying* (17)*, Let h_{HE} be the Hermitian-Einstein metric obtained in Proposition* 2.31*. Then we have $M\big(h_0,h_{HE}\big)\leq 0$.*

Proof Recall that h_{HE} is obtained as the limit $h_0\cdot s_\infty$ of some sequence $\{h_0s_{t_i}\}$, and we have $M(h_0,h_0\cdot s_{t_i})\leq 0$. We use the formula (25). Let Z be any compact subset of X. The sequence $\{s_{t_i}\}$ converges to s_∞ in C^0 on Z. The sequence $\{\Lambda_\omega G(h_{t_i})\}$ converges to $\Lambda_\omega G(h_{HE})$ weakly in L^2 on Z. Therefore we have the convergence:

$$\lim_{t_i\to\infty}\int_Z \mathrm{tr}\big(u_{t_i}\cdot\Lambda_\omega G(h_{t_i})\big)\,\mathrm{dvol}_\omega=\int_Z \mathrm{tr}\big(u_\infty\cdot\Lambda_\omega G(h_{HE})\big)\,\mathrm{dvol}_\omega\,.$$

Here u_t are given by $\exp(u_t)=s_t$. Since $\sup_X|s_t|$ and $\sup_X|\Lambda G(h_t)|$ are bounded independently of t, we can easily obtain the convergence:

$$\lim_{t_i\to\infty}\int_X \mathrm{tr}\big(u_{t_i}\cdot\Lambda_\omega G(h_{t_i})\big)\,\mathrm{dvol}_\omega=\int_X \mathrm{tr}\big(u_\infty\cdot\Lambda_\omega G(h_{HE})\big)\,\mathrm{dvol}_\omega\,.$$

We have the C^0-convergence of the sequence $\{\mathbb{D}^\lambda u_{t_i}\}$ to $\mathbb{D}^\lambda u_\infty$. Hence we have the following inequality, due to Fatou's lemma:

$$\int_X\big(\Psi(u_\infty)\mathbb{D}^\lambda u_\infty,\ \mathbb{D}^\lambda u_\infty\big)\,\mathrm{dvol}_\omega\leq\underline{\lim}\int_X\big(\Psi(u_{t_i})\mathbb{D}^\lambda u_{t_i},\ \mathbb{D}^\lambda u_{t_i}\big)\,\mathrm{dvol}_\omega\,.$$

Then we obtain the desired inequality. ∎

2.5 Review of Donaldson Functional

We recall the Donaldson functional, by following Donaldson and Simpson ([4] and [34]).

2.5.1 Functions of self adjoint endomorphisms

Let V be a vector space over $\boldsymbol{C}$ with a hermitian metric h. Let $S(V,h)$ denote the set of the endomorphisms of V which are self-adjoint with respect to h. Let $\varphi:\boldsymbol{R}\longrightarrow\boldsymbol{R}$ be a continuous function. Then $\varphi(s)$ is naturally defined for any $s\in S(V,h)$. Namely, let $v_1,\ldots,v_r$ be the orthogonal base which consists of the eigen vectors of s, and let $v_1^\vee,\ldots,v_r^\vee$ be the dual base. Then we have the description $s=\sum\kappa_i\cdot v_i^\vee\otimes v_i$, and we put $\varphi(s):=\sum\varphi(\kappa_i)\cdot v_i^\vee\otimes v_i$. Thus we obtain the induced map $\varphi:S(V,h)\longrightarrow S(V,h)$, which is well known to be continuous. To see the continuity, for example, we can argue as follows: Let $U(h)$ denote the unitary group with respect to h. Take $\boldsymbol{e}=(e_1,\ldots,e_r)$ be an orthogonal base of V. Let T denote the set of endomorphisms of V which is diagonal with respect to the base $\boldsymbol{e}$. Then we have the continuous surjective map $\pi:U(h)\times T\longrightarrow S(V,h)$ given by $(u,t)\longmapsto u\cdot t\cdot u^{-1}$. It is easy to check the continuity of the composite $\varphi\circ\pi$. Since the topology of $S(V,h)$ is same as the induced topology via π, we obtain the continuity. When φ is real analytic given by the convergent power series $\sum a_j\cdot t^j$, then $\varphi(s)=\sum a_j\cdot s^j$. The induced map is real analytic in this case.

Let $\Psi:\boldsymbol{R}\times\boldsymbol{R}\longrightarrow\boldsymbol{R}$ be a continuous function. For a self adjoint map $s\in S(V,h)$, let $v_1,\ldots,v_r$ and $v_1^\vee,\ldots,v_r^\vee$ be as above. Then we put $\Psi(s)(A)=\sum\Psi(\kappa_i,\kappa_j)\cdot A_{i,j}\cdot v_i^\vee\otimes v_j$ for any endomorphism $A=\sum A_{i,j}\cdot v_i^\vee\otimes v_j$ of V. Thus we obtain $\Psi:S(V,h)\longrightarrow S(\mathrm{End}(V),h)$, which is also well known to be continuous. Here $S(\mathrm{End}(E),h)$ denotes the set of the self adjoint endomorphisms of $\mathrm{End}(V)$ with respect to the metric induced by h. To see the continuity, we can use the same argument as above. When Ψ is real analytic given by a power series, $\sum b_{m,n}t_1^mt_2^n$, then we have $\Psi(s)(A)=\sum b_{m,n}s^m\cdot A\cdot s^n$, and the induced map is real analytic.

When φ is C^1, the continuous function $d\varphi:\boldsymbol{R}^2\longrightarrow\boldsymbol{R}$ is given by $d\varphi(t_1,t_2)=(t_1-t_2)^{-1}\big(\varphi(t_1)-\varphi(t_2)\big)$ $(t_1\neq t_2)$ and $d\varphi(t_1,t_1)=\varphi'(t_1)$. In this case, $\varphi:S(V,h)\longrightarrow S(V,h)$ is also C^1, and the derivative at s is given by $d\varphi(s)$. To see it, we can argue as follows: When φ is real analytic, the claim can be checked by a direct calculation. In general, we can take an approximate sequence $\varphi_i\longrightarrow\varphi$ by real analytic functions on an appropriate compact neighbourhoods of the eigenvalues of $s \in S(V,h)$. The induced maps $\varphi_i : S(V,h) \longrightarrow S(V,h)$ and $d\varphi_i : S(V,h) \longrightarrow S(\mathrm{End}(E),h)$ uniformly converge to φ and $d\varphi$ on an appropriate compact neighbourhoods of s. Then we can derive that φ is the integral of the form $d\varphi$ by a general fact.

The construction can be done on manifolds. Namely, let E be a C^∞-vector bundle with a hermitian metric h. Let $S_h(E)$ (or simply S_h) be the bundle of the self-adjoint endomorphisms of (E,h), and let $S_h(\mathrm{End}(E))$ be the bundle of the self-adjoint endomorphisms of $(\mathrm{End}(E),h)$. Then a continuous function $\varphi : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ induces $\varphi : S_h(E) \longrightarrow S_h(E)$, and $\Psi : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$ induces $\Psi : S_h(E) \longrightarrow S_h(\mathrm{End}(E))$. We have $\mathbb{D}^\lambda\varphi(s) = d\varphi(s)\big(\mathbb{D}^\lambda s\big)$, when φ is C^1.

2.5.2 A closed one form

Let (X,ω) and $(E,\mathbb{D}^\lambda)$ be as in the subsection 2.4.1. Following Simpson [34], we introduce the space $P(S_h)$, which consists of sections s of $S_h(E)$ satisfying the following finiteness:

$$\|s\|_{h,\omega,P} := \sup_X |s|_h + \|\mathbb{D}^\lambda s\|_{2,h,\omega} + \|\Delta^\lambda_{h,\omega} s\|_{1,h,\omega} < \infty.$$

Here $\|\cdot\|_{p,h,\omega}$ denote the L^p-norm with respect to (h,ω). We will omit to denote ω and h, when there are no confusion. The following lemma corresponds to Proposition 4.1 (d) in [34]. The proof is same.

Lemma 2.33 *Let φ and Ψ are analytic functions on $\boldsymbol{R}$ with infinite radius of convergence. Then $\varphi : P(S_h) \longrightarrow P(S_h)$ and $\Psi : P(S_h) \longrightarrow P(S_h(\mathrm{End}(E)))$ are analytic.* ∎

Let h be a metric satisfying (18). Let $\mathcal{P}_+(S_h)$ denote the set of the self adjoint positive definite endomorphisms s with respect to h such that $\|s\|_{h,P} < \infty$ and $\|s^{-1}\|_{h,P} < \infty$. We put $\mathcal{P}_h := \big\{h \cdot s \,\big|\, s \in \mathcal{P}_+(S_h)\big\}$. It is easy to see that any $h_1 \in \mathcal{P}_h$ also satisfies (18) and $\mathcal{P}_h = \mathcal{P}_{h_1}$, due to (14). It is also easy to see $\mathcal{P}_h = \mathcal{P}_{h_1}$ for $h_1 \in \mathcal{P}_h$.

Let $\mathcal{P}(S_h)$ denote the space of the self adjoint endomorphisms s with respect to h such that $\|s\|_{P,h} < \infty$. It is easy to see that $\mathcal{P}_+(S_h)$ is open in $\mathcal{P}(S_h)$. In particular, we obtain the Banach manifold structure of $\mathcal{P}_+(S_h)$. By the natural bijection $\mathcal{P}_h \simeq \mathcal{P}_+(S_{h_1})$ for $h_1 \in \mathcal{P}_h$, we also obtain the Banach manifold structure of $\mathcal{P}_h$, which is independent of a choice of $h_1 \in \mathcal{P}_h$. We have the map $\mathcal{P}(S_{h_1}) \longrightarrow \mathcal{P}_+(S_{h_1})$ given by $s \longmapsto e^s$ (Lemma 2.33). It gives a diffeomorphism around $0 \in \mathcal{P}(S_{h_1})$ and $1 \in \mathcal{P}_+(S_{h_1})$. Therefore the map $\mathcal{P}(S_{h_1}) \longrightarrow \mathcal{P}_h$ by $s \longmapsto h_1 \cdot e^s$ gives a diffeomorphism around 0 and h_1. In particular, the tangent space $T_{h_1}\mathcal{P}_h$ can be naturally identified with $\mathcal{P}(S_{h_1})$ for any $h_1 \in \mathcal{P}_h$. We also have the natural isomorphism $\mathcal{P}(S_{h_1}) \simeq \mathcal{P}(S_h)$ given by $t \longmapsto u \cdot t$ for $h_1 = h \cdot u \in \mathcal{P}_h$, which gives the local trivialization of the tangent bundle.

For any $h_1 \in \mathcal{P}_h$ and $s \in T_{h_1}\mathcal{P}_h$, we put as follows:

$$\Phi_{h_1}(s) := \int_X \Phi'_{h_1}(s)\,\mathrm{dvol}_\omega \in \boldsymbol{C}, \quad \Phi'_{h_1}(s) := \sqrt{-1}\,\mathrm{tr}\big(s \cdot \Lambda_\omega G(\mathbb{D}^\lambda, h_1)\big).$$

Then Φ' gives the $L^1(X,\Omega^{1,1}_X)$-valued one form on $\mathcal{P}_h$, and Φ gives the one form of $\mathcal{P}_h$. The differentiability of Φ is easy to see.

Lemma 2.34 *Φ is a closed one form.*

Proof In the following argument, we use the notation $\mathbb{D}^{\lambda\star}$ instead of $\mathbb{D}^{\lambda\star}_h$. Let $k_1, k_2 \in \mathcal{P}_h$. They naturally give the vector field by addition. At any point $h_1 \in \mathcal{P}_h$, they give the tangent vectors $\sigma = h_1^{-1}k_1$ and $\tau = h_1^{-1}k_2$ in $T_{h_1}\mathcal{P}_h = \mathcal{P}(S_{h_1})$. Hence we have the following at $h + \epsilon k_1$:

$$\Phi_{h+\epsilon k_1}(k_2) = \sqrt{-1}\int \mathrm{tr}\Big((h+\epsilon k_1)^{-1} \cdot k_2 \cdot G(h+\epsilon k_1)\Big) \cdot \omega^{n-1}.$$

We have $(h+\epsilon k_1)^{-1}k_2 = (1+\epsilon\sigma)^{-1}\tau = \tau - \epsilon\sigma\tau + (1+\epsilon\sigma)^{-2}\epsilon^2\sigma^2\tau$. Remark $\sigma^2\tau$ is bounded. We also have the following:

$$\begin{aligned}(1+\epsilon\sigma)\big(G(h+\epsilon k_1) - G(h)\big) &= \mathbb{D}^\lambda\mathbb{D}^{\lambda\star}(1+\epsilon\sigma) - \mathbb{D}^\lambda(1+\epsilon\sigma)\cdot(1+\epsilon\sigma)^{-1}\mathbb{D}^{\lambda\star}(1+\epsilon\sigma)\\ &= \epsilon\mathbb{D}^\lambda\mathbb{D}^{\lambda\star}\sigma - \epsilon^2\mathbb{D}^\lambda\sigma\cdot(1+\epsilon\sigma)^{-1}\mathbb{D}^{\lambda\star}\sigma. \quad (22)\end{aligned}$$

Hence we have $G(h+\epsilon k_1)-G(h)=\epsilon\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}\sigma+\epsilon^2R_0(\epsilon,\sigma,\tau)$, where $R_0(\epsilon,\sigma,\tau)$ is an L^1-section of $\mathrm{End}(E)\otimes\Omega^2$, and the L^1-norm is bounded independently of ϵ. Therefore we obtain the following:

$$\begin{aligned}\Phi_{h+\epsilon k_1}(k_2)-\Phi_h(k_2)&=\sqrt{-1}\int\mathrm{tr}\big((h+\epsilon k_1)^{-1}\cdot k_2\cdot G(h+\epsilon k_1)\big)\cdot\omega^{n-1}-\sqrt{-1}\int\mathrm{tr}\big(h^{-1}\cdot k_2\cdot G(h)\big)\cdot\omega^{n-1}\\&=\sqrt{-1}\int\mathrm{tr}\big(\tau G(h+\epsilon k_1)-\tau G(h)\big)\cdot\omega^{n-1}-\epsilon\sqrt{-1}\int\mathrm{tr}\big(\sigma\tau G(h+\epsilon k_1)\big)\cdot\omega^{n-1}+\epsilon\cdot R_1(\epsilon,\sigma,\tau)\\&=\epsilon\left(\sqrt{-1}\int\mathrm{tr}\big(\tau\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}\sigma\big)\cdot\omega^{n-1}-\sqrt{-1}\int\mathrm{tr}\big(\sigma\cdot\tau\cdot G(h)\big)\cdot\omega^{n-1}\right)+\epsilon R_2(\epsilon,\sigma,\tau).\end{aligned}\tag{23}$$

Here we have $R_i(\epsilon,\sigma,\tau)\longrightarrow 0$ $(i=1,2)$ in $\epsilon\to 0$, due to $\|\sigma\|_P$ and $\|\tau\|_P<\infty$. Hence we obtain the following equality:

$$d_h\Phi(\sigma,\tau)=\sqrt{-1}\int\Big(\mathrm{tr}\big(\tau\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}\sigma\big)-\mathrm{tr}\big(\sigma\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}\tau\big)\Big)\cdot\omega^{n-1}-\sqrt{-1}\int\mathrm{tr}\big([\sigma,\tau]\cdot G(h)\big)\cdot\omega^{n-1}.$$

We have the following equality, due to $[\mathbb{D}^{\lambda},\mathbb{D}^{\lambda\star}]=G(h)$:

$$\begin{aligned}(-\overline{\lambda\partial}+\partial)\,\mathrm{tr}(\tau\mathbb{D}^{\lambda}\sigma)+(\lambda\partial+\overline{\partial})\,\mathrm{tr}(\sigma\mathbb{D}^{\lambda\star}\tau)&=\mathrm{tr}(\mathbb{D}^{\lambda\star}\tau\mathbb{D}^{\lambda}\sigma)+\mathrm{tr}(\tau\mathbb{D}^{\lambda\star}\mathbb{D}^{\lambda}\sigma)+\mathrm{tr}(\mathbb{D}^{\lambda}\sigma\mathbb{D}^{\lambda\star}\tau)+\mathrm{tr}(\sigma\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}\tau)\\&=-\mathrm{tr}(\tau\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}\sigma)+\mathrm{tr}(\tau\cdot[G(h),\sigma])+\mathrm{tr}(\sigma\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}\tau)=-\mathrm{tr}\big(\tau\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}\sigma\big)+\mathrm{tr}(\sigma\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}\tau)+\mathrm{tr}\big([\sigma,\tau]\cdot G(h)\big)\end{aligned}\tag{24}$$

Hence we obtain $d_h\Phi(\sigma,\tau)=-\sqrt{-1}\int_X\Big((-\overline{\lambda\partial}+\partial)\,\mathrm{tr}(\tau\mathbb{D}^{\lambda}\sigma)+(\lambda\partial+\overline{\partial})\,\mathrm{tr}(\sigma\mathbb{D}^{\lambda\star}\tau)\Big)\cdot\omega^{n-1}$. By using $\|\sigma\|_P<\infty$ and $\|\tau\|_P<\infty$, we obtain the vanishing of $d_h\Phi(\sigma,\tau)$, due to Lemma 5.2 of [34]. ∎

2.5.3 Donaldson functional

For $h_1,h_2\in\mathcal{P}_h$, take a differentiable path $\gamma:[0,1]\longrightarrow\mathcal{P}_h$ such that $\gamma(0)=h_1$ and $\gamma(1)=h_2$, and the Donaldson functional is defined to be $M(h_1,h_2):=\int_\gamma\Phi$. It is independent of a choice of a base metric ω, in the case $\dim X=1$. We have $M(h_1,h_2)+M(h_2,h_3)=M(h_1,h_3)$ by the construction.

Lemma 2.35 *When $h_2=h_1\cdot e^s$ for $s\in\mathcal{P}(S_{h_1})$, we have the following formula:*

$$M\big(h_1,h_2\big)=\sqrt{-1}\int_X\mathrm{tr}\big(s\Lambda_\omega G(h_1)\big)\,\mathrm{dvol}_\omega+\int_X\big(\Psi(s)\mathbb{D}^{\lambda}s,\mathbb{D}^{\lambda}s\big)_{\omega,h_1}\,\mathrm{dvol}_\omega\,.\tag{25}$$

Here $(\cdot,\cdot)_{\omega,h_1}$ denotes the hermitian product induced by ω and h_1, and Ψ is given as follows:

$$\Psi(t_1,t_2)=\frac{e^{t_2-t_1}-(t_2-t_1)-1}{(t_2-t_1)^2}.\tag{26}$$

See the subsection 2.5.1 for the meaning of $\Psi(s)(\mathbb{D}^{\lambda}s)$.

Proof Let $M'(h_1,h_2)$ denote the right hand side of (25). The following formula immediately follows from the definition:

$$\frac{\partial}{\partial u}M'(h_1e^{ts},h_1e^{(t+u)s})_{|u=0}=\int_X\sqrt{-1}\,\mathrm{tr}\big(s\Lambda_\omega G(h_1e^{ts})\big).$$

The following formula can be shown:

$$\frac{\partial^2}{\partial t\partial u}M'(h_1e^{ts},h_1e^{(t+u)s})_{|u=0}=\frac{\partial^2}{\partial t\partial u}M'(h_1,h_1e^{(t+u)s})_{|u=0}.\tag{27}$$

We omit to give the argument to show (27), because it is same as that in the page 883 of [34] to show the following equality:

$$\frac{\partial^2}{\partial t^2}M(h,he^{ts})_{|t=1,u=0}=\frac{\partial^2}{\partial t\partial u}M(he^{ts},he^{(t+u)s})_{|t=1,u=0}.$$

We have the obvious equality:

$$\frac{\partial}{\partial u} M'(h_1 e^{ts}, h_1 e^{(t+u)s})_{|t=0,u=0} = \frac{\partial}{\partial u} M'(h_1, h_1 e^{(t+u)s})_{|t=0,u=0}.$$

Hence we obtain the following:

$$\frac{\partial}{\partial t} M'(h_1, h_1 e^{ts}) = \int_X \sqrt{-1}\,\mathrm{tr}\big(s\Lambda_\omega G(h_1 e^{ts})\big).$$

Thus $M'(h_1, h_1 e^s)$ is the integral of Φ' along the path $\gamma(t) = h_1 e^{ts}$, and hence $M'(h_1, h_2) = M(h_1, h_2)$. ∎

Remark 2.36 *In* [34]*, the formula* (25) *is adopted to be the definition of the functional. We follow the original definition of Donaldson* ([4])*.* ∎

We obtain the following corollary due to the positivity of the function Ψ.

Corollary 2.37 *When* $\sup |\Lambda_\omega G(h)|_h < B$*, we have the following inequality:*

$$M(h, he^s) \geq \sqrt{-1} \int \mathrm{tr}\big(s\Lambda_\omega G(h)\big) \cdot \mathrm{dvol}_\omega \geq -B \int |s|_h \cdot \mathrm{dvol}_\omega\,.$$

In particular, the upper bound of s gives the lower bound of $M(h, he^s)$. ∎

2.5.4 Main estimate

The following key estimate is the counterpart of Proposition 5.3 in [34]. The proof is same.

Proposition 2.38 *Fix $B > 0$. Let $(E, \mathbb{D}^\lambda)$ be a flat λ-connection. Let h be a hermitian metric of E such that $\sup\big|\Lambda_\omega G(h, \mathbb{D}^\lambda)\big|_h \leq B$. Let $(E, \mathbb{D}^\lambda, h)$ be analytically stable with respect to ω. Then there exist positive constants C_i $(i = 1, 2)$ with the following property:*

- *Let s be any self adjoint endomorphism satisfying $\|s\|_{P,h} < \infty$, $\mathrm{tr}(s) = 0$ and $\sup\big|\Lambda_\omega G(h \cdot e^s, \mathbb{D}^\lambda)\big| \leq B$. Then the following inequality holds:*

$$\sup_X |s|_h \leq C_1 + C_2 \cdot M(h, he^s)$$

(Sketch of the proof) The excellent argument given in [34] works in the case of λ-connection without any essential change. Since we would like to use some minor variants of the proposition (the subsections 2.5.5–2.5.6), we recall an outline of the proof for the convenience of the reader. To begin with, we remark that we have only to show the following inequality due to Corollary 2.37:

$$\sup_X |s|_h \leq C_1' + C_2' \cdot \max\big\{0, M(h, he^s)\big\},$$

As is noticed in the subsubsection 2.2.5, the inequality $\Delta_\omega^\lambda \log \mathrm{tr}(e^s) \leq C_\lambda \cdot \Big(\big|\Lambda G(h)\big|_h + \big|\Lambda G(he^s)\big|_{he^s}\Big) \leq 2BC_\lambda$ holds. Hence there exist some constants C_i $(i = 3, 4)$ such that the inequality $\log \mathrm{tr}(e^s) \leq C_3 + C_4 \cdot \int \log \mathrm{tr}(e^s)$ holds for any s as above, due to Condition 2.28. Since we have $C_5 + C_6 \cdot |s|_h \leq \log \mathrm{tr}\, e^s \leq C_7 + C_8 \cdot |s|_h$ for some positive constants C_i $(i = 5, 6, 7, 8)$, there exist some constants C_i $(i = 9, 10)$ such that the following holds for any s as above:

$$\sup |s|_h \leq C_9 + C_{10} \cdot \int |s|_h. \tag{28}$$

Assume that the claim of the proposition does not hold, and we will derive a contradiction. Under the assumption, either one of the following occurs:

Case 1. There exists a sequence $\{s_i \in \mathcal{P}(S_h) \,|\, i = 1, 2, \cdots, \}$ such that $\sup |s_i|_h \longrightarrow \infty$ and $M(h, he^{s_i}) \leq 0$.

Case 2. There exist sequences $\{s_i \in \mathcal{P}(S_h)\}$ and $\{C_{2,i} \in \boldsymbol{R}\}$ with the following properties:

$$\sup_X |s_i| \longrightarrow \infty, \quad C_{2,i} \longrightarrow \infty, \quad (i \longrightarrow \infty)$$

$$M(h, he^{s_i}) > 0, \quad \sup |s_i|_h \geq C_{2,i} M(h, he^{s_i})$$

In both cases, we have $\|s_i\|_{L^1} \longrightarrow \infty$. We put $\ell_i := \|s_i\|_{L^1}$ and $u_i := s_i/\ell_i$. Clearly we have $\|u_i\|_{L^1} = 1$, and uniform boundedness $\sup_X |u_i| < C$ due to (28). In the following, let $L^2(S_h)$ (resp. $L_1^2(S_h)$) denote the space of L^2-sections (resp. L_1^2-sections) of S_h. The following lemma is one of the keys in the proof of Proposition 2.38.

Lemma 2.39 *After going to an appropriate subsequence, $\{u_i\}$ weakly converges to some $u_\infty \neq 0$ in $L_1^2(S_h)$. Moreover, we have the following inequality, for any C^∞-function $\Phi : \boldsymbol{R} \times \boldsymbol{R} \longrightarrow \boldsymbol{R}_{\geq 0}$ such that $\Phi(y_1, y_2) \leq (y_1 - y_2)^{-1}$ for $y_1 > y_2$:*

$$\sqrt{-1} \int \operatorname{tr}\bigl(u_\infty \Lambda_\omega G(h)\bigr) + \int_X \bigl(\Phi(u_\infty) \mathbb{D}^\lambda u_\infty, \mathbb{D}^\lambda u_\infty\bigr)_{h,\omega} \leq 0.$$

Proof By considering $\Phi - \epsilon$ for any small positive number ϵ, we have only to consider the case $\Phi(y_1, y_2) < (y_1 - y_2)^{-1}$ for $y_1 > y_2$. In the both cases, we have the inequalities for some constant C, from the formula (27):

$$\ell_i \sqrt{-1} \int_X \operatorname{tr}\bigl(u_i \Lambda_\omega G(h, \mathbb{D}^\lambda)\bigr) + \ell_i^2 \int \bigl(\Psi(\ell_i u_i) \mathbb{D}^\lambda u_i, \mathbb{D}^\lambda u_i\bigr)_h \leq \ell_i \cdot \frac{C}{C_{2,i}}.$$

(In the case 1, we take any sequence $\{C_{2,i}\}$ such that $C_{2,i} \longrightarrow \infty$). Let Φ be as above. Due to the uniform boundedness of u_i, we may assume that Φ has the compact support. Then if ℓ is sufficiently large, we have $\Phi(\lambda_1, \lambda_2) < \ell \Psi(\ell\lambda_1, \ell\lambda_2)$. Therefore, we obtain the following inequality:

$$\sqrt{-1} \int_X \operatorname{tr}\bigl(u_i \Lambda_\omega G(h, \mathbb{D}^\lambda)\bigr) + \int_X \bigl(\Phi(u_i) \mathbb{D}^\lambda u_i, \mathbb{D}^\lambda u_i\bigr)_{h,\omega} \leq \frac{C}{C_{2,i}}.$$

Since $\sup_X |u_i|$ is bounded independently of i, there exists a function Φ as above which satisfies $\Phi(u_i) = c \cdot \mathrm{id}$, moreover, for some small positive number $c > 0$. Therefore, we obtain the boundedness of $\{u_i\}$ in L_1^2. By taking an appropriate subsequence, $\{u_i\}$ is weakly convergent in L_1^2. Let u_∞ denote the weak limit. Let Z be any compact subset of X. Then $\{u_i\}$ is convergent to u_∞ on Z in L^2, and hence $\int_Z |u_i| \to \int_Z |u_\infty|$. Since $\sup |u_i|$ are uniformly bounded, we obtain $\int_Z |u_\infty| \neq 0$, if the volume of $X - Z$ is sufficiently small. Thus $u_\infty \neq 0$. Similarly, we can show the convergence $\int \operatorname{tr}\bigl(u_i \Lambda G(h, \mathbb{D}^\lambda)\bigr) \longrightarrow \int \operatorname{tr}\bigl(u_\infty \Lambda G(h, \mathbb{D}^\lambda)\bigr)$. Since $\{u_i\}$ are weakly convergent to u_∞ in L_1^2, we have the almost everywhere convergence of $\{u_i\}$ and $\{\mathbb{D}^\lambda u_i\}$ to u_∞ and $\mathbb{D}^\lambda u_\infty$ respectively. Therefore $\{\Phi(u_i) \mathbb{D}^\lambda u_i\}$ converges to $\Phi(u_\infty) \mathbb{D}^\lambda u_\infty$ almost everywhere. Therefore we have

$$\int \bigl(\Phi(u_\infty) \mathbb{D}^\lambda u_\infty, u_\infty\bigr)_{h,\omega} \leq \underline{\lim} \int \bigl(\Phi(u_i) \mathbb{D}^\lambda u_i, u_i\bigr)_{h,\omega}$$

due to Fatou's lemma. Thus we obtain the desired inequality, and the proof of Lemma 2.39 is finished. ∎

We reproduce the rest of the excellent argument given in [34] just for the completeness. We do not use it in the later argument. The point is that we can derive a contradiction from the existence of the non-trivial section u_∞ as in Lemma 2.39.

Lemma 2.40 *The eigenvalues of u_∞ are constant, and u_∞ has at least two distinct eigenvalues.*

Proof To show the constantness of the eigenvalues, we have only to show the constantness of $\operatorname{tr}\bigl(\varphi(u_\infty)\bigr)$ for any C^∞-function $\varphi : \boldsymbol{R} \longrightarrow \boldsymbol{R}$. We have $\bigl(\overline{\partial} + \lambda\partial\bigr) \operatorname{tr} \varphi(u_\infty) = \operatorname{tr}\bigl(\mathbb{D}^\lambda \varphi(u_\infty)\bigr) = \operatorname{tr}\bigl(d\varphi(u_\infty) \mathbb{D}^\lambda u_\infty\bigr)$. Let N be any large number. We can take a C^∞-function $\Phi : \boldsymbol{R} \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ such that $\Phi(y_1, y_1) = d\varphi(y_1, y_1)$ and $N\Phi^2(y_1, y_2) < (y_1 - y_2)^{-1}$ for $y_1 > y_2$. We obtain $\operatorname{tr} d\varphi(u_\infty)(\mathbb{D}^\lambda u_\infty) = \operatorname{tr}\bigl(\Phi(u_\infty) \mathbb{D}^\lambda u_\infty\bigr)$ due to the first condition. We obtain the following inequality from Lemma 2.39:

$$\int_X |\Phi(u_\infty) \mathbb{D}^\lambda u_\infty|^2 \leq -\frac{\sqrt{-1}}{N} \int_X \operatorname{tr}\bigl(u_\infty \Lambda G(h)\bigr).$$

Therefore $\left|(\overline{\partial}+\lambda\partial)\operatorname{tr}\varphi(u_\infty)\right|^2_{L^2}=0$. Thus the eigenvalues of u_∞ are constant. Since $\operatorname{tr}(u_\infty)=0$ and $u_\infty\neq 0$, u_∞ has at least two distinct eigenvalues. ∎

Let $\kappa_1<\kappa_2<\cdots<\kappa_w$ denote the constant distinct eigenvalues of u_∞. Then $\varphi(u_\infty)$ and $\Phi(u_\infty)$ depend only on the values $\varphi(\kappa_i)$ and $\varphi(\kappa_i,\kappa_j)$ respectively.

Lemma 2.41 *Let $\Phi:\boldsymbol{R}^2\longrightarrow\boldsymbol{R}$ be a C^∞-function such that $\Phi(\kappa_i,\kappa_j)=0$ for $\kappa_i>\kappa_j$. Then $\Phi(u_\infty)(\mathbb{D}^\lambda u_\infty)=0$.*

Proof We may replace Φ with Φ_1 satisfying $\Phi_1(\kappa_i,\kappa_j)=0$ for $\kappa_i>\kappa_j$ and $N\Phi_1^2(y_1,y_2)<(y_1-y_2)^{-1}$ for $y_1>y_2$. Then we obtain $\left\|\Phi_1(u_\infty)\mathbb{D}^\lambda u_\infty\right\|^2_{L^2}\leq C/N$ due to Lemma 2.39, and hence we obtain $\Phi(u_\infty)\mathbb{D}^\lambda u_\infty=\Phi_1(u_\infty)\mathbb{D}^\lambda u_\infty=0$. ∎

Let γ_i denote the open interval $]\kappa_i,\kappa_{i+1}[$. Let $p_\gamma:\boldsymbol{R}\longrightarrow[0,1]$ be any decreasing C^∞-function such that $p_\gamma(\kappa_i)=1$ and $p_\gamma(\kappa_{i+1})=0$. We put $\pi_\gamma=p_\gamma(u_\infty)$. It is easy to see that π_γ is L_1^2. Due to $p_\gamma^2=p_\gamma$, we have $\pi_\gamma^2=\pi_\gamma$. We have $\mathbb{D}^\lambda\pi_\gamma=dp(u_\infty)\mathbb{D}^\lambda u_\infty$. We put $\Phi_\gamma(y_1,y_2)=(1-p_\gamma)(y_2)\cdot dp_\gamma(y_1,y_2)$, and then we have $(1-\pi_\gamma)\circ\mathbb{D}^\lambda\pi_\gamma=\Phi_\gamma(u_\infty)\circ\mathbb{D}^\lambda u_\infty$. On the other hand, since we have $\Phi_\gamma(\kappa_i,\kappa_j)=0$ $(\kappa_i>\kappa_j)$, we obtain $\Phi_\gamma(u_\infty)\mathbb{D}^\lambda u_\infty=0$ due to Lemma 2.41. Therefore we obtain $(1-\pi_\gamma)\circ\mathbb{D}^\lambda\pi_\gamma=0$.

From $(1-\pi_\gamma)d''\pi_\gamma=0$, we obtain a saturated coherent subsheaf V_γ such that π_γ is the orthogonal projection on V_γ due to the result of Uhlenbeck-Yau [43]. From $(1-\pi_\gamma)d'\pi_\gamma=0$, the bundle V_γ is $\mathbb{D}^\lambda$-invariant. Since we consider the case $\lambda\neq 0$, it is easy to see that V_γ is indeed a subbundle of E. Namely, we obtain the λ-flat subbundle $(V_\gamma,\mathbb{D}^\lambda_{V_\gamma})\subset(E,\mathbb{D}^\lambda)$.

Let us show $\deg_\omega(V_\gamma,h_\gamma)/\operatorname{rank}V_\gamma>\deg_\omega(E,h)/\operatorname{rank}E$ for some γ, which contradicts the stability assumption of $(E,\mathbb{D}^\lambda,h)$, where $h_\gamma:=h_{|V_\gamma}$. From Lemma 2.29, we have

$$\deg(V_\gamma)=\frac{1}{2\pi}\frac{1}{1+|\lambda|^2}\left(\sqrt{-1}\int\operatorname{tr}\bigl(\pi_\gamma G(h)\bigr)-\int\|\mathbb{D}^\lambda\pi_\gamma\|^2\right).$$

We have $u_\infty=\kappa_w-\sum|\gamma|\cdot\pi_\gamma$, where $|\gamma|$ denotes the length of γ. We put

$$W=\kappa_w\deg(E)-\sum|\gamma|\cdot\deg(V_\gamma)=\frac{1}{2\pi}\frac{1}{1+|\lambda|^2}\left(\sqrt{-1}\int\operatorname{tr}\bigl(u_\infty\Lambda G(h)\bigr)+\int\sum|\gamma|\cdot\left|\mathbb{D}^\lambda\pi_\gamma\right|^2.\right).$$

Since $\mathbb{D}^\lambda\pi_\gamma=dp_\gamma(u_\infty)\mathbb{D}^\lambda u_\infty$, we have

$$W=\frac{1}{2\pi}\frac{1}{1+|\lambda|^2}\left(\sqrt{-1}\int\operatorname{tr}\bigl(u_\infty\Lambda G(h)\bigr)+\int\Bigl(\sum|\gamma|\cdot dp_\gamma(u_\infty)^2\cdot\mathbb{D}^\lambda u_\infty,\ \mathbb{D}^\lambda u_\infty\Bigr)\right).$$

We can check $\sum|\gamma|(dp_\gamma)(\kappa_i,\kappa_j)=(\kappa_i-\kappa_j)^{-1}$ for $\kappa_i>\kappa_j$ by a direct argument. Therefore we obtain $W\leq 0$, due to Lemma 2.39. Namely we obtain $a\cdot\deg E\leq\sum|\gamma|\cdot\deg(V_\gamma)$. On the other hand, we have $0=\operatorname{tr}(u_\infty)=a\cdot\operatorname{rank}E-\sum|\gamma|\cdot\operatorname{rank}V_\gamma$. Therefore, we obtain $\deg(V_\gamma)/\operatorname{rank}V_\gamma\geq\deg(E)/\operatorname{rank}E$ for one of γ, which contradicts with the stability of $(E,\mathbb{D}^\lambda,h)$. Thus the proof of Proposition 2.38 is finished. ∎

2.5.5 Variant 1 of Proposition 2.38

Let C be a smooth projective curve, and D be a simple divisor. Let $(E,\mathbb{D}^\lambda,\boldsymbol{F})$ be a λ-flat bundle on (C,D). Let η be a sufficiently small positive number such that $10\cdot\eta<\operatorname{gap}(E,\boldsymbol{F})$. Let ϵ_0 be a sufficiently smaller number than η, for example $10\operatorname{rank}(E)\epsilon_0<\eta$. Let ω_ϵ $(0\leq\epsilon<\epsilon_0)$ be a Kahler metric of $C-D$ with the following properties:

- Let $P\in D$. Let (U,z) be a holomorphic coordinate around P such that $z(P)=0$. Then the following holds for some positive constants C_i $(i=1,2)$:

$$C_1\cdot\omega_\epsilon\leq\epsilon^2|z|^{2\epsilon}\frac{dz\cdot d\overline{z}}{|z|^2}+\eta^2|z|^{2\eta}\frac{dz\cdot d\overline{z}}{|z|^2}\leq C_2\cdot\omega_\epsilon$$

- $\omega_\epsilon \longrightarrow \omega_0$ for $\epsilon \to 0$ in the C^∞-sense locally on $C - D$.

Let $\boldsymbol{F}^{(\epsilon)}$ be an ϵ-perturbation of $\boldsymbol{F}$. See the subsubsection 2.1.5 for the notion of ϵ-perturbation. We discuss the surface case there, but it can be applied in the curve case. Suppose that we are given hermitian metrics $h^{(\epsilon)}$ for $(E, \boldsymbol{F}^{(\epsilon)})$ with the following properties:

- $\big|\Lambda_{\omega_\epsilon} G(h^{(\epsilon)}, \mathbb{D}^\lambda)\big|_{h^{(\epsilon)}} \leq C_1$, where the constant C_1 is independent of ϵ.
- $\{h^{(\epsilon)}\}$ converges to $h^{(0)}$ for $\epsilon \to 0$ in the C^∞-sense locally on $C - D$.

Lemma 2.42 *Let $s^{(\epsilon)}$ be self adjoint endomorphisms of $(E, h^{(\epsilon)})$ satisfying $\operatorname{tr} s^{(\epsilon)} = 0$ and the following properties:*

- $\|s^{(\epsilon)}\|_{P,h^{(\epsilon)},\omega_\epsilon} < \infty$. *But we do not assume the uniform boundedness.*
- $\big|\Lambda_{\omega_\epsilon} G(h^{(\epsilon)} e^{s^{(\epsilon)}}, \mathbb{D}^\lambda)\big|_{h^{(\epsilon)}} \leq C_1$. *The constant C_1 is independent of ϵ.*

Then there exist constants C_i $(i = 3, 4)$, which is independent of ϵ, with the following property:

$$\sup |s^{(\epsilon)}|_{h^{(\epsilon)}} \leq C_3 + C_4 \cdot M(h^{(\epsilon)},\ h^{(\epsilon)} e^{s^{(\epsilon)}}).$$

(Sketch of a proof) The argument is essentially same as the proof of Proposition 2.38. We assume that the claim does not hold, and we will derive a contradiction. After going to an appropriate subsequence, either one of the following holds:

Case 1. $M(h^{(\epsilon)}, h^{(\epsilon)} e^{s^{(\epsilon)}}) \leq 0$ and $\sup_{C-D} |s^{(\epsilon)}| \longrightarrow \infty$ for $\epsilon \to 0$.

Case 2. $M(h^{(\epsilon)}, h^{(\epsilon)} e^{s^{(\epsilon)}}) > 0$, $\sup |s^{(\epsilon)}| \geq C_2^{(\epsilon)} M(h^{(\epsilon)}, h^{(\epsilon)} e^{s^{(\epsilon)}})$, $\sup_{C-D} |s^{(\epsilon)}| \longrightarrow \infty$ and $C_2^{(\epsilon)} \longrightarrow \infty$ for $\epsilon \to 0$.

By using Lemma 2.44 (given below) and the argument given in the first part of Proposition 2.38, we can show that there exist positive constants C_i $(i = 5, 6)$, which are independent of ϵ, with the following property:

$$\sup_{C-D} |s^{(\epsilon)}|_{h^{(\epsilon)}} \leq C_5 + C_6 \cdot \int |s^{(\epsilon)}|_{h^{(\epsilon)}} \operatorname{dvol}_{\omega_\epsilon}.$$

We put $\ell^{(\epsilon)} := \|s^{(\epsilon)}\|_{L^1}$ and $u^{(\epsilon)} := s^{(\epsilon)}/\ell^{(\epsilon)}$. The following lemma is the counterpart of Lemma 2.39.

Lemma 2.43 *We have a non-trivial L_1^2-section u_∞ of $S_{h^{(0)}}$ with the following property:*

- *The following inequality holds for any C^∞-function $\Phi : \boldsymbol{R} \times \boldsymbol{R} \longrightarrow \boldsymbol{R}_{\geq 0}$ such that $\Phi(y_1, y_2) \leq (y_1 - y_2)^{-1}$ for $y_1 > y_2$:*

$$\sqrt{-1} \int_{C-D} \operatorname{tr}\big(u_\infty \Lambda_{\omega_0} G(h^{(0)})\big) \operatorname{dvol}_{\omega_0} + \int_{C-D} \big(\Phi(u_\infty) \mathbb{D}^\lambda u_\infty, \mathbb{D}^\lambda u_\infty\big)_{h,\omega_0} \operatorname{dvol}_{\omega_0} \leq 0.$$

Proof The argument is essentially same as the proof of Lemma 2.39. We have the following, for some constant C_5:

$$\sqrt{-1} \int_{C-D} \operatorname{tr}\big(u^{(\epsilon)} \Lambda_{\omega_\epsilon} G(h^{(\epsilon)})\big) \operatorname{dvol}_{\omega_\epsilon} + \int_{C-D} \big(\Phi(u^{(\epsilon)}) \mathbb{D}^\lambda u^{(\epsilon)}, \mathbb{D}^\lambda u^{(\epsilon)}\big)_{h^{(\epsilon)}} \operatorname{dvol}_{\omega_\epsilon} \leq \frac{C_5}{C_2^{(\epsilon)}}.$$

(In the case 1, we take any sequence $\{C_2^{(\epsilon)}\}$ such that $C_2^{(\epsilon)} \longrightarrow \infty$.) From this, we obtain the following boundedness as in the proof of Lemma 2.39:

$$\int_{C-D} \big|\mathbb{D}^\lambda u^{(\epsilon)}\big|^2_{h^{(\epsilon)}} \operatorname{dvol}_{\omega_\epsilon} < C_{10}.$$

Let us take a sequence of C^∞-isometries $F_\epsilon : (E, h^{(\epsilon)}) \longrightarrow (E, h^{(0)})$ which converges to the identity of E, in the C^∞-sense locally on $C - D$. Remark that the sequence $\{F_\epsilon(\mathbb{D}^\lambda)\}$ converges to $\mathbb{D}^\lambda$ for $\epsilon \to 0$ in the C^∞-sense locally on $C-D$. The sequence $\{F_\epsilon(u^{(\epsilon)})\}$ is bounded on L_1^2 locally on $C-D$. By going to an appropriate subsequence, we may assume that the sequence $\{u^{(\epsilon)}\}$ is weakly convergent in L_1^2 locally on $C-D$, and hence it is convergent in L^2 on any compact subset $Z\subset C-D$. Let u_∞ denote the weak limit. We have $\int_Z |u^{(\epsilon)}|\longrightarrow \int_Z |u_\infty|$. Hence $\int_Z |u_\infty|\neq 0$, when the volume of $C-Z\cup D$ is sufficiently small, due to the boundedness of $\{\sup|u^{(\epsilon)}|\,\big|\,\epsilon>0\}$. In particular, $u_\infty\neq 0$. Similarly, we obtain $\int_{C-D}\mathrm{tr}(u^{(\epsilon)}G(h^{(\epsilon)}))\longrightarrow\int_{C-D}\mathrm{tr}(u_\infty G(h^{(0)}))$. Since we can derive the almost everywhere convergence $\Phi(u^{(\epsilon)})\mathbb{D}^\lambda u^{(\epsilon)}\longrightarrow\Phi(u_\infty)\mathbb{D}^\lambda u_\infty$ and $u^{(\epsilon)}\longrightarrow u_\infty$, we obtain $\int_{C-D}\bigl(\Phi(u_\infty)\mathbb{D}^\lambda u_\infty,\mathbb{D}^\lambda u_\infty\bigr)\leq \underline{\lim}\int_{C-D}\bigl(\Phi(u^{(\epsilon)})\mathbb{D}^\lambda u^{(\epsilon)},\mathbb{D}^\lambda u^{(\epsilon)}\bigr)$ due to Fatou's lemma. Thus the proof of Lemma 2.43 is finished. ∎

The rest of the proof of Lemma 2.42 is completely same as the argument for Proposition 2.38. ∎

We have used the following lemma in the proof.

Lemma 2.44 *For any positive number B, there exist positive constants C_i $(i=1,2)$ with the following property:*

- *Let ϵ be any positive number such that $\epsilon<1/2$. For any non-negative function f such that $\Delta_{\omega_\epsilon}f\leq B$, the inequality $\sup(f)\leq C_1+C_2\int f\cdot \mathrm{dvol}_{\omega_\epsilon}$ holds.* ∎

Proof Let (U_P,z) be as above for $P\in D$. On U_P, the inequality $\Delta_{\omega_\epsilon}f\leq B$ is equivalent to the following:

$$\Delta_{g_0}f\leq B\cdot\left(\epsilon^2\frac{|z|^{2\epsilon}}{|z|^2}+\eta^2\frac{|z|^{2\eta}}{|z|^2}\right).$$

Here $g_0:=dz\cdot d\bar{z}$. Then we obtain the following inequality on U_P:

$$\Delta_{g_0}\bigl(f-B\cdot\phi\bigr)\leq 0,\quad \phi=|z|^{2\epsilon}+|z|^{2\eta}.$$

For any point $Q\in\Delta(P,1/2)$, we have the following:

$$(f-\phi)(Q)\leq\frac{4}{\pi}\int_{\Delta(Q,1/2)}(f-\phi)\cdot\mathrm{dvol}_{g_0}.$$

Therefore there exist some constants C_i $(i=3,4)$ which are independent of ϵ, such that the following holds:

$$f(Q)\leq C_3+C_4\int f\cdot\mathrm{dvol}_{\omega_\epsilon}.$$

Thus we obtain the upper bound of $f(Q)$, when Q is close to a point of D. We can obtain such an estimate when Q is far from D, similarly and more easily. ∎

2.5.6 Variant 2 of Proposition 2.38

We will use another variant. Let $\pi:\mathcal{C}\longrightarrow\Delta$ be a holomorphic family of smooth projective curves. Let $\mathcal{D}\subset\mathcal{C}$ be a relative divisor. Let $(E,\mathbb{D}^\lambda,\boldsymbol{F})$ be a λ-flat parabolic bundle on $(\mathcal{C},\mathcal{D})$. We denote the fiber $\pi^{-1}(t)$ by $\mathcal{C}_t$ for $t\in\Delta$. The restriction $(E,\mathbb{D}^\lambda,\boldsymbol{F})_{|\mathcal{C}_t}$ is denoted by $(E_t,\mathbb{D}^\lambda_t,\boldsymbol{F}_t)$. Let ω be a metric of the relative tangent bundle of $\mathcal{C}/\Delta$ such that $\omega\sim\eta^2|z|^{2\eta-2}dz\cdot d\bar{z}$ around $\mathcal{D}$. Here η denotes a small positive number such that $10\,\mathrm{rank}(E)\cdot\eta<\mathrm{gap}(E,\boldsymbol{F})$, and z is holomorphic function such that $z^{-1}(0)=\mathcal{D}$ and $dz\neq 0$. The restriction $\omega_{|\mathcal{C}_t}$ is denoted by ω_t for $t\in\Delta$. Let h be a C^∞-hermitian metric of E adapted to $\boldsymbol{F}$ such that $\bigl|\Lambda_{\omega_t}G(\mathbb{D}^\lambda_t,h_t)\bigr|_{h_t}\leq C_1$ for any $t\in\Delta$, where a constant C_1 is independent of t, and h_t denotes the restriction $h_{|\mathcal{C}_t}$. The following lemma can be shown by an argument similar to the proof of Lemma 2.44.

Lemma 2.45 *There exist positive constants C_i $(i=3,4)$, which are independent of t, with the following property.*

- *Let $s^{(t)}$ be an element of $\mathcal{P}_{h_t}(E_t)$ satisfying $\mathrm{tr}\,s^{(t)}=0$, $\|s^{(t)}\|_{h_t,P}<\infty$ and $\bigl|\Lambda_{\omega_t}G(\mathbb{D}^\lambda_t,h_te^{s^{(t)}})\bigr|\leq C_1$. Then the inequality $\sup|s^{(t)}|\leq C_3+C_4\cdot M(h_t,h_te^{s^{(t)}})$ holds.* ∎

2.6 The Integral of the Pseudo Curvature of Non-flat λ-connection on a Curve

Let Y be a smooth projective curve, and let D be a divisor. Let $(E,\boldsymbol{F})$ be a parabolic bundle on (Y,D). Let $\mathbb{D}^{\lambda}$ be a C^{∞} λ-connection on $E_{|Y-D}$. In this subsection, we do not assume $\mathbb{D}^{\lambda}$ is flat, i.e., $(\mathbb{D}^{\lambda})^2$ may not be 0. But we assume that it is flat around an appropriate neighbourhood U_P of each $P\in D$, and $(E,\boldsymbol{F},\mathbb{D}^{\lambda})_{|U_P}$ is a parabolic λ-flat bundle. In particular, we have $\mathrm{Res}_P(\mathbb{D}^{\lambda})\in\mathrm{End}(E_{|P})$. We assume that it is graded semisimple, for simplicity. For the later use (the subsection 3.5), we calculate the integral of the trace of the pseudo curvature.

For each $P\in D$, we have the generalized eigen decomposition $E_{|P}:=\bigoplus {}^P\mathbb{E}_{\alpha}$ of $\mathrm{Res}_P(\mathbb{D}^{\lambda})$. We also have the filtration ${}^PF$ of $E_{|P}$. Let us take a holomorphic frame $\boldsymbol{v}$ of $E_{|U_P}$, which is compatible with $({}^P\mathbb{E},{}^PF)$. We put $\alpha(v_i):=\deg^{\mathbb{E}}(v_i)$ and $a(v_i):=\deg^{F}(v_i)$. Let h be a C^{∞}-metric of $E_{|Y-D}$ such that $h(v_i,v_j)=|z|^{-2a(v_i)}$ $(i=j)$ and 0 $(i\neq j)$. Let us decompose $\mathbb{D}^{\lambda}=d''+d'$. Let us take a $(1,0)$-operator d_0' such that $d''+d_0'$ is C^{∞} λ-connection of E on Y. We also assume $d_0'\boldsymbol{v}=0$. We put $A:=d'-d_0'$, which is a C^{∞}-section of $\mathrm{End}(E)\otimes\Omega^{1,0}(\log D)$ on Y, and holomorphic around D. We have $\mathrm{tr}\,\mathrm{Res}_P(A)=\mathrm{tr}\,\mathrm{Res}_P(\mathbb{D}^{\lambda})$.

Let h_0 be a C^{∞}-metric of E on Y such that $h_0(v_i,v_j)$ is 1 $(i=j)$ or 0 $(i\neq j)$ on U_P $(P\in D)$. Let s be the endomorphism determined by $h=h_0\cdot s$. Then s is described by the diagonal matrix $\mathrm{diag}\bigl(|z|^{-2a(v_1)},\ldots,|z|^{-2a(v_r)}\bigr)$ with respect to the frame $\boldsymbol{v}$ on U_P.

Although $\mathbb{D}^{\lambda}$ is not necessarily flat, we obtain the operators δ_h', δ_h'', $\overline{\partial}_h$, ∂_h, θ_h and $\theta_h^{\dagger}$ as in the subsection 2.2.1. We also have $\mathbb{D}_h^{\lambda\star}=\delta_h'-\delta_h''$. Then we put $G(\mathbb{D}^{\lambda},h):=[\mathbb{D}^{\lambda},\mathbb{D}_h^{\lambda\star}]$ for the non-flat λ-connection $\mathbb{D}^{\lambda}$.

Remark 2.46 *Since $\mathbb{D}^{\lambda}$ is not assumed to be flat, $G(h)=-\bigl(1+|\lambda|^2\bigr)^2\lambda^{-1}\bigl(\overline{\partial}_h\theta_h\bigr)$ does not hold in general.* ∎

Lemma 2.47 *We have the following formula:*

$$\frac{\sqrt{-1}}{2\pi}\int\mathrm{tr}\,G(\mathbb{D}^{\lambda},h)=(1-|\lambda|^2)\cdot\deg(E)-\sum_P\Bigl(2\,\mathrm{Re}(\overline{\lambda}\cdot\mathrm{tr}\,\mathrm{Res}_P\,\mathbb{D}^{\lambda})+(1+|\lambda|^2)\cdot\mathrm{wt}(E,\boldsymbol{F},P)\Bigr). \tag{29}$$

We also have the following formula:

$$\frac{\sqrt{-1}}{2\pi}\int_Y\overline{\partial}\,\mathrm{tr}\,\theta=\frac{\lambda}{1+|\lambda|^2}\sum_P\Bigl(\lambda^{-1}\cdot\mathrm{tr}\,\mathrm{Res}_P\,\mathbb{D}^{\lambda}+\mathrm{wt}(E,\boldsymbol{F},P)\Bigr). \tag{30}$$

Proof By a direct calculation, we have $\mathbb{D}^{\lambda\star}=\lambda^{-1}d'-\overline{\lambda}d''-(1+|\lambda|^2)\cdot\bigl(\lambda^{-1}\cdot\theta_h+\theta_h^{\dagger}\bigr)$. Hence we obtain

$$G(\mathbb{D}^{\lambda},h)=[\mathbb{D}^{\lambda},\mathbb{D}^{\lambda\star}]=\frac{1-|\lambda|^2}{\lambda}[d',d'']+(1+|\lambda|^2)\cdot\mathbb{D}^{\lambda}(-\lambda^{-1}\theta_h+\theta_h^{\dagger}).$$

Therefore we have

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\int_Y\mathrm{tr}\,G(\mathbb{D}^{\lambda},h)&=\frac{\sqrt{-1}}{2\pi}\left(\int_Y\frac{1-|\lambda|^2}{\lambda}\,\mathrm{tr}[d',d'']+(1+|\lambda|^2)\int(\overline{\partial}+\lambda\partial)\,\mathrm{tr}\bigl(-\lambda^{-1}\theta_h+\theta_h^{\dagger}\bigr)\right)\\&=\frac{\sqrt{-1}}{2\pi}\left(\int_Y\frac{1-|\lambda|^2}{\lambda}\,\mathrm{tr}[d',d'']+(1+|\lambda|^2)\int\bigl(-\lambda^{-1}\overline{\partial}\,\mathrm{tr}\,\theta_h+\lambda\partial\,\mathrm{tr}\,\theta_h^{\dagger}\bigr)\right).\end{aligned} \tag{31}$$

Recall $d''+\lambda^{-1}d_0'$ gives the C^{∞}-connection of $(E,\boldsymbol{F})$ in the usual sense. Hence $\lambda^{-1}\,\mathrm{tr}[d'',d_0']$ gives the first Chern class of E. Therefore we have

$$\frac{\sqrt{-1}}{2\pi}\int_Y\mathrm{tr}[d',d'']=\frac{\sqrt{-1}}{2\pi}\int_Y\mathrm{tr}[d_0',d'']+\frac{\sqrt{-1}}{2\pi}\int_Y\overline{\partial}\,\mathrm{tr}\,A=\lambda\deg(E)+\sum_P\mathrm{tr}\,\mathrm{Res}_P\,\mathbb{D}^{\lambda}. \tag{32}$$

Let us consider the integral of $\overline{\partial}\,\mathrm{tr}\,\theta_h$. Let δ_{h_0}' denote the $(1,0)$-operator obtained from d'' and h_0 as in the subsection 2.2.1. Then we have

$$\theta_h=\frac{1}{1+|\lambda|^2}(d'-\lambda\cdot\delta_h')=\frac{1}{1+|\lambda|^2}(d_0'-\lambda\cdot\delta_{h_0}')+\frac{1}{1+|\lambda|^2}(A-\lambda\cdot s^{-1}\delta_{h_0}'s).$$

We would like to apply the Stokes formula to the integral of $\overline{\partial}\operatorname{tr}\theta_h$. If we do so, $d_0'-\lambda\delta_{h_0}'$ does not contribute, because it is the C^∞-section of $\mathrm{End}(E)\otimes\Omega^{1,0}$. We have

$$\frac{\sqrt{-1}}{2\pi}\int_Y\overline{\partial}\operatorname{tr}(A)=\sum_P\operatorname{tr}\mathrm{Res}_P\,\mathbb{D}^\lambda.$$

Since $s^{-1}\delta_{h_0}'s$ is described by $\mathrm{diag}\bigl(-a(v_1),\ldots,-a(v_r)\bigr)\cdot dz/z$ with respect to $\boldsymbol{v}$ on U_P $(P\in D)$, we have

$$\frac{\sqrt{-1}}{2\pi}\int_Y\overline{\partial}\operatorname{tr}(s^{-1}\delta_{h_0}'s)=\sum_P\sum_{i=1}^{\operatorname{rank}E}-a(v_i)=-\sum_P\mathrm{wt}(E,\boldsymbol{F},P).$$

Therefore, we obtain the following formula:

$$\frac{\sqrt{-1}}{2\pi}\frac{1+|\lambda|^2}{\lambda}\int\overline{\partial}\operatorname{tr}\theta_h=\sum_P\Bigl(\lambda^{-1}\operatorname{tr}\mathrm{Res}_P\,\mathbb{D}^\lambda+\mathrm{wt}(E,\boldsymbol{F},P)\Bigr).\tag{33}$$

In particular, we obtain (30).

Let us consider the integral of $\partial\operatorname{tr}\theta_h^\dagger$. Let δ_{0,h_0}'' (resp. $\delta_{0,h}''$) denote the operator obtained from d_0' and h_0 (resp. h) as in the subsection 2.2.1. We have $\delta_h''=\delta_{0,h}''-A_h^\dagger=\delta_{0,h_0}''+s^{-1}\delta_{0,h_0}''s-A_h^\dagger$. (See the proof of Lemma 2.21 for the notation "$A_h^\dagger$".) Hence we have

$$\theta_h^\dagger=\frac{1}{1+|\lambda|^2}(\overline{\lambda}d''-\delta_{0,h_0}'')+\frac{1}{1+|\lambda|^2}(A_h^\dagger-s^{-1}\delta_{0,h_0}''s).$$

Again, we would like to apply the Stokes formula to the integral of $\partial\operatorname{tr}\theta^\dagger$. Since $\overline{\lambda}d''-\delta_{0,h_0}''$ is a C^∞-section of $\mathrm{End}(E)\otimes\Omega^{0,1}$, the contribution is 0. As for the other terms, we have the following:

$$\frac{\sqrt{-1}}{2\pi}\int\partial\operatorname{tr}A_h^\dagger=-\sum_P\overline{\operatorname{tr}\mathrm{Res}_P\,\mathbb{D}^\lambda},\qquad\frac{\sqrt{-1}}{2\pi}\int_Y\partial\bigl(\operatorname{tr}s^{-1}\delta_{0,h_0}''s\bigr)=\sum_P\overline{\lambda}\,\mathrm{wt}(E,\boldsymbol{F},P).$$

Hence we obtain the following formula:

$$\frac{\sqrt{-1}}{2\pi}(1+|\lambda|^2)\lambda\int_Y\partial\operatorname{tr}\theta_h^\dagger=-\sum_P\Bigl(\lambda\cdot\overline{\operatorname{tr}\mathrm{Res}_P\,\mathbb{D}^\lambda}+|\lambda|^2\,\mathrm{wt}(E,\boldsymbol{F},P)\Bigr).\tag{34}$$

The formula (29) immediately follows from (32), (33) and (34). ∎

Remark 2.48 *When $\mathbb{D}^\lambda$ is flat, we have the relation $G(h)=-(1+|\lambda|^2)^2\lambda^{-1}\overline{\partial}_h\theta_h$, and hence the formulas* (29) *and* (30) *give some equality. But we obtain only the well known formulas.*

$$\deg(E)+\sum\mathrm{Re}\bigl(\lambda^{-1}\operatorname{tr}\mathrm{Res}_P\,\mathbb{D}^\lambda\bigr)=0,\quad\sum_P\mathrm{Im}(\lambda^{-1}\operatorname{tr}\mathrm{Res}_P\,\mathbb{D}^\lambda)=0.$$

Such a consideration leads us some results in the surface case. See the subsection 3.5. ∎

3 Ordinary metric and some consequences

We give a construction of an ordinary metric for a parabolic λ-flat bundle on a surface, and we give the estimate for the pseudo curvature. The construction is essentially same as that for the parabolic Higgs bundle, given in the section 4 of [30]. Namely, we give the constructions and the estimates around the intersection of the divisor (the subsection 3.1) and the smooth point of the divisor (the subsection 3.2), and then we immediately obtain a global construction and estimate (the subsection 3.3). However, we can derive some additional information about the characteristic numbers in the case $\lambda\neq 0$. Hence we give the detail.

In the following of this paper, "parabolic" means "$\boldsymbol{c}$-parabolic" for some tuple of real numbers $\boldsymbol{c}$, and we prefer to use the notation $(E,\boldsymbol{F})$ or E_* to denote a parabolic bundle instead of $({}_{\boldsymbol{c}}E,\boldsymbol{F})$ or ${}_{\boldsymbol{c}}E_*$, as is explained in the subsection 2.1.6.

3.1 Around the Intersection of the Divisor

3.1.1 Construction of a metric

We put $X := \Delta^2$, $D_i := \{z_i = 0\}$, and $D := D_1 \cup D_2$. We use the metric $\omega_\epsilon = \sum_{i=1,2} \epsilon^{N+2}|z_i|^{2\epsilon-2} \cdot dz_i \cdot d\overline{z}_i$ of $X - D$ in this subsection. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a *graded semisimple* λ-flat parabolic bundle on (X, D). Assume $10\epsilon < \mathrm{gap}(E, \boldsymbol{F})$. We have the endomorphism $\mathrm{Res}_i(\mathbb{D}^\lambda)$ of $E_{|D_i}$. We also have the naturally induced flat λ-connection ${}^i\mathbb{D}^\lambda$ of $E_{|D_i}$, i.e., for a section f of $E_{|D_i}$, let us take a section $\widetilde{f}$ of E such that $\widetilde{f}_{|D_i} = f$ and ${}^i\mathbb{D}^\lambda f := \mathbb{D}^\lambda \widetilde{f}_{|D_i}$.

When $\lambda \neq 0$, the eigenvalues of $\mathrm{Res}_i(\mathbb{D}^\lambda)$ are constant, since $\mathrm{Res}_i(\mathbb{D}^\lambda)$ is flat with respect to ${}^i\mathbb{D}^\lambda$. We have the generalized eigen decomposition $E_{|D_i} = \bigoplus {}^i\mathbb{E}_\alpha$. The tuple $({}^1F, {}^2F, {}^1\mathbb{E}, {}^2\mathbb{E})$ is compatible in the sense of [28], i.e., we have the following:

$$ {}^iF_a = \bigoplus_\alpha {}^iF_a \cap {}^i\mathbb{E}_\alpha, \quad \underline{2}F_{\boldsymbol{a}}(E_{|O}) = \bigoplus \underline{2}F_{\boldsymbol{a}}(E_{|O}) \cap \underline{2}\mathbb{E}_{\boldsymbol{\alpha}}(E_{|O}). $$

Here we put $\underline{2}F_{(a_1,a_2)}(E_{|O}) = {}^1F_{a_1\,|\,O} \cap {}^2F_{a_2\,|\,O}$ and $\underline{2}\mathbb{E}_{(\alpha_1,\alpha_2)}(E_{|O}) = {}^1\mathbb{E}_{\alpha_1\,|\,O} \cap {}^2\mathbb{E}_{\alpha_2\,|\,O}$. Let us take a holomorphic decomposition:

$$ E = \bigoplus_{(\boldsymbol{a},\boldsymbol{\alpha}) \in \boldsymbol{R}^2 \times \boldsymbol{C}^2} U_{\boldsymbol{a},\boldsymbol{\alpha}} \quad \text{such that} \quad {}^i\mathbb{E}_\alpha \cap {}^iF_a = \bigoplus_{\substack{q_i(\boldsymbol{a}) \leq a \\ q_i(\boldsymbol{\alpha}) = \alpha}} U_{\boldsymbol{a},\boldsymbol{\alpha}\,|\,D_i}, \quad \underline{2}\mathbb{E}_{\boldsymbol{\alpha}} \cap \underline{2}F_{\boldsymbol{a}} = \bigoplus_{\boldsymbol{b} \leq \boldsymbol{a}} U_{\boldsymbol{b},\boldsymbol{\alpha}\,|\,O}. $$

Let us take a holomorphic frame $\boldsymbol{v}$ of E compatible with the decomposition. We put $\alpha(v_i) := {}^i\deg^{\mathbb{E}}(v_i)$ and $a(v_i) := {}^i\deg^F(v_i)$. Let h be a metric such that $h(v_i, v_j) = \delta_{i,j}|z_1|^{-2a_1(v_i)} \cdot |z_2|^{-2a_2(v_i)}$, where $\delta_{i,j}$ denotes 1 $(i = j)$ or 0 $(i \neq j)$.

One of our main purpose of this subsection (Lemma 3.1) is to show the boundedness of $G(h, \mathbb{D}^\lambda)$ with respect to ω_ϵ and h. However, we will also need more close estimate, which relate $G(h, \mathbb{D}^\lambda)$ and the pseudo curvatures on the metrized λ-connection on the divisors, which we explain in the next subsubsection.

3.1.2 Objects on the divisors

Let ${}^1\mathrm{Gr}^F(E_{|D_1})$ denote the graded bundle associated to the filtration ${}^1F$. We have the generalized eigen decomposition ${}^1\mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}(E_{|D_1})$ with respect to $\mathrm{Res}_1(\mathbb{D}^\lambda)$. We put ${}^1\widehat{E}_u := {}^1\mathrm{Gr}^{F,\mathbb{E}}_u(E_{|D_1})$, and ${}^1\widehat{E} := \bigoplus {}^1\widehat{E}_u$. Due to the graded semisimplicity assumption, the residue $\mathrm{Res}_1(\mathbb{D}^\lambda)$ induces the endomorphism $\bigoplus_{(a,\alpha)} \alpha \cdot \mathrm{id}_{{}^1\widehat{E}_{(a,\alpha)}}$. Since ${}^1\mathbb{D}^\lambda$ preserves ${}^1F$ and ${}^1\mathbb{E}$, the flat λ-connection ${}^1\widehat{\mathbb{D}}^\lambda$ of ${}^1\widehat{E}$ is induced. We put as follows:

$$ {}^1U_{a,\alpha} := \bigoplus_{q_1(\boldsymbol{a},\boldsymbol{\alpha}) = (a,\alpha)} U_{\boldsymbol{a},\boldsymbol{\alpha}}. $$

Then we have the natural isomorphism ${}^1U_{u\,|\,D_1} \simeq {}^1\widehat{E}_u$, which induces the identification of the holomorphic bundles ${}^1\widehat{E} \simeq E_{|D_1}$. Let ${}^1h$ be a metric of $E_{|D_1}$ given by ${}^1h\big(v_{i\,|\,D_1}, v_{j\,|\,D_1}\big) := \delta_{i,j}|z_2|^{-a_2(v_i)}$. Then the λ-connection ${}^1\widehat{\mathbb{D}}^\lambda$ and the metric ${}^1h$ induce the operators ${}^1\partial$, ${}^1\theta$ and $G({}^1\widehat{\mathbb{D}}^\lambda, {}^1h)$ on ${}^1E$. Similarly, we obtain the metric ${}^2h$, ${}^2\partial$, ${}^2\theta$ and $G({}^2\widehat{\mathbb{D}}^\lambda, {}^2h)$ for $E_{|D_2}$,

3.1.3 Estimate

Let us estimate $G(h) = -(1 + |\lambda|^2)^2\lambda^{-1}\big(\overline{\partial}_h^2 + \overline{\partial}_h\theta_h + \theta_h^2\big)$ and $\overline{\partial}_h\theta_h$. For the projection $X \longrightarrow D_i$, we give the isomorphism $\pi_i^* E_{|D_i} \simeq E$ via the frames $\pi^{-1}\boldsymbol{v}_{|D_i}$ and $\boldsymbol{v}$. We put $\Gamma_i := \bigoplus a_i \cdot \mathrm{id}_{U_{\boldsymbol{a},\boldsymbol{\alpha}}}$ and $\mathcal{Q}_i := \bigoplus \alpha_i \cdot \mathrm{id}_{U_{\boldsymbol{a},\boldsymbol{\alpha}}}$.

Lemma 3.1 *$G(\mathbb{D}^\lambda, h)$ is bounded with respect to (ω_ϵ, h). More closely, we have the following estimate:*

$$ G(\mathbb{D}^\lambda, h) = \pi_1^* G({}^1\mathbb{D}^\lambda, {}^1h) + \pi_2^* G({}^2\mathbb{D}^\lambda, {}^2h) + O(|z_1|^{3\epsilon}|z_2|^{3\epsilon}). \tag{35} $$

In particular, $\overline{\partial}_h\theta_h$ is bounded with respect to (ω_ϵ, h). More closely, we have the following estimate:

$$\overline{\partial}_h\theta_h = \pi_1^*\big({}^1\overline{\partial}({}^1\theta)\big) + \pi_2^*\big({}^2\overline{\partial}({}^2\theta_2)\big) + O(|z_1|^{3\epsilon}|z_2|^{3\epsilon}). \tag{36}$$

We also have the following estimate with respect to ω_ϵ and h:

$$\theta = \sum_{i=1,2} \pi_i^*\big({}^i\theta\big) + \frac{1}{1+|\lambda|^2}\sum_{i=1,2}\big(\mathcal{Q}_i + \lambda\Gamma_i\big)\cdot\frac{dz_i}{z_i} + O(1) \tag{37}$$

Lemma 3.2 *We put $Y_\delta := \{|z_1| = \delta,\ |z_2| \geq \delta\} \cup \{|z_2| = \delta\ |z_1| \geq \delta\}$. Then we obtain the following:*

$$\lim_{\delta\to 0}\int_{Y_\delta} \mathrm{tr}\big(\theta_h^2\cdot\theta_h^\dagger\big) = 0.$$

The proof is given in the rest of this subsection.

3.1.4 Preliminary

The diagonal matrix valued functions H_k $(k = 1,2)$ are given by $H_k := \mathrm{diag}\Big(|z_k|^{-2a_k(v_1)}, \ldots, |z_k|^{-2a_k(v_r)}\Big)$. We also put $H := H_1 \cdot H_2$, and then we have $H = \big(h(v_i, v_j)\big)$, and $\partial_h \boldsymbol{v} = \boldsymbol{v}\cdot H^{-1}\partial H$. We also have $H_k = \big({}^l h(v_{i\,|\,D_l},\, v_{j\,|\,D_l})\big)$ and ${}^l\partial\boldsymbol{v}_{|D_l} = \boldsymbol{v}_{|D_l}\cdot H_k^{-1}\partial H_k$, for $l \neq k$. We also remark $H^{-1}\cdot\big(\partial H/\partial z_k\big) = H_k^{-1}\cdot\big(\partial H_k/\partial z_k\big)$.

The matrix-valued functions A_i are determined by $\mathbb{D}^\lambda\boldsymbol{v} = \boldsymbol{v}\sum A_i\cdot dz_i/z_i$. Then $\mathrm{Res}_i(\mathbb{D}^\lambda)\boldsymbol{v}_{|D_i} = \boldsymbol{v}_{|D_i}\cdot A_{i\,|\,D_i}$ and ${}^i\mathbb{D}^\lambda\boldsymbol{v}_{|D_i} = \boldsymbol{v}_{|D_i}\cdot A_{j\,|\,D_i}\cdot dz_j/z_j$ for $j \neq i$. The diagonal matrix-valued function A_k^d $(k = 1,2)$ are given by $A_k^d := \mathrm{diag}\big(\alpha_k(v_1), \ldots, \alpha_k(v_r)\big)$. We put $N_i := A_i - A_i^d$, and $N = \sum N_i\cdot dz_i/z_i$.

Lemma 3.3 *Let $\mathcal{N}_i$ denote the endomorphism of E determined by $\mathcal{N}_i\boldsymbol{v} = \boldsymbol{v}\cdot N_i$.*

- *We have the estimate $\big|\mathcal{N}_i\big|_h = O(|z_i|^{5\epsilon})$.*
- *Let $\mathcal{F}$ be an endomorphism of E of the form $\mathcal{F} = \bigoplus\kappa(\boldsymbol{a},\boldsymbol{\alpha})\cdot\mathrm{id}_{U_{\boldsymbol{a},\boldsymbol{\alpha}}}$, where $\kappa(\boldsymbol{a},\boldsymbol{\alpha})$ denote complex numbers. Then we have $\big|[\mathcal{F},\mathcal{N}_i]\big|_h = O\big(|z_1|^{5\epsilon}|z_2|^{5\epsilon}\big)$.*
- $\big|[\mathcal{N}_1,\mathcal{N}_2]\big|_h = O(|z_1|^{5\epsilon}\cdot|z_2|^{5\epsilon})$.

Proof $\mathcal{N}_{i\,|\,D_j}$ preserves ${}^j\mathbb{E}$ and ${}^jF$. If $i = j$, we have $\mathcal{N}_{i\,|\,D_i}({}^iF_a) \subset {}^iF_{<a}$ due to the graded semisimplicity assumption. Then the first claim follows. The third claim immediately follows from the first one. The second claim follows from $[\mathcal{F},\mathcal{N}_i]_{|D_k}({}^kF_a) \subset {}^kF_{<a}$ for $k = 1,2$. ∎

Remark 3.4 *In the following argument, the norm of the matrix is taken for the metric h. Namely, for a matrix valued function B, we have the endomorphism F_B determined by $F_B(\boldsymbol{v}) = \boldsymbol{v}\cdot B$. And, the norm of B with respect to h means the norm of F_B with respect to h.* ∎

Lemma 3.5 *Lemma 3.3 can be restated as follows.*

- *$N_i = O\big(|z_i|^{5\epsilon}\big)$ with respect to the metric h.*
- *Let T be a constant diagonal matrix. Then $[T, N_i] = O\big(|z_1|^{5\epsilon}|z_2|^{5\epsilon}\big)$ with respect to h.*
- *$[N_1, N_2] = O\big(|z_1|^{5\epsilon}|z_2|^{5\epsilon}\big)$ with respect to h.* ∎

By a similar argument, we can show the following lemma.

Lemma 3.6 *We have the following estimates for $(i,j) = (1,2)$ or $(2,1)$, with respect to (ω_ϵ, h):*

$$\frac{\partial N_i}{\partial z_j}\cdot dz_j\cdot\frac{dz_i}{z_i} = O\big(|z_1|^{5\epsilon}|z_2|^{5\epsilon}\big)\frac{dz_1\cdot dz_2}{z_1\cdot z_2},$$

∎

The matrix-valued $(1,0)$-forms $Q(\delta_h')$ and $Q(\overline{\partial}_h)$ are determined by $\delta_h'\boldsymbol{v} = \boldsymbol{v}\cdot Q(\delta_h')$ and $\overline{\partial}_h\boldsymbol{v} = \boldsymbol{v}\cdot Q(\overline{\partial}_h)$ respectively. Then we have $Q(\delta_h') = H^{-1}\partial H$ and $Q(\overline{\partial}_h) = \lambda(1+|\lambda|^2)^{-1}Q(\delta_h'')$. The matrix-valued $(0,1)$-forms $Q(\delta_h'')$ and Θ are determined by $\delta_h''\boldsymbol{v} = \boldsymbol{v}\cdot Q(\delta_h'')$ and $\theta_h\boldsymbol{v} = \boldsymbol{v}\cdot\Theta$ respectively. Then we have $Q(\delta_h'') = \overline{\lambda}H^{-1}\overline{\partial}H - A_h^\dagger$ and $\Theta = (1+|\lambda|^2)^{-1}(A - \lambda Q(\delta_h'))$, where $A_h^\dagger = H^{-1}\,{}^t\overline{A}H$.

3.1.5 Proof of the estimate (37) and Lemma 3.2

The estimate (37) is easy to see, as follows:

$$\Theta=\frac{1}{1+|\lambda|^2}\bigl(A-\lambda Q(\delta'_h)\bigr)=\frac{1}{1+|\lambda|^2}\bigl(A^d-\lambda H^{-1}\partial H\bigr)+\sum_{i=1,2}O(|z_i|^{5\epsilon})\frac{dz_i}{z_i}$$
$$=\frac{1}{1+|\lambda|^2}\sum_{i=1,2}(\mathcal{Q}_i+\lambda\Gamma_i)\frac{dz_i}{z_i}+\sum_{i=1,2}O(|z_i|^{5\epsilon})\frac{dz_i}{z_i}. \quad (38)$$

Then we obtain $\theta^2=O(|z_1|^{5\epsilon}|z_2|^{5\epsilon})\cdot dz_1\cdot dz_2/z_1\cdot z_2$ with respect to h, from Lemma 3.5 and (38). Thus we obtain the following estimate:

$$\theta^2\theta^\dagger=O\bigl(|z_1|^{5\epsilon}|z_2|^{5\epsilon}\bigr)\frac{dz_1\cdot d\overline{z}_1}{|z_1|^2}\frac{dz_2}{z_2}+O\bigl(|z_1|^{5\epsilon}|z_2|^{5\epsilon}\bigr)\frac{dz_1}{z_1}\frac{dz_2\cdot d\overline{z}_2}{|z_2|^2}$$

Then the claim of Lemma 3.2 immediately follows. ∎

3.1.6 Proof of Lemma 3.1

Let us show that $\overline{\partial}_h^2$ is small. We have $\overline{\partial}_h^2\boldsymbol{v}=\boldsymbol{v}\bigl(\overline{\partial}Q(\overline{\partial}_h)+Q(\overline{\partial}_h)\circ Q(\overline{\partial}_h)\bigr)$. Let us see $\overline{\partial}Q(\overline{\partial}_h)$:

$$\overline{\partial}Q(\overline{\partial}_h)=\frac{\lambda}{1+|\lambda|^2}\bigl(\overline{\lambda}\overline{\partial}(H^{-1}\overline{\partial}H)-\overline{\partial}A_h^\dagger\bigr)=\frac{-\lambda}{1+|\lambda|^2}\overline{\partial}A_h^\dagger.$$

We have $\overline{\partial}A_h^\dagger=\overline{\partial}\bigl(H^{-1}{}^t\overline{A}H\bigr)=H^{-1}\overline{\partial}({}^t\overline{A})H+\bigl[A_h^\dagger,H^{-1}\overline{\partial}H\bigr]=H^{-1}{}^t\overline{(\partial N)}H+\bigl[N_h^\dagger,H^{-1}\overline{\partial}H\bigr]$. Then we know the following estimate with respect to the metric h, due to Lemma 3.5 and Lemma 3.6:

$$\partial N=O\bigl(|z_1|^{5\epsilon}\cdot|z_2|^{5\epsilon}\bigr)\frac{dz_1\cdot dz_2}{z_1\cdot z_2},\qquad \bigl[N,H^{-1}\partial H\bigr]=O\bigl(|z_1|^{5\epsilon}\cdot|z_2|^{5\epsilon}\bigr)\frac{dz_1\cdot dz_2}{z_1\cdot z_2}.$$

Therefore we obtain $\overline{\partial}A_h^\dagger=O\bigl(|z_1|^{3\epsilon}\cdot|z_2|^{3\epsilon}\bigr)$ with respect to (ω_ϵ,h). Let us see the term $Q(\overline{\partial}_h)\circ Q(\overline{\partial}_h)$:

$$Q(\overline{\partial}_h)\circ Q(\overline{\partial}_h)=\frac{1}{2}\left(\frac{\lambda}{1+|\lambda|^2}\right)^2\Bigl[\overline{\lambda}H^{-1}\overline{\partial}H-A_h^\dagger,\ \overline{\lambda}H^{-1}\overline{\partial}H-A_h^\dagger\Bigr]$$
$$=\frac{1}{2}\left(\frac{\lambda}{1+|\lambda|^2}\right)^2\Bigl(-2\overline{\lambda}\bigl[H^{-1}\overline{\partial}H,\ N_h^\dagger\bigr]+2\bigl[A_h^{d\,\dagger},N_h^\dagger\bigr]+\bigl[N_h^\dagger,N_h^\dagger\bigr]\Bigr) \quad (39)$$

As in the case of $\overline{\partial}Q(\overline{\partial}_h)$, we obtain $Q(\overline{\partial}_h)\circ Q(\overline{\partial}_h)=O(|z_1|^{3\epsilon}|z_2|^{3\epsilon})$ with respect to (ω_ϵ,h), from Lemma 3.5.

We have $(1+|\lambda|^2)^2\Theta\circ\Theta=\bigl[A-\lambda H^{-1}\partial H,\ A-\lambda H^{-1}\partial H\bigr]=2\bigl[N,A^d-\lambda H^{-1}\partial H\bigr]+[N,N]$. This can be estimated in the same way as $Q(\overline{\partial}_h)\circ Q(\overline{\partial}_h)$, due to Lemma 3.5. Thus we obtain $\theta_h\circ\theta_h=O(|z_1|^{3\epsilon}|z_2|^{3\epsilon})$.

Let us see $\overline{\partial}_h\theta_h$. We have $(\overline{\partial}_h\theta_h)\boldsymbol{v}=\boldsymbol{v}\bigl(\overline{\partial}\Theta+[Q(\overline{\partial}),\Theta]\bigr)$. We have $\overline{\partial}\Theta=-\lambda\cdot(1+|\lambda|^2)^{-1}\cdot\overline{\partial}Q(\delta'_h)=0$ by a direct calculation. We also have the following:

$$[Q(\overline{\partial}_h),\Theta]=\frac{1}{(1+|\lambda|^2)^2}\bigl[\lambda Q(\delta''_h),A-\lambda Q(\delta'_h)\bigr]=\frac{\lambda}{(1+|\lambda|^2)^2}\bigl[\overline{\lambda}H^{-1}\overline{\partial}H-A_h^\dagger,\ A-\lambda H^{-1}\partial H\bigr]$$
$$=\frac{\lambda}{(1+|\lambda|^2)^2}\Bigl(\bigl[\overline{\lambda}H^{-1}\overline{\partial}H-A_h^{d\,\dagger},\ N\bigr]-\bigl[N_h^\dagger,A^d-\lambda H^{-1}\partial H\bigr]-\bigl[N_h^\dagger,N\bigr]\Bigr). \quad (40)$$

The boundedness of the right hand side easily follows from Lemma 3.5. Moreover, we can see that the terms containing $dz_i\cdot d\overline{z}_j$ $(i\neq j)$ are dominated as $O(|z_1|^{3\epsilon}|z_2|^{3\epsilon})$ with respect to (ω_ϵ,h).

Let us see the terms containing $dz_i\cdot d\overline{z}_i$ more closely. Let $A_2=\sum A_{2,u,u'}$ be the decomposition corresponding to $E=\bigoplus{}^1U_u$. We put $A_2^p:=\sum A_{2,u,u}$ and $A_2^r:=A_2-A_2^p$. Then we have ${}^1\widehat{\mathbb{D}}^\lambda\boldsymbol{v}_{|D_1}=\boldsymbol{v}_{|D_1}\cdot A^p_{2\,|\,D_1}\cdot dz_2/z_2$. We have the following with respect to h:

$$A_2=\pi_1^*(A^p_{2|D_1})+O\bigl(|z_1|^{5\epsilon}|z_2|^{5\epsilon}\bigr)\frac{dz_2}{z_2}$$

Hence we obtain the following estimates:

$$\Big[\overline{\lambda} H^{-1}\overline{\partial}_2 H-(A_2)^{\dagger}_h\frac{d\overline{z}_2}{\overline{z}_2},\ A_2\frac{dz_2}{z_2}-\lambda H^{-1}\partial_2 H\Big]$$
$$=\pi_1^{*}\Big[\overline{\lambda} H_2^{-1}\overline{\partial}_2 H_2-(A^p_{2\,|\,D_1})^{\dagger}_{1h}\frac{d\overline{z}_2}{\overline{z}_2},\ A^p_{2\,|\,D_1}\frac{dz_2}{z_2}-\lambda H_2^{-1}\partial_2 H_2\Big]+O\big(|z_1|^{5\epsilon}|z_2|^{5\epsilon}\big)\frac{dz_2\cdot d\overline{z}_2}{|z_2|^2}. \quad (41)$$

Here $\overline{\partial}_2 H$ denotes $(\partial H/\partial\overline{z}_2)\cdot d\overline{z}_2$, and the meaning of $\partial_2 H$ is similar. We have a similar estimate for $\big[\overline{\lambda} H^{-1}\overline{\partial}_1 H-(A_1)^{\dagger}_h,\ A_1-\lambda H^{-1}\partial_1 H\big]$. On the other hand, we have the following formula for ${}^1\overline{\partial}({}^1\theta)$, as in the case of $\overline{\partial}_h\theta_h$ (see (40), for example):

$$\big({}^1\overline{\partial}({}^1\theta)\big)\boldsymbol{v}_{|D_1}=\boldsymbol{v}_{|D_1}\Big[\overline{\lambda} H_2^{-1}\overline{\partial}_2 H_2-(A^p_{2\,|\,D_1})^{\dagger}_{1h}\frac{d\overline{z}_2}{\overline{z}_2},\ A^p_{2\,|\,D_1}\frac{dz_2}{z_2}-\lambda H_2^{-1}\partial_2 H_2\Big]. \quad (42)$$

Thus we obtain the estimate (35). Since we have already shown that $\overline{\partial}^2$ and θ^2 are sufficiently small, we also obtain (36). ∎

3.2 Around the Smooth Part of the Divisor

3.2.1 Construction of a metric

Let Y be a complex curve, and let $\pi: L\longrightarrow Y$ be a line bundle over Y. Let $|\cdot|$ denote a hermitian metric of L. We use the same notation to denote the induced hermitian metric on π^*L. Let $DL:=\{(y,s)\,|\,|s|\leq 1\}\subset L$. Let σ denote the canonical section $L\longrightarrow \pi^*L$.

Let J_{DL} denote the natural complex structure of DL. We denote by $\overline{\partial}$ and ∂ the natural $(0,1)$-operator and the $(1,0)$-operator respectively. Let J denote a given integrable complex structure of DL such that $J_{DL}-J=O(|\sigma|)$. We use the notation $\widetilde{\overline{\partial}}$ and $\widetilde{\partial}$ to denote the corresponding $(0,1)$-operator and $(1,0)$-operator. We put $s_Y:=\widetilde{\overline{\partial}}-\overline{\partial}=-\widetilde{\partial}+\partial$. For any point $Q\in Y$, we take a holomorphic coordinate (U_Q,z_1,z_2) around Q with respect to J such that $z_1^{-1}(0)=U_Q\cap Y$. For a real coordinate (x_1,x_2,x_3,x_4) given by $z_1=x_1+\sqrt{-1}x_2$, $z_2=x_3+\sqrt{-1}x_4$, we have the expression $s_Y=\sum f_{i,j}\cdot dx_j\cdot\partial_{x_i}+\sum g_j\cdot dx_j$, where $f_{i,j}$ and g_j are C^∞-functions such that $O(|z_1|)$.

Let $(E,F,\mathbb{D}^\lambda)$ be a parabolic flat λ-connection on (DL,Y) with respect to J, which is assumed to be *graded semisimple*. We have the decomposition $\mathbb{D}^\lambda_E=\widetilde{d}''_E+\widetilde{d}'_E$. We put $E_Y:=E_{|Y}$. Because of $\lambda\neq 0$, the eigenvalues of $\mathrm{Res}\,\mathbb{D}^\lambda$ are constant, and hence we have the decomposition $E_Y=\bigoplus\mathbb{E}_\alpha$. We have the parabolic filtration F of E_Y. We put $\widehat{E}_{Y,(a,\alpha)}:=\mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}(E_Y)$ and $\widehat{E}_Y:=\bigoplus\widehat{E}_{Y,(a,\alpha)}$. We have the surjection $\mathbb{E}_\alpha\cap F_a\longrightarrow\widehat{E}_{Y,(a,\alpha)}$. By taking a C^∞-splitting for each (a,α), we obtain the C^∞-identification $E_Y\simeq\widehat{E}_Y$. We put $S'':=d''_{E_Y}-d''_{\widehat{E}_Y}$.

We can take a C^∞-isomorphism $\Phi: E\simeq\pi^*E_Y$ for which $T:=\Phi(\widetilde{d}''_E)-d''_{\pi^*E_Y}$ is small in the following sense.

(T is small) For each $Q\in Y$ with the holomorphic coordinate (U_Q,z_1,z_2) as above, we have the following expression, where $F_{i,j}$ and G_j are C^∞-sections of $\mathrm{End}(\pi^*E_Y)$ such that $F_{i,j\,|\,Y}=G_{j\,|\,Y}=0$:

$$T=\sum F_{i,j}\cdot dx_i\otimes\frac{\partial}{\partial x_j}+\sum G_j\cdot dx_j,$$

In the following argument, we identify E and π^*E_Y via Φ. Let us take a C^∞-metric $h_{Y,(a,\alpha)}$ of $\widehat{E}_{Y,(a,\alpha)}$, and we put $h_Y:=\bigoplus_{(a,\alpha)}h_{Y,a,\alpha}$. We put as follows:

$$h'=\pi^*h_Y=\bigoplus\pi^*h_{Y,(a,\alpha)},\qquad h=\bigoplus\pi^*h_{Y,(a,\alpha)}\cdot|\sigma|^{-2a}.$$

Let ω be a Kahler form with respect to J. We put $\omega_\epsilon=\omega+C\cdot\epsilon^N\sqrt{-1}\widetilde{\partial}\widetilde{\overline{\partial}}|\sigma|^{2\epsilon}$, where ϵ be a small positive number such that $10\cdot\epsilon<\mathrm{gap}(E,\boldsymbol{F})$.

Our main purpose of this subsection is to show that $G(h,\mathbb{D}^\lambda)$ is bounded with respect to h and ω_ϵ (Lemma 3.8). However, we would like to derive a more detailed estimate relating $G(h,\mathbb{D}^\lambda)$ and the pseudo curvature of the λ-connection on Y, which we will explain in the next subsubsection.

3.2.2 The induced λ-connection on the divisor

We will often use the index u to denote an element $(a,\alpha)\in \boldsymbol{R}\times\boldsymbol{C}$. We put $E_u:=\pi^*\widehat{E}_{Y,u}$. We also put $\mathcal{Q}:=\bigoplus \alpha\cdot\mathrm{id}_{E_u}$ and $\Gamma:=\bigoplus_{(a,\alpha)} a\cdot\mathrm{id}_{E_u}$, We have the λ-connection $\widehat{\mathbb{D}}^{\lambda}:=\mathbb{D}^{\lambda}-\mathcal{Q}\cdot\widetilde{\partial}\log|\sigma|^2$, which is not necessarily flat. It gives the map:

$$C^{\infty}(E)\longrightarrow C^{\infty}\big(E\otimes\Omega^{1,0}(\log Y)\big)\oplus C^{\infty}(E\otimes\Omega^{0,1}).$$

Lemma 3.7 *$\widehat{\mathbb{D}}^{\lambda}$ induces the λ-connection $\mathbb{D}^{\lambda}_{\widehat{E}_Y}$ of $\widehat{E}_Y$, which is also not necessarily flat.*

Proof Let Q be a point of Y, and (U,z_1,z_2) be the holomorphic coordinate as above. Let f be a C^{∞}-section of E on U_Q such that $f_{|Y\cap U_Q}\in F_a(E)$. Let us decompose:

$$\widehat{\mathbb{D}}^{\lambda}v=f_1\frac{dz_1}{z_1}+f_{1'}d\overline{z}_1+f_2dz_2+f_{2'}d\overline{z}_2.$$

Due to the graded semisimplicity assumption, we obtain the following:

$$f_{1\,|\,Y\cap U_Q}\in F_{<a}(E),\quad f_{\kappa\,|\,Y\cap U_Q}\in F_a(E),\ (\kappa=1',2,2').$$

Let us see that $(f_2\cdot dz_2+f_{2'}\cdot d\overline{z}_2)_{|Y\cap U_Q}$ is well defined, i.e., it is independent of a choice of the coordinate (z_1,z_2). Let (w_1,w_2) be another coordinate such that $w_1^{-1}(0)=Y\cap U_Q$. Then we have $dz_1/z_1=dw_1/w_1+g_1$, where g_1 is a C^{∞}-one form. We also have $d\bar{z}_1=d\bar{w}_1+g_{1'}$, where $g_{1'}$ is C^{∞} such that $g_{1'\,|\,U_Q\cap Y}=0$. Then the claim immediately follows from $g_{1'\,|\,U_Q\cap Y}=0$ and $f_{1\,|\,U_Q\cap Y}\in F_{<a}$. Therefore the λ-connection $\mathbb{D}^{\lambda}_{\widehat{E}_Y}$ on $\widehat{E}_Y$ are induced. ∎

3.2.3 Estimate

Lemma 3.8 *$G(\mathbb{D}^{\lambda},h)$ is bounded with respect to (ω_{ϵ},h). More closely, we have the following estimate:*

$$G(\mathbb{D}^{\lambda},h)=\pi^*G\big(\mathbb{D}^{\lambda}_{\widehat{E}_Y},h_Y\big)-\big(\lambda\overline{\mathcal{Q}}+\overline{\lambda}\mathcal{Q}+(1+|\lambda|^2)\Gamma\big)\cdot\widetilde{\overline{\partial}}\widetilde{\partial}\log|\sigma|^2+O(|\sigma|^{5\epsilon}).\tag{43}$$

We also have the following estimate with respect to (ω_{ϵ},h):

$$\overline{\partial}_{E,h}\theta_{E,h}=\pi^*(\overline{\partial}_{\widehat{E}_Y,h_Y}\theta_{\widehat{E}_Y,h_Y})+\frac{1}{1+|\lambda|^2}(\mathcal{Q}+\lambda\Gamma)\cdot\widetilde{\overline{\partial}}\widetilde{\partial}\log|\sigma|^2+O(|\sigma|^{5\epsilon})\tag{44}$$

We also have the following estimate with respect to ω_{ϵ} and h:

$$\theta_h=\pi^*\theta_{\widehat{E}_Y,h_Y}+\frac{1}{1+|\lambda|^2}(\mathcal{Q}+\lambda\Gamma)\cdot\widetilde{\partial}\log|\sigma|^2+O(1).\tag{45}$$

Lemma 3.9 *We have the estimate $\theta^2_{E,h}=O(|\sigma|^{5\epsilon})$ with respect to (ω_{ϵ},h). In particular, $\lim_{\delta\to 0}\int_{Y_{\delta}}\mathrm{tr}(\theta^2\theta^{\dagger})=0$, where we put $Y_{\delta}:=\{|\sigma|=\delta\}$.*

The proof of these lemmas are given in the rest of this subsection.

3.2.4 Preliminary

Let $\mathbb{D}^{\lambda,f}$ denote the associated connection to $\mathbb{D}^{\lambda}$, i.e., $\mathbb{D}^{\lambda,f}=d''_E+\lambda^{-1}d'_E$. Let $\mathbb{D}^{\lambda,f}_{\widehat{E}_Y}$ be similar. Let $\pi^*\mathbb{D}^{\lambda,f}_{\widehat{E}_Y}$ denote the connection of E induced by $\mathbb{D}^{\lambda,f}_{\widehat{E}_Y}$. Then $\mathbb{D}^{\lambda,f}-\pi^*\mathbb{D}^{\lambda,f}_{\widehat{E}_Y}$ is a C^{∞}-section of $\mathrm{End}(E)\otimes\Omega^{1,0}(\log Y)$. We have the expression $\mathbb{D}^{\lambda,f}-\pi^*\mathbb{D}^{\lambda,f}_{\widehat{E}_Y}=\lambda^{-1}\mathcal{Q}\cdot\widetilde{\partial}\log|\sigma|^2+M$. Then M is a C^{∞}-section of $\mathrm{End}(E)\otimes\Omega^{1,0}(\log Y)$. Moreover, we decompose it as follows, around $Q\in Y$:

$$M=M_1\cdot\frac{dz_1}{z_1}+M_{1'}\cdot\frac{d\overline{z}_1}{\overline{z}_1}+M_2\cdot dz_2+M_{2'}\cdot d\overline{z}_2,\quad M_{\kappa}=\sum M_{\kappa,u,u'},\quad M_{\kappa,u,u'}\in Hom(E_u,E_{u'}).$$

Then we have $M_{\kappa,u,u'|Y\cap U_Q}=0$ for any κ, unless $a>a'$ and $\alpha=\alpha'$, due to our construction of $\mathbb{D}^{\lambda}_{\pi^*\widehat{E}_Y}$.

Recall $\widetilde{d}''_E=d''_{\pi^*\widehat{E}_Y}+T+\pi^*S''$. Hence we have $\mathbb{D}^{\lambda\,f}-\pi^*\mathbb{D}^{\lambda\,f}_{\widehat{E}_Y}=T+\pi^*S''+\lambda^{-1}\big(\widetilde{d}'_E-d'_{\pi^*\widehat{E}_Y}\big)$, and we obtain the following:

$$\widetilde{d}'_E=d'_{\pi^*\widehat{E}_Y}+\mathcal{Q}\cdot\widetilde{\partial}\log|\sigma|^2+\lambda(M-T-\pi^*S'').$$

Let $T=\sum T_{u,u'}$ be the decomposition corresponding to $E=\bigoplus E_u$. We put $T^p:=\sum T_{u,u}$ and $T^r=T-T^p$. Then T^p is a differential operator of order 1, which satisfies the twisted Leibniz rule $T^p(fv)-fT^p(v)=s_Y(f)v$, and T^r is a C^∞-section of $\bigoplus_{u\neq u'}\mathrm{Hom}(E_u,E_{u'})$. Let $T^{p\,\circ}_h$ be the operator determined by $s_Y\big(h(u,v)\big)=h\big(T^pu,v\big)+h(u,T^{p\,\circ}_hv)$. We remark the twisted Leibniz rule $T^{p\,\circ}_h(fv)-fT^{p\,\circ}_h(v)=-s_Y(f)v$. Similarly the operator $T^{p\,\circ}_{h'}$ is defined from T^p and h'. Then we have the relation:

$$T^{p\,\circ}_h=T^{p\,\circ}_{h'}+\Gamma\cdot s_Y\big(\log|\sigma|^2\big).$$

Since we have $\widetilde{d}'_E=d'_{\pi^*\widehat{E}_Y}-\lambda T^p-\mathcal{Q}\widetilde{\partial}\log|\sigma|^2+\lambda\big(M-T^r-\pi^*S''\big)$ and $\widetilde{d}''_E=d''_{\pi^*\widehat{E}_Y}+T^p+T^r+\pi^*S''$, we obtain the following formula:

$$\widetilde{\delta}''_{E,h}=\delta''_{\pi^*\widehat{E}_Y,h'}-\overline{\lambda}T^{p\,\circ}_{h'}-\overline{\mathcal{Q}}\widetilde{\overline{\partial}}\log|\sigma|^2-\Gamma\overline{\lambda}\widetilde{\overline{\partial}}\log|\sigma|^2-\overline{\lambda}(M-T^r-\pi^*S'')^\dagger_h,$$

$$\widetilde{\delta}'_{E,h}=\delta'_{\pi^*\widehat{E}_Y,h'}+T^{p\,\circ}_{h'}-\Gamma\widetilde{\partial}\log|\sigma|^2-(T^r+\pi^*S'')^\dagger_h.$$

3.2.5 Proof of the estimate (43)

We put as follows:

$$\mathcal{D}_1:=\mathbb{D}^{\lambda}_{\pi^*\widehat{E}_Y}+(1-\lambda)T^p,\quad \mathcal{D}_2:=\mathbb{D}^{\lambda\,\star}_{\pi^*\widehat{E}_Y,h'}+(1+\overline{\lambda})T^{p\,\circ}_{h'},$$

$$\mathcal{R}_1:=\lambda M+(1-\lambda)(T^r+\pi^*S''),\quad \mathcal{R}_2:=-(1-\overline{\lambda})(T^r+\pi^*S'')^\dagger_h+\overline{\lambda}M^\dagger_h.$$

Then, we have the following equality:

$$G(\mathbb{D}^\lambda,h)=[\mathbb{D}^\lambda,\mathbb{D}^{\lambda\,\star}_h]=\big[\mathcal{D}_1+\mathcal{Q}\widetilde{\partial}\log|\sigma|^2+\mathcal{R}_1,\ \mathcal{D}_2+\overline{\mathcal{Q}}\widetilde{\overline{\partial}}\log|\sigma|^2-\Gamma(\widetilde{\partial}-\overline{\lambda}\widetilde{\overline{\partial}})\log|\sigma|^2+\mathcal{R}_2\big].\tag{46}$$

Let us see the right hand side of (46). We have the following:

$$[\mathcal{D}_1,\mathcal{D}_2]=\pi^*G(\mathbb{D}^{\lambda}_{\widehat{E}_Y},h_Y)+\big[(1-\lambda)T^p,\mathbb{D}^{\lambda\,\star}_{\pi^*\widehat{E},h'}\big]+\big[\mathbb{D}^{\lambda}_{\pi^*\widehat{E}_Y},\ (1+\overline{\lambda})T^{p\circ}_{h'}\big]+\big[(1-\lambda)T^p,\ (1+\overline{\lambda})T^{p\circ}_{h'}\big].$$

Since $G(\mathbb{D}^\lambda,h)$ and $\pi^*G(\mathbb{D}^\lambda_{\widehat{E}_Y},h_Y)$ are C^∞-sections, it is easy to see that the summation of the last three terms is also just a C^∞-section of $\bigoplus\mathrm{End}(E_u)\otimes\Omega^2$. Moreover it is $O(|\sigma|^{5\epsilon})$ with respect to ω_ϵ and h, since T is small.

By a direct calculation, we obtain the following equality:

$$\begin{aligned}&\big[\mathcal{D}_1,\ \overline{\mathcal{Q}}\cdot\widetilde{\overline{\partial}}\log|\sigma|^2-\Gamma(\widetilde{\partial}-\overline{\lambda}\cdot\widetilde{\overline{\partial}})\log|\sigma|^2\big]+\big[\mathcal{Q}\widetilde{\partial}\log|\sigma|^2,\ \mathcal{D}_2\big]\\&\qquad=\big(\lambda\widetilde{\partial}+\widetilde{\overline{\partial}}\big)\widetilde{\overline{\partial}}\log|\sigma|^2\cdot\overline{\mathcal{Q}}-(\lambda\widetilde{\partial}+\widetilde{\overline{\partial}})(\widetilde{\partial}-\overline{\lambda}\cdot\widetilde{\overline{\partial}})\log|\sigma|^2\cdot\Gamma+(\widetilde{\partial}-\overline{\lambda}\widetilde{\overline{\partial}})\widetilde{\partial}\log|\sigma|^2\cdot\mathcal{Q}\\&\qquad=-\big(\lambda\overline{\mathcal{Q}}+\overline{\lambda}\mathcal{Q}+(1+|\lambda|^2)\Gamma\big)\cdot\widetilde{\overline{\partial}}\widetilde{\partial}\log|\sigma|^2.\end{aligned}\tag{47}$$

We decompose $\mathcal{R}_i=\sum\mathcal{R}_{i,u,u'}$ corresponding to $\bigoplus E_u$, and we decompose $\mathcal{R}_{i,u,u'}$ as follows:

$$\mathcal{R}_{i,u,u'}=\mathcal{R}_{i,u,u';1}\frac{dz_1}{z_1}+\mathcal{R}_{i,u,u';1'}\frac{d\overline{z}_1}{\overline{z}_1}+\mathcal{R}_{i,u,u';2}dz_2+\mathcal{R}_{i,u,u';2'}d\overline{z}_2.$$

Let us see $\mathcal{R}_1$. Then $\mathcal{R}_{1,u,u';\kappa}$ is a C^∞-section of $Hom(E_u,E_{u'})$, and we have $\mathcal{R}_{1,(a,\alpha),(a',\alpha');\kappa\,|\,Y\cap U_Q}=0$ unless $a>a'$ and $\alpha=\alpha'$. Hence we obtain $\mathcal{R}_1=O(|\sigma|^{5\epsilon})$ with respect to (ω_ϵ,h), and it is easy to check $[\mathcal{D}_2,\mathcal{R}_1]=O(|\sigma|^{5\epsilon})$ with respect to (ω_ϵ,h). On the other hand, $\mathcal{R}_{2,u,u';\kappa}=\mathcal{R}'_{2,u,u';\kappa}\cdot|\sigma|^{-2(a-a')}$, where $\mathcal{R}'_{2,u,u';\kappa}$ are C^∞-sections of $Hom(E_u,E_{u'})$ and $\mathcal{R}'_{2,(a,\alpha),(a',\alpha');\kappa\,|\,Y\cap U_Q}=0$ unless $a<a'$ and $\alpha=\alpha'$. Hence we can easily

obtain $\mathcal{R}_2 = O(|\sigma|^{5\epsilon})$ and $[\mathcal{D}_1, \mathcal{R}_2] = O(|\sigma|^{5\epsilon})$ with respect to (ω_ϵ, h). In particular, we also obtain the following estimate with respect to (ω_ϵ, h):

$$[\mathcal{R}_1, \mathcal{R}_2] = O(|\sigma|^{5\epsilon}), \quad [\lambda \cdot \mathcal{Q} \cdot \widetilde{\partial} \log |\sigma|^2, \mathcal{R}_2] = O(|\sigma|^{5\epsilon}),$$

$$\big[\mathcal{R}_1,\ \overline{\lambda} \cdot \overline{\mathcal{Q}} \cdot \widetilde{\overline{\partial}} \log |\sigma|^2 - \Gamma(\widetilde{\partial} - \overline{\lambda \widetilde{\partial}}) \log |\sigma|^2\big] = O(|\sigma|^{5\epsilon}).$$

In all, we obtain (43). ∎

3.2.6 Proof of the estimates (44) and (45)

We put $\mathcal{S}_1 := -\lambda\Big(T^p + T^{p\circ}_{h'} - M + T^r + \pi^* S'' + (T^r + \pi^* S'')^{\dagger}_h\Big)$. Then $\theta_{E,h}$ can be described as follows:

$$\theta_{E,h} = \frac{1}{1+|\lambda|^2}(\widetilde{d}'_E - \lambda \widetilde{\delta}'_{E,h}) = \pi^* \theta_{\widehat{E}_Y, h_Y} + \frac{1}{1+|\lambda|^2}\Big((\mathcal{Q} + \lambda \Gamma) \cdot \widetilde{\partial} \log |\sigma|^2 - \mathcal{S}_1\Big). \tag{48}$$

We put as follows:

$$\mathcal{D}_3 := \overline{\partial}_{\pi^* \widehat{E}_Y, h'} + \frac{1}{1+|\lambda|^2}(T^p - |\lambda|^2 \cdot T^{p\circ}_{h'}), \quad \mathcal{S}_2 := \frac{1}{1+|\lambda|^2}\big(T^r + \pi^* S'' - |\lambda|^2 (M - T^r - \pi^* S'')^{\dagger}_h\big).$$

Then $\overline{\partial}_{E,h}$ is described as follows:

$$\overline{\partial}_{E,h} = \frac{1}{1+|\lambda|^2}(\widetilde{d}''_E + \lambda \delta''_h) = \mathcal{D}_3 - \frac{1}{1+|\lambda|^2}\big(\overline{\mathcal{Q}} + \overline{\lambda}\Gamma\big)\widetilde{\overline{\partial}} \log |\sigma|^2 + \mathcal{S}_2.$$

Therefore we obtain the following:

$$\begin{aligned}\overline{\partial}_{E,h}\theta_{E,h} = {} & \big[\mathcal{D}_3, \pi^* \theta_{\widehat{E}_Y, h_Y}\big] + \frac{1}{1+|\lambda|^2}\big(\mathcal{Q} + \lambda \Gamma\big) \cdot \widetilde{\overline{\partial}}\widetilde{\partial} \log |\sigma|^2 + \big[\mathcal{D}_3, \mathcal{S}_1\big] \\ & - \frac{1}{1+|\lambda|^2}\big[(\overline{\mathcal{Q}} + \overline{\lambda}\Gamma)\widetilde{\overline{\partial}} \log |\sigma|^2, \mathcal{S}_1\big] + \big[\mathcal{S}_2, \pi^* \theta_{\widehat{E}_Y, h_1}\big] + \frac{1}{1+|\lambda|^2}\big[\mathcal{S}_2, (\mathcal{Q} + \lambda\Gamma)\widetilde{\partial} \log |\sigma|^2\big] + [\mathcal{S}_2, \mathcal{S}_1].\end{aligned} \tag{49}$$

We have $[\mathcal{D}_3, \pi^* \theta_{\widehat{E}_Y, h_Y}] = \pi^*(\overline{\partial}_{\widehat{E}_Y} \theta_{\widehat{E}_Y, h_1}) + (1+|\lambda|^2)^{-1}\big[T^p - |\lambda|^2 \cdot T^{p\circ}_{h'},\ \pi^* \theta_{\widehat{E}_Y, h_Y}\big]$. Since T is small, the second term is $O(|\sigma|^{5\epsilon})$ with respect to (ω_ϵ, h). Since $\mathcal{S}_1$ is a sum of the small diagonal term $T^p + T^{p\circ}_{h'}$ and the term of the forms which are similar to $\mathcal{R}_1$ and $\mathcal{R}_2$, we can obtain $\mathcal{S}_1 = O(|\sigma|^{5\epsilon})$ and $[\mathcal{D}_3, \mathcal{S}_1] = O(|\sigma|^{5\epsilon})$ with respect to (ω_ϵ, h) similarly. In particular, we obtain (45) from (48). We obtain a similar estimate for $\mathcal{S}_2$. Hence the remaining terms are also $O(|\sigma|^{5\epsilon})$ with respect to (ω_ϵ, h). In all, we obtain (44), and thus the proof of Lemma 3.8 is finished. ∎

3.2.7 Proof of Lemma 3.9

We have $(\pi^* \theta_{\widehat{E}, h_1})^2 = 0$ due to $\dim Y = 1$, and hence we obtain the following from (48):

$$\theta^2_{E,h} = 2[\pi^* \theta_{\widehat{E}_Y, h_Y}, \mathcal{S}_1] + 2(1+|\lambda|^2)^{-1}[(\mathcal{Q} + \lambda\Gamma)\widetilde{\partial} \log |\sigma|^2, \mathcal{S}_1] + [\mathcal{S}_1, \mathcal{S}_1].$$

Thus we obtain Lemma 3.9. ∎

3.3 Global Construction of a Metric

3.3.1 Setting

Let X be a smooth projective surface, and D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. Let L be an ample line bundle on X, and ω be a Kahler form which represents $c_1(L)$. For any point $P \in D_i \cap D_j$, we take a holomorphic coordinate (U_P, z_i, z_j) around P such that $U_P \cap D_k = \{z_k = 0\}$ $(k = i, j)$ and $U_P \simeq \Delta^2$ by the coordinate. Let us take a hermitian metric g_i of $\mathcal{O}(D_i)$ and the canonical section

$\mathcal{O} \longrightarrow \mathcal{O}(D_i)$ is denoted by σ_i. We may assume $|\sigma_k|^2_{g_k} = |z_k|^2$ $(k = i, j)$ on U_P for $P \in D_i \cap D_j$. Let us take a hermitian metric g of the tangent bundle TX such that $g = dz_i \cdot d\bar{z}_i + dz_j \cdot d\bar{z}_j$ on U_P. It is not necessarily same as ω, and not necessarily Kahler. The metric g induces the exponential map $\exp : TX \longrightarrow X$.

Let $N_{D_i}X$ denote the normal bundle of D_i in X. We can take a sufficiently small neighbourhood U_i' of D_i in $N_{D_i}X$ such that the restriction of $\exp_{|U_i'}$ gives the diffeomorphism of U_i' and the neighbourhood U_i of D_i in X. We may assume $U_i \cap U_j = \coprod_{P \in D_i \cap D_j} U_P$ and $U_i = \big\{ |\sigma_i|_{g_i} < 1 \big\}$.

Let p_i denote the diffeomorphism $\exp_{|U_i} : U_i \longrightarrow U_i'$. Let π_i denote the natural projection $U_i' \longrightarrow D_i$. Via the diffeomorphism p_i, we also have the C^∞-map $U_i \longrightarrow D_i$, which is also denoted by π_i. On U_P, π_i is same as the natural projection $(z_i, z_j) \longmapsto z_j$.

Via p_i, we have two complex structure $J_{U_i'}$ and J_{U_i} on U_i. Due to our choice of the hermitian metric g, p_i preserves the holomorphic structure (i.e., $J_{U_i'} - J_{U_i} = 0$) on U_P. The derivative of p_i gives the isomorphism of the complex bundles $T(N_{D_i}(X))_{|D_i} \simeq TD_i \oplus N_{D_i}X \simeq TX_{|D_i}$ on D_i. Hence we have the estimate $J_{U_i'} - J_{U_i} = O\big(|\sigma_i|\big)$.

Let ϵ be any number such that $0 < \epsilon < 1/2$. Let us fix a sufficiently large number N, for example $N > 10$. We put as follows, for some positive number $C > 0$:

$$\omega_\epsilon := \omega + \sum_i C \cdot \epsilon^N \cdot \sqrt{-1}\partial\bar{\partial}|\sigma_i|^{2\epsilon}_{g_i}. \tag{50}$$

It can be shown that ω_ϵ are Kahler metrics of $X - D$ for any $0 < \epsilon < 1/2$, if C is sufficiently small,

Remark 3.10 *Let τ be a closed 2-form on $X - D$ which is bounded with respect to ω_ϵ. Then the following formula holds:*

$$\int_{X-D} \omega \cdot \tau = \int_{X-D} \omega_\epsilon \cdot \tau.$$

In particular, we also have $\int_{X-D} \omega^2 = \int_{X-D} \omega_\epsilon^2$. ∎

In the case $\epsilon = 1/m$ for some positive integer m, it can be shown that the metric ω_ϵ satisfies Condition 2.28. The Kahler forms ω_ϵ behave well around any point of D in the following sense, which is clear from the construction:

- Let P be any point of $D_i \cap D_j$. Then there exist positive constants $C_i(\epsilon)$ $(i = 1, 2)$ such that the following holds on U_P, for any $0 < \epsilon < 1/2$

$$C_1 \cdot \omega_\epsilon \leq \sqrt{-1} \cdot \epsilon^{N+2} \cdot \left(\frac{dz_i \cdot d\bar{z}_i}{|z_i|^{2-2\epsilon}} + \frac{dz_j \cdot d\bar{z}_j}{|z_j|^{2-2\epsilon}} \right) + \sqrt{-1}\big(dz_i \cdot d\bar{z}_i + dz_j \cdot d\bar{z}_j \big) \leq C_2 \cdot \omega_\epsilon.$$

- Let Q be any point of $D_i \setminus \bigcup_{j \neq i} D_j$, and (U, w_1, w_2) be a holomorphic coordinate around Q such that $U \cap D_i = \{w_1 = 0\}$. Then there exist positive constants C_i $(i = 1, 2)$ such that the following holds for any $0 < \epsilon < 1/2$ on U:

$$C_1 \cdot \omega_\epsilon \leq \sqrt{-1} \cdot \epsilon^{N+2} \cdot \left(\frac{dw_1 \cdot d\bar{w}_1}{|w_1|^{2-2\epsilon}} \right) + \sqrt{-1}\big(dw_1 \cdot d\bar{w}_1 + dw_2 \cdot d\bar{w}_2 \big) \leq C_2 \cdot \omega_\epsilon.$$

3.3.2 Construction of a metric

Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a *graded semisimple* parabolic λ-flat bundle. For simplicity, we consider only the case $\lambda \neq 0$. We will recall the construction of an ordinary metric h_0 for $(E, \boldsymbol{F})$ ([30], for example). For each point $P \in D_i \cap D_j$, we may assume that there is a decomposition, as in the subsection 3.1:

$$E_{|U_P} = \bigoplus {}^P U_{\boldsymbol{a}, \boldsymbol{\alpha}}. \tag{51}$$

We take a holomorphic frame $\boldsymbol{v}$ of $E_{|U_P}$ compatible with the decomposition (51) for each P. We can take a C^∞-isomorphism ${}^i\Phi : \pi_i^*\big(E_{|D_i}\big) \simeq E$ on U_i, satisfying the following:

- ${}^{i}\Phi(d''_{\pi_i^*(E_{|D_i})}) - d''_E$ is small in the sense of the subsection 3.2.1.
- The restriction of ${}^{i}\Phi$ to D_i is the identity.
- For $P \in D_i \cap D_j$, the restriction of ${}^{i}\Phi$ to U_P is holomorphic such that ${}^{i}\Phi\big(\pi_i^*(\boldsymbol{v}_{|D_i \cap U_P})\big) = \boldsymbol{v}$.

We take the C^∞-decomposition $E_{|U_i} = \bigoplus {}^{i}E_u$, as in the subsection 3.2. We may assume the following on U_P:

$$ {}^{i}E_{u\,|\,U_P} = \bigoplus_{q_i(\boldsymbol{a},\boldsymbol{\alpha})=u} {}^{P}U_{\boldsymbol{a},\boldsymbol{\alpha}}. $$

Here $(\boldsymbol{a},\boldsymbol{\alpha})$ denotes an element $(a_i, a_j, \alpha_i, \alpha_j) \in \boldsymbol{R}^2 \times \boldsymbol{C}^2$, and $q_i(\boldsymbol{a},\boldsymbol{\alpha})$ denotes (a_i, α_i).

We can take a hermitian metric h_0' of E satisfying the following conditions:

- We have $h_0'(v_k, v_l) = \delta_{k,l}$, i.e., it is 1 $(k = l)$ or 0 $(k \neq l)$ on U_P for $P \in D_i \cap D_j$. In particular, the decomposition $E_{|U_P} = \bigoplus {}^{P}U_{\boldsymbol{a},\boldsymbol{\alpha}}$ is orthogonal.
- $E_{|U_i} = \bigoplus {}^{i}E_u$ is orthogonal with respect to h_0'. Thus we have the decomposition $h_0' = \bigoplus {}^{i}h_u'$ on U_i.
- We put $h'_{0\,D_i} := h'_{0\,|\,D_i}$. Then we have ${}^{i}\Phi(\pi_i^* h'_{0\,D_i}) = h_0'$ on U_i. Note that we have the decomposition $h'_{0\,D_i} = \bigoplus h'_{u\,D_i}$.

We put $D_i^\circ := D_i \setminus \bigcup_{j\neq i} D_j$. By modifying $h'_{0\,D_i}$, we take a C^∞-hermitian metric $h_{0\,D_i}$ of $E_{|D_i^\circ}$ satisfying the following:

- The decomposition $E_{|D_i^\circ} = \bigoplus {}^{i}E_{u\,|\,D_i^\circ}$ is orthogonal. Hence we have the decomposition $h_{0\,D_i} = \bigoplus h_{u\,D_i}$.
- For $P \in D_i \cap D_j$, we have $h_{0\,D_i}(v_{k\,|\,D_i}, v_{l\,|\,D_i}) = \delta_{k.l}|z_j|^{-2a_j(v_k)}$ on $U_P \cap D_i^\circ$.

Then we can take a C^∞-metric h_0 of E on $X - D$ satisfying the following conditions:

- $h_0(v_k, v_l) = \delta_{k,l}|z_i|^{-2a_i(v_k)}|z_j|^{-2a_j(v_k)}$ on $U_P \setminus D$ for $P \in D_i \cap D_j$.
- The decomposition $E_{|U_i \setminus D} = \bigoplus {}^{i}U_{a,\alpha\,|\,U_i\setminus D}$ is orthogonal with respect to h_0. In particular, we have the decomposition $h_0 = \bigoplus {}^{i}h_u$ on $U_i \setminus D$.
- ${}^{i}h_{(a,\alpha)} = \pi_i^* h_{(a,\alpha),D_i} \cdot |\sigma_i|_{g_i}^{-2a}$.

Such a hermitian metric h_0 is called an ordinary metric of (E, θ).

3.3.3 Estimate and some formula

We put $\mathcal{Q}_i := \bigoplus_{(a,\alpha)} \alpha \cdot \mathrm{id}_{{}^{i}E_{a,\alpha}}$ and $\Gamma_i := \bigoplus_{(a,\alpha)} a \cdot \mathrm{id}_{{}^{i}E_{a,\alpha}}$ on U_i. We put $\widehat{E}_{u\,D_i} := {}^{i}\,\mathrm{Gr}_u^{F,\mathbb{E}}(E_{|D_i})$ and $\widehat{E}_{D_i} := \bigoplus \widehat{E}_{u\,D_i}$. Now it has been identified with $E_{|D_i}$ in the C^∞-category. Recall that we also obtain the λ-connection $\widehat{\mathbb{D}}^\lambda_{D_i}$ of $\widehat{E}_{D_i}$, which is constructed as in the subsubsection 3.2.2. It is flat around $P \in D_i \cap D_j$. As before, $\widehat{\mathbb{D}}^\lambda_{D_i}$ and the metric h_{0,D_i} induce the operators ${}^{i}\overline{\partial}$ and ${}^{i}\theta$ of $\widehat{E}_{D_i}$.

Let ϵ be a sufficiently small positive number such that $10\cdot\epsilon < \mathrm{gap}(E, \boldsymbol{F})$. Combining Lemma 3.1 and Lemma 3.8, we obtain the following proposition.

Proposition 3.11 *$G(\mathbb{D}^\lambda, h_0)$ is bounded with respect to (ω_ϵ, h_0). Moreover, we have the following estimate with respect to (ω_ϵ, h_0), on $U_i \setminus (\bigcup U_P \cup D)$:*

$$ G(\mathbb{D}^\lambda, h_0) = \pi_i^* G\big(\widehat{\mathbb{D}}^\lambda_{D_i}, h_{0\,D_i}\big) - \big(\lambda\overline{\mathcal{Q}}_i + \overline{\lambda}\mathcal{Q}_i + (1+|\lambda|^2)\Gamma_i\big)\cdot\overline{\partial}\partial\log|\sigma_i|^2_{g_i} + O(|\sigma_i|^{3\epsilon}_{g_i}), $$

$$ \overline{\partial}_{h_0}\theta_{h_0} = \pi_i^*\big({}^{i}\overline{\partial}({}^{i}\theta)\big) + \frac{1}{1+|\lambda|^2}\big(\mathcal{Q}_i + \lambda\Gamma_i\big)\cdot\overline{\partial}\partial\log|\sigma_i|^2 + O(|\sigma_i|^{3\epsilon}_{g_i}). $$

$$ \theta_{h_0} = \pi_i^*\big({}^{i}\theta\big) + \frac{1}{1+|\lambda|^2}\big(\mathcal{Q}_i + \lambda\Gamma_i\big)\partial\log|\sigma_i|^2 + O(1). $$

On $U_P \setminus D$ for $P \in D_i \cap D_j$, we have the following estimate:

$$G(\mathbb{D}^\lambda, h_0) = \pi_i^* G\big(\widehat{\mathbb{D}}^\lambda_{D_i}, h_{0\,D_i}\big) + \pi_j^* G\big(\widehat{\mathbb{D}}^\lambda, h_{0\,D_j}\big) + O(|\sigma_i|^{3\epsilon}|\sigma_j|^{3\epsilon}),$$

$$\overline{\partial}_{h_0}\theta_{h_0} = \sum_{k=i,j} \pi_k^*\big({}^k\overline{\partial}({}^k\theta)\big) + O\big(|\sigma_i|^{3\epsilon}|\sigma_j|^{3\epsilon}\big).$$

$$\theta_{h_0} = \sum_{k=i,j} \pi_k^*\big({}^k\theta\big) + \sum_{k=i,j} \frac{1}{1+|\lambda|^2}(\mathcal{Q}_k + \lambda\Gamma_k)\cdot\partial\log|\sigma_k|^2.$$

We remark $\overline{\partial}\partial\log|\sigma_i|^2_{g_i} = 0$ on $U_P \setminus D$ for $P \in D_i \cap D_j$. ▮

From the lemmas 3.2 and 3.9, we obtain the following proposition.

Proposition 3.12 *We put $Y_\delta := \big\{x \in X \,\big|\, \min_i |\sigma_i|(x) = \delta\big\}$, and then we have $\int_{Y_\delta} \mathrm{tr}\big(\theta^2_{h_0}\cdot\theta^\dagger_{h_0}\big) \longrightarrow 0$ for $\delta \longrightarrow 0$.* ▮

Corollary 3.13 *The following equality holds:*

$$\int_{X-D} \mathrm{tr}\big(R(h_0)^2\big) = \frac{1}{(1+|\lambda|^2)^2}\int_{X-D}\mathrm{tr}\big(G(h_0)^2\big).$$

As a result, we have the following formula:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \frac{1}{(1+|\lambda|^2)^2}\int_{X-D}\mathrm{tr}\big(G(h_0)^2\big) = \int_X \text{par-ch}_2(E,\boldsymbol{F}). \tag{52}$$

Proof The second equality follows from the first equality and the proof of Proposition 4.4 of [30]. Due to Lemma 2.25, we have only to show the vanishing $\int \overline{\partial}\,\mathrm{tr}\big(\theta^2_{h_0}\theta^\dagger_{h_0}\big) = 0$, which is given in Proposition 3.12. ▮

Remark we can show the following equality similarly:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_{X-D}\left(\frac{\mathrm{tr}\,G(h_0)}{1+|\lambda|^2}\right)^2 = \left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_{X-D}\big(\mathrm{tr}\,R(h_0)\big)^2 = \int_X \text{par-c}_1(E,\boldsymbol{F})^2.$$

Corollary 3.14 *Let τ be any C^∞ two form on X. Then we have the following equalities:*

$$\frac{\sqrt{-1}}{2\pi}\int_{X-D}\mathrm{tr}\,R(h_0)\cdot\tau = \int_X\big(\text{par-c}_1(E,\boldsymbol{F})\cdot[\tau]\big) = \int_X\left(c_1(E)\cdot\tau - \sum_{i\in S}\mathrm{wt}(E,\boldsymbol{F},i)\cdot[D_i]\cdot\tau\right).$$

$$\frac{\sqrt{-1}}{2\pi}\int_{X-D}\frac{\mathrm{tr}\,G(h_0)}{1+|\lambda|^2}\cdot\tau = -\int_X\sum_{i\in S}\big(\lambda^{-1}\,\mathrm{tr}\,\mathrm{Res}_i\,\mathbb{D}^\lambda + \mathrm{wt}(E,\boldsymbol{F},i)\big)\cdot[D_i]\cdot\tau.$$

Proof The first equality follows from the estimate of $\mathrm{tr}\,R(h_0)$ given in the proof of Lemma 4.28 of [30]. The second equality follows from the relation of $G(h_0)$ and $\overline{\partial}\theta_0$ and the estimates of θ_0 given in Proposition 3.11. ▮

Corollary 3.15

$$\text{par-deg}_\omega(E,\boldsymbol{F}) = \frac{\sqrt{-1}}{2\pi}\int_{X-D}\mathrm{tr}\,R(h_0)\cdot\omega_\epsilon = \frac{\sqrt{-1}}{2\pi}\int_{X-D}\frac{\mathrm{tr}\,G(h_0)}{1+|\lambda|^2}\cdot\omega_\epsilon.$$

Proof The second follows from Lemma 2.24. The first equality follows from Lemma 4.18 of [30] and Corollary 3.14. ▮

In particular, we obtain the following equality of the cohomology classes from Corollary 3.14:

$$c_1(E) + \sum_{i\in S}\mathrm{Re}\big(\lambda^{-1}\,\mathrm{tr}\,\mathrm{Res}_i\,\mathbb{D}^\lambda\big)\cdot[D_i] = 0, \qquad \sum_{i\in S}\mathrm{Im}\big(\lambda^{-1}\,\mathrm{tr}\,\mathrm{Res}_i\,\mathbb{D}^\lambda\big)\cdot[D_i] = 0. \tag{53}$$

The first equality implies the following:

$$\text{par-c}_1(E_*) = -\sum \Big(\mathrm{Re}\big(\lambda^{-1} \operatorname{tr} \mathrm{Res}_i(\mathbb{D}^\lambda)\big) + \mathrm{wt}\big(E_*, i\big) \Big) \cdot [D_i]. \tag{54}$$

Especially, we obtain the following formula:

$$\text{par-deg}_\omega(E_*) = -\sum_i \mathrm{Re}\big(\lambda^{-1} \operatorname{tr} \mathrm{Res}_i(\mathbb{D}^\lambda)\big) + \mathrm{wt}\big(E_*, i\big) \Big) \cdot (D_i, \omega). \tag{55}$$

Remark 3.16 *It can be shown that these equalities hold for any parabolic λ-flat bundle $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ which are not necessarily graded semisimple, by using the method of perturbation of the parabolic structures. They can also be derived from a similar formula for the curve case and the fact that the Neron-Severi group $NS^1(X) \otimes \boldsymbol{R}$ can be embedded into the rational cohomology group $H^2(X, \boldsymbol{R})$.* ▮

3.3.4 The relation between the pseudo curvature and the data at the divisor

Recall $\int_X \operatorname{tr} G(h)^l = 0$, when X is compact ([36]). In the case where X is not compact, such a vanishing does not hold, in general. But we can derive some formulas for $\int_{X-D} \operatorname{tr}\big(G(h_0)^2\big)$ by the same way. For simplicity of the description, we put as follows:

$$u = (a, \alpha), \quad r(i, u) := \mathrm{rank}_{D_i} \widehat{E}_{u\, D_i} = \mathrm{rank}_{D_i} {}^i \mathrm{Gr}^{F,\mathbb{E}}_u (E_{|D_i}), \quad d(i, u) := \deg_{D_i} \widehat{E}_{u\, D_i}. \tag{56}$$

We also put $r(P, u_i, u_j) := \mathrm{rank}\, {}^P \mathrm{Gr}^{F,\mathbb{E}}_{(u_i,u_j)}(E_{|P})$. Let $\mathcal{KMS}(i)$ denote the set of the KMS-spectrum of $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ at D_i. Let $\mathcal{KMS}(P)$ denote the set of the KMS-spectrum at $P \in D_i \cap D_j$. (See the subsection 2.1.2 for the KMS-spectrum.)

Lemma 3.17 *We have the following formula:*

$$\begin{aligned}
&\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int \operatorname{tr} G(h_0)^2 = \\
&\sum_{i \in S} \sum_{u \in \mathcal{KMS}(i)} -\big(\lambda^{-1}\alpha + \lambda\overline{\alpha} + (1+|\lambda|^2)a\big) \Big((1-|\lambda|^2) \cdot d(i,u) - r(i,u) \cdot \big((1+|\lambda|^2)a + 2\,\mathrm{Re}(\overline{\lambda}\alpha) \big) [D_i]^2 \Big) \\
&\quad + \sum_{i \in S} \sum_{\substack{j \neq i \\ P \in D_i \cap D_j}} \sum_{(u_i,u_j) \in \mathcal{KMS}(P)} \big(\lambda^{-1}\alpha_i + \lambda\overline{\alpha}_i + (1+|\lambda|^2)a_i\big) \cdot \big((1+|\lambda|^2)a_j + 2\,\mathrm{Re}(\overline{\lambda}\alpha_j) \big) \cdot r(P, u_i, u_j).
\end{aligned} \tag{57}$$

Proof Since we have $\mathbb{D}^\lambda G(h_0) = 0$ and $G(h_0) = (1+|\lambda|^2) \cdot \mathbb{D}^\lambda\big(-\lambda^{-1}\theta_{h_0} + \theta^\dagger_{h_0}\big)$, we obtain the following equality:

$$\operatorname{tr}\big(G(h_0)^2\big) = (1+|\lambda|^2) \cdot (\overline{\partial} + \lambda\partial) \operatorname{tr}\Big((-\lambda^{-1}\theta_{h_0} + \theta^\dagger_{h_0}) \cdot G(h_0) \Big).$$

We decompose $G(h_0) = G^{2,0} + G^{1,1} + G^{0,2}$, where $G^{p,q}$ are (p,q)-forms. Then we have the following:

$$\frac{\operatorname{tr} G(h_0)^2}{1+|\lambda|^2} = d \operatorname{tr}\big(-\lambda^{-1}\theta_{h_0} \cdot G^{1,1}\big) - d \operatorname{tr}\big(\theta_{h_0} \cdot G^{0,2}\big) + d \operatorname{tr}\big(\lambda\theta^\dagger_{h_0} \cdot G^{1,1}\big) + d \operatorname{tr}\big(\theta^\dagger_{h_0} \cdot G^{2,0}\big).$$

We would like to apply the Stokes formula to the integral of $\operatorname{tr} G(h_0)^2$. Since $G(h_0)$ is bounded with respect to h_0 and ω_ϵ, and since we need $(1,1)$-form for the integration over complex curves D_i, it is easy to see that the only terms $d \operatorname{tr}\big(-\lambda^{-1}\theta_{h_0} \cdot G^{1,1}\big)$ and $d \operatorname{tr}\big(\lambda\theta^\dagger_{h_0} \cdot G^{1,1}\big)$ can contribute. Namely, we have the following equality:

$$\int \operatorname{tr}\big(G(h_0)^2\big) = (1+|\lambda|^2) \int d \operatorname{tr}\Big(\big(-\lambda^{-1}\theta_{h_0} + \lambda\theta^\dagger_{h_0}\big) \cdot G^{1,1} \Big).$$

We also remark the following estimate on $U_i \setminus U_P$ with respect to (ω_ϵ, h_0), which follows from Proposition 3.11:

$$-\lambda^{-1}\theta_{h_0} + \lambda \cdot \theta^\dagger_{h_0} = \frac{1}{1+|\lambda|^2} \Big((-\lambda^{-1}\mathcal{Q}_i - \Gamma_i) \cdot \partial \log |\sigma_i|^2_{g_i} + \big(\lambda\overline{\mathcal{Q}}_i + |\lambda|^2\Gamma_i\big) \cdot \overline{\partial} \log |\sigma_i|^2_{g_i} \Big) + O(1).$$

Similarly, on U_P ($P \in D_i \cap D_j$), we have the following:

$$-\lambda^{-1}\theta_{h_0} + \lambda \cdot \theta_{h_0}^{\dagger} = \frac{1}{1+|\lambda|^2} \sum_{k=i,j} \Big((-\lambda^{-1}\mathcal{Q}_k - \Gamma_k) \cdot \partial \log |\sigma_k|^2_{g_k} + \big(\lambda\overline{\mathcal{Q}}_k + |\lambda|^2\Gamma_k\big) \cdot \overline{\partial} \log |\sigma_k|^2_{g_k} \Big) + O(1).$$

Then we obtain the following equality due to Proposition 3.11:

$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int \operatorname{tr}\big(G(h_0)^2\big) = \sum_{i\in S} \sum_{u \in \mathcal{KMS}(i)} \big(-\lambda^{-1}\alpha - \lambda\overline{\alpha} - (1+|\lambda|^2)a\big) \times$$
$$\frac{\sqrt{-1}}{2\pi} \Big(\int_{D_i} \operatorname{tr} G\big(\mathbb{D}^{\lambda}_{\widehat{E}_{D_i,u}}, h_{u\,D_i}\big) - r(i,u) \cdot \big(\lambda\overline{\alpha} + \overline{\lambda}\alpha + (1+|\lambda|^2)a\big) \int_{D_i} \overline{\partial}\partial \log |\sigma_i|^2_{g_i} \Big). \quad (58)$$

Recall we have $\sqrt{-1}\cdot(2\pi)^{-1}\int_{D_i} \overline{\partial}\partial \log |\sigma_i|^2_{g_i} = [D_i]^2$. Due to the formula (29), the right hand side of (58) is rewritten as follows:

$$-\sum_{i,u} \big(\lambda^{-1}\alpha + \lambda\overline{\alpha} + (1+|\lambda|^2)a\big) \cdot \Big((1-|\lambda|^2) \cdot d(i,u) - r(i,u) \cdot \big((1+|\lambda|^2)a + 2\operatorname{Re}\overline{\lambda}\alpha\big) \cdot [D_i]^2 \Big)$$
$$+ \sum_{i,u} \sum_{P \in D_i} \big(\lambda^{-1}\alpha + \lambda\overline{\alpha} + (1+|\lambda|^2)a\big) \cdot \Big(2\operatorname{Re}(\overline{\lambda} \operatorname{tr} \operatorname{Res}_P \mathbb{D}^{\lambda}_{\widehat{E}_{D_i,u}}) + (1+|\lambda|^2) \operatorname{wt}\big(\widehat{E}_{D_i,u\,*}, P\big) \Big). \quad (59)$$

Here $\widehat{E}_{D_i\,u\,*}$ denotes the parabolic bundle which is a pair of the vector bundle $E_{D_i\,u\,*}$ with the naturally induced parabolic structure. We remark the following equality:

$$\sum_{i\in S} \sum_{u\in\mathcal{KMS}(i)} \sum_{P\in D_i} \big(\lambda^{-1}\alpha + \lambda\overline{\alpha} + (1+|\lambda|^2)a\big) \cdot \Big(2\operatorname{Re}(\overline{\lambda} \operatorname{tr} \operatorname{Res}_P \mathbb{D}^{\lambda}_{\widehat{E}_{D_i,u}}) + (1+|\lambda|^2) \operatorname{wt}\big(\widehat{E}_{D_i,u*}, P\big) \Big)$$
$$= \sum_i \sum_{\substack{j\neq i \\ P\in D_i\cap D_j}} \sum_{(u_i,u_j)\in\mathcal{KMS}(P)} \big(\lambda^{-1}\alpha_i + \lambda\overline{\alpha}_i + (1+|\lambda|^2)a_i\big) \cdot \big((1+|\lambda|^2)a_j + 2\operatorname{Re}(\overline{\lambda}\alpha_j) \big) \cdot r(P,u_i,u_j). \quad (60)$$

Then we obtain (57). ∎

Lemma 3.18 *We have the following equality:*

$$\operatorname{tr}\big(G(h_0)^2\big) = \lambda^{-2} \cdot (1+|\lambda|^2)^4 \cdot \overline{\partial} \operatorname{tr}(\theta_{h_0} \cdot \overline{\partial}_{h_0}\theta_{h_0}). \quad (61)$$

Proof We have $\operatorname{tr}\big(G(h_0)^2\big) = (1+|\lambda|^2)^4 \cdot \lambda^{-2} \Big(\operatorname{tr}(\overline{\partial}_{h_0}\theta_{h_0})^2 + 2\operatorname{tr}(\overline{\partial}^2_{h_0} \cdot \theta^2_{h_0}) \Big)$. We also have the following:

$$\overline{\partial} \operatorname{tr}\big(\theta_{h_0} \cdot \overline{\partial}_{h_0}\theta_{h_0}\big) = \operatorname{tr}\big((\overline{\partial}_{h_0}\theta_{h_0})^2\big) - \operatorname{tr}\big(\theta_{h_0} \cdot [\overline{\partial}^2_{h_0}, \theta_{h_0}]\big) = \operatorname{tr}\big((\overline{\partial}_{h_0}\theta_{h_0})^2\big) + 2\operatorname{tr}\big(\overline{\partial}^2_{h_0} \cdot \theta^2_{h_0}\big),$$

Then (61) follows. ∎

Lemma 3.19 *The following formula holds:*

$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int \frac{\operatorname{tr}(G(h_0)^2)}{(1+|\lambda|^2)^2} = \sum_{i\in S} \sum_{\substack{j\neq i \\ P\in D_i\cap D_j}} \sum_{(u_i,u_j)\in\mathcal{KMS}(P)} (\lambda^{-1}\alpha_i + a_i)(\lambda^{-1}\alpha_j + a_j) \cdot r(P,u_i,u_j)$$
$$+ \sum_{i\in S} \sum_{u\in\mathcal{KMS}(i)} r(i,u) \cdot (\lambda^{-1}\alpha + a)^2 \cdot [D_i]^2. \quad (62)$$

Proof From Proposition 3.11, we obtain the following:

$$\Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int \overline{\partial} \operatorname{tr}\big(\theta_{h_0} \cdot \overline{\partial}_{h_0}\theta_{h_0}\big) = \sum_i \sum_u \frac{\alpha + \lambda a}{1+|\lambda|^2} \frac{\sqrt{-1}}{2\pi} \Big(\int_{D_i} \operatorname{tr}({}^i\overline{\partial}{}^i\theta_u) + \frac{\alpha+\lambda a}{1+|\lambda|^2} \cdot r(i,u) \cdot \int_{D_i} \overline{\partial}\partial \log|\sigma_i|^2 \Big) \quad (63)$$

By using (30), the right hand side hand can be rewritten as follows:

$$\sum_i \sum_u \frac{\alpha+\lambda a}{(1+|\lambda|^2)^2}\left(\sum_P \Big(\operatorname{tr}\operatorname{Res}_P {}^i\mathbb{D}^\lambda_u + \lambda\,\operatorname{wt}(\widehat{E}_{D_i\,u\,*},P)\Big) + (\alpha+\lambda a)\cdot r(i,u)\cdot [D_i]^2\right)$$
$$= \sum_i \sum_{\substack{j\neq i\\ P\in D_i\cap D_j}} \sum_{(u_i,u_j)\in\mathcal{KMS}(P)} \frac{(\alpha_i+\lambda a_i)(\alpha_j+\lambda a_j)}{(1+|\lambda|^2)^2}\cdot r(P,u_i,u_j)$$
$$+\sum_i \sum_{u\in\mathcal{KMS}(i)} \frac{(\alpha+\lambda a)^2}{(1+|\lambda|^2)^2}\cdot r(i,u)\cdot [D_i]^2. \quad (64)$$

Then (62) follows from (61). ∎

3.4 Preliminary Existence Result of a Hermitian-Einstein Metric

3.4.1 Hermitian-Einstein metric for graded semisimple λ-flat parabolic bundle on surface

We use the setting in the subsection 3.3.1. Let X be a smooth projective surface with an ample line bundle L and a simple normal crossing divisor D. Let $(E,\boldsymbol{F},\mathbb{D}^\lambda)$ be a λ-flat bundle on (X,D), which is *graded semisimple* and μ_L-stable. Let ω be the Kahler form representing $c_1(L)$. Let ϵ be a small positive number such that $10\cdot\epsilon < \operatorname{gap}(E,\boldsymbol{F})$. The metric ω_ϵ is given by (50). Let h_0 be an ordinary metric constructed in the subsection 3.3.2.

Lemma 3.20 *We can construct a hermitian metric h_{in} for $E_{|X-D}$ which satisfies the following conditions:*

- *h_{in} is adapted to the parabolic structure $\boldsymbol{F}$.*
- *$G(h_{in},\mathbb{D}^\lambda)$ is bounded with respect to h_{in} and ω_ϵ.*
- *Let V be any saturated coherent subsheaves $E_{|X-D}$, and let π_V denote the orthogonal projection of $E_{|X-D}$ onto V, which is defined outside a finite subset. Then $\mathbb{D}^\lambda\pi_V$ is L^2 with respect to h_{in} and ω_ϵ, if and only if there exists a coherent subsheaf $\widetilde{V}$ of E such that $\widetilde{V}_{|X-D}=V$. Moreover we have* $\operatorname{par-deg}_\omega(\widetilde{V},\boldsymbol{F}_V) = \deg_{\omega_\epsilon}(V,h_{in,V})$.
- $\operatorname{tr} G(h_{in},\mathbb{D}^\lambda)\cdot\omega_\epsilon = (1+|\lambda|^2)\cdot a\cdot\omega_\epsilon^2$ *for some constant a. The constant a is determined by the following condition:*
$$a\cdot\frac{\sqrt{-1}}{2\pi}\int_{X-D}\omega_\epsilon^2 = a\cdot\frac{\sqrt{-1}}{2\pi}\int_X\omega^2 = \operatorname{par-deg}_\omega(E,\boldsymbol{F}). \quad (65)$$
- *The following equalities hold:*
$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_{X-D}\frac{\operatorname{tr}\big(G(h_{in})^2\big)}{(1+|\lambda|^2)^2} = \int_X 2\operatorname{par-ch}_2(E,\boldsymbol{F}),\quad \left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_{X-D}\frac{\operatorname{tr}\big(G(h_{in})\big)^2}{(1+|\lambda|^2)^2} = \int_X \operatorname{par-c}_1^2(E,\boldsymbol{F}).$$
- *Let s be determined by $h_{in}=h_0\cdot s$. Then s and s^{-1} is bounded, and $\mathbb{D}^\lambda s$ is L^2 with respect to h_0 and ω_ϵ.*

Due to the third condition, (E,h_{in},θ) is analytic stable with respect to ω_ϵ, if and only if $(E,\boldsymbol{F},\mathbb{D}^\lambda)$ is μ_L-stable. The metric h_{in} is called an initial metric.

Proof We have only to modify an ordinary metric h_0 to $h_0\cdot e^\chi$ for some positive scalar function χ so that $\operatorname{tr} G(h_0)\cdot\omega_\epsilon = a\cdot\omega_\epsilon^2$ holds. Once we have obtained the estimate as in Proposition 3.11, it can be shown by the argument given in the subsection 6.1.3 of [30]. (In the case $\lambda\neq 0$, we can use Lemma 2.7 for the prolongation of subsheaves to show the third property, and hence the proof is a little easier.) ∎

Proposition 3.21 *There exists a hermitian metric h_{HE} of $(E,\mathbb{D}^\lambda)$ with respect to ω_ϵ satisfying the following properties:*

- *Hermitian-Einstein condition* $\Lambda_{\omega_\epsilon} G(h_{HE}) = a$ *holds for the constant* a *determined by* (65).
- $\text{par-deg}_L(E, \boldsymbol{F}) = \deg_\omega(E, h_{HE})$.
- *We have the following formulas:*

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \frac{\operatorname{tr}\bigl(G(h_{HE})^{\perp\, 2}\bigr)}{(1+|\lambda|^2)^2} = \int_X \left(2\, \text{par-ch}_2(E, \boldsymbol{F}) - \frac{\text{par-c}_1^2(E, \boldsymbol{F})}{\operatorname{rank} E}\right) \tag{66}$$

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \frac{\operatorname{tr}\bigl(G(h_{HE})^2\bigr)}{(1+|\lambda|^2)^2} = \int_X 2\, \text{par-ch}_2(E, \boldsymbol{F}). \tag{67}$$

- h_{HE} *is adapted to the parabolic structure* $\boldsymbol{F}$. *More strongly, let* s *be determined by* $h_{HE} = h_{in} \cdot s$. *Then* s *and* s^{-1} *are bounded with respect to* h_{in}, *and* $\mathbb{D}^\lambda s$ *is* L^2 *with respect to* h_{in} *and* ω_ϵ.

Proof It follows from Lemma 3.20 and Proposition 2.31. ▮

3.4.2 Bogomolov-Gieseker inequality

Let Y be a smooth projective variety of *any* dimension Let L be an ample line bundle on Y, and let D be a simple normal crossing divisor.

Corollary 3.22 *Let* $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ *be a* μ_L*-stable regular filtered* λ*-flat bundle on* (Y, D) *in codimension two. Then, Bogomolov-Gieseker inequality holds for* $\boldsymbol{E}_*$. *Namely, we have the following inequality:*

$$\int_Y \text{par-ch}_{2,L}(\boldsymbol{E}_*) \leq \frac{\int_Y \text{par-c}_{1,L}^2(\boldsymbol{E}_*)}{2 \operatorname{rank} E}.$$

Proof Similar to Theorem 6.1 of [30]. Namely, since we have the Mehta-Ramanathan type theorem (Proposition 2.8), we have only to prove the claim in the case $\dim Y = 2$. Due to the method of perturbation of parabolic structure, we have only to prove the inequality in the case $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ is a graded semisimple μ_L-stable parabolic λ-flat bundle on (Y, D). Then we can take a Hermitian-Einstein metric h_{HE} as in Proposition 3.21, for which we have the standard inequality (See Proposition 3.4 of [34]):

$$\int_{Y-D} \operatorname{tr}\bigl(G(h_{HE}, \mathbb{D}^\lambda)^{\perp\, 2}\bigr) \geq 0. \tag{68}$$

Here $G(h_{HE}, \mathbb{D}^\lambda)^\perp$ denotes the trace free part of $G(h_{HE}, \mathbb{D}^\lambda)$. Hence we obtain the desired inequality from (68). ▮

3.5 Some Formulas and Vanishings of Characteristic Numbers

Let X be a smooth projective surface, and let D be a simple normal crossing divisor of X. We obtained some formulas for $\int_X \text{par-ch}_2(E, \boldsymbol{F})$ when $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ is a graded semisimple parabolic λ-flat bundle on (X, D). By using them, we will derive some formulas and vanishings for $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ which is *not* necessarily graded semisimple in this subsection. We will use the notation given by (56).

Remark 3.23 *We restrict ourselves to the case* $\dim X = 2$ *just for simplicity. The formula can be obviously generalized for* $\int_X \text{par-ch}_{2,L}(E, \boldsymbol{F})$ *of regular* λ*-flat parabolic bundles* $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ *on* (X, D) *in codimension two for* $\dim X > 2$, *where* L *denotes a line bundle on* X. ▮

3.5.1 Formulas of $\int_X \text{par-ch}_2(E, \boldsymbol{F})$ in terms of the data at the divisor

To begin with, we remark that we have only to show such formulas for *graded semisimple* parabolic λ-flat bundles, due to the method of perturbation of the parabolic structure (the subsection 2.1.5). We will use it without mention in the following argument.

Proposition 3.24

$$\int_X 2\,\text{par-ch}_2(E,\boldsymbol{F}) = \sum_{i\in S}\sum_{u\in\mathcal{KMS}(i)} \big(\mathrm{Re}(\lambda^{-1}\alpha)+a\big)^2\cdot r(i,u)\cdot[D_i]^2 \\ +\sum_{i\in S}\sum_{\substack{j\neq i\\ P\in D_i\cap D_j}}\sum_{(u_i,u_j)\in\mathcal{KMS}(P)} \big(\mathrm{Re}\,\lambda^{-1}\alpha_i+a_i\big)\big(\mathrm{Re}\,\lambda^{-1}\alpha_j+a_j\big)\cdot r(P,u_i,u_j). \quad (69)$$

We also have the following:

$$2\,\text{par-ch}_2(E,\boldsymbol{F}) = \sum_{i\in S}\sum_{i\in\mathcal{KMS}(i)} \mathrm{Re}\big(\lambda^{-1}\alpha+a\big)\cdot\text{par-deg}(E_{u\,D_i\,*}). \quad (70)$$

Here $E_{u\,D_i\,*}$ denote the parabolic bundle which is the pair of $E_{u\,D_i}$ with the naturally induced parabolic structure.

Proof From (52) and (57), we obtain the following equality:

$$\int_X 2\,\text{par-ch}_2(E,\boldsymbol{F}) = -\sum_{i,u}\left(\frac{\lambda^{-1}\alpha+\lambda\overline{\alpha}}{1+|\lambda|^2}+a\right)\left(\frac{1-|\lambda|^2}{1+|\lambda|^2}\cdot d(i,u)-r(i,u)\cdot\left(a+\frac{2\,\mathrm{Re}\,\lambda\overline{\alpha}}{1+|\lambda|^2}\right)\cdot[D_i]^2\right) \\ +\sum_i\sum_{\substack{j\neq i\\ P\in D_i\cap D_j}}\sum_{(u_i,u_j)}\left(\frac{\lambda^{-1}\alpha_i+\lambda\overline{\alpha}_i}{1+|\lambda|^2}+a_i\right)\left(\frac{2\,\mathrm{Re}\,\overline{\lambda}\alpha_j}{1+|\lambda|^2}+a_j\right)\cdot r(P,u_i,u_j) \\ = -\sum_{i,u}\left(\frac{\lambda^{-1}\alpha+|\lambda|^2\overline{\lambda^{-1}\alpha}}{1+|\lambda|^2}+a\right)\left(\frac{1-|\lambda|^2}{1+|\lambda|^2}d(i,u)-r(i,u)\cdot\left(a+\frac{2|\lambda|^2\,\mathrm{Re}\,\lambda^{-1}\alpha}{1+|\lambda|^2}\right)\cdot[D_i]^2\right) \\ +\sum_i\sum_{\substack{j\neq i\\ P\in D_i\cap D_j}}\sum_{(u_i,u_j)}\left(\frac{\lambda^{-1}\alpha_i+|\lambda|^2\overline{\lambda^{-1}\alpha_i}}{1+|\lambda|^2}+a_i\right)\left(\frac{2|\lambda|^2\,\mathrm{Re}\,\lambda^{-1}\alpha_j}{1+|\lambda|^2}+a_j\right)r(P,u_i,u_j). \quad (71)$$

By taking the real part, we obtain

$$\int_X 2\,\text{par-ch}_2(E,\boldsymbol{F}) = -\sum_{i,u}(\mathrm{Re}\,\lambda^{-1}\alpha+a)\left(\frac{1-|\lambda|^2}{1+|\lambda|^2}d(i,u)-r(i,u)\left(a+\frac{2|\lambda|^2\,\mathrm{Re}(\lambda^{-1}\alpha)}{1+|\lambda|^2}\right)[D_i]^2\right) \\ +\sum_i\sum_{\substack{j\neq i\\ P\in D_i\cap D_j}}\sum_{(u_i,u_j)}(\mathrm{Re}\,\lambda^{-1}\alpha_i+a_i)\left(\frac{2|\lambda|^2}{1+|\lambda|^2}\,\mathrm{Re}\,\lambda^{-1}\alpha_j+a_j\right)r(P,u_i,u_j) \\ = -\sum_{i,u}\big(\mathrm{Re}(\lambda^{-1}\alpha)+a\big)\left(\frac{1-|\lambda|^2}{1+|\lambda|^2}d(i,u)-r(i,u)\left(a+\mathrm{Re}\,\lambda^{-1}\alpha-\frac{1-|\lambda|^2}{1+|\lambda|^2}\,\mathrm{Re}\,\lambda^{-1}\alpha\right)[D_i]^2\right) \\ +\sum_i\sum_{\substack{j\neq i\\ P\in D_i\cap D_j}}\sum_{(u_i,u_j)}(\mathrm{Re}\,\lambda^{-1}\alpha_i+a_i)\left(\left(1-\frac{1-|\lambda|^2}{1+|\lambda|^2}\right)\mathrm{Re}\,\lambda^{-1}\alpha_j+a_j\right)\cdot r(P,u_i,u_j) \quad (72)$$

Let us make the following observation. For the decomposition $\mathbb{D}^\lambda = d''+d'$, we put $\mathbb{D}^{\lambda_1} := d''+(\lambda_1/\lambda)d'$. Then $\mathbb{D}^{\lambda_1}$ is the flat λ_1-connection of the parabolic bundle $(E,\boldsymbol{F})$. If (a,α) is a KMS-spectrum for $\mathbb{D}^\lambda$, then $(a',\alpha') = (a,\lambda_1\alpha/\lambda)$ is a KMS-spectrum for $\mathbb{D}^{\lambda_1}$. Under the correspondence, we have $\lambda^{-1}\alpha = \lambda_1^{-1}\alpha'$. (Indeed, it is the eigenvalues of the residue of the associated flat connection $\mathbb{D}^{\lambda f}$.) Therefore, we obtain the following formula by considering the formula (72) for $\lambda_1 \neq 0$:

$$\int_X 2\,\text{par-ch}_2(E,\boldsymbol{F}) = \sum_i \sum_{\substack{j\neq i\\ P\in D_i\cap D_j}} \sum_{(u_i,u_j)} (\operatorname{Re}\lambda^{-1}\alpha_i + a_i)\left(\left(1-\frac{1-|\lambda_1|^2}{1+|\lambda_1|^2}\right)\operatorname{Re}\lambda^{-1}\alpha_j + a_j\right)\cdot r(P,u_i,u_j)$$
$$-\sum_{i,u}\big(\operatorname{Re}(\lambda^{-1}\alpha)+a\big)\left(\frac{1-|\lambda_1|^2}{1+|\lambda_1|^2}\cdot d(i,u) - r(i,u)\left(a+\operatorname{Re}\lambda^{-1}\alpha - \frac{1-|\lambda_1|^2}{1+|\lambda_1|^2}\operatorname{Re}\lambda^{-1}\alpha\right)[D_i]^2\right) \tag{73}$$

We can regard the formula (73) as a polynomial of $t=(1-|\lambda_1|^2)(1+|\lambda_1|^2)^{-1}$. Therefore we obtain (69) by taking the degree 0-part of (73). By considering the coefficients of the degree one part of (73), we obtain the following:

$$\sum_{i,u}\operatorname{Re}(\lambda^{-1}\alpha+a)\Big(d(i,u)+r(i,u)\cdot\operatorname{Re}(\lambda^{-1}\alpha)[D_i]^2\Big)$$
$$+\sum_i \sum_{\substack{j\neq i\\ P\in D_i\cap D_j}} \sum_{(u_i,u_j)} \big(\operatorname{Re}(\lambda^{-1}\alpha_i)+a_i\big)\operatorname{Re}(\lambda^{-1}\alpha_j)\cdot r(P,u_i,u_j)=0. \tag{74}$$

Subtracting (74) from (69), we obtain (70). ∎

Remark 3.25 *The formula* (69) *can be regarded as the equality of parabolic second Chern character numbers for* $(E,\boldsymbol{F},\mathbb{D}^\lambda)$ *and the corresponding filtered local system. See the section* 6. ∎

3.5.2 Some vanishing

Proposition 3.26 *We have the following vanishing:*

$$\sum_{i\in S}\sum_{u\in\mathcal{KMS}(i)}\operatorname{Im}(\lambda^{-1}\alpha)\Big(d(i,u)+r(i,u)\cdot\operatorname{Re}(\lambda^{-1}\alpha)\cdot[D_i]^2\Big)$$
$$+\sum_{i\in S}\sum_{\substack{j\neq i\\ P\in D_i\cap D_j}}\sum_{(u_i,u_j)\in\mathcal{KMS}(P)}\operatorname{Im}(\lambda^{-1}\alpha_i)\cdot\operatorname{Re}(\lambda^{-1}\alpha_j)\cdot r(P,u_i,u_j)=0. \tag{75}$$

We also have the following vanishing:

$$\sum_i\sum_u \operatorname{Im}(\lambda^{-1}\alpha)\cdot\Big(\deg\widehat{E}_{u\,D_i,*} - r(i,u)\cdot a\cdot[D_i]^2\Big)=0. \tag{76}$$

Proof We obtain the following, by taking the imaginary part of (71):

$$\sum_i\sum_u -\left(\frac{1-|\lambda|^2}{1+|\lambda|^2}\operatorname{Im}(\lambda^{-1}\alpha)\right)\left(\frac{1-|\lambda|^2}{1+|\lambda|^2}d(i,u)-r(i,u)\left(a+\left(1-\frac{1-|\lambda|^2}{1+|\lambda|^2}\right)\right)\operatorname{Re}\lambda^{-1}\alpha\cdot[D_i]^2\right)$$
$$+\sum_i\sum_{\substack{j\neq i\\ P\in D_i\cap D_j}}\sum_{(u_i,u_j)}\frac{1-|\lambda|^2}{1+|\lambda|^2}\operatorname{Im}\lambda^{-1}\alpha_i\left(\left(1-\frac{1-|\lambda|^2}{1+|\lambda|^2}\right)\operatorname{Re}\lambda^{-1}\alpha_j+a_j\right)r(P,u_i,u_j)=0. \tag{77}$$

By the same consideration, we can regard (77) as a polynomial of $t=(1-|\lambda|^2)(1+|\lambda|^2)^{-1}$. By taking the degree two part, we obtain (75). By taking the degree one part, we obtain the following:

$$\sum_i\sum_u\operatorname{Im}(\lambda^{-1}\alpha)\cdot r(i,u)\cdot\big(a+\operatorname{Re}\lambda^{-1}\alpha\big)\cdot[D_i]^2+\sum_i\sum_{\substack{j\neq i\\ P\in D_i\cap D_j}}\sum_{(u_i,u_j)}\operatorname{Im}(\lambda^{-1}\alpha_i)\big(a_j+\operatorname{Re}\lambda^{-1}\alpha_j\big)\cdot r(P,u_i,u_j)=0. \tag{78}$$

From (75) and (78), we obtain the following:

$$\sum_i \sum_u \mathrm{Im}(\lambda^{-1}\alpha)\cdot\Big(d(i,u)-r(i,u)\cdot a\cdot[D_i]^2\Big)-\sum_i\sum_{j\neq i}\sum_{(u_i,u_j)}(\mathrm{Im}\,\lambda^{-1}\alpha_i)\cdot a_j\cdot r(P,u_i,u_j)=0. \tag{79}$$

It is equivalent to (76). ∎

Proposition 3.27 *We have the following formula:*

$$\sum_i\sum_{j\neq i}\sum_{(u_i,u_j)}\mathrm{Im}(\lambda^{-1}\alpha_i)\cdot\mathrm{Im}(\lambda^{-1}\alpha_j)\cdot r(P,u_i,u_j)+\sum_i\sum_u\big(\mathrm{Im}(\lambda^{-1}\alpha)\big)^2 r(i,u)\cdot[D_i]^2=0. \tag{80}$$

Proof From (52) and (62), we obtain the following:

$$\int_X 2\,\text{par-ch}_2(E,\boldsymbol{F})=\sum_i\sum_{j\neq i}\sum_{(u_i,u_j)}(\lambda^{-1}\alpha_i+a_i)(\lambda^{-1}\alpha_j+a_j)\cdot r(P,u_i,u_j)$$
$$+\sum_i\sum_{u\in\mathcal{KMS}(i)}(\lambda^{-1}\alpha+a)^2\cdot r(i,u)\cdot[D_i]^2. \tag{81}$$

Let us take the real part of (81), and compare it with (69). Then we obtain (80). ∎

3.5.3 Remark on the vanishing of the parabolic Chern character numbers

Recall the formulas for $\int_X \text{par-ch}_2(E,\boldsymbol{F})$ (Proposition 3.24, for example) and the formula for $\text{par-c}_1(E,\boldsymbol{F})$ (see (55) and Remark 3.16). Then we immediately obtain the following corollary.

Corollary 3.28 *When $a+\mathrm{Re}\,\lambda^{-1}\alpha=0$ for any KMS-spectrum (a,α) of $(E,\boldsymbol{F},\mathbb{D}^\lambda)$, the characteristic numbers $\text{par-deg}_\omega(E,\boldsymbol{F})$ and $\int_X\text{par-ch}_2(E,\boldsymbol{F})$ automatically vanish.* ∎

Remark 3.29 *Let E be a vector bundle on $X-D$ with a flat connection ∇. We have the Deligne extension $(\widetilde{E},\nabla)$. (See the subsection 2.1.3, for example.) Then we have the canonically defined parabolic structure $\boldsymbol{F}$ such that $\mathrm{Re}\,\alpha+a=0$ for any KMS-spectrum. In that case, the stability of $(\widetilde{E},\boldsymbol{F},\nabla)$ and the semisimplicity of (E,∇) is equivalent. The corollary means $\int_X\text{par-c}_2(\widetilde{E},\boldsymbol{F})=\text{par-deg}_\omega(\widetilde{E},\boldsymbol{F})=0$.*

When (E,∇) is semisimple, we know that there exists the Corlette-Jost-Zuo metric of (E,∇) which is a pure imaginary tame pluri-harmonic metric adapted to the parabolic structure $\boldsymbol{F}$ (See [2] for the case $D=\emptyset$ and [14] for the general case. See also [29].) To show such an existence theorem from the Kobayashi-Hitchin correspondence, we have to show the vanishing of the characteristic numbers which is "the obstruction on the way from harmonicity to pluri-harmonicity". Corollary 3.28 clarifies the point. ∎

4 Continuity of some families of harmonic metrics

4.1 Statements

In this section, we will show continuity of two kinds of families of harmonic metrics on curves, i.e., Proposition 4.1 and Proposition 4.2. We will give a detailed proof of the first one. Because the second one can be proved similarly and more easily, we just give some remarks in the end of this section.

4.1.1 Continuity for ϵ-perturbation

Let C be a smooth projective curve with a simple divisor D. Let $(E,\boldsymbol{F},\mathbb{D}^\lambda)$ be a parabolic flat λ-connection over (C,D), which is stable and $\text{par-deg}(E,\boldsymbol{F})=0$. Let $\boldsymbol{F}^{(\epsilon)}$ be the ϵ-perturbation of the parabolic structures. (See the subsection 2.1.5.) We remark $\det(E,\boldsymbol{F})=\det(E,\boldsymbol{F}^{(\epsilon)})$. Let $h^{(\epsilon)}$ be the harmonic metric for $(E,\boldsymbol{F}^{(\epsilon)},\mathbb{D}^\lambda)$ for $0\le\epsilon\le\epsilon_0$. Let $\theta^{(\epsilon)}$ denote the Higgs fields for the harmonic bundles $(E,\mathbb{D}^\lambda,h^{(\epsilon)})$.

Proposition 4.1 *The sequences* $\{h^{(\epsilon)} \mid \epsilon > 0\}$ *and* $\{\theta^{(\epsilon)}\}$ *converge to* $h^{(0)}$ *and* $\theta^{(0)}$ *respectively, in the* C^∞*-sense locally on* $C - D$.

The proof is given in the subsection 4.5 after the preparation given in the subsections 4.2–4.4. Before going into the proof of Proposition 4.1, we give a similar statement for another family in the next subsubsection.

4.1.2 Continuity for a holomorphic family

Let $\mathcal{C} \longrightarrow \Delta$ be a holomorphic family of smooth projective curve, and $\mathcal{D} \longrightarrow \Delta$ be a relative divisor. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a parabolic flat bundle on $(\mathcal{C}, \mathcal{D})$. Let t be any point of Δ. We denote the fibers by $\mathcal{C}_t$ and $\mathcal{D}_t$, and the restriction of $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ to $(\mathcal{C}_t, \mathcal{D}_t)$ is denoted by $(E_t, \boldsymbol{F}_t, \mathbb{D}^\lambda_t)$. We assume par-deg$(E_t, \boldsymbol{F}_t) = 0$ and that $(E_t, \boldsymbol{F}_t, \mathbb{D}^\lambda_t)$ is stable for each t. For simplicity, we also assume that we are given a pluri harmonic metric $h_{\det(E)}$ of $\det(E, \mathbb{D}^\lambda)_{|\mathcal{C}-\mathcal{D}}$ which is adapted to the induced parabolic structure.

Let $h_{H,t}$ be a harmonic metric of $(E_t, \boldsymbol{F}_t, \mathbb{D}^\lambda_t)$ such that $\det(h_{H,t}) = h_{\det(E)\,|\,\mathcal{C}_t}$. They give the metric h_H of E. Let $\theta_{H,t}$ be the Higgs filed obtained from $(E_t, \mathbb{D}^\lambda, h^{(\epsilon_t)})$, which is a section of $\mathrm{End}(E_t) \otimes \Omega^{1,0}_{\mathcal{C}_t}(\log \mathcal{D}_t)$. They give the section θ_H of $\mathrm{End}(E) \otimes \Omega^{1,0}_{\mathcal{C}/\Delta}(\log \mathcal{D})$, where $\Omega^{1,0}_{\mathcal{C}/\Delta}(\log \mathcal{D})$ denotes the sheaf of the logarithmic relative $(1,0)$-forms.

Proposition 4.2 h_H *and* θ_H *are continuous. Their derivatives of any degree along the fiber directions are continuous.*

Since Proposition 4.2 can be proved similarly and more easily, we will not give a detailed proof. See Remark 4.14.

4.2 Preliminary from Elementary Calculus

For any $z \in \Delta^* = \{z \in \boldsymbol{C} \mid |z| < 1\}$ and $\epsilon > 0$, we put as follows:

$$L_\epsilon(z) := \frac{|z|^{-\epsilon} - |z|^\epsilon}{\epsilon}, \quad K_\epsilon(z) := \frac{|z|^{-\epsilon} + |z|^\epsilon}{2}, \quad M_\epsilon(z) := |z|^{4\epsilon}(1 - \log |z|^{4\epsilon}).$$

We also put $L_0(z) := -\log |z|^2$, $K_0(z) = 1$ and $M_0(z) = 1$. Then they are continuous with respect to $(z, \epsilon) \in \Delta^* \times \boldsymbol{R}_{\geq 0}$.

Lemma 4.3 *For any* $(z, \epsilon) \in \Delta^* \times \boldsymbol{R}_{\geq 0}$*, we have* $L_0(z) \leq L_\epsilon(z)$.

Proof We put $g(\epsilon) := a^{-\epsilon} - a^\epsilon + \epsilon \cdot \log a^2$ for $0 < a < 1$ and $0 \leq \epsilon$. Taking the derivative with respect to ϵ, we obtain the following:

$$g'(\epsilon) = -\big(a^{-\epsilon} + a^\epsilon\big) \log a + \log a^2, \quad g''(\epsilon) = (a^{-\epsilon} - a^\epsilon)(\log a)^2 \geq 0.$$

Since we have $g(0) = g'(0) = 0$, the claim of the lemme follows. ∎

Lemma 4.4 $(K_\epsilon(z) - 1) \cdot \big(L_\epsilon(z)^2 \cdot \epsilon^2 \cdot |z|^\epsilon\big)^{-1}$ *are bounded on* Δ^**, independently of* ϵ*. We also have* $K_\epsilon(z) - 1 \geq 0$.

Proof The second claim is clear. Let us check the first claim. We put as follows, for $0 < a < 1$ and $0 \leq \epsilon \leq 1$:

$$g_1(\epsilon) := a^{-\epsilon} - 2 + a^\epsilon, \quad g_2(\epsilon) := (a^{-\epsilon} - a^\epsilon)^2 a^\epsilon = a^{-\epsilon} - 2a^\epsilon + a^{3\epsilon}.$$

We have only to show that $g_2(\epsilon) \geq g_1(\epsilon)$. We put $g(\epsilon) := g_2(\epsilon) - g_1(\epsilon) = 2 + a^{3\epsilon} - 3a^\epsilon$. By taking the derivative with respect to ϵ, we obtain the following:

$$g'(\epsilon) = 3a^{3\epsilon} \cdot \log a - 3a^\epsilon \cdot \log a = 3(-a^{3\epsilon} + a^\epsilon)(-\log a) \geq 0.$$

Since we have $g(0) = 0$, we obtain $g(\epsilon) \geq 0$. Thus we are done. ∎

Lemma 4.5 *$\big(1-M_\epsilon(z)\big)\cdot\big(L_\epsilon(z)^2\cdot\epsilon^2\cdot|z|^\epsilon\big)^{-1}$ are bounded on Δ^*, independently of ϵ. We also have $1-M_\epsilon(z)\geq 0$.*

Proof We have only to show the following inequalities for $0<a<1$ and $0\leq\epsilon<1$:

$$0\leq 1-a^{4\epsilon}(1-\log a^{4\epsilon})\leq 3(a^{-\epsilon}-a^{\epsilon})^2a^{\epsilon}.$$

To show the left inequality, we put $h(\epsilon):=1-a^{4\epsilon}(1-\log a^{4\epsilon})$. By taking the derivative with respect to ϵ, we have $h'(\epsilon)=-a^{4\epsilon}\log a^4(1-\log a^{4\epsilon})+a^{4\epsilon}\log a^4=\epsilon a^{4\epsilon}(\log a^4)^2\geq 0$. We also have $h(0)=0$. Hence we obtain $h(\epsilon)\geq 0$. To show the right inequality, we put as follows:

$$g(\epsilon):=a^{-4\epsilon}\Big(3(a^{-\epsilon}-a^{\epsilon})^2a^{\epsilon}-\big(1-a^{4\epsilon}(1-\log a^{4\epsilon})\big)\Big)=3(a^{-5\epsilon}-2a^{-3\epsilon}+a^{-\epsilon})+(1-\log a^{4\epsilon})-a^{-4\epsilon}.$$

By taking the derivative with respect to ϵ, we obtain the following:

$$g'(\epsilon)=3\big(a^{-5\epsilon}(-5\log a)-2a^{-3\epsilon}(-3\log a)+a^{-\epsilon}(-\log a)\big)-4\log a-a^{-4\epsilon}(-4\log a)$$

$$g''(\epsilon)=\big(75a^{-5\epsilon}-16a^{-4\epsilon}-54a^{-3\epsilon}+3a^{-\epsilon}\big)\cdot(\log a)^2.$$

It is easy to check $g''(\epsilon)\geq 0$ by using $a^{-5\epsilon}\geq a^{-k\epsilon}$ $(k=3,4)$. Since we have $g'(0)=g(0)=0$, we obtain $g(\epsilon)\geq 0$. Thus we are done. ∎

Lemma 4.6 *Let $P(t)$ be a polynomial with variable t, and let b be any fixed positive number. Then we have the boundedness of $|z|^{b\epsilon}P\big(\epsilon L_0(z)\big)$ on Δ^*, independently of $0\leq\epsilon\leq 1/2$.*

Proof We put $u:=|z|^\epsilon$, and then $|z|^{b\epsilon}P(\epsilon L_0(z))=u^b\cdot P\big(L_0(u)\big)$. Hence we have only to show the boundedness of $u^b\cdot P\big(L_0(u)\big)$ when $0<u<1$, but it is easy. ∎

4.3 A Family of the Metrics for Logarithmic flat λ-bundle of Rank Two on a Disc

4.3.1 Construction of a family of metrics

We put $X=\Delta=\{z\,\big|\,|z|<1\}$. Let O denote the origin, and we put $X^*:=X-\{O\}$. We use the Kahler form $\omega_\epsilon:=\epsilon^2|z|^\epsilon dz\cdot d\overline{z}/|z|^2+dz\cdot d\overline{z}$ in this subsection. We will use the notation in the subsection 4.2.

To begin with, we recall an example of a harmonic bundle on a punctured disc. Let $E=\mathcal{O}_X\cdot v_1\oplus\mathcal{O}_X\cdot v_2$ be a holomorphic vector bundle on a disc. Let θ be a Higgs bundle such that $\theta\cdot v_1=v_2\cdot dz/z$ and $\theta\cdot v_2=0$. Let h be the metric of $E_{|X^*}$ such that $h(v_1,v_1)=L_0$, $h(v_2,v_2)=L_0^{-1}$ and $h(v_i,v_j)=0$ $(i\neq j)$. Recall that the tuple $(E,\overline{\partial}_E,\theta,h)$ is a harmonic bundle. Let us see the associated λ-connection. We put $u_1:=v_1$ and $u_2:=v_2-\lambda\cdot L_0^{-1}\cdot v_1$. Then it can be shown by a direct calculation that $(\overline{\partial}_E+\lambda\theta^\dagger)u_i=0$ $(i=1,2)$, $\mathbb{D}^\lambda u_1=u_2\cdot dz/z$ and $\mathbb{D}^\lambda u_2=0$. We also have the following:

$$h(u_1,u_1)=L_0,\quad h(u_2,u_2)=(1+|\lambda|^2)\cdot L_0^{-1},\quad h(u_1,u_2)=-\overline{\lambda},\quad h(u_2,u_1)=-\lambda.$$

Motivated by this example, we consider the following family of the metrics h_ϵ on the λ-connection $(E,\mathbb{D}^\lambda)$ given as follows:

$$h_\epsilon(u_1,u_1)=L_\epsilon,\quad h_\epsilon(u_2,u_2)=\big(1+|\lambda|^2\big)^{-1}\cdot L_\epsilon,\quad h_\epsilon(u_1,u_2)=-\overline{\lambda}\cdot M_\epsilon,\quad h_\epsilon(u_2,u_1)=-\lambda\cdot M_\epsilon.$$

The λ-connection $\mathbb{D}^\lambda$ and the metric h_ϵ induce the operators $\overline{\partial}_\epsilon$ and θ_ϵ (the subsection 2.2.1). The main purpose of this subsection is to show the following proposition.

Proposition 4.7 *There exists a some positive constant C such that $\big|\overline{\partial}_\epsilon\theta_\epsilon\big|_{h_\epsilon,\omega_\epsilon}\leq C$ for any $0\leq\epsilon<1/2$.*

Although the proof of the proposition is just a calculation, we will give the detail in the rest of this subsection.

Remark 4.8 *Let h'_ϵ be the metric determined by $h'_\epsilon(u_1,u_1)=L_\epsilon$, $h'_\epsilon(u_2,u_2)=L_\epsilon^{-1}$ and $h'_\epsilon(u_i,u_j)=0$ $(i\neq j)$. Then there exist positive constants C_i $(i=1,2)$ such that $C_1\cdot h'_\epsilon\leq h_\epsilon\leq C_2\cdot h'_\epsilon$ for any $0\leq\epsilon\leq 1/2$. Hence we have only to consider the norms for h'_ϵ instead of those for h_ϵ.* ∎

4.3.2 Preliminary

Let H_ϵ be the hermitian matrix valued function given by $H_\epsilon := H(h_\epsilon, \boldsymbol{u})$, i.e.,

$$H_\epsilon := \begin{pmatrix} L_\epsilon & -\overline{\lambda}\cdot M_\epsilon \\ -\lambda\cdot M_\epsilon & (1+|\lambda|^2)L_\epsilon^{-1} \end{pmatrix}.$$

Let N be determined by $\mathbb{D}^\lambda \boldsymbol{u} = \boldsymbol{u}\cdot N\cdot dz/z$, and let $N_\epsilon^\dagger$ denote the adjoint of N with respect to the metric H_ϵ, i.e.,

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad N_\epsilon^\dagger = \overline{H}_\epsilon^{-1}\cdot {}^t\overline{N}\cdot \overline{H}_\epsilon = \frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\begin{pmatrix} -\overline{\lambda}(1+|\lambda|^2)L_\epsilon^{-1}M_\epsilon & (1+|\lambda|^2)^2L_\epsilon^{-2} \\ -\overline{\lambda}^2M_\epsilon^2 & \overline{\lambda}(1+|\lambda|^2)M_\epsilon L_\epsilon^{-1} \end{pmatrix}.$$

Recall the calculation given in the subsubsection 2.2.2. Then $\overline{\partial}_\epsilon$ and θ_ϵ can be described with respect to $\boldsymbol{u}$ as follows:

$$\overline{\partial}_\epsilon\boldsymbol{u} = \boldsymbol{u}\cdot\frac{\lambda}{1+|\lambda|^2}\Big(\overline{\lambda}\cdot\overline{H}_\epsilon^{-1}\overline{\partial}\overline{H}_\epsilon - N_\epsilon^\dagger\frac{d\overline{z}}{\overline{z}}\Big),\quad \theta_\epsilon\boldsymbol{u} = \boldsymbol{u}\frac{1}{1+|\lambda|^2}\Big(N\frac{dz}{z} - \lambda\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon\Big).$$

Therefore $\overline{\partial}_\epsilon(\theta_\epsilon)$ is described by the following 2×2-matrix valued 2-form with respect to $\boldsymbol{u}$:

$$\frac{1}{1+|\lambda|^2}\overline{\partial}\Big(-\lambda\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon\Big) + \frac{\lambda}{(1+|\lambda|^2)^2}\left(\Big[\overline{\lambda}\cdot\overline{H}_\epsilon^{-1}\overline{\partial}\overline{H}_\epsilon, N\frac{dz}{z}\Big] - \Big[N_\epsilon^\dagger\frac{d\overline{z}}{\overline{z}},\, N\frac{dz}{z}\Big] + \Big[N_\epsilon^\dagger\frac{d\overline{z}}{\overline{z}},\, \lambda\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon\Big]\right). \tag{82}$$

Here we have used $\big[\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon,\ \overline{H}_\epsilon^{-1}\overline{\partial}\overline{H}_\epsilon\big] = 0$, which can be checked easily.

Lemma 4.9 *To show Proposition* 4.7*, we have only to show the uniform boundedness of* $(1,1)$*-entry,* $(2,2)$*-entry,* $L_\epsilon\times$ $(1,2)$*-entry and* $L_\epsilon^{-1}\times$ $(2,1)$*-entry, in the matrix valued function* (82).

Proof It follows from Remark 4.8. ∎

In the following calculation, we often use the notation L and M instead of L_ϵ and M_ϵ, if there are no confusion. Let us see $\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon$. We have the following equality:

$$\overline{H}_\epsilon^{-1} = \frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\begin{pmatrix} (1+|\lambda|^2)\cdot L_\epsilon^{-1} & \lambda\cdot M_\epsilon \\ \overline{\lambda}\cdot M_\epsilon & L_\epsilon \end{pmatrix},\quad \partial\overline{H}_\epsilon = \begin{pmatrix} \partial L_\epsilon & -\lambda\cdot\partial M_\epsilon \\ -\overline{\lambda}\cdot\partial M_\epsilon & (1+|\lambda|^2)\cdot\partial L_\epsilon^{-1} \end{pmatrix}.$$

Then we obtain the following formula for $\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon$:

$$\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon = \frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\begin{pmatrix} (1+|\lambda|^2)L^{-1}\partial L - |\lambda|^2M\partial M & \lambda(1+|\lambda|^2)\big(-L^{-1}\partial M + M\partial L^{-1}\big) \\ \overline{\lambda}(M\partial L - L\partial M) & (1+|\lambda|^2)L\partial L^{-1} - |\lambda|^2M\cdot\partial M \end{pmatrix}. \tag{83}$$

We also have a similar formula for $\overline{H}_\epsilon^{-1}\overline{\partial}\overline{H}_\epsilon$. We obtain the following formula for $\overline{\partial}\big(\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon\big)$:

$$\begin{aligned}\overline{\partial}\big(\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon\big) &= \frac{2|\lambda|^2M\overline{\partial}M}{1+|\lambda|^2(1-M^2)}\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon \\ &+ \frac{1}{1+|\lambda|^2(1-M^2)}\begin{pmatrix} (1+|\lambda|^2)\overline{\partial}\partial\log L - 2^{-1}|\lambda|^2\overline{\partial}\partial M^2 & \lambda(1+|\lambda|^2)(M\overline{\partial}\partial L^{-1} - L^{-1}\overline{\partial}\partial M) \\ \overline{\lambda}(M\overline{\partial}\partial L - L\overline{\partial}\partial M) & (1+|\lambda|^2)\overline{\partial}\partial\log L^{-1} - 2^{-1}|\lambda|^2\overline{\partial}\partial M^2 \end{pmatrix}.\end{aligned} \tag{84}$$

The commutator of $\overline{H}_\epsilon^{-1}\overline{\partial}\overline{H}_\epsilon$ and $N\cdot dz/z$ is as follows:

$$\Big[\overline{H}_\epsilon^{-1}\overline{\partial}\overline{H}_\epsilon, N\cdot\frac{dz}{z}\Big] = \frac{(1+|\lambda|^2)}{1+|\lambda|^2(1-M^2)}\begin{pmatrix} \lambda(-L^{-1}\overline{\partial}M + M\overline{\partial}L^{-1}) & 0 \\ 2L\overline{\partial}L^{-1} & -\lambda(-L^{-1}\overline{\partial}M + M\overline{\partial}L^{-1}) \end{pmatrix}\frac{dz}{z}. \tag{85}$$

Let us see the commutator of $\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon$ and $N_\epsilon^\dagger$. By direct calculations, we have the following equality:

$$\begin{aligned}\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon\cdot N_\epsilon^\dagger &= \frac{1}{1+|\lambda|^2(1-M^2)}\begin{pmatrix} -\overline{\lambda}(1+|\lambda|^2)L^{-2}M\partial L & (1+|\lambda|^2)^2L^{-3}\partial L \\ \overline{\lambda}^2\cdot M\partial M & -\overline{\lambda}(1+|\lambda|^2)L^{-1}\partial M \end{pmatrix} \\ &+ \frac{1}{\big(1+|\lambda|^2(1-M^2)\big)^2}\begin{pmatrix} 2|\lambda|^2\overline{\lambda}(1+|\lambda|^2)M^2L^{-1}\partial M & -2|\lambda|^2(1+|\lambda|^2)^2ML^{-2}\partial M \\ 2M^3\partial M\overline{\lambda}^2|\lambda|^2 & -2\overline{\lambda}|\lambda|^2(1+|\lambda|^2)M^2L^{-1}\partial M \end{pmatrix}.\end{aligned} \tag{86}$$

We also have the following:

$$N_\epsilon^\dagger \cdot \overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon = \frac{1}{1+|\lambda|^2(1-M^2)}\begin{pmatrix} -\overline{\lambda}(1+|\lambda|^2)L^{-1}\partial M & (1+|\lambda|^2)^2L^{-1}\partial L^{-1} \\ -\overline{\lambda}^2 M\partial M & \overline{\lambda}(1+|\lambda|^2)M\partial L^{-1} \end{pmatrix}. \tag{87}$$

Therefore we obtain the following formula:

$$\begin{aligned}
&\Big[N_\epsilon^\dagger \frac{d\overline{z}}{\overline{z}}, \overline{H}_\epsilon\partial\overline{H}_\epsilon\Big] \\
&= \frac{1}{1+|\lambda|^2(1-M^2)}\frac{d\overline{z}}{\overline{z}}\begin{pmatrix} -\overline{\lambda}(1+|\lambda|^2)(L^{-1}\partial M - L^{-2}M\partial L) & -2(1+|\lambda|^2)^2L^{-3}\partial L \\ -2\overline{\lambda}^2 M\partial M & \overline{\lambda}(1+|\lambda|^2)(M\partial L^{-1}+L^{-1}\partial M) \end{pmatrix} \\
&\qquad - \frac{2|\lambda|^2}{\big(1+|\lambda|^2(1-M^2)\big)^2}\frac{d\overline{z}}{\overline{z}}\begin{pmatrix} \overline{\lambda}(1+|\lambda|^2)M^2L^{-1}\partial M & -(1+|\lambda|^2)^2ML^{-2}\partial M \\ \overline{\lambda}^2M^3\partial M & -\overline{\lambda}(1+|\lambda|^2)M^2L^{-1}\partial M \end{pmatrix}
\end{aligned} \tag{88}$$

The commutator of N and $N_\epsilon^\dagger$ is as follows:

$$\big[N_\epsilon^\dagger, N\big] = \frac{1}{1+|\lambda|^2(1-M^2)}\begin{pmatrix} (1+|\lambda|^2)^2L^{-2} & 0 \\ 2\overline{\lambda}(1+|\lambda|^2)ML^{-1} & -(1+|\lambda|^2)^2L^{-2} \end{pmatrix}. \tag{89}$$

4.3.3 Estimate

We have the following:

$$\partial L_\epsilon = -K_\epsilon\frac{dz}{z}, \quad \partial K_\epsilon = -\frac{\epsilon^2}{4}L_\epsilon\frac{dz}{z}, \quad \partial M_\epsilon = 4\epsilon^2\cdot|z|^{4\epsilon}\cdot L_0\cdot\frac{dz}{z}. \tag{90}$$

In particular, we have the following estimate:

$$M_\epsilon\partial M_\epsilon = O\Big(\epsilon^2\cdot|z|^{8\epsilon}\cdot L_0\cdot\big(1+\epsilon L_0\big)\frac{dz}{z}\Big).$$

Let us see the first term in the right hand side of (84):

$$\frac{2|\lambda|^2M_\epsilon\overline{\partial}M_\epsilon}{1+|\lambda|^2(1-M_\epsilon^2)}H_\epsilon^{-1}\partial H_\epsilon \tag{91}$$

For the $(1,1)$-entry and $(2,2)$-entry, we have the following estimates:

$$M_\epsilon\overline{\partial}M_\epsilon\cdot L_\epsilon^{-1}\partial L_\epsilon = O\left(\epsilon^2\cdot L_0\cdot|z|^{8\epsilon}(1+\epsilon L_0)\frac{K_\epsilon}{L_\epsilon}\right)\frac{d\overline{z}\cdot dz}{|z|^2} = O\left(|z|^{5\epsilon}(1+\epsilon L_0)\frac{L_0}{L_\epsilon}\right)\cdot\omega_\epsilon$$

$$M_\epsilon\overline{\partial}M_\epsilon\cdot M_\epsilon\partial M_\epsilon = O\Big(\epsilon^4\cdot|z|^{16\epsilon}\cdot(1+\epsilon L_0)^2L_0^2\Big)\frac{dz\cdot d\overline{z}}{|z|^2} = O\Big(|z|^{15\epsilon}\cdot(1+\epsilon L_0)^2(\epsilon L_0)^2\Big)\cdot\omega_\epsilon.$$

They are bounded with respect to ω_ϵ due to Lemma 4.3 and Lemma 4.6. Hence the $(1,1)$-entry and the $(2,2)$-entry of (91) are bounded independently of ϵ. Let us see the $(1,2)$-entry. Recall Lemma 4.9. Hence we have only to see the following:

$$L_\epsilon\times(M_\epsilon\overline{\partial}M_\epsilon)\cdot\big(L_\epsilon^{-1}\partial M_\epsilon - M_\epsilon\partial L_\epsilon^{-1}\big) = M_\epsilon\overline{\partial}M_\epsilon\partial M_\epsilon + M_\epsilon^2\overline{\partial}M_\epsilon L_\epsilon^{-1}\partial L_\epsilon.$$

Both terms in the right hand side can be estimated as in the previous case, by using Lemma 4.3 and Lemma 4.6:

$$M_\epsilon\overline{\partial}M_\epsilon\partial M_\epsilon = O\Big(|z|^{10\epsilon}(1+\epsilon L_0)(\epsilon L_0)^2\Big)\cdot\omega_\epsilon = O(1)\cdot\omega_\epsilon.$$

$$M_\epsilon^2\overline{\partial}M_\epsilon L_\epsilon^{-1}\partial L_\epsilon = O\Big(|z|^{11\epsilon}(1+\epsilon L_0)^2\frac{L_0}{L_\epsilon}\Big)\cdot\omega_\epsilon = O(1)\cdot\omega_\epsilon$$

The $(2,1)$-entry can be estimated similarly:

$$L_\epsilon^{-1} \times (M_\epsilon \overline{\partial} M_\epsilon)\big(M_\epsilon \partial L_\epsilon - L_\epsilon \partial M_\epsilon\big) = M_\epsilon^2 L_\epsilon^{-1} \overline{\partial} M_\epsilon \partial L_\epsilon - M_\epsilon \cdot \overline{\partial} M_\epsilon \partial M_\epsilon = O(1) \cdot \omega_\epsilon.$$

Let us see the second term in the right hand side of (84):

$$\frac{1}{1+|\lambda|^2(1-M^2)} \begin{pmatrix} (1+|\lambda|^2)\overline{\partial}\partial \log L - 2^{-1}|\lambda|^2 \overline{\partial}\partial M^2 & \lambda(1+|\lambda|^2)(M\overline{\partial}\partial L^{-1} - L^{-1}\overline{\partial}\partial M) \\ \overline{\lambda}(1+|\lambda|^2)(M\overline{\partial}\partial L - L\overline{\partial}\partial M) & (1+|\lambda|^2)\overline{\partial}\partial \log L^{-1} - 2^{-1}|\lambda|^2 \overline{\partial}\partial M^2 \end{pmatrix}. \tag{92}$$

It is easy to see the following estimate:

$$\overline{\partial}\partial M_\epsilon^2 = O\Big(\epsilon^2 \cdot |z|^{6\epsilon}(1+\epsilon L_0)^2\Big) \cdot \omega_\epsilon = O(\epsilon^2) \cdot \omega_\epsilon. \tag{93}$$

Hence it is bounded with respect to ω_ϵ independently of ϵ. We remark that $L_\epsilon^{-1} M_\epsilon \overline{\partial}\partial L_\epsilon$ is also bounded independently of ϵ:

$$L_\epsilon^{-1} M_\epsilon \cdot \overline{\partial}\partial L_\epsilon = \frac{\epsilon^2}{4} M_\epsilon \cdot \frac{d\overline{z} \cdot dz}{|z|^2} = O(1) \cdot \omega_\epsilon.$$

Hence we have the following, modulo the uniformly bounded term with respect to $(h_\epsilon, \omega_\epsilon)$:

$$\overline{\partial}\big(\overline{H}_\epsilon^{-1} \partial \overline{H}_\epsilon\big) \equiv \frac{(1+|\lambda|^2)}{1+|\lambda|^2(1-M_\epsilon^2)} \begin{pmatrix} \overline{\partial}\partial \log L_\epsilon & \lambda M_\epsilon \overline{\partial}\partial L_\epsilon^{-1} \\ 0 & -\overline{\partial}\partial \log L_\epsilon \end{pmatrix}. \tag{94}$$

Let us see (85). By the same argument, we have the following uniform boundedness:

$$L_\epsilon^{-1} \overline{\partial} M_\epsilon \cdot \frac{dz}{z} = O\left(\epsilon^2 |z|^{4\epsilon} \frac{L_0}{L_\epsilon}\right) \cdot \frac{dz \cdot d\overline{z}}{|z|^2} = O(1) \cdot \omega_\epsilon.$$

Hence we have the following, modulo the uniformly bounded terms with respect to $(h_\epsilon, \omega_\epsilon)$:

$$\left[\overline{H}_\epsilon^{-1} \overline{\partial} \overline{H}_\epsilon,\ N \cdot \frac{dz}{z}\right] \equiv \frac{(1+|\lambda|^2)}{1+|\lambda|^2(1-M_\epsilon^2)} \begin{pmatrix} \lambda M_\epsilon \overline{\partial} L_\epsilon^{-1} & 0 \\ 2L_\epsilon \overline{\partial} L_\epsilon^{-1} & -\lambda M_\epsilon \overline{\partial} L_\epsilon^{-1} \end{pmatrix} \cdot \frac{dz}{z}. \tag{95}$$

Let us see (88). We remark the following, for any $k \geq 1$:

$$\frac{d\overline{z}}{\overline{z}} \frac{M_\epsilon^k \partial M_\epsilon}{L_\epsilon} = O\left(\epsilon^2 |z|^{4(k+1)\epsilon}(1+\epsilon L_0)^k \frac{L_0}{L_\epsilon}\right) \cdot \frac{d\overline{z} \cdot dz}{|z|^2} = O(1) \cdot \omega_\epsilon.$$

Hence the terms containing ∂M in the right hand side of (88) can be ignored. Hence we obtain the following, modulo the uniformly bounded terms with respect to $(h_\epsilon, \omega_\epsilon)$:

$$\left[N_\epsilon^\dagger \frac{d\overline{z}}{\overline{z}},\ \overline{H}_\epsilon^{-1} \partial \overline{H}_\epsilon\right] \equiv \frac{(1+|\lambda|^2)}{1+|\lambda|^2(1-M_\epsilon^2)} \frac{d\overline{z}}{\overline{z}} \begin{pmatrix} \overline{\lambda} L_\epsilon^{-2} M_\epsilon \partial L_\epsilon & -2(1+|\lambda|^2) L_\epsilon^{-3} \partial L_\epsilon \\ 0 & \overline{\lambda} M_\epsilon \partial L_\epsilon^{-1} \end{pmatrix}. \tag{96}$$

In all, (82) is same as the following, modulo uniformly bounded terms due to (89), (94), (95) and (96):

$$\begin{aligned} &\frac{1}{1+|\lambda|^2(1-M_\epsilon^2)} \begin{pmatrix} -\lambda \overline{\partial}\partial \log L_\epsilon & -\lambda^2 M_\epsilon \cdot \overline{\partial}\partial L_\epsilon^{-1} \\ 0 & \lambda \overline{\partial}\partial \log L_\epsilon \end{pmatrix} \\ &\quad + \frac{1}{1+|\lambda|^2(1-M_\epsilon^2)} \frac{|\lambda|^2}{1+|\lambda|^2} \frac{d\overline{z} \cdot dz}{|z|^2} \begin{pmatrix} \lambda \cdot M_\epsilon \cdot K_\epsilon \cdot L_\epsilon^{-2} & 0 \\ 2K_\epsilon \cdot L_\epsilon^{-1} & -\lambda \cdot M_\epsilon \cdot K_\epsilon \cdot L_\epsilon^{-2} \end{pmatrix} \\ &\quad + \frac{1}{1+|\lambda|^2(1-M_\epsilon^2)} \frac{\lambda^2}{1+|\lambda|^2} \frac{d\overline{z} \cdot dz}{|z|^2} \begin{pmatrix} -\overline{\lambda} \cdot M_\epsilon \cdot K_\epsilon \cdot L_\epsilon^{-2} & 2(1+|\lambda|^2) L_\epsilon^{-3} \cdot K_\epsilon \\ 0 & \overline{\lambda} \cdot M_\epsilon \cdot K_\epsilon \cdot L_\epsilon^{-2} \end{pmatrix} \\ &\quad - \frac{\lambda}{1+|\lambda|^2(1-M_\epsilon^2)} \frac{d\overline{z} \cdot dz}{|z|^2} \begin{pmatrix} L_\epsilon^{-2} & 0 \\ 2\overline{\lambda}(1+|\lambda|^2)^{-1} M_\epsilon \cdot L_\epsilon^{-1} & -L_\epsilon^{-2} \end{pmatrix}. \end{aligned} \tag{97}$$

The summation of the last three term in (97) is as follows:

$$\frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\frac{d\overline{z}\cdot dz}{|z|^2}\begin{pmatrix} -\lambda L_\epsilon^{-2} & 2\lambda^2 L_\epsilon^{-3}K_\epsilon \\ 2|\lambda|^2(1+|\lambda|^2)^{-1}(K_\epsilon - M_\epsilon)L_\epsilon^{-1} & \lambda L_\epsilon^{-2}\end{pmatrix}. \tag{98}$$

By a direct calculation, we can show the following equalities:

$$\overline{\partial}\partial\log L_\epsilon = -\frac{1}{L_\epsilon^2}\frac{d\overline{z}\cdot dz}{|z|^2}, \qquad \overline{\partial}\partial L_\epsilon^{-1} = \frac{2}{L_\epsilon^3}\frac{d\overline{z}\cdot dz}{|z|^2} - \frac{\epsilon^2}{2}\frac{1}{L_\epsilon}\frac{d\overline{z}\cdot dz}{|z|^2}.$$

Therefore, (97) can be rewritten as follows:

$$\frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\begin{pmatrix} 0 & 2\lambda^2 L_\epsilon^{-3}(K_\epsilon - M_\epsilon) \\ 2|\lambda|^2(1+|\lambda|^2)^{-1}L_\epsilon^{-1}(K_\epsilon - M_\epsilon) & 0\end{pmatrix}\cdot\frac{d\overline{z}\cdot dz}{|z|^2} + \frac{1}{1+|\lambda|^2(1-M_\epsilon^2)}\begin{pmatrix} 0 & \lambda^2\epsilon^2 M_\epsilon(2L_\epsilon)^{-1} \\ 0 & 0\end{pmatrix}\cdot\frac{d\overline{z}\cdot dz}{|z|^2}. \tag{99}$$

Due to $M_\epsilon = O\big(|z|^{4\epsilon}(1+\epsilon L_0)\big)$, the second term in (99) can be ignored. Due to Lemma 4.5 and Lemma 4.4, we have the uniform boundedness of $(M_\epsilon - 1)\cdot L_\epsilon^{-2}\cdot dz\cdot d\overline{z}/|z|^2$ and $(K_\epsilon - 1)\cdot L_\epsilon^{-2}\cdot dz\cdot d\overline{z}/|z|^2$. Thus the proof of Proposition 4.7 is finished. ∎

4.4 A Family of Metrics of a Parabolic Flat Bundle on a Disc

4.4.1 Simple case

We put $X := \Delta = \{z\in \boldsymbol{C}\,|\,|z|<1\}$ and $X^* := \Delta - \{O\}$. Let V_l be a vector space over $\boldsymbol{C}$ with a base $\boldsymbol{e} = (e_1,\ldots,e_l)$, and let N_l be the nilpotent endomorphism of V_l given by $N_l\cdot e_{i+1} = e_i$ for $i=1,\ldots,l-1$ and $N_l\cdot e_l = 0$. We put $E_l := \mathcal{O}_X\otimes V_l$. Then e_i naturally induce the frame of E_l, which we denote by $\boldsymbol{v} = (v_1,\ldots,v_l)$. The fiber $E_{|O}$ is naturally identified with V, and we have $\boldsymbol{v}_{|O} = \boldsymbol{e}$. We have the logarithmic λ-connection $\mathbb{D}_l^\lambda$ of E_l given by $\mathbb{D}_l^\lambda v_i = v_{i+1}\cdot dz/z$ for $i=1,\ldots,l-1$ and $\mathbb{D}_l^\lambda v_l = 0$. The residue $\mathrm{Res}(\mathbb{D}^\lambda)$ is given by N_l. We have the weight filtration W of $E_{|O}$ with respect to N_l.

We have the trivial parabolic structure F of E_l. Take a sufficiently small positive number ϵ. The ϵ-perturbation $F^{(\epsilon)}$ is given by $F^{(\epsilon)}_{k\epsilon} = W_k$ for $k = -l+1, -l+3\ldots, l-1$ in this case.

Let us fix a sufficiently small positive number ϵ_0 such that $\mathrm{rank}\,E\cdot\epsilon_0 < \eta/10$. In the previous subsection, we have constructed a family of metrics $h_2^{(\epsilon)}$ $(0\le\epsilon\le\epsilon_0)$. It induces the metric of $\mathrm{Sym}^{l-1}(E_2,\mathbb{D}_2^\lambda)\simeq(E_l,\mathbb{D}_l)$, which we denote by $h_l^{(\epsilon)}$. The following property can be shown by reducing to the case $l=2$.

- $h_l^{(0)}$ is the harmonic metric.
- $h_l^{(\epsilon)}\longrightarrow h_l^{(0)}$ for $\epsilon\to 0$, in the C^∞-sense locally on X^*.
- $\big|\Lambda_{\omega_\epsilon}G(h_l^{(\epsilon)})\big|_{h_l^{(\epsilon)}} < C$.
- $h_l^{(\epsilon)}$ is adapted to the parabolic structure $F_l^{(\epsilon)}$.
- Let $t_\epsilon := \det(h_l^{(\epsilon)})\big/\det(h_l^{(0)})$. Then t_ϵ and t_ϵ^{-1} are bounded, independently of ϵ.

Lemma 4.10 *Let $H_\epsilon = \big(h^{(\epsilon)}(v_i,v_j)\big)$. Then, we have the following estimate with respect to $h_l^{(\epsilon)}$:*

$$\overline{H}_\epsilon^{-1}\cdot\big(\overline{\partial}+\lambda\partial\big)\overline{H}_\epsilon = O(1)\cdot\frac{dz}{z} + O(1)\cdot\frac{d\overline{z}}{\overline{z}}$$

Proof We see only $\overline{H}_\epsilon^{-1}\partial\overline{H}_\epsilon$. The term $\overline{H}_\epsilon^{-1}\overline{\partial}\overline{H}_\epsilon$ can be discussed in the same way. We have only to check the case $l=2$. As in Lemma 4.9, we have only to see the $(1,1)$-entry, $(2,2)$-entry, $L_\epsilon\times$ $(1,2)$-entry and $L_\epsilon^{-1}\times$ $(2,1)$-entry in the matrix valued function (83). As is seen in the subsubsection 4.3.3, the term containing ∂M_ϵ is bounded with respect to ω_ϵ, and hence we can ignore them. Therefore we have only to show that $L_\epsilon^{-1}\partial L_\epsilon = -L_\epsilon\partial L_\epsilon^{-1}$ is $O(1)\cdot dz/z$, but it can be checked by a direct calculation. ∎

4.4.2 General case

Let $(E,\boldsymbol{F},\mathbb{D}^{\lambda})$ be a parabolic flat λ-connection on (X,O). Take a positive number η such that $10\cdot\eta<\mathrm{gap}(E,\boldsymbol{F})$. We will use the metrics:

$$\omega_{\epsilon}=\epsilon^2|z|^{\epsilon}\frac{dz\cdot d\overline{z}}{|z|^2}+|z|^{2\eta}\frac{dz\cdot d\overline{z}}{|z|^2}. \tag{100}$$

Here ϵ will be a small positive number such that $10\,\mathrm{rank}(E)\cdot\epsilon<\eta$. We take the ϵ-perturbation $\boldsymbol{F}^{(\epsilon)}$ as in the subsection 2.1.5.

We have the endomorphism $\mathrm{Res}(\mathbb{D}^{\lambda})$ of Gr^F_a. It induces the generalized eigen decomposition $\mathrm{Gr}^F_a(E)=\bigoplus_{\alpha\in\boldsymbol{C}}\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(E)$. On $\mathrm{Gr}^{F,\mathbb{E}}_u(E)$, the endomorphism $\mathrm{Res}(\mathbb{D}^{\lambda})$ is decomposed as $\alpha\cdot\mathrm{id}+N_u$, where $u=(a,\alpha)\in\boldsymbol{R}\times\boldsymbol{C}$. Let W be the weight filtration of N_u on $\mathrm{Gr}^{F,\mathbb{E}}_u(E)$. They induce the filtration W of $\mathrm{Gr}^F_a(E)$. Recall that the ϵ-perturbation is constructed from W and F.

For $u\in\boldsymbol{R}\times\boldsymbol{C}$, we put $V_u:=\mathrm{Gr}^{F,\mathbb{E}}_u(E)$ with the induced nilpotent map N_u. Then we can take an isomorphism:

$$(V_u,N_u)\simeq\bigoplus_{i=1}^{m(u)}\bigl(V_{l(u,i)},N_{l(u,i)}\bigr).$$

We put $(E_u,\mathbb{D}^{\lambda}_u):=\bigoplus\bigl(E_{l(u,i)},\mathbb{D}^{\lambda}_{l(u,i)}\bigr)$. Let $h'^{(\epsilon)}_u$ denote the metric of E_u induced by $h^{(\epsilon)}_{l(u,i)}$ (see the subsubsection 4.4.1).

Let $Q(u)$ denote harmonic bundle of rank one for $u=(a,\alpha)$, which is given by $\mathcal{O}_X\cdot e$ with the λ-connection $\mathbb{D}^{\lambda}e=e\cdot\alpha\cdot dz/z$ and the metric $h''_u(e,e)=|z|^{-2a}$. Then we obtain the vector bundle E_0 with the λ-connection $\mathbb{D}^{\lambda}_0$ and the parabolic structure F, as follows:

$$(E_0,\mathbb{D}^{\lambda}_0)=\bigoplus_u(E_u,\mathbb{D}^{\lambda}_u)\otimes Q(u),\qquad F_b(E_{0\,|\,O})=\bigoplus_{a\leq b}E_{(a,\alpha)}\otimes Q(a,\alpha)_{|O}.$$

The metrics $h'^{(\epsilon)}_u$ and h''_u induce the metric $h^{(\epsilon)}_u$ of $E_u\otimes Q(u)$. Let $h^{(\epsilon)}_0$ denote the direct sum of them. We can take a holomorphic isomorphism $\Psi:E_0\longrightarrow E$ satisfying the following conditions:

- It preserves the filtration F.
- $\mathrm{Gr}^F(\Psi)\circ\mathrm{Gr}^F\mathrm{Res}\,\mathbb{D}^{\lambda}=\mathrm{Gr}^F\mathrm{Res}\,\mathbb{D}^{\lambda}_0\,\mathrm{Gr}^F(\Psi)$.

We identify E_0 and E via Ψ. The naturally induced metric is denoted by the same notation $h^{(\epsilon)}_0$.

Lemma 4.11 *The family* $\bigl\{h^{(\epsilon)}_0\,\big|\,0\leq\epsilon\leq\epsilon_0\bigr\}$ *of the hermitian metrics has the following properties:*

- $G(\mathbb{D}^{\lambda},h^{(\epsilon)}_0)$ *is uniformly bounded with respect to* $(\omega_{\epsilon},h^{(\epsilon)}_0)$.
- $\{h^{(\epsilon)}_0\,|\,\epsilon>0\}$ *converges to* $h^{(0)}_0$ *in the* C^{∞}*-sense locally on* X^*.
- $h^{(\epsilon)}_0$ *is adapted to the perturbed parabolic structure* $F^{(\epsilon)}$.
- *Let* t_{ϵ} *be determined by* $\det(h^{(\epsilon)}_0)\big/\det(h^{(0)}_0)$. *Then* t_{ϵ} *and* t_{ϵ}^{-1} *are bounded, independently of* ϵ.

Proof We check only the first claim. The other claims are easy to see. Let f be determined by $f\cdot dz/z=\mathbb{D}^{\lambda}-\mathbb{D}^{\lambda}_0$, and we put $f^{\dagger}_{\epsilon}:=f^{\dagger}_{h^{(\epsilon)}}$. We put $\mathbb{D}^{\lambda\star}_{\epsilon}:=\mathbb{D}^{\lambda\star}_{h^{(\epsilon)}}$ and $\mathbb{D}^{\lambda\star}_{0,\epsilon}:=\mathbb{D}^{\lambda\star}_{0,h^{(\epsilon)}}$. Then we have the following:

$$G(\mathbb{D}^{\lambda},h^{(\epsilon)}_0)=\bigl[\mathbb{D}^{\lambda},\mathbb{D}^{\lambda\star}_{\epsilon}\bigr]=\Bigl[\mathbb{D}^{\lambda}_0+f\frac{dz}{z},\ \mathbb{D}^{\lambda\star}_{0,\epsilon}+f^{\dagger}_{\epsilon}\frac{d\overline{z}}{\overline{z}}\Bigr]$$
$$=G(\mathbb{D}^{\lambda}_0,h^{(\epsilon)}_0)+\mathbb{D}^{\lambda\star}_{0,\epsilon}(f)\frac{dz}{z}+\mathbb{D}^{\lambda}_0(f^{\dagger}_{\epsilon})\frac{d\overline{z}}{\overline{z}}+[f,f^{\dagger}_{\epsilon}]\frac{dz\cdot d\overline{z}}{|z|^2}. \tag{101}$$

We have the decomposition $f = \sum f_{u,u'}$, where $f_{u,u'} \in Hom\big(E_u \otimes Q(u),\ E_{u'} \otimes Q(u')\big)$. We have $f_{u,u'\,|\,O} = 0$ unless $\alpha = \alpha'$ and $a > a'$. Hence there exist positive constants C and N such that the following holds for $0 < \epsilon < \epsilon_0$:

$$|f|_{h_0^{(\epsilon)}} \leq C \cdot |z|^{2\eta} L_\epsilon^N,$$

Here $N \cdot \epsilon$ is sufficiently smaller than η. Hence, we have the following:

$$|f|_{h_0^{(\epsilon)}} \leq C \cdot |z|^{\eta}, \qquad [f, f_\epsilon^\dagger] = O\big(|z|^{2\eta}\big).$$

We have the induced frames $\boldsymbol{v}_u$ of $E_u \otimes Q(u)$. They induce the frame $\boldsymbol{v}$ of E_0. Let B and A_0 be determined by $F\boldsymbol{v} = \boldsymbol{v} \cdot B \cdot dz/z$ and $\mathbb{D}_0^\lambda \boldsymbol{v} = \boldsymbol{v} A_0 \cdot dz/z$. Then we have the following:

$$\big[\mathbb{D}_0^\lambda, F^\dagger\big]\boldsymbol{v} = \boldsymbol{v}\left(\mathcal{D}B_\epsilon^\dagger \frac{d\overline{z}}{\overline{z}} + [A_0, B_\epsilon^\dagger]\frac{dz \cdot d\overline{z}}{|z|^2}\right).$$

Here we put $\mathcal{D} = \overline{\partial} + \lambda\partial$ and $B_\epsilon^\dagger = \overline{H}_\epsilon^{-1} \cdot {}^t\overline{B} \cdot \overline{H}_\epsilon$, where $H = H(h_0^{(\epsilon)}, \boldsymbol{u})$. Since $B_\epsilon^\dagger$ is sufficiently small with respect to $(\omega_\epsilon, h_0^{(\epsilon)})$, $[A_0, B_\epsilon^\dagger]$ is also sufficiently small. Corresponding to the decomposition $f = \sum f_{u,u'}$, we have $B = \sum B_{u,u'}$. Then the following holds:

$$\big(B_\epsilon^\dagger\big)_{u,u'} = \overline{H}_{u',\epsilon}^{-1}\, {}^t\overline{B}_{u',u} \overline{H}_{u,\epsilon}.$$

Here $H_{u,\epsilon} := H(h_u^{(\epsilon)}, \boldsymbol{v}_u)$. Hence we obtain the following:

$$\mathcal{D}B_{u,u'}^\dagger \frac{d\overline{z}}{\overline{z}} = \overline{H}_{u',\epsilon}^{-1}(\mathcal{D}^t\overline{B}_{u',u})\overline{H}_{u,\epsilon} - \overline{H}_{u',\epsilon}^{-1}\mathcal{D}\overline{H}_{u',\epsilon}(B_\epsilon^\dagger)_{u,u'} + (B_\epsilon^\dagger)_{u,u'}\overline{H}_{u,\epsilon}^{-1}\mathcal{D}\overline{H}_{u,\epsilon}.$$

Since B is holomorphic, we have $\overline{H}_{u',\epsilon}^{-1} \cdot \big(\mathcal{D}^t\overline{B}_{u',u}\big) \cdot \overline{H}_{u,\epsilon} \cdot d\overline{z}/\overline{z} = 0$. We put $H'_{u\,\epsilon} := H(h_u'^{(\epsilon)}, \boldsymbol{v}_u)$. Then we have $H_{u,\epsilon} = |z|^{-2a} H'_{u,\epsilon}$, and the following holds with respect to $h_0^{(\epsilon)}$ due to Lemma 4.10:

$$\overline{H}_{u,\epsilon}^{-1}\mathcal{D}\overline{H}_{u,\epsilon} = -a\left(\lambda\frac{dz}{z} + \frac{d\overline{z}}{\overline{z}}\right) + \overline{H}_{u,\epsilon}'^{-1}\mathcal{D}\overline{H}'_{u,\epsilon} = O(1)\frac{dz}{z} + O(1)\frac{d\overline{z}}{\overline{z}}.$$

Since $(B_\epsilon^\dagger)_{u,u'}$ is small with respect to $(\omega_\epsilon, h_0^{(\epsilon)})$, $(B_\epsilon^\dagger)_{u,u'} \cdot \overline{H}_{u,\epsilon}^{-1}\partial\overline{H}_{u,\epsilon}$ is also small. Therefore, $\mathbb{D}_0^\lambda F^\dagger \cdot d\overline{z}/\overline{z}$ is small with respect to $(\omega_\epsilon, h_0^{(\epsilon)})$. It also follows that $\mathbb{D}_{0,\epsilon}^{\lambda\,\star} F \cdot dz/z$ is small. Thus we are done. ∎

4.5 Proof of Proposition 4.1

4.5.1 Construction of a family of initial metrics

Let η be a small positive number such that $\eta < \mathrm{gap}(E, \boldsymbol{F})/10$. Let ϵ_0 be a small positive number such that $\operatorname{rank} E \cdot \epsilon_0 < \eta$. For any $0 < \epsilon < \epsilon_0$, let us take ω_ϵ be the Kahler forms of $C - D$ with the following properties:

- Let (U_P, z) be a holomorphic coordinate around $P \in D$ such that $z(P) = 0$, and then ω_ϵ is given by (100).
- $\omega_\epsilon \longrightarrow \omega_0$ for $\epsilon \longrightarrow 0$ in the C^∞-sense locally on $X - D$.

Lemma 4.12 *We can construct a family of metrics $h_0^{(\epsilon)}$ of $E_{|C-D}$ with the following properties:*

- $h_0^{(\epsilon)}$ *is adapted to the perturbed parabolic structure $\boldsymbol{F}^{(\epsilon)}$.*
- $h_0^{(\epsilon)} \longrightarrow h_0^{(0)}$ *in the C^∞-sense locally on $C - D$.*
- $G(h_0^{(\epsilon)})$ *is uniformly bounded with respect to $(\omega_\epsilon, h_0^{(\epsilon)})$.*
- *We put $t_\epsilon := \det(h_0^{(\epsilon)})/\det(h_0^{(0)})$. Then t_ϵ and t_ϵ^{-1} are bounded independently of ϵ.*

Proof We construct a C^{∞}-metric of E on $\bigcup_{P\in D}(U_P-\{P\})$, by applying the construction given in the subsection 4.4.2 to $(E,\boldsymbol{F},\mathbb{D}^{\lambda})_{|U_P}$ for each $P\in D$, and then we prolong it to a C^{∞}-metric of E on $C-D$. ∎

Let $R(\det h_0^{(0)})$ denote the curvature of the metrized holomorphic bundle $\det(E,d'',h_0^{(0)})$, where d'' denote the $(0,1)$-part of $\mathbb{D}^{\lambda}$. Since $\det h_0^{(0)}$ gives the harmonic metric around D due to our construction, $R(\det h_0^{(0)})$ vanishes around D. We also have $\int R(\det h_0^{(0)})=-2\pi\sqrt{-1}\cdot\text{par-deg}(E,\boldsymbol{F})=0$. Let us take the C^{∞}-function χ_0 on C and satisfies the equality $\mathrm{rank}(E)\cdot\overline{\partial}\partial\chi_0+R\bigl(\det(h_0^{(0)})\bigr)=0$. We put $h_{in}^{(0)}:=h_0^{(0)}\cdot\exp\bigl(\chi_0\bigr)$. Then $R\bigl(\det h_{in}^{(0)}\bigr)=0$, i.e., $\det h_{in}^{(0)}$ is a harmonic metric of $\det(E,\mathbb{D}^{\lambda})$. Let χ_{ϵ} be the functions determined by $\det(h_{in}^{(0)})=\det(h_0^{(\epsilon)})\cdot\exp\Bigl(\mathrm{rank}(E)\cdot\chi_{\epsilon}\Bigr)$. The following claims immediately follows from Lemma 4.12.

- χ_{ϵ} and $-\chi_{\epsilon}$ are bounded on C, independently of ϵ.
- $\chi_{\epsilon}\longrightarrow 0$ in the C^{∞}-sense on $C-D$.

We put $h_{in}^{(\epsilon)}:=h_0^{(\epsilon)}\cdot\exp\bigl(\chi_{\epsilon}\bigr)$, which is the metric of $E_{|C-D}$.

Lemma 4.13 *The following claims are easy to check.*

- $h_{in}^{(\epsilon)}$ *is adapted to the parabolic structure* $\boldsymbol{F}^{(\epsilon)}$.
- $h_{in}^{(\epsilon)}\longrightarrow h_{in}^{(0)}$ *in the* C^{∞}*-sense locally on* $C-D$.
- $G(h_{in}^{(\epsilon)})$ *is uniformly bounded with respect to* $(\omega_{\epsilon},h_{in}^{(\epsilon)})$.
- $\det h_{in}^{(\epsilon)}$ *is harmonic, and we have* $\det h_{in}^{(\epsilon)}=\det h_{in}^{(0)}$.

In other words, they give initial metrics for $(E,\boldsymbol{F}^{(\epsilon)},\mathbb{D}^{\lambda})$ *in the sense of Lemma* 3.20*, and their pseudo curvature satisfy some uniform finiteness.* ∎

4.5.2 L_1^2-finiteness of the sequence

Due to Proposition 2.31, we obtain harmonic metrics $h^{(\epsilon)}$ for $(E,\boldsymbol{F}^{(\epsilon)},\mathbb{D}^{\lambda})$. Due to Lemma 2.32, we have the following inequalities for any ϵ:

$$M_{\omega_{\epsilon}}(h_{in}^{(\epsilon)},h^{(\epsilon)})\leq 0. \tag{102}$$

Let $s^{(\epsilon)}$ be determined by $h^{(\epsilon)}=h_{in}^{(\epsilon)}s^{(\epsilon)}$. Due to Lemma 2.42, (102) and $\det s^{(\epsilon)}=1$, there exists a positive constant A which is independent on ϵ, with the following property:

$$\bigl|s^{(\epsilon)}\bigr|_{h_{in}^{(\epsilon)}}\leq A,\quad \bigl|s^{(\epsilon)\,-1}\bigr|_{h_{in}^{(\epsilon)}}\leq A. \tag{103}$$

Let $\mathbb{D}_{in}^{\lambda\star}$ be the operator obtained from $\mathbb{D}^{\lambda}$, ω_{ϵ} and $h_{in}^{(\epsilon)}$ as in the subsection 2.2.1. We have the following equalities:

$$\Delta_{\omega_{\epsilon}}^{\lambda}\,\mathrm{tr}\,s^{(\epsilon)}=-\sqrt{-1}\,\mathrm{tr}\bigl(s^{(\epsilon)}\Lambda_{\omega_{\epsilon}}G(h_{in}^{(\epsilon)})\bigr)+\sqrt{-1}\,\mathrm{tr}\Bigl(\Lambda_{\omega_{\epsilon}}\mathbb{D}^{\lambda}s^{(\epsilon)}\cdot(s^{(\epsilon)})^{-1}\cdot\mathbb{D}_{in}^{\lambda\star}s^{(\epsilon)}\Bigr).$$

See Remark 2.20 for $\Delta_{\omega_{\epsilon}}^{\lambda}$. Since we have $\int\Delta_{\omega_{\epsilon}}^{\lambda}\,\mathrm{tr}\,s^{(\epsilon)}=0$, there exists a positive constant A' such that the following holds:

$$\int|\mathbb{D}^{\lambda}s^{(\epsilon)}\cdot s^{(\epsilon)\,-1/2}|^2_{h_{in}^{(\epsilon)},\omega_{\epsilon}}\,\mathrm{dvol}_{\omega_{\epsilon}}\leq A'. \tag{104}$$

In particular, we obtain $\bigl\|\mathbb{D}^{\lambda}s^{(\epsilon)}\bigr\|_{L^2,\omega_{\epsilon},h_{in}^{(\epsilon)}}$ is bounded for $0<\epsilon<\epsilon_0$.

4.5.3 The end of the proof of Proposition 4.1

Let Q be a point of $C-D$. Let (U,z) be a holomorphic coordinate around Q such that $z(Q)=0$ and $U\simeq\Delta=\{z\,|\,|z|<1\}$. We use the standard metric $g=dz\cdot d\overline{z}$ of U. The harmonic bundle $(E,\mathbb{D}^{\lambda},h^{(\epsilon)})$ induces the Higgs bundle $(E,\overline{\partial}_{\epsilon},\theta_{\epsilon})$. We have $\theta_{\epsilon}=f_{\epsilon}\cdot dz$ on U. On the other hand, we also obtain $\overline{\partial}_{in,\epsilon}$ and $\theta_{in,\epsilon}$ from $(E,\mathbb{D}^{\lambda},h^{(\epsilon)}_{in})$, although $\overline{\partial}_{in,\epsilon}(\theta_{in,\epsilon})=0$ is not satisfied, in general. Let $\delta'_{in,\epsilon}$ be the $(1,0)$-operator obtained from $h^{(\epsilon)}_{in}$ and d'', as in the subsubsection 2.2.1. Then we have the relation:

$$\theta_{\epsilon}=\theta_{in,\epsilon}-\frac{\lambda}{1+|\lambda|^2}\big(s^{(\epsilon)\,-1}\cdot\delta'_{in,\epsilon}s^{(\epsilon)}\big). \tag{105}$$

Due to (103), (104) and (105), there exists a positive constant C_0 such that $\int_U|f_{\epsilon}|^2\cdot\mathrm{dvol}_g<C_0$ holds for any $0<\epsilon<\epsilon_0$. Hence the following inequality holds for some positive constants C_i $(i=1,2,3)$ and for any $0<\epsilon<\epsilon_0$:

$$\int_U\log|f_{\epsilon}|^2\cdot\mathrm{dvol}_g\leq C_1+\int_U C_2\cdot|f_{\epsilon}|^2\cdot\mathrm{dvol}_g\leq C_3. \tag{106}$$

Recall the fundamental inequality for the Higgs field of a harmonic bundle ([35]):

$$\Delta_g\log|f_{\epsilon}|^2\leq-\frac{\big|[f_{\epsilon},f_{\epsilon}^{\dagger}]\big|^2}{|f_{\epsilon}|^2}\leq 0. \tag{107}$$

Due to (106) and (107), there exists a positive constant C_4 such that the following holds for any $Q'\in U(1/2):=\{|z|<1/2\}$:

$$|f_{\epsilon}(Q')|^2_{h^{(\epsilon)}_{in}}\leq C_4. \tag{108}$$

By using (105), we obtain that $\delta'_{in,\epsilon}s^{(\epsilon)}$ is uniformly bounded with respect to $(\omega_{\epsilon},h^{(\epsilon)}_{in})$ on $U(1/2)$.

Since $\theta^{\dagger}_{\epsilon}$ is the adjoint of θ_{ϵ}, we obtain the uniform boundedness of $\theta^{\dagger}_{\epsilon}$ on $U(1/2)$. Let $\delta''_{in,\epsilon}$ be the operator obtained from $h^{(\epsilon)}_{in}$ and d' as in the subsubsection 2.2.1, where d' denotes the $(1,0)$-part of $\mathbb{D}^{\lambda}$. Then we also obtain the uniform boundedness of $\delta''_{in,\epsilon}s^{(\epsilon)}$ on $U(1/2)$. Hence $\mathbb{D}^{\lambda\star}_{in,\epsilon}s^{(\epsilon)}$ is uniformly bounded on $U(1/2)$, where $\mathbb{D}^{\lambda\star}_{in,\epsilon}=\delta'_{in,\epsilon}-\delta''_{in,\epsilon}$. Since we have $d''=\overline{\lambda}^{-1}\big(\delta''_{in,\epsilon}+(1+|\lambda|^2)\theta^{\dagger}_{in,\epsilon}\big)$ and $d'=\lambda\delta'_{in,\epsilon}+(1+|\lambda|^2)\theta$, we also obtain $\mathbb{D}^{\lambda}s^{(\epsilon)}$ is uniformly bounded on $U(1/2)$. Recall the formula $\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}_{in}s^{(\epsilon)}=s^{(\epsilon)}\cdot G(h^{(\epsilon)}_{in})+\mathbb{D}^{\lambda}s^{(\epsilon)}\cdot s^{(\epsilon)\,-1}\cdot\mathbb{D}^{\lambda\star}_{in}s^{(\epsilon)}$. Thus $\mathbb{D}^{\lambda}\mathbb{D}^{\lambda\star}_{in}s^{(\epsilon)}$ is also uniformly bounded on $U(1/2)$. Therefore $\{s^{(\epsilon)}\}$ is L^p_2-bounded for any $p>1$ and $U(1/2)$. By taking an appropriate subsequence (ϵ_i), $s^{(\epsilon_i)}$ weakly converges to some $\widetilde{s}$ in L^p_2 locally on $C-D$.

It is easy to see that $h^{(0)}_{in}\cdot\widetilde{s}$ is a harmonic metric. We have $\det\widetilde{s}=1$. We also have the boundedness of $\widetilde{s}$ and $\widetilde{s}^{-1}$ with respect to $h^{(0)}_{in}$. Thus we have $h^{(0)}_{in}\cdot\widetilde{s}=h^{(0)}$, i.e., the sequence $\{h^{(\epsilon_i)}\}$ converges to $h^{(0)}$ weakly in L^p_2 locally on $C-D$.

Although we take a subsequence in the above argument, we can conclude that $h^{(\epsilon)}$ converges to $h^{(0)}$ weakly in L^p_2 locally on $C-D$, due to a general argument. We can also obtain the C^{∞}-convergence by a standard bootstrapping argument. In the above argument, the convergence of $\{\theta^{(\epsilon)}\}$ is also proved. ∎

Remark 4.14 *As for the proof of Proposition* 4.2*, we take a C^{∞}-metric h_{in} of $(E,\boldsymbol{F},\mathbb{D}^{\lambda})$ such that each restriction $h_{in\,|\,C_t}$ is an initial metric. Let s be determined by $h_H=h_{in}\cdot s$. By applying the same argument, we obtain the continuity of s. Similarly for θ_H.* ∎

5 The existence of a pluri-harmonic metric

We will prove our main existence theorem of pluri-harmonic metric for parabolic λ-flat bundle, which is adapted to the parabolic structure. (See the subsection 3.3 of [30] for the adaptedness.)

5.1 Preliminary

Let C be a smooth projective curve with a simple effective divisor D. Let $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ be a stable parabolic λ-flat bundle on (C, D) with par-deg$(E, \boldsymbol{F}) = 0$. For each $P \in D$, let (U_P, z) be a holomorphic coordinate around P such that $z(P) = 0$. Let $\boldsymbol{F}^{(\epsilon)}$ be an ϵ-perturbation. We have $h_0^{(\epsilon_i)}$ be a harmonic metric for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^\lambda)$ for some sequence $\{\epsilon_i\}$ such that $\epsilon_i \longrightarrow 0$. For simplicity of the description, we use ϵ instead of ϵ_i. We assume $\det h_0^{(\epsilon)} = \det h_0^{(0)}$. Let N be a large positive number, for example $N > 10$. In this subsection, we use Kahler metrics g_ϵ ($\epsilon \geq 0$) of $C - D$ which are as follows on U_P for each $P \in D$:

$$\big(\epsilon^{N+2}|z|^{2\epsilon} + |z|^2\big)\frac{dz \cdot d\overline{z}}{|z|^2}.$$

We assume that $\{g_\epsilon\}$ converges to g_0 for $\epsilon \longrightarrow 0$ in the C^∞-sense locally on $C - D$.

Proposition 5.1 *Let $h^{(\epsilon)}$ ($\epsilon > 0$) be hermitian metrics of $E_{|C-D}$ with the following properties:*

1. *Let $s^{(\epsilon)}$ be determined by $h^{(\epsilon)} = h_0^{(\epsilon)} \cdot s^{(\epsilon)}$. Then $s^{(\epsilon)}$ is bounded with respect to $h_0^{(\epsilon)}$, and we have $\det s^{(\epsilon)} = 1$. We also have the finiteness $\big\|\mathbb{D}^\lambda s^{(\epsilon)}\big\|_{2,h_0^{(\epsilon)},g_\epsilon} < \infty$. (The estimates may depend on ϵ.)*
2. *We have $\|G(h^{(\epsilon)})\|_{2,h^{(\epsilon)},g_\epsilon} < \infty$ and $\lim_{\epsilon\to 0} \|G(h^{(\epsilon)})\|_{2,h^{(\epsilon)},g_\epsilon} = 0$.*

Then the following claims hold.

- *The sequence $\{s^{(\epsilon)}\}$ is weakly convergent to the identity in L_1^2 locally on $C - D$.*
- *$\big\{\sup_{P\in C-D} |s^{(\epsilon)}_{|P}|_{h_0^{(\epsilon)}} \,\big|\, \epsilon > 0\big\}$ and $\big\{\sup_{P\in C-D} |s^{(\epsilon)\,-1}_{|P}|_{h_0^{(\epsilon)}} \,|\, \epsilon > 0\big\}$ are bounded.*

Proof To begin with, we remark that we have only to show the existence of a subsequence $\{s^{(\epsilon_i)}\}$ with the desired properties as above. We put $\|s^{(\epsilon)}\|_{\infty,h_0^{(\epsilon)}} := \sup_{P\in C-D}\big|s^{(\epsilon)}_{|P}\big|_{h_0^{(\epsilon)}}$. For any point $P \in C - D$, let $SE(s^{(\epsilon)})(P)$ denote the maximal eigenvalue of $s^{(\epsilon)}_{|P}$. There exists a constant $0 < C_1 < 1$ such that $C_1 \cdot |s^{(\epsilon)}_{|P}|_{h_0^{(\epsilon)}} \leq SE(s^{(\epsilon)})(P) \leq |s^{(\epsilon)}_{|P}|_{h_0^{(\epsilon)}}$. We have $\det s^{(\epsilon)}_{|P} = 1$. Hence it is easy to see $\log \operatorname{tr} s^{(\epsilon)}_{|P} \geq \log r \geq 0$. We also have $SE(s^{(\epsilon)})(P) \geq 1$ for any P.

Let us take $b_\epsilon > 0$ satisfying $2 \leq b_\epsilon \cdot \sup SE(s^{(\epsilon)})(P) \leq 2 + \epsilon$. We put $\widetilde{s}^{(\epsilon)} = b_\epsilon s^{(\epsilon)}$ and $\widetilde{h}^{(\epsilon)} := h_0^{(\epsilon)} \cdot \widetilde{s}^{(\epsilon)}$. Then $\widetilde{s}^{(\epsilon)}$ are uniformly bounded with respect to $h_0^{(\epsilon)}$. We remark $G(\widetilde{h}^{(\epsilon)}) = G(h^{(\epsilon)})$. We also remark that $h^{(\epsilon)}$ and $\widetilde{h}^{(\epsilon)}$ induce the same metric of $\operatorname{End}(E)$.

Lemma 5.2 *After going to an appropriate subsequence, $\big\{\widetilde{s}^{(\epsilon_i)}\big\}$ converges to a positive constant multiplication, weakly in L_1^2 locally on $C - D$.*

Proof We have the following (the subsection 2.2.5):

$$\Delta^\lambda_{g_0,h_0^{(\epsilon)}} \widetilde{s}^{(\epsilon)} = \widetilde{s}^{(\epsilon)}\sqrt{-1}\Lambda_{g_0} G(\widetilde{h}^{(\epsilon)}) + \sqrt{-1}\Lambda_{g_0}\mathbb{D}^\lambda \widetilde{s}^{(\epsilon)}\big(\widetilde{s}^{(\epsilon)\,-1}\big)\mathbb{D}^{\lambda\star}_{h_0^{(\epsilon)}}\widetilde{s}^{(\epsilon)}. \qquad (109)$$

Since we have $\int \Delta^\lambda_{g_0} \operatorname{tr} \widetilde{s} \cdot \mathrm{dvol}_{g_0} = 0$, we obtain the following inequality from (109) and the uniform boundedness of $\widetilde{s}^{(\epsilon)}$:

$$\int \big|\mathbb{D}^\lambda \widetilde{s}^{(\epsilon)} \cdot \widetilde{s}^{(\epsilon)\,-1/2}\big|^2_{g_0,h_0^{(\epsilon)}} \mathrm{dvol}_{g_0} \leq A \cdot \int \big|\operatorname{tr} \Lambda_{g_0} G(\widetilde{h}^{(\epsilon)})\big| \cdot \mathrm{dvol}_{g_0}$$
$$= A \cdot \int \big|\operatorname{tr} \Lambda_{g_\epsilon} G(\widetilde{h}^{(\epsilon)})\big| \cdot \mathrm{dvol}_{g_\epsilon} \leq A' \cdot \big\|G(\widetilde{h}^{(\epsilon)})\big\|_{2,\widetilde{h}^{(\epsilon)},g_\epsilon}. \qquad (110)$$

In particular, we obtain the uniform finiteness $\big\|\mathbb{D}^\lambda \widetilde{s}^{(\epsilon)}\big\|_{2,g_0,h_0^{(\epsilon)}} \leq A'' \cdot \big\|G(\widetilde{h}^{(\epsilon)})\big\|_{2,\widetilde{h}^{(\epsilon)},g_\epsilon}$. Therefore the sequence $\big\{\widetilde{s}^{(\epsilon)}\big\}$ is L_1^2-bounded on any compact subset of $C - D$. By taking an appropriate subsequence, it is weakly

L_1^2-convergent locally on $C-D$. Let $\widetilde{s}^{(\infty)}$ denote the weak limit. We obtain $\mathbb{D}^\lambda \widetilde{s}^{(\infty)}=0$. We also know that $\widetilde{s}^{(\infty)}$ is bounded with respect to $h_0^{(0)}$. Therefore $\widetilde{s}^{(\infty)}$ gives an automorphism of $(E,\boldsymbol{F},\mathbb{D}^\lambda)$. Due to the stability of $(E,\boldsymbol{F},\mathbb{D}^\lambda)$, $\widetilde{s}^{(\infty)}$ is a constant multiplication.

We would like to show $\widetilde{s}^{(\infty)}\neq 0$. Let us take any point $Q_\epsilon\in C-D$ satisfying the following:

$$SE(s^{(\epsilon)})(Q_\epsilon)\geq \frac{9}{10}\cdot \sup_{P\in C-D} SE(s^{(\epsilon)})(P).$$

Then we have $\log\operatorname{tr}\widetilde{s}^{(\epsilon)}(Q_\epsilon)\geq \log(9/5)$. By taking an appropriate subsequence, we may assume the sequence $\{Q_\epsilon\}$ converges to a point Q_∞. We have two cases (i) $Q_\infty\in D$ (ii) $Q_\infty\not\in D$. We discuss only the case (i). The other case is similar and easier.

We use the coordinate neighbourhood (U,z) such that $z(Q_\infty)=0$. For any point $P\in U$, we put $\Delta(P,r):=\{Q\in U\,|\,|z(P)-z(Q)|<r\}$. When ϵ is sufficiently small, Q_ϵ is contained in $\Delta(Q_\infty,1/2)=\{|z|<1/2\}$. Let $g=dz\cdot d\overline{z}$ denote the standard metric of U. We have the following inequality on U (see the subsubsection 2.2.5):

$$\Delta_g^\lambda \log\operatorname{tr}\widetilde{s}^{(\epsilon)}\leq \big|\Lambda_g G(\widetilde{h}^{(\epsilon)})\big|_{\widetilde{h}^{(\epsilon)}}.$$

Let $B^{(\epsilon)}$ be the endomorphism of E determined as follows:

$$G(\widetilde{h}^{(\epsilon)})=G(h^{(\epsilon)})=B^{(\epsilon)}\cdot\frac{dz\cdot d\overline{z}}{|z|^2},$$

Then we have the following estimate, which is independent of ϵ:

$$\int\big|B^{(\epsilon)}\big|^2_{\widetilde{h}_0^{(\epsilon)}}\big(\epsilon^{N+1}|z|^{2\epsilon}+|z|^2\big)^{-1}\frac{\mathrm{dvol}_g}{|z|^2}\leq A\int\big|G(\widetilde{h}^{(\epsilon)})\big|^2_{\widetilde{h}^{(\epsilon)},g_\epsilon}\cdot\mathrm{dvol}_{g_\epsilon}.$$

Here A denotes a constant independent of ϵ. Due to Proposition 2.5 in our previous paper [30], there exist $v^{(\epsilon)}$ such that the following inequalities hold for some positive constant A' which is independent of ϵ:

$$\overline{\partial}\partial v^{(\epsilon)}=\big|B^{(\epsilon)}\big|_{\widetilde{h}^{(\epsilon)}}\frac{dz\cdot d\overline{z}}{|z|^2},\qquad \big|v^{(\epsilon)}(z)\big|\leq A'\cdot\big(\epsilon^{(N-1)/2}|z|^\epsilon+|z|^{1/2}\big)\cdot\big\|G(\widetilde{h}^{(\epsilon)})\big\|_{2,\widetilde{h}^{(\epsilon)},g_\epsilon}$$

Then we have $\Delta_g^\lambda\big(\log\operatorname{tr}\widetilde{s}^{(\epsilon)}-v^{(\epsilon)}\big)\leq 0$. Therefore, we obtain the following:

$$\log\operatorname{tr}\widetilde{s}^{(\epsilon)}(Q_\epsilon)-v^{(\epsilon)}(Q_\epsilon)\leq A''\cdot\int_{\Delta(Q_\epsilon,1/2)}\Big(\log\operatorname{tr}\widetilde{s}^{(\epsilon)}-v^{(\epsilon)}\Big)\cdot\mathrm{dvol}_g.$$

Here A'' denotes a positive constant which is independent of ϵ. Then we obtain the following inequalities, for some positive constants C_i $(i=1,2)$ which are independent of ϵ:

$$\log(9/5)\leq\log\operatorname{tr}\widetilde{s}^{(\epsilon)}(Q_\epsilon)\leq C_1\cdot\int_{\Delta(Q_\epsilon,1/2)}\log\operatorname{tr}\widetilde{s}^{(\epsilon)}\cdot\mathrm{dvol}_g+C_2.$$

Recall that $\log\operatorname{tr}\widetilde{s}^{(\epsilon)}$ are uniformly bounded from above. Therefore there exists a positive constant C_3 such that the following holds for any sufficiently small $\epsilon>0$:

$$\int_{\Delta(Q_\epsilon,1/2)}-\min(0,\log\operatorname{tr}\widetilde{s}^{(\epsilon)})\cdot\mathrm{dvol}_g\leq C_3.$$

Due to Fatou's lemma, we obtain the following:

$$\int_{\Delta(Q_\infty,1/2)}-\min\big(0,\log\operatorname{tr}\widetilde{s}^{(\infty)}\big)\cdot\mathrm{dvol}_g\leq C_3.$$

It means $\widetilde{s}^{(\infty)}$ is not constantly 0 on $\Delta(Q_\infty,1/2)$. In all, we can conclude that $\widetilde{s}^{(\infty)}$ is a positive constant multiplication. Thus the proof of Lemma 5.2 is accomplished. ∎

Let $\left\{\widetilde{s}^{(\epsilon_i)}\right\}$ be a subsequence as in Lemma 5.2. It is almost everywhere convergent to some constant multiplication. Then we obtain that the sequence $\left\{\det \widetilde{s}^{(\epsilon_i)} = b_{\epsilon_i}^{\operatorname{rank} E} \cdot \mathrm{id}_{\det(E)}\right\}$ converges to the positive constant. In particular, $\{b_{\epsilon_i}\}$ is convergent. Therefore, the sequence $\left\{s^{(\epsilon_i)}\right\}$ is convergent to the identity. Thus we are done. ∎

Corollary 5.3

- *The sequence $\left\{h^{(\epsilon)}\right\}$ is convergent to $h_0^{(0)}$ weakly in L_1^2 locally on $C - D$.*
- *The sequence $\left\{\mathbb{D}^{\lambda} s^{(\epsilon)}\right\}$ is weakly convergent to 0 in L^2 locally on $C - D$.*
- *The sequence $\{\theta^{(\epsilon)}\}$ converges to $\theta^{(0)}$ is weakly convergent to 0 in L^2 locally on $C - D$.*
- *In particular, the sequences are convergent almost everywhere.* ∎

5.2 The Surface Case

5.2.1 Statement

Let X be a smooth projective surface with an ample line bundle L, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. We put $X^* := X - D$. Let $\boldsymbol{c}$ be any element of $\boldsymbol{R}^S$. Let $(E, \boldsymbol{F}, \mathbb{D}^{\lambda})$ be a μ_L-stable $\boldsymbol{c}$-parabolic flat λ-connection on (X, D) with trivial characteristic numbers $\text{par-deg}_L(E, \boldsymbol{F}) = \int_X \text{par-ch}_2(E, \boldsymbol{F}) = 0$. Recall that we have already known $\text{par-c}_1(E, \boldsymbol{F}) = 0$ due to Bogomolov-Gieseker inequality and Hodge index theorem (See Corollary 6.2 of [30].) Hence we can take the pluri-harmonic metric $h_{\det(E)}$ of the determinant bundle $\det(E, \boldsymbol{F}, \mathbb{D}^{\lambda})$. The purpose of this subsection is to show the following existence theorem.

Theorem 5.4 *There exists a tame pluri-harmonic metric h of $(E, \mathbb{D}^{\lambda})_{|X^*}$ with $\det(h) = h_{\det E}$ which is adapted to the parabolic structure.*

The proof will be given in the rest of this subsection.

5.2.2 The sequence of Hermitian-Einstein metrics for the ϵ-perturbations

Let $\boldsymbol{F}^{(\epsilon)}$ be an ϵ-perturbation as in the subsection 2.1.5. If ϵ is sufficiently small, $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^{\lambda})$ is also μ_L-stable. We also have $\text{par-c}_1(E, \boldsymbol{F}^{(\epsilon)}) = \text{par-c}_1(E, \boldsymbol{F}) = 0$. Since $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^{\lambda})$ is graded semisimple, we can apply Proposition 3.21. Let $h^{(\epsilon)}$ be the Hermitian-Einstein metric for $(E, \boldsymbol{F}^{(\epsilon)}, \mathbb{D}^{\lambda})$ with respect to ω_{ϵ}, such that $\det h^{(\epsilon)} = h_{\det(E)}$ and $\Lambda_{\omega_\epsilon} G(h^{(\epsilon)}) = 0$ (Proposition 3.21).

Remark 5.5 *Since $\operatorname{gap}(E, \boldsymbol{F}^{(\epsilon)}) \sim \epsilon$, we have to take a smaller number ϵ_1, for example $\epsilon/100$, and use ω_{ϵ_1}. However, we use the notation ω_ϵ for simplicity.* ∎

Since $h_{\det(E)}$ is pluri-harmonic, we also have $\operatorname{tr} G(h^{(\epsilon)}) = 0$. Therefore, we have the following convergence:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int \left|G(h^{(\epsilon)})\right|^2_{h^{(\epsilon)}, \omega_\epsilon} \cdot \mathrm{dvol}_{\omega_\epsilon} = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int \operatorname{tr}\Big(G(h^{(\epsilon)})^2\Big) = 2\big(1 + |\lambda|^2\big)^2 \cdot \text{par-ch}_2(E, \boldsymbol{F}^{(\epsilon)}) \longrightarrow 0. \quad (111)$$

We would like to discuss the limit of $h^{(\epsilon)}$ for $\epsilon \to 0$.

5.2.3 Convergence on almost every curve

Let L^m be sufficiently ample. We put $\mathbb{P}_m := \mathbb{P}\big(H^0(X, L^m)^{\vee}\big)$. For any $s \in \mathbb{P}_m$, we put $X_s := s^{-1}(0)$. Recall Proposition 2.8, and let $\mathcal{U}$ denote the Zariski open subset of $\mathbb{P}_m$ which consists of the points s with the following properties:

- X_s is smooth, and $X_s \cap D$ is a simple normal crossing divisor.

- $(E, \boldsymbol{F}, \mathbb{D}^\lambda)_{|X_s}$ is μ_L-stable.

We will use the notation $X_s^* := X_s \setminus D$ and $D_s := X_s \cap D$. We have the metric $\omega_{\epsilon,s}$ of X_s^*, induced by ω_ϵ. The induced volume form is denoted by dvol_s. We put $(E_s, \boldsymbol{F}_s, \mathbb{D}^\lambda_s) := (E, \boldsymbol{F}, \mathbb{D}^\lambda)_{|X_s}$. We have the metric $h^{(\epsilon)}_{|X_s^*}$ of $E_{s\,|\,X_s^*}$. Since $(E_s, \boldsymbol{F}^{(\epsilon)}_s, \mathbb{D}^\lambda_s)$ is stable for any point $s \in \mathcal{U}$, we have the harmonic metric $h_s^{(\epsilon)}$ of $(E_s, \boldsymbol{F}^{(\epsilon)}_s, \mathbb{D}^\lambda_s)$ with $\det h_s^{(\epsilon)} = h_{\det E\,|\,X_s^*}$. Let $u_s^{(\epsilon)}$ be the endomorphism of $E_{|X_s^*}$ determined by $h^{(\epsilon)}_{|X_s^*} = h_s^{(\epsilon)} \cdot u_s^{(\epsilon)}$. For a point $x \in X^*$, we put $\mathcal{U}_x := \{s \in \mathcal{U} \,|\, x \in X_s\}$. We put $Z := \{x \in X^* \,\big|\, \mathcal{U}_x = \emptyset\}$. We remark that Z is a finite set. Let us fix a sequence $\epsilon_i \longrightarrow 0$. We often use the notation "ϵ" instead of "ϵ_i", for simplicity of the description.

Lemma 5.6 *For almost every $s \in \mathcal{U}$, the following holds:*

- *We have the following convergence when $\epsilon_i \longrightarrow 0$:*

$$\int_{X_s} \big|G(h^{(\epsilon)}_{|X_s})\big|^2_{h_s^{(\epsilon)},\omega_\epsilon} \mathrm{dvol}_s \longrightarrow 0. \tag{112}$$

- *For each ϵ_i, we have the finiteness:*

$$\big\|\mathbb{D}^\lambda_{X_s^*} u_s^{(\epsilon)}\big\|_{L^2, h_s^{(\epsilon)},\omega_\epsilon} < \infty. \tag{113}$$

Let $\widetilde{\mathcal{U}}$ denote the set of s for which both of (112) *and* (113) *hold.*

Proof We discuss only the condition (112). The other one can be discussed similarly by using the fourth property in Proposition 3.21. Let us fix $s_1 \in \mathcal{U}$. We take generic $s_i \in \mathcal{U}$ $(i = 2, 3)$, i.e., X_{s_1} is transversal with X_{s_i} $(i = 2, 3)$ and $X_{s_1} \cap X_{s_2} \cap X_{s_3} = \emptyset$. We also fix the lifts of s_i to $H^0(X, L^m)$, and denote them by the same notation. Take open subsets $W_i^{(j)}$ $(j = 1, 2)$ such that $X_{s_1} \cap X_{s_i} \subset W_i^{(1)} \subset W_i^{(2)}$. Moreover, we assume that the closure of $W_i^{(1)}$ in X are contained in $W_i^{(2)}$. Take an open neighbourhood U_1 of s_1, which is relatively compact in $\mathcal{U}$, with the following property:

- For any $s' \in U_1$, $X_{s'}$ is transversal with X_{s_i} $(i = 2, 3)$ and $X_{s'} \cap X_{s_i} \subset W_i^{(1)}$.

We also take an embedding $U_1 \longrightarrow H^0(X, L^m)$ which is a lift of $U_i \subset \mathcal{U}$, compatible with the lift of s_1.

We have the line bundle $q_1^* L^m \otimes q_3^* \mathcal{O}_{\mathbb{P}^1}(1)$ on $X \times U_1 \times \mathbb{P}^1$, where q_i denote the projections onto the i-th components. We have the section Ψ of $q_1^* L^m$ given by $\Psi(x, s', p) = s'(x)$. The section Φ of $q_1^* L^m \otimes q_3^* \mathcal{O}_{\mathbb{P}^1}(1)$ is given by $\Phi = q_3^* t_0 \cdot q_1^* s_2 + q_3^*(t_\infty - t_0) \cdot q_1^* \Psi$, where $[t_0 : t_\infty]$ is a homogeneous coordinate of $\mathbb{P}^1$. Then $\mathcal{Z}_2$ denote the zero set of Φ. In other words, we put $\mathcal{Z}_2 := \big\{(x, s', t) \in X \times U_1 \times \mathbb{P}^1 \,\big|\, (t s_2 + (1-t)s')(x) = 0\big\}$. The fiber over $s' \in U_1$ via $q_{2\,|\,\mathcal{Z}_2}$ is the Lefschetz pencil of s' and s_2.

We fix any Kahler forms ω_{U_1} and $\omega_{\mathbb{P}^1}$ of U_1 and $\mathbb{P}^1$. The induced volume forms are denoted by dvol_{U_1} and $\mathrm{dvol}_{\mathbb{P}^1}$. Then we have the following convergence for $\epsilon \to 0$:

$$\int_{\mathcal{Z}_2} q_1^*\Big(\big|G(h^{(\epsilon)})\big|^2 \cdot \mathrm{dvol}_X\Big) \cdot \mathrm{dvol}_{U_1} \longrightarrow 0.$$

We put $\mathcal{Z}_2' := \mathcal{Z}_2 \setminus q_1^{-1}(W_2^{(2)})$. Then the following convergence is obtained, in particular:

$$\int_{\mathcal{Z}_2'} q_1^*\Big(\big|G(h^{(\epsilon)})\big|^2 \cdot \mathrm{dvol}_X\Big) \cdot \mathrm{dvol}_{U_1} \longrightarrow 0. \tag{114}$$

Let $\psi : \mathcal{Z}_2 \longrightarrow U_1 \times \mathbb{P}^1$ denote the projection. For $(s', t) \in U_1 \times \boldsymbol{C}$, we put $X_{(s',t)} := \psi^{-1}(s', t) = \big(t s_2 + (1-t)s'\big)^{-1}(0) = X_{t s_2 + (1-t)s'}$. On $X_{(s',t)}$, we have the induced volume forms $\mathrm{dvol}_{(s',t)}$. The family $\{\mathrm{dvol}_{(s',t)} \,|\, (s', t) \in U_1 \times \boldsymbol{C}\}$ gives the C^∞-relative volume form $\mathrm{dvol}_{\mathcal{Z}_2'/U_1 \times \mathbb{P}^1}$ of $\mathcal{Z}_2' \longrightarrow U_1 \times \mathbb{P}^1$. There exists a constant A such that the following holds on $U_1 \times \mathbb{P}^1$, under the isomorphism:

$$A \cdot q_1^* \big|G(h^{(\epsilon)})\big|^2 \cdot \mathrm{dvol}_X \geq \big|G(h^{(\epsilon)}_{|X_{(s',t)}})\big|^2 \cdot \mathrm{dvol}_{\mathcal{Z}_2'/U_1 \times \mathbb{P}^1} \cdot \mathrm{dvol}_{\mathbb{P}^1}.$$

Therefore, we obtain the following convergence for almost every $(s',t) \in U_1 \times \mathbb{P}^1$, from (114):

$$\int_{X^*_{(s',t)} \setminus W_2^{(2)}} \big|G(h^{(\epsilon)}_{X_{(s',t)}})\big|^2 \cdot \mathrm{dvol}_{(s',t)} \longrightarrow 0. \tag{115}$$

Let $\mathcal{S}$ denote the set of the points $(s',t) \in U_1 \times \mathbb{P}^1$ such that the above convergence (115) does not hold. The measure of $\mathcal{S}$ is 0 with respect to $\mathrm{dvol}_{U_1} \times \mathrm{dvol}_{\mathbb{P}^1}$.

Let $\mathcal{J} : U_1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}_m$ denote the map given by $(s',t) \longmapsto ts_2 + (1-t)s'$. We have the open subset $\mathcal{J}^{-1}(U_1) \subset U_1 \times \mathbb{P}^1$ and the measure of $\mathcal{S} \cap \mathcal{J}^{-1}(U_1)$ is 0 with respect to $\mathrm{dvol}_{U_1} \cdot \mathrm{dvol}_{\mathbb{P}^1}$. We have $\mathcal{S} \cap \mathcal{J}^{-1}(U_1) = \mathcal{J}^{-1}\big(\mathcal{J}(\mathcal{S}) \cap U_1\big)$, and hence the measure of $\pi(\mathcal{S})$ is 0 with respect to ω_{U_1}. Namely, we have the following convergence for almost every $s \in U_1$:

$$\int_{X_s^* \setminus W_2^{(2)}} \big|G(h^{(\epsilon)}_{|X_s^*})\big|^2 \cdot \mathrm{dvol}_s \longrightarrow 0.$$

Similarly, we can show the following convergence for almost every $s \in U_1$:

$$\int_{X_s^* \setminus W_3^{(2)}} \big|G(h^{(\epsilon)}_{|X_s^*})\big|^2 \cdot \mathrm{dvol}_s \longrightarrow 0$$

Now the claim of the lemma immediately follows. ∎

We obtain the following claims from Proposition 5.1 and Corollary 5.3.

Corollary 5.7 *For any $s \in \widetilde{\mathcal{U}}$, the sequence $\{h^{(\epsilon)}_{|X_s^*}\}$ converges to $h_s^{(0)}$ weakly in L_1^2 locally on X_s^*, and $\{\theta^{(\epsilon)}_{|X_s^*}\}$ converges to $\theta_s^{(0)}$ weakly in L^2 locally on X_s^*. In particular, they are almost everywhere convergent.*

Proof It follows from Lemma 5.6 and Proposition 5.1 ∎

5.2.4 The construction of a metric defined almost everywhere

Let us take any Kahler form $\omega_{\mathbb{P}_m}$ of $\mathbb{P}_m$. Then we obtain the induced metric of $X \times \mathbb{P}_m$. We put $\mathcal{Z} := \{(s,x) \in \mathcal{U} \times X^* \,|\, x \in X_s\}$. Then we have the induced metric of $\mathcal{Z}$. The induced volume form is denoted by $\mathrm{dvol}_{\mathcal{Z}}$. Let $\mathcal{T}$ denote the set of $(s,x) \in \widetilde{\mathcal{U}} \times X$ such that $(s,x) \in \mathcal{Z}$ and $\lim_{\epsilon \to 0} h^{(\epsilon)}_{|x} = h^{(0)}_{s|x}$.

Lemma 5.8 *The measure of $\mathcal{T}^c := \mathcal{Z} - \mathcal{T}$ is 0 with respect to* $\mathrm{dvol}_{\mathcal{Z}}$.

Proof Let us consider the naturally defined fibration $\mathcal{Z} \longrightarrow \mathcal{U}$. Then the claim follows from Corollary 5.7 and Fubini's theorem. ∎

Lemma 5.9 *For almost every $x \in X^*$ and almost every $s \in \mathcal{U}_x$, the sequence $\{h^{(\epsilon)}_{|x}\}$ converges to $h^{(0)}_{s\,|\,x}$.*

Proof Let us consider the naturally defined fibration $\mathcal{T} \longrightarrow X^*$. Then the claim follows from Lemma 5.8 and Fubini's theorem. ∎

Let $\mathcal{V}$ denote the set of $x \in X^*$ such that the sequence $\{h^{(\epsilon)}_{|x}\}$ converges to $h^{(0)}_{s\,|\,x}$ for almost $s \in \mathcal{U}_x$. For any $x \in \mathcal{V}$, let $\widetilde{\mathcal{U}}_x$ denote the set of s such that $\{h^{(\epsilon)}_{|x}\}$ converges to $h^{(0)}_{s\,|\,x}$.

Lemma 5.10 *For any $x \in \mathcal{V}$ and for any $s_i \in \widetilde{\mathcal{U}}_x$ $(i = 1,2)$, we have $h^{(0)}_{s_1\,|\,x} = h^{(0)}_{s_2\,|\,x}$.*

Proof Both of them are same as the limit $\lim_{\epsilon \to 0} h^{(\epsilon)}_x$. ∎

Let us take any $x \in \mathcal{V}$ and any $s \in \widetilde{\mathcal{U}}_x$. Then the metric h_x of $E_{|x}$ is given by $h_x := h_{s\,|\,x}$. Due to Lemma 5.10, it is well defined. Thus we obtain the metric $h_{\mathcal{V}} := (h_x \,|\, x \in \mathcal{V})$ of $E_{|\mathcal{V}}$.

5.2.5 The C^1-property

We would like to show that $h_{\mathcal{V}}$ is C^1 on X^*-Z in other words, we would like to show the existence of a C^1-metric h of $E_{|X^*-Z}$ such that $h=h_{\mathcal{V}}$ on $\mathcal{V}$. Let us begin with a preparation.

Lemma 5.11 *Let $x\in X^*-Z$. Let us take any $s\in\mathcal{U}_x$. Then there exists a Lefschetz fibration $\varphi:\widetilde{X}\longrightarrow\mathbb{P}^1$ with the following properties:*

- *x is not a singular point of φ.*
- *$\varphi^{-1}(0)=X_s$.*
- *Almost every $t\in\mathbb{P}^1$ belongs to $\widetilde{\mathcal{U}}$.*

Proof Let $\mathcal{M}$ denote the set of the lines ℓ of $\mathbb{P}_m$ which contain s. We put as follows:

$$\widehat{\mathbb{P}}_m=\big\{(\ell,s')\in\mathcal{M}\times\mathbb{P}_m\,\big|\,s'\in\ell\big\}\subset\mathcal{M}\times\mathbb{P}_m.$$

It is the blow up of $\mathbb{P}_m$ at s. We have the projection $\pi_2:\widehat{\mathbb{P}}_m\longrightarrow\mathbb{P}_m$. We put $\widehat{\mathcal{U}}:=\pi_2^{-1}(\mathcal{U})$ and $\widehat{\widetilde{\mathcal{U}}}:=\pi_2^{-1}(\widetilde{\mathcal{U}})$. Since $\mathcal{U}-\widetilde{\mathcal{U}}$ has measure 0, the measure of $\widehat{\mathbb{P}}_m-\widehat{\widetilde{\mathcal{U}}}$ is also 0. Let us consider the projection $\pi_1:\widehat{\mathbb{P}}_m\longrightarrow\mathcal{M}$, and apply Fubini's theorem. Then we obtain $s_1\in\widehat{\widetilde{\mathcal{U}}}$ for almost every $\ell\in\mathcal{M}$ and for almost every $s_1\in\ell$. Thus we are done. ∎

Let x be any point of X^*-Z. Let us take a Lefschetz fibration $\pi_i:\widetilde{X}_i\longrightarrow\mathbb{P}^1$ $(i=1,2)$ with the following properties:

- Both of them satisfy the properties in Lemma 5.11.
- Around x, the fibers of π_1 and π_2 are transversal. Then two fibrations give the holomorphic coordinate (z_1,z_2) of an appropriate neighbourhood U_x of x, such that $\{z_i=a\}=\pi_i^{-1}(a)\cap U_x$.

For any $t_i\in\mathbb{P}^1$, let $X_{t_i}:=\pi_i^{-1}(t_i)$. If t_i are close to 0, $(E,\boldsymbol{F},\mathbb{D}^\lambda)_{|X_{t_i}}$ are stable, and hence there exist tame harmonic bundles h_{t_i} for $(E,\boldsymbol{F},\mathbb{D}^\lambda)_{|X_{t_i}}$ such that $\det(h_{t_i})=h_{\det(E)|X_{t_i}}$. Let θ_{t_i} denote the operator obtained from $\mathbb{D}^\lambda_{|X_{t_i}}$ and h_{t_i} as in the subsection 2.2.1.

Let us take an appropriate neighbourhoods $B_i\subset\mathbb{P}^1$ of 0. Recall Proposition 4.2. Then $\big\{h_{t_1}\,\big|\,t_1\in B_1\big\}$ are C^∞-along z_2, and it is continuous with respect to (z_1,z_2). The family $\big\{\theta_{t_1}\,\big|\,t_1\in B_1\big\}$ has a similar property. Thus we obtain a continuous metric $h^{(1)}$ and the continuous section $\theta^{(1)}$ of $\mathrm{End}(E)\otimes\Omega^{1,0}$ around x. Similarly $\big\{h_{t_2}\,\big|\,t_2\in B_2\big\}$ is C^∞ along z_1 and it is continuous with respect to (z_1,z_2). The family $\big\{\theta_{t_2}\,\big|\,t_2\in B_2\big\}$ has a similar property. Thus we obtain a continuous metric $h^{(2)}$ and the continuous section $\theta^{(2)}$ of $\mathrm{End}(E)\otimes\Omega^{1,0}$ around x.

We remark that $h^{(1)}=h_{\mathcal{V}}=h^{(2)}$ on $U_x\cap\mathcal{V}$ due to our construction of $h_{\mathcal{V}}$. Since $h^{(i)}$ are continuous, we obtain $h^{(1)}=h^{(2)}$ on U_x. Then we obtain that $h^{(i)}$ are C^1 on U_x, due to the continuity of $\theta^{(i)}$.

Therefore we obtain the C^1-metric h of E on X^*-Z with the following properties:

- $h_{|\mathcal{V}}=h_{\mathcal{V}}$
- For any $s\in\mathcal{U}$, we have $h_{|X_s^*}=h_s$ and $\theta_{h\,|\,X_s^*}=\theta_{h_s}$.

5.2.6 Pluri-harmonicity

We would like to show that h is pluri-harmonic. By the formalism explained in the subsection 2.2.1, the operators $\overline{\partial}_h$ and θ_h are given on $X-(D\cup Z)$ from h and $\mathbb{D}^\lambda$. Let us take any C^∞ metric h' of E on $X-D$, and let s' be the endomorphism determined by $h=h'\cdot s'$. Then s' is C^1, and we have the following relation:

$$\overline{\partial}_h=\overline{\partial}_{h'}+\frac{\lambda}{1+|\lambda|^2}s'^{-1}\delta''_{h'}s',\quad \theta_h=\theta_{h'}-\frac{\lambda}{1+|\lambda|^2}s'^{-1}\delta'_{h'}s'.$$

Then we obtain $\overline{\partial}_h\theta_h$ as a distribution:

$$\overline{\partial}_h\theta_h=\overline{\partial}_{h'}\theta_{h'}-\frac{\lambda}{1+|\lambda|^2}\overline{\partial}_{h'}\big(s'^{\,-1}\delta'_{h'}s'\big)+\frac{\lambda}{1+|\lambda|^2}\big[s'^{\,-1}\delta''_{h'}s',\,\theta_{h'}\big]-\left(\frac{\lambda^2}{1+|\lambda|^2}\right)^2\big[s'^{\,-1}\delta''_{h'}s',\;s'^{\,-1}\delta'_{h}s'\big].$$

Similarly, we obtain $G(h)$ as a distribution.

Lemma 5.12 $\overline{\partial}_h\theta_h=0$.

Proof For any point $x\in X^*-D$, let us take the holomorphic coordinate (z_1,z_2) as before. We remark that the curves $\{z_i=a\}$ $(i=1,2)$, $\{z_1+z_2=b\}$, $\{z_1+\sqrt{-1}z_2=c\}$ can be regarded as parts of $X_{s'}$ for some $s'\in\mathcal{U}$. We have the expression $\theta=f_1\cdot dz_1+f_2\cdot dz_2$, where f_i are continuous sections of $\mathrm{End}(E)$. We have already known $\partial f_1/\partial\overline{z}_1=\partial f_2/\partial\overline{z}_2=0$. Thus we have only to show $\partial f_i/\partial\overline{z}_j=0$ for $i\neq j$. Let us consider the change of the coordinate given by $w_1=z_1+z_2$ and $w_2=z_1-z_2$. Then we have the following:

$$f_1\cdot dz_1+f_2\cdot dz_2=\frac{1}{2}(f_1+f_2)\cdot dw_1+\frac{1}{2}(f_1-f_2)\cdot dw_2.$$

Thus we obtain the following:

$$0=\frac{\partial}{\partial\overline{w}_1}(f_1+f_2)=\frac{1}{2}\left(\frac{\partial}{\partial\overline{z}_1}+\frac{\partial}{\partial\overline{z}_2}\right)(f_1+f_2)=\frac{1}{2}\left(\frac{\partial f_2}{\partial\overline{z}_1}+\frac{\partial f_1}{\partial\overline{z}_2}\right). \tag{116}$$

Let us consider the change of the coordinate given by $u_1=z_1+\sqrt{-1}z_2$ and $u_2=z_1-\sqrt{-1}z_2$. Then we have the following:

$$f_1\cdot dz_1+f_2\cdot dz_2=\frac{1}{2}\left(f_1+\frac{1}{\sqrt{-1}}f_2\right)du_1+\frac{1}{2}\left(f_1-\frac{1}{\sqrt{-1}}f_2\right)du_2.$$

Thus we obtain the following:

$$0=\frac{\partial}{\partial\overline{u}_1}\left(f_1+\frac{1}{\sqrt{-1}}f_2\right)=\frac{1}{2}\left(\frac{\partial}{\partial\overline{z}_1}-\frac{1}{\sqrt{-1}}\frac{\partial}{\partial\overline{z}_2}\right)\left(f_1+\frac{1}{\sqrt{-1}}f_2\right)=\frac{1}{2}\left(\frac{1}{\sqrt{-1}}\frac{\partial f_2}{\partial\overline{z}_1}-\frac{1}{\sqrt{-1}}\frac{\partial f_1}{\partial\overline{z}_2}\right). \tag{117}$$

From (116) and (117), we obtain $\partial f_i/\partial\overline{z}_j=0$ for $i\neq j$. Thus we obtain $\overline{\partial}_h\theta_h=0$, and the proof of Lemma 5.12 is accomplished. ∎

Lemma 5.13 *h is a harmonic metric for $(E,\mathbb{D}^\lambda)$ with respect to ω_0 on X^*-Z. (Recall $Z=\big\{x\in X^*\,\big|\,\mathcal{U}_x=\emptyset\big\}$.)*

Proof Due to Lemma 5.12, we have $\Lambda_\omega G(h)=\Lambda_\omega(\overline{\partial}_h\theta_h)=0$. Hence we have only to show that h is C^∞. We obtain the following formula in the level of distribution, by the formalism explained in the subsection 2.2.5:

$$\Delta^\lambda_{h',\omega}(s')=s'\big(-\Lambda_\omega G(h')\big)+\sqrt{-1}\Lambda_\omega\mathbb{D}^\lambda s'\cdot s'^{\,-1}\cdot\mathbb{D}^{\lambda\star}_{h'}s'.$$

The right hand side is C^0. Hence by using the elliptic regularity and the standard boot strapping argument, we obtain that s' is C^∞. Thus we obtain Lemma 5.13. ∎

Lemma 5.14 *h is pluri-harmonic metric of $E_{|X^*-Z}$.*

Proof We have already shown $\overline{\partial}_h\theta_h=0$ in Lemma 5.12. Recall Corollary 2.23. Then we have only to show $\theta_h^2=0$. Due to Corollary 5.7 and $\theta_{h\,|\,X_s}=\theta_s$, we know that the sequence $\{\theta^{(\epsilon)}\}$ converges to θ_h almost everywhere. In particular, we obtain the almost everywhere convergence of $\{\theta^{(\epsilon)\,2}\}$ to θ_h^2. On the other hand, we know the almost everywhere convergence $G(h^{(\epsilon)})\longrightarrow 0$, due to (111). We have $G(h^{(\epsilon)})=\overline{\partial}^{(\epsilon)\,2}+\overline{\partial}^{(\epsilon)}\theta^{(\epsilon)}+\theta^{(\epsilon)\,2}$, which is the decomposition into $(2,0)$, $(1,1)$ and $(0,2)$-forms. Therefore we obtain $\theta_h^2=0$, almost everywhere. Thus we obtain Lemma 5.14. ∎

Lemma 5.15 *h gives a pluri-harmonic metric of $E_{|X^*}$.*

Proof We have only to check that h gives a C^∞-metric of $E_{|X^*}$. Let Q be a point of Z. Let (U, z_1, z_2) be a holomorphic coordinate around Q such that $z_1(Q) = z_2(Q) = 0$. The pluri-harmonic metric h of $(E, \mathbb{D}^\lambda)_{|U-\{Q\}}$ is given. We would like to show that h is naturally extended to the pluri-harmonic metric of $(E, \mathbb{D}^\lambda)_{|U}$.

We have $\theta = f_1 \cdot dz_1 + f_2 \cdot dz_2$ defined on $U - \{Q\}$. Let us consider the characteristic polynomials $\det(t - f_i)$ for $i = 1, 2$. The coefficients are holomorphic on $U - \{Q\}$, and thus on U due to the theorem of Hartogs. Hence the eigenvalues of f_i are bounded on U. Let us consider the restriction of $(E, \mathbb{D}^\lambda, h)$ to the discs $C(a_j) := \{z_j = a_j\}$ $(a_j \neq 0)$ for $j = 1, 2$. Then it can be shown that the norms $\left|f_{i\,|\,C(a_j)}\right|_h \leq C$ $(i \neq j)$ can be dominated independently of a_j. (See Lemma 2.7 in [36], for example.) Thus f_i are bounded with respect to h on $U - \{Q\}$. In other words, θ is bounded on $U - \{Q\}$.

Let $E' := E_{|U-\{z_1 \cdot z_2 = 0\}}$. Let us consider the sheaf ${}^\diamond E'$ on U of the sections satisfying the growth condition $|g|_h = O(\prod |z_i|^{-\epsilon})$ for any $\epsilon > 0$ (the subsection 2.3.2). By using the result of the asymptotic behaviour of tame harmonic bundle at λ ([28]), ${}^\diamond E'$ is locally free on U. Since ${}^\diamond E'$ and $E_{|U-\{Q\}}$ are naturally isomorphic on $U - \{Q\}$, they are isomorphic on U. Let h' be any C^∞-metric of $E_{|U}$, and let s' be the endomorphism determined by $h = h' \cdot s'$. Due to the norm estimate given in ([28]), the metrics h and h' are mutually bounded. Hence s' and s'^{-1} are bounded on U. Due to the boundedness of θ, $s'^{-1}\mathbb{D}^\lambda s'$ is also bounded on $U - \{Q\}$ (See the subsubsection 2.2.5, for example.) Since we have the formula $\Delta^\lambda_{h',\omega_0} s' = s'(-\Lambda_{\omega_0} G(h')) + \Lambda_{\omega_0} \mathbb{D}^\lambda_{h'} s' \cdot s'^{-1} \cdot \mathbb{D}^{\lambda\star}_{h'} s'$, we can conclude that s' is C^∞ due to the standard bootstrapping argument. Namely, h is extended to the C^∞-metric of $E_{|U}$. ∎

5.2.7 The end of the proof of Theorem 5.4

Now, we have only to show that h is tame and adapted to the parabolic structure. Since $h_{|X_s} = h_s$ for any $s \in \mathcal{U}$, the tameness immediately follows from the curve test shown in [28]. Then we obtain the prolongment $\widetilde{E} := {}_{\boldsymbol{c}}E_{|X^*}$ with the induced parabolic structure $\boldsymbol{F}$ (the subsection 2.3.2). We would like to show that $(E, \boldsymbol{F}, \mathbb{D}^\lambda)$ and $(\widetilde{E}, \boldsymbol{F}, \mathbb{D}^\lambda)$ are isomorphic. For that purpose, we see that the identity $E_{|X^*} \longrightarrow E_{|X^*}$ can be prolonged to the homomorphism $\Psi : E \longrightarrow \widetilde{E}$. Let Q be any smooth point of $D_i \subset D$. We take a holomorphic coordinate (U_Q, z_1, z_2) with the following property:

- The curve $z_1^{-1}(0)$ is same as $U_Q \cap D$.
- The curves $C(b) := z_2^{-1}(b)$ are parts of $X_{s(b)}$ for $s(b) \in \mathcal{U}$.

Let f be a holomorphic section of $E_{|U}$. Since the restriction $h_{|X_{s(b)}}$ is same as $h_{s(b)}$, we have $|f_{|C(b)}|_h = O(|z_1|^{-c_i-\epsilon})$ for any $\epsilon > 0$. Then we obtain $|f|_h = O(|z_1|^{-c_i-\epsilon})$ for any $\epsilon > 0$, due to the result given in [28]. Thus f naturally gives the section of $\widetilde{E}_{|X^*}$ on U. Therefore, we obtain the morphism $E \longrightarrow \widetilde{E}_{|X^*}$ on $X - \left(\cup_{i\neq j} D_i \cap D_j\right)$. It naturally prolongs to the morphism $E \longrightarrow \widetilde{E}_{|X^*}$.

Recall that the restriction of $\widetilde{E} = {}_{\boldsymbol{c}}E_{|X^*}$ to X_s is same as ${}_{\boldsymbol{c}}(E_{|X_s^*})$. (See [28].) Therefore, the restrictions of Ψ to X_s are isomorphic, due to the hypothesis of the induction. Hence Ψ is isomorphic on $X - \left(\cup_{i\neq j} D_i \cap D_j\right)$, and thus on X. By a similar argument, we can show that the parabolic structures are also same. Thus the proof of Theorem 5.4 is finished. ∎

5.3 Correspondences

5.3.1 Kobayashi-Hitchin correspondence in the higher dimensional case

Let X be a smooth projective variety of dimension n $(n \geq 3)$ with an ample line bundle L, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i\in S} D_i$. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a μ_L-stable regular filtered λ-flat bundle on (X, D) in codimension two with trivial characteristic numbers $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$, and we put $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-D}$. Recall $\text{par-c}_1(\boldsymbol{E}) = 0$ due to the Bogomolov-Gieseker inequality and the Hodge index theorem. For each $\boldsymbol{c} \in \boldsymbol{R}^S$, we have the determinant line bundle $\det({}_{\boldsymbol{c}}E)$ of torsion-free sheaf ${}_{\boldsymbol{c}}E$, on which we have the induced parabolic structure and the induced flat λ-connection. Thus we obtain the canonically determined regular filtered λ-flat bundle $\left(\det \boldsymbol{E}_*, \mathbb{D}^\lambda\right)$ on (X, D) of rank one. We also have $\text{par-c}_1\left(\det \boldsymbol{E}_*\right) = \text{par-c}_1\left(\boldsymbol{E}_*\right) = 0$. Therefore, we can take a pluri-harmonic metric

$h_{\det E}$ of $(\det(E), \mathbb{D}^\lambda)$ which is adapted to the parabolic structure of $\det \boldsymbol{E}_*$. Recall that we have a subset $Z \subset D$ with $\operatorname{codim}_X(Z) \geq 3$ such that $(\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-Z}$ is a regular filtered λ-flat *bundle*.

Theorem 5.16 *There exists the unique tame pluri-harmonic metric h of $(E, \mathbb{D}^\lambda)$ with the following properties:*

- $\det(h) = h_{\det E}$.
- *It is adapted on the parabolic structure of $\boldsymbol{E}_*$ on $X - Z$. Namely, $(\boldsymbol{E}_*(h), \mathbb{D}^\lambda)_{|X-Z} \simeq (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-Z}$, where $(\boldsymbol{E}_*(h), \mathbb{D}^\lambda)$ denotes the regular λ-flat bundle on (X, D) obtained from $(E, \mathbb{D}^\lambda, h)$. (See the subsection* 2.3.*)*

Proof Due to Mehta-Ramanathan type theorem (Proposition 2.8), the uniqueness can be easily reduced to the $\dim X = 1$ case, by considering the restriction to the generic curves $C \subset X$. We have already known it (Proposition 2.27).

We will use the induction on the dimension n to show the existence. The case $n = 2$ has already been shown (Theorem 5.4). Assume that L^m is sufficiently ample. We put $\mathbb{P}_m := \mathbb{P}(H^0(X, L^m)^\vee)$. For any $s \in \mathbb{P}_m$, we put $X_s := s^{-1}(0)$. Recall Proposition 2.8. Let $\mathcal{U}$ be the Zariski open subset of $\mathbb{P}_m$ which consists of $s \in \mathbb{P}_m$ with the following properties:

- X_s is smooth, and $D_s := X_s \cap D$ is a normal crossing divisor.
- The codimension of $W_s = W \cap X_s$ in X_s is larger than 3.
- $(\boldsymbol{E}, \mathbb{D}^\lambda)_{|X_s}$ is μ_L-stable.

We use the existence hypothesis in the $(n-1)$-dimensional case of the induction. Then we may have the tame pluri-harmonic metric h_s of $(E, \mathbb{D}^\lambda)_{|X_s \setminus D}$ with $\det(h_s) = h_{\det E\,|\,X_s \setminus D}$ which is adapted to the parabolic structure on $X_s \setminus W$. We also use the uniqueness result in the $(n-2)$-dimensional case. Then we can show the existence of a finite subset $Z' \subset X - D$ and a metric h of $E_{|X-D}$ such that $h_{s\,|\,P} = h_{|P}$. By the arguments given in the subsections 5.2.5–5.2.7, we can show that h is the desired metric. The only different point is the argument to show the vanishing of $G(h) = 0$. Due to $\dim X_s \geq 2$, it can be shown easier. ∎

Theorem 5.17 *Let X, D and L be as above. Let $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ be a saturated μ_L-stable regular filtered λ-flat sheaf on (X, D) with the trivial characteristic numbers* $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$. *We put $(E, \mathbb{D}^\lambda) := (\boldsymbol{E}_*, \mathbb{D}^\lambda)_{|X-D}$. Then there exists a pluri-harmonic metric h of $(E, \mathbb{D}^\lambda)$ such that the induced regular filtered λ-flat bundle $\big(\boldsymbol{E}_*(h), \mathbb{D}^\lambda\big)$ is isomorphic to $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$. Such a metric is unique up to positive constant multiplication. In particular, $\boldsymbol{E}_*$ is a filtered bundle.*

Proof Since a saturated regular filtered λ-flat sheaf is a regular filtered λ-flat bundle in codimension two (Lemma 2.11), we may apply Theorem 5.16. Then there exists a pluri-harmonic metric h and a subset $W \subset D$ with $\operatorname{codim}_X(W) \geq 3$ such that the induced regular filtered λ-flat bundle $(\boldsymbol{E}_*(h), \mathbb{D}^\lambda)$ is isomorphic to $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ on $X - W$. Since both of $(\boldsymbol{E}_*(h), \mathbb{D}^\lambda)$ and $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ are saturated, they are isomorphic on X. ∎

5.3.2 The equivalence of the categories

Let $\mathcal{C}_\lambda^{poly}$ denote the category of μ_L-stable regular filtered λ-flat bundles $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ on (X, D) with the trivial characteristic numbers $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$. Morphisms $f : (\boldsymbol{E}_{1\,*}, \mathbb{D}_1^\lambda) \longrightarrow (\boldsymbol{E}_{2\,*}, \mathbb{D}_2^\lambda)$ are defined to be $\mathcal{O}_X$-homomorphism $f : \boldsymbol{E}_1 \longrightarrow \boldsymbol{E}_2$ satisfying $\mathbb{D}_2^\lambda \circ f = f \circ \mathbb{D}_1^\lambda$ and $f\big({}_{\boldsymbol{c}}E_1\big) \subset {}_{\boldsymbol{c}}E_2$ for any $\boldsymbol{c}$.

Corollary 5.18 *Let λ_i $(i = 1, 2)$ be two complex numbers. We have the natural functor $\Xi_{\lambda_1,\lambda_2} : \mathcal{C}_{\lambda_1}^{poly} \longrightarrow \mathcal{C}_{\lambda_2}^{poly}$, which is equivalent. It preserves direct sums, tensor products and duals.*

Proof Let $(\boldsymbol{E}_*^{\lambda_1}, \mathbb{D}^{\lambda_1})$ be an object of $\mathcal{C}_{\lambda_1}^{poly}$. We put $E^{\lambda_1} := \boldsymbol{E}^{\lambda_1}_{|D}$. Then we have a pluri-harmonic metric h of $(E^{\lambda_1}, \mathbb{D}^{\lambda_1})$, which is adapted to the parabolic structure. Then we obtain the operators $\overline{\partial}_h, \partial_h, \theta_h, \theta_h^\dagger$, as in the subsection 2.2.1. Note that the holomorphic structure of E^{λ_1} is given by $\overline{\partial}_h + \lambda_1 \theta_h^\dagger$. The $(0,1)$-operator $\overline{\partial}_h + \lambda_2 \theta_h^\dagger$ also gives a holomorphic structure of C^∞-bundle E^{λ_1}. To distinguish them, we use the notation

E^{λ_2}, when we consider the holomorphic structure $\overline{\partial}_h + \lambda_2\theta_h^{\dagger}$. We put $\mathbb{D}^{\lambda_2} := \overline{\partial}_h + \theta_h + \lambda_2(\partial_h + \theta_h^{\dagger})$, which gives a flat λ_2-connection of E^{λ_2}. The metric h is pluri-harmonic for $(E^{\lambda_2}, \mathbb{D}^{\lambda_2})$. Since the corresponding Higgs bundle for $(E^{\lambda_1}, \mathbb{D}^{\lambda_1}, h)$ and $(E^{\lambda_2}, \mathbb{D}^{\lambda_2}, h)$ are same, we obtain the tameness of $(E^{\lambda_2}, \mathbb{D}^{\lambda_2}, h)$. Therefore, we obtain the prolongment $(\boldsymbol{E}^{\lambda_2}, \mathbb{D}^{\lambda})$, which are μ_L-polystable regular filtered λ_2-flat bundle on (X, D) with trivial characteristic numbers (Proposition 2.26).

We remark that $(\boldsymbol{E}^{\lambda_2}, \mathbb{D}^{\lambda_2})$ is independent of a choice of h, due to the uniqueness in Proposition 2.27. Therefore we put $\Xi_{\lambda_1,\lambda_2}(\boldsymbol{E}^{\lambda_1}, \mathbb{D}^{\lambda_1}) := (\boldsymbol{E}^{\lambda_2}, \mathbb{D}^{\lambda_2})$. It is easy to see that $\Xi_{\lambda_1,\lambda_2}$ gives a functor. It is also easy to see that $\Xi_{\lambda_2,\lambda_1} \circ \Xi_{\lambda_1,\lambda_2}(\boldsymbol{E}^{\lambda_1}, \mathbb{D}^{\lambda_1})$ is naturally isomorphic to $(\boldsymbol{E}^{\lambda_1}, \mathbb{D}^{\lambda_1})$. The compatibility with the direct sums, duals and tensor products are obtained from the corresponding compatibility statements of the prolongments for tame harmonic bundles ([28]). ∎

Remark 5.19 *From a λ_1-connection $\mathbb{D}^{\lambda_1} = d'' + d'$, a λ_2-connection is given $d'' + (\lambda_2/\lambda_1) \cdot d'$. Hence we have the obvious functor* $\mathrm{Obv} : \mathcal{C}^{poly}_{\lambda_1} \longrightarrow \mathcal{C}^{poly}_{\lambda_2}$. *This is not same as the above functor* $\Xi_{\lambda_1,\lambda_2}$. ∎

6 Filtered local system

6.1 Definition

6.1.1 Filtered structure

Let X be a complex manifold, and let D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. We will use the notation $D^{[2]} := \bigcup_{i \neq j} D_i \cap D_j$ and $D_i^{\circ} := D_i \setminus \bigcup_{j \neq i} D_j$. Let $\mathcal{L}$ be a local system on $X - D$. A filtered structure of $\mathcal{L}$ at D is a tuple of increasing filtrations ${}^i\mathcal{F}$ $(i \in S)$ of $\mathcal{L}_{|U_i \setminus D}$ indexed by $\boldsymbol{R}$, where U_i denotes an appropriate open neighbourhood of D_i. Let U_i' be an open neighbourhood of D_i such that $U_i' \subset U_i$, then we have the induced filtration ${}^i\mathcal{F}_{|U_i'}$, and the filtration ${}^i\mathcal{F}$ can be reconstructed from ${}^i\mathcal{F}_{|U_i'}$. Hence we define two filtered structures $({}^i\mathcal{F}, U_i \,|\, i \in S)$ and $({}^i\mathcal{F}', U_i' \,|\, i \in S)$ are equivalent, if there exists an open neighbourhood U_i'' of D_i such that $U_i'' \subset U_i \cap U_i'$ and ${}^i\mathcal{F}_{|U_i''} = {}^i\mathcal{F}'_{|U_i''}$. A tuple of a local system $\mathcal{L}$ and an equivalence class of filtered system $({}^i\mathcal{F}, U_i)$ is called a filtered local system, and it is denoted by $\mathcal{L}_*$. We do not have to care about a choice of open neighbourhoods U_i.

Morphisms of filtered local systems $f : \mathcal{L}_{1*} \longrightarrow \mathcal{L}_{2*}$ are defined to be a morphism $f : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ of local systems preserving the filtered structures in an obvious sense. We denote by $\widetilde{\mathcal{C}}(X, D)$ the category of filtered local systems on (X, D).

6.1.2 Characteristic numbers

We put $U_i^* := U_i \setminus D$ and ${}^i\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*}) := {}^i\mathcal{F}_a(\mathcal{L}_{|U_i^*}) / {}^i\mathcal{F}_{<a}(\mathcal{L}_{|U_i^*})$. Since the local monodromy around D_i preserves the filtration ${}^i\mathcal{F}$, we obtain the induced endomorphism of ${}^i\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*})$, and thus the generalized eigen decomposition:

$$
{}^i\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*}) = \bigoplus_{\omega} {}^i\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{(a,\omega)}(\mathcal{L}_{|U_i^*}).
$$

We put as follows:

$$
\mathcal{P}ar(\mathcal{L}_*, i) := \bigl\{a \in \boldsymbol{R} \,\big|\, {}^i\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*}) \neq 0\bigr\}, \quad \mathcal{KMS}(\mathcal{L}_*, i) := \bigl\{(a, \omega) \in \boldsymbol{R} \times \boldsymbol{C}^* \,\big|\, {}^i\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{(a,\omega)}(\mathcal{L}_{|U_i^*}) \neq 0\bigr\}.
$$

The parabolic first Chern class is defined as follows:

$$
\operatorname{par-c}_1(\mathcal{L}_*) := -\sum_{i \in S} \operatorname{wt}(\mathcal{L}_*, i) \cdot [D_i] \in H^2(X, \boldsymbol{R}), \qquad \operatorname{wt}(\mathcal{L}_*, i) := \sum_{a \in \mathcal{P}ar(\mathcal{L}_*, i)} a \cdot \operatorname{rank} {}^i\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{L}_{|U_i^*}). \tag{118}
$$

Here $[D_i]$ denotes the cohomology class representing D_i.

Let $\operatorname{Irr}(D_i \cap D_j)$ denote the set of the irreducible components of $D_i \cap D_j$. For each $P \in \operatorname{Irr}(D_i \cap D_j)$, let U_P be an appropriate open neighbourhood of P in X such that $U_P \subset U_i \cap U_j$. We put $U_P^* := U_P \setminus D$. We have the two filtrations ${}^{i}\mathcal{F}$ and ${}^{j}\mathcal{F}$ of $\mathcal{L}_{|U_P^*}$. The naturally induced graded local system is denoted as follows:

$$ {}^{P}\mathrm{Gr}^{\mathcal{F}}(\mathcal{L}_{|U_P^*}) = \bigoplus_{(a_i,a_j)\in \boldsymbol{R}^2} {}^{P}\mathrm{Gr}^{\mathcal{F}}_{(a_i,a_j)}(\mathcal{L}_{|U_P^*}), \quad {}^{P}\mathrm{Gr}^{\mathcal{F}}_{(a_i,a_j)}(\mathcal{L}_{|U_P^*}) := \frac{{}^{i}\mathcal{F}_{a_i}\cap {}^{j}\mathcal{F}_{a_j}}{\sum_{(b_i,b_j)\lneq(a_i,a_j)} {}^{i}\mathcal{F}_{b_i}\cap {}^{j}\mathcal{F}_{b_j}}. $$

Here $(b_i,b_j)\lneq(a_i,a_j)$ means "$b_i\leq a_i$, $b_j\leq a_j$ and $(b_i,b_j)\neq(a_i,a_j)$". We have the two endomorphisms induced by the local monodromies around $U_P\cap D_i$ and $U_P\cap D_j$, which are commutative. Hence we obtain the generalized eigen decomposition:

$$ {}^{P}\mathrm{Gr}^{\mathcal{F}}_{\boldsymbol{a}}(\mathcal{L}_{|U_P^*}) = \bigoplus_{\boldsymbol{\omega}\in \boldsymbol{C}^{*\,2}} {}^{P}\mathrm{Gr}^{\mathcal{F},\mathbb{E}}_{\boldsymbol{a},\boldsymbol{\omega}}(\mathcal{L}_{|U_P^*}). $$

We put as follows:

$$ Par(\mathcal{L}_*,P) := \bigl\{(a_i,a_j)\in\boldsymbol{R}^2 \bigm| {}^{P}\mathrm{Gr}^{\mathcal{F}}_{(a_i,a_j)}(\mathcal{L}_{|U_P^*})\neq 0\bigr\}, $$
$$ \mathcal{KMS}(\mathcal{L}_*,P) := \bigl\{(\boldsymbol{a},\boldsymbol{\omega})\in\boldsymbol{R}^2\times\boldsymbol{C}^{*\,2} \bigm| {}^{P}\mathrm{Gr}^{\mathcal{F},\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\omega})}(\mathcal{L}_{|U_P^*})\neq 0\bigr\}. $$

The parabolic second Chern character is defined as follows:

$$ \begin{aligned} \mathrm{par\text{-}ch}_2(\mathcal{L}_*) := &\frac{1}{2}\sum_{i\in S}\sum_{a\in\mathcal{P}ar(\mathcal{L}_*,i)} a^2\cdot \mathrm{rank}\,{}^{i}\mathrm{Gr}^{\mathcal{F}}_a(\mathcal{L})\cdot[D_i]^2 \\ &+\frac{1}{2}\sum_{i\in S}\sum_{j\neq i}\sum_{P\in\mathrm{Irr}(D_i\cap D_j)}\sum_{(a_i,a_j)\in\mathcal{P}ar(\mathcal{L}_*,P)} a_i\cdot a_j\cdot\mathrm{rank}\,{}^{P}\mathrm{Gr}^{\mathcal{F}}_{(a_i,a_j)}\bigl(\mathcal{L}_{|U_P^*}\bigr)\cdot[P]. \end{aligned} \tag{119} $$

When X is a smooth projective variety with an ample line bundle L, we put as follows:

$$ \mathrm{par\text{-}deg}_L(\mathcal{L}_*) := \int_X \mathrm{par\text{-}c}_1(\mathcal{L}_*)\cdot c_1(L)^{\dim X-1}, \quad \mu_L(\mathcal{L}_*) := \frac{\mathrm{par\text{-}deg}_L(\mathcal{L}_*)}{\mathrm{rank}\,\mathcal{L}}. $$

Then the notion of μ_L-stability, μ_L-semistability, and μ_L-polystability for filtered local systems on (X,D) are defined in the standard manner. We also put as follows:

$$ \int_X \mathrm{par\text{-}c}^2_{1,L}(\mathcal{L}_*) := \int_X \mathrm{par\text{-}c}_1(\mathcal{L}_*)^2\cdot c_1(L)^{\dim X-2}, \quad \int_X \mathrm{par\text{-}ch}_{2,L}(\mathcal{L}_*) := \int_X \mathrm{par\text{-}ch}_{2,L}(\mathcal{L}_*)\cdot c_1(L)^{\dim X-2}. $$

6.2 Correspondence

In this subsection, we give the correspondence of filtered local systems on (X,D) and saturated regular filtered λ-flat sheaves ($\lambda\neq 0$). See the subsection 2.1.4 for saturated regular filtered λ-flat sheaves. Since we have the obvious correspondence between flat λ-connection and flat 1-connection, we only discuss the case $\lambda=1$, i.e. ordinary flat connections.

Let $\mathcal{C}_1^{sat}(X,D)$ denote the category of saturated regular filtered flat sheaves on (X,D). Let us see briefly that we have the equivalent functor $\Phi:\widetilde{\mathcal{C}}(X,D)\longrightarrow\mathcal{C}_1^{sat}(X,D)$. Since it is given by Simpson in [35] essentially in the curve case, we give only an outline.

6.2.1 Construction of Φ

First we give a construction of Φ. Let $\mathcal{L}_*$ be a filtered local system on (X,D). Let (E,∇) be the corresponding flat bundle on $X-D$. We have the Deligne extension $(\widetilde{E},\nabla)$ on (X,D). We put $\boldsymbol{E}:=\widetilde{E}\otimes\mathcal{O}(*D)$. Thus we have only to give the way of the construction of the $\mathcal{O}_X$-coherent submodules ${}_{\boldsymbol{a}}E\subset\boldsymbol{E}$ such that $\nabla{}_{\boldsymbol{a}}E\subset{}_{\boldsymbol{a}}E\otimes\Omega^{1,0}(\log D)$ and $\bigcup_{\boldsymbol{a}\in\boldsymbol{R}^S}{}_{\boldsymbol{a}}E=\boldsymbol{E}$. Let us consider the case $X=\Delta^n=\{(z_1,\ldots,z_n)\,|\,|z_i|<1\}$ and $D=\{z_1=0\}$. Then the construction is essentially same as that for the case $\dim X=1$ given by Simpson [35]. We briefly recall it. Let $H(\mathcal{L})$ denote the space of the multi-valued flat sections of $\mathcal{L}$. We have the induced filtration $\mathcal{F}H(\mathcal{L})$ and the generalized eigen decomposition $H(\mathcal{L})=\bigoplus_{\omega}\mathbb{E}_{\omega}(H(\mathcal{L}))$, which are compatible in the sense $\mathcal{F}_a=\bigoplus_{\omega}\mathcal{F}_a\cap\mathbb{E}_{\omega}$. Let $\boldsymbol{u}=(u_1,\ldots,u_r)$ be a frame compatible of $H(\mathcal{L})$, compatible with $(\mathcal{F},\mathbb{E})$. Then for each u_i, the numbers $\omega(u_i) \in \boldsymbol{C}$ and $a(u_i) \in \boldsymbol{R}$ are determined by $u_i \in \mathbb{E}_{\omega(u_i)}$ and $u_i \in \mathcal{F}_{a(u_i)} - \mathcal{F}_{<a(u_i)}$. The complex number $\alpha(u_i)$ is determined by the conditions $\exp(-2\pi\alpha(u_i)) = \omega(u_i)$ and $0 \leq \operatorname{Re}\alpha(u_i) < 1$. Let M^u denote the endomorphism of $H(\mathcal{L})$ or $\mathcal{L}$, which is the unipotent part of the monodromy around D, and we put $N := -(2\pi\sqrt{-1})^{-1}\log M^u$. We regard u_i as a multi-valued C^∞-section of E. Then it is standard that $v_i := \exp\bigl(\log z_1(\alpha(u_i)+N)\bigr)\cdot u_i$ gives a holomorphic section of E. Moreover, $\boldsymbol{v} = (v_1, \ldots, v_r)$ gives a frame of the Deligne extension $\widetilde{E}$. Let b be any real number. Then we put $n(b, u_i) := \max\bigl\{n \in \mathbb{Z} \bigm| a(u_i) - \operatorname{Re}\alpha(u_i) + n \leq b\bigr\}$, and we put $v_i(b) := z_1^{-n_i(b,u_i)} \cdot v_i$. Let ${}_bE$ denote the $\mathcal{O}_X$-submodule of $\boldsymbol{E}$ generated by $v_1(b), \ldots, v_r(b)$. It is easy to check that ${}_bE$ is locally free and independent of a choice of $\boldsymbol{u}$. It is also easy to see $\boldsymbol{E} = \bigcup_{b\in\boldsymbol{R}} {}_bE$. Thus we obtain the filtration in the case $X = \Delta^n$ and $D = \{z_1 = 0\}$. It can be checked that the filtration is independent of a choice of the coordinate $(z_1, z_2, \ldots, z_n)$ satisfying $D = \{z_1 = 0\}$. Then we obtain the ${}_{\boldsymbol{a}}E$ on $X - D^{[2]}$ by gluing them. The subsheaves ${}_{\boldsymbol{a}}E$ are determined by the condition (4). It is easy to see that ${}_{\boldsymbol{a}}E$ is the saturation of a finitely generated submodules of $\widetilde{E} \otimes \mathcal{O}_X(N\cdot D)$ for some large integer N, and hence we have the $\mathcal{O}_X$-coherence of ${}_{\boldsymbol{a}}E$.

Let $f : \mathcal{L}_{1\,*} \longrightarrow \mathcal{L}_{2\,*}$ be a morphism. Let $(\boldsymbol{E}_{i\,*}, \nabla_i) := \Phi(\mathcal{L}_i)$. We have the induced map $\widetilde{f} : \boldsymbol{E}_1 \longrightarrow \boldsymbol{E}_2$. It is easy to see that ${}_{\boldsymbol{c}}E_{1\,|\,X-D^{[2]}} \longrightarrow {}_{\boldsymbol{c}}E_{2\,|\,X-D^{[2]}}$ is induced. Due to saturatedness of $(\boldsymbol{E}_{2\,*}, \nabla)$, we obtain maps ${}_{\boldsymbol{c}}E_1 \longrightarrow {}_{\boldsymbol{c}}E_2$, and thus $\Phi(f) : (\boldsymbol{E}_{1\,*}, \nabla_1) \longrightarrow (\boldsymbol{E}_{2\,*}, \nabla_2)$.

6.2.2 Equivalence

Let us show that Φ is equivalent. To begin with, we consider the case $X = \Delta^n$ and $D = \{z_1 = 0\}$. Let $\mathcal{C}_1^{vb}(X, D)$ denote the category of regular filtered flat bundles on (X, D), which is the subcategory of $\mathcal{C}_1^{sat}(X, D)$. By the construction, the image of Φ is contained in $\mathcal{C}_1^{vb}(X, D)$. The following lemma can be shown as in [35].

Lemma 6.1 *The functor Φ gives the equivalence of $\widetilde{\mathcal{C}}_1(X, D)$ and $\mathcal{C}_1^{vb}(X, D)$. It is also compatible with direct sums, duals, and tensor products.* ∎

Lemma 6.2 *In the case $X = \Delta^n$ and $D = \{z_1 = 0\}$, we have $\mathcal{C}_1^{vb}(X, D) \simeq \mathcal{C}_1^{sat}(X, D)$ naturally. In particular, Φ gives the equivalence $\widetilde{\mathcal{C}}_1(X, D) \simeq \mathcal{C}_1^{sat}(X, D)$.*

Proof Let $(\boldsymbol{E}_*, \nabla)$ be a saturated regular filtered flat sheaf on (X, D). We put $(E, \nabla) := (\boldsymbol{E}_*, \nabla)_{|X-D}$, and let $\mathcal{L}$ denote the corresponding local system on $X - D$. Let $H(\mathcal{L})$ denote the space of the multi-valued flat sections of $\mathcal{L}$.

Recall that there exists a subset $W \subset D$ with $\operatorname{codim}_X(W) \geq 3$ such that $(\boldsymbol{E}_*, \nabla)_{|X-W}$ is regular filtered flat bundle on $(X - W, D - W)$ (Lemma 2.11). Let P be any point of $D - W$, and let $(U_P, z_1, \ldots, z_n)$ be a holomorphic coordinate neighbourhood such that $z_1^{-1}(0) = U_P \cap D$ and $U_P \cap W = \emptyset$. Due to Lemma 6.1, we have the unique filtration $\mathcal{F}$ of $H(\mathcal{L}_{|U_P\setminus D}) \simeq H(\mathcal{L})$ corresponding to $(\boldsymbol{E}_*, \nabla)_{|U_P}$. Due to the uniqueness, it is independent of a choice of P and U_P.

Let $\boldsymbol{u} = (u_1, \ldots, u_r)$ be a frame of $H(\mathcal{L})$ compatible with the filtration $\mathcal{F}$ and the generalized eigen decomposition with respect to the monodromy around D. For any real number $b \in \boldsymbol{R}$, we construct $\boldsymbol{v}(b) = \bigl(v_1(b), \ldots, v_r(b)\bigr)$ as above. Then, for any $P \in D - W$, $\boldsymbol{v}(b)$ gives a holomorphic frame of ${}_bE_{|U_P}$ compatible with the filtration due to Lemma 6.2. Hence each $v_i(b)$ gives a section of ${}_bE_{|X-W}$. Due to the saturatedness of $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$, $v_i(b)$ gives a section of ${}_bE$ on X. Now it is easy to see that $\boldsymbol{v}(b)$ gives a frame of ${}_bE$, and in particular, ${}_bE$ is locally free. Hence $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ is a regular filtered flat bundle on (X, D). ∎

Now, it is easy to see that Φ is equivalent for general (X, D). Let us see the fully faithfulness of Φ. The faithfulness is obvious. Let $f : \Phi(\mathcal{L}_{1\,*}) \longrightarrow \Phi(\mathcal{L}_{2\,*})$ be a morphism in $\mathcal{C}_1^{sat}(X, D)$. We have the map $g : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ corresponding to f. We would like to check that g preserves the filtrations ${}^i\mathcal{F}$. Let P be any point of D_i°, and $(U, z_1, \ldots, z_n)$ be any coordinate neighbourhood such that $U \cap D = z_1^{-1}(0)$. Applying Lemma 6.2, we obtain that g preserves the filtration ${}^i\mathcal{F}$ on $U \setminus D_i$. Thus we obtain the fully faithfulness.

Let us show the essential surjectivity. Let $(\boldsymbol{E}_*, \nabla)$ be a saturated filtered flat sheaf on (X, D). Let $\mathcal{L}$ denote the local system corresponding to $(\boldsymbol{E}_*, \nabla)_{|X-D}$. We have only to construct appropriate filtrations ${}^i\mathcal{F}$ of $\mathcal{L}_{|U_i\setminus D}$ on appropriate neighbourhoods of D_i. Let P be any point of D_i°, and $(U_P, z_1, \ldots, z_n)$ denote any coordinate neighbourhood around P such that $z_1^{-1}(0) = U_P \cap D$. Due to Lemma 6.1, we obtain the unique filtration ${}^i\mathcal{F}$

of $\mathcal{L}_{|U_P\setminus D}$. We obtain the filtration ${}^i\mathcal{F}$ on $\bigcup_{P\in D_i^\circ} U_P$ by gluing them, due to the uniqueness. Thus we obtain that Φ is essentially surjective, and hence equivalent.

6.2.3 The parabolic first Chern class

We have the $\mathbb{Z}$-action on $\boldsymbol{R}\times\boldsymbol{C}$ given by $n\cdot(a,\alpha)=(a+n,\alpha-n)$. It induces the action of $\mathbb{Z}$ on $\mathcal{KMS}(\boldsymbol{E}_*,i)$. The following lemma is clear from the construction of Φ.

Lemma 6.3 *We have the bijective correspondence of the sets* $\mathcal{KMS}(\Phi(\mathcal{L}_*),i)/\mathbb{Z}$ *and* $\mathcal{KMS}(\mathcal{L}_*,i)$*, which is given by* $(a,\alpha)\longmapsto(b,\omega)=\Bigl(a+\operatorname{Re}\alpha,\exp\bigl(-2\pi\sqrt{-1}\alpha\bigr)\Bigr)$ *for* $(a,\alpha)\in\mathcal{KMS}(\Phi(\mathcal{L}_*),i)$*. Moreover,* $\operatorname{rank}{}^i\operatorname{Gr}^{F,\mathbb{E}}_{(a,\alpha)}=\operatorname{rank}{}^i\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{(b,\omega)}$. ∎

Corollary 6.4 *We have the equality of the parabolic first Chern class* $\text{par-c}_1(\mathcal{L}_*)=\text{par-c}_1(\Phi(\mathcal{L}_*))$*. In particular, when* X *is a smooth projective variety with an ample line bundle* L*, the* μ_L*-stability of* $\mathcal{L}_*$ *and* μ_L*-stability of* $\Phi(\mathcal{L}_*)$ *are equivalent.*

Proof Recall the formula (54). It is shown for the case where $(\boldsymbol{E}_*,\nabla)$ is graded semisimple and $\dim X$ is two dimensional. However, the graded semisimplicity condition is not necessary as is explained in Remark 3.16. The assumption $\dim X=2$ is also not necessary, due to the Lefschetz theorem. Then the claim of the corollary follows from the formula (54) and the correspondence of the KMS-spectrums given in Lemma 6.3. ∎

6.2.4 The second parabolic Chern character

Lemma 6.5 *Let* $X=\Delta^n=\{(z_1,\ldots,z_n)\,|\,|z_i|<1\}$*, and* $D=D_1\cup D_2$*, where* $D_i=\{z_i=0\}$*. Let* $(\boldsymbol{E}_*,\nabla)$ *be a saturated regular filtered flat sheaf on* (X,D)*.*

- $(\boldsymbol{E}_*,\nabla)$ *is a regular filtered flat bundle on* (X,D)*.*
- *Let* $\boldsymbol{c}$ *be any element of* $\boldsymbol{R}^2$*, and let* ${}_{\boldsymbol{c}}E$ *denote the* $\boldsymbol{c}$*-truncation. Let* $\mathcal{L}_*$ *be the corresponding filtered local system on* (X,D)*. Then we have the equality:*

$$\operatorname{rank}{}^{\underline{2}}\operatorname{Gr}^{\mathcal{F},\mathbb{E}}_{(\boldsymbol{b},\boldsymbol{\omega})}(\mathcal{L})=\operatorname{rank}{}^{\underline{2}}\operatorname{Gr}^{F,\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\alpha})}({}_{\boldsymbol{c}}E).$$

  Here the meaning of the notation is as follows:

  - $\boldsymbol{b}=(b_1,b_2)$ *and* $\boldsymbol{\omega}=(\omega_1,\omega_2)$ *denote elements of* $\boldsymbol{R}^2$ *and* $\boldsymbol{C}^{*\,2}$ *respectively.*
  - $\boldsymbol{a}=(a_1,a_2)$ *and* $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ *denote elements of* $\boldsymbol{R}^2$ *and* $\boldsymbol{C}^2$ *respectively, determined by the conditions* $c_i-1<a_i\leq c_i$*,* $\exp(-2\pi\sqrt{-1}\alpha_i)=\omega_i$ *and* $a_i+\operatorname{Re}\alpha_i=b_i$*.*

Proof Let $\mathcal{L}_*=(\mathcal{L},{}^1\mathcal{F},{}^2\mathcal{F})$ be as above. Let $\boldsymbol{u}$ be a frame of $H(\mathcal{L})$ compatible with ${}^1\mathcal{F}$ and ${}^2\mathcal{F}$. For each u_j and the divisor D_k, the complex number $\alpha_k(u_j)$ and $a_k(u_j)$ are determined as before. For the monodromies around D_k, we obtain the nilpotent endomorphism N_k as before. The holomorphic section v_j is given by $v_j:=\exp\Bigl(\sum\log z_k\bigl(\alpha_k(u_j)+N_k\bigr)\Bigr)$. Let $n_k(u_j)$ be the numbers determined by the condition $c_k-1<n_k(u_j)+a_k(u_j)-\operatorname{Re}\alpha_k(u_j)\leq c_k$. We put $\widetilde{v}_j:=\prod z_k^{-n_k(u_j)}\cdot v_j$. Then $\widetilde{\boldsymbol{v}}=(\widetilde{v}_1,\ldots,\widetilde{v}_r)$ gives the frame of ${}_{\boldsymbol{c}}E_{|X-(D_1\cap D_2)}$. Due to the saturatedness, $\widetilde{\boldsymbol{v}}=(\widetilde{v}_1,\ldots,\widetilde{v}_r)$ gives the frame of ${}_{\boldsymbol{c}}E$, and hence ${}_{\boldsymbol{c}}E$ are locally free. Thus the first claim is proved. The frame $\widetilde{\boldsymbol{v}}$ is compatible with ${}^i\mathbb{E}$ and ${}^iF$, and we have ${}^k\deg^F(\widetilde{v}_j)=a_k(u_j)-\operatorname{Re}\alpha_k(u_j)+n_k(u_j)$ and $\widetilde{v}_{j\,|\,D_k}\in{}^k\mathbb{E}(\alpha_k(u_j)-n_k(u_j))$. Thus the second claim follows. ∎

Corollary 6.6 *Let* X *be a complex manifold, and let* D *be a simple normal crossing divisor. Let* $(\boldsymbol{E}_*,\nabla)$ *be a saturated regular filtered flat sheaf on* (X,D)*. Then it is a regular filtered flat bundle in codimension two, and we have the equality of the parabolic second Chern character numbers* $\int_X\text{par-ch}_{2,L}(\mathcal{L}_*)=\int_X\text{par-ch}_{2,L}(\boldsymbol{E}_*)$*. Here* $\mathcal{L}_*$ *denotes the corresponding filtered local system.* ∎

Corollary 6.7 *Let X be a smooth projective variety with an ample line bundle L, and let D be a simple normal crossing divisor. Let $\mathcal{L}_*$ be a μ_L-stable filtered local system on (X, D). Then the Bogomolov-Gieseker inequality for $\mathcal{L}_*$ holds:*

$$\int_X \text{par-ch}_{2,L}(\mathcal{L}_*) \leq \frac{\int_X \text{par-c}_{1,L}^2(\mathcal{L}_*)}{2 \operatorname{rank} \mathcal{L}}.$$

Proof Recall that saturated regular filtered flat shaves are regular filtered flat bundles in codimension two (Lemma 2.11). Hence the claim follows from Corollary 6.4, Corollary 6.6 and Corollary 3.22. ▮

Corollary 6.8 *Let X be a smooth projective variety with an ample line bundle L, and let D be a simple normal crossing divisor. Let $\mathcal{C}_1^{poly}$ be the category of μ_L-polystable regular filtered flat bundle on (X, D) with trivial characteristic numbers, and let $\widetilde{\mathcal{C}}_1^{poly}$ be the category of μ_L-polystable filtered local system on (X, D) with trivial characteristic numbers. Then the functor Φ naturally gives the equivalence of them.*

Proof We have only to remark that saturated μ_L-stable regular filtered flat sheaves are regular filtered bundles (Theorem 5.17). ▮

Remark 6.9 *Due to the result in* [28] *and the existence of a pluri-harmonic metric for $\Phi(\mathcal{L}_*)$, the filtrations ${}^i\mathcal{F}$ for μ_L-stable filtered local systems $\mathcal{L}_*$ satisfy some compatibility around the intersection points of D.* ▮

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