ELLIPTIC GENUS OF CALABI–YAU MANIFOLDS AND JACOBI AND SIEGEL MODULAR FORMS

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Abstract. In the paper we study two types of relations: a one is between the elliptic genus of Calabi–Yau manifolds and Jacobi modular forms, another one is between the second quantized elliptic genus, Siegel modular forms and Lorentzian Kac–Moody Lie algebras. We also determine the structure of the graded ring of the weak Jacobi forms with integral Fourier coefficients. It gives us a number of applications to the theory of elliptic genus and of the second quantized elliptic genus.

Introduction

For a compact complex manifold one can define its elliptic genus as a function in two complex variables. If the first Chern class $c_1(M)$ of the complex manifold is equal to zero in $H^2(M,\mathbb{R})$, then the elliptic genus is an automorphic form in variables $\tau \in \mathbb{H}_1$ ($\mathbb{H}_1$ is the upper-half plane) and $z \in \mathbb{C}$. More exactly, it is a Jacobi modular form with integral Fourier coefficients of weight 0 and index $d/2$, where $d = \dim_{\mathbb{C}}(M)$. The q^0-term of the Fourier expansion ($q = e^{2\pi i \tau}$) of the elliptic genus is essentially equal to the Hirzebruch χ_y-genus. Thus we can analyze the arithmetic properties of the χ_y-genus of the Calabi–Yau manifolds and its special values such as signature ($y = 1$) and Euler number ($y = -1$) in terms of Jacobi modular forms. The famous Rokhlin theorem about divisibility by 16 of the signature of a compact, oriented, differentiable spin manifold of dimension 4 was one of the starting points of the theory of elliptic genera. Ochanine generalized this Rokhlin result to the manifolds of $\dim_{\mathbb{R}} M \equiv 4 \mod 8$. One can find an elegant proof the Ochanine's theorem using modular forms in one variable with respect to $\Gamma_0(2)$ in the lectures of Hirzebruch [HBJ, Chapter 8].

In this paper we study the $\mathbb{Z}$-structure of the graded ring $J_{0,*}^{\mathbb{Z}} = \oplus_{m\ge 1} J_{0,m}^{\mathbb{Z}}$ of all weak Jacobi forms with integral Fourier coefficients (Theorem 1.9). We prove that this ring has four generators

$$J_{0,*}^{\mathbb{Z}} = \mathbb{Z}[\phi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4}]$$

where $\phi_{0,1}, \ldots, \phi_{0,4}$ are some fundamental Jacobi forms related to Calabi–Yau manifolds of dimension $d = 2, 3, 4, 8$. We consider some applications of this result to Calabi–Yau manifolds. Properties of the signature (i.e. the value of χ_y-genus at $y = 1$) modulo some powers of 2 are well known (see (2.8)). We analyze properties of the value of the elliptic

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genus and the Hirzebruch χ_y-genus at $y=-1$, $y=\zeta_3=e^{2\pi i/3}$ and $y=i$. For example, we prove that the Euler number of a Calabi-Yau manifold M_d of dimension d satisfies

$$e(M_d)\equiv 0 \mod 8 \qquad \text{if} \quad d\equiv 2 \mod 8$$

(see Proposition 2.4) and

$$\chi_{y=\zeta_3}(M_d)\equiv 0 \mod 9 \qquad \text{if} \quad d\equiv 2 \mod 6$$

(see Proposition 2.5 for details).

The same Jacobi forms $\phi_{0,1},\dots,\phi_{0,4}$ are generating functions for the multiplicities of all positive roots of the four generalized Lorentzian Kac–Moody Lie algebras of Borcherds type constructed in [GN1–GN4]. In §3 we show that the second quantized elliptic genus (introduced by R. Dijkgraaf, E. Verlinde and H. Verlinde in [DVV]) of an arbitrary Calabi–Yau manifolds of dimension 4 and 6 can be written as a product of some powers of the denominator functions of the Lorentzian Kac–Moody algebras constructed in [GN4].

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§1. Elliptic genus and weak Jacobi forms

Let M be a compact complex manifold of complex dimension d and T_M be the holomorphic tangent bundle of M. One defines the formal power series

$$\mathbb{E}_{q,y}=\bigotimes_{n\ge 0}\bigwedge\nolimits_{-y^{-1}q^n}T_M\otimes\bigotimes_{n\ge 1}\bigwedge\nolimits_{-yq^n}T_M^*\otimes\bigotimes_{n\ge 1}S_{q^n}(T_M\oplus T_M^*)$$

where $\wedge^k$ and S^k denote the k-th exterior product and k-th symmetric tensor product respectively and T_M^* denotes the dual bundle. The elliptic genus of the complex manifold is the holomorphic Euler characteristic of this formal power series with vector bundle coefficients. This definition is standard in physical literature (see, for example, [W], [EOTY], [KYY], [AYS], [D]).

Definition. The *elliptic genus* of M is defined as follows

$$\chi(M;\,\tau,z):=y^{\frac{d}{2}}\int_M \mathrm{ch}(\mathbb{E}_{q,y})\,\mathrm{td}(M) \tag{1.1}$$

where $\mathrm{td}(M)$ is the Todd class of M. One applies the Chern character in $\mathrm{ch}(\mathbb{E}_{q,y})$ to all coefficients of $\mathbb{E}_{q,y}$ and the integral $\int_M$ denotes the evaluation of the top degree differential form on the fundamental cycle of the manifold.

One can consider this definition as a limit case of the level N elliptic genus of Hirzebruch (see [H], [HBJ], [K], [Hö]). See also §3 for a physical definition. The coefficient $f(m,l)$ of the elliptic genus

$$\chi(M;\,\tau,z)=\sum_{n\ge 0,l} f(n,l)q^ny^l$$

is equal to the index of the Dirac operator (see, for example, [HBJ, Appendix II]) twisted with the vector bundle $E_{m,l-\frac{d}{2}}$, where $\mathbb{E}_{q,y} = \bigoplus_{m,\,n} E_{m,n}\cdot q^m y^n$. In particular all coefficients of the elliptic genus are integral. According to the Riemann–Roch–Hirzebruch theorem one can see that the q^0-term of $\chi(M;\,\tau,z)$ is essentially the Hirzebruch χ_y-genus of the manifold M:

$$\chi(M;\,\tau,z)=\sum_{p=0}^{d}(-1)^p\chi_p(M)\,y^{\frac{d}{2}-p}+ \tag{1.2}$$

$$q\Bigl(\sum_{p=-1}^{d+1}(-1)^p y^{-p}\bigl(\chi_p(M,T_M^*)-\chi_{p-1}(M,T_M^*)+\chi_p(M,T_M)-\chi_{p+1}(M,T_M)\bigr)\Bigr)+\dots$$

where $\chi(M,E)=\sum_{q=0}^{d}(-1)^q\dim H^q(M,E)$ and $\chi^p(M,E)=\chi(M,\wedge^pT_M^*\otimes E)$ or, for a Kähler manifold, $\chi_p(M)=\sum_q(-1)^q h^{p,q}(M)$.

It turns out that the elliptic genus of a Calabi–Yau d-fold is a weak Jacobi form of weight zero and index $d/2$. The Jacobi modular forms are modular forms with respect to the Jacobi group or in other words with respect to a parabolic subgroup of the integral symplectic group. In this paper we consider the Jacobi forms with respect to a maximal parabolic subgroup $\Gamma_\infty\subset Sp_2(\mathbb{Z})$, which consists of all element preserving an isotropic line. The Jacobi group $\Gamma^J:=\Gamma_\infty/\{\pm E_4\}\cong SL_2(\mathbb{Z})\ltimes H(\mathbb{Z})$. is the semi-direct product of $SL_2(\mathbb{Z})$ and the integral Heisenberg group $H(\mathbb{Z})$ which is the central extension of $\mathbb{Z}\times\mathbb{Z}$. The binary character $v_H([\lambda,\mu;\,\kappa]):=(-1)^{\lambda+\mu+\lambda\mu+\kappa}$ can be extended to a binary character v_J of the Jacobi group if we put $v_J|_{SL_2(\mathbb{Z})}=1$. In this paper we use Jacobi forms of integral or half integral index.

Definition. A holomorphic function $\phi(\tau,z)$ on $\mathbb{H}_1\times\mathbb{C}$ is called a *weak Jacobi form of weight* $k\in\mathbb{Z}/2$ *and index* $t\in\mathbb{Z}/2$ *with respect to the Jacobi group* Γ^J if the function $\widetilde{\phi}(Z):=\phi(\tau,z)\exp(2\pi i t\omega)$ on the Siegel upper half-plane $\mathbb{H}_2=\{Z=\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right)\in M_4(\mathbb{C}),\ \operatorname{Im}(Z)>0\}$ is a Γ^J-modular form of weight k with character (or multiplier system) v_J^{2t}, i.e., if it satisfies the functional equations

$$\phi(\frac{a\tau+b}{c\tau+d},\,\frac{z}{c\tau+d})=(c\tau+d)^k e^{2\pi i t c z^2/(c\tau+d)}\phi(\tau,z)\qquad (\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})) \tag{1.3a}$$

and

$$\phi(\tau,z+\lambda\tau+\mu)=(-1)^{2t(\lambda+\mu)}e^{-2\pi i t(\lambda^2\tau+2\lambda z)}\phi(\tau,z)\qquad(\lambda,\mu\in\mathbb{Z}). \tag{1.3b}$$

and it has the Fourier expansion of the type

$$\phi(\tau,z)=\sum_{n\ge 0,\ l\in t+\mathbb{Z}} f(n,l)\exp(2\pi i(n\tau+lz)).$$

We denote the space of all weak Jacobi forms of weight k and index t by $J_{k,t}$.

If the Fourier coefficients satisfy $f(n,l)=0$ unless $4tn-l^2\ge 0$, then ϕ is holomorphic at infinity. We call such Jacobi forms *holomorphic* Jacobi forms. Weak Jacobi forms of integral index were firstly defined in [EZ]. The main advantage of this notion is that $J_{k,t}$ is still finite dimensional and the graded ring of all weak Jacobi forms is finitely generated (over $\mathbb{C}$). In [EZ] only Jacobi forms of integral index were considered. In the next example we define the main week Jacobi forms of half-integral index (see Lemma 1.4 bellow).

Example 1.1. *Weak Jacobi forms $\phi_{0,\frac{3}{2}}$ and $\phi_{-1,\frac{1}{2}}$.* Let us define a Jacobi theta-series

$$\vartheta(\tau,z)=\sum_{n\equiv 1\,mod\,2}(-1)^{\frac{n-1}{2}}\exp\big(\frac{\pi i\,n^2}{4}\tau+\pi i\,nz\big)=\sum_{m\in\mathbb{Z}}\Big(\frac{-4}{m}\Big)q^{m^2/8}\,y^{m/2}$$
$$=-q^{1/8}y^{-1/2}\prod_{n\ge 1}(1-q^{n-1}y)(1-q^ny^{-1})(1-q^n) \tag{1.4}$$

where we use the formal variables $q=e^{2\pi i\tau}$, $y=e^{2\pi iz}$. This is a holomorphic Jacobi form of weight $1/2$ and index $1/2$ with multiplier system $v_\eta^3\times v_H$ (v_η is the multipier system of the Dedekind eta-function $\eta(\tau)$). One can check that

$$\phi_{0,\frac{3}{2}}(\tau,z)=\frac{\vartheta(\tau,2z)}{\vartheta(\tau,z)}=y^{-\frac{1}{2}}\prod_{n\ge 1}(1+q^{n-1}y)(1+q^ny^{-1})(1-q^{2n-1}y^2)(1-q^{2n-1}y^{-2})\in J_{0,\frac{3}{2}}$$
$$\phi_{-1,\frac{1}{2}}(\tau,z)=\frac{\vartheta(\tau,z)}{\eta(\tau)^3}=-y^{-1/2}\prod_{n\ge 1}(1-q^{n-1}y)(1-q^ny^{-1})(1-q^n)^{-2}\in J_{-1,\frac{1}{2}}$$

are weak Jacobi forms with integral Fourier coefficients.

The next proposition is well-known in physical literature (see [KMY], [AYS]). See also [Hö] where another normalization for the elliptic genus function was used.

Proposition 1.2. *If M_d is a compact complex manifold of dimension d with $c_1(M)=0$ (over $\mathbb{R}$), then its elliptic genus $\chi(M_d;\,q,y)$ is a weak Jacobi form of weight 0 and index $d/2$.*

Proof. For the convenience of the readers we give here a proof which is similar to the proof of the modular behavior of the level N elliptic genus (see [H] and [HBJ, Appendix III]). Let us represent $\chi(M;\tau,z)$ in terms of the theta-series. Let $c_i(T_M)\in H^{2i}(M,\mathbb{Z})$ be the Chern class of T_M and let $c(T_M)$ and $\mathrm{ch}(T_M)$ be the total Chern class and the Chern character of M and $x_i=2\pi i\xi_i$ ($1\le i\le d$) be the formal Chern parameters of T_M:

$$c(T_M)=\sum_{i=0}^{r}c_i(T_M)=\prod_{i=1}^{r}(1+x_i),\qquad \mathrm{ch}(T_M)=\sum_{i=1}^{r}e^{x_i}.$$

We recall that

$$\mathrm{ch}\,(\textstyle\bigwedge_t E)=\displaystyle\prod_{i=1}^{r}(1+te^{x_i}),\qquad \mathrm{ch}\,(S_tE)=\prod_{i=1}^{r}\frac{1}{1-te^{x_i}}.$$

Then we have

$$\mathrm{ch}\,(\mathbb{E}_{q,y})\,\mathrm{td}\,(T_M)=\prod_{i=1}^{d}\prod_{n=1}^{\infty}\frac{(1-q^{n-1}y^{-1}e^{-x_i})(1-q^nye^{x_i})}{(1-q^{n-1}e^{-x_i})(1-q^nye^{x_i})}x_i\,.$$

Therefore

$$\Phi(\tau,z,\xi_1,\dots,\xi_d)=y^{d/2}\mathrm{ch}\,(\mathbf{E}_{q,y})\mathrm{td}\,(T_M)=\prod_{i=1}^{d}\frac{\vartheta(\tau,z+\xi_i)}{\vartheta(\tau,\xi_i)}(2\pi i\xi_i)$$

is holomorphic function in τ, z and ξ_i. (We recall that $\vartheta(\tau, z)$ has zero of order 1 along $z = 0$.) For arbitrary $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ one has

$$\frac{\vartheta(\gamma < \tau >, \frac{z+\xi_i}{c\tau+d})}{\vartheta(\gamma < \tau >, \frac{\xi_i}{c\tau+d})} = e^{\pi i c(z^2+2z\xi_i)/(c\tau+d)} \frac{\vartheta(\tau,\ z+\xi_i)}{\vartheta(\tau, \xi_i)}.$$

Thus $\Phi(\tau, z, \xi_1, \dots, \xi_d)$ transforms like a Jacobi form of weight d (due to d factors ξ_i after the quotient of the theta-series) and index $\frac{d}{2}$ if $c_1(M) = x_1 + \dots x_d = 0$ (over $\mathbb{R}$!). Therefore the coefficient of arbitrary monom $\xi_1^{n_1} \cdot \dots \cdot \xi_d^{n_d}$ of total degree d in the Taylor expansion of $\Phi(\tau, z, \xi_1, \dots, \xi_d)$ with respect to ξ_i is a weak Jacobi form of weight 0 and index $\frac{d}{2}$. It follows that after integration over M one gets a Jacobi form of weight zero and index $d/2$. The claim that the elliptic genus does not contain Fourier coefficients with negative powers of q follows directly from the definition (1.1).

Remark. The definition (1.1) has sense and Proposition 1.2 is still true for arbitrary compact manifold with a $Spin^c$-structure. Moreover one can give a similar definition for an arbitrary complex vector bundle E with $c_1(E) = 0$. A generalization of this staff for vector bundles with $c_1(E) \neq 0$ see in [G2].

Example 1.3. *Elliptic genus of Enriques and $K3$ surfaces.* Let us consider the case of a complex surface M_2 with $c_1(M_2) = 0$. Then

$$\chi(M_2;\, \tau, z) \in J_{0,1} = \mathbb{C}\,\phi_{0,1}(\tau, z)$$

where

$$\phi_{0,1}(\tau, z) = (y + 10 + y^{-1}) + q(10y^{-2} - 64y^{-1} + 108 - 64y + 10y^2) + q^2(\dots) \tag{1.5}$$

is the unique, up to a constant, weak Jacobi form of weight 0 and index 1. The value of arbitrary weak Jacobi form of weight 0 at $z = 0$ is a holomorphic $SL_2(\mathbb{Z})$-modular form of weight 0, i.e. it is a constant. According (1.2)

$$\chi(M_d; \tau, 0) = \sum_{p=0}^{d} (-1)^p \chi_p(M) = e(M_d) \tag{1.6}$$

is the Euler number of M_d. Therefore we have

$$\chi(\text{Enriques};\, \tau, z) = \phi_{0,1}(\tau, z), \quad \chi(K3;\, \tau, z) = 2\phi_{0,1}(\tau, z)$$

for arbitrary Enriques and $K3$ surfaces.

The structure of the bigraded algebra over $\mathbb{C}$ of all Jacobi forms of integral index was determined in [EZ]. We describe the structure of the graded $\mathbb{Z}$-algebra of all Jacobi forms of weight zero with integral Fourier coefficients. Firstly we show how the weak Jacobi forms of integral and half-integral indices are related.

Lemma 1.4. *Let m be integral, then we have for the weak Jacobi forms*

$$J_{2k,m+\frac12}=\phi_{0,\frac32}\cdot J_{2k,m-1},\qquad J_{2k+1,m+\frac12}=\phi_{-1,\frac12}\cdot J_{2k+2,m}$$

where $\phi_{0,\frac32}$ and $\phi_{-1,\frac12}$ are defined in Example 1.1.

Proof. From the functional equation (1.3b) one has

$$\operatorname{div}(\phi_{2k,m+\frac12})\supset\{z\equiv\frac12,\frac{\tau}{2},\frac{\tau+1}{2}\mod \mathbb{Z}\tau+\mathbb{Z}\},\qquad \operatorname{div}(\phi_{2k+1,m+\frac12})\supset\{\mathbb{Z}\tau+\mathbb{Z}\}.$$

Thus any Jacobi form of half-integral index and even (resp. odd) weight is divisible by $\phi_{0,\frac32}$ (resp. $\phi_{-1,\frac12}$).

Example 1.5. *Calabi–Yau 3-folds and 5-folds.* Let $c_1(M_3)=0$ and $c_1(M_5)=0$. From Lemma 1.4 it follows that

$$J_{0,\frac32}=\mathbb{C}\phi_{0,\frac32},\qquad J_{0,\frac42}=\mathbb{C}\phi_{0,\frac33}\phi_{0,1}.$$

Thus

$$\chi(M_3;\,q,y)=\frac{e(M_3)}{2}\left[(y^{1/2}+y^{-1/2})+q(-y^{5/2}+y^{1/2}+y^{-1/2}-y^{-5/2})+q^2(\dots)\right],$$
$$\chi(M_5;\,q,y)=\frac{e(M_5)}{24}\left[(y^{\pm\frac32}+11y^{\pm\frac12})+q(-y^{\pm\frac72}+\dots)+q^2(-11y^{\pm\frac92}+\dots)\right]$$

($y^{\pm l}$ means that we have two summands with y^l and y^{-l} respectively). As a simple corollary we have that *the Euler characteristic of an arbitrary Calabi–Yau 5-fold is divisible by* 24 and

$$\chi_1(CY_5)=\frac{1}{24}\,e(CY_5),\qquad \chi_2(CY_5)=\frac{11}{24}\,e(CY_5).$$

In particular, for the Hodge numbers of an arbitrary strict Calabi–Yau 5-fold we have

$$11(h^{1,1}-h^{1,4})=h^{2,2}-h^{2,3}+10(h^{2,1}-h^{3,1}).$$

To calculate the elliptic genus of a Calabi–Yau manifold of even dimension we introduce some other basic Jacobi forms. Let us denote by $J^{\mathbb{Z}}_{0,m}$ the $\mathbb{Z}$-module of all weak Jacobi forms of weight 0 and index m with integral Fourier coefficients. We consider the graded ring

$$J^{\mathbb{Z}}_{0,*}=\bigoplus_{m\in\mathbb{Z}_{\ge0}}J^{\mathbb{Z}}_{0,m}$$

of all Jacobi forms of integral index with integral Fourier coefficients and its ideal

$$J^{\mathbb{Z}}_{0,*}(q)=\{\phi\in J^{\mathbb{Z}}_{0,*}\,|\,\phi(\tau,z)=\sum_{n\ge1,\ l\in\mathbb{Z}}a(n,l)q^ny^l\}$$

of the Jacobi forms without q^0-term.

Lemma 1.6. *The ideal $J_{0,*}^{\mathbb{Z}}(q)$ is principal. It is generated by a weak Jacobi form of weight* 0 *and index* 6

$$\xi_{0,6}(\tau,z)=\Delta(\tau)\phi_{-1,\frac12}(\tau,z)^{12}=\frac{\vartheta(\tau,z)^{12}}{\eta(\tau)^{12}}=q(y^{\frac12}-y^{-\frac12})^{12}+q^2(\dots).$$

Proof. Let $\psi\in J_{k,m}$ be an arbitrary Jacobi form. The product

$$\Psi(\tau,z)=\exp\bigl(-8\pi^2 m\,G_2(\tau)z^2\bigr)\psi(\tau,z)$$

transforms like a $SL_2(\mathbb{Z})$-modular form of weight k. We recall that $G_2(\tau)=-\frac{1}{24}+\sum_{n=1}^{\infty}\sigma_1(n)\,q^n$ is a quasi-modular form of weight 2, i.e. it satisfies

$$G_2(\gamma<\tau>)=(c\tau+d)^2G_2(\tau)-\frac{c(c\tau+d)}{4\pi i}.$$

Thus we have $\Psi(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d})=(c\tau+d)^k\Psi(\tau,z)$. Therefore the coefficients $f_n(\tau)$ in the Taylor expansion

$$\Psi(\tau,z)=\sum_{n\in\mathbb{Z}}f_n(\tau)z^n \tag{1.7}$$

are $SL_2(\mathbb{Z})$-modular forms of weight $k+n$. If $\psi\in J_{0,m}^{\mathbb{Z}}(q)$ has no q^0-term, then $f_{2n}(\tau)$ is a cusp form of weight $2n$. Thus $n\ge 6$ and $\psi(\tau,z)$ has zero of order at least 12 along $z=0$. Therefore $\psi(\tau,z)/\xi_{0,6}(\tau,z)$ is holomorphic on $\mathbb{H}_1\times\mathbb{C}$.

Corollary 1.7. *A weak Jacobi form of weight* 0 *is uniquely determined by its q^0-term if its index is less than* 6 *or equal to* $\frac{13}{2}$.

Example 1.8. *Calabi–Yau manifolds of dimension* 4, 6, 8 *and* 10. The last corollary shows us that for these values of dimension of a Calabi–Yau manifold M_d one can define the elliptic genus of M_d if one knows its Hirzebruch χ_y-genus. In [GN1–GN4] we used the following important Jacobi forms of small index

$$\begin{aligned}\phi_{0,2}(\tau,z)&=\tfrac12\eta(\tau)^{-4}\sum_{m\,,n\in\mathbb{Z}}(3m-n)\left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)q^{\frac{3m^2+n^2}{24}}y^{\frac{m+n}{2}}\\&=(y+4+y^{-1})+q(y^{\pm3}-8y^{\pm2}-y^{\pm1}+16)+q^2(\dots),\qquad(1.8)\\ \phi_{0,3}(\tau,z)&=\phi_{0,\frac32}^2(\tau,z)=(y+2+y^{-1})+q(-2y^{\pm3}-2y^{\pm2}+2y^{\pm1}+4)+q^2(\dots),\\ \phi_{0,4}(\tau,z)&=\frac{\vartheta(\tau,3z)}{\vartheta(\tau,z)}=(y+1+y^{-1})-q(y^{\pm4}+y^{\pm3}-y^{\pm1}-2)+q^2(\dots).\qquad(1.9)\end{aligned}$$

Using these Jacobi forms together with $\phi_{0,1}$ (see (1.5)) one can construct a basic of the module of the weak Jacobi forms of weight 0 and index 2, 3, 4 and 5 and to write a formula for the elliptic genus of M_d in terms of the cohomological invariants $\chi_p(M_d)$:

$$\psi_{0,2}^{(1)} = \phi_{0,2} = y + 4 + y^{-1} + q(\dots)$$
$$\psi_{0,2}^{(2)} = \phi_{0,1}^2 - 20\phi_{0,2} = (y^2 + 22 + y^{-2}) + q(\dots)$$

$$\begin{aligned} \psi_{0,3}^{(1)} &= \phi_{0,3} = \phi_{0,\frac{3}{2}}^2 = y + 2 + y^{-1} + q(\dots) \\ \psi_{0,3}^{(2)} &= \phi_{0,1}\phi_{0,2} - 15\phi_{0,3} = (y^{\pm 2} - y^{\pm 1} + 12) + q(\dots) \\ \psi_{0,3}^{(3)} &= \phi_{0,1}^3 - 30\phi_{0,1}\phi_{0,2} + 117\phi_{0,3} = \psi_{0,1}|T_-(3) - 3\phi_{0,1} = (y^{\pm 3} + 34) + q(\dots) \end{aligned} \tag{1.10}$$

$$\begin{aligned} \psi_{0,4}^{(1)} &= \phi_{0,4} = y + 1 + y^{-1} + q(\dots) \\ \psi_{0,4}^{(2)} &= \phi_{0,1}\phi_{0,3} - 12\phi_{0,4} = (y^{\pm 2} + 10) + q(\dots) \\ \psi_{0,4}^{(3)} &= \phi_{0,2}\psi_{0,2}^{(2)} - 4\psi_{0,4}^{(2)} - 24\phi_{0,4} = (y^{\pm 3} - y^{\pm 1} + 24) + q(\dots) \\ \psi_{0,4}^{(4)} &= \psi_{0,2}^{(2)}|T_-(2) - 2\phi_{0,4}^{(2)} = (y^{\pm 4} + 46) + q(\dots), \end{aligned} \tag{1.11}$$

where $T_-(m)$ is the standard Hecke operator $\big(\phi_{0,t}|T_-(m)\big)(\tau,z) = \sum_{n,l} f_m(n,l)q^n y^l$ with $f_m(N,L) = m\sum_{a|(N,L,m)} a^{-1} f(\frac{Nm}{a^2},\frac{L}{a})$.

Fourier coefficient $f(n,l)$ of a weak Jacobi form of weight 0 and index t depends only on the norm $4nt - l^2$ of its index and $\pm l \mod 2t$. For every Jacobi form written above its q^0-term contains all "orbits" of non-zero Fourier coefficients with negative norm $4nt - l^2$ of its index. This fact will be important in §3 bellow. For $d = 10$ the situation is a little bit more complicated. After some calculation we obtain the following Jacobi forms

$$\begin{aligned} \psi_{0,5}^{(1)} &= 5y + 2 + 5y^{-1} + q(-y^5 + \dots) \\ \psi_{0,5}^{(2)} &= (y^{\pm 2} + y^{\pm 1} + 8) + q(y^5 + \dots) \\ \psi_{0,5}^{(3)} &= (y^{\pm 3} + y^{\pm 1} + 20) + q(\dots) \\ \psi_{0,5}^{(4)} &= (y^{\pm 4} - y^{\pm 1} + 36) + q(-3y^5 + \dots) \\ \psi_{0,5}^{(5)} &= (y^{\pm 5} + 58) + q(y^{\pm 5} + \dots) \end{aligned} \tag{1.12}$$

where we include in the formulae all "orbits" of non-zero Fourier coefficients with negative norm $20n - l^2$. One knows (see [EZ]) that $\dim J_{0,m} = m$ for $m = 1,\ 2,\ 3,\ 4$ and 5. Thus the functions $\psi_{0,m}^{(p)}$ constructed above ($1 \le p \le m$) form a basis of $J_{0,m}^{\mathbb{Z}}$. Thus for $d = 2,\ 4,\ 6,\ 8,\ 10$ we can represent the elliptic genus of a compact complex manifold M_d with $c_1(M_d) = 0$ as a linear combination

$$\chi(M_d;\tau,z) = \sum_{p=0}^{\frac{d}{2}-2} \chi_p(M_d)\psi_{0,\frac{d}{2}}^{(\frac{d}{2}-p)}(\tau,z) + c\psi_{0,\frac{d}{2}}^{(1)}(\tau,z)$$

with $c \in \mathbb{Z}$. In particular, the coefficients χ_p of the Hirzebruch y-genus of the manifold M_d satisfy the following relations

$$\begin{aligned} &\chi_2 = 22\chi_0 - 4\chi_1 && \text{and} && e(M_4) \equiv 0 \mod 6 \text{ (for CY}_4) \\ &\chi_3 = 34\chi_0 - 14\chi_1 + 2\chi_2 && \text{and} && e(M_6) \equiv 0 \mod 4 \text{ (for CY}_6) \\ &\chi_4 = 46\chi_0 - 25\chi_1 + 10\chi_2 - \chi_3 && \text{and} && e(M_8) \equiv 0 \mod 3 \text{ (for CY}_8). \end{aligned} \tag{1.13}$$

For a Calabi–Yau 10-fold we have

$$\chi_5 = 58\chi_0 - 36\chi_1 + 20\chi_2 - 8\chi_3 + \frac{2}{5}(\chi_4 + \chi_3 - \chi_2 - \chi_1).$$

We note that for dimension $d \ge 12$ ($d \ne 13$) an additional term $c\xi_{0,6}$ could appear in the formula for the elliptic genus. The odd dimensions $d = 7$, 9, 11 and 13 are related to the even dimensions considered above by the factor $\phi_{0,3/2}$. For example,

$$\chi(M_7; \tau, z) = \chi_2(M_7)\phi_{0,\frac{3}{2}}\phi_{0,2} - \chi_1(M_7)\phi_{0,\frac{3}{2}}\psi_{0,2}^{(2)}, \qquad e(M_7) = 12(\chi_2 - 4\chi_1) \equiv 0 \mod 12.$$

These examples show us that information about Jacobi forms provides some information about Calabi–Yau manifolds. In the next theorem we construct a basis of the module $J_{0,m}^{\mathbb{Z}}$ and we find generators of the graded ring $J_{0,*}$.

Theorem 1.9. 1. *Let m be a positive integer. The module*

$$J_{0,m}^{\mathbb{Z}}/J_{0,m}^{\mathbb{Z}}(q) = \mathbb{Z}\,[\psi_{0,m}^{(1)}, \dots, \psi_{0,m}^{(m)}]$$

is a free $\mathbb{Z}$-module of rank m. Moreover there is a basis consists of $\psi_{0,m}^{(n)}$ ($1 \le n \le m$) with the following q^0-term

$$\begin{aligned}
[\psi_{0,m}^{(n)}]_{q^0} &= y^n + \dots + y^{-n} && (3 \le n \le m)\\
[\psi_{0,m}^{(2)}]_{q^0} &= y^2 - 4y + 6 - 4y^{-1} + y^{-2}\\
[\psi_{0,m}^{(1)}]_{q^0} &= \frac{1}{(12,m)}\big(my + (12-2m) + my^{-1}\big)
\end{aligned}$$

where $(12, m)$ is the greatest common divisor of 12 and m.

2. *The graded ring of all weak Jacobi forms of weight 0 with integral coefficients is finitely generated*

$$J_{0,*}^{\mathbb{Z}} = \bigoplus_m J_{0,m}^{\mathbb{Z}} = \mathbb{Z}\,[\phi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4}]$$

where the corresponding Jacobi forms are defined in (1.8)–(1.9). *$\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3}$ are algebraicly independent and*

$$4\phi_{0,4} = \phi_{0,1}\phi_{0,3} - \phi_{0,2}^2.$$

To prove Theorem 1.9 we need

Lemma 1.10. *For arbitrary weak Jacobi form $\phi_{0,m} = \sum_{n,l} f(n,l)q^n y^l$ the following identities are valid*

$$m\sum_l f(0,l) = 6\sum_l l^2 f(0,l), \qquad 24m\sum_l f(0,l) = \sum_n (m - 6n^2) f(1,n).$$

Proof. We consider the Taylor expansion (1.7)

$$\exp\big(-8\pi^2 m\, G_2(\tau)z^2\big)\phi_{0,m}(\tau,z) = f_0 + \sum_{n\ge 1} f_n(\tau)z^n$$

where $f_n(\tau)$ is a $SL_2(\mathbb{Z})$-modular form of weight n. Thus $f_2(\tau) \equiv 0$. The left hand side of the identities of the lemma are equal to the first two coefficients of $f_2(\tau)$. This proves the lemma. We remark that one can continue the list of similar identities calculating the second, third, ... coefficients of $f_2(\tau)$.

Using (1.2) we get

$$\frac{e(M)d}{12} = \sum_p (-1)^p \chi^p(M)(\frac{d}{2} - p)^2 \tag{1.14}$$

if $c_1(M) = 0$ over $\mathbb{R}$.

Remark. Identity (1.14) was rediscovered several times in the mathematical and physical literature. See, for example, [LW], [S] for a proof based on the Riemann–Roch–Hirzebruch formula and [AYS] where (1.14) is related to the sum rule for the charges in $N = 2$ superconformal field theory. Lemma 1.11 provides us with an automorphic proof of (1.14) (see [GN3, Lemma 2.2] for a more general statement).

Proof of Theorem 1.9. For a weak Jacobi form the polynom $[\phi]_{q^0}$ is of order not greater than m (see [EZ]). It can not be a constant according to the first identity of Lemma 1.10. Moreover if $[\phi]_{q^0}$ is "linear", i.e. $[\phi]_{q^0} = ay + b + ay^{-1}$, then a is divisible by $\frac{m}{(m,12)}$. It follows that m Jacobi forms of type dicribed in the first statement of the theorem, if they would exist, form a basis of the module $J^{\mathbb{Z}}_{0,m}/J^{\mathbb{Z}}_{0,m}(q)$.

Let us construct a basis by induction. For $m = 1, 2, 3, 4, 6, 8, 12$ we set $\psi^{(1)}_{0,m} = \phi_{0,m}$ where $\phi_{0,1}, \ldots, \phi_{0,4}$ were defined above and

$$\begin{aligned}
\phi_{0,6}(\tau, z) &= \phi_{0,2}\phi_{0,4} - \phi_{0,3}^2 = (y + y^{-1}) + q(\ldots),\\
\phi_{0,8}(\tau, z) &= \phi_{0,2}\phi_{0,6} - \phi_{0,4}^2 = (2y - 1 + 2y^{-1}) + q(\ldots),\\
\phi_{0,12}(\tau, z) &= \phi_{0,4}\phi_{0,8} - 2\phi_{0,6}^2 = (y - 1 + y^{-1}) + q(\ldots).
\end{aligned}$$

Now we can define the weak Jacobi forms $\psi^{(1)}_{0,m}$ using the following procedure. (We omit the weight 0 bellow to simplify the notation). Let $m \ge 5$. We set

$$\psi_{m,I} = \widetilde{\psi}_{m-4}\phi_4 + \widetilde{\psi}_{m-2}\phi_2 - 2\widetilde{\psi}_{m-3}\phi_3 = my^{\pm 1} + (12 - 2m) + q(\ldots)$$

where by definition $\widetilde{\psi}_m := (12, m)\psi^{(1)}_m$. We can take $\psi_{m,I}$ as $\psi^{(1)}_m$ if m is coprime with 12. If $m \equiv 0 \mod 2$ and $m \ge 6$ we can define a form $\psi_{m,II} = \frac{1}{2}\psi_{m,I}$ with integral Fourier coefficients. If $m \equiv 0 \mod 3$ and $m \ge 9$ we define

$$\psi_{m,III} = \frac{2}{3}\widetilde{\psi}_{m-3}\phi_3 + \frac{1}{3}\widetilde{\psi}_{m-6}\phi_6 - \widetilde{\psi}_{m-4}\phi_4 = \frac{m}{3}y^{\pm 1} + (4 - \frac{2m}{3}) + q(\ldots).$$

For $m \equiv 0 \mod 4$ and $m > 12$ we take

$$\psi_{m,IV} = \frac{1}{4}[\widetilde{\psi}_{m-12}\phi_{12} + \widetilde{\psi}_{m-4}\phi_4 - \widetilde{\psi}_{m-8}\phi_8] = \frac{m}{4}y^{\pm 1} + (3 - \frac{m}{2}) + q(\ldots).$$

In the remain cases $m \equiv 0 \mod 6$ and $m \equiv 0 \mod 12$ $(m > 12)$ we put

$$\begin{aligned}
\psi_{m,VI} &= \frac{1}{2}\psi_{m,III} = \frac{m}{6}y^{\pm 1} + (2 - \frac{m}{3}) + q(\ldots),\\
\psi_{m,XII} &= \psi_{m,III} - \psi_{m,IV} = \frac{m}{12}y^{\pm 1} + \frac{6 - m}{6} + q(\ldots).
\end{aligned}$$

Thus we finish the construction of functions $\psi_m^{(1)}$ if we put $\psi_m^{(1)} := \psi_{m,D}$ for $(m, 12) = D$.

Next we construct $\psi_m^{(2)}$. We put

$$\psi_{0,m}^{(2)} = \widetilde{\psi}_{0,m-3}\phi_{0,3} - \widetilde{\psi}_{0,m-4}\phi_{0,4} - \widetilde{\psi}_{0,m} \qquad (m \ge 5)$$

and

$$\psi_{0,2}^{(2)} = \phi_{0,1}^2 - 24\phi_{0,2}, \quad \psi_{0,3}^{(2)} = \phi_{0,1}\phi_{0,2} - 18\phi_{0,3}, \quad \psi_{0,4}^{(2)} = \phi_{0,1}\phi_{0,3} - 16\phi_{0,4}.$$

One can see that these forms have the q^0-term equal to $y^2 - 4y + 6 - 4y^{-1} + y^{-2}$. For $3 \le n \le m-2$ one can use $\phi_{0,3}$ and the forms $\phi_{0,m-3}^{(n-1)}$ in order to construct $\phi_{0,m}^{(n)}$. To finish the proof of the first statement of the theorem we put

$$\phi_{0,m}^{(m-1)} = \phi_{0,1}^{m-2} \cdot \phi_{0,2}, \qquad \phi_{0,m}^{(m)} = \phi_{0,1}^m.$$

We note that each function of the basis $\{\psi_m^{(n)}\}_{n=1}^m$ of the module $J_{0,m}^{\mathbb{Z}}/J_{0,m}^{\mathbb{Z}}(q)$ constructed above is a polynom in the basic Jacobi forms $\phi_{0,1}, \dots, \phi_{0,4}$. The torsion relation $4\phi_{0,4} = \phi_{0,1}\phi_{0,3} - \phi_{0,2}^2$ follows from Corollary 1.7. The form $\xi_{0,6}$ generating the principle ideal $J_{0,*}^{\mathbb{Z}}(q)$ is also a polynom in $\phi_{0,1}, \dots, \phi_{0,4}$:

$$\xi_{0,6} = -\phi_{0,1}^2\phi_{0,4} + 9\phi_{0,1}\phi_{0,2}\phi_{0,3} - 8\phi_{0,2}^3 - 27\phi_{0,3}^2. \tag{1.15}$$

(To prove the last formula one needs to check that the q-constant term of the write hand side is zero and to compair coefficients at the first power q.)

Let us prove that $\phi_{0,1}$, $\phi_{0,2}$ and $\phi_{0,3}$ are algebraicly independent. For this end we consider its values at $z = \frac{1}{2}$. We have

$$\phi_{0,2}(\tau, \frac{1}{2}) \equiv 2, \quad \phi_{0,3}(\tau, \frac{1}{2}) \equiv 0, \quad \phi_{0,4}(\tau, \frac{1}{2}) \equiv -1.$$

(The two last identities follow from definition and the first one is a corollary of the torsion relation.) The restriction of

$$\phi_{0,1}(\tau, \frac{1}{2}) = \alpha(\tau) = 8 + 2^8 q + 2^{11}q^2 + 11 \cdot 2^{10}q^3 + 3 \cdot 2^{14}q^4 + 359 \cdot 2^9 q^5 + \dots \tag{1.16}$$

is a modular function with respect to $\Gamma_0(2)$ with a character of order 2 (see §2). The square of this function is, up to factor 2^{12}, the "Hauptmodul" for the congruence subgroup group $\Gamma_0(2)$. Since only one function obtained from the Jacobi forms of different indices is a non-constant function for $z = 1/2$, then they are algebraicly independent.

§2. Special values of the elliptic genus

In this section we analyze the value of the elliptic genus at the following special points $z = 0$ (Euler number), $z = \frac{1}{2}$ (signature), $z = \frac{\tau+1}{2}$ ($\hat{A}$-genus) and $z = \frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$. For this end we have to study the restriction of the main generators of the graded ring of the integral week Jacobi forms. A special value of a Jacobi form is a modular form in τ. In the next lemma we give a little more precise statement than in [EZ, Theorem 1.3].

Lemma 2.1. *Let $\phi \in J_{0,t}$ $(t \in \mathbb{Z}/2)$ and $X = (\lambda, \mu) \in \mathbb{Q}^2$. Then*

$$\phi|_X(\tau, 0) = \phi(\tau, \lambda\tau + \mu)\exp(2\pi i t(\lambda^2\tau + \lambda\mu))$$

is an automorphic form of weight 0 *with a character with respect to the subgroup*

$$\Gamma_X = \{M \in SL_2(\mathbb{Z}) \mid XM - X \in \mathbb{Z}^2\}.$$

Let us consider some particular examples which we need in this section.

Example 2.2. Let $X = (0, \frac{1}{N})$. Then

$$\Gamma_X = \Gamma_0^{(1)}(N) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid M \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

If t is integer, then the character of the function

$$\phi|_X(\tau, 0) = \phi(\tau, \frac{1}{N})$$

is given by $v(M) = \exp(-2\pi i t c/N^2)$ and it has has order $\frac{N}{(t,N)}$.

It is easy to see that if $\phi \in J_{k,m}^{\mathbb{Z}}$ with integral m, then the form $\phi(\tau, \frac{1}{N})$ still has integral Fourier coefficients if $N = 1, \ldots, 6$. In particular, the value of $\xi_6(\tau, z)$ at these points is related to the "Hauptmodule" for the corresponding group $\Gamma_0(N)$:

$$\begin{aligned} \xi_6(\tau, \frac{1}{2}) &= 2^{12}\frac{\Delta(2\tau)}{\Delta(\tau)}, & \xi_6(\tau, \frac{1}{4}) &= 2^6\left(\frac{\Delta(4\tau)}{\Delta(\tau)}\right)^{1/2}, \\ \xi_6(\tau, \frac{1}{3}) &= 3^6\left(\frac{\Delta(3\tau)}{\Delta(\tau)}\right)^{1/2}, & \xi_6(\tau, \frac{1}{6}) &= \left(\frac{\Delta(\tau)\Delta(6\tau)}{\Delta(2\tau)\Delta(3\tau)}\right)^{1/2}. \end{aligned} \tag{2.1}$$

Let us analyze the corresponding values of the four generators $\phi_{0,n}$ of the graded ring $J_{0,*}^{\mathbb{Z}}$. From the definition (see (1.8)–(1.9)) and the identity $4\phi_{0,4} = \phi_{0,1}\phi_{0,3} - \phi_{0,2}^2$ we obtain

$$\phi_{0,1}(\tau, 0) = 12, \quad \phi_{0,2}(\tau, 0) = 6, \quad \phi_{0,3}(\tau, 0) = 4, \quad \phi_{0,4}(\tau, 0) = 3 \tag{2.2}$$

and

$$\begin{aligned} &\phi_{0,1}(\tau, \frac{1}{2}) = \alpha(\tau) && \phi_{0,1}(\tau, \frac{1}{3}) = \beta^2(\tau) && \phi_{0,1}(\tau, \frac{1}{4}) = \frac{\gamma(\tau)^4 + 4}{\gamma(\tau)} \\ &\phi_{0,2}(\tau, \frac{1}{2}) = 2 && \phi_{0,2}(\tau, \frac{1}{3}) = \beta(\tau) && \phi_{0,2}(\tau, \frac{1}{4}) = 4\gamma^2(\tau) \\ &\phi_{0,3}(\tau, \frac{1}{2}) = 0 && \phi_{0,3}(\tau, \frac{1}{3}) = 1 && \phi_{0,3}(\tau, \frac{1}{4}) = 2\gamma(\tau) \\ &\phi_{0,4}(\tau, \frac{1}{2}) = -1 && \phi_{0,4}(\tau, \frac{1}{3}) = 0 && \phi_{0,4}(\tau, \frac{1}{4}) = 1. \end{aligned} \tag{2.3}$$

The automorphic functions $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$ are automorphic forms of weight 0 with respect to the group $\Gamma_0(2)$, $\Gamma_0^{(1)}(3)$ and $\Gamma_0^{(1)}(4)$ respectively. These functions have integral

Fourier coefficients. The identity (1.15) gives us the following relations for the modular forms α, β and γ

$$2^{12}\frac{\Delta(2\tau)}{\Delta(\tau)}=\alpha(\tau)^2-64,\qquad 3^6\left(\frac{\Delta(3\tau)}{\Delta(\tau)}\right)^{1/2}=\beta(\tau)^3-27$$
$$2^6\left(\frac{\Delta(4\tau)}{\Delta(\tau)}\right)^{1/2}=4\Big(\left(\frac{\gamma(\tau)}{2}\right)^2-\left(\frac{2}{\gamma(\tau)}\right)^2\Big).$$

It follows that

$$\alpha(\tau)-8\equiv 0 \mod 2^8,\qquad \beta(\tau)-3\equiv 0 \mod 3^3 \tag{2.4}$$

(compare with (1.16)). Using the definition of $\phi_{0,3}$ and $\gamma(\tau)$ and the relations between the Jacobi theta-series ϑ_{ab} of level 2 we have

$$\gamma(\tau)=\frac{\vartheta_{00}(2\tau)}{\vartheta_{01}(2\tau)}=\frac{\vartheta_{00}(2\tau,0)}{\vartheta_{01}(2\tau,0)}. \tag{2.5}$$

Using Corollary 1.7 we check that

$$\phi_{0,1}(\tau,2z)=\phi_{0,2}^2(\tau,z)-8\phi_{0,4}(\tau,z).$$

Thus

$$\alpha(\tau)=16\gamma(\tau)^4-8=16\frac{\vartheta_{00}^4(2\tau)}{\vartheta_{01}^4(2\tau)}-8. \tag{2.6}$$

In particular *all Fourier coefficients of* $\gamma(\tau)$ *and* $\alpha(\tau)$ *are positive.*

In connection with (2.5)–(2.6) we note that one can write the generators of the graded ring as symmetric polynomials in $\xi_{ab}(\tau,z)=\vartheta_{ab}(\tau,z)/\vartheta_{ab}(\tau,0)$:

$$\phi_{0,1}=4(\xi_{00}^2+\xi_{10}^2+\xi_{01}^2),\qquad \phi_{0,\frac{3}{2}}=4\xi_{00}\xi_{10}\xi_{01}$$
$$\phi_{0,2}=2\big((\xi_{00}\xi_{10})^2+(\xi_{00}\xi_{01})^2+(\xi_{10}\xi_{01})^2\big). \tag{2.7}$$

To check these formulae one can use Corollary 1.7 and the fact that the generators of modular group transform ξ_{ab} to each other.

Example 2.3. Let $X=(\frac{1}{N},\frac{1}{N})$. Then Γ_X contains the principle congruence subgroup $\Gamma_1(N)$. In some cases Γ_X will be strictly larger. For example, if $X_2=(\frac{1}{2},\frac{1}{2})$, then

$$\phi|_{X_2}(\tau,0)=\phi(\tau,\frac{\tau+1}{2})\exp(\frac{\pi i}{2}(\tau+1))$$

is an automorphic form with respect of the so-called theta-group

$$\Gamma_\theta=\left\{M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})\mid M\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\ \text{or}\ \begin{pmatrix}0&1\\1&0\end{pmatrix}\mod 2\right\}.$$

The corresponding character is given by $\epsilon_2(M)=\exp(2\pi i m(d+b-a-c)/4)=\pm 1$. This character is trivial if the index m of Jacobi form is even and it has order 2 if the index is odd. Let us consider Γ_θ-automorphic function

$$\hat\phi_m(\tau)=q^{-\frac{m}{4}}\phi_{0,m}(\tau,-\frac{\tau+1}{2}).$$

We have

$$\hat\phi_3=0,\quad \hat\phi_4=-1,\quad \hat\phi_2=-2,\quad \hat\xi_6=\hat\phi_1^2+64=\left(\frac{\vartheta_{00}}{\eta}\right)^{12}$$

where

$$\hat\phi_1(\tau)=4\frac{\vartheta_{10}^4-\vartheta_{01}^4}{\vartheta_{01}^2\vartheta_{10}^2}=-q^{-\frac14}+20q^{\frac14}+\dots\in\mathfrak{M}_0^{\mathbb{Z}}(\Gamma_\theta,\epsilon_2).$$

Let us analyze some special values of the elliptic genus. As it easy follows from (1.2) we get Euler number and signature for $z=0$ (d is arbitrary) and $z=\frac12$ (d is even)

$$\chi(M_d,\tau,0)=e(M_d),$$
$$\chi(M_d,\tau,\frac12)=\sigma_M(\tau)=(-1)^{\frac d2}s(M_d)+q(\dots)\in\mathfrak{M}_0^{\mathbb{Z}}(\Gamma_0(2),v_2),\quad v_2(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right))=e^{\pi i m\frac c2}.$$

In §1 we calculated Euler number $e(M_d)$ for small d in terms of Jacobi forms. It gave us some divisibility of Euler number of Calai–Yau manifolds. We note that the quotient $e(M)/24$ appears in physics as obstruction to cancelling the tadpole (see [SVW] where it was proved that $e(M_4)\equiv 0 \mod 6$).

Proposition 2.4. *Let M_d be an almost complex manifold of complex dimension d such that $c_1(M)=0$ in $H^2(M,\mathbb{R})$. Then*

$$d\cdot e(M_d)\equiv 0\mod 24.$$

If $c_1(M)=0$ in $H^2(M,\mathbb{Z})$, then we have a more strong congruence

$$e(M)\equiv 0\mod 8\qquad \textit{if }\ d\equiv 2\mod 8.$$

Proof. The first fact follows simply from (2.2) or from (1.14). If $d\equiv 2\mod 8$ one can write the elliptic genus as a polynom over $\mathbb{Z}$ in the generators $\phi_{0,*}$

$$e(M_d)\equiv P(\phi_{0,1},\phi_{0,2},\phi_{0,3},\phi_{0,4})|_{z=0}\equiv c_{1,m}(\phi_{0,1}|_{z=0})(\phi_{0,4}|_{z=0})^{\frac{d-2}{8}}\mod 8.$$

If one put $z=-\frac{\tau+1}{2}$, i.e., $y=-q^{1/2}$ (see Example 2.3), then one obtain that the series

$$\mathbb{E}_{q,-q^{1/2}}=\bigotimes_{n\ge1}\bigwedge\nolimits_{q^{n/2}}T_M\otimes\bigotimes_{n\ge1}\bigwedge\nolimits_{q^{n/2}}T_M^*\otimes\bigotimes_{n\ge1}S_{q^n}(T_M\oplus T_M^*)$$

is $*$-symmetric. According to the Serre duality all Fourier coefficients of $\hat\chi(M_d,\tau)$ are even. The constant $c_{1,m}$ from the last congruence is equal to the coefficient of $\hat\chi(M_d,\tau)$ at the minimal negative power of q. Therefore $c_{1,m}$ is even and we obtain divisibility of $e(M_{8m+2})$ by 8.

We note that divisibility of $d \cdot e(M)$ by 3 was proved by F. Hirzebruch in 1960. For a hyper-Kähler compact manifold the claim of the proposition above was proved by S. Salamon in [S]. After my talk on the elliptic genus at a seminar of MPI in Bonn in April 1997 Professor F. Hirzebruch informed me that the result of Proposition 2.4 was known for him (non-published). Using some natural examples he also proved that this property of divisibility of the Euler number modulo 24 is strict (see [H2]).

Formulae (2.3) provide us with a formula for the signature $\chi(M_d; \tau, \frac{1}{2})$ as a polynom in $\alpha(\tau)$. As a corollary of (2.3) and Theorem 1.9 we have that for an arbitrary Jacobi form of integral index

$$\begin{aligned} &\phi_{0,4m}(\tau, \frac{1}{2}) = c + 2^{13}q(\dots) \qquad &&\phi_{0,4m+1}(\tau, \frac{1}{2}) = 8c + 2^{8}q(\dots) \\ &\phi_{0,4m+2}(\tau, \frac{1}{2}) = 2c + 2^{12}q(\dots) \qquad &&\phi_{0,4m+3}(\tau, \frac{1}{2}) = 16c + 2^{9}q(\dots). \end{aligned} \tag{2.8}$$

Similar to the proof of Proposition 2.4 we obtain a better congruence for the signature of a manifold with dim$\equiv 2 \mod 8$ and $c_1(M) = 0$:

$$\chi(M_{8m+2}; \tau, z) = 16c + 2^{9}q(\dots). \tag{2.9}$$

This gives us another proof of the Oshanine's theorem in this particular case.

This is interesting that the values of the Hirzebruch y-genus at $y = e^{2\pi i/3}$ and $y = i$ also have some properties of divisibility. For $z = \frac{1}{3}$ (resp. $z = \frac{1}{4}$) we can write $\phi_{0,m}(\tau, \frac{1}{3})$ (resp. $\phi_{0,m}(\tau, \frac{1}{4})$) as a polynom in $\beta(\tau) = 3 + 27(q + \dots)$ (resp. in $\gamma(\tau)^{\pm 1}$). This gives us the following results

$$\begin{gathered} \phi_{0,3m}(\tau, \frac{1}{3}) = c + 3^{6}q(\dots), \qquad \phi_{0,3m+1}(\tau, \frac{1}{3}) = 9c + 3^{4}q(\dots) \\ \phi_{0,3m+2}(\tau, \frac{1}{3}) = 3c + 3^{3}q(\dots). \end{gathered} \tag{2.10}$$

Thus we have

Proposition 2.5. *If $c_1(M) = 0$ (over $\mathbb{R}$), then*

$$\begin{gathered} \chi(M_{6m}; \tau, \frac{1}{3}) = c_1 \mod 3^{6}, \qquad \chi(M_{6m+2}; \tau, \frac{1}{3}) = 9c_2 \mod 3^{4}, \\ \chi(M_{6m+4}; \tau, \frac{1}{3}) = 3c_3 \mod 3^{3}. \end{gathered}$$

We finish with some relations for $z = \frac{1}{4}$:

$$\begin{gathered} \chi(M_{8m+2}; \tau, \frac{1}{4}) = 4c + 2^{4}q(\dots), \qquad \phi_{0,4m+2}(\tau, \frac{1}{4}) = 4c + 2^{5}q(\dots) \\ \phi_{0,4m+3}(\tau, \frac{1}{4}) = 2c + 2^{8}q(\dots). \end{gathered} \tag{2.11}$$

§3. Second quantized elliptic genus (SQEG) of Calabi–Yau manifolds

3.1. SQEG of Calabi–Yau manifolds and Siegel modular forms. The notion of elliptic genus of $N=2$ super-symmetric theories was introduced more than 10 years ago (see, for example, [W1], [W2], [EOTY], [AYS], [KYY]). In physics the elliptic genus of a Calabi–Yau manifold M_d is defined as the genus one partition function of the super-symmetric sigma model whose target space is M_d. By definition this is the trace of an operator over the Ramond-Ramond sector of the sigma model

$$\chi(M_d;\,\tau,z)=\mathrm{Tr}_{\mathcal{H}}\,(-1)^F q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}y^{F_L}$$

where $q^{L_0-\frac{c}{24}}$ is the evolution operator, F_L is the fermion number and L_0 ($\overline{L_0}$) is Virasoro generator of of left (right) movers, $(-1)^F$ is the fermion parity operator in $N=2$ super-symmetric theory and c is the central charge.

We can consider n-fold symmetric product of the manifold M, i.e., the orbifold space $S^nM=M^n/S_n$, where S_n is the symmetric group of n elements. This is a singular manifold but one can define the *orbifold* elliptic genus of S^nM (see for details the talk of R. Dijkgraaf at ICM-1998 in Berlin [D]). It gives us a string version of the elliptic genus of the symmetric product of a Calabi–Yau manifold. One can compare this construction with the definition of the string Euler number of symmetric products (see [HH]). Using some arguments from the conformal field theory on orbifolds it was proved in [DVV] and [DMVV] that the string elliptic genus of the second quantization $\cup_{n\ge 1}S^nM$ of a Calabi–Yau manifold M coincides with the second quantized elliptic genus of the given manifold:

$$\sum_{n=0}^{\infty}p^n\chi_{orb}(S^nM;\,q,y)=\prod_{m\ge 0,\,l,\,n>0}\frac{1}{(1-q^my^lp^n)^{f(mn,l)}} \tag{3.1}$$

where

$$\chi(M,\tau,z)=\sum_{m\ge 0,\ l\in\mathbb{Z}\ (or\,\mathbb{Z}/2)}f(m,l)q^my^l$$

is the elliptic genus of M.

For a $K3$ surface, the product in the left hand side of (3.1) is essentially the power -2 of the infinite product expansion of the product of all even theta-constants (see [GN1])

$$\Delta_5(Z)^2=2^{-12}\prod_{{}^tab=0\,mod\,2}\Theta_{a,b}(Z)^2=(qpr)\prod_{(m,l,n)>0}(1-q^my^lp^n)^{f(mn,l)}$$

where

$$Z=\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix}\in\mathbb{H}_2,\quad q=e^{2\pi i\tau},\ y=e^{2\pi ir},\ p=e^{2\pi i\omega}$$

belongs to the Siegel upper half-plane, $\Theta_{a,b}(Z)$ are the even Siegel theta-constants of level 2 and $f(m,l)$ is the Fourier coefficient of $2\phi_{0,1}(\tau,z)$ (see Example 1.3). The modular form $\Delta_5(Z)$ is the first cusp form with respect to the Siegel modular group $Sp_4(\mathbb{Z})$ (with a non-trivial binary character). In [DVV] (3.1) was conjectured as a generalization of a formula for the orbifold Euler numbers

$$\sum_{n\ge 1}e_{orb}(S^nM)\,p^n=\prod_{n>0}(1-p^n)^{-24}=\prod_{m\ge 0,\,l,\,n>0}(1-q^my^lp^n)^{-f(mn,l)}|_{y=1}.$$

In the case of an algebraic $K3$ surface $e_{orb}(S^nM)$ is the topological Euler characteristic of the Hilbert scheme $M^{(n)}$ of zero dimension subschemas of length n (see [Gö]).

Following [DVV, §4] we call the product

$$\Psi(M;\,Z)=\prod_{m\ge 0,\,l,\,n>0}(1-q^my^lp^n)^{-f(mn,l)},\tag{3.2}$$

where $f(m,l)$ are coefficients of the elliptic genus of M, the *second-quantized elliptic genus* (SQEG) of the manifold M.

In this section we prove that the SQEG multiplied by a factor (*Hodge anomaly*) depending only on invariants $\chi_p(M)$ of the Calabi–Yau d-fold

$$E(CY_d;\,Z)=\text{Hodge anomaly}(CY_d;\,Z)\cdot\Psi(CY_d;\,Z)$$

is a Siegel automorphic form with respect to the paramodular group of polarization $(1,d)$ (resp. $(1,2d)$) for even (resp. odd) dimension d. We also calculate $E(CY_d;\,Z)$ for small d in terms of the basic Siegel modular forms for the corresponding paramodular groups. For small dimensions $d=2,\ 3,\ 4,\ 6$ the function $E(CY_d;\,Z)$ is product of some powers of the denominator function of generalized Lorentzian Kac–Moody Lie algebras constructed in [GN1–GN4].

We considered in [GN1]–[GN4] the following infinite product for an arbitrary nearly holomorphic Jacobi form $\phi_{0,t}(\tau,z)=\sum_{n,l}c(n,l)q^ny^l$ of weight 0 and integral index t

$$\text{Exp-Lift}(\phi_{0,t})(Z)=q^Ay^Bs^C\prod_{\substack{n,l,m\in\mathbb{Z}\\(n,l,m)>0}}(1-q^ny^ls^{tm})^{c(nm,l)},\tag{3.3}$$

where

$$A=\frac{1}{24}\sum_l f(0,l),\quad B=\frac12\sum_{l>0}lf(0,l),\quad C=\frac14\sum_l l^2f(0,l)$$

and $(n,l,m)>0$ means that if $m>0$, then l and n are arbitrary integers, if $m=0$, then $n>0$ and $l\in\mathbb{Z}$ or $l<0$ if $n=m=0$.

In order to define the corresponding product for Calabi–Yau manifolds of odd dimension it is useful to modify the definition of SQEG. We remark that for arbitrary $\phi_{0,\frac d2}(\tau,z)\in J_{0,\frac d2}$ the Jacobi form $\phi_{0,\frac d2}(\tau,2z)\in J_{0,2d}$ has even index. To see this one can apply symplectic transformation $\Lambda_2=\operatorname{diag}(1,2,1,2^{-1})$ to the corresponding Γ^J-modular form:

$$(F|\Lambda_2)\left(\begin{smallmatrix}\tau&z\\z&\omega\end{smallmatrix}\right)=F\begin{pmatrix}\tau&2z\\2z&4\omega\end{pmatrix}.$$

The next theorem is a particular case of Theorem 2.1 from [GN4].

Theorem 3.1. *Let $M=M_d$ be a compact complex manifold of dimension d with trivial $c_1(M)$,*

$$\chi(M;\,\tau,z)=\sum_{m\ge 0,\ l\in\mathbb{Z}\ (or\ \mathbb{Z}/2)}f(m,l)q^my^l$$

be its elliptic genus and $\mathrm{SQEG}(M;\,Z)$ *($Z\in\mathbb{H}_2$) be its second quantized elliptic genus* (3.3)*. We define a factor*

$$H(M;\,Z)=\begin{cases}\eta(\tau)^{-\frac12(e-3\chi'_{d_0})}\prod_{p=1}^{d_0}\bigl(\vartheta(\tau,pz)\,e^{\pi i p^2\omega}\bigr)^{-\chi'_{d_0-p}} & \text{if } d=2d_0\\ \eta(\tau)^{-\frac12 e}\prod_{p=1}^{d_0}\bigl(\vartheta(\tau,\frac{2p-1}{2}z)\,e^{\frac14\pi i(2p-1)^2\omega}\bigr)^{-\chi'_{d_0-p+1}} & \text{if } d=2d_0+1\end{cases}$$

where $e=e(M)$ is Euler number of M and $\chi'_p=(-1)^p\chi_p(M)$ (see (1.2)*). Then the product*

$$E(M;\,Z)=\Psi(M;\,Z)\cdot\mathrm{SQEG}(M;\,Z)\qquad (d=2d_0)$$
$$E^{(2)}(M;\,Z)=(E|\Lambda_2)(M;\,Z)\qquad (d=2d_0+1)$$

determines a Siegel automorphic form of weight $-\frac12\chi'_{d_0}(M)$ if d is even and of of weight 0 if d is odd with a character or a multiplier system of order $24/(24,e)$ with respect to a double extension of the paramodular group Γ_d^+ (resp. Γ_{2d}^+), if d is even (resp. d is odd). The divisor of $E(M;\,Z)$ (resp. of $E^{(2)}(M;\,Z)$) on $\mathcal{A}_d^+=\Gamma_d\setminus\mathbb{H}_2$ (resp. $\mathcal{A}_{2d}^+$) is the union of a finite number of Humbert modular surfaces $H_D(b)$ of discriminant $D=b^2-4ad$ (resp. $D=4b^2-8ad$) with multiplicities

$$m_{D,b}=-\sum_{n>0}f(n^2a,nb).$$

Remark. We call the factor $H(M;\,Z)$ defined above Hodge anomaly of SQEG. The divisor of $\vartheta(\tau,pz)$ is $pz\in\tau\mathbb{Z}+\mathbb{Z}$. Thus the zeros and poles of $H(M;\,Z)$ (but not of $E(M;\,Z)$!) are completely defined by $\chi_p(M)$.

We would like to consider applications of this result to Calabi–Yau manifolds of dimension 3, 5 and 2, 4, … , 10. Before doing this we recall the definitions of the notions we used above. The paramodular group Γ_t is isomorphic to the integral symplectic group of the skew-symmetric form with elementary divisors $(1,t)$. It can be realized as a subgroup of $Sp_4(\mathbb{Q})$

$$\Gamma_t:=\left\{\begin{pmatrix} * & t* & * & * \\ * & * & * & t^{-1}* \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix}\in Sp_4(\mathbb{Q})\mid \text{ all } * \text{ are integral}\right\}.$$

If $t\neq 1$, the group Γ_t is not a maximal discrete subgroup and it has normal extensions (see, for example, [GH1]). By definition,

$$\Gamma_t^+=\Gamma_t\cup\Gamma_tV_t,\qquad V_t=\frac{1}{\sqrt t}\begin{pmatrix}0&t&0&0\\1&0&0&0\\0&0&0&1\\0&0&t&0\end{pmatrix}.$$

The double quotient

$$\mathcal{A}_t\xrightarrow{2:1}\mathcal{A}_t^+=\Gamma_t^+\setminus\mathbb{H}_2$$

of $\mathcal{A}_t$ can be interpreted as a moduli space of lattice-polarized $K3$ surfaces for arbitrary t or as the moduli space of Kummer surfaces of $(1,p)$-polarized Abelian surfaces for a prime

$t = p$ (see [GH1, Theorem 1.5]). Any Humbert surface in $\mathcal{A}_t^+$ of discriminant D can be represented in the form

$$H_D^+(b) = \pi_t^+ \Bigl(\bigcup_{g\in\Gamma_t^+} g^*(\{Z \in \mathbb{H}_2 \mid a\tau + bz + t\omega = 0\}) \Bigr)$$

where $a, b \in \mathbb{Z}$, $D = b^2 - 4ta$, $0 \le b < 2t$ and $\pi_t^+ : \mathbb{H}_2 \to \mathcal{A}_t^+$ is the natural projection.

According (3.3) the infinite product $E(M;\, Z)$ can be written as follows

$$E(M_d;\, Z) = \begin{cases} \text{Exp-Lift}(-\chi(M_d;\, \tau, z)) & \text{if } d \text{ is even} \\ \text{Exp-Lift}(-\chi(M;\, \tau, 2z)) & \text{if } d \text{ is odd.} \end{cases}$$

We note that for odd d the product $E(M;\, Z)$ (without Λ_2-modification) is an automorphic function with respect to the group $\Lambda_2\Gamma_{2d}\Lambda_2^{-1}$ conjugated to the paramodular group.

3.2. SQEG of Calabi–Yau manifolds of even dimension and Lorentzian Kac–Moody algebras.

1. The case of $\mathbf{CY_2}$. One of the starting points for considerations in [DVV] was the infinite product expansion formula for the modular form $\Delta_5(Z)$. According to Example 1.3 we have for the SQEG of surfaces with trivial c_1

$$\begin{aligned} E(\mathit{Enriques\ surface};\, Z) &= \text{Exp-Lift}(-\phi_{0,1})(Z) = \Delta_5(Z)^{-1}, \\ E(K3;\, Z) &= \text{Exp-Lift}(-2\phi_{0,1})(Z) = \Delta_5(Z)^{-2}. \end{aligned}$$

In [GN1] we proved that the modular form $\Delta_5(Z)$ defines an automorphic correction of the Kac–Moody algebra of hyperbolic type with the Cartan matrix

$$A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

It means that the Fourier expansion of the modular form $\Delta_5(Z)$ determines a generalized Kac–Moody Lie super-algebra with a system of the real simple roots of type $A_{1,II}$. We would like to note that the Lorentzian Kac–Moody algebras with Cartan matrix of types $A_{1,0}$ and $A_{1,I}$ constructed in [GN2] are related to SQEG of *some vector bundles of rank* 2 *over a manifold of dimension* 14 (see [G2]).

2. The case of $\mathbf{CY_4}$. The basic Jacobi modular forms for this dimension are the Jacobi forms $\phi_{0,2}$ and $\psi_{0,2}^{(2)}$ (see (1.8) and (1.10)). They correspond to the following cusp forms for the paramodular group Γ_2 (see [GN1] and [GN4]):

$$\begin{aligned} \Delta_2(Z) &= \text{Exp-Lift}(\phi_{0,2}(\tau, z)) = \text{Lift}(\eta(\tau)^3\vartheta(\tau, z)) \\ &= \sum_{N\ge 1} \sum_{\substack{n,\, m>0,\, l\in\mathbb{Z} \\ n,\, m\equiv 1 \bmod 4 \\ 2nm-l^2=N^2}} N\Bigl(\frac{-4}{Nl}\Bigr) \sum_{a \mid (n,l,m)} \Bigl(\frac{-4}{a}\Bigr) q^{n/4}y^{l/2}s^{m/2} \in \mathfrak{M}_2^{cusp}(\Gamma_2, v_\eta^6 \times v_H) \end{aligned}$$

and

$$\Delta_{11}(Z) = \operatorname{Lift}(\eta(\tau)^{21}\vartheta(\tau, 2z)) = \operatorname{Exp-Lift}(\psi_{0,2}^{(2)}(\tau, z)) \in \mathfrak{N}_{11}(\Gamma_2).$$

For an arbitrary Calabi–Yau 4-fold M_4 we have the following formula for its SQEG

$$E(M_4;\, Z) = \Delta_{11}(Z)^{-\chi_0(M)}\Delta_2(Z)^{\chi_1(M)}. \tag{3.4}$$

Its divisor is equal to $(\chi_1(M) - \chi_0(M))H_1 - \chi_0 H_4$. We note that $\Delta_2(Z)^4$ is the first Γ_2-cusp form with trivial character and $\Delta_{11}(Z)$ is the first cusp form of odd weight with respect to Γ_2. Thus, if the Euler number of M_4 is divisible by 24, then the automorphic form $E(M_4;\, Z)$ has trivial character. The Fourier expansion of the cusp forms $\Delta_2(Z)$, $\Delta_{11}(Z)$ and $\frac{\Delta_{11}(Z)}{\Delta_2(Z)}$ coincide with the Weyl–Kac–Borcherds denominator formula of generalized Kac–Moody super-algebras with generalized Cartan matrix $A_{1,II}$, $A_{2,II}$ and $A_{2,0}$ respectively:

$$A_{2,II} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix}, \quad A_{2,0} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}, \quad A_{2,I} = \begin{pmatrix} 2 & -2 & -4 & 0 \\ -2 & 2 & 0 & -4 \\ -4 & 0 & 2 & -2 \\ 0 & -4 & -2 & 2 \end{pmatrix}$$

(see [GN1]–[GN4]). Thus, the formula (3.4) gives us three particular cases of Calabi–Yau 4-folds of Kac–Moody type when the second quantized elliptic genus is a power of the denominator function of the corresponding Lorentzian Kac–Moody algebra:

$$\begin{aligned} E(M_4;\, Z) &= \Delta_{11}(Z)^{-\chi_0} && \text{if } \chi_1 = 0 \\ E(M_4;\, Z) &= \left(\frac{\Delta_{11}(Z)}{\Delta_2(Z)}\right)^{-\chi_0} && \text{if } \chi_0(M) = -\chi_1(M) \\ E(M_4;\, Z) &= \Delta_2(Z)^{\chi_1} && \text{if } \chi_0(M) = 0. \end{aligned}$$

At this point we would like to discuss some relations of Hecke type between the basic Siegel modular forms occurred in SQEG-construction. It would be interesting to find a geometric or a physical interpretation of such relations. We recall that the Hecke operators $T_-(m)$ on the space of Jacobi forms of weight zero play the main role in the construction of the Borcherds products (3.3) (see [B2], [GN1], [GN3]) and in the proof of the identity (3.1) for the second quantized elliptic genus (see [DMVV]). We start with a relation

$$\psi_{0,2}^{(2)} = \phi_{0,1}|T_-(2) - 2\phi_{0,2}.$$

Thus, according to [GN4, Theorem 3.3] we can represent $\Delta_{11}(Z)$ in the following form

$$\Delta_{11}(Z) = \Delta_5(Z)\Delta_5(\begin{pmatrix} \tau & 2z \\ 2z & 4\omega \end{pmatrix})\Delta_5(\begin{pmatrix} \tau & z \\ z & \omega+\frac{1}{2} \end{pmatrix})\Delta_2(Z)^{-2}.$$

Moreover we can write Δ_{11}^2 only in terms of the cusp form Δ_2 since

$$2\psi_{0,2}^{(2)} = \phi_{0,2}|T_0(2),$$

where $T_0(2)$ is another Hecke operator which does not change the index of Jacobi forms. If $f(n, l) = g(8n - l^2)$ is the Fourier coefficient of $\phi_{0,2}$ (for a prime index t Fourier coefficients depend only on the norm $4tn-l^2$), then the Fourier coefficient $f_2(n,l)=g_2(8n-l^2)$ of $\phi_{0,2}|T_0(2)$ is given by the formula

$$g_2(N)=8g(4N)+2\left(\frac{-N}{2}\right)g(N)+g\left(\frac{N}{4}\right).$$

According [GN2, Theorem A.7], $\Delta_{11}(Z)^2=[\Delta_2(Z)]_{T(2)}/\Delta_2(Z)^4$, where

$$[\Delta_2(Z)]_{T(2)}=\prod_{a,b,c\, mod\, 2}\Delta_2(\tfrac{z_1+a}{2},\tfrac{z_2+b}{2},\tfrac{z_3+c}{2})\prod_{a\, mod\, 2}\Delta_2(\tfrac{z_1+a}{2},z_2,2z_3)\,\Delta_2(2z_1,z_2,\tfrac{z_3+a}{2})$$
$$\times\Delta_2(2z_1,2z_2,2z_3)\prod_{b\, mod\, 2}\Delta_2(2z_1,-z_1+z_2,\tfrac{z_1-2z_2+z_3+b}{2}).$$

3. The case of $\mathbf{CY_6}$. In this case the elliptic genus of a Calabi–Yau 6-fold is a sum of three basic Jacobi forms $\psi_{0,3}^{(i)}$ (see (1.11)). Thus there are three basic Siegel modular forms

$$\begin{aligned}\Delta_1(Z)&=\text{Exp-Lift}(\psi_{0,3}^{(1)})=\text{Lift}(\eta(\tau)\vartheta(\tau,z))\\&=\sum_{M\ge 1}\ \sum_{\substack{m>0,\, l\in\mathbb{Z}\\ n,\, m\equiv 1\, mod\, 6\\ 4nm-3l^2=M^2}}\left(\frac{-4}{l}\right)\left(\frac{12}{M}\right)\sum_{a|(n,l,m)}\left(\frac{6}{a}\right)q^{n/6}y^{l/2}s^{m/2},\end{aligned}$$

$$D_6(Z)=\text{Exp-Lift}(\psi_{0,3}^{(2)})=\text{Lift}\left(\eta(\tau)^{12}\phi_{0,\frac{3}{2}}\right),\quad \Delta_{17}(Z)=\text{Exp-Lift}(\psi_{0,3}^{(3)})$$

with divisors H_1, H_4 and H_1+H_9 respectively. (These Humbert modular surfaces have only one component in $\mathcal{A}_6^+$). The Fourier expansion of the cusp forms Δ_1, D_6 and $\Delta_7(Z)=\Delta_1(Z)D_6(Z)$ determine three generalized Kac–Moody Lie super-algebras with the following Cartan matrix of the real simple roots:

$$A_{3,II}=\begin{pmatrix}2&-2&-10&-14&-10&-2\\-2&2&-2&-10&-14&-10\\-10&-2&2&-2&-10&-14\\-14&-10&-2&2&-2&-10\\-10&-14&-10&-2&2&-2\\-2&-10&-14&-10&-2&2\end{pmatrix},\ A_{3,I}=\begin{pmatrix}2&-2&-5&-1\\-2&2&-1&-5\\-5&-1&2&-2\\-1&-5&-2&2\end{pmatrix},\ A_{3,0}=\begin{pmatrix}2&-2&-2\\-2&2&-1\\-2&-1&2\end{pmatrix}.$$

It follows that SQEG of an arbitrary Calabi–Yau 6-fold can be represented as

$$E(M_6,Z)=\Delta_{17}(Z)^{-\chi_0}\Delta_7(Z)^{\chi_1}\Delta_1(Z)^{-\chi_2}.$$

Thus there are three types of manifolds of Kac–Moody type: ($\chi_0=0$, $\chi_1=0$) or $\chi_2=-\chi_1$ or $\chi_2=0$. Similar to Δ_{11} the cusp form Δ_{17} can be represented using a Hecke type product of Δ_5 (see [GN4, Theorem 3.3]):

$$\Delta_{17}(Z)=\left[\Delta_5(\left(\begin{smallmatrix}\tau&3z\\3z&9\omega\end{smallmatrix}\right))\prod_{b\, mod\, 3}\Delta_5(\left(\begin{smallmatrix}\tau&z\\z&\omega+\frac{b}{3}\end{smallmatrix}\right))\right]\Delta_1(Z)^{-3}.$$

4. The case of $\mathbf{CY_8}$ and $\mathbf{CY_{10}}$. In the case $d = 8$ we have four basic weak Jacobi forms of index 4 (see (1.9)). The exponential lifting of $\phi_{0,4}(\tau, z)$ is the "most odd" even Siegel theta-constant

$$\text{Exp-Lift}(\phi_{0,4})(Z) = \Delta_{1/2}(Z) = \frac{1}{2}\sum_{n,\,m\in\mathbb{Z}} \left(\frac{-4}{n}\right)\left(\frac{-4}{m}\right) q^{n^2/8} y^{nm/4} s^{m^2/8}$$

which is the denominator function of a generalized Lorentzian Kac–Moody super-algebra with a real root system of *parabolic type* (see [GN2–GN3]). For arbitrary Calabi–Yau 8-fold we have (see (1.11)):

$$E(CY_8;\, Z) = \Delta_{1/2}(Z)^{\chi_1-\chi_3} \cdot \Delta_5(\left(\begin{smallmatrix} \tau & 2z \\ 2z & 4\omega \end{smallmatrix}\right))^{-\chi_2} \cdot \Delta_{12}(Z)^{\chi_1} \cdot \Delta_{23}(Z)^{-\chi_0},$$

where

$$\Delta_{12}(Z) = \text{Exp-Lift}(\psi_{0,4}^{(3)}) \in \mathfrak{M}_{12}(\Gamma_4^+), \quad \Delta_{23}(Z) = \text{Exp-Lift}(\psi_{0,4}^{(4)}) \in \mathfrak{M}_{23}(\Gamma_4^+)$$

are Γ_4^+-modular forms with trivial character. We remark that all forms above have multiplicities one along divisors! From the formulae above it follows that for $d = 2,\ 4,\ 6,\ 8$ the divisor of $E(M_d; Z)$ is determined by the divisor of Hodge anomaly of M_d. For $d = 10$ this is not the case. From (1.12) we see that for $d = 10$ one new divisor appears. This is the Humbert surface $H_5 = \pi_5^+\{\tau + 5z + 5\omega = 0\} \subset \mathcal{A}_5^+$. The second quantized elliptic genus $E(M_{11}; Z)$ *is anti-invariant with respect to the involution defined by this rational quadratic divisor.*

We remark also that for $d < 12$ (or $d = 13$) we have

$$\begin{aligned} E(CY_d;\, Z) &= E(CY_d^{mir};\, Z)^{-1} && \text{(if } d \text{ is odd)}\\ E(CY_d;\, Z) &= E(CY_d^{mir};\, Z) && \text{(if } d \text{ is even)} \end{aligned}$$

where CY_d^{mir} is the mirror partner of Calabi–Yau d-fold.

3.3. SQEG of Calabi–Yau 3-folds and 5-folds. The elliptic genus of a Calabi–Yau 3-folds M_3 is defined uniquely up to a constant (see Example 1.5):

$$\chi(M_3, \tau, z) = \frac{1}{2} e(M_3)\phi_{0,\frac{3}{2}}(\tau, z).$$

According (3.3) and Theorem 3.1 the product

$$\Phi_3(Z) = \text{Exp-Lift}(\phi_{0,\frac{3}{2}}(\tau, 2z)) = \\ q^{\frac{1}{12}} y^{\frac{1}{2}} s^{\frac{1}{2}} \prod_{\substack{n,l,m\in\mathbb{Z}\\ (n,l,m)>0}} (1 - q^n y^l s^{tm})^{c(nm,l)} \in \mathfrak{M}_0(\Gamma_6^+, v_{12}) \qquad (3.5)$$

is an automorphic function of weight 0 with respect to Γ_6^+ with a character of order 12. (This character is induced by the square v_η^2 of the multiplier system of the Dedekind eta-function $\eta(\tau)$ and by the binary character v_H of the Heisenberg group defined in §1.) Moreover

$$\operatorname{div}_{\mathcal{A}_6^+}(\Phi_3(Z)) = H_1(0) - H_1(5) = \pi_6^+(\{Z \in \mathbb{H}_2 \mid z = 0\}) - \pi_6^+(\{Z \in \mathbb{H}_2 \mid \tau + 5z + 6\omega = 0\}).$$

The divisor $H_1(5)$ appears because the Fourier expansion of $\phi_{0,\frac{3}{2}}(\tau, 2z)$ contains the term $-qy^5$. We remark that $\Phi_3(Z)$ is anti-invariant with respect to the reflection in the vectors which define the Humbert surfaces $H_1(0)$ and $H_1(5)$. The character of the automorphic function $\Phi_3(Z)$ has the maximal possible order 12 since $\Gamma_6^+/(\Gamma_6^+)^{com} = \mathbb{Z}/2 \times \mathbb{Z}/12$ (see [GH2, Theorem 2.1]). One can say that $\Phi_3(Z)$ *is a Siegel modular function with the simplest possible divisor*. Such a function should have a zero and pole inside $\mathbb{H}_2$. The simplest divisor is a rational quadratic divisor (a Humbert surface) with discriminant one. The case $t = 6$ is the first case when there are two divisors of discriminant one. $\Phi_3(Z)$ has zero and poles of order one along these surfaces. Thus *we can consider* $\Phi_3(Z)$ *as the best analogue of the modular invariant* $j(\tau)$ *in the case of Siegel modular functions!*

It is interesting that $\Phi_3(Z)$ is related to Siegel modular threefolds of geometric genus 1 and 2. More exactly, the square of $\Phi_3(Z)$ can be written as quotient of two Γ_6^+-cusp forms constructed in [GN2, Example 4.6]

$$\Phi_3(Z)^2 = \frac{\operatorname{Lift}\big(\eta(\tau)^3\vartheta(\tau,z)^2\vartheta(\tau,2z)\big)}{\operatorname{Lift}(\eta(\tau)^5\vartheta(\tau,2z))} = \frac{\text{Exp-Lift}(3\phi_{0,3}^2 - 2\phi_{0,2}\phi_{0,4})}{\text{Exp-Lift}(5\phi_{0,3}^2 - 4\phi_{0,2}\phi_{0,4})}.$$

The modular form in the numerator $F_1(Z) = \operatorname{Lift}\big(\eta(\tau)^3\vartheta(\tau,z)^2\vartheta(\tau,2z)\big) \in \mathfrak{N}_3(\Gamma_6, v_2)$ is cusp form of weight 3 with divisor $3H_1(1) + 2H_1(5) + H_4$. Let us introduce the subgroup $\Gamma_6(v_2) = \operatorname{Ker}(v_2) \subset \Gamma_6$ of index 2 (v_2 is character of order 2) and the covering

$$\mathcal{A}_6(v_2) = \Gamma_6(v_2) \setminus \mathbb{H}_2 \xrightarrow{2:1} \mathcal{A}_6.$$

It is known that $h^{3,0}(\mathcal{A}_6(v_2)) = 1$ and the cusp form

$$F_1(Z) d\tau \wedge dz \wedge d\omega \in H^{3,0}(\mathcal{A}_6^o(v_2), \mathbb{C})$$

defines the unique canonical differential form on a smooth model of $\mathcal{A}_6(v_2)$ (see [GH2, Theorem 3.1]). The form in the numerator $F_2(Z) = \operatorname{Lift}(\eta(\tau)^5\vartheta(\tau,2z)) \in \mathfrak{N}_3(\Gamma_6, v_3)$ is a cusp form of weight 3 with a character of order 3. The Siegel threefold

$$\mathcal{A}_6(v_3) = \Gamma_6(v_3) \setminus \mathbb{H}_2 \xrightarrow{3:1} \mathcal{A}_6$$

has $h^{3,0} = 2$ and $F_2(Z)$ defines one of the canonical differential forms on it.

Let us consider an arbitrary Calabi–Yau 3-fold M_3 with Hodge numbers $h^{1,1}$ and $h^{2,1}$. Its modified SQEG is the modular function

$$E^{(2)}(M_3;\, Z) = \Phi_3(Z)^{h^{2,1} - h^{1,1}}.$$

In particular $\Phi_3(Z)$ is the modified SQEG of a Calabi–Yau 3-fold with Euler number -2. The first divisor $(h^{2,1} - h^{2,2})H_1(0)$ of $E^{(2)}(M_3;\, Z)$ comes from the Hodge anomaly

$H(M_3;\, Z)$, the second $(h^{2,2}-h^{2,1})H_1(5)$ is the additional divisor which is, in some sense, "mirror symmetric" to the first one, because if M_3^{mir} is a mirror partner of M_3, then

$$E^{(2)}(M_3^{mir};\, Z)=E^{(2)}(M_3;\, Z)^{-1}.$$

It turns out (see [GN5]) that *the functions $\Phi_3(Z)^{\pm 1}$ determine Lorentzian Kac–Moody super-algebras with an infinite system of real simple roots of the third possible type.* This is the so-called *hyperbolic type* when the infinite system of real simple roots has a "limit" line (see [N]). The Lorentzian Kac–Moody algebras related to $\mathrm{SQEG}(CY_d)$ ($d=2,\,4,\,6,\,8$) have elliptic type (the system of real simple roots is finite) or parabolic type (the system of real simple roots has geometry similar to the real simple roots of the fake monster Lie algebra constructed by Borcherds in [B1]).

In the case $d=5$ the elliptic genus is again defined uniquely by the Euler number. The basic Jacobi form is $\phi_{0,\frac52}=\phi_{0,\frac32}\phi_{0,1}$ (see Example 1.5). We have that

$$\Phi_5(Z)=\text{Exp-Lift}(\phi_{0,\frac52}(\tau,2z))\in\mathfrak{M}_0(\Gamma_{10}^+)$$

is an automorphic function of weight 0 with respect to Γ_{10}^+ with trivial character. The automorphic function $\Phi_5(Z)$ has the divisor consisting of four irreducible components

$$\mathrm{div}_{\mathcal{A}_{10}^+}\big(\Phi(Z)\big)=H_9(3)-H_9(7)+12H_1(1)-12H_1(9).$$

For arbitrary Calabi–Yau fivefold M_5 with Euler number $\chi(M_5)$ we have

$$E^{(2)}(M_5;\, Z)=\Phi_5(Z)^{-e(M_5)/24}.$$

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