ON K_3 OF WITT VECTORS OF LENGTH TWO OVER FINITE FIELDS

Thomas Geisser

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
GERMANY

MPI/95-129

ON K_3 OF WITT VECTORS OF LENGTH TWO OVER FINITE FIELDS

THOMAS GEISSER

ABSTRACT. We prove that for $W_2(\mathbb{F}_q)$ the Witt vectors of length two over the finite field $\mathbb{F}_q$, we have $K_3(W_2(\mathbb{F}_{p^f})) = (\mathbb{Z}/p^2)^f \oplus \mathbb{Z}/(p^{2f}-1)$ in characteristic at least 5 and $K_3(W_2(\mathbb{F}_{3^f})) = (\mathbb{Z}/9)^{f-1} \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/(3^{2f}-1)$ for $(3,f)=1$. The result is proved by using the identity $K_3(W_2(\mathbb{F}_q)) = H_3(SL(W_2(\mathbb{F}_q)))$ and calculating the right term with a group homology spectral sequence. Some information on the spectral sequence is achieved by using the action of the outer automorphism of SL on the homology groups and recent results on K-groups of local rings and the ring of dual numbers over finite fields.

1. INTRODUCTION

Some of the higher algebraic K-groups which can be explicitly calculated are the groups $K_i(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$ for $\mathcal{O}_\mathfrak{p}$ a local field with prime $\mathfrak{p}$. The prime-to-p part is given by the prime-to-p part of $K_i(\mathcal{O}_\mathfrak{p}/\mathfrak{p})$ by Suslin [21]. The groups $K_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$ have been calculated by Dennis and Stein [5]. In the totally ramified case, the groups $K_i(\mathbb{F}_q[t]/t^2)$ have been determined by Hesselholt and Madsen [10] and in the unramified case, Evens and Friedlander [7] proved $K_3(\mathbb{Z}/p^2)_p = \mathbb{Z}/p^2$ for $p \geq 5$. In this paper we extend this result in two ways. The main theorem is, see 6.2, 7.2:

Theorem 1.1. *a) Let $p \geq 5$ then*

$$K_3(W_2(\mathbb{F}_{p^f})) = (\mathbb{Z}/p^2)^f \oplus \mathbb{Z}/(p^{2f}-1).$$

b) Let $(3,f)=1$ then

$$K_3(W_2(\mathbb{F}_{3^f})) = (\mathbb{Z}/9)^{f-1} \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/(3^{2f}-1).$$

The characteristic 3 case is of particular interest. It is known that $\pi_3(\operatorname{im} J)_3$, the homotopy group of the image of the J homomorphism, gives a direct summand $\mathbb{Z}/3$ of $K_3(\mathbb{Z}) = \mathbb{Z}/48$ and of $K_3(\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3 \oplus \mathbb{Z}/3$. On the other hand one knows by Panin [15] that $K_3(\mathbb{Z}_3, \mathbb{Z}/3) = \varprojlim K_3(\mathbb{Z}/3^n, \mathbb{Z}/3)$. So the question arises at which

Key words and phrases. Higher algebraic K-theory, Hochschild Serre spectral sequence, Charlap-Vasquez theory, group cohomology, linear groups, local rings.

* supported by Deutsche Forschungsgemeinschaft

level the image of J occurs for the first time in the inverse system. The above theorem says that it arises at the earliest possible level $n = 2$.
The proof of the theorem uses the identity

$$\mathrm{K}_3(W_2(\mathbb{F}_q)) = H_3(SL(W_2(\mathbb{F}_q))).$$

The right hand term is then calculated as in [7], [1], [14] and [19], using the Hochschild Serre spectral sequence to the extension

$$0 \to V \to SL(W_2(\mathbb{F}_q)) \to SL(\mathbb{F}_q) \to 0.$$

Some E^2-terms in this spectral sequence have been calculated by Lluis-Puebla [14] and Friedlander and Parshall [8]. We need the following additional results.
On the one hand, a main lemma 4.2 tells us that the map $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) \to \mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^2)$ is surjective. This gives an upper bound on the number of generators of $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^2)$, because the groups $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p)$ have been calculated by Levine [13] and Bökstedt and Madsen [2].
On the other hand, we use the action of the outer automorphism of SL on the terms of the spectral sequence to show that some differentials vanish. Using the calculation of $\mathrm{K}_3(\mathbb{F}_q[t]/t^2)$ of [10], which admits a spectral sequence with the same E_2-terms, this suffices to calculate $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^2)$ in characteristic at least 5.
In characteristic 3 we have to calculate an explicit differential in the spectral sequence. This takes the second half of the paper and follows ideas of [7].
Notation: $\mathbb{F}_q$ denotes the field with $q = p^f$ elements, $W_n(R)$ the Witt vectors of length n over R and $W(R)$ all Witt vectors. For a group V, V^* denotes the dual group $\mathrm{Hom}(V, \mathbb{Q}/\mathbb{Z})$ and V_p the p-part of V. $V_n(\mathbb{F}_q)$ are the $n \times n$-matrices of trace zero over $\mathbb{F}_q$, and $V(\mathbb{F}_q)$ is the direct limit of the $V_n(\mathbb{F}_q)$. We will sometimes write V if the field in question is clear from the context. For an R-module V over the ring R, $\Lambda^n_R V$ is the n-th exterior power and $S^n_R V$ the n-th symmetric power.
I would like to thank the following people for many helpful conversations and their patience in listening to me: B.Gross, L.Hesselholt, M.Levine, V.Snaith. I would also like to thank C.Stahlke for his help with the computer calculations and the Harvard Department of Mathematics for its hospitality during my stay.

2. K-groups and group cohomology

For any ring R and $n \geq 1$ the K-groups are defined to be

$$\mathrm{K}_n(R) = \pi_n(BGL^+(R)),$$

where $GL(R) = \varinjlim GL_n(R)$, B is the classifying space and $+$ Quillen's plus-construction. As $BSL(R)^+$ is the universal covering of $BGL^+(R)$, we get for $n \geq 2$: $\mathrm{K}_n(R) = \pi_n(BSL^+(R))$.

If $K_2(R)_p = 0$, we get from the spectral sequence to the exact sequence $0 \to K_2(R) \to St(R) \to SL(R) \to 0$:

$$K_3(R)_p = H_3(St(R))_p = H_3(SL(R))_p$$
$$K_3(R, \mathbb{Z}/p) = H_3(St(R), \mathbb{Z}/p) = H_3(SL(R), \mathbb{Z}/p)$$

and the latter sequence determines the number of generators of the former. Thus we will be interested in the low dimensional homology groups of $SL(R)$.
Note that by duality we have

$$H_n(SL(R), \mathbb{Z}/p) = H^n(SL(R), \mathbb{Z}/p)^*.$$

If G is torsion, we get from the long exact sequence to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and duality that $H^1(G) = 0$ and that for $n \geq 2$

$$H^n(G) = H^{n-1}(G, \mathbb{Q}/\mathbb{Z}) = H_{n-1}(G)^*.$$

If R is finite, the groups $SL_n(R)$ are finite and thus $SL(R)$ is torsion, so we can also use cohomology groups to calculate K-groups.
For V an abelian group, we have [3, theorem 6.6]

$$H_n(V, \mathbb{Z}/p) = \bigoplus_{a+2b=n} \Lambda^a_{\mathbb{Z}/p} V \otimes S^b_{\mathbb{Z}/p} V.$$

If V is p-torsion, we have $H_1(V) = V$, $H_2(V) = \Lambda^2 V$ [3, V 6.4]. From the long cohomology sequence to the sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0$ we get

$$V^* = H^1(V, \mathbb{Z}/p) \underset{\sim}{\xrightarrow{\delta}} H^2(V),$$

and the two dual sequences

$$0 \to H^2(V) = \delta V^* \to H^2(V, \mathbb{Z}/p) \xrightarrow{\delta} H^3(V) = \Lambda^2 V^* \to 0$$

and

$$0 \to H_2(V) = \Lambda^2 V \xrightarrow{p} H_2(V, \mathbb{Z}/p) \xrightarrow{\partial} H_1(V) = V \to 0.$$

In terms of the bar resolution the map p is given by $p(u \wedge v) = [u|v] - [v|u]$ for $u \wedge v \in \Lambda^2 V = H_2(V)$ and ∂ is given by $\partial[u|v] = \frac{[u]-[u+v]+[v]}{p}$. The map p is split by $[u|v] \mapsto \frac{u \wedge v}{2}$ and ∂ is split by $\rho : [v] \mapsto \sum_{j=0}^{p-1} [v|jv]$.
Finally we get

$$H_3(V) = \Lambda^3 V \oplus S^2 V = H^4(V)^*.$$

As we are interested in Witt vectors of length two $W_2(\mathbb{F}_q)$ over finite fields, we will consider the low terms of the spectral sequences associated to the short exact sequence induced by reduction modulo p:

$$0 \to K \to SL(W_2(\mathbb{F}_q)) \to SL(\mathbb{F}_q) \to 0.$$

One easily verifies that $X \mapsto 1 + pX$ identifies matrices of trace zero $V(\mathbb{F}_q)$ with K. We will sometimes switch between the additive and multiplicative notation for K. The sequence gives rise to the Hochschild-Serre spectral sequences

$$E^2_{p,q}(\mathbb{Z}) = H_p(SL(\mathbb{F}_q), H_q(V(\mathbb{F}_q))) \Rightarrow H_{p+q}(SL(W_2(\mathbb{F}_q)))$$
$$E^2_{p,q}(\mathbb{Z}/p) = H_p(SL(\mathbb{F}_q), H_q(V(\mathbb{F}_q), \mathbb{Z}/p)) \Rightarrow H_{p+q}(SL(W_2(\mathbb{F}_q)), \mathbb{Z}/p)$$

and similarly for cohomology.

Lemma 2.1. *Let M be the group of all matrices over $\mathbb{F}_q$ and V be the trace zero matrices.*
a)
$$H_*(GL(\mathbb{F}_q), M) = H_*(SL(\mathbb{F}_q), M)$$
b)
$$H_*(GL(\mathbb{F}_q), M) = H_*(GL(\mathbb{F}_q), V) \oplus H_*(GL(\mathbb{F}_q), \mathbb{F}_q)$$

Proof: a) If $(n, q-1) = 1$, the map $\det : GL_n(\mathbb{F}_q) \to \mathbb{F}_q^*$ is split by $x \to \mathrm{diag}(x, x, \ldots, x)$ and the action of $\mathbb{F}_q^*$ on $H_*(SL_n(\mathbb{F}_q), M_n)$ induced by conjugation is trivial. As $\mathbb{F}_q^*$ has order prime to p and M_n is p-torsion, the spectral sequence

$$H_i(\mathbb{F}_q^*, H_j(SL_n(\mathbb{F}_q), M_n) \Rightarrow H_{i+j}(GL_n(\mathbb{F}_q), M_n)$$

shows that $H_*(GL_n(\mathbb{F}_q), M_n) = H_*(SL_n(\mathbb{F}_q), M_n)$ and this carries over to the limit.
b) If $(n, p) = 1$, then the trace map is split as a $GL(\mathbb{F}_q)$-map by $x \mapsto \mathrm{diag}(\frac{x}{n}, \ldots \frac{x}{n})$, and we have

$$H_*(GL_n(\mathbb{F}_q), M_n) = H_*(GL_n(\mathbb{F}_q), V_n) \oplus H_*(GL_n(\mathbb{F}_q), \mathbb{F}_q),$$

which again carries over to the limit. □

The following terms of the above spectral sequence are known:

Proposition 2.2.

a)	$H_i(SL(\mathbb{F}_q), \mathbb{Z})_p =$	0	$i > 0$
b)	$H_i(SL(\mathbb{F}_q), V) =$	$(\mathbb{Z}/p)^f$	$i \geq 2$ *even*
		0	*otherwise*
c)	$H_i(SL(\mathbb{F}_q), \Lambda^2 V) =$	0	$i = 0, 1$
d)	$H_0(SL(\mathbb{F}_q), S^2 V) =$	$(\mathbb{Z}/p)^f$	
e)	$H_0(SL(\mathbb{F}_q), \Lambda^3 V) =$	$(\mathbb{Z}/p)^f$	

Proof: a) [16, theorem 6]
b) By lemma 2.1, a) and duality we have

$$H_i(SL(\mathbb{F}_q), V) = H_i(GL(\mathbb{F}_q), V) = H_i(GL(\mathbb{F}_q), M) = H^i(GL(\mathbb{F}_q), M^*)^*.$$

As $M = M^*$, we get the claimed result from [8, prop. 1.6]
c) [14, theorems 2.3e, 2.4b] or [12, théorème 3.4]

d), e) [14, theorem 2.4c] □

Remark: As [14] only contains sketches of proofs, we like to mention that the results of this paper remain valid if in d) and e) we only know that the homology groups have p-rank at least f.

But we have the $SL(\mathbb{F}_q)$-invariant linear forms $S^2V \xrightarrow{ab} V \xrightarrow{tr} \mathbb{F}_q \rightarrow \mathbb{Z}/3$ and $\Lambda^3 V \xrightarrow{abc-bac} V \xrightarrow{tr} \mathbb{F}_q \rightarrow \mathbb{Z}/3$, proving that $H^0(SL(\mathbb{F}_q), S^2V^*)$ and $H^0(SL(\mathbb{F}_q), \Lambda^3 V^*)$ have dimension at least f over $\mathbb{Z}/p$.

If we denote $H_2(SL(\mathbb{F}_q), \Lambda^2 V)$ by H we thus get

Corollary 2.3. *a) The low terms in the spectral sequence $H_p(SL(\mathbb{F}_q), H_q(V))_p \Rightarrow H_{p+q}(SL(W_2(\mathbb{F}_q)))_p$ are*

3	$(\mathbb{Z}/p)^{2f}$				
2	0	0	H		
1	0	0	$(\mathbb{Z}/p)^f$	0	$(\mathbb{Z}/p)^f$
0	$\mathbb{Z}$	0	0	0	0
	0	1	2	3	4

b) The low terms in the spectral sequence $H_p(SL(\mathbb{F}_q), H_q(V, \mathbb{Z}/p)) \Rightarrow H_{p+q}(SL(W_2(\mathbb{F}_q)), \mathbb{Z}/p)$ are

3	$(\mathbb{Z}/p)^{2f}$				
2	0	0	$(\mathbb{Z}/p)^f \oplus H$		
1	0	0	$(\mathbb{Z}/p)^f$	0	$(\mathbb{Z}/p)^f$
0	$\mathbb{Z}/p$	0	0	0	0
	0	1	2	3	4

3. K-GROUPS OF LOCAL RINGS

In this section we will recall some results on K-groups of dual numbers and local rings and relate them to the Lichtenbaum-Quillen conjectures.

By Suslin [21] we know that for a local ring $\mathcal{O}_\mathfrak{p}$ with quotient field $\mathbb{F}_q$ and m prime to p we have

$$K_i(\mathcal{O}_\mathfrak{p}, \mathbb{Z}/m) = K_i(\mathbb{F}_q, \mathbb{Z}/m).$$

Thus we will be only interested in the p-part of K-groups, as the prime to p-part is known by Quillen [16, theorem 8].

Similarly, Panin [15] has shown that

$$K_i(\mathcal{O}_\mathfrak{p}, \mathbb{Z}/p^n) = \varprojlim K_i(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r, \mathbb{Z}/p^n),$$

which allows us to relate K-groups of local rings to K-groups of their quotients.

The following two theorems have been proved by comparison of K-theory with topological cyclic homology:

Theorem 3.1. [10] *Let k be a finite field of characteristic $p \neq 2$, then*

$$K_{2n}(k[t]/(t^2))_p = 0$$
$$K_{2n-1}(k[t]/(t^2))_p = \bigoplus_{(i,2)=1} W_{s_i}(k).$$

Here s_i is given by $ip^{s_i-1} \leq n < ip^{s_i}$.

Theorem 3.2. [2] *Let $\mathcal{O}_\mathfrak{p}$ be an unramified extension of $\mathbb{Z}_p$, $p \geq 3$, of degree f. Then we have*

$$K_{2n}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) = \pi_{2n-1}(\operatorname{im} J)_p$$
$$K_{2n-1}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) = \mathbb{Z}_p^f \oplus \pi_{2n-1}(\operatorname{im} J)_p.$$

Here $\operatorname{im} J$ *is the image of the J-spectrum, i.e.* $\pi_{4n-1}(\operatorname{im} J)_p = (\mathbb{Z}/d_n)_p$*, where d_n is the denominator of the Bernoulli-number $\frac{B_n}{n}$.*

For K_3, the last theorem has also been proven by Levine [13].
Let compare the last theorem with the Lichtenbaum-Quillen conjectures:
Since we have by the localization sequence for $n \geq 2$

$$K_n(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) = K_n(K_\mathfrak{p}, \mathbb{Z}_p)$$

for $K_\mathfrak{p}$ the quotient field of $\mathcal{O}_\mathfrak{p}$, we can consider the K-groups of $K_\mathfrak{p}$.
One formulation of the Lichtenbaum-Quillen conjectures in this case is that that natural surjection [6]

$$\rho : K_i(K_\mathfrak{p}, \mathbb{Z}_p) \to K_i^{et}(K_\mathfrak{p})$$

is an isomorphism for sufficiently large i. By the splitting of the Dwyer-Friedlander spectral sequence for K_*^{et}, [20, theorem 1], we have

$$K_{2n}^{et}(K_\mathfrak{p}) = H^0(K_\mathfrak{p}, \mathbb{Z}_p(n)) \oplus H^2(K_\mathfrak{p}, \mathbb{Z}_p(n+1))$$
$$K_{2n-1}^{et}(K_\mathfrak{p}) = H^1(K_\mathfrak{p}, \mathbb{Z}_p(n)).$$

Now one can conclude from the results in [17, par. 3] that

$$H^0(K_\mathfrak{p}, \mathbb{Z}_p(n)) = 0$$
$$H^1(K_\mathfrak{p}, \mathbb{Z}_p(n)) = {\mathbb{Z}_p}^f \oplus \mathbb{Z}/w_n(K_\mathfrak{p})$$
$$H^2(K_\mathfrak{p}, \mathbb{Z}_p(n+1)) = H^0(K_\mathfrak{p}, \mathbb{Q}_p/\mathbb{Z}_p(-n))^* = \mathbb{Z}/w_{-n}(K_\mathfrak{p}).$$

Here $w_n(K_\mathfrak{p}) = \max\{p^j : [K_\mathfrak{p}(\mu_{p^j}) : K_\mathfrak{p}] | n\}$.

Conjecture 3.3. *(Lichtenbaum-Quillen conjecture for local fields)*

$$K_{2n}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) = \mathbb{Z}/w_n(K_\mathfrak{p})$$
$$K_{2n-1}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) = \mathbb{Z}_p^f \oplus \mathbb{Z}/w_n(K_\mathfrak{p}).$$

If the field $K_\mathfrak{p}$ is unramified, we have $[K_\mathfrak{p}(\mu_{p^j}) : K_\mathfrak{p}] = (p-1)p^{j-1}$, so that

$$w_n(K_\mathfrak{p}) = \#\pi_{2n-1}(\operatorname{im} J)_p = \begin{cases} 1 & \text{for } (p-1) \nmid n \\ p^{\operatorname{ord}_p(n)+1} & \text{for } (p-1) | n \end{cases}$$

In particular we see that the above surjections ρ must be isomorphisms.
We also have an action of Adams operators on both the K-groups and on the constituents of the Dwyer-Friedlander spectral sequence. The Adams operator ψ^k acts like k^n on $H^i(K_\mathfrak{p}, \mathbb{Z}_p(n)) = E_2^{i,-2n} = E_\infty^{i,-2n}$, see [20, prop. 2, theorem 1], so we get:

Proposition 3.4. *Let $\mathcal{O}_\mathfrak{p}$ be an unramified extension of $\mathbb{Z}_p$ and $p \geq 3$. Then we have*

$$\mathrm{K}_{2n}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) = \mathrm{K}_{2n}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p)^{(n+1)}$$
$$\mathrm{K}_{2n-1}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) = \mathrm{K}_{2n-1}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p)^{(n)}$$

4. The cokernel of $K_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r) \to K_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$

Let $\mathcal{O}_\mathfrak{p}$ be a finite extension of $\mathbb{Z}_p$ with ramification index e and residue degree f. We will examine the cokernel C_n^r of the maps $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r) \to \mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$. We assume for simplicity $\mathrm{K}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r) = 0$, which is for example true in case $\mathcal{O}_\mathfrak{p}$ does not contain p-th roots of unity or if $r < \frac{p}{p-1}e$, [5, theorem 5.1]. This implies that $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r) = H_3(SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r))$ and similarly for $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$.

Proposition 4.1. *Let $n \leq r \leq 2n$ and $\mathrm{K}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r) = 0$. Then the cokernel of the map $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r) \to \mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$ equals $\Omega_{\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n} \otimes_{\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n} \mathfrak{p}^n/\mathfrak{p}^r = \mathcal{O}_\mathfrak{p}/\mathfrak{p}^c$, where $c = \min(r-n, d, (n-1) + v_\mathfrak{p}(n))$, d the exponent of the discriminant of $\mathcal{O}_\mathfrak{p}$.*

Proof: Consider the spectral sequence of homology groups for the short exact sequence of groups

$$0 \to N_n^r \to SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r) \to SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n) \to 0.$$

Since $r \leq 2n$, the map $A \mapsto 1 + A$ induces an isomorphism between $V(\mathfrak{p}^n/\mathfrak{p}^r)$, the trace zero matrices with entries in $\mathfrak{p}^n/\mathfrak{p}^r$, and N_n^r. Thus $H_1(N_n^r) = V(\mathfrak{p}^n/\mathfrak{p}^r)$ and we have $H_2(N_n^r) = \Lambda^2 V(\mathfrak{p}^n/\mathfrak{p}^r)$, [3, theorem 6.4]. This gives us

$$\begin{aligned}
E^2_{1,0} &= H_1(SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)) = SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)^{ab} = 0 \\
E^2_{2,0} &= H_2(SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)) = \mathrm{K}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n) = 0 \\
E^2_{3,0} &= H_3(SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)) = \mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n) \\
E^2_{0,1} &= H_0(SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n), V(\mathfrak{p}^n/\mathfrak{p}^r)) = 0 \qquad \text{[12, prop. 1.2]} \\
E^2_{0,2} &= H_0(SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n), \Lambda^2 V(\mathfrak{p}^n/\mathfrak{p}^r)) = 0 \qquad \text{[12, théorème 3.4]}
\end{aligned}$$

2	0			
1	0	$E^2_{1,1}$		
0	$\mathbb{Z}$	0	0	$\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$
	0	1	2	3

So we get the short exact sequence

$$\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r) \to \mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n) \xrightarrow{d^2_{3,0}} E^2_{1,1} \to 0.$$

By [12, théorème 2.16] we have:

$$E^2_{1,1} = H_1(SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n), V(\mathfrak{p}^n/\mathfrak{p}^r)) = \Omega_{\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n} \otimes_{\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n} \mathfrak{p}^n/\mathfrak{p}^r.$$

For the last equation of the proposition we have $\Omega_{\mathcal{O}_\mathfrak{p}} = \mathcal{O}_\mathfrak{p}/\mathfrak{p}^d d\pi$ for π a uniformizer of $\mathcal{O}_\mathfrak{p}$, d the valuation of the discriminant, and $e-1 \le d \le e-1+v_\mathfrak{p}(e)$, [18, prop.13,14]. We have the exact sequence

$$\mathfrak{p}^n/\mathfrak{p}^{2n} \xrightarrow{\delta} \Omega_{\mathcal{O}_\mathfrak{p}} \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{O}_\mathfrak{p}/\mathfrak{p}^n \to \Omega_{\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n} \to 0,$$

where $\delta(x) = dx \otimes 1$. From $d\pi^n = n\pi^{n-1}d\pi$ we get

$$\Omega_{\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n} = \frac{\mathcal{O}_\mathfrak{p}/\mathfrak{p}^d d\pi \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{O}_\mathfrak{p}/\mathfrak{p}^n}{\mathcal{O}_\mathfrak{p} d\pi^n \otimes 1} = \mathcal{O}_\mathfrak{p}/\mathfrak{p}^{\min(n,d,(n-1)+v_\mathfrak{p}(n))}$$

hence

$$\Omega_{\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n} \otimes \mathfrak{p}^n/\mathfrak{p}^r = \mathcal{O}_\mathfrak{p}/\mathfrak{p}^c$$

with $c = \min(n, r-n, d, (n-1)+v_\mathfrak{p}(v)) = \min(r-n, d, (n-1)+v_\mathfrak{p}(n))$ as $r \le 2n$.
□

Corollary 4.2. *If $\mathcal{O}_\mathfrak{p}$ is unramified, then $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r) \to \mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$ is surjective for all $r > n$. Consequently $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p)$ surjects onto $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)_p$.*

Proof: Since $\Omega_{\mathcal{O}_\mathfrak{p}} = 0$, the map $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^{n+1}) \to \mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$ is surjective and the first claim follows. As the map is an isomorphism outside p and for the p-part surjectivity and surjectivity mod p are equivalent, the second claim follows from (see [15]) $\mathrm{K}_3(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p)/p = \mathrm{K}_3(\mathcal{O}_\mathfrak{p}, \mathbb{Z}/p) = \varprojlim \mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r, \mathbb{Z}/p) = \varprojlim \mathrm{K}_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r)/p$, because if all maps in an inverse system are surjective then the map from the inverse limit to a member of the system is surjective. □

More generally, for r not necessarily less than or equal $2n$, the term $E^2_{0,2}/\operatorname{im} d^2_{2,1} = H_0(SL(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n), H_2(V(\mathfrak{p}^n/\mathfrak{p}^r)))/\operatorname{im} d^2_{2,1}$ gives an extra contribution to the cokernel. For example for $e > r$, the groups C^r_n grow regularly by $(\mathbb{Z}/p)^f$ for $r = n+1, \dots, \min(2n, 2n-1+v_\mathfrak{p}(n))$ (because $\mathcal{O}_\mathfrak{p}/\mathfrak{p}^c = \mathbb{F}_q^c$) until they reach $\mathbb{F}_q^{\min(n,n-1+v_\mathfrak{p}(n))}$

and the $E_{1,1}$-contribution is exhausted. Then there is an irregular contribution coming from $E_{0,2}/\operatorname{im} d^2_{2,1}$. In case $\mathcal{O}_\mathfrak{p}$ sufficiently ramified (i.e. $e > r$), we eventually get $K_3(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n) = C^r_n$, and the precise pattern can be read of from [11, 3.4].
For example C^r_n grows for the following r:

$$p = 3, \quad n = 5: \qquad 6, 7, 8, 9, 12, 18, 27, 81$$

$$p = 3, \quad n = 9: \qquad 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 24, 27, 36, 45, 54, 81$$

$$p = 5, \quad n = 5: \qquad 6, 7, 8, 9, 10, 15, 20, 25$$

$$p = 5, \quad n = 6: \qquad 7, 8, 9, 10, 11, 15, 20, 25, 50, 125$$

5. The outer automorphism

The outer automorphism

$$\tau : SL(R) \to SL(R)$$
$$A \mapsto {}^tA^{-1}$$

induces an automorphism of order 2 on homology groups with coefficients in any self-dual representation. For $R = \mathbb{F}_q$ and as coefficients the homology groups of the adjoint representation V, the automorphism is compatible with the stabilization maps $SL_n(R) \to SL_{n+1}(R)$.
For the extension $1 \to V(\mathbb{F}_q) \to SL(W_2(\mathbb{F}_q)) \to SL(\mathbb{F}_q) \to 1$ the induced action on V is given by $A \mapsto -{}^tA$. The automorphism induces a map on the spectral sequences, all terms of the spectral sequence decompose into +- and −-eigenspaces and the differentials respect this decomposition. The action corresponds to the Adams operator ψ_{-1} on K-groups, because changing the R-module structure on a projective module by τ corresponds to going to the dual module. Thus the +-eigenspaces under τ correspond to even Adams eigenspaces and the −-eigenspaces correspond to odd Adams eigenspaces.
We will determine the action of τ on some of the E_2-terms:

Proposition 5.1. *a) The automorphism τ acts like $+1$ on $H_0(SL(\mathbb{F}_q), \Lambda^3 V)$ and on $H_0(SL(\mathbb{F}_q), S^2 V)$.*
b) For $n \geq 2$ and $p \geq n$, τ acts like $(-1)^n$ on $H_{2n-2}(SL(\mathbb{F}_q), V) = \mathbb{F}_q$.

Proof: a) We prove the dual cohomological result. The stabilization maps

$$H^0(SL_n(\mathbb{F}_q), \Lambda^3 V_n^*) \to H^0(SL_2(\mathbb{F}_q), \Lambda^3 V_2^*)$$

are isomorphisms, as one sees with the diagram

$$\begin{array}{ccccccc}
\Lambda^3 V_2 & \xrightarrow{abc-bac} & V_2 & \xrightarrow{tr} & \mathbb{F}_q & \longrightarrow & \mathbb{Z}/3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Lambda^3 V_n & \xrightarrow{abc-bac} & V_n & \xrightarrow{tr} & \mathbb{F}_q & \longrightarrow & \mathbb{Z}/3
\end{array}$$

and similarly for S^2V^*. But on the SL_2-level τ is an inner automorphism, thus the action must be trivial.
b) Will be proved in the remainder of this section.
By [19, theorem 7.6] we can always go to a bigger field and thus assume that $2n-2 < f(2p-3)-2$. By duality and lemma 2.1 we have

$$H_{2n-2}(SL(\mathbb{F}_q), V) = H^{2n-2}(SL(\mathbb{F}_q), V^*)^* = H^{2n-2}(GL(\mathbb{F}_q), M^*)^*$$

and since we assume $2n-2 < \min(2p-1, f(2p-3)-2)$ we know by [8] that we have stably

$$H^{2n-2}(GL(\mathbb{F}_q), M) = H^{2n-2}(B_n(\mathbb{F}_q), M_n) = \mathbb{F}_q,$$

where $B_n(\mathbb{F}_q)$ is the Borel subgroup of upper triangular matrices.
Instead of τ we consider the composition σ of τ with conjugation by g, where $g = (a_{i,j})$ with $a_{i,j} = 1$ for $j+i = n+1$ and 0 otherwise, because σ respects the Borel subgroup. An easy calculation shows that σ acts on $M_n = \ker GL_n(W_2(\mathbb{F}_q)) \to GL_n(\mathbb{F}_q)$ by $(a_{i,j}) \to (-a_{n+1-j,n+1-i})$ (i.e. -1 times the reflection on the diagonal $(1,n)\dots(n,1)$), since $\tau(a_{i,j}) = -(a_{i,j})^t$ and Int$g$ induces a turn by 180 degree.
We define the following σ-invariant descending filtration on M_n:

$$F^s M_n = \{(a_{i,j}) | a_{i,j} = 0 \quad \text{for} \quad i-j \geq n-s\}.$$

The associatec graded pieces are isomorphic to

$$\operatorname{gr}^s M_n = \{(a_{i,j}) | a_{i,j} = 0 \quad \text{for} \quad i-j \neq n-s-1\}.$$

Lemma 5.2.

$$H^{2n-2}(B_n, \operatorname{gr}^s M_n) = \begin{cases} \mathbb{F}_q & \textit{for} \quad s = 2n-2 \\ 0 & \textit{otherwise} \end{cases}$$

Proof: To compute the cohomology of B_n with coefficients in the graded pieces we use the "symbolic weight equations" of [8]:
First note that for U_n the unipotent subgroup of B_n and T_n its torus, we have

$$H^{2n-2}(B_n, \operatorname{gr}^s M_n) = H^{2n-2}(U_n, \operatorname{gr}^s M_n)^{T_n} = (H^{2n-2}(U_n, \mathbb{F}_q) \otimes_{\mathbb{F}_q} \operatorname{gr}^s M_n)^{T_n}.$$

The first equation follows because the order of T_n is prime to p and $\operatorname{gr}^s M_n$ is a p-torsion group. The second equation follows because U_n acts trivially on $\operatorname{gr}^s M_n$.

In [8] one sees that U_n admits a filtration such that we have for the graded pieces $\mathrm{gr}\, U_n = \mathbb{F}_q^{n(n-1)/2}$ and for the cohomology $H^{2n-2}(U_n, \mathbb{F}_q) = H^{2n-2}(\mathrm{gr}\, U_n, \mathbb{F}_q)$. On the other hand the cohomology of $\mathrm{gr}\, U_n$ is given by

$$H^*(\mathrm{gr}\, U_n, \mathbb{F}_q) = \Lambda^*_{\mathbb{F}_q}(V_n) \otimes_{\mathbb{F}_q} S^*_{\mathbb{F}_q}(W_n),$$

where V_n has a basis $\{a^s_{i,j} | 1 \leq i < j \leq n, 0 \leq s < f\}$ and is of degree 1, and W_n has a basis $\{b^s_{i,j} | 1 \leq i < j \leq n, 0 \leq s < f\}$ and is of degree 2. The T_n-action on this ring is given by the condition that $a^s_{i,j}$ and $b^s_{i,j}$ have weight $-p^s \alpha_{i,j}$, where $\alpha_{i,j}$ is the character $(t_1, \ldots, t_n) \mapsto t_i/t_j$. We write this symbolically as

$$[a^s_{i,j}] = [b^s_{i,j}] = -p^s[i] + p^s[j].$$

The T_n-action on $e_{u,v} \in \mathrm{gr}^s M_n$ $(u - v = n - s - 1)$ is given by $\alpha_{u,v}$, so it has symbolic weight $[u] - [v]$. We want to determine

$$(H^{2n-2}(\mathrm{gr}\, U_n, \mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathrm{gr}^s M_n)^{T_n}.$$

As T_n acts like scalars on all basis elements of $H^{2n-2}(\mathrm{gr}\, U_n, \mathbb{F}_q)$ and $\mathrm{gr}^s M_n$, it suffices to consider monomials of the form

$$z = a^{s_1}_{i_1,j_1} \wedge \cdots \wedge a^{s_m}_{i_m,j_m} \otimes b^{t_1}_{k_1,l_1} \otimes \cdots \otimes b^{t_r}_{k_r,l_r} \otimes e_{u,v} \in \Lambda^m_{\mathbb{F}_q} V_n \otimes S^r_{\mathbb{F}_q} W_n \otimes \mathrm{gr}^s M_n$$

for $m + 2r = 2n - 2$ and $u - v = n - s - 1$ in order to get all T_n-invariant elements. The monomial z has symbolic weight

$$[z] = -p^{s_1}[i_1] + p^{s_1}[j_1] - \cdots + p^{t_r}[l_r] + [u] - [v] =: \sum_{e=1}^{n} g_e[e].$$

Obviously the sum of the positive g_e equals minus the sum of the negative g_e. In order for z to be T_n-invariant, we need $g_e \equiv 0 \mod p^f - 1$ for all e.

Let l_1 be the smallest subscript occurring. If $g_{l_1} = 0$, we must have $u = l_1$ and the only a and b occurring with l_1 as the first subscript is $a^0_{l_1,k_1}$ or $b^0_{l_1,k_1}$ for some k_1. In this case let i_2 be the next smallest subscript occurring. Again, if $g_{l_2} = 0$, then $l_2 = k_1$ and there is at most one $a^0_{l_2,k_2}$ or $b^0_{l_2,k_2}$ occurring for some k_2. Continuing in this fashion, we either find a smallest l such that $g_l \neq 0$, and all but one coefficients of $[l]$ are negative (and the positive coefficient can only be 1), or all $g_e = 0$ and z is made from elements $c^0_{l_1,l_2}, c^0_{l_2,l_3}, \cdots, c^0_{l_m,l_{m+1}}$ with $u = l_1 < l_2 \cdots < l_{m+1} = v$. Clearly $m+1 \leq l_{m+1} \leq n$, on the other hand $\deg z = 2n-2 \leq 2m$, so we conclude $m = n-1$, $l_i = i$ and

$$z = b^0_{1,2} \otimes \cdots \otimes b^0_{n-1,n} \otimes e_{1,n}.$$

Thus we find a unique basic element in $H^{2n-2}(B_n, \mathrm{gr}^s M)$ for $s = 2n - 2$.

In case there is a smallest l such that $g_l \neq 0$ we similarly find a largest j such that $g_j \neq 0$, and all but one coefficients of j are positive (and the one exception can only be -1).

Consider the minimal p-adic expression

$$|g_e| = \sum_{\nu=0}^{f-1} g_{e,\nu} p^\nu,$$

where minimal means that $\sum g_{e,\nu}$ is minimal. We have

$$-g_l = \sum_{\nu=0}^{f-1} g_{l,\nu} p^\nu \equiv 0 \mod p^f - 1.$$

Because $g_l \neq 0$ and z has $2n-2$ factors with coefficients at most p^{f-1}, we have $-g_l \leq (2n-2)p^{f-1} \leq (2p-2)p^{f-1} < 2(p^f-1)$. So $-g_l = p^f - 1$ and we can conclude $\sum g_{l,\nu} \geq f(p-1)$. Similarly we get $\sum g_{j,\nu} \geq f(p-1)$.
Let I be the number of factors of z of the form $a^s_{l,j}$ and $b^s_{l,j}$, then the sum of degrees of these terms is $f+2(I-f)$, as there are at most f factors of this form of cohomological degree 1. Since the number of factors with an l occurring as a subscript is at least $\sum g_{l,\nu}$, we get

$$2n-2 = \deg z \geq \left(\sum g_{l,\nu} - I\right) + \left(\sum g_{j,\nu} - I\right) + (f + 2(I-f)) \geq f(2p-3),$$

contradicting $2n-2 < f(2p-3) - 2$. □

We now consider the spectral sequence to the filtration $F^s M_n$,

$$E_1^{s,t} = H^{s+t}(B_n, \operatorname{gr}^s M_n) \Rightarrow H^{s+t}(B_n, M_n).$$

From

$$H^{2n-2}(B_n, \operatorname{gr}^s M_n) = \begin{cases} \mathbb{F}_q & \text{for} \quad s = 2n-2 \\ 0 & \text{otherwise} \end{cases}$$

we conclude that we have

$$H^{2n-2}(B_n, \operatorname{gr}^{2n-2} M_n) = E_1^{2n-2,0} = E_\infty^{2n-2,0} = H^{2n-2}(B_n, M_n) = \mathbb{F}_q$$

and we can calculate the action of σ on $H^{2n-2}(B_n, \operatorname{gr}^s M_n)$.
But as $H^{2n-2}(B_n, M_n) = H^{2n-2}(B_n, \mathbb{F}_q e_{1,n})$ is generated by the cocycle

$$z = b^0_{1,2} \otimes \cdots \otimes b^0_{n-1,n} \otimes e_{1,n},$$

we have to calculate the action of σ on z. An easy calculation shows that $\sigma(e_{1,n}) = -e_{1,n}$ and $\sigma(b_{i,j}) = -b_{n+1-j,n+1-i}$. As the $b_{i,j}$ commute we get $\sigma(z) = (-1)^{n-1+1} z$, which was to be proven. □

6. $K_3(W_2(\mathbb{F}_q))$ FOR CHAR $\mathbb{F}_q \neq 3$

Proposition 6.1. *For $p \geq 3$ we have the following +-eigenspaces under τ in the spectral sequence $H_i(SL(\mathbb{F}_{p^f}), H_j(V))_p \Rightarrow H_{i+j}(SL(W_2(\mathbb{F}_{p^f})))_p$:*

3	$(\mathbb{Z}/p)^{2f}$				
2	0	0	$(\mathbb{Z}/p)^f$		
1	0	0	$(\mathbb{Z}/p)^f$	0	0
0	$\mathbb{Z}$	0	0	0	0
	0	1	2	3	4

Proof: This is an immediate consequence of 2.3 and 5.1 except from the identity $(E^2_{2,2})^+ = H^+ = (\mathbb{Z}/p)^f$. For this consider the extension

$$0 \to V \to SL(\mathbb{F}_{p^f}[t]/t^2) \to SL(\mathbb{F}_{p^f}) \to 0.$$

The corresponding spectral sequence has the same E_2-terms as the spectral sequence to the extension $0 \to V \to SL(\mathrm{W}_2(\mathbb{F}_q)) \to SL(\mathbb{F}_{p^f}) \to 0$, since the action of $SL(\mathbb{F}_{p^f})$ on V is the adjoint action in both cases. The differentials are different, however, as the latter sequence does not split whereas the former does.
From $\# \mathrm{K}_3(\mathbb{F}_{p^f}[t]/t^2) = p^{2f}$ we conclude that $E^\infty_{0,3} = (\mathbb{Z}/p)^f$ and thus that $d^2_{2,2}$ has rank f. On the other hand we know that $\mathrm{K}_4(\mathbb{F}_{p^f}[t]/t^2) = 0$, so $E^\infty_{2,2} = 0$. As there are no nonzero differentials ending in $E_{2,2}$, we conclude $H^+ = (\mathbb{Z}/p)^f$. □

Theorem 6.2. *Let $p \geq 5$ then*

$$\mathrm{K}_3(W_2(\mathbb{F}_{p^f})) = (\mathbb{Z}/p^2)^f \oplus \mathbb{Z}/(p^{2f} - 1).$$

Proof: By Suslins result the prime to p-part is the same as for $\mathbb{F}_q$. For the p-part let us first determine the +-eigenspaces. By 4.2 and 3.2 we know that $\mathrm{K}_3(W_2(\mathbb{F}_{p^f}))_p$ has at most f generators. This forces the differential $d^2_{2,2}$ in 6.1 to be injective. Thus we are left with a group with f generators and two graded pieces isomorphic to $(\mathbb{Z}/p)^f$, giving the desired result for the +-eigenspaces.
As $\mathrm{K}_3(W_2(\mathbb{F}_{p^f}))_p$ has at most f generators and the +-eigenspace already has f generators, we conclude that the −-eigenspace is trivial. □

7. $K_3(W_2(\mathbb{F}_q))$ FOR CHAR $\mathbb{F}_q = 3$

In this section we determine $\mathrm{K}_3(W_2(\mathbb{F}_{3^f}))$ for $(3, f) = 1$. The problem in characteristic 3 is that there might be $f+1$ generators instead of f generators and so the differential $d^2_{2,2}$ in 6.1 may not be injective (and similar in the mod 3 spectral sequence).

It turns out that cohomological calculations are easier than homological calculations, so from now on we work with cohomology groups. The dual of 6.1 gives us the following E_2-terms in the spectral sequences:

$H^i(SL(\mathbb{F}_{3^f}), H^j(V,\mathbb{Z}/3))$:

3	$(\mathbb{Z}/3)^{2f}$				
2	0	0	$(\mathbb{Z}/3)^{2f}$		
1	0	0	$(\mathbb{Z}/3)^{f}$	0	0
0	$\mathbb{Z}$	0	0	0	0
	0	1	2	3	4

$H^i(SL(\mathbb{F}_{3^f}), H^j(V))_3$:

4	$(\mathbb{Z}/3)^{2f}$					
3	0	0	$(\mathbb{Z}/3)^{f}$			
2	0	0	$(\mathbb{Z}/3)^{f}$	0	0	
1	0	0	0	0	0	0
0	$\mathbb{Z}$	0	0	0	0	0
	0	1	2	3	4	5

In order to determine $H^3(SL(W_2(\mathbb{F}_{3^f})),\mathbb{Z}/3)$ and $H^4(SL(W_2(\mathbb{F}_{3^f})))_3$, we have to calculate the differentials

$$\begin{array}{ccc} d_2^{0,3}: H^0(SL(\mathbb{F}_{3^f}), H^3(V,\mathbb{Z}/3)) & \longrightarrow & H^2(SL(\mathbb{F}_{3^f}), H^2(V,\mathbb{Z}/3)) \\ \cong\downarrow\delta & & \downarrow\delta \\ d_2^{0,4}: H^0(SL(\mathbb{F}_{3^f}), H^4(V)) & \longrightarrow & H^2(SL(\mathbb{F}_{3^f}), H^3(V)) \end{array}$$

The calculations will be similar to the calculations in [7, par.9-11]. The idea is to use stability to reduce to the SL_2-level first, and then make the calculations for a 3-Sylow group. However, as we are in characteristic 3, the short exact sequence

$$1 \to V_2 \to SL_2(\mathbb{Z}/9) \to SL_2(\mathbb{Z}/3) \to 1$$

splits. Thus we would have to work on the SL_3-level. Instead we make calculations for $W_2(\mathbb{F}_9)$ and deduce results for $\mathbb{F}_3$, because the 3-Sylow group of $SL_2(\mathbb{F}_9)$ is abelian and has only rank 2.

We choose a basis $\{1,z\}$ of $\mathbb{F}_9$ over $\mathbb{F}_3$ such that $z^2=-1$, and consider the following short exact sequence

$$1 \to V_2 \to U \to P \to 1,$$

where U is the 3-Sylow subgroup of $SL_2(W_2(\mathbb{F}_9))$ consisting of matrices

$$\begin{pmatrix} 1+3a & b \\ 3c & 1+3d \end{pmatrix}, \qquad a+d-bc \equiv 0 \mod 3.$$

We get the following diagram

$$\begin{array}{ccccc}
H^0(SL(\mathbb{F}_9),H^3(V,\mathbb{Z}/3)) & \xrightarrow{d} & H^2(SL(\mathbb{F}_9),H^2(V,\mathbb{Z}/3)) & \xrightarrow{\delta} & H^2(SL(\mathbb{F}_9),H^3(V)) \\
\cong\big\downarrow\alpha_0 & & \cong\big\downarrow\alpha_1 & & \cong\big\downarrow\alpha_2 \\
H^0(SL_2(\mathbb{F}_9),H^3(V_2,\mathbb{Z}/3)) & \xrightarrow{d} & H^2(SL_2(\mathbb{F}_9),H^2(V_2,\mathbb{Z}/3)) & \xrightarrow{\delta} & H^2(SL_2(\mathbb{F}_9),H^3(V_2)) \\
\big\downarrow\beta_0 & & \big\downarrow & & \big\downarrow\beta_2 \\
H^0(P,H^3(V_2,\mathbb{Z}/3)) & \xrightarrow{d} & H^2(P,H^2(V_2,\mathbb{Z}/3)) & \xrightarrow{\delta} & H^2(P,H^3(V_2))
\end{array}$$

The map α_0 is an isomorphism as in the proof of 5.1(a). To show that α_1 and α_2 are isomorphisms consider the following diagram:

$$\begin{array}{ccc}
H^2(SL(\mathbb{F}_9),V^*) & \longrightarrow & H^2(SL(\mathbb{F}_9),\Lambda^2V^*) \\
\cong\big\downarrow\gamma & & \big\downarrow\alpha_2 \\
H^2(SL_2(\mathbb{F}_9),V_2^*) & \xrightarrow{\eta} & H^2(SL_2(\mathbb{F}_9),\Lambda^2V_2^*).
\end{array}$$

By [8, prop 1.6] and 2.1 the map γ is an isomorphism. On the other hand the lower map η, induced by $a\wedge b\mapsto ab-ba$, is split by the map induced by $e_{ij}\mapsto\frac{1}{2}\sum_k e_{ik}\wedge e_{kj}$ and thus injective. Since all groups in the diagram equal $\mathbb{F}_9$, we see that α_2 is an isomorphism. As α_1 is the direct sum of α_2 and γ, it must be an isomorphism too. The maps β_0 and β_2 are injective as P is a 3-Sylow group of $SL_2(\mathbb{F}_9)$.
These considerations show that we can calculate the differential in the lower row of the image of $\beta_0\circ\alpha_0$.
From now on we will write V for V_2, as there is no danger of confusion.
We have

$$H^0(SL_2(\mathbb{F}_9),H^3(V,\mathbb{Z}/3))=(\Lambda^3V^*)^{SL_2(\mathbb{F}_9)}\oplus(S^2V^*)^{SL_2(\mathbb{F}_9)}.$$

A basis of invariants is given by

$$\begin{array}{lccccccc}
\varphi:\Lambda^3V & \xrightarrow{abc-bac} & V & \xrightarrow{tr} & \mathbb{F}_9 & \xrightarrow{\chi} & \mathbb{Z}/3 \\
\psi:S^2V & \xrightarrow{ab} & V & \xrightarrow{tr} & \mathbb{F}_9 & \xrightarrow{\chi} & \mathbb{Z}/3,
\end{array}$$

where χ runs through a basis of linear forms. If we choose the linear forms χ_1 : $a+bz\mapsto a$ and $\chi_2 : a+bz\mapsto b$ as a basis, we find the following basic invariant forms:

$$A:=-\chi_1\circ\varphi,\quad B:=\chi_2\circ\varphi,\quad C:=\chi_1\circ\psi,\quad D:=\chi_2\circ\psi.$$

Proposition 7.1. *a)*

$$\mathrm{K}_3(W_2(\mathbb{F}_9))=\mathbb{Z}/9\oplus\mathbb{Z}/3\oplus\mathbb{Z}/3\oplus\mathbb{Z}/80$$

b)

$$\mathrm{K}_3(\mathbb{Z}/9)=\mathbb{Z}/3\oplus\mathbb{Z}/3\oplus\mathbb{Z}/8.$$

Proof: a) Consider the spectral sequence

$$E_2^{i,j}(\mathbb{Z}/3) = H^i(P, H^j(V, \mathbb{Z}/3)) \Rightarrow H^{i+j}(U, \mathbb{Z}/3)$$

and its differential

$$d_2^{0,3} : E_2^{0,3}(\mathbb{Z}/3) = (\mathbb{Z}/3)^4 \to E_2^{2,2}(\mathbb{Z}/3) = (\mathbb{Z}/3)^4.$$

We will see in 10.1 that $d_2^{0,3}(A + C) = 0$, so $d_2^{0,3}$ has rank at most 3. On the other hand it has rank at least 3, because $\mathrm{K}_3(W_2(\mathbb{F}_q))$ has at most 3 generators by 4.2 and 3.2. So the number of generators of $\mathrm{K}_3(W_2(\mathbb{F}_9))$ is 3.
Now consider the spectral sequence

$$E_2^{i,j}(\mathbb{Z}) = H^i(P, H^j(V)) \Rightarrow H^{i+j}(U)$$

with differential

$$d_2^{0,4} : E_2^{0,4}(\mathbb{Z}) = (\mathbb{Z}/3)^4 \to E_2^{2,3}(\mathbb{Z}) = (\mathbb{Z}/3)^2.$$

We will see in 10.1 that $d_2^{0,4}(A)$ and $d_2^{0,4}(B)$ are linearly independent, so $d_2^{0,4}(\mathbb{Z})$ has rank 2, and the cardinality of $\mathrm{K}_3(W_2(\mathbb{F}_9))_3$ is 3^4.
b) The inclusion $i : \mathbb{Z}/9 \to W_2(\mathbb{F}_{3^f})$ induces the natural map $i_* : \mathrm{K}_3(\mathbb{Z}/9) \to \mathrm{K}_3(W_2(\mathbb{F}_{3^f}))$. On the other hand we have the transfer map $i^* : \mathrm{K}_3(W_2(\mathbb{F}_{3^f})) \to \mathrm{K}_3(\mathbb{Z}/9)$ induced by considering a $W_2(\mathbb{F}_{3^f})$-module as a $\mathbb{Z}/9$-module. As $W_2(\mathbb{F}_{3^f})$ is a free $\mathbb{Z}/9$-module of rank f, we have that $i^* \circ i_*$ is multiplication by f.
Consider now the following diagram

$$\begin{array}{ccc} \mathrm{K}_3(\mathbb{Z}_3, \mathbb{Z}/3) & \xrightarrow{i_*} & \mathrm{K}_3(W(\mathbb{F}_9), \mathbb{Z}/3) \\ \big\downarrow & & \simeq\big\downarrow \\ \mathrm{K}_3(\mathbb{Z}/9, \mathbb{Z}/3) & \xrightarrow{i_*} & \mathrm{K}_3(W_2(\mathbb{F}_9), \mathbb{Z}/3). \end{array}$$

As the upper horizontal arrow is injective and the right vertical arrow is an isomorphism by a), the left vertical surjection must be an isomorphism and thus $\mathrm{K}_3(\mathbb{Z}/9)$ has 2 generators.
For the number of elements we use the following diagram:

$$\begin{array}{ccc} H^0(SL(\mathbb{F}_9), H^4(V(\mathbb{F}_9))) & \xrightarrow{d_2} & H^2(SL(\mathbb{F}_9), H^3(V(\mathbb{F}_9))) = (\mathbb{Z}/3)^2 \\ i^*\big\downarrow & & i^*\big\downarrow \\ H^0(SL(\mathbb{F}_3), H^4(V(\mathbb{F}_3))) & \xrightarrow{d_2} & H^2(SL(\mathbb{F}_3), H^3(V(\mathbb{F}_3))) = \mathbb{Z}/3. \end{array}$$

By the dual of [19, theorem 7.6] the vertical maps are surjective. And according to (a) the upper horizontal map is surjective, so the lower horizontal map must be surjective as well and thus the cardinality of $\mathrm{K}_3(\mathbb{Z}/9)_3$ is 9. □

Theorem 7.2. *Let* $(3, f) = 1$, *then we have*

$$K_3(W_2(\mathbb{F}_{3^f})) = (\mathbb{Z}/9)^{f-1} \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/(3^{2f} - 1).$$

Proof: As in the above proposition we can conclude from $(3, f) = 1$ that the natural map i_* maps $(\mathbb{Z}/3)^2 = K_3(\mathbb{Z}/9)_3$ to a direct summand of $K_3(W_2(\mathbb{F}_{3^f}))_3$. We know by 6.1 that it is 9-torsion and has at least 3^{2f} elements. As it has at most $f + 1$ generators by 4.2 and 3.2, the theorem follows. □

Remark: The result $K_3(\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3 \oplus \mathbb{Z}/3$ of [2] contradicts the results of [1]. Similarly, our result on $K_3(\mathbb{Z}/9)$ contradicts the result $K_3(\mathbb{Z}/9)_3 = \mathbb{Z}/9$ of [1]. The problem seems to be in [1, prop.II 4.5].

8. CALCULATION OF THE DIFFERENTIAL $d_2^{0,3}$ IN CHARACTERISTIC 3

Recall that we want to calculate a differential in a spectral sequence for the extension

$$1 \to V \to U \to P \to 1,$$

where U is the 3-Sylow subgroup of $SL_2(W_2(\mathbb{F}_9))$ such that P consists of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{F}_9.$$

We choose for P the basis

$$t = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix}.$$

We also choose inverse images of t and s in U of the same form. For V we take as a basis the matrices (written multiplicatively)

$$\begin{array}{llll} x_1 = & \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \bar{x}_1 = & \begin{pmatrix} 1 & 0 \\ 3z & 1 \end{pmatrix} \\ x_2 = & \begin{pmatrix} 1+3 & -3 \\ 0 & 1-3 \end{pmatrix} & \bar{x}_2 = & \begin{pmatrix} 1+3z & -3z \\ 0 & 1-3z \end{pmatrix} \\ x_3 = & \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} & \bar{x}_3 = & \begin{pmatrix} 1 & 3z \\ 0 & 1 \end{pmatrix}. \end{array}$$

If we order this basis as $(x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3)$, then the action of t^{-1} and s^{-1} is given by the matrices

$$t^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \qquad s^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

for example the second column in s^{-1} is obtained by

$$s^{-1}\bar{x}_1 s = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3z & 1 \end{pmatrix} \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-3 & 3z \\ 3z & 1+3 \end{pmatrix} = \bar{x}_1 - x_2 - x_3 + \bar{x}_3.$$

If we denote the dual basis of V^* by $\alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2, \alpha_3, \bar{\alpha}_3$, then the action of t and s on V^* is given by the transpose of the above matrices.

Proposition 8.1. *The following are bases and dual bases for homology and cohomology groups of V:*

$$\begin{array}{lcr}
H_1(V, \mathbb{Z}/p): & x_i, \bar{x}_i & \\
H^1(V, \mathbb{Z}/p): & \alpha_i, \bar{\alpha}_i & \\
H_2(V, \mathbb{Z}/p): & x_i \cap x_j, \bar{x}_i \cap \bar{x}_j & i < j \\
 & x_i \cap \bar{x}_j & \\
 & \rho(x_i), \rho(\bar{x}_i) & \\
H^2(V, \mathbb{Z}/p): & \alpha_i \cup \alpha_j, \bar{\alpha}_i \cup \bar{\alpha}_j & i < j \\
 & \alpha_i \cup \bar{\alpha}_j & \\
 & \delta(\alpha_i), \delta(\bar{\alpha}_i) & \\
H_3(V, \mathbb{Z}/p): & x_1 \cap x_2 \cap x_3, \bar{x}_1 \cap \bar{x}_2 \cap \bar{x}_3 & \\
 & x_i \cap x_j \cap \bar{x}_k, \bar{x}_i \cap \bar{x}_j \cap x_k & i < j \\
 & x_i \cap \rho(x_j), \bar{x}_i \cap \rho(x_j), x_i \cap \rho(\bar{x}_j), \bar{x}_i \cap \rho(\bar{x}_j) & \\
H^3(V, \mathbb{Z}/p): & \alpha_1 \cup \alpha_2 \cup \alpha_3, \bar{\alpha}_1 \cup \bar{\alpha}_2 \cup \bar{\alpha}_3 & \\
 & \alpha_i \cup \alpha_j \cup \bar{\alpha}_k, \bar{\alpha}_i \cup \bar{\alpha}_j \cup \alpha_k & i < j \\
 & \alpha_i \cup \delta(\alpha_j), \bar{\alpha}_i \cup \delta(\alpha_j), \alpha_i \cup \delta(\bar{\alpha}_j), \bar{\alpha}_i \cup \delta(\bar{\alpha}_j) & .
\end{array}$$

Proof: This follows from explicit formulas for the cup and the Pontrjagin product, see [3, V.3,V.5]. An analogue result is [7, prop 10.3,10.4]. □

We will frequently use the graded commutativity of the cup and Pontrjagin product and identify terms, e.g. when we write $x_3 \cap \bar{x}_2 \cap \bar{x}_1$ we mean the basis element $-\bar{x}_1 \cap \bar{x}_2 \cap x_3$.

Note that by construction of the Bockstein homomorphism we have for $v, v' \in V$, $v = i_1x_1+i_2x_2+i_3x_3+j_1\bar{x}_1+j_2\bar{x}_2+j_3\bar{x}_3$ and $v' = i'_1x_1+i'_2x_2+i'_3x_3+j'_1\bar{x}_1+j'_2\bar{x}_2+j'_3\bar{x}_3$:

$$\delta\alpha_k([v|v']) = [\tfrac{i_k+i'_k}{3}], \qquad \delta\bar{\alpha}_k([v|v']) = [\tfrac{j_k+j'_k}{3}].$$

Proposition 8.2. *The following are a description of the $SL(\mathbb{F}_9)$-invariant forms A, B, C and D in terms of our basis of $H^3(V, \mathbb{Z}/3)$:*

$$\begin{aligned}
A &= \alpha_1\alpha_2\alpha_3 - \bar{\alpha}_1\bar{\alpha}_2\alpha_3 - \bar{\alpha}_1\alpha_2\bar{\alpha}_3 - \alpha_1\bar{\alpha}_2\bar{\alpha}_3\\
B &= \bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3 - \bar{\alpha}_1\alpha_2\alpha_3 - \alpha_1\alpha_2\bar{\alpha}_3 - \alpha_1\bar{\alpha}_2\alpha_3\\
C &= \alpha_3\delta\alpha_1 + \alpha_1\delta\alpha_3 - \alpha_2\delta\alpha_2 - \alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1\\
&\quad - \bar{\alpha}_3\delta\bar{\alpha}_1 - \bar{\alpha}_1\delta\bar{\alpha}_3 + \bar{\alpha}_2\delta\bar{\alpha}_2 + \bar{\alpha}_1\delta\bar{\alpha}_2 + \bar{\alpha}_2\delta\bar{\alpha}_1\\
D &= \alpha_3\delta\bar{\alpha}_1 + \alpha_1\delta\bar{\alpha}_3 - \alpha_2\delta\bar{\alpha}_2 - \alpha_1\delta\bar{\alpha}_2 - \alpha_2\delta\bar{\alpha}_1\\
&\quad + \bar{\alpha}_3\delta\alpha_1 + \bar{\alpha}_1\delta\alpha_3 - \bar{\alpha}_2\delta\alpha_2 - \bar{\alpha}_1\delta\alpha_2 - \bar{\alpha}_2\delta\alpha_1
\end{aligned}$$

Proof: Recall that A, B, C and D are expressions of the form

$$\begin{array}{ccccccc}
\Lambda^3V & \xrightarrow{abc-bac} & V & \xrightarrow{tr} & \mathbb{F}_9 & \to & \mathbb{Z}/3\\
S^2V & \xrightarrow{ab} & V & \xrightarrow{tr} & \mathbb{F}_9 & \to & \mathbb{Z}/3.
\end{array}$$

Now we just have to calculate the effect of these maps on our basis of Λ^3V respectively S^2V (written additively). For example

$$\begin{aligned}
&A(\bar{x}_1 \cap x_2 \cap \bar{x}_3) =\\
&-\chi_1 \circ tr\left(\begin{pmatrix} 0 & 0\\ z & 0\end{pmatrix}\begin{pmatrix} 1 & -1\\ 0 & -1\end{pmatrix}\begin{pmatrix} 0 & z\\ 0 & 0\end{pmatrix} - \begin{pmatrix} 1 & -1\\ 0 & -1\end{pmatrix}\begin{pmatrix} 0 & 0\\ z & 0\end{pmatrix}\begin{pmatrix} 0 & z\\ 0 & 0\end{pmatrix}\right)\\
&\qquad = -\chi_1 \circ tr\left(\begin{pmatrix} 0 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right) = -\chi_1(-2) = -1,
\end{aligned}$$

so we get a contribution $-\bar{\alpha}_1\alpha_2\bar{\alpha}_3$ for A. □

9. The Charlap-Vasques description of the differential

In a situation like ours, Charlap and Vasquez [4] described the differential

$$d_2^{p,q} : E_2^{p,q}(\mathbb{Z}/p) \to E_2^{p+2,q-1}(\mathbb{Z}/p)$$

as follows:
Considering the following cup product

$$H^p(P, H^q(V, \mathbb{Z}/3)) \otimes H^2(P, H^{q-1}(V, \mathbb{Z}/3) \otimes H_q(V, \mathbb{Z}/3)) \xrightarrow{\cup} H^{p+2}(P, H^{q-1}(V, \mathbb{Z}/3)),$$

the differential is given by

$$d_2^{p,q}(\xi) = (-1)^p \xi \cup (V^q - Q_*(\chi)).$$

Here $\chi \in H^2(P, V)$ is the cohomology class of the extension and Q_* the functor $H^2(P, -)$ applied to the following map Q induced by Pontrjagin multiplication from the right:

$$V = H_1(V, \mathbb{Z}/3) \xrightarrow{\cap} \mathrm{Hom}_{\mathbb{Z}/3}(H_{q-1}(V, \mathbb{Z}/3), H_q(V, \mathbb{Z}/3)) = H^{q-1}(V, \mathbb{Z}/3) \otimes H_q(V, \mathbb{Z}/3).$$

On the other hand V^q is universal in the sense that it only depends on the action of P on V and not on the specific extension. We will calculate the term $\xi \cup V^q$ in the next section by explicitly calculating the differential in the spectral sequence for the split extension.
In this section we are going to calculate the term $\xi \cup Q_*(\chi)$. To do this we have to determine the class χ of the extension, calculate Q_* of χ and calculate the above cup product.

For the cohomology of P we have the following results:
As P is the direct product of the cyclic groups $T = \langle t \rangle$ and $S = \langle s \rangle$, we will use the tensor product of the minimal resolutions of T and S as our resolution of P: the minimal resolution of T is given by

$$E. = \ldots \xrightarrow{N_t} \mathbb{Z}[T] \xrightarrow{t-1} \mathbb{Z}[T] \xrightarrow{N_t} \mathbb{Z}[T] \xrightarrow{t-1} \mathbb{Z}[T] \to 0$$

where $N_t = 1 + t + t^2$, and similarly we have the minimal resolution $F.$ for S. The tensor product of the two resolutions is given by

$$(E. \otimes F.)_n = \bigoplus_{p+q=n} E_p \otimes F_q, \qquad d(e \otimes f) = de \otimes f + (-1)^{\deg e} e \otimes df.$$

Note that $\mathbb{Z}[P] = \mathbb{Z}[T \times S] = \mathbb{Z}[T] \otimes \mathbb{Z}[S]$, so in low degrees the resolution is given by

$$\ldots \to \mathbb{Z}[P]^3 \xrightarrow{(N_t,0),(-s+1,t-1),(0,N_s)} \mathbb{Z}[P]^2 \xrightarrow{t-1,s-1} \mathbb{Z}[P] \to 0.$$

The cohomology of P with coefficients in the module M is given by the homology of the complex $Y_q = \mathrm{Hom}_{\mathbb{Z}[P]}(\mathbb{Z}[P]^{q+1}, M)$. We will identify a $\mathbb{Z}[P]$-linear homomorphism $\mathbb{Z}[P]^{q+1}$ with the $q+1$-tupels of images of 1, ordered in the following way: $E_q \otimes F_0, E_{q-1} \otimes F_1, \ldots$

Lemma 9.1. *a)* $H^2(P, V) = (\mathbb{Z}/3)^2$, *a basis for cycles is given by*

$$(x_3, 0, 0), (\bar{x}_3, 0, 0), (0, 0, x_3), (0, 0, \bar{x}_3)$$

and a basis of boundaries is given by $(x_3, 0, -\bar{x}_3), (\bar{x}_3, 0, x_3)$.
b) $H^2(P, V^*) = (\mathbb{Z}/3)^2$, *a basis for cycles is given by*

$$(\alpha_1, 0, 0), (\bar{\alpha}_1, 0, 0), (0, 0, \alpha_1), (0, 0, \bar{\alpha}_1)$$

and a basis of boundaries is given by $(\alpha_1, 0, \bar{\alpha}_1), (\bar{\alpha}_1, 0, -\alpha_1)$.
c) $H^2(P, \Lambda^2 V^*) = (\mathbb{Z}/3)^2$, *a basis for cycles is given by*

$$(\alpha_1\bar{\alpha}_2 + \bar{\alpha}_1\alpha_2, 0, 0), (\alpha_1\alpha_2 - \bar{\alpha}_1\bar{\alpha}_2, 0, 0), (0, 0, \alpha_1\alpha_2 - \bar{\alpha}_1\bar{\alpha}_2), (0, 0, \alpha_1\bar{\alpha}_2 + \bar{\alpha}_1\alpha_2)$$

and a basis of boundaries is given by

$$(\alpha_1\bar\alpha_2+\bar\alpha_1\alpha_2, 0, -\alpha_1\alpha_2+\bar\alpha_1\bar\alpha_2), (\alpha_1\alpha_2-\bar\alpha_1\bar\alpha_2, 0, \alpha_1\bar\alpha_2+\bar\alpha_1\alpha_2).$$

Proof: The cycles are given by triples (a,b,c) such that

$$0=(t-1)a=(s-1)a+N_tb=-N_sb+(t-1)c=(s-1)c$$

and the boundaries are given by triples

$$\left(N_tx,(t-1)y-(s-1)x,N_sy\right).$$

The action of P on V and V^* is given by t and s resp. ${}^tt^{-1}$ and ${}^ts^{-1}$, the action of P on Λ^2V^* has to be calculated. We have chosen representants such that the second component is always trivial. □

Since the cocyle of our extension is most easily given in terms of the bar resolution, we need a comparison between the minimal and bar resolution for cyclic groups:

Lemma 9.2. *The following is an augmentation preserving chain map from the minimal to the bar resolution of a cyclic group of order m with generator t (necessarily being a homotopy equivalence* [3, I 7.5]*): In odd degree we take the map*

$$\begin{array}{rcc} \mathbb{Z}[T] & \to & \mathbb{Z}[T][T^{2n+1}] \\ 1 & \mapsto & \sum[t|t^{i_1}|t|t^{i_2}|\dots|t^{i_n}|t] \end{array}$$

and in even degree

$$\begin{array}{rcc} \mathbb{Z}[T] & \to & \mathbb{Z}[T][T^{2n}] \\ 1 & \mapsto & \sum[t^{i_1}|t|t^{i_2}|\dots|t^{i_n}|t] \end{array}$$

The sum goes over all n-tuples $(i_1,\dots,i_n)\in\{0,\dots,m-1\}^n$.

Proof: Easy verification by induction. □

Let U be an extension of P by V and choose a lift $\tilde a$ of each element a of P in U. Then the cocycle corresponding to the extension is given by

$$[a|b]\mapsto \tilde a\tilde b(\tilde{ab})^{-1}.$$

Lemma 9.3. *A representant of the class χ of our extension in* $\mathrm{Hom}_{\mathbb{Z}[P]}(\mathbb{Z}[P]^3,V)$ *is given by* $(-x_3,0,-\bar x_3)$.

Proof: We have to take the tensor product of the above maps from the minimal to the bar resolution for the groups T and S and calculate the class of the cocycle in the bar resolution. For the first component we get

$$\begin{array}{rcc} \mathbb{Z}[P] & \to & \mathbb{Z}[P][P\times P] \\ 1 & \mapsto & \sum_{i=0}^2[t^i|t] \end{array}$$

and for our choice of the lift of t we have

$$[t^i|t] \mapsto \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i+1 \\ 0 & 1 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = -x_3 & \text{for } i = 2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{otherwise .} \end{cases}$$

Similarly, we get for the second component

$$1 \mapsto [t|s] \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z+1 \\ 0 & 1 \end{pmatrix} = 1$$

and for the third component

$$1 \mapsto \sum_{i=0}^{2} [s^i|s] \mapsto \sum_{i=0}^{2} \begin{pmatrix} 1 & -iz \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (i+1)z \\ 0 & 1 \end{pmatrix} = -\bar{x}_3.$$

□

The next step is to calculate $Q_*(\chi)$ of this element.

Lemma 9.4. *The element $Q_*(\chi) \in H^2(P, H_3(V, \mathbb{Z}/3) \otimes H^2(V, \mathbb{Z}/3))$ is represented by $(u, 0, v)$, where*

$$u = -(x_1 \cap x_2 \cap x_3) \otimes \alpha_1 \cup \alpha_2 - \sum_{i \neq 3} (x_i \cap \bar{x}_j \cap x_3) \otimes \alpha_i \cup \bar{\alpha}_j$$
$$- \sum_{i<j} (\bar{x}_i \cap \bar{x}_j \cap x_3) \otimes \bar{\alpha}_i \cup \bar{\alpha}_j - \sum x_3 \cap \rho(x_i) \otimes \delta\alpha_i - \sum x_3 \cap \rho(\bar{x}_i) \otimes \delta\bar{\alpha}_i,$$

$$v = -(\bar{x}_1 \cap \bar{x}_2 \cap \bar{x}_3) \otimes \bar{\alpha}_1 \cup \bar{\alpha}_2 - \sum_{i \neq 3} (\bar{x}_i \cap x_j \cap \bar{x}_3) \otimes \bar{\alpha}_i \cup \alpha_j$$
$$- \sum_{i<j} (x_i \cap x_j \cap \bar{x}_3) \otimes \alpha_i \cup \alpha_j - \sum \bar{x}_3 \cap \rho(x_i) \otimes \delta\alpha_i - \sum \bar{x}_3 \cap \rho(\bar{x}_i) \otimes \delta\bar{\alpha}_i.$$

Proof: To get the components of $Q_*(\chi)$, we have to determine what the cup product with $(-x_3, 0, -\bar{x}_3)$ does on a basis of $H_2(V, \mathbb{Z}/3)$. For example, $-x_3$ sends $x_i \cap \bar{x}_j$ to $-x_i \cap \bar{x}_j \cap x_3$ and thus gives a contribution $-x_i \cap \bar{x}_j \cap x_3 \otimes \alpha_i \cup \bar{\alpha}_j$ to u, or $-\bar{x}_3$ sends ρx_i to $-\rho x_i \cap \bar{x}_3 = -\bar{x}_3 \cap \rho x_i$ and thus gives a contribution $-\bar{x}_3 \cap \rho x_i \otimes \delta\alpha_i$ to v. □

Finally we have to calculate the cup product $\xi \cup Q_*(\chi)$. For this we have to go into the definition of the cup product:

The cup product of two cocycles $a \in \mathrm{Hom}(Y_i, M)$ and $b \in \mathrm{Hom}(Y_j, N)$ is represented by the map

$$a \cup b : Y_{i+j} \xrightarrow{\Delta} Y_i \otimes Y_j \to M \otimes N,$$

where Δ is a "diagonal approximation",[3, V.3]. For a cyclic group with generator t a diagonal approximation is given in [3, V 1]:

$$\Delta_{ij}(1) = \begin{cases} 1 \otimes 1 & i \text{ even} \\ 1 \otimes t & i \text{ even }, j \text{ odd} \\ \sum_{i<j} t^i \otimes t^j & i, j \text{ odd} \end{cases}$$

We have to work with the tensor product of the approximations for T and S: let $E.$ be the resolution for T and $F.$ be the resolution for S. Then an elements $\xi \in H^0(P, H^3(V, \mathbb{Z}/3))$ is represented by a map sending $1 \otimes 1 \in E_0 \otimes F_0$ to some cocycle ω in $H^3(V, \mathbb{Z}/3)$. On the other hand we just calculated that $Q_*(\chi)$ is represented by the map sending $(1\otimes 1, 1\otimes 1, 1\otimes 1) \in (E_2 \otimes F_0) \oplus (E_1 \otimes F_1) \oplus (E_0 \otimes F_2)$ to $(u, 0, v)$ in $H^2(V, \mathbb{Z}/3) \otimes H_3(V, \mathbb{Z}/3)$. Thus a representant of the cup product has the following three components:

$$\begin{array}{ccccc} E_2 \otimes F_0 & \xrightarrow{\Delta\otimes\Delta} & E_2 \otimes F_0 \otimes E_0 \otimes F_0 & \to & H^2(V,\mathbb{Z}/3) \otimes H_3(V,\mathbb{Z}/3) \otimes H^3(V,\mathbb{Z}/3) \\ 1 \otimes 1 & \mapsto & 1 \otimes 1 \otimes 1 \otimes 1 & \mapsto & u \otimes \omega \\ E_1 \otimes F_1 & \xrightarrow{\Delta\otimes\Delta} & E_1 \otimes F_1 \otimes E_0 \otimes F_0 & \to & H^2(V,\mathbb{Z}/3) \otimes H_3(V,\mathbb{Z}/3) \otimes H^3(V,\mathbb{Z}/3) \\ 1 \otimes 1 & \mapsto & 1 \otimes 1 \otimes t \otimes s & \mapsto & 0 \otimes ts\omega \\ E_0 \otimes F_2 & \xrightarrow{\Delta\otimes\Delta} & E_0 \otimes F_2 \otimes E_0 \otimes F_0 & \to & H^2(V,\mathbb{Z}/3) \otimes H_3(V,\mathbb{Z}/3) \otimes H^3(V,\mathbb{Z}/3) \\ 1 \otimes 1 & \mapsto & 1 \otimes 1 \otimes 1 \otimes 1 & \mapsto & v \otimes \omega \end{array}$$

Evaluating $u \otimes \omega$ and $v \otimes \omega$ we get

Proposition 9.5. *The second term $\xi \cup (-Q_*(\chi))$ in the Charlap Vasquez description of the differential $d_2^{0,3}$ is given by*

$$\begin{array}{lcl} A & \mapsto & (-\alpha_1 \cup \alpha_2 + \bar\alpha_1 \cup \bar\alpha_2 \;, 0, \; \bar\alpha_1 \cup \alpha_2 + \alpha_1 \cup \bar\alpha_2) \\ B & \mapsto & (\alpha_1 \cup \bar\alpha_2 + \bar\alpha_1 \cup \alpha_2 \;, 0, \; \alpha_1 \cup \alpha_2 - \bar\alpha_1 \cup \bar\alpha_2) \\ C & \mapsto & (\delta\alpha_1 \;, 0, \; -\delta\bar\alpha_1) \\ D & \mapsto & (\delta\bar\alpha_1 \;, 0, \; \delta\alpha_1) \end{array}$$

10. The differential for the split extension

Let $\bar U$ be the split extension of P by V. Let $X_n = \mathbb{Z}[V][V^n]$ be the bar resolution of V. There is an action of P on $X.$ by

$$p * v[v_1|v_2|\ldots|v_n] = p(v)[p(v_1)|p(v_2)|\ldots|p(v_n)],$$

which is compatible with the differential and the augmentation. Let $Y.$ be the minimal resolution of P, i.e. the tensor product of the minimal resolutions of T and S. Then $Y. \otimes X.$ is a $\mathbb{Z}[\bar U]$-module via the natural action

$$p(y \otimes x) = p(y) \otimes p(x), \quad v(y \otimes x) = y \otimes vx.$$

Furthermore $Y. \otimes X.$ is a $\mathbb{Z}[\bar U]$-free resolution of $\mathbb{Z}$, [7, prop 11.1].

Thus we can calculate the cohomology $H^*(\bar{U}, \mathbb{Z}/3)$ as the homology of the double complex

$$C_{..} = \mathrm{Hom}_{\mathbf{Z}[\bar{U}]}(Y. \otimes X., \mathbb{Z}/3) = \mathrm{Hom}_{\mathbf{Z}[P]}(Y., \mathrm{Hom}_{\mathbf{Z}[V]}(X., \mathbb{Z}/3)).$$

This double complex yields a spectral sequence with

$$E_0^{p,q} = \mathrm{Hom}_{\mathbf{Z}[P]}(Y_p, \mathrm{Hom}_{\mathbf{Z}[V]}(X_q, \mathbb{Z}/3))$$
$$E_2^{p,q} = H^p(P, H^q(V, \mathbb{Z}/3))$$

and limit $H^*_{total}(C) = H^*(\bar{U}, \mathbb{Z}/3)$. One sees as in [7, prop 11.2] that this spectral sequence is the same as the Hochschild-Serre spectral sequence to the extension $1 \to V \to \bar{U} \to P \to 1$.

The differential for the spectral sequence to the above double complex is calculated as follows see [9, 4.8]:

Let d_{II} be the vertical and d_I be the horizontal differential.

$$\begin{array}{ccc} E^{0,3} & \xrightarrow{d_I = d_P} & E^{1,3} \\ & & \uparrow {\scriptstyle d_{II} = -d_V} \\ & & E^{1,2} \xrightarrow{d_I = d_P} E^{2,2} \end{array}$$

Elements of $Z_2^{0,3}$ are of the form $x = x^{0,3} + x^{1,2}$ such that $d_{II}x^{0,3} = 0$ and $d_{II}x^{1,2} + d_I x^{0,3} = 0$. They can be identified modulo boundaries with $H^0(P, H^3(V, \mathbb{Z}/3))$ by projection to $x^{0,3}$. The differential of x is given by

$$d(x) = (d_I + d_{II})(x^{0,3} + x^{1,2}) = d_I x^{0,3} + d_{II}x^{0,3} + d_I x^{1,2} + d_{II}x^{1,2} = d_I x^{1,2}.$$

In our case $\alpha \in \{A, B, C, D\}$ we have $d_{II}\alpha = 0$ and we have to find an element $\beta \in E_0^{1,2}$ such that $d_{II}\beta + d_I\alpha = 0$. Then we have to calculate $d_I\beta$ and the resulting element of $E_0^{2,2}$ will represent an element of $H^2(P, H^2(V, \mathbb{Z}/3))$.

As we have

$$E_0^{p,q} = \mathrm{Hom}_{\mathbf{Z}[P]}(\mathbb{Z}[P]^{p+1}, \mathrm{Hom}_{\mathbf{Z}[V]}(X_q, \mathbb{Z}/3)),$$

we will identify a $\mathbb{Z}[P]$-linear homomorphism $\mathbb{Z}[P]^{p+1} \to \mathrm{Hom}_{\mathbf{Z}[P]}(X_q, \mathbb{Z}/3)$ with the $p+1$-tupel of images of 1. Similarly we have $\mathrm{Hom}_{\mathbf{Z}[V]}(X_q, \mathbb{Z}/3) = \mathrm{Hom}_{\mathbf{Z}[V]}(\mathbb{Z}[V][V^q], \mathbb{Z}/3)$ and we will identify an element of this group with a map $V^q \to \mathbb{Z}/3$.

So for a representant of α we have to calculate $d_I\alpha = d_P\alpha$. With the above identifications this element has components $d_P(\alpha)_1 = (t-1)\alpha$, and $d_P(\alpha)_2 = (s-1)\alpha$.

Then we have to find an element β of $E_0^{1,2}$ such that $-d_{II}\beta = d_V\beta = d_I\alpha$. The differential d_V is given by

$$d_V(f)[a|b|c] = f[a|b] - f[a+b|c] + f[a|b+c] - f[b|c]$$

on each component.

The next step is to calculate the differential $d_I = d_P$ of $\beta = (\beta_1, \beta_2)$: With the above identifications it has the three components $N_t(\beta_1), -(s-1)\beta_1 + (t-1)\beta_2$ and $N_s(\beta_2)$ respectively.
Finally we will show that that some of the resulting cocycles become zero in $E_2^{2,2} = H^2(P, H^2(V, \mathbb{Z}/3))$ by exhibiting them as boundaries from $E_0^{2,1}$.
We will proceed for $A, B \in (\Lambda^3 V^*)^P$ and $C, D \in (S^2 V^*)^P$ separately. We will only give the results of the calculation and indicate how the calculations can be done. All verifications are left to the reader.
The following will be the result of the next sections:

Proposition 10.1. *Let A, B, C and D as in proposition 8.2. Then the three components for the differential in $E_2^{2,2}(\mathbb{Z}/3)$ are:*

$$\begin{array}{llll} A: & (-\alpha_1\alpha_2 + \bar\alpha_1\bar\alpha_2 - \delta\alpha_1 & ,0, & \bar\alpha_1\alpha_2 + \alpha_1\bar\alpha_2 + \delta\bar\alpha_1) \\ B: & (\alpha_1\bar\alpha_2 + \bar\alpha_1\alpha_2 - \delta\bar\alpha_1 & ,0, & \alpha_1\alpha_2 - \bar\alpha_1\bar\alpha_2 - \delta\alpha_1) \\ C: & (\delta\alpha_1 + \alpha_1\alpha_2 - \bar\alpha_1\bar\alpha_2 & ,0, & -\delta\bar\alpha_1 - \alpha_1\bar\alpha_2 - \bar\alpha_1\alpha_2) \\ D: & (\delta\bar\alpha_1 + \alpha_1\bar\alpha_2 + \bar\alpha_1\alpha_2 & ,0, & \delta\alpha_1 + \alpha_1\alpha_2 - \bar\alpha_1\bar\alpha_2) \end{array}$$

b) The three components of the differential in $E_2^{2,2}(\mathbb{Z})$ are given by

$$\begin{array}{llll} A: & (-\alpha_1\alpha_2 + \bar\alpha_1\bar\alpha_2 & ,0, & \bar\alpha_1\alpha_2 + \alpha_1\bar\alpha_2) \\ B: & (\alpha_1\bar\alpha_2 + \bar\alpha_1\alpha_2 & ,0, & \alpha_1\alpha_2 - \bar\alpha_1\bar\alpha_2) \\ C: & (\alpha_1\alpha_2 - \bar\alpha_1\bar\alpha_2 & ,0, & -\alpha_1\bar\alpha_2 - \bar\alpha_1\alpha_2) \\ D: & (\alpha_1\bar\alpha_2 + \bar\alpha_1\alpha_2 & ,0, & \alpha_1\alpha_2 - \bar\alpha_1\bar\alpha_2) \end{array}$$

Proof: a) 9.5, A.5, B.4
b) Obvious from a). □

Appendix A. The differential for $\Lambda^3 V^*$

Let a_i be α_i considered as a map $V \to \mathbb{Z}/3$. Then $a_i a_j a_k : V^3 \to \mathbb{Z}/3$ represents $\alpha_i\alpha_j\alpha_k$ etc.

Lemma A.1. *The image of A and B in $E_0^{1,3}$ are given by*

$$A_1^{1,3} = a_1a_1a_2 + a_1a_1a_3 + a_1a_2a_2 - a_1\bar a_1\bar a_2 - a_1\bar a_1\bar a_3 - a_1\bar a_2\bar a_2 - \bar a_1 a_1\bar a_2 - \bar a_1 a_1\bar a_3 \\ - \bar a_1 a_2\bar a_2 - \bar a_1\bar a_1 a_2 - \bar a_1\bar a_1 a_3 - \bar a_1\bar a_2 a_2$$

$$A_2^{1,3} = -a_1a_1a_1 - a_1a_1a_2 - a_1a_1\bar a_1 - a_1a_1\bar a_3 + a_1a_2a_1 - a_1a_2\bar a_1 - a_1a_2\bar a_2 - a_1\bar a_1 a_1 \\ - a_1\bar a_1 a_3 + a_1\bar a_1\bar a_1 + a_1\bar a_1\bar a_2 - a_1\bar a_2 a_1 - a_1\bar a_2 a_2 - a_1\bar a_2\bar a_1 - \bar a_1 a_1 a_1 - \bar a_1 a_1 a_3 \\ + \bar a_1 a_1\bar a_1 + \bar a_1 a_1\bar a_2 - \bar a_1 a_2 a_1 - \bar a_1 a_2 a_2 - \bar a_1 a_2\bar a_1 + \bar a_1\bar a_1 a_1 + \bar a_1\bar a_1 a_2 + \bar a_1\bar a_1\bar a_1 \\ + \bar a_1\bar a_1\bar a_3 - \bar a_1\bar a_2 a_1 + \bar a_1\bar a_2\bar a_1 + \bar a_1\bar a_2\bar a_2$$

$$B_1^{1,3} = -a_1a_1\bar{a}_2 - a_1a_1\bar{a}_3 - a_1a_2\bar{a}_2 - a_1\bar{a}_1a_2 - a_1\bar{a}_1a_3 - a_1\bar{a}_2a_2 - \bar{a}_1a_1a_2 - \bar{a}_1a_1a_3 - \bar{a}_1a_2a_2 + \bar{a}_1\bar{a}_1\bar{a}_2 + \bar{a}_1\bar{a}_1\bar{a}_3 + \bar{a}_1\bar{a}_2\bar{a}_2$$

$$B_1^{1,3} = -a_1a_1a_1 - a_1a_1a_3 + a_1a_1\bar{a}_1 + a_1a_1\bar{a}_2 - a_1a_2a_1 - a_1a_2a_2 - a_1a_2\bar{a}_1 + a_1\bar{a}_1a_1 + a_1\bar{a}_1a_2 + a_1\bar{a}_1\bar{a}_1 + a_1\bar{a}_1\bar{a}_3 - a_1\bar{a}_2a_1 + a_1\bar{a}_2\bar{a}_1 + a_1\bar{a}_2\bar{a}_2 + \bar{a}_1a_1a_1 + \bar{a}_1a_1a_2 + \bar{a}_1a_1\bar{a}_1 + \bar{a}_1a_1\bar{a}_3 - \bar{a}_1a_2a_1 + \bar{a}_1a_2\bar{a}_1 + \bar{a}_1a_2\bar{a}_2 + \bar{a}_1\bar{a}_1a_1 + \bar{a}_1\bar{a}_1a_3 - \bar{a}_1\bar{a}_1\bar{a}_1 - \bar{a}_1\bar{a}_1\bar{a}_2 + \bar{a}_1\bar{a}_2a_1 + \bar{a}_1\bar{a}_2a_2 + \bar{a}_1\bar{a}_2\bar{a}_1.$$

Proof: One has to calculate $t-1$ and $s-1$ of A and B. □

Lemma A.2. *Define the following maps $V \to \mathbb{Z}/3$ for $v = i_1x_1 + i_2x_2 + i_3x_3 + j_1\bar{x}_1 + j_2\bar{x}_2 + j_3\bar{x}_3$*

$$\begin{aligned} u_n(v) &= i_n^2, & \bar{u}_n(v) &= j_n^2 \\ w_{n,m}(v) &= -i_ni_m, & \bar{w}_{n,m}(v) &= -j_nj_m \\ v_{n,m}(v) &= -i_nj_m \end{aligned}$$

Then the following are lifts of A and B to $E_0^{1,2}$, i.e. $d_VA^{1,2} = d_PA^{0,3}$ and $d_VB^{1,2} = d_PB^{0,3}$:

$$A_1^{1,2} = a_1\bar{u}_2 - a_1u_2 + \bar{a}_1v_{2,2} - \bar{u}_1a_2 - \bar{u}_1a_3 + u_1a_2 + u_1a_3 - v_{1,1}\bar{a}_2 - v_{1,1}\bar{a}_3$$

$$A_2^{1,2} = -a_1\bar{u}_1 + a_1v_{2,1} + a_1v_{2,2} + a_2u_1 - \bar{a}_1\bar{u}_2 + \bar{a}_1u_2 - \bar{a}_1v_{1,1} + \bar{a}_1v_{2,1} + \bar{a}_1w_{12} + \bar{a}_2\bar{u}_1 - \bar{a}_2u_1 - \bar{a}_2v_{1,1} - \bar{u}_1a_2 + \bar{u}_1\bar{a}_1 + \bar{u}_1\bar{a}_3 - \bar{w}_{1,2}a_1 + \bar{w}_{1,2}\bar{a}_1 - u_1a_1 - u_1a_2 - u_1\bar{a}_1 - u_1\bar{a}_3 - v_{1,1}a_1 + v_{1,1}a_2 - v_{1,1}a_3 + v_{1,1}\bar{a}_2 - v_{1,2}a_1 - v_{1,2}\bar{a}_1 + w_{1,2}a_1$$

$$B_1^{1,2} = a_1v_{2,2} - \bar{a}_1\bar{u}_2 + \bar{a}_1u_2 + \bar{u}_1\bar{a}_2 + \bar{u}_1\bar{a}_3 - u_1\bar{a}_2 - u_1\bar{a}_3 - v_{1,1}a_2 - v_{1,1}a_3$$

$$B_2^{1,2} = -a_1\bar{u}_1 - a_1\bar{u}_2 + a_1u_2 + a_1v_{2,1} - a_2u_1 - \bar{a}_1v_{1,1} - \bar{a}_1v_{2,1} - \bar{a}_1v_{2,2} + \bar{a}_1w_{1,2} + \bar{a}_2\bar{u}_1 - \bar{a}_2u_1 + \bar{a}_2v_{1,1} - \bar{u}_1a_2 + \bar{u}_1a_3 - \bar{u}_1\bar{a}_1 - \bar{u}_1\bar{a}_2 + \bar{w}_{1,2}a_1 + \bar{w}_{1,2}\bar{a}_1 - u_1a_1 - u_1a_3 + u_1\bar{a}_1 + u_1\bar{a}_2 + v_{1,1}a_1 - v_{1,1}a_2 + v_{1,1}\bar{a}_3 - v_{1,2}a_1 + v_{1,2}\bar{a}_1 - w_{1,2}a_1$$

Proof: First one has to verify the following equations of functions $V^2 \to \mathbb{Z}/3$:

$$\begin{aligned} a_na_n &= d_Vu_n \\ \bar{a}_n\bar{a}_n &= d_V\bar{u}_n \\ a_n\bar{a}_m + a_m\bar{a}_n &= d_Vv_{n,m} \\ a_na_m + a_ma_n &= d_Vw_{n,m} \\ \bar{a}_n\bar{a}_m + \bar{a}_m\bar{a}_n &= d_V\bar{w}_{n,m}. \end{aligned}$$

Then one uses these equations and $d_V(a_n) = d_V(\bar{a}_n) = 0$ to write the expressions of the last lemma as images of $-d_V$, for example

$$a_1 a_1 a_2 = d_V(u_1)a_2 = d_V(u_1 a_2).$$

□

Lemma A.3. *The three components of $d_P A^{1,2}$ and $d_P B^{1,2}$ in $E_0^{2,2}$ are given as follows:*

$$\begin{aligned}
A_1^{2,2} &= -a_1\bar{u}_1 + a_1 u_1 - \bar{a}_1 v_{1,1} - \bar{u}_1 a_1 + u_1 a_1 - v_{1,1}\bar{a}_1\\
A_2^{2,2} &= a_1\bar{u}_1 - a_1 v_{1,1} - \bar{a}_1 u_1 + \bar{a}_1 v_{1,1} + \bar{u}_1 a_1 - u_1\bar{a}_1 - v_{1,1}a_1 + v_{1,1}\bar{a}_1\\
A_3^{2,2} &= a_1 v_{1,1} - \bar{a}_1\bar{u}_1 + \bar{a}_1 u_1 - \bar{u}_1\bar{a}_1 + u_1\bar{a}_1 + v_{1,1}a_1\\
B_1^{2,2} &= -a_1 v_{1,1} + \bar{a}_1\bar{u}_1 - \bar{a}_1 u_1 + \bar{u}_1\bar{a}_1 - u_1\bar{a}_1 - v_{1,1}a_1\\
B_2^{2,2} &= -a_1\bar{u}_1 - a_1 v_{1,1} - \bar{a}_1 u_1 - \bar{a}_1 v_{1,1} - \bar{u}_1 a_1 - u_1\bar{a}_1 - v_{1,1}a_1 - v_{1,1}\bar{a}_1\\
B_3^{2,2} &= -a_1\bar{u}_1 + a_1 u_1 - \bar{a}_1 v_{1,1} - \bar{u}_1 a_1 + u_1 a_1 - v_{1,1}\bar{a}_1
\end{aligned}$$

Proof: We have to calculate $N_t A_1^{1,2}$, $-(s-1)A_1^{1,2}+(t-1)A_2^{1,2}$ and $N_s A_2^{1,2}$ and similarly for B. The action of t and s on a_1, $\bar{a}_1$, u_1, $\bar{u}_1$ and $v_{1,1}$ is trivial and on the other terms given as follows:

$$\begin{array}{rclrcl}
ta_2 &=& a_2 + a_1 & sa_2 &=& a_2 + 2\bar{a}_1\\
t\bar{a}_2 &=& \bar{a}_2 + \bar{a}_1 & s\bar{a}_2 &=& \bar{a}_2 + a_1\\
ta_3 &=& a_3 + a_2 & sa_3 &=& a_3 + a_1 + 2\bar{a}_1 + 2\bar{a}_2\\
t\bar{a}_3 &=& \bar{a}_3 + \bar{a}_2 & s\bar{a}_3 &=& \bar{a}_3 + \bar{a}_1 + a_1 + a_2\\
tu_2 &=& u_2 + u_1 + w_{1,2} & su_2 &=& u_2 + \bar{u}_1 - v_{2,1}\\
t\bar{u}_2 &=& \bar{u}_2 + \bar{u}_1 + \bar{w}_{1,2} & s\bar{u}_2 &=& \bar{u}_2 + u_1 + v_{1,2}\\
tw_{1,2} &=& w_{1,2} + 2u_1 & sw_{1,2} &=& w_{1,2} + 2v_{1,1}\\
t\bar{w}_{1,2} &=& \bar{w}_{1,2} + 2\bar{u}_1 & s\bar{w}_{1,2} &=& \bar{w}_{1,2} + v_{1,1}\\
tv_{1,2} &=& v_{1,2} + v_{1,1} & sv_{1,2} &=& v_{1,2} + 2u_1\\
tv_{2,1} &=& v_{2,1} + v_{1,1} & sv_{2,1} &=& v_{2,1} + \bar{u}_1\\
tv_{2,2} &=& v_{2,2} + v_{1,1} + v_{1,2} + v_{2,1} & sv_{2,2} &=& v_{2,2} + w_{1,2} + 2\bar{w}_{1,2} + 2v_{1,1}
\end{array}$$

For example

$$s\bar{w}_{1,2}(v) = \bar{w}_{1,2}(s^{-1}v) = -j_1(j_2 + i_1) = \bar{w}_{1,2}(v) + v_{1,1}(v).$$

□

Lemma A.4. *Let $ch_{n,m}$ be the characteristic function which is 1 on $nx_1 + m\bar{x}_1$ and 0 on all other elements of V. Let $d_V : E_0^{2,1} \to E_0^{2,2}$ be the boundary given by*

$d_V(f)[v|w] = f[w] - f[v+w] + f[v]$ on each component. Then we have the following equations in $E_2^{2,2}$:

$$\begin{aligned}
A_1^{2,2} &= -\delta a_1 + d_V(ch_{1,1} + ch_{1,2} + ch_{2,0}) \\
A_2^{2,2} &= d_V(ch_{1,2} - ch_{2,1}) \\
A_3^{2,2} &= \delta\bar{a}_1 - d_V(ch_{0,2} + ch_{1,1} + ch_{2,1}) \\
B_1^{2,2} &= -\delta\bar{a}_1 + d_V(ch_{0,2} + ch_{1,1} + ch_{2,1}) \\
B_2^{2,2} &= d_V(ch_{2,2} - ch_{1,1}) \\
B_3^{2,2} &= -\delta a_1 + d_V(ch_{1,1} + ch_{1,2} + ch_{2,0})
\end{aligned}$$

Proof: This has to be proved by inspection. Since only the coefficients of x_1 and $\bar{x}_1$ of elements in V are involved, one has to check that the above functions agree on all 81 elements of $\langle x_1, \bar{x}_1 \rangle^2 \subseteq V^2$. □

Finally the lemmas prove the following proposition:

Proposition A.5. *The first term $\xi \cup V^3$ in the Charlap Vasquez description of the differential $d_2^{0,3}$ is given by*

$$\begin{aligned}
A &\mapsto (-\delta\alpha_1, 0, \delta\bar{\alpha}_1) \\
B &\mapsto (-\delta\bar{\alpha}_1, 0, -\delta\alpha_1)
\end{aligned}$$

Appendix B. The differential for S^2V^*

Lemma B.1. *Let $[\frac{a}{3}]$ be the largest integer less than or equal to $\frac{a}{3}$ and define the following functions on $v \in V$, $v = i_1x_1 + i_2x_2 + i_3x_3 + j_1\bar{x}_1 + j_2\bar{x}_2 + j_3\bar{x}_3$:*

$$\begin{aligned}
p_2 &= -[\tfrac{i_1+i_2}{3}] & q_2 &= -[\tfrac{i_2+2j_1}{3}] \\
\bar{p}_2 &= -[\tfrac{j_1+j_2}{3}] & \bar{q}_2 &= -[\tfrac{j_2+i_1}{3}] \\
p_3 &= -[\tfrac{i_3+i_2}{3}] & q_3 &= -[\tfrac{i_3+2j_2+2j_1+i_1}{3}] \\
\bar{p}_3 &= -[\tfrac{j_3+j_2}{3}] & \bar{q}_3 &= -[\tfrac{j_3+i_2+j_1+i_1}{3}]
\end{aligned}$$

Then the action of t and s on terms of the form δa_i and $\delta\bar{a}_i$ can be described as follows:

$$\begin{aligned}
t\delta a_1 &= \delta a_1 & s\delta a_1 &= \delta a_1 \\
t\delta\bar{a}_1 &= \delta\bar{a}_1 & s\delta\bar{a}_1 &= \delta\bar{a}_1 \\
t\delta a_2 &= \delta a_2 + \delta a_1 + d_V p_2 & s\delta a_2 &= \delta a_2 - \delta\bar{a}_1 + d_V q_2 \\
t\delta\bar{a}_2 &= \delta\bar{a}_2 + \delta\bar{a}_1 + d_V\bar{p}_2 & s\delta\bar{a}_2 &= \delta\bar{a}_2 + \delta a_1 + d_V\bar{q}_2 \\
t\delta a_3 &= \delta a_3 + \delta a_2 + d_V p_3 & s\delta a_3 &= \delta a_3 - \delta\bar{a}_2 - \delta\bar{a}_1 + \delta a_1 + d_V q_3 \\
t\delta\bar{a}_3 &= \delta\bar{a}_3 + \delta\bar{a}_2 + d_V\bar{p}_3 & s\delta\bar{a}_3 &= \delta\bar{a}_3 + \delta a_2 + \delta a_1 + \delta\bar{a}_1 + d_V\bar{q}_3
\end{aligned}$$

Proof: As in [7, 11.9]: define $a \mod 3 \in \{0,1,2\}$ as usual and let v, v' be two elements of V, $v = i_1x_1 + i_2x_2 + i_3x_3 + j_1\bar{x}_1 + j_2\bar{x}_2 + j_3\bar{x}_3$, $v' = i'_1x_1 + i'_2x_2 + i'_3x_3 + j'_1\bar{x}_1 + j'_2\bar{x}_2 + j'_3\bar{x}_3$. We will calculate the example $s\delta a_3$. We have

$$i_3 + i'_3 + 2j_2 + 2j'_2 + 2j_1 + 2j'_1 + i_1 + i'_1 = 3[\tfrac{i_3+2j_2+2j_1+i_1}{3}] + 3[\tfrac{i'_3+2j'_2+2j'_1+i'_1}{3}] + (i_3 + 2j_2 + 2j_1 + i_1) \mod 3 + (i'_3 + 2j'_2 + 2j'_1 + i'_1) \mod 3$$

and thus

$$[\tfrac{i_3+i'_3+2j_2+2j'_2+2j_1+2j'_1+i_1+i'_1}{3}] = [\tfrac{i_3+2j_2+2j_1+i_1}{3}] + [\tfrac{i'_3+2j'_2+2j'_1+i'_1}{3}] + [\tfrac{(i_3+2j_2+2j_1+i_1) \mod 3+(i'_3+2j'_2+2j'_1+i'_1) \mod 3}{3}].$$

Similarly

$$[\tfrac{i_3+i'_3+2j_2+2j'_2+2j_1+2j'_1+i_1+i'_1}{3}] = [\tfrac{i_3+i'_3}{3}] + 2[\tfrac{j_2+j'_2}{3}] + 2[\tfrac{j_1+j'_1}{3}] + [\tfrac{i_1+i'_1}{3}] + [\tfrac{(i_3+i'_3) \mod 3+2((j_2+j'_2) \mod 3)+2((j_1+j'_1) \mod 3)+(i_1+i'_1) \mod 3}{3}].$$

Finally, using these equations and recalling the definition of d_V, we get

$$\begin{aligned} s\delta a_3(v,v') &= \delta a_3(s^{-1}v, s^{-1}v') = [\tfrac{(i_3+2j_2+2j_1+i_1) \mod 3+(i'_3+2j'_2+2j'_1+i'_1) \mod 3}{3}] \\ &= [\tfrac{i_3+i'_3}{3}] + 2[\tfrac{j_2+j'_2}{3}] + 2[\tfrac{j_1+j'_1}{3}] + [\tfrac{i_1+i'_1}{3}] - [\tfrac{i_3+2j_2+2j_1+i_1}{3}] - [\tfrac{i'_3+2j'_2+2j'_1+i'_1}{3}] \\ &\quad + [\tfrac{(i_3+i'_3) \mod 3+2((j_2+j'_2) \mod 3)+2((j_1+j'_1) \mod 3)+(i_1+i'_1) \mod 3}{3}] \\ &= \delta a_3(v,v') + 2\delta\bar{a}_2(v,v') + 2\delta\bar{a}_1(v,v') + \delta a_1(v,v') + d_V q_3(v,v'). \end{aligned}$$

□

Lemma B.2. *a) The image of C and D in $E_0^{1,3}$ are given by*

$$C_1^{1,3} = a_1d_Vp_2 + a_1d_Vp_3 - a_2d_Vp_2 - \bar{a}_1d_V\bar{p}_2 - \bar{a}_1d_V\bar{p}_3 + \bar{a}_2d_V\bar{p}_2$$

$$C_2^{1,3} = -a_1d_Vq_2 + a_1d_Vq_3 + a_1d_V\bar{q}_2 - a_2d_Vq_2 + \bar{a}_1d_Vq_2 + \bar{a}_1d_V\bar{q}_2 - \bar{a}_1d_V\bar{q}_3 + \bar{a}_2d_V\bar{q}_2$$

$$D_1^{1,3} = a_1d_V\bar{p}_2 + a_1d_V\bar{p}_3 - a_2d_V\bar{p}_2 + \bar{a}_1d_Vp_2 + \bar{a}_1d_Vp_3 - \bar{a}_2d_Vp_2$$

$$D_2^{1,3} = -a_1d_Vq_2 - a_1d_V\bar{q}_2 + a_1d_V\bar{q}_3 - a_2d_V\bar{q}_2 - \bar{a}_1d_Vq_2 + \bar{a}_1d_Vq_3 + \bar{a}_1d_V\bar{q}_2 - \bar{a}_2d_Vq_2$$

b) The lifts of C and D to $E_0^{1,2}$ are given as follows:

$$C_1^{1,2} = -(a_1p_2 + a_1p_3 - a_2p_2 - \bar{a}_1\bar{p}_2 - \bar{a}_1\bar{p}_3 + \bar{a}_2\bar{p}_2)$$

$$C_2^{1,2} = -(-a_1q_2 + a_1q_3 + a_1\bar{q}_2 - a_2q_2 + \bar{a}_1q_2 + \bar{a}_1\bar{q}_2 - \bar{a}_1\bar{q}_3 + \bar{a}_2\bar{q}_2)$$

$$D_1^{1,2} = -(a_1\bar{p}_2 + a_1\bar{p}_3 - a_2\bar{p}_2 + \bar{a}_1p_2 + \bar{a}_1p_3 - \bar{a}_2p_2)$$

$$D_2^{1,2} = -(-a_1q_2 - a_1\bar{q}_2 + a_1\bar{q}_3 - a_2\bar{q}_2 - \bar{a}_1q_2 + \bar{a}_1q_3 + \bar{a}_1\bar{q}_2 - \bar{a}_2q_2)$$

Proof: a) We have to calculate $t-1$ respectively $s-1$ of the terms of C and D. This is done easily with the last lemma.
b) Use the fact that $d_V a_i = d_V \bar{a}_i = 0$ and that d_V is a derivation, e.g $d_V(a_1 p_2) = d_V(a_1)p_2 - a_1 d_V(p_2) = -a_1 d_V(p_2)$. □

Lemma B.3. *Let x be any function on V, then we have*
a)

$$\begin{aligned} t(a_1x) &= a_1tx & t(\bar{a}_1x) &= \bar{a}_1tx \\ s(a_1x) &= a_1sx & s(\bar{a}_1x) &= \bar{a}_1sx \end{aligned}$$

b)

$$\begin{aligned} N_t(a_2x) &= a_2N_tx + a_1tx + 2a_1t^2x & N_t(\bar{a}_2x) &= \bar{a}_2N_tx + \bar{a}_1tx + 2\bar{a}_1t^2x \\ (s-1)(a_2x) &= a_2(s-1)x - \bar{a}_1sx & (s-1)(\bar{a}_2x) &= \bar{a}_2(s-1)x + a_1sx \\ (t-1)(a_2x) &= a_2(t-1)x + a_1tx & (t-1)(\bar{a}_2x) &= \bar{a}_2(t-1)x + \bar{a}_1tx \\ N_s(a_2x) &= a_2N_sx + \bar{a}_1sx + \bar{a}_1s^2x & N_s(\bar{a}_2x) &= \bar{a}_2N_sx + a_1sx - a_1s^2x \end{aligned}$$

c) We have the following identity of functions on V:

$$\begin{aligned} N_tp_2 &= -a_1 & N_t\bar{p}_2 &= -\bar{a}_1 \\ sp_2 - p_2 + q_2 - tq_2 &= 0 & s\bar{p}_2 - \bar{p}_2 + \bar{q}_2 - t\bar{q}_2 &= 0 \\ N_sq_2 &= \bar{a}_1 & N_s\bar{q}_2 &= -a_1 \end{aligned}$$

$$p_3 + tp_3 + t^2p_3 - tp_2 + t^2p_2 = a_1 - a_2$$
$$\bar{p}_3 + t\bar{p}_3 + t^2\bar{p}_3 - t\bar{p}_2 + t^2\bar{p}_2 = \bar{a}_1 - \bar{a}_2$$
$$q_3 + sq_3 + s^2q_3 + s\bar{q}_2 - s^2\bar{q}_2 = a_1 + \bar{a}_1 + \bar{a}_2$$
$$\bar{q}_3 + s\bar{q}_3 + s^2\bar{q}_3 - sq_2 + s^2q_2 = \bar{a}_1 - a_1 - a_2$$
$$p_3 - s\bar{p}_2 - q_2 - \bar{q}_2 + t\bar{q}_2 - sp_3 - q_3 + tq_3 = 0$$
$$\bar{p}_3 + sp_2 + q_2 - \bar{q}_2 + tq_2 - s\bar{p}_3 - q_3 + tq_3 = 0.$$

Proof: a) follows because t and s act trivially on a_1 and $\bar{a}_1$.
b) is easily verified, for example

$$\begin{aligned} N_t(a_2x) &= t^2(a_2)t^2(x) + t(a_2)t(x) + a_2x \\ &= (a_2 + 2a_1)t^2(x) + (a_2 + a_1)t(x) + a_2x = a_2N_tx + 2a_1t^2x + a_1tx. \end{aligned}$$

c) is explicitly verified on the 3^6 elements of V (the identities have been found by taking the vectors of values of these functions and finding linear relations between them). □

Proposition B.4. *The first term $\xi \cup V^3$ in the Charlap Vasquez description of the differential $d_2^{0,3}$ is given by*

$$\begin{array}{lcl} C & \mapsto & (\alpha_1\alpha_2 - \bar{\alpha}_1\bar{\alpha}_2, 0, -\alpha_1\bar{\alpha}_2 - \bar{\alpha}_1\alpha_2) \\ D & \mapsto & (\alpha_1\bar{\alpha}_2 + \bar{\alpha}_1\alpha_2, 0, \alpha_1\alpha_2 - \bar{\alpha}_1\bar{\alpha}_2) \end{array}$$

Proof: The three components are given by $N_t C_1^{1,2}$, $-(s-1)C_1^{1,2} + (t-1)C_2^{1,2}$ and $N_s C_2^{1,2}$ and similarly for D. This calculation and the simplifications are done with the help of the last lemma. Finally we need that $a_1 a_2$ and $-a_2 a_1$ represent $\alpha_1\alpha_2$ and similarly for the other parts. □

REFERENCES

1. J. AISBETT, On $K_3(\mathbb{Z}/p^n)$ and $K_4(\mathbb{Z}/p^n)$, Memoirs AMS 329 (1985), 1-90
2. M.BÖKSTEDT, I.MADSEN, Algebraic K-theory of local number fields: the unramified case, To appear.
3. K.S.BROWN, Cohomology of groups, GTM 87 (1982)
4. L.S.CHARLAP, A.T. VASQUEZ, The cohomology of group extensions, Trans. AMS 124 (1966), 24-40
5. R.K.DENNIS, M.R.STEIN, K_2 of discrete valuation rings, Adv.Math. 18 (1975), 182-238
6. W.DWYER, E.FRIEDLANDER, Algebraic and étale K-theory, Trans. AMS 292 (1985)
7. L.EVENS, E.M.FRIEDLANDER, On $K_*(\mathbb{Z}/p^2\mathbb{Z})$ and related homology groups, Trans. AMS 270 (1982), 1-46
8. E.M.FRIEDLANDER, B.J.PARSHALL, On the cohomology of algebraic and related finite groups, Inv.Math. 74 (1983), 83-117
9. R.GODEMENT, Topologie algébrique et théorie des faisceaux, Hermann (1958)
10. L.HESSELHOLT, I.MADSEN, Topological cyclic homology of perfect fields and their dual numbers, To appear.
11. W. VAN DER KALLEN, J.STIENSTRA, The relative K_2 of truncated polynomial rings, J.Pure Appl.Math. 34 (1984), 277-289
12. C.KASSEL, Calcul algébrique de l'homologie de certains groupes de matrices, Journal of Algebra 90 (1983), 235-260
13. M.LEVINE, The indecomposable K_3 of fields, Ann.Scient.Ec.Norm.Sup. 22 (1989), 255-344
14. E.LLUIS-PUEBLA, On $K_3(\mathbb{F}_{p^l}[t]/(t^2))$ and $K_3(\mathbb{Z}/9)$, p an odd prime, Memoirs AMS 329 (1985), 91-100
15. I.A.PANIN, On a theorem of Hurewicz and K-theory of complete discrete valuation rings, Math. USSR Izvestija 29 (1987), 119-131
16. D.QUILLEN, On the cohomology and K-theory of the general linear groups over a finite field, Annals Math. 96 (1972), 552-586
17. P.SCHNEIDER, Über gewisse Galoiscohomologiegruppen, Math. Zeitschrift 168 (1979), 181-205
18. J.P.SERRE, Local fields, GTM 67
19. V.SNAITH, On K_3 of dual numbers, Memoirs AMS 329 (1985), 101-200
20. C.SOULÉ, Operations on étale K-theory. Applications, SLN 966 (1982), 271-303
21. A.SUSLIN, On the K-theory of local fields, J.Pure Appl.Alg. 34 (1984), 301-318