SMASH PRODUCTS FOR SECONDARY HOMOTOPY GROUPS

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Abstract. We construct a smash product operation on secondary homotopy groups yielding the structure of a lax symmetric monoidal functor. Applications on cup-one products, Toda brackets and Whitehead products are considered.

Contents

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Introduction

The classical homotopy groups $\pi_n X$, $n \geq 0$, of a pointed space X give rise to a graded abelian group $\Pi_* X$ obtained by additivization in low dimensions. In particular $\Pi_n X = \pi_n X$ for $n \geq 2$, $\Pi_1 X = (\pi_1 X)_{ab}$ is the abelianized fundamental group, and $\Pi_0 X = \mathbb{Z}[\pi_0 X]$ is the free abelian group on the pointed set of path components of X. The smash product on homotopy groups induces a natural homomorphism of graded abelian groups

$$\Pi_* X \otimes \Pi_* Y \xrightarrow{\wedge} \Pi_*(X \wedge Y), \tag{1}$$

which carries $f \otimes g$ with $f\colon S^n \to X$ and $g\colon S^m \to Y$ to $f \wedge g\colon S^{n+m} \to X \wedge Y$. This shows that Π_* is a lax symmetric monoidal functor from pointed spaces to graded abelian groups.

The smash product (1) can be used for example to define the Whitehead product on homotopy groups, compare Section 9.

The purpose of this paper is to generalize these properties of primary homotopy groups on the level of secondary homotopy theory.

Secondary homotopy operations like Toda brackets [Tod62] or cup-one products [BJM83], [HM93], are defined by pasting tracks, where tracks are homotopy classes of homotopies. Since secondary homotopy operations play a crucial role in homotopy theory it is of importance to develop the algebraic theory of tracks. We do this by introducing secondary homotopy groups of a pointed space X

$$\Pi_{n,*} X = \left(\Pi_{n,1} X \xrightarrow{\partial} \Pi_{n,0} X\right)$$

which have the structure of a quadratic pair module, see Sections 1 and 6. Here ∂ is a group homomorphism with cokernel $\Pi_n X$ for $n \geq 0$ and kernel $\pi_{n+1} X$ for $n \geq 3$.

We define $\Pi_{n,*} X$ for $n \geq 2$ directly in terms of maps $S^n \to X$ and tracks from such maps to the trivial map. For $n \geq 0$ the functor $\Pi_{n,*}$ is an additive version of the functor $\pi_{n,*}$ studied in [BM05a].

We introduce and study the smash product morphism for additive secondary homotopy groups

$$\Pi_{*,*} X \odot \Pi_{*,*} Y \xrightarrow{\wedge} \Pi_{*,*}(X \wedge Y). \tag{2}$$

Here one needs the symmetric monoidal structure $\odot$ of the category of quadratic pair modules **qpm** which is based on the symmetric monoidal structure on the category of square groups constructed in [BJP05]. The smash product morphism (2) is compatible with the associativity isomorphisms but it is not directly compatible with the commutativity isomorphisms.

In order to deal with commutativity we need the action of the symmetric track group $\mathrm{Sym}_\square(n)$ on $\Pi_{n,*} X$ in [BM05b]. We show that $\wedge$ in (2) is equivariant with respect to this action, and is commutative up to the action of a shuffle permutation. This leads to the definition of the symmetric monoidal category $\mathbf{qpm}^{\mathrm{Sym}_\square}$ with objects given by symmetric sequences of quadratic pair modules with extra structure. Then the morphism (2) induces a morphism in $\mathbf{qpm}^{\mathrm{Sym}_\square}$ for which the associativity and commutativity isomorphisms are compatible with the symmetric monoidal structure of $\mathbf{qpm}^{\mathrm{Sym}_\square}$. Therefore $\Pi_{*,*}$ considered as a functor to the category $\mathbf{qpm}^{\mathrm{Sym}_\square}$ is, in fact, a lax symmetric monoidal functor.

The smash product (2) is used to define the Whitehead product on secondary homotopy groups, compare Section 9.

As an illustrating application of the results in this paper we prove a formula of Barratt-Jones-Mahowald on unstable cup-one products, see Section 10. This formula was stated in [BJM83], but a proof did not appear in the literature. A further application yields a formula for a triple Toda bracket which generalizes a well-known formula in [Tod62], see Section 11.

The paper consists of three parts. The first part is concerned with the algebra needed for the statements of the main theorems. In Part 2 we present our main results and we give applications. Part 3 contains the construction of the smash product operation for additive secondary homotopy groups. There we prove all the properties which imply our main results.

Part 1. Quadratic pair modules and their tensor product

In this part we describe the algebraic concepts needed for the structure of secondary homotopy groups. We introduce the category of quadratic pair modules and we show that this category is symmetric monoidal. The tensor product of quadratic pair modules is related to the exterior cup-products in the category **Top***.

1. Square groups and quadratic pair modules

We first recall the notion of square group, see [BP99] and [BJP05].

Definition 1.1. A *square group* X is a diagram

$$X = (X_e \underset{H}{\overset{P}{\leftrightarrows}} X_{ee})$$

where X_e is a group with an additively written group law, X_{ee} is an abelian group, P is a homomorphism, H is a quadratic map, i. e. a function such that the cross effect

$$(a|b)_H = H(a + b) - H(b) - H(a)$$

is linear in $a, b \in X_e$, and the following relations are satisfied, $x, y \in X_{ee}$,

(1) $(Px|b)_H = 0$, $(a|Py) = 0$,
(2) $P(a|b)_H = [a, b]$,
(3) $PHP(x) = P(x) + P(x)$.

Here $[a, b] = -a - b + a + b$ is the commutator bracket. The function

$$T = HP - 1 \colon X_{ee} \longrightarrow X_{ee}$$

is an involution, i. e. a homomorphism with $T^2 = 1$. Moreover,

$$\Delta \colon X_e \longrightarrow X_{ee} \colon x \mapsto (x|x)_H - H(x) + TH(x)$$

is a homomorphism which satisfies $T\Delta = -\Delta$.

A morphism of square groups $f \colon X \to Y$ is given by homomorphisms $f_e \colon X_e \to Y_e$, $f_{ee} \colon X_{ee} \to Y_{ee}$, commuting with P and H.

As an example of square group we can consider

$$\mathbb{Z}_{nil} = (\mathbb{Z} \underset{H}{\overset{P}{\leftrightarrows}} \mathbb{Z})$$

with $P = 0$ and $H(n) = \binom{n}{2} = \frac{n(n-1)}{2}$. This is the unit object of the symmetric monoidal structure defined in the next section.

We refer the reader to [BJP05] where the quadratic algebra of square groups is developed. We need square groups for the definition of quadratic pair modules as follows.

Definition 1.2. A *quadratic pair module* C is a morphism $\partial\colon C_{(1)} \to C_{(0)}$ between square groups

$$C_{(0)} = (C_0 \underset{H}{\overset{P_0}{\leftrightarrows}} C_{ee}),$$
$$C_{(1)} = (C_1 \underset{H_1}{\overset{P}{\leftrightarrows}} C_{ee}),$$

such that $\partial_{ee} = 1\colon C_{ee} \to C_{ee}$ is the identity homomorphism. In particular C is completely determined by the diagram

(1.3)
$$\begin{tikzcd} & C_{ee} \arrow[dl, "P"'] & \\ C_1 \arrow[rr, "\partial"'] & & C_0 \arrow[ul, "H"'] \end{tikzcd}$$

where $\partial = \partial_e$, $H_1 = H\partial$ and $P_0 = \partial P$.

Morphisms of quadratic pair modules $f\colon C \to D$ are therefore given by group homomorphisms $f_0\colon C_0 \to D_0$, $f_1\colon C_1 \to D_1$, $f_{ee}\colon C_{ee} \to D_{ee}$, commuting with H, P and ∂ in (1.3). They form a category denoted by **qpm**.

Quadratic pair modules are also the objects of a larger category **wqpm** given by weak morphisms. A *weak morphism* $f\colon C \to D$ between quadratic pair modules is given by three homomorphisms f_0, f_1, f_{ee} as above, but we only require compatibility with T, $(-|-)_H$, P and ∂. Therefore $\mathbf{qpm} \subset \mathbf{wqpm}$ is a subcategory with the same objects.

Let $(\mathbb{Z}, \cdot)$ be the multiplicative (abelian) monoid of the integers $\mathbb{Z}$. We recall from [BM05b] 1.5 the definition of the $(\mathbb{Z}, \cdot)$-action on quadratic pair modules. For any additively written group G and any $n \in \mathbb{Z}$, $g \in G$ we denote

$$n \cdot g = \begin{cases} g + \overset{n}{\cdots} + g, & \text{if } n \geq 0; \\ \\ -g - \overset{-n}{\cdots} - g, & \text{if } n < 0. \end{cases}$$

Definition 1.4. Any quadratic pair module C admits an action of $(\mathbb{Z}, \cdot)$ given by the morphisms $n^*\colon C \to C$ in **wqpm**, $n \in \mathbb{Z}$, defined by the equations

- $n^*x = n \cdot x + \binom{n}{2}\partial PH(x)$ for $x \in C_0$,
- $n^*y = n \cdot y + \binom{n}{2}PH\partial(y)$ for $y \in C_1$,
- $n^*z = n^2 z$ for $z \in C_{ee}$.

We point out that $n^*\colon C \to C$ is an example of a weak morphism which is not a morphism in **qpm** since n^* is not compatible with H. The existence of this action should be compared to the fact that abelian groups are $\mathbb{Z}$-modules.

Remark 1.5. The category **squad** of stable quadratic modules is described in [Bau91] IV.C and [BM05a]. Quadratic modules in general are discussed in [Bau91] and [Ell93], they are special 2-crossed modules in the sense of [Con84]. There is a faithful forgetful functor **wqpm** $\to$ **squad** sending C to the stable quadratic module

$$\otimes^2(C_0)_{ab} \xrightarrow{P(-|-)_H} C_1 \xrightarrow{\partial} C_0.$$

Here G_{ab} denotes the abelianization of a group G, and $\otimes^2 A = A \otimes A$ is the tensor square of an abelian group A. Such a stable quadratic module gives rise to a crossed module $\partial\colon C_1 \to C_0$ where C_0 acts on the right of C_1 by the formula, $x \in C_1$, $y \in C_0$,

$$x^y = x + P(\partial(x)|y)_H,$$

so that we also get a forgetful functor **wqpm** $\to$ **cross** to the track category of crossed modules. Tracks in **qpm** and **wqpm** (i. e. invertible 2-morphisms) will be considered in Section 5 below by using this forgetful functor.

2. The tensor product of square groups

We now recall the notion of tensor product of square groups which is essential for this paper. This tensor product, first defined in [BJP05], originates from properties of the exterior cup-products in the next section and in Part 3.

Definition 2.1. The *tensor product* $X \odot Y$ of square groups X, Y is defined as follows. The group $(X \odot Y)_e$ is generated by the symbols $x\underline{\circledcirc}y$, $x \circledcirc y$, $a\bar{\otimes}b$ for $x \in X_e$, $y \in Y_e$, $a \in X_{ee}$ and $b \in Y_{ee}$, subject to the following relations

1. the symbol $a\bar{\otimes}b$ is bilinear and central,
2. the symbol $x\underline{\circledcirc}y$ is right linear, $x\underline{\circledcirc}(y_1 + y_2) = x\underline{\circledcirc}y_1 + x\underline{\circledcirc}y_2$,
3. the symbol $x \circledcirc y$ is left linear, $(x_1 + x_2) \circledcirc y = x_1 \circledcirc y + x_2 \circledcirc y$,
4. $P(a)\underline{\circledcirc}y = a\bar{\otimes}\Delta(y)$,
5. $x \circledcirc P(b) = \Delta(x)\bar{\otimes}b$,
6. $T(a)\bar{\otimes}T(b) = -a\bar{\otimes}b$,
7. $x\underline{\circledcirc}y - x \circledcirc y = H(x)\bar{\otimes}TH(y)$.

The abelian group $(X \odot Y)_{ee}$ is defined as the tensor product $X_{ee} \otimes Y_{ee}$. The homomorphism

$$P\colon (X \odot Y)_{ee} \longrightarrow (X \odot Y)_e$$

is $P(a \otimes b) = a\bar{\otimes}b$, and

$$H\colon (X \odot Y)_e \longrightarrow (X \odot Y)_{ee}$$

is the unique quadratic map satisfying

$$\begin{aligned}
H(x\underline{\circledcirc}y) &= (x|x)_H \otimes H(y) + H(x) \otimes \Delta(y), \\
H(x \circledcirc y) &= \Delta(x) \otimes H(y) + H(x) \otimes (y|y)_H, \\
H(a\bar{\otimes}b) &= a \otimes b - T(a) \otimes T(b), \\
(a\bar{\otimes}b|-)_H &= (-|a\bar{\otimes}b)_H = 0, \\
(a\underline{\circledcirc}b|c\underline{\circledcirc}d)_H &= (a\underline{\circledcirc}b|c \circledcirc d)_H \\
&= (a \circledcirc b|c\underline{\circledcirc}d)_H \\
&= (a \circledcirc b|c \circledcirc d)_H \\
&= (a|c)_H \otimes (b|d)_H.
\end{aligned}$$

Relation (7) above shows that $(X \odot Y)_e$ is actually generated just by $x\underline{\circledcirc}y$ and $a\bar{\otimes}b$. A complete list of relations for these generators is given by (1), (2), (4) and (6) above together with

(8) $(x_1+x_2)\underline{\circledcirc}y = x_1\underline{\circledcirc}y + x_2\underline{\circledcirc}y + (x_2|x_1)_H\bar{\otimes}H(y)$,
(9) $x\underline{\circledcirc}P(b) = (x|x)_H\bar{\otimes}b$.

Similarly $(X\odot Y)_e$ is also generated by just $x\circledcirc y$ and $a\bar{\otimes}b$ with relations (1), (3), (5) and (6) above together with

(10) $x\circledcirc(y_1+y_2) = x\circledcirc y_1 + x\circledcirc y_2 + H(x)\bar{\otimes}(y_2|y_1)_H$,
(11) $P(a)\circledcirc y = a\bar{\otimes}(y|y)_H$.

As proved in [BJP05] the tensor product of square groups is a symmetric monoidal structure on the category of square groups with unit $\mathbb{Z}_{nil}$ in Definition 1.1. The associativity isomorphism

$$(X\odot Y)\odot Z \cong X\odot(Y\odot Z)$$

is given by $(x\underline{\circledcirc}y)\underline{\circledcirc}z \mapsto x\underline{\circledcirc}(y\underline{\circledcirc}z)$, $(a\bar{\otimes}b)\underline{\circledcirc}z \mapsto a\bar{\otimes}(b\otimes\Delta(z))$ and $(a\otimes b)\bar{\otimes}c \mapsto a\bar{\otimes}(b\otimes c)$ at the e-level and by the associativity isomorphism

$$(X_{ee}\otimes Y_{ee})\otimes Z_{ee} \cong X_{ee}\otimes(Y_{ee}\otimes Z_{ee})$$

for the tensor product of abelian groups at the ee-level. The symmetry isomorphism

$$\tau_\odot\colon X\odot Y\cong Y\odot X$$

is defined on e-groups by $x\underline{\circledcirc}y\mapsto y\circledcirc x$, $x\circledcirc y\mapsto y\underline{\circledcirc}x$, and $a\bar{\otimes}b\mapsto b\bar{\otimes}a$, and on ee-groups by the standard symmetry isomorphism

$$\tau_\otimes\colon X_{ee}\otimes Y_{ee}\cong Y_{ee}\otimes X_{ee}$$

for the tensor product of abelian groups. The unit isomorphism

$$\mathbb{Z}_{nil}\odot X\cong X$$

is defined on e-groups by the formulas $n\circledcirc x\mapsto n\cdot x$ and $n\bar{\otimes}a\mapsto n\cdot P(a)$.

3. Exterior cup-products

We will work with the track category $\mathbf{Top}^*$ of compactly generated pointed spaces. A track category is a category enriched in groupoids, i. e. a 2-category where all 2-morphisms are vertically invertible. A 2-morphism in a track category is also termed a *track*, and a 2-functor between track categories is called a *track functor*. Tracks in $\mathbf{Top}^*$ are homotopy classes of homotopies between pointed maps. The identity track on a pointed map $f\colon X\to Y$, also called the trivial track, will be denoted by $0^\square_f$. We use the symbol $\square$ for the vertical composition, and $F^\boxminus$ denotes the vertical inverse of a track F. Horizontal composition is denoted by juxtaposition.

The smash product of pointed spaces X,Y in $\mathbf{Top}^*$ is the quotient space

$$X\wedge Y = X\times Y/X\vee Y \tag{3.1}$$

where the coproduct $X\vee Y$ in $\mathbf{Top}^*$ admits the canonical inclusion $X\vee Y\subset X\times Y$ to the product. If σ is a permutation of $\{1,\dots,n\}$ the map

$$\sigma\colon X_1\wedge\cdots\wedge X_n \longrightarrow X_{\sigma^{-1}(1)}\wedge\cdots\wedge X_{\sigma^{-1}(n)}, \tag{3.2}$$

induced by the permutation of coordinates according to σ, is also denoted by σ. For the sake of simplicity we will ocasionally omit the permutation σ in the diagrams and equations where it is understood. Given a subset $\{i_1,\dots,i_k\}\subset\{1,\dots,n\}$ we denote by $\sigma=(i_1\ \dots\ i_k)$ the permutation defined by $\sigma(i_s)=i_{s+1}$ if $1\le s<k$, $\sigma(i_k)=i_1$ and $\sigma(m)=m$ otherwise.

The smash product is a track functor

$$\wedge\colon \mathbf{Top}^* \times \mathbf{Top}^* \longrightarrow \mathbf{Top}^*.$$

It is defined as usually at the level of pointed spaces and pointed maps. The pointed space I_+ is the disjoint union of the interval $I = [0,1]$ with an outer base point $*$. Let $F\colon f \Rightarrow g$ be a track between maps $f, g\colon A \to B$ represented by a homotopy $F\colon I_+ \wedge A \to B$, and let $h\colon X \to Y$ be another map. Then the track $F \wedge h\colon f \wedge h \Rightarrow g \wedge h$ is represented by a homotopy $F \wedge h\colon I_+ \wedge A \wedge X \to B \wedge Y$ and $h \wedge F\colon h \wedge f \Rightarrow h \wedge g$ is represented by the composite

$$I_+ \wedge X \wedge A \cong X \wedge I_+ \wedge A \xrightarrow{h \wedge F} Y \wedge B.$$

If $G\colon h \Rightarrow k$ is another track then

$$F \wedge G = (g \wedge G)\square(F \wedge h) = (F \wedge k)\square(f \wedge G)\colon f \wedge h \Rightarrow g \wedge k.$$

The smash product defines a symmetric monoidal structure in the category $\mathbf{Top}^*$. The unit object is the 0-sphere S^0.

Definition 3.3. Given maps $f\colon \Sigma A \to \Sigma B$ and $g\colon \Sigma X \to \Sigma Y$ the *left exterior cup-product* $f\#g$ is the composite

$$S^1 \wedge A \wedge X \xrightarrow{f \wedge X} S^1 \wedge B \wedge X \cong B \wedge S^1 \wedge X \xrightarrow{B \wedge g} B \wedge S^1 \wedge Y \cong S^1 \wedge B \wedge Y.$$

Similarly the *right exterior cup-product* $f\underline{\#}g$ is the composite

$$S^1 \wedge A \wedge X \cong A \wedge S^1 \wedge X \xrightarrow{A \wedge g} A \wedge S^1 \wedge Y \cong S^1 \wedge A \wedge Y \xrightarrow{f \wedge Y} S^1 \wedge B \wedge Y.$$

The equality

$$f\#g = (2\ 3)(g\underline{\#}f)(2\ 3) \tag{3.4}$$

is always satisfied.

These constructions give rise to homotopy operations called *exterior cup-products*

$$\#, \underline{\#}\colon [\Sigma A, \Sigma B] \times [\Sigma X, \Sigma Y] \longrightarrow [\Sigma A \wedge X, \Sigma B \wedge Y].$$

see [Bau81] II.1.14. The operation $\#$ is left-linear and $\underline{\#}$ is right-linear,

$$\begin{aligned}(f_1 + f_2)\#g &= f_1\#g + f_2\#g,\\ f\underline{\#}(g_1 + g_2) &= f\underline{\#}g_1 + f\underline{\#}g_2.\end{aligned}$$

Given a pointed discrete set E we denote $\vee_E S^n = \Sigma^n E$. Then

$$\pi_1(\vee_E S^1) = \langle E\rangle$$

is the free group with basis $E - *$, and

$$\pi_n(\vee_E S^n) = \mathbb{Z}[E]$$

is the free abelian group with basis $E - *$ for $n \geq 2$. We write

$$\langle E\rangle_{nil}$$

for the free group of nilpotency class 2 (*nil-group* for short) with basis $E - *$, which is obtained from $\langle E\rangle$ by dividing out triple commutators.

If $A = X = S^0$, and $B = E$, $Y = \bar{E}$ are pointed sets then the exterior cup-products are functions

$$\#, \underline{\#}\colon \langle E\rangle \times \langle \bar{E}\rangle \longrightarrow \langle E \wedge \bar{E}\rangle.$$

These functions factor in a unique way through the natural projection onto the nilization in the following way

$$\begin{tikzcd} \langle E\rangle \times \langle \bar{E}\rangle \arrow[r, "\#,\underline{\#}"] \arrow[d, two heads] & \langle E\wedge \bar{E}\rangle \arrow[d, two heads] \\ \langle E\rangle_{nil} \times \langle \bar{E}\rangle_{nil} \arrow[r, "\#,\underline{\#}"] & \langle E\wedge \bar{E}\rangle_{nil} \end{tikzcd} \tag{3.5}$$

A free nil-group $\langle E\rangle_{nil}$ on a pointed set E gives rise to a square group

$$\mathbb{Z}_{nil}[E] = (\langle E\rangle_{nil} \underset{H}{\overset{P}{\leftrightarrows}} \otimes^2\mathbb{Z}[E]) \tag{3.6}$$

defined by $P(a\otimes b) = [b,a]$, $H(e) = 0$ for any $e \in E$ and $(s|t)_H = t\otimes s$ so that $\mathbb{Z}_{nil}[S^0] = \mathbb{Z}_{nil}$ in Definition 1.1. For (3.6) the involution T is up to sign the interchange of factors in the tensor square $T(a\otimes b) = -b\otimes a$ and Δ is defined by $\Delta(e) = e\otimes e$ for $e \in E$. Recall that we denote

$$\tau_\otimes \colon A\otimes B \cong B\otimes A$$

to the symmetry isomorphism for the tensor product of abelian groups, which should not be confused with $T = -\tau_\otimes$ in this case.

The next proposition is essentially [BJP05] 34. It shows the connection between the tensor product of square groups and the exterior cup-products.

Proposition 3.7. *Given two pointed sets E and $\bar{E}$ there is a square group isomorphism*

$$\mathbb{Z}_{nil}[E]\odot\mathbb{Z}_{nil}[\bar{E}] \xrightarrow{\cong} \mathbb{Z}_{nil}[E\wedge\bar{E}]$$

defined on the e-groups by $x\underline{\circledcirc}y \mapsto x\underline{\#}y$, $x\circledcirc y\mapsto x\#y$, and on the ee-groups by

$$\begin{array}{rcl} 1\otimes\tau_\otimes\otimes 1\colon \mathbb{Z}[E]\otimes\mathbb{Z}[E]\otimes\mathbb{Z}[\bar{E}]\otimes\mathbb{Z}[\bar{E}] & \cong & \mathbb{Z}[E]\otimes\mathbb{Z}[\bar{E}]\otimes\mathbb{Z}[E]\otimes\mathbb{Z}[\bar{E}], \\ a\otimes b\otimes c\otimes d & \mapsto & a\otimes c\otimes b\otimes d. \end{array}$$

This is the quadratic analogue of the well-known fact that free abelian groups have the tensor product

$$\mathbb{Z}[E]\otimes\mathbb{Z}[\bar{E}] \cong \mathbb{Z}[E\wedge\bar{E}].$$

4. The symmetric monoidal category of quadratic pair modules

A *pair* in a category $\mathbf{C}$ is a morphism $f\colon X\to Y$ in $\mathbf{C}$. Let $\mathbf{Pair}(\mathbf{C})$ be the category of such pairs. Morphisms $(\alpha,\beta)\colon f\to f'$ in $\mathbf{Pair}(\mathbf{C})$ are given by morphisms $\alpha\colon X\to X'$ and $\beta\colon Y\to Y'$ in $\mathbf{C}$ satisfying $\beta f = f'\alpha$. A quadratic pair module is a special pair in the category $\mathbf{SG}$ of square groups and the category $\mathbf{qpm}$ of quadratic pair modules is a full reflective subcategory of $\mathbf{Pair}(\mathbf{SG})$. The left adjoint to the inclusion $\mathbf{qpm}\subset\mathbf{Pair}(\mathbf{SG})$, i. e. the reflection functor, is denoted by

$$\Phi\colon \mathbf{Pair}(\mathbf{SG}) \longrightarrow \mathbf{qpm}.$$

Given a pair $f\colon D\to C$ in $\mathbf{SG}$ we have $\Phi(f)_{(0)} = C$, so that $\Phi(f)_0 = C_e$ and $\Phi(f)_{ee} = C_{ee}$. Moreover, $\Phi(f)_1$ is the quotient group

$$\Phi(f)_1 = D_e\times C_{ee}/\sim,$$

$$\begin{array}{rcl} (0, f_{ee}(d)) & \sim & (P(d),0),\ \ d\in D_{ee}, \\ (0, HP(c)) & \sim & (0,2c),\ \ c\in C_{ee}. \end{array}$$

The operators P and H for $\Phi(f)_{(1)}$ and the homomorphism $\partial\colon \Phi(f)_1 \to \Phi(f)_0 = C_e$ are defined by the formulas, $c \in C_{ee}$, $d \in D_e$,

$$\begin{array}{rcl} P(c) & = & (0, c), \\ H(d,c) & = & f_{ee}H(d) + HP(c), \\ \partial(d,c) & = & f_e(d) + P(c). \end{array}$$

The unit of this adjunction is a natural morphism in **Pair**(**SG**)

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\upsilon"'] & C \arrow[d, equal] \\ \Phi(f)_{(1)} \arrow[r, "\partial"] & \Phi(f)_{(0)} \end{tikzcd} \tag{4.1}$$

and is given by $\upsilon_e(d) = (d, 0)$ for $d \in D_e$ and $\upsilon_{ee}(d') = f_{ee}(d')$ for $d' \in D_{ee}$. We use the functor Φ for the following definition of the tensor product in **qpm**.

The category **qpm** is a symmetric monoidal category. This structure is inherited from the tensor product in **SG** described above. More precisely, the *tensor product* $C \odot D$ of two quadratic pair modules $\partial\colon C_{(1)} \to C_{(0)}, \partial\colon D_{(1)} \to D_{(0)}$ is given as follows. Consider the push-out diagram

$$\begin{tikzcd} C_{(1)} \odot D_{(1)} \arrow[r, "1\odot\partial"] \arrow[d, "\partial\odot 1"'] \arrow[dr, phantom, "\text{push}"] & C_{(1)} \odot D_{(0)} \arrow[d, "\zeta"] \arrow[ddr, bend left, "\partial\odot 1"] & \\ C_{(0)} \odot D_{(1)} \arrow[r, "\xi"'] \arrow[drr, bend right, "1\odot\partial"'] & C\overline{\odot}D \arrow[dr, "\overline{\partial}"] & \\ & & C_{(0)} \odot D_{(0)} \end{tikzcd} \tag{4.2}$$

in the category **SG**. Here $\overline{\partial}$ is a pair in **SG** for which we derive the tensor product in **qpm** by the functor Φ above, that is,

$$C \odot D = \Phi(\overline{\partial}\colon C\overline{\odot}D \to C_{(0)} \odot D_{(0)}),$$

is particular $(C \odot D)_{(0)} = C_{(0)} \odot D_{(0)}$ and $(C \odot D)_{ee} = C_{ee} \otimes D_{ee}$. Moreover, notice that $\upsilon\zeta$ and $\upsilon\xi$ are both the identity on ee-groups. The unit element for this tensor product is the quadratic pair module $\overline{\mathbb{Z}}_{nil} = \Phi(0 \to \mathbb{Z}_{nil})$ given by

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl, two heads] & \\ \mathbb{Z}/2 \arrow[rr, "0"] & & \mathbb{Z} \arrow[ul, "H"'] \end{tikzcd} \tag{4.3}$$

where $H(n) = \binom{n}{2}$.

Obviously the tensor product of quadratic pair modules is a functor

$$\odot\colon \mathbf{qpm} \times \mathbf{qpm} \longrightarrow \mathbf{qpm}. \tag{4.4}$$

Proposition 4.5. *The symmetric monoidal category* **qpm** *acts on* **wqpm** *via*

$$\odot\colon \mathbf{wqpm} \times \mathbf{qpm} \longrightarrow \mathbf{wqpm}.$$

One can use the symmetry isomorphism $\tau_\odot$ for the tensor product of quadratic pair modules to define also an action from the left $\odot\colon \mathbf{qpm} \times \mathbf{wqpm} \to \mathbf{wqpm}$.

Proof of Proposition 4.5. Let $f\colon C \to D$ be a weak morphism and let $g\colon X \to Y$ be a morphism in **qpm**. The weak morphism

$$f \odot g\colon C \odot X \longrightarrow D \odot Y$$

is induced by the square group morphisms

$$(f \odot g)_{i,j}\colon C_{(i)} \odot X_{(j)} \longrightarrow D_{(i)} \odot Y_{(j)},\ \ 0 \leq i,j \leq 1,$$

defined as follows: given $c \in C_i$ and $x \in X_j$

$$(f \odot g)_{i,j}(c \underline{\circledcirc} x) = f_i(c) \underline{\circledcirc} g_i(x),$$

and given $a \in C_{ee}$ and $b \in B_{ee}$

$$(f \odot g)_{i,j}(a \otimes b) = f_{ee}(a) \otimes g_{ee}(b).$$

This is well-defined since looking at the definition of the tensor product of square groups in terms only of the generators of type $\underline{\circledcirc}$ and $\bar{\otimes}$, see Definition 2.1, we observe that on the left hand side we only have the structure homomorphisms involved in the definition of a weak morphism between quadratic pair modules, i. e. full compatibility with H on the left hand side is not required to obtain induced morphisms. Compare [BJP05] 6.7. $\square$

The next lemma can be derived from [BJP05] 28.

Lemma 4.6. *Given two quadratic pair modules C, D and an integer $n \in \mathbb{Z}$ the morphisms n^* in* **wqpm** *in Definition 1.4 satisfy*

$$C \odot n^* = n^* \odot D = n^*.$$

5. The track category of quadratic pair modules

Given a quadratic pair module C the group C_0 acts on C_1 by the formula, $x \in C_1$, $y \in C_0$,

$$x^y = x + P(\partial(x)|y)_H,$$

see Remark 1.5. We define tracks in **wqpm** as follows.

Definition 5.1. A *track* $\alpha\colon f \Rightarrow g$ between two morphisms $f, g\colon C \to D$ in **wqpm** is a function

$$\alpha\colon C_0 \longrightarrow D_1$$

satisfying the equations, $x, y \in C_0$, $z \in C_1$,

1. $\alpha(x+y) = \alpha(x)^{f_0(y)} + \alpha(y)$,
2. $g_0(x) = f_0(x) + \partial\alpha(x)$,
3. $g_1(z) = f_1(z) + \alpha\partial(z)$.

Tracks in **qpm** are tracks in **wqpm** between morphisms in the subcategory **qpm** $\subset$ **wqpm**.

These tracks are pulled back from the track category of crossed modules along the forgetful functor in Remark 1.5. The track structure for crossed modules is described in [BM05a] 7. In particular we obtain the following result.

Proposition 5.2. *The categories* **wqpm** *and* **qpm** *are track categories.*

The vertical composition of tracks α, β is defined by addition $(\alpha\Box\beta)(x) = \beta(x)+\alpha(x)$. The horizontal composition of a track α and a map f, g is defined as $(f\alpha)(x) = f_1\alpha(x)$ and $(\alpha g)(x) = \alpha g_0(x)$. A trivial track $0^{\Box}_f\colon f \Rightarrow f$ is always defined as $0^{\Box}_f(x) = 0$.

One can use the *interval quadratic pair module* $\mathbb{I}$ to characterize tracks in **wqpm** and **qpm** in some cases. This quadratic pair module $\mathbb{I}$ is defined as follows.

$$\mathbb{I} = \left(\begin{tikzcd} & \otimes^2\mathbb{Z}[i_0,i_1] \arrow[dl, "P"'] & \\ \mathbb{Z}\bar{\imath}\oplus\mathbb{Z}/2P(i_0|i_0)_H\oplus\mathbb{Z}P(i_0|i_1)_H \arrow[rr, "\partial"'] & & \langle i_0,i_1\rangle_{nil} \arrow[ul, "H"'] \end{tikzcd}\right)$$

The quadratic map H is defined as in (3.6). The structure homomorphisms P and ∂ are completely determined by the laws of a quadratic pair module and the equality $\partial(\bar{\imath}) = -i_0+i_1$. There are two obvious inclusions $i_0, i_1\colon \overline{\mathbb{Z}}_{nil}\to\mathbb{I}$ and a projection $p\colon \mathbb{I}\to\overline{\mathbb{Z}}_{nil}$ defined by $p(i_0) = p(i_1) = 1$ and $p(\bar{\imath}) = 0$.

Lemma 5.3. *Let $f, g\colon C\to D$ be morphisms in* **wqpm**. *Assume that $(-|-)_H\colon \otimes^2 (C_0)_{ab}\cong C_{ee}$ is an isomorphism. Then tracks $\alpha\colon f\Rightarrow g$ are in bijection with morphisms $\bar\alpha\colon \mathbb{I}\odot C\to D$ with $f = \bar\alpha i_0$ and $g = \bar\alpha i_1$. The equivalence is given by the formula $\alpha(c) = \bar\alpha \upsilon_e\zeta_e(\bar{\imath}\circledcirc c)$ for $c\in C_0$. Here we use the square group morphisms υ and ζ in (4.1) and (4.2).*

This lemma can be derived from the definition of the tensor product of square groups.

Proposition 5.4. *The functor*

$$\odot\colon \mathbf{wqpm}\times\mathbf{qpm}\longrightarrow\mathbf{wqpm}$$

in Proposition 4.5 extends to a track functor.

Proof. Given a track $\alpha\colon f\Rightarrow g\colon C\to D$ in **wqpm** and a track $\beta\colon h\Rightarrow k\colon X\to Y$ in **qpm**, the track $\alpha\odot\beta\colon f\odot h\Rightarrow g\odot k\colon C\odot X\to D\odot Y$ in **wqpm** with

$$\alpha\odot\beta\colon (C\odot X)_0 = (C_{(0)}\odot X_{(0)})_e\longrightarrow (D\odot Y)_1$$

is defined as follows. Given $c\in C_0$ and $x\in X_0$

$$\begin{aligned}(\alpha\odot\beta)(c\underline{\circledcirc}x) &= \upsilon_e\xi_e(f_0(c)\underline{\circledcirc}\beta(x)) + \upsilon_e\zeta_e(\alpha(c)\underline{\circledcirc}k_0(x)) \\ &\quad +(-f_0+g_0|f_0(c))_H\bar\otimes Hk_0(x) \\ &\overset{\text{(a)}}{=} \upsilon_e\zeta_e(\alpha(c)\underline{\circledcirc}h_0(x)) + \upsilon_e\xi_e(g_0(c)\underline{\circledcirc}\beta(x)) \\ &\quad +(-f_0(c)+g_0(c)|f_0(c))_H\bar\otimes Hh_0(x),\end{aligned}$$

and given $a\in C_{ee}$ and $b\in C_{ee}$

$$(\alpha\odot\beta)(a\bar\otimes b) = -f_{ee}(a)\bar\otimes h_{ee}(b) + g_{ee}(a)\bar\otimes k_{ee}(b).$$

Here we use the square group morphisms υ, ζ, and ξ in (4.1) and (4.2). For (a) we use

$$\upsilon_e\zeta_e(\partial\alpha(c)\underline{\circledcirc}\beta(x)) = \upsilon_e\xi_e(\alpha(c)\underline{\circledcirc}\partial\beta(x)).$$

This equality follows from the fact that the square in (4.2) commutes. We leave the reader to check that $\alpha\odot\beta$ is indeed a track $f\odot h\Rightarrow g\odot k$ and that the axioms of a track functor are satisfied. $\Box$

Using the symmetry isomorphism $\tau_\odot$ for the tensor product of quadratic pair modules we deduce from Proposition 5.4 a track functor

$$\odot\colon \mathbf{qpm}\times\mathbf{wqpm}\longrightarrow\mathbf{wqpm}. \tag{5.5}$$

We also have a well-defined track functor

$$\odot\colon \mathbf{qpm}\times\mathbf{qpm}\longrightarrow\mathbf{qpm} \tag{5.6}$$

which can be defined as the restriction of either (5.5) or the track functor in Proposition 5.4. Both definitions yield the same track functor (5.6) since the following lemma holds.

Lemma 5.7. *Given tracks α, β in* **qpm** *the equation $\tau_\odot(\alpha\odot\beta)=(\beta\odot\alpha)\tau_\odot$ holds.*

The proof is a straightforward but somewhat lengthy computation. One can also use the track functor $\odot$ in (5.6) to show that **qpm** is indeed a symmetric monoidal 2-category, compare [BN96] and [Cra98].

Part 2. Secondary homotopy groups as a lax symmetric monoidal functor

In this part we introduce the additive secondary homotopy group as a quadratic pair module and we formulate our main results on the smash product for additive secondary homotopy groups leading to a lax symmetric monoidal functor. We also give applications to unstable cup-one products, Toda brackets, and secondary Whitehead products.

6. Homotopy groups and secondary homotopy groups

Let **Ab** be the category of abelian groups. Using classical homotopy groups π_nX we obtain for $n\geq 0$ the functor

$$\Pi_n\colon \mathbf{Top}^*\longrightarrow\mathbf{Ab}$$

with

$$\Pi_nX=\begin{cases}\pi_nX, & n\geq 2,\\ (\pi_1X)_{ab}, & n=1,\\ \mathbb{Z}[\pi_0X], & n=0,\end{cases} \tag{6.1}$$

termed *additive homotopy group.*

One readily checks that the smash product $f\wedge g\colon S^{n+m}\longrightarrow X\wedge Y$ of maps $\{f\colon S^n\to X\}\in\pi_nX$ and $\{g\colon S^m\to Y\}\in\pi_mY$ induces a well-defined homomorphism

$$\wedge\colon \Pi_nX\otimes\Pi_mY\longrightarrow\Pi_{n+m}(X\wedge Y). \tag{6.2}$$

This homomorphism is symmetric in the sense that the symmetry isomorphism $\tau_\wedge\colon X\wedge Y\to Y\wedge X$ yields the equation in $\Pi_{n+m}(Y\wedge X)$

$$(\tau_\wedge)_*(f\wedge g)=(-1)^{nm}g\wedge f. \tag{6.3}$$

The sign $(-1)^{nm}$ is given by the degree of the symmetry isomorphism

$$\tau_{n,m}=\tau_\wedge\colon S^{n+m}=S^n\wedge S^m\longrightarrow S^m\wedge S^n=S^{m+n}. \tag{6.4}$$

Here

$$\tau_{n,m}\in\mathrm{Sym}(n+m) \tag{6.5}$$

is the shuffle permutation of $n+m$ elements which exchanges the blocks $\{1,\dots,n\}$ and $\{n+1,\dots,n+m\}$. For this we recall that the symmetric group of k letters $\mathrm{Sym}(k)$ acts on the k-sphere

$$S^k = S^1 \wedge \overset{k}{\cdots} \wedge S^1$$

by permutation of coordinates according to (3.2).

The main purpose of this paper is the generalization of the smash product operator (6.2) for additive secondary homotopy groups.

Definition 6.6. Let $n \geq 2$. For a pointed space X we define the *additive secondary homotopy group* $\Pi_{n,*}X$ which is the quadratic pair module given by the diagram

$$\Pi_{n,*}X = \left(\begin{tikzcd} & \Pi_{n,ee}X = \otimes^2\mathbb{Z}[\Omega^n X] \arrow[dl, "P"'] & \\ \Pi_{n,1}X \arrow[rr, "\partial"'] & & \Pi_{n,0}X = \langle \Omega^n X\rangle_{nil} \arrow[ul, "H"'] \end{tikzcd}\right)$$

Here $\Omega^n X$ is the discrete pointed set of maps $S^n \to X$ in $\mathbf{Top}^*$ and H is defined as in (3.6).

We describe the group $\Pi_{n,1}X$ and the homomorphisms P and ∂ as follows. The group $\Pi_{n,1}X$ is given by the set of equivalence classes $[f,F]$ represented by a map $f\colon S^1 \to \vee_{\Omega^n X} S^1$ and a track

$$\begin{tikzcd} S^n \arrow[r, "\Sigma^{n-1}f"'] \arrow[rr, bend left, "0"] & S^n_X \arrow[r, "ev"'] \arrow[u, phantom, "\Uparrow F"] & X. \end{tikzcd}$$

Here the pointed space

$$S^n_X = \vee_{\Omega^n X} S^n = \Sigma^n \Omega^n X$$

is the n-fold suspension of the discrete pointed set $\Omega^n X$, which is the coproduct of n-spheres indexed by the set of non-trivial maps $S^n \to X$. The map $ev\colon S^n_X \to X$ is the obvious evaluation map. Given a map $f\colon S^1 \to \vee_{\Omega^n X} S^1$ we will denote $f_{ev} = ev(\Sigma^{n-1}f)$, so that F in the previous diagram is a track $F\colon f_{ev} \Rightarrow 0$.

The equivalence relation $[f,F] = [g,G]$ holds provided there is a track

$$N\colon \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}g$$

with $\overline{Hopf}(N) = 0$, see (6.8) and (6.9) below, such that the composite track in the following diagram is the trivial track.

(6.7)
$$\begin{tikzcd} S^n \arrow[r, bend left, "\Sigma^{n-1}f"{name=A}] \arrow[r, bend right, "\Sigma^{n-1}g"'{name=B}] \arrow[rr, bend left=60, "0"] \arrow[rr, bend right=60, "0"'] \arrow[Rightarrow, from=A, to=B, "N"] & S^n_X \arrow[r, "ev"] \arrow[u, phantom, "\Downarrow F^{\boxminus}"] \arrow[d, phantom, "\Downarrow G"] & X \end{tikzcd}$$

That is $F = G\square(ev\,N)$. The map ∂ is defined by the formula

$$\partial[f,F] = (\pi_1 f)_{nil}(1),$$

where $1 \in \pi_1 S^1 = \mathbb{Z}$.

The *Hopf invariant* $Hopf(N)$ of a track $N\colon \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}g$ between maps as above is defined in [BM05a] 3.3 by the homomorphism

$$(6.8)\quad H_2(IS^1, S^1 \vee S^1) \xrightarrow{ad(N)_*} H_2(\Omega^{n-1}S^n_X, \vee_{\Omega^n X} S^1) \cong \begin{cases} \hat{\otimes}^2 \mathbb{Z}[\Omega^n X], & n \geq 3, \\ \otimes^2 \mathbb{Z}[\Omega^n X], & n = 2. \end{cases}$$

which carries the generator $1 \in \mathbb{Z} \cong H_2(IS^1, S^1 \vee S^1)$ to $Hopf(N)$. Here the isomorphism is induced by the Pontrjagin product and $ad(N)_*$ is the homomorphism induced in homology by the adjoint of

$$S^{n-1} \wedge I_+ \wedge S^1 \overset{(1\,2)}{\cong} I_+ \wedge S^n \xrightarrow{N} S^n_X.$$

The reduced tensor square $\hat{\otimes}^2$ in (6.8) is given by

$$\hat{\otimes}^2 A = \frac{A \otimes A}{a \otimes b + b \otimes a \sim 0},$$

and $\bar{\sigma}\colon \otimes^2 A \twoheadrightarrow \hat{\otimes}^2 A$ is the natural projection. We define

$$(6.9)\qquad \overline{Hopf} = \begin{cases} Hopf, & \text{for } n \geq 3, \\ \bar{\sigma} Hopf, & \text{for } n = 2. \end{cases}$$

We refer the reader to [BM05a] 3 for the elementary properties of *Hopf* which will be used in this paper.

This completes the definition of $\Pi_{n,1}X$, $n \geq 2$, as a set. The group structure of $\Pi_{n,1}X$ is induced by the comultiplication $\mu\colon S^1 \to S^1 \vee S^1$, compare [BM05a] 4.4.

We now define the homomorphism P. Consider the diagram

$$\begin{tikzcd} S^n \arrow[r, "\Sigma^{n-1}\beta"'] \arrow[r, bend left=60, "0"] \arrow[r, phantom, bend left=30, "\Uparrow B"] & S^n \vee S^n \end{tikzcd}$$

where $\beta\colon S^1 \to S^1 \vee S^1$ is given such that $(\pi_1\beta)_{nil}(1) = [a, b] \in \langle a, b\rangle_{nil}$ is the commutator of the generators. The track B is any track with $\overline{Hopf}(B) = -\bar{\sigma}(a \otimes b) \in \hat{\otimes}^2 \mathbb{Z}[a, b]$. Given $x \otimes y \in \otimes^2 \mathbb{Z}[\Omega^n X]$ let $\tilde{x}, \tilde{y}\colon S^1 \to \vee_{\Omega^n X} S^1$ be maps with $(\pi_1 \tilde{x})_{ab}(1) = x$ and $(\pi_1 \tilde{y})_{ab}(1) = y$. Then the diagram

$$(6.10)\qquad \begin{tikzcd} S^n \arrow[r, "\Sigma^{n-1}\beta"'] \arrow[r, bend left=60, "0"] \arrow[r, phantom, bend left=30, "\Uparrow B"] & S^n \vee S^n \arrow[r, "\Sigma^{n-1}(\tilde{y}, \tilde{x})"'] & S^n_X \arrow[r, "ev"'] & X \end{tikzcd}$$

represents an element

$$P(x \otimes y) = [(\tilde{y}, \tilde{x})\beta, (\tilde{y}_{ev}, \tilde{x}_{ev})B] \in \Pi_{n,1}X.$$

This completes the definition of the quadratic pair module $\Pi_{n,*}X$ for $n \geq 2$. For $n = 0, 1$ we define the additive secondary homotopy groups $\Pi_{n,*}X$ by Remark 6.13 below. In this way we get for $n \geq 0$ a functor

$$\Pi_{n,*}\colon \mathbf{Top}^* \longrightarrow \mathbf{qpm}$$

which is actually a track functor.

For a quadratic pair module C we denote $h_0C = \operatorname{Coker}\partial$, $h_1C = \operatorname{Ker}\partial$. There are natural isomorphisms

(6.11)
$$\begin{array}{rcl} h_0\Pi_{n,*}X & \cong & \pi_nX,\ \ n \geq 2, \\ h_1\Pi_{n,*}X & \cong & \pi_{n+1}X,\ \ n \geq 3, \\ h_1\Pi_{2,*}X & \cong & \pi_3X/[\pi_2X, \pi_2X], \end{array}$$

where $[-,-]$ denotes here the Whitehead product. Here we use [BM05a] 5.1 and also [BM05a] 6.11 for the case $n = 2$. Furthermore, the following property is crucial.

Proposition 6.12. *The homomorphism*

$$h_0\Pi_{n,*}X \to h_1\Pi_{n,*}X : x \mapsto P(x|x)_H$$

coincides via (6.11) with the homomorphism $\eta^* : \pi_nX \otimes \mathbb{Z}/2 \to \pi_{n+1}X$ *if* $n \geq 3$ *and* $\eta^* : \pi_2X \otimes \mathbb{Z}/2 \to \pi_3X/[\pi_2X, \pi_2X]$ *if* $n = 2$. *Here* η^* *is induced by precomposition with* $\Sigma^{n-2}\eta$ *where* $\eta : S^3 \to S^2$ *is the Hopf map.*

This follows from [BM05a] 8.2.

Remark 6.13. Considering maps $f : S^n \to X$ together with tracks of such maps to the trivial map, we introduced in [BM05a] the secondary homotopy group $\pi_{n,*}X$, which is a groupoid for $n = 0$, a crossed module for $n = 1$, a reduced quadratic module for $n = 2$, and a stable quadratic module for $n \geq 3$.

Then using the adjoint functors Ad_n of the forgetful functors ϕ_n as discussed in [BM05a] 6 we get the *additive secondary homotopy group* track functor

$$\Pi_{n,*} : \mathbf{Top}^* \longrightarrow \mathbf{squad}$$

given by

$$\Pi_{n,*}X = \begin{cases} \pi_{n,*}X, & \text{for } n \geq 3, \\ \mathsf{Ad}_3\pi_{2,*}X, & \text{for } n = 2, \\ \mathsf{Ad}_3\mathsf{Ad}_2\pi_{1,*}X, & \text{for } n = 1, \\ \mathsf{Ad}_3\mathsf{Ad}_2\mathsf{Ad}_1\pi_{0,*}X, & \text{for } n = 0. \end{cases}$$

This is the secondary analogue of (6.1).

Here the category **squad** of stable quadratic modules is not appropriate to study the smash product of secondary homotopy groups since we do not have a symmetric monoidal structure in **squad**. Therefore we introduced above the category **qpm** of quadratic pair modules and we observe that $\Pi_{n,*}X$ in **squad** yields a functor to the category **qpm** in the following way. As a stable quadratic module $\Pi_{n,*}X$ looks as follows

$$\otimes^2\mathbb{Z}[\Omega^nX] = \otimes^2(\Pi_{n,0}X)_{ab} \xrightarrow{P(-|-)_H} \Pi_{n,1}X \xrightarrow{\partial} \Pi_{n,0}X = \langle\Omega^nX\rangle_{nil}.$$

In the quadratic pair module $\Pi_{n,*}X$ the quadratic map H is defined as in (3.6) and the homomorphism $(-|-)_H : \otimes^2 (\Pi_{n,0}X)_{ab} = \Pi_{n,ee}X$ is the identity. A map $f : X \to Y$ in $\mathbf{Top}^*$ induces a homomorphism $\Pi_{n,0}f : \Pi_{n,0}X \to \Pi_{n,0}Y$ between free nil-groups which carries generators in $\Pi_{n,0}X$ to generators in $\Pi_{n,0}Y$ and therefore $\Pi_{n,*}f$ is compatible with H. This shows that there is a canonical lift

$$\begin{tikzcd} & \mathbf{qpm} \arrow[d] \\ \mathbf{Top}^* \arrow[ur, "\Pi_{n,*}"] \arrow[r, "\Pi_{n,*}"'] & \mathbf{squad} \end{tikzcd}$$

Compare [BM05b] 1.15.

The definition of $\Pi_{2,*}X$ given above coincides with the lifting of $\mathsf{Ad}_3\pi_{2,*}X$ to **qpm** by the claim (*) in the proof of [BM05a] 4.9.

Generalizing (6.11) we have for all $n \geq 0$ a natural isomorphism

$$h_0\Pi_{n,*}X \quad \cong \Pi_n X, \tag{6.14}$$

see [BM05a] 6.10.

In this paper we are concerned with the properties of the track functor $\Pi_{n,*}$ mapping to the category **qpm**. In order to simplify notation given a map $f\colon X \to Y$ in $\mathbf{Top}^*$ we will just denote

$$f_* = \Pi_{n,i}f\colon \Pi_{n,i}X \longrightarrow \Pi_{n,i}Y, \text{ for } i = 0, 1, ee \text{ and } n \geq 0.$$

Moreover, given a track $\alpha\colon f \Rightarrow g$ between maps $f, g\colon X \to Y$ we denote by

$$\alpha_* = \Pi_{n,*}\alpha\colon \Pi_{n,0}X \longrightarrow \Pi_{n,1}Y, \;\; n \geq 0,$$

the induced track $\alpha_*\colon f_* \Rightarrow g_*$ in **qpm**.

7. Smash product for secondary homotopy groups

In this section we describe our main results connecting the tensor product of quadratic pair modules and the smash product of pointed spaces. The smash product operator in the next theorem is the canonical analogue of the smash product for classical homotopy groups in (6.2) above.

Theorem 7.1. *The functor of additive secondary homotopy groups* $\Pi_{n,*}\colon \mathbf{Top}^* \to$ **qpm** *admits a well-defined smash product operator*

$$\wedge\colon \Pi_{n,*}X \odot \Pi_{m,*}Y \longrightarrow \Pi_{n+m,*}(X \wedge Y),$$

which is a morphism in **qpm**, *inducing the smash product of classical homotopy groups in (6.2) on* h_0. *This operator is natural in* X *and* Y *with respect to maps and tracks.*

This smash product operator is given in the (0)-level by the following morphism of square groups

$$\mathbb{Z}_{nil}[\Omega^n X] \odot \mathbb{Z}_{nil}[\Omega^m Y] \cong \mathbb{Z}_{nil}[\Omega^n X \wedge \Omega^m Y] \stackrel{\mathbb{Z}_{nil}[\wedge]}{\longrightarrow} \mathbb{Z}_{nil}[\Omega^{n+m}(X \wedge Y)].$$

Here

$$\wedge\colon (\Omega^n X) \wedge (\Omega^m Y) \longrightarrow \Omega^{m+n}(X \wedge Y), \;\; n, m \geq 0,$$

is the map between discrete pointed sets defined by

$$(f\colon S^n \to X) \wedge (g\colon S^m \to Y) \mapsto (f \wedge g\colon S^{n+m} \to X \wedge Y).$$

On the (1)-level the definition of the smash product operator in Theorem 7.1 is more elaborate, see Definition 14.3 below.

Proof of Theorem 7.1. The first part of Theorem 7.1 follows from Lemmas 18.2, 18.3, 18.5 and 19.1 in Part 3. For the naturality we use Lemma 19.2 in Part 3 and Theorems 7.2 and 7.3. □

We will use the following notation for the image of generators in the tensor product by the smash product morphism in Theorem 7.1. Given $x \in \Pi_{n,i}X$ and $y \in \Pi_{m,j}Y$ for $0 \leq i, j, i+j \leq 1$ we denote by

$$x \wedge y \in \Pi_{n+m,i+j}(X \wedge Y)$$

to the image by $\wedge$ of the element $x \circledcirc y \in \Pi_{n+m,0}(X \wedge Y)$ if $i = j = 0$, of $\upsilon_e \zeta_e(x \circledcirc y) \in \Pi_{n+m,1}(X \wedge Y)$ if $i = 1$ and $j = 0$, and of $\upsilon_e \xi_e(x \circledcirc y) \in \Pi_{n+m,1}(X \wedge Y)$ if $i = 0$ and $j = 1$. Here we use the square group morphisms υ, ζ and ξ in (4.1) and (4.2). Similarly for $x \underline{\circledcirc} y$ and $x \underline{\wedge} y$. Moreover, given $a \in \Pi_{n,ee}X$ and $b \in \Pi_{m,ee}Y$ we denote by

$$a \wedge b \in \Pi_{n+m,ee}(X \wedge Y)$$

to the image of $a \otimes b \in (\Pi_{n,*}X \odot \Pi_{m,*}Y)_{ee}$ by $\wedge$.

Theorem 7.2. *The smash product operator endows $\Pi_{*,*}$ with the structure of a lax monoidal functor from $\mathbf{Top}^*$ to the category of graded quadratic pair modules. That is, the following diagram commutes*

$$\begin{tikzcd}
\Pi_{n,*}X \odot (\Pi_{m,*}Y \odot \Pi_{l,*}Z) \arrow[r, "\cong"] \arrow[dd, "1\odot\wedge"] & (\Pi_{n,*}X \odot \Pi_{m,*}Y) \odot \Pi_{l,*}Z \arrow[d, "\wedge\odot 1"] \\
 & \Pi_{n+m,*}(X \wedge Y) \odot \Pi_{l,*}Z \arrow[d, "\wedge"] \\
\Pi_{n,*}X \odot \Pi_{m+l,*}(Y \wedge Z) \arrow[r, "\wedge"] & \Pi_{n+m+l,*}(X \wedge Y \wedge Z)
\end{tikzcd}$$

and for the unit S^0 of the symmetric monoidal category $(\mathbf{Top}^*, \wedge)$ and the unit $\overline{\mathbb{Z}}_{nil}$ of $(\mathbf{qpm}, \odot)$ there is an isomorphism $u\colon \overline{\mathbb{Z}}_{nil} \cong \Pi_{0,*}S^0$ such that the following diagrams commute.

$$\begin{tikzcd}
\Pi_{0,*}S^0 \odot \Pi_{n,*}X \arrow[r, "\wedge"] & \Pi_{n,*}(S^0 \wedge X) \arrow[d, "\cong"] \\
\overline{\mathbb{Z}}_{nil} \odot \Pi_{n,*}X \arrow[u, "u\odot 1", "\cong"'] \arrow[r, "\cong"] & \Pi_{n,*}X
\end{tikzcd}$$

$$\begin{tikzcd}
\Pi_{n,*}X \odot \Pi_{0,*}S^0 \arrow[r, "\wedge"] & \Pi_{n,*}(X \wedge S^0) \arrow[d, "\cong"] \\
\Pi_{n,*}X \odot \overline{\mathbb{Z}}_{nil} \arrow[u, "1\odot u", "\cong"'] \arrow[r, "\cong"] & \Pi_{n,*}X
\end{tikzcd}$$

The isomorphism $u\colon \overline{\mathbb{Z}}_{nil} \cong \Pi_{0,*}S^0$ is the unique one sending $1 \in \mathbb{Z} = (\overline{\mathbb{Z}}_{nil})_0$ to $u(1) = 1_{S^0}\colon S^0 \to S^0$ in $\Pi_{0,0}S^0$.

Proof of Theorem 7.2. In dimensions ≥ 1 the associativity property in Theorem 7.2 follows from Lemma 18.6 in Part 3. In case dimension 0 is involved we use the more algebraic Lemma 19.1. The commutativity of the squares with the isomorphism u is easy to check. This is left to the reader. $\square$

The graded commutativity equation (6.3) for the smash product for classical homotopy groups has a secondary analogue as follows.

Theorem 7.3. *The following diagram commutes in* **qpm**.

$$\begin{tikzcd}
\Pi_{n,*}X \odot \Pi_{m,*}Y \arrow[r, "\wedge"] \arrow[dd, "\tau_\odot"', "\cong"] & \Pi_{n+m,*}(X \wedge Y) \arrow[d, "\cong"', "(\tau_{X,Y})_*"] \\
& \Pi_{n+m,*}(Y \wedge X) \\
\Pi_{m,*}Y \odot \Pi_{n,*}X \arrow[r, "\wedge"] & \Pi_{m+n,*}(Y \wedge X) \arrow[u, "\cong", "\tau^*_{n,m}"']
\end{tikzcd}$$

Here $\tau_\odot$ is the symmetry isomorphism of the tensor product $\odot$ in **qpm**, *and $\tau^*_{n,m}$ is given by the right action of the shuffle permutation $\tau_{n,m} \in \mathrm{Sym}(n+m)$ in (6.5) on $\Pi_{m+n,*}(Y \wedge X)$, see the next section.*

This follows from Lemma 18.7 in Part 3.

8. The symmetric action on smash products

Secondary homotopy groups, regarded as a functor from pointed spaces to graded quadratic pair modules

$$\Pi_{*,*} \colon \mathbf{Top}^* \longrightarrow \mathbf{qpm}^{\mathbb{N}}$$

is a lax monoidal functor, see Theorem 7.2. The monoidal structure in $\mathbf{qpm}^{\mathbb{N}}$ is the usual graded extension of the tensor product $\odot$ of quadratic pair modules, and $\mathbf{Top}^*$ has the monoidal structure given by the smash product $\wedge$. Both monoidal structures are symmetric, however Theorem 7.3 shows that $\Pi_{*,*}$ is not lax symmetric monoidal since the action of the shuffle permutation $\tau_{n,m}$ in (6.5) is involved. This inconvenience is solved by enriching the structure of secondary homotopy groups with the sign action of the symmetric track groups constructed in [BM05b].

Definition 8.1. Let $\{\pm 1\}$ be the multiplicative group of order 2. A *sign group* $G_\square$ is a diagram of group homomorphisms

$$\{\pm 1\} \overset{\imath}{\hookrightarrow} G_\square \overset{\partial}{\twoheadrightarrow} G \overset{\varepsilon}{\longrightarrow} \{\pm 1\}$$

where the first two morphisms $\imath$ and ∂ form an extension. Here all groups have a multiplicative group law and the composite $\varepsilon\partial$ is also denoted by $\varepsilon \colon G_\square \to \{\pm 1\}$.

A morphism $f_\square \colon G_\square \to K_\square$ of sign groups is a commutative diagram of group homomorphisms

$$\begin{tikzcd}
\{\pm 1\} \arrow[r, hook] \arrow[d, equal] & G_\square \arrow[r, two heads] \arrow[d, "f_\square"] & G \arrow[r] \arrow[d, "f"] & \{\pm 1\} \arrow[d, equal] \\
\{\pm 1\} \arrow[r, hook] & K_\square \arrow[r, two heads] & K \arrow[r] & \{\pm 1\}
\end{tikzcd}$$

This defines the category $\mathbf{Gr}_\pm$ of sign groups. The initial object in this category given by $G = \{1\}$ will be termed the *trivial sign group*.

Remark 8.2. Recall from [BM05b] 3.6 that a sign group $G_\square$ gives rise to a crossed module

$$\partial_\square = (\varepsilon, \partial) \colon G_\square \longrightarrow \{\pm 1\} \times G.$$

where $\{\pm 1\} \times G$ acts on $G_\square$ by the formula

$$g^{(x,h)} = \bar{h}^{-1} g \bar{h} \imath \left(\varepsilon(g)^{\binom{x}{2}} \right).$$

Here $g \in G_\square$, $x \in \{\pm 1\}$, $h \in G$, and $\bar{h} \in G_\square$ is any element with $\partial(\bar{h}) = h$. This is a well-defined crossed module since $G_\square$ is a central extension of G by $\{\pm 1\}$.

The main examples of sign groups are the symmetric track groups

$$\{\pm 1\} \hookrightarrow \mathrm{Sym}_\square(n) \overset{\delta}{\twoheadrightarrow} \mathrm{Sym}(n) \xrightarrow{\mathrm{sign}} \{\pm 1\}$$

defined as follows. The symmetric group $\mathrm{Sym}(n)$ acts on the left of the n-sphere

$$S^n = S^1 \wedge \overset{n}{\cdots} \wedge S^1,$$

see (3.2). The elements of the symmetric track group $\mathrm{Sym}_\square(n)$ for $n \geq 2$ are tracks $\alpha\colon \sigma \Rightarrow (\cdot)_n^{\mathrm{sign}(\sigma)}$ between maps $\sigma, (\cdot)_n^{\mathrm{sign}(\sigma)}\colon S^n \to S^n$, with $\sigma \in \mathrm{Sym}(n)$ and $(\cdot)_n^{\mathrm{sign}(\sigma)} = \Sigma^{n-1}(\cdot)^{\mathrm{sign}(\sigma)}$, where

$$(\cdot)^k\colon S^1 \longrightarrow S^1\colon z \mapsto z^k, \quad k \in \mathbb{Z},$$

is given by the (multiplicative) topological abelian group structure of S^1. The group law in $\mathrm{Sym}_\square(n)$ is given by the horizontal composition of tracks. For $n = 0, 1$, let $\mathrm{Sym}_\square(n)$ be the trivial sign group. Compare [BM05b] 5 and 6.

The smash product $S^m \wedge -$ induces a sign group morphism

$$S^m \wedge -\colon \mathrm{Sym}_\square(n) \longrightarrow \mathrm{Sym}_\square(m+n) \tag{8.3}$$

sending a track $\alpha\colon \sigma \Rightarrow (\cdot)_n^{\mathrm{sign}(\sigma)}$ as above to $S^m \wedge \alpha$. This morphism is given on symmetric groups by the usual inclusion $\mathrm{Sym}(n) \subset \mathrm{Sym}(m+n)$ obtained by regarding $\mathrm{Sym}(n)$ as the subgroup of permutations of $m+n$ elements fixing the first m elements.

One can not directly define a sign group morphism

$$- \wedge S^n\colon \mathrm{Sym}_\square(m) \to \mathrm{Sym}_\square(m+n)$$

in a geometric way as above since $(\cdot)_m^k \wedge S^n \neq (\cdot)_{m+n}^k$. With the help of the crossed module structure for sign groups described in Remark 8.2 and the shuffle permutation $\tau_{n,m}$ in (6.5) we define $- \wedge S^n$ as the following composite

$$- \wedge S^n\colon \mathrm{Sym}_\square(m) \xrightarrow{S^n \wedge -} \mathrm{Sym}_\square(n+m) \xrightarrow{(\cdot)^{(1,\tau_{n,m})}} \mathrm{Sym}_\square(m+n) \tag{8.4}$$

This morphism is given on symmetric groups by the inclusion $\mathrm{Sym}(m) \subset \mathrm{Sym}(m+n)$ obtained by identifying $\mathrm{Sym}(m)$ with the group of permutations of $m+n$ elements which only permute the first m ones.

Definition 8.5. A *twisted bilinear morphism* of sign groups

$$(f_\square, g_\square)\colon G_\square \times L_\square \longrightarrow K_\square$$

is given by a pair of sign group morphisms $f_\square\colon G_\square \to K_\square$, $g_\square\colon L_\square \to K_\square$, such that given $a \in G$ and $b \in L$ the equality

$$f(a)g(b) = g(b)f(a)$$

holds in K, and therefore the group homomorphism

$$(f,g)\colon G \times L \longrightarrow K\colon (a,b) \mapsto f(a)g(b)$$

is defined, and given $x \in G_\square$, $y \in L_\square$ the following equality is satisfied in $K_\square$

$$f_\square(x)g_\square(y) = g_\square(y)f_\square(x)\imath\left((-1)^{\binom{\varepsilon(x)}{2}\binom{\varepsilon(y)}{2}}\right).$$

The *twisted product* $G_\square \tilde{\times} L_\square$ of sign groups $G_\square$, $L_\square$ is a sign group

$$\{\pm 1\} \overset{\imath}{\hookrightarrow} G_\square \tilde{\times} L_\square \twoheadrightarrow G \times L \overset{\varepsilon}{\twoheadrightarrow} \{\pm 1\}$$

together with a universal twisted bilinear morphism

$$(i_{G_\square}, i_{L_\square})\colon G_\square \times L_\square \longrightarrow G_\square \tilde{\times} L_\square. \tag{8.6}$$

The group $G_\square \tilde{\times} L_\square$ is generated by the symbols $\overline{s}$, $\overline{t}$ and ω, for $t \in G_\square$ and $s \in L_\square$, with the following relations:

1. ω is central,
2. $\overline{\imath(-1)} = \omega$ for both $\imath\colon \{\pm 1\} \hookrightarrow G_\square$ and $\imath\colon \{\pm 1\} \hookrightarrow L_\square$,
3. $\overline{r_1 r_2} = \overline{r_1} \cdot \overline{r_2}$ when r_1, r_2 lie both in the group $G_\square$ or both in $L_\square$,
4. $\overline{t}\overline{s} = \overline{s}\overline{t}\omega^{\binom{\varepsilon(t)}{2}\binom{\varepsilon(s)}{2}}$ for $t \in G_\square$ and $s \in L_\square$.

The homomorphism $\partial\colon G_\square \tilde{\times} L_\square \twoheadrightarrow G \times L$ is defined by $\partial(\overline{t}) = (\partial(t), 0)$ for $t \in G_\square$, $\partial(\overline{s}) = (0, \partial(s))$ for $s \in L_\square$, and $\partial(\omega) = 0$. The universal bilinear morphism is given by $i_{G_\square}(t) = \overline{t}$ and $i_{L_\square}(s) = \overline{s}$. The twisted product induces a symmetric monoidal structure in the category $\mathbf{Gr}_\pm$ of sign groups.

Proposition 8.7. *The morphisms in (8.3) and (8.4) induce a morphism of sign groups*

$$\mathrm{Sym}_\square(m) \tilde{\times} \mathrm{Sym}_\square(n) \longrightarrow \mathrm{Sym}_\square(m+n).$$

This proposition can be derived easily from the presentation of the symmetric track groups given in [BM05b] 6.11.

We now introduce the action of a sign group on a quadratic pair module. In [BM05b] we show that the sign group $\mathrm{Sym}_\square(n)$ acts in this sense on $\Pi_{n,*}X$.

Definition 8.8. A sign group $G_\square$ *acts on the right* of a quadratic pair module C if G acts on the right of C by morphisms $g^*\colon C \to C$, $g \in G$, in **qpm**, and there is a bracket

$$\langle -, - \rangle = \langle -, - \rangle_G\colon C_0 \times G_\square \longrightarrow C_1$$

satisfying the following properties, $x, y \in C_0$, $z \in C_1$, $s, t \in G_\square$, with $\varepsilon(t)^*$ given by the action of $(\mathbb{Z}, \cdot)$ in Definition 1.4.

1. $\langle x + y, t \rangle = \langle x, t \rangle + \langle y, t \rangle + P(-\partial(t)^*(x) + \varepsilon(t)^*(x)|\partial(t)^*(y))_H$,
2. $\varepsilon(t)^*(x) = \partial(t)^*(x) + \partial\langle x, t \rangle$,
3. $\varepsilon(t)^*(z) = \partial(t)^*(z) + \langle \partial(z), t \rangle$,
4. $\langle x, s \cdot t \rangle = \langle \partial(s)^*(x), t \rangle + \langle \varepsilon(t)^* x, s \rangle$,
5. for the element $\omega = \imath(-1) \in G_\square$ we have the *ω-formula*:

   $$\langle x, \omega \rangle = P(x|x)_H.$$

   Notice that the ω-formula corresponds to the homomorphism in Proposition 6.12.

The reader can easily check that the trivial sign group acts on any quadratic pair module in a unique way.

Lemma 8.9. *If the sign group $G_\square$ acts on the quadratic pair module C, then for any quadratic pair module X the induced action of G on $C \odot X$ and $X \odot C$ has a canonical extension to a sign group action of $G_\square$.*

Proof. This proposition, up to the verification of ω-formula, is just a consequence of Propositions 4.5 and 5.4, Lemma 4.6, and the fact that a sign group action can be regarded as a monoid-groupoid action in the track category **wqpm**, see [BM05b] 3.7.

Before checking the ω-formula for the actions in the tensor products we specify for the convenience of the reader how the action of G and the bracket operation are defined.

The group G acts on $C \odot X$ and $X \odot C$ by the morphisms $g^* \odot X$ and $X \odot g^*$ in **qpm**, $g \in G$, respectively.

Following Proposition 5.4 the bracket operation

$$\langle -, - \rangle \colon (C \odot X)_0 \times G_\square \longrightarrow (C \odot X)_1$$

is defined as follows: given $t \in G_\square$, $c \in C_0$ and $x \in X_0$ we have

$$\langle c \underline{\circledcirc} x, t\rangle = v_e \zeta_e(\langle c, t\rangle \underline{\circledcirc} x) + (-\partial(t)^*(c) + \varepsilon(t)^*(c)|\partial(t)^*(c))_H \bar{\otimes} H(x),$$

and for $a \in C_{ee}$ and $b \in X_{ee}$

$$\langle a \bar{\otimes} b, t\rangle = -(\partial(t)^*(a))\bar{\otimes} b + (\varepsilon(t)^*(a))\bar{\otimes} b.$$

Here we use the square group morphisms in (4.1) and (4.2).

Similarly the bracket operation

$$\langle -, - \rangle \colon (X \odot C)_0 \times G_\square \longrightarrow (X \odot C)_1$$

is defined as follows: given $t \in G_\square$, $x \in X_0$ and $c \in C_0$ we have

$$\langle x \circledcirc c, t\rangle = v_e \xi_e(x \circledcirc \langle c, t\rangle) + H(x)\bar{\otimes}(-\partial(t)^*(c) + \varepsilon(t)^*(c)|\partial(t)^*(c))_H,$$

and for $a \in X_{ee}$ and $b \in C_{ee}$

$$\langle a \bar{\otimes} b, t\rangle = -a\bar{\otimes}(\partial(t)^*(b)) + a\bar{\otimes}(\varepsilon(t)^*(b)).$$

Let us now check that the ω-formula is satisfied. We will do it only in the first case. The second case follows by symmetry.

By Definition 8.8 the symbol $\langle x, \omega\rangle$ is linear in x since $\partial(\omega) = 1$ and $\varepsilon(\omega) = 1$. Moreover, it is not difficult to check that $P(x|x)_H$ is also linear in x for any quadratic pair module, therefore we only have to check the ω-formula for the 0-level generators of $C \odot X$. Given $c \in C_0$ and $x \in X_0$ we have

$$\begin{aligned}
P(c\underline{\circledcirc} x|c\underline{\circledcirc} x)_H &\overset{(a)}{=} (c|c)_H \bar{\otimes} (x|x)_H \\
&\overset{(b)}{=} (c|c)_H \bar{\otimes} (\Delta(x) + H(x) - TH(x)) \\
&\overset{(c)}{=} (c|c)_H \bar{\otimes} \Delta(x) + (c|c)_H \bar{\otimes} (H(x) - TH(x)) \\
&\overset{(d)}{=} P(c|c)_H \underline{\circledcirc} x + c\underline{\circledcirc}(PH(x) - PTH(x)) \\
&\overset{(e)}{=} P(c|c)_H \underline{\circledcirc} x \\
&\overset{(f)}{=} \langle c, \omega\rangle \underline{\circledcirc} x \\
&\overset{(g)}{=} \langle c\underline{\circledcirc} x, \omega\rangle.
\end{aligned}$$

In (a) we use the definition of P and $(-|-)_H$ for tensor products of square groups, see Definition 2.1. In (b) we use the definition of Δ in Definition 1.1. For (c) we use that the symbol $\bar{\otimes}$ is bilinear, see Definition 2.1 (1). In (d) we use (4) and (9) in Definition 2.1. For (e) we use (3) in Definition 1.1. In (f) we use the ω-formula for the action of $G_\square$ on C, and finally in (g) we use that $\partial(\omega) = 1$ and $\varepsilon(\omega) = 1$. The ω-formula is trivial for the generators $a\bar{\otimes}b$ in the 0-level of $C \odot X$. □

The following lemma is the original motivation for the definition of the twisted product of sign groups.

Lemma 8.10. *If $G_\square$ and $L_\square$ are sign groups acting on the quadratic pair modules C and D, respectively, then $G_\square \tilde{\times} L_\square$ acts on $C \odot D$ as follows: the group $G \times L$ acts on $C \odot D$ by morphisms in* **qpm**

$$(g,k)^* = g^* \odot k^* \colon C \odot D \longrightarrow C \odot D, \;\; g \in G, k \in L,$$

and the bracket operation

$$\langle -,-\rangle_{G\times L} \colon (C \odot D)_0 \times (G_\square \tilde{\times} L_\square) \longrightarrow (C \odot D)_1$$

is determined by the axioms in Definition 8.8 and the formulas

$$\begin{aligned} \langle -, i_{G_\square}(s)\rangle_{G\times L} &= \langle -, s\rangle_G, \;\; s \in G_\square; \\ \langle -, i_{L_\square}(t)\rangle_{G\times L} &= \langle -, t\rangle_L, \;\; t \in L_\square. \end{aligned}$$

Here we use the actions of $G_\square$ and $L_\square$ on $C \odot D$ given by Lemma 8.9 and the universal bilinear morphism of sign groups in (8.6).

Proof. The unique non-obvious formula that one needs to prove is the following one, $t \in G_\square$, $s \in L_\square$, $x \in (C \odot D)_0$,

$$\begin{aligned} &\langle(\partial(t)^* \odot D)x, s\rangle_L \\ &\quad +\langle \varepsilon(s)^* x, t\rangle_G = \binom{\varepsilon(t)}{2}\binom{\varepsilon(s)}{2} P((\partial(t)^* \odot \partial(s)^*)x|(\partial(t)^* \odot \partial(s)^*)x)_H \\ &\qquad\qquad\qquad\qquad +\langle (C \odot \partial(s)^*)x, t\rangle_G + \langle \varepsilon(t)^* x, s\rangle_L \end{aligned}$$

By Definition 8.8 (4) and (5) and Lemma 4.6 this equality corresponds to relation (4) in the definition of twisted product of sign groups. By [BM05a] 3.7 and 3.4 (4)

$$\begin{aligned} \langle (C \odot \partial(s)^*)x, t\rangle_G + \langle(\varepsilon(t)^* \odot D)x, s\rangle_L &= (\partial(t)^* \odot D)\varepsilon(t)^*\langle \varepsilon(t)^* x, s\rangle_L \\ &\quad +\langle \varepsilon(s)^* x, t\rangle_G. \end{aligned}$$

By [BM05a] 1.16

$$\varepsilon(t)^*\langle \varepsilon(t)^* x, s\rangle_L = \langle x, s\rangle_L + \binom{\varepsilon(t)}{2}\binom{\varepsilon(s)}{2} P(\varepsilon(t)^* x|\varepsilon(t)^* x)_H.$$

The map $x \mapsto P(x|x)_H$ defines a homomorphism $\operatorname{Coker}\partial \otimes \mathbb{Z}/2 \to \operatorname{Ker}\partial$, therefore by using Definition 8.8 (2)

$$\begin{aligned} P((\partial(t)^* \odot \partial(s)^*)x|(\partial(t)^* \odot \partial(s)^*)x)_H &= P(x|x)_H, \\ P(\varepsilon(t)^* x|\varepsilon(t)^* x)_H &= P(x|x)_H. \end{aligned}$$

In particular the right hand side of the first equation is

$$(\partial(t)^* \odot D)\langle x, s\rangle_L + \langle \varepsilon(s)^* x, t\rangle_G.$$

Now we only need to check the formula

$$(\partial(t)^* \odot D)\langle x, s\rangle_L = \langle(\partial(t)^* \odot D)x, s\rangle_L.$$

This can be easily done by using the definition of $\langle -,-\rangle_L \colon (C\odot D)_0 \times L_\square \to (C\odot D)_1$ in the proof of Proposition 8.9 and the fact that $\partial(t)^*$ is a morphism in **qpm**. □

Since the sign group $\mathrm{Sym}_\square(n)$ acts on $\Pi_{n,*}X$ by the results in [BM05b] we get the following result as a corollary.

Theorem 8.11. *The sign group* $\mathrm{Sym}_\square(n)\tilde{\times}\,\mathrm{Sym}_\square(m)$ *acts on* $\Pi_{n,*}X\odot\Pi_{m,*}Y$.

We will now consider the compatibility of the smash product operation in Theorem 7.1 with the sign group actions.

Definition 8.12. Let $G_\square$, $K_\square$ be sign groups acting on quadratic pair modules C, D respectively. Suppose that $f_\square\colon G_\square\to K_\square$ is a sign group morphism and $\varphi\colon C\to D$ is a morphism in **qpm**. We say that φ is $f_\square$*-equivariant* if φ is f-equivariant in the usual sense, i. e. given $g\in G$ and $a\in C_i$ for $i=0,1,ee$, then

$$\varphi_i(g^*(a))=(f(g))^*(\varphi_i(a)).$$

In addition, if $x\in C_0$ and $t\in G_\square$ we require

$$\varphi_1(\langle x,t\rangle_G)=\langle\varphi_0(x),f_\square(t)\rangle_K.$$

Theorem 8.13. *The smash product morphism*

$$\wedge\colon\Pi_{n,*}X\odot\Pi_{m,*}Y\longrightarrow\Pi_{n+m,*}(X\wedge Y)$$

in Theorem 7.1 is equivariant with respect to the action of $\mathrm{Sym}_\square(n)\tilde{\times}\,\mathrm{Sym}_\square(m)$ *on* $\Pi_{n,*}X\odot\Pi_{m,*}Y$ *defined by Theorem 8.11 and the sign group morphism*

$$\mathrm{Sym}_\square(n)\tilde{\times}\,\mathrm{Sym}_\square(m)\longrightarrow\mathrm{Sym}_\square(n+n)$$

in Proposition 8.7.

This theorem follows from Lemma 18.8 in Part 3.

Since the secondary homotopy groups $\Pi_{n,*}X$ have a canonical action of the sign group $\mathrm{Sym}_\square(n)$ we are led to consider the following category of symmetric sequences in **qpm** (this is similar to the treatment of symmetric spectra in [HSS00]).

Definition 8.14. An object X in the category $\mathbf{qpm}^{\mathrm{Sym}_\square}$ of symmetric sequences is a sequence of quadratic pair modules X_n endowed with a sign group action of the symmetric track group $\mathrm{Sym}_\square(n)$, $n\geq 0$. A morphism $f\colon X\to Y$ in $\mathbf{qpm}^{\mathrm{Sym}_\square}$ is a sequence of $\mathrm{Sym}_\square(n)$-equivariant morphisms $f_n\colon X_n\to Y_n$ in **qpm**. The results in [BM05b] show that secondary homotopy groups yield a functor

$$\Pi_{*,*}\colon\mathbf{Top}^*\longrightarrow\mathbf{qpm}^{\mathrm{Sym}_\square}.$$

The category $\mathbf{qpm}^{\mathrm{Sym}_\square}$ has a symmetric monoidal structure denoted by $\odot_{\mathrm{Sym}_\square}$. The tensor product $X\odot_{\mathrm{Sym}_\square}Y$ of two symmetric sequences X,Y of quadratic pair modules is characterized by the following universal property: for any symmetric sequence Z of quadratic pair modules there is a natural bijection

$$\mathrm{Hom}_{\mathbf{qpm}^{\mathrm{Sym}_\square}}(X\odot_{\mathrm{Sym}_\square}Y,Z)\cong\prod_{p,q\in\mathbb{N}}\mathrm{Hom}_{\mathrm{Sym}_\square(p)\tilde{\times}\,\mathrm{Sym}_\square(q)}(X_p\odot Y_q,Z_{p+q}).$$

Here $\mathrm{Hom}_{\mathrm{Sym}_\square(p)\tilde{\times}\,\mathrm{Sym}_\square(q)}$ denotes the set of morphisms in **qpm** which are equivariant with respect to the sign group morphism in Proposition 8.7. The symmetry isomorphism

$$X\odot_{\mathrm{Sym}_\square}Y\cong Y\odot_{\mathrm{Sym}_\square}X$$

is induced by the morphisms in **qpm**

$$X_p\odot Y_q\cong Y_q\odot X_p\longrightarrow(Y\odot_{\mathrm{Sym}_\square}X)_{q+p}\xrightarrow{\tau_{p,q}^*}(Y\odot_{\mathrm{Sym}_\square}X)_{p+q}.$$

Here the first morphism is the symmetry isomorphism for $\odot$, the second one is induced by the universal property of $Y \odot_{\mathrm{Sym}_\square} X$ and in the third morphism we use the sign group action of $\mathrm{Sym}_\square(p+q)$ and the shuffle permutation $\tau_{p,q} \in \mathrm{Sym}(p+q)$ in (6.5). The associativity isomorphism is defined by using the universal property of the 3-fold tensor product, which is analogous to the 2-fold case above. The unit element is $\overline{\mathbb{Z}}_{nil}$ concentrated in degree 0.

Now Theorems 7.1, 8.13, 7.2 and 7.3 can be restated as follows.

Theorem 8.15. *The smash product operator induces a natural morphism in the category* $\mathbf{qpm}^{\mathrm{Sym}_\square}$

$$\wedge \colon \Pi_{*,*}X \odot_{\mathrm{Sym}_\square} \Pi_{*,*}Y \longrightarrow \Pi_{*,*}(X \wedge Y)$$

which is compatible with the associativity, commutativity and unit isomorphisms for the symmetric monoidal structures $\wedge$ *and* $\odot_{\mathrm{Sym}_\square}$ *in* $\mathbf{Top}^*$ *and* $\mathbf{qpm}^{\mathrm{Sym}_\square}$*, respectively. Equivalently the functor*

$$\Pi_{*,*} \colon \mathbf{Top}^* \longrightarrow \mathbf{qpm}^{\mathrm{Sym}_\square}$$

given by secondary homotopy groups is lax symmetric monoidal.

9. Secondary Whitehead products

The smash product may be used for the definition of Whitehead products in ordinary homotopy groups. In fact, any path connected space X is homotopy equivalent to the classifying space of a topological group G so that $\pi_n G = \pi_{n+1} X$. We consider the additive homotopy groups $\Pi_n G$ which satisfy

$$\Pi_n G = \pi_n G \text{ for } n \geq 1.$$

Using the smash product operator $\wedge$ in (6.2) for the functor Π_* and the commutator map $c \colon G \wedge G \to G$ with $c(a \wedge b) = a^{-1}b^{-1}ab$ we obtain the composite

$$[-,-] \colon \Pi_* G \otimes \Pi_* G \xrightarrow{\wedge} \Pi_*(G \wedge G) \xrightarrow{\Pi_* c} \Pi_* G$$

which corresponds to the Whitehead product in $\pi_* X$. It is well known that $(\Pi_* G, [-,-])$ has the structure of a graded Lie algebra if X is simply connected.

In a similar way we now define the *secondary Whitehead product* for the additive secondary homotopy groups $\Pi_{n,*}G$ by the composite

$$[-,-] \colon \Pi_{*,*} G \otimes \Pi_{*,*} G \xrightarrow{\wedge} \Pi_{*,*}(G \wedge G) \xrightarrow{\Pi_{*,*} c} \Pi_{*,*} G.$$

Marcum defines in [Mar93] the partial Whitehead product of a map α and a track F as in the following diagram

$$\begin{tikzcd}
\Sigma A \arrow[r, "\alpha"'] & X & B \arrow[l, "\beta"] \arrow[u, phantom, "\Uparrow F"] & \tilde{B}. \arrow[l, "e"] \arrow[ll, bend right=60, "0"']
\end{tikzcd}$$

Marcum's partial Whitehead product lives in the group of homotopy classes $[\Sigma A \wedge \tilde{B}, X]$. It can be obtained from the secondary Whitehead product for additive secondary homotopy groups in case A and $\tilde{B}$ are spheres.

We will explore this connection in a sequel of this paper where we shall discuss the algebraic properties of the structure $(\Pi_{*,*}G, [-,-])$ which leads to the notion of a secondary Lie algebra. This should be compared with the notion of secondary Hopf algebra discussed in [Bau06].

10. Cup-one products

Let $n \geq m > 1$ be even integers. The cup-one product operation

$$\pi_n S^m \longrightarrow \pi_{2n+1} S^{2m} \colon \alpha \mapsto \alpha \cup_1 \alpha = Sq_1(\alpha)$$

is defined in the following way, compare [HM93] 2.2.1. Let k be any positive integer and let $\tau_k \in \mathrm{Sym}(2k)$ be the permutation exchanging the first and the second block of k elements in $\{1, \ldots, 2k\}$. If k is even then $\operatorname{sign} \tau_k = 1$. We choose for any even integer $k > 1$ a track $\hat{\tau}_k \colon \tau_k \Rightarrow 1_{S^{2k}}$ in $\mathrm{Sym}_\square(2k)$. Consider the following diagram in the track category $\mathbf{Top}^*$ of pointed spaces where $a \colon S^n \to S^m$ represents α.

(10.1)
$$\begin{tikzcd}
S^{2n} \arrow[r, "a\wedge a"] \arrow[d, "\tau_n"] \arrow[d, bend right=70, "1_{S^{2n}}"'] & S^{2m} \arrow[d, "\tau_m"'] \arrow[d, bend left=70, "1_{S^{2m}}"] \\
S^{2n} \arrow[r, "a\wedge a"] & S^{2m}
\end{tikzcd}$$

By pasting this diagram we obtain a self-track of $a \wedge a$

(10.2) $$(\hat{\tau}_m(a \wedge a))\square((a \wedge a)\hat{\tau}_n^{\boxminus}) \colon a \wedge a \Rightarrow a \wedge a.$$

The set of self-tracks $a \wedge a \Rightarrow a \wedge a$ is the automorphism group of the map $a \wedge a$ in the track category $\mathbf{Top}^*$. The element $\alpha \cup_1 \alpha \in \pi_{2n+1} S^{2m}$ is given by the track (10.2) via the well-known Barcus-Barratt-Rutter isomorphism

$$\mathrm{Aut}(a \wedge a) \cong \pi_{2n+1} S^{2m},$$

see [BB58], [Rut67] and also [Bau91] VI.3.12 and [BJ01] for further details.

The following proposition yields a description of the cup-one product in terms of the structure of additive secondary homotopy groups.

Proposition 10.3. *Let n and m be even positive integers. For $\alpha \in \Pi_n S^m$ we choose $a \in \Pi_{n,0} S^m$ representing α and we define in $\Pi_{2n,1} S^{2m}$*

$$Sq_1(\alpha) = -\langle a \wedge a, \hat{\tau}_n \rangle + (\hat{\tau}_m)_*(a \underline{\triangle} a) - P(H(a) \wedge TH(a)).$$

Then $\partial Sq_1(\alpha) = 0$ so that $Sq_1(\alpha) \in h_1 \Pi_{2n,1} S^{2m} = \pi_{2n+1} S^{2m}$. Moreover, $Sq_1(\alpha) = \alpha \cup_1 \alpha$.

Proof. The track $\hat{\tau}_m \colon \tau_m \Rightarrow 1$ induces a track $(\hat{\tau}_m)_* \colon (\tau_m)_* \Rightarrow 1$ in $\mathbf{qpm}$ satisfying

$$\partial(\hat{\tau}_m)_*(a \underline{\triangle} a) = -(\tau_m)_*(a \underline{\triangle} a) + a \underline{\triangle} a.$$

By the symmetric action we have the element $\langle a \wedge a, \hat{\tau}_n \rangle \in \Pi_{2n,1} S^{2m}$ satisfying

$$-\partial \langle a \wedge a, \hat{\tau}_n \rangle = -a \wedge a + \tau_n^*(a \wedge a).$$

Hence we get

$$\partial Sq_1(\alpha) = -a \wedge a + (\tau_n)^*(a \wedge a) - (\tau_m)_*(a \underline{\triangle} a) + a \underline{\triangle} a - \partial P(H(a) \wedge TH(a))$$

where $(\tau_n)^*(a \wedge a) = (\tau_m)_*(a \underline{\triangle} a)$ and $a \underline{\triangle} a - a \wedge a = \partial P(H(a) \wedge TH(a))$. This shows $\partial Sq_1(\alpha) = 0$. Using the definition of secondary homotopy groups as track functors in [BM05a] and the symmetric actions in [BM05b] we see that $Sq_1(\alpha)$ coincides with the track definition of $\alpha \cup_1 \alpha$. $\square$

Theorem 10.4. *Let n and m be even positive integers and $\alpha, \beta \in \pi_n S^m$. Then*

$$Sq_1(\alpha|\beta) = Sq_1(\alpha+\beta) - Sq_1(\alpha) - Sq_1(\beta) = \left(\frac{n-m}{2}+1\right)(\alpha\wedge\beta)(\Sigma^{2n-3}\eta).$$

This result is stated in [BJM83], but a proof did not appear in the literature.

Proof of Theorem 10.4. We choose representatives $a, b \in \Pi_{n,0}S^m$ of α, β with

$$H(a) = 0 = H(b).$$

Then we have

$$a \wedge b = a\underline{\triangle} b$$

and we get

$$\begin{array}{rcl} (a+b)\wedge(a+b) &=& a\wedge(a+b) + b\wedge(a+b) \\ &=& a\wedge a + a\wedge b + b\wedge a + b\wedge b \\ &=& x+u+y, \\ (a+b)\underline{\triangle}(a+b) &=& (a+b)\underline{\triangle} a + (a+b)\underline{\triangle} b \\ &=& a\wedge a + b\wedge a + a\wedge b + b\wedge b \\ &=& x+v+y. \end{array}$$

Here we set $x = a\wedge a$, $y = b\wedge b$ and

$$\begin{array}{rcl} u &=& a\wedge b + b\wedge a, \\ v &=& b\wedge a + a\wedge b, \end{array}$$

so that

$$u = v + \partial P(a\wedge b|b\wedge a)_H.$$

Now the formula for $Sq_1(\alpha)$ yields:

$$\begin{array}{rcl} Sq_1(\alpha) &=& -\langle a\wedge a, \hat{\tau}_n\rangle + (\hat{\tau}_m)_*(a\wedge a), \\ Sq_1(\beta) &=& -\langle b\wedge b, \hat{\tau}_n\rangle + (\hat{\tau}_m)_*(b\wedge b). \end{array}$$

Moreover for $\gamma = \alpha+\beta$ represented by $c = a+b$ we have

$$Sq_1(\gamma) = -\langle c\wedge c, \hat{\tau}_n\rangle + (\hat{\tau}_m)_*(c\underline{\triangle} c) - P(H(c)\wedge TH(c)).$$

The summands of $Sq_1(\gamma)$ satisfy the formulas:

$$\begin{array}{rcl} \langle c\wedge c, \hat{\tau}_n\rangle &=& \langle x+u+y, \hat{\tau}_n\rangle \\ &=& \langle x, \hat{\tau}_n\rangle^{\tau_n^*(u+y)} + \langle u, \hat{\tau}_n\rangle^{\tau_n^*(y)} + \langle y, \hat{\tau}_n\rangle, \\ (\hat{\tau}_m)_*(c\underline{\triangle} c) &=& (\hat{\tau}_m)_*(x+v+y) \\ &=& (\hat{\tau}_m)_*(x)^{(\tau_m)_*(v+y)} \\ && +(\hat{\tau}_m)_*(v)^{(\tau_m)_*(y)} + (\hat{\tau}_m)_*(y). \end{array}$$

Here we have

$$\begin{array}{rcl} (\tau_n)^*(a\wedge b) &=& (\tau_m)_*(b\underline{\triangle} a) \\ &=& (\tau_m)_*(b\wedge a), \\ (\tau_n)^*(u) &=& (\tau_m)_*(v). \end{array}$$

Hence we get

$$\begin{aligned} Sq_1(\gamma) &= -\langle y, \hat{\tau}_n\rangle - \langle u, \hat{\tau}_n\rangle^{\tau_n^*(y)} \\ &\quad + (-\langle x, \hat{\tau}_n\rangle + (\hat{\tau}_m)_*(x))^{(\tau_m)_*(v+y)} \\ &\quad +(\hat{\tau}_m)_*(v)^{(\tau_m)_*(y)} + (\hat{\tau}_m)_*(y) - P(H(c)\wedge TH(c)). \end{aligned}$$

Since the action on $\operatorname{Ker}\partial$ is trivial and since the image of P and $\operatorname{Ker}\partial$ are both central we thus get

$$\begin{aligned} Sq_1(\gamma) - Sq_1(\beta) &= -\langle u, \hat{\tau}_n\rangle^{\tau_n^*(y)} + Sq_1(\alpha) + (\hat{\tau}_m)_*(v)^{(\tau_m)_*(y)} \\ &\quad -P(H(c)\wedge TH(c)). \end{aligned}$$

Therefore we have

$$\begin{aligned} Sq_1(\alpha|\beta) &= -\langle u, \hat{\tau}_n\rangle^{\tau_n^*(y)} + (\hat{\tau}_m)_*(v)^{(\tau_m)_*(y)} - P(H(c)\wedge TH(c)) \\ &= (-\langle u, \hat{\tau}_n\rangle + (\hat{\tau}_m)_*(v))^{(\tau_m)_*(y)} + P(a\wedge b|b\wedge a)_H \end{aligned}$$

since $\partial(-\langle u, \hat{\tau}_n\rangle + (\hat{\tau}_m)_*(v))$ is a commutator, and hence in the image of ∂P. Here we have

$$\begin{aligned} \langle u, \hat{\tau}_n\rangle &= \langle a\wedge b, \hat{\tau}_n\rangle + \langle b\wedge a, \hat{\tau}_n\rangle + P(-\tau_n^*(a\wedge b) + a\wedge b|\tau_n^*(b\wedge a))_H, \\ (\hat{\tau}_m)_*(v) &= (\hat{\tau}_m)_*(b\wedge a) + (\hat{\tau}_m)_*(a\wedge b) \\ &\quad +P(-(\tau_m)_*(b\wedge a) + b\wedge a|(\tau_m)_*(a\wedge b))_H. \end{aligned}$$

Hence we obtain

$$\begin{aligned} Sq_1(a|b) &= -\langle b\wedge a, \hat{\tau}_n\rangle - \langle a\wedge b, \hat{\tau}_n\rangle + (\hat{\tau}_m)_*(b\wedge a) + (\hat{\tau}_m)_*(a\wedge b) + (\mathrm{c}), \\ (\mathrm{c}) &= P(b\wedge a - a\wedge b|(\hat{\tau}_m)_*(a\wedge b)) + P(a\wedge b|b\wedge a)_H. \end{aligned}$$

Now we consider the following formulas with $\partial(\mathrm{a}) = 0 = \partial(\mathrm{b})$.

$$\begin{aligned} (\mathrm{a}) &= \langle a\wedge b, \hat{\tau}_n\hat{\tau}_n\rangle \\ &= \langle \tau_n^*(a\wedge b), \hat{\tau}_n\rangle + \langle a\wedge b, \hat{\tau}_n\rangle, \\ (\mathrm{b}) &= (\hat{\tau}_m\hat{\tau}_m)_*(a\wedge b) \\ &= (\hat{\tau}_m)_*((\tau_m)_*(a\wedge b)) + (\hat{\tau}_m)_*(a\wedge b). \end{aligned}$$

For (b) we use that $\Pi_{2n,*}$ is a track functor and

$$\begin{aligned} \hat{\tau}_m\hat{\tau}_m &= \hat{\tau}_m\Box(\tau_m\hat{\tau}_m) \\ &= \hat{\tau}_m\Box(\hat{\tau}_m\tau_m). \end{aligned}$$

Now we have the following equations.

$$\begin{aligned} \langle a\wedge b, \hat{\tau}_n\rangle + \langle b\wedge a, \hat{\tau}_n\rangle &= (\mathrm{a}) - \langle \tau_n^*(a\wedge b), \hat{\tau}_n\rangle + \langle b\wedge a, \hat{\tau}_n\rangle \\ &= (\mathrm{a}) + \langle -(\tau_m)_*(b\wedge a) + b\wedge a, \hat{\tau}_n\rangle - (\mathrm{d}) \\ &= (\mathrm{a}) + \langle \partial(\hat{\tau}_m)_*(b\wedge a), \hat{\tau}_n\rangle - (\mathrm{d}) \\ &= (\mathrm{a}) - \tau_n^*(\hat{\tau}_m)_*(b\wedge a) + (\hat{\tau}_m)_*(b\wedge a) - (\mathrm{d}) \\ &= (\mathrm{a}) - (\hat{\tau}_m)_*\tau_n^*(b\wedge a) + (\hat{\tau}_m)_*(b\wedge a) - (\mathrm{d}) \\ &= (\mathrm{a}) - (\hat{\tau}_m)_*(\tau_m)_*(a\wedge b) + (\hat{\tau}_m)_*(b\wedge a) - (\mathrm{d}). \end{aligned}$$

Here (d) is given by the following formula.

$$\begin{aligned}
\text{(d)} &= -P(\tau_n^*(\tau_m)_*(b\wedge a)-(\tau_m)_*(b\wedge a))|\tau_n^*(\tau_m)_*(b\wedge a))_H \\
&\quad +P(\tau_n^*(\tau_m)_*(b\wedge a)-(\tau_m)_*(b\wedge a))|\tau_n^*(b\wedge a))_H \\
&= -P(a\wedge b-(\tau_m)_*(b\wedge a)|a\wedge b)_H \\
&\quad +P(a\wedge b-(\tau_m)_*(b\wedge a)|\tau_n^*(b\wedge a))_H \\
&= -P(a\wedge b|a\wedge b)_H+P((\tau_m)_*(b\wedge a)|a\wedge b)_H \\
&\quad +P(a\wedge b|(\tau_m)_*(b\wedge a))_H-P((\tau_m)_*(b\wedge a)|(\tau_m)_*(a\wedge b))_H.
\end{aligned}$$

On the other hand we get

$$\begin{aligned}
(\hat{\tau}_m)_*(b\wedge a)+(\hat{\tau}_m)_*(a\wedge b) &= (\hat{\tau}_m)_*(b\wedge a) \\
&\quad -(\hat{\tau}_m)_*((\tau_m)_*(a\wedge b))+\text{(b)}.
\end{aligned}$$

Now we get

$$\begin{aligned}
Sq_1(\alpha|\beta) &= -\text{(a)}-\text{(d)}-(\hat{\tau}_m)_*(b\wedge a)+(\hat{\tau}_m)_*((\tau_m)_*(a\wedge b)) \\
&\quad +(\hat{\tau}_m)_*(b\wedge a)-(\hat{\tau}_m)_*((\tau_m)_*(a\wedge b))+\text{(b)}+\text{(c)} \\
&= -\text{(a)}+\text{(d)}+\text{(e)}+\text{(b)}+\text{(c)}
\end{aligned}$$

where (e) is the commutator:

$$\begin{aligned}
\text{(e)} &= P((\hat{\tau}_m)_*(b\wedge a)|-(\hat{\tau}_m)_*((\tau_m)_*(a\wedge b))_{H\partial} \\
&= P(-(\tau_m)_*(b\wedge a)+b\wedge a|(\tau_m\tau_m)_*(a\wedge b) \\
&\quad -(\tau_m)_*(a\wedge b))_H \\
&= -P((\tau_m)_*(b\wedge a)|a\wedge b)_H+P(b\wedge a|a\wedge b)_H \\
&\quad +P((\tau_m)_*(b\wedge a)|(\tau_m)_*(a\wedge b))_H \\
&\quad -P(b\wedge a|(\tau_m)_*(a\wedge b))_H.
\end{aligned}$$

$$\begin{aligned}
\text{(d)}+\text{(e)}+\text{(c)} &= -P(a\wedge b|a\wedge b)_H+P((\tau_m)_*(b\wedge a)|a\wedge b)_H \\
&\quad +P(a\wedge b|(\tau_m)_*(a\wedge b))_H \\
&\quad -P((\tau_m)_*(b\wedge a)|(\tau_m)_*(a\wedge b))_H \\
&\quad -P((\tau_m)_*(b\wedge a)|a\wedge b)_H+P(b\wedge a|a\wedge b)_H \\
&\quad +P((\tau_m)_*(b\wedge a)|(\tau_m)_*(a\wedge b))_H \\
&\quad -P(b\wedge a|(\tau_m)_*(a\wedge b))_H \\
&\quad +P(b\wedge a|(\tau_m)_*(a\wedge b))_H-P(a\wedge b|(\tau_m)_*(a\wedge b))_H \\
&\quad +P(a\wedge b|b\wedge a)_H \\
&= -P(a\wedge b|a\wedge b)_H.
\end{aligned}$$

Hence we get

$$Sq_1(\alpha|\beta)=-\text{(a)}+\text{(b)}-P(a\wedge b|a\wedge b)_H,$$

and this implies the result by use of Proposition 6.12. □

11. Toda brackets

For a pointed space X we use the suspension $\Sigma X = S^1 \wedge X$ and the *E-suspension* $EX = X \wedge S^1$. Here Σ and E are isomorphic endofunctors of $\mathbf{Top}^*$. The E-suspension is for example used by Toda in his book [Tod62].

Definition 11.1. Let $n \geq k \geq 0$ and consider morphisms in $\mathbf{Top}^*/\simeq$

$$Z \overset{\alpha}{\leftarrow} E^kY \overset{E^k\beta}{\leftarrow} E^kX,\ \ Y \overset{\beta}{\leftarrow} X \overset{\gamma}{\leftarrow} S^{n-k}$$

with $\alpha(E^k\beta) = 0$ and $\beta\gamma = 0$. Then the *Toda bracket*

$$\left\{\alpha, E^k\beta, E^k\gamma\right\}_k \subset \pi_{n+1}Z$$

is the subset of all elements in $\pi_{n+1}Z$ obtained by pasting tracks as in the diagram

$$\begin{tikzcd}
Z & E^kY \arrow[l, "a"'] \arrow[d, phantom, ""] & E^kX \arrow[l, "E^kb"'] \arrow[ll, bend right=60, "0"'] \arrow[ll, phantom, bend right=30, "\Downarrow A"] & S^n \arrow[l, "E^kc"'] \arrow[ll, bend left=60, "0"] \arrow[ll, phantom, bend left=30, "\Downarrow E^kB"]
\end{tikzcd}$$

where a, b, c represent α, β, γ and $B\colon bc \Rightarrow 0$ and $A\colon 0 \Rightarrow a(E^kb)$.

Let $\imath_k \in \pi_k(S^k)$ be the element represented by the identity of S^k, $k \geq 0$. Moreover, let

$$\imath_{k,0} \in \Omega^k S^k \subset \Pi_{k,0}(S^k) = \langle \Omega^k S^k \rangle_{nil}$$

be given by the identity of S^k. This element yields a quadratic pair module morphism

$$\imath_{k,0}\colon \overline{\mathbb{Z}}_{nil} \longrightarrow \Pi_{k,*}S^k.$$

We define the morphisms $\bar{E}^k$ in $\mathbf{qpm}$, $n \geq k$,

$$\bar{E}^k\colon \Pi_{n-k,*}X \cong \Pi_{n-k,*}X \odot \overline{\mathbb{Z}}_{nil} \overset{1\otimes\imath_{k,0}}{\to} \Pi_{n-k,*}X \odot \Pi_{k,*}S^k \overset{\wedge}{\to} \Pi_{n,*}(E^kX). \tag{11.2}$$

Let α, β, γ be given as in Definition 11.1 with $\alpha(E^k\beta) = 0$ and $\beta\gamma = 0$. We choose maps a, b representing α, β and we choose a track $A\colon 0 \Rightarrow a(E^kb)$ as in Definition 11.1. Moreover, let

$$\bar{c} \in \Pi_{n-k,0}X$$

be an element representing $\gamma \in \pi_{n-k}X$ with $n - k \geq 2$ and let

$$\bar{B} \in \Pi_{n-k,1}Y \tag{11.3}$$

be an element with $\partial\bar{B} = b_*(\bar{c}) \in \Pi_{n-k,0}Y$. Then $\bar{E}^k\bar{B} \in \Pi_{n,1}E^kY$ satisfies

$$\partial\bar{E}^k\bar{B} = \partial(\bar{B} \wedge \imath_{k,0}) = (\partial\bar{B}) \wedge \imath_{k,0} = b_*(\bar{c}) \wedge \imath_{k,0} = (E^kb)_*(\bar{c} \wedge \imath_{k,0}).$$

Moreover, the track A induces a track in $\mathbf{qpm}$ which is given by a map

$$A_*\colon \Pi_{n,0}E^kX \longrightarrow \Pi_{n,1}Z$$

with $\partial A_*(x) = (a(E^kb))_*(x)$. Therefore the element

$$t = a_*(E^k\bar{B}) - A_*(\bar{c} \wedge \imath_{k,0}) \in \Pi_{n,1}Z$$

satisfies $\partial(t)=0$ and hence t is an element in $h_1\Pi_{n,*}Z$. Recall from (6.11) that $h_1\Pi_{n,*}Z$ is naturally isomorphic to $\pi_{n+1}Z$ for $n\geq 3$ and to $\pi_3Z/[\pi_2Z,\pi_2Z]$ for $n=2$, where $[-,-]$ is the Whitehead product.

Lemma 11.4. *For $n\geq 3$, $a_*\bar{E}^k(\bar{B})-A_*(\bar{c}\wedge\imath_{k,0})\in\left\{\alpha,E^k\beta,E^k\gamma\right\}_k$. Moreover, all elements in $\iota\left\{\alpha,E^k\beta,E^k\gamma\right\}_k$ can be obtained in this way. The same equality holds for $n=2 \mod [\pi_2Z,\pi_2Z]$, the image of the Whitehead product.*

Proposition 11.5. *Let Y be a pointed space and let $r\in\mathbb{Z}$ and $\beta\in\pi_{n-1}Y$ with $r\beta=0$. In the group of homotopy classes $[EY,EY]$ let $r1_{EY}$ be the r-fold sum of the identity 1_{EY}. Then the Toda bracket $\{r1_{EY},E\beta,r\imath_n\}_1\subset\pi_{n+1}EY$ is defined and for $n\geq 3$*

$$\{r1_{EY},E\beta,r\imath_n\}_1\ni\begin{cases}0, & \text{if } r \text{ is odd,}\\ \frac{r}{2}(E\beta)(\Sigma^{n-2}\eta) & \text{if } r \text{ is even.}\end{cases}$$

Here $\eta\colon S^3\to S^2$ is the Hopf map. For $n=2$ the same formula holds in the quotient $\pi_3EY/[\pi_2EY,\pi_2EY]$ where $[-,-]$ is the Whitehead product.

In [Tod62] 3.7 Toda proves this result in case Y is a sphere. Toda's proof uses different methods relying on the assumption that Y is a sphere.

Proof of 11.5. Let $(\cdot)^r\colon S^1\to S^1$ be the degree r map $z\mapsto z^r$ so that $r1_{EY}$ is represented by $a=Y\wedge(\cdot)^r$. Let b be a map representing β. We choose $\bar{c}\in\Pi_{n-1,0}S^{n-1}$ representing $r\imath_n$ by $\bar{c}=r\imath_{n-1,0}$ and we choose

$$\bar{B}\in\Pi_{n-1,1}Y$$

with $\partial\bar{B}=b_*(\bar{c})=r\,b_*(\imath_{n-1,0})$. Then we get as in (11.3)

$$\partial\bar{E}(\bar{B})=(Eb)_*(\bar{c}\wedge\imath_{1,0})$$

where $\bar{c}\wedge\imath_{1,0}=r(\imath_{n-1,0}\wedge\imath_{1,0})=r\imath_{n,0}$. Now we choose $A\colon 0\Rightarrow a(Eb)$ in such a way that the induced track A_* satisfies

$$A_*(\imath_{n,0})=\bar{E}(\bar{B}).$$

In fact, the boundary of $A_*(\imath_{n,0})$ is

$$\partial A_*(\imath_{n,0})=(a(Eb))_*(\imath_{n,0})=\partial\bar{E}(\bar{B})$$

where $a(Eb)=(Y\wedge(\cdot)^r)(b\wedge S^1)=b\wedge(\cdot)^r=(Eb)(\Sigma^{n-1}(\cdot)^r)$ and $(\Sigma^{n-1}(\cdot)^r)_*(\imath_{n,0})=r^*\imath_{n,0}=r\imath_{n,0}$. since $H(i_{n,0})=0$.

Now we can compute the element in Lemma 11.4

$$t=a_*\bar{E}(\bar{B})-A_*(\bar{c}\wedge\imath_{1,0})\in\{r1_{EY},E\beta,r\imath_n\}_1.$$

Here A is a track $f_0\Rightarrow g_0$ with $f_0=0$ so that

$$\begin{aligned}A_*(\bar{c}\wedge\imath_{1,0}) &= A_*(r\imath_{n,0})\\ &= r\,A_*(\imath_{n,0})\\ &= r\bar{E}\bar{B}.\end{aligned}$$

On the other hand we get

$$\begin{aligned} a_*(\bar{E}\bar{B}) &= (Y \wedge (\cdot)^r)_*(\bar{B} \wedge \imath_{1,0}) \\ &= \bar{B} \wedge (((\cdot)^r)_*(\imath_{1,0})) \\ &= \bar{B} \wedge (r^* \imath_{1,0}) \\ &= \bar{B} \wedge (r\imath_{1,0}) \\ &= r(\bar{B} \wedge \imath_{1,0}) + t \\ &= r\bar{E}(\bar{B}) + t \end{aligned}$$

where

$$\begin{aligned} t &= \binom{r}{2} P(H\partial(\bar{B}) \wedge (\imath_{1,0}|\imath_{1,0})_H) \\ &= \binom{r}{2} P(H(r\, b_*(\imath_{n-1,0})) \wedge (\imath_{1,0}|\imath_{1,0})_H) \\ &= \binom{r}{2}^2 P((b_*(\imath_{n-1,0})|b_*(\imath_{n-1,0}))_H \wedge (\imath_{1,0}|\imath_{1,0})_H) \\ &= \binom{r}{2}^2 P(b_*(\imath_{n-1,0}) \wedge \imath_{1,0}|b_*(\imath_{n-1,0}) \wedge \imath_{1,0})_H. \end{aligned}$$

Therefore t represents $\binom{r}{2}^2(E\beta)(\Sigma^{n-2}\eta)$. If r is odd we see that $t = 0$ since β has odd order and $\Sigma\eta$ has even order. □

Part 3. The construction of the smash product for secondary homotopy groups

In this part we define the smash product operator for secondary homotopy groups and we prove the results described in Part 2. A crucial step for this definition will be the construction of canonical tracks

$$\Sigma(f\underline{\#}g) \overset{\mathbb{H}^{\underline{\#}}}{\Rightarrow} f \wedge g \overset{\mathbb{H}^{\#}}{\Leftarrow} \Sigma(f\#g)$$

termed the *exterior tracks*, connecting the exterior cup-products and the smash product of two maps. Then we use $\mathbb{H}^{\#}$ and $\mathbb{H}^{\underline{\#}}$ for the definition of the smash product operator on the (1)-level. Some of the algebraic properties of the smash product are then derived from formulas concerning the Hopf invariant of the track $(\mathbb{H}^{\underline{\#}})^{\boxminus}\square\mathbb{H}^{\#}$ and of some other tracks between suspensions built out of the exterior tracks.

12. Exterior cup-products for higher suspensions and tracks

We begin this section by stating the basic properties of the exterior cup-product operations.

Lemma 12.1. *We have the following formulas for suspensions*

(1) $f\#(\Sigma g') = f \wedge g' = f\underline{\#}(\Sigma g')$,
(2) $(\Sigma f')\#g = (1\ 2)(f' \wedge g)(1\ 2) = (\Sigma f')\underline{\#}g$,

coproducts

(3) $(f_1, f_2)\#g = (f_1\#g, f_2\#g)$,
(4) $(f_1, f_2)\underline{\#}g = (f_1\underline{\#}g, f_2\underline{\#}g)$,
(5) $f\#(g_1, g_2) = (f\#g_1, f\#g_2)$,

(6) $f\underline{\#}(g_1,g_2)=(f\underline{\#}g_1,f\underline{\#}g_2)$,

and compositions

(7) $(f_1f_2)\#(g_1(\Sigma g_2'))=(f_1\#g_1)(f_2\#(\Sigma g_2'))$,
(8) $((\Sigma f_1')f_2)\#(g_1g_2)=((\Sigma f_1')\#g_1)(f_2\#g_2)$,
(9) $(f_1(\Sigma f_2'))\underline{\#}(g_1g_2)=(f_1\underline{\#}g_1)((\Sigma f_2')\underline{\#}g_2)$,
(10) $(f_1f_2)\underline{\#}((\Sigma g_1')g_2)=(f_1\underline{\#}(\Sigma g_1'))(f_2\underline{\#}g_2)$.

The exterior cup-products are associative

(11) $f\#(g\#h)=(f\#g)\#h$,
(12) $f\underline{\#}(g\underline{\#}h)=(f\underline{\#}g)\underline{\#}h$.

The proof of this lemma is straightforward.

In order to define the exterior cup-products

$$f\#g,f\underline{\#}g\colon \Sigma^{n+m-1}A\wedge Y\to\Sigma^{n+m-1}B\wedge Y$$

of maps between higher suspensions $f\colon\Sigma^nA\to\Sigma^nB$, $g\colon\Sigma^mX\to\Sigma^mY$ we take the first spherical coordinates to the end of the smash product

$$\bar f\colon S^1\wedge A\wedge S^{n-1}\cong S^{n-1}\wedge S^1\wedge A\xrightarrow{f}S^{n-1}\wedge S^1\wedge B\cong S^1\wedge B\wedge S^{n-1},$$

$$\bar g\colon S^1\wedge X\wedge S^{m-1}\cong S^{m-1}\wedge S^1\wedge X\xrightarrow{g}S^{m-1}\wedge S^1\wedge Y\cong S^1\wedge Y\wedge S^{m-1},$$

then we perform the usual exterior cup product on these maps, and we recollect the permuted spherical coordinates at the beginning of the smash product in an ordered way,

$$\begin{array}{rcl} f\#g\colon S^{n-1}\wedge S^{m-1}\wedge S^1\wedge A\wedge X & \cong & S^1\wedge A\wedge S^{n-1}\wedge X\wedge S^{m-1}\\ & \xrightarrow{\bar f\#\bar g} & S^1\wedge B\wedge S^{n-1}\wedge Y\wedge S^{m-1}\\ & \cong & S^{n-1}\wedge S^{m-1}\wedge S^1\wedge B\wedge Y, \end{array}$$

and the same for $\underline{\#}$. These exterior cup-products generalize the classical ones in the following sense. If $f=\Sigma^{n-1}f'$ and $g=\Sigma^{m-1}g'$ for $f\colon\Sigma A\to\Sigma B$ and $g\colon\Sigma X\to\Sigma Y$ then

$$\begin{array}{rcl} (\Sigma^{n-1}f')\#(\Sigma^{m-1}g') & = & \Sigma^{n+m-1}(f'\#g'),\\ (\Sigma^{n-1}f')\underline{\#}(\Sigma^{m-1}g') & = & \Sigma^{n+m-1}(f'\underline{\#}g'). \end{array}$$

The properties of the classical exterior cup-products in Lemma 12.1 can be accordingly restated for the exterior cup-product of maps between higher suspensions.

Let $F\colon f\Rightarrow g$, $G\colon h\Rightarrow k$ be now tracks between maps $f,g\colon\Sigma^nA\to\Sigma^nB$, $h,k\colon\Sigma^mX\to\Sigma^mY$. The exterior products of a track with a map

$$\begin{array}{c} F\#h\colon f\#h\Rightarrow g\#h,\\ F\underline{\#}h\colon f\underline{\#}h\Rightarrow g\underline{\#}h,\\ f\#G\colon f\#h\Rightarrow f\#k,\\ f\underline{\#}G\colon f\underline{\#}h\Rightarrow f\underline{\#}k, \end{array}$$

are defined by exchanging the interval I_+ with the spherical coordinates and using the exterior cup-products of maps between higher suspensions as defined above. For example the track $F\#h$ is represented by the homotopy

$$I_+\wedge S^{n+m-1}\wedge A\wedge X\cong S^{n+m-1}\wedge I_+\wedge A\wedge X\xrightarrow{\tilde F\#h}S^{n+m-1}\wedge B\wedge Y,$$

where

$$\tilde F\colon S^{n-1}\wedge I_+\wedge A\cong I_+\wedge S^{n-1}\wedge A\xrightarrow{F}S^{n-1}\wedge B$$

is defined from a homotopy F representing the corresponding track. Now one can define the exterior products of two tracks as the vertical composition

$$\begin{aligned} F\#G &= (g\#G)\square(F\#h) = (F\#k)\square(f\#G), \\ F\underline{\#}G &= (g\underline{\#}G)\square(F\underline{\#}h) = (F\underline{\#}k)\square(f\underline{\#}G). \end{aligned}$$

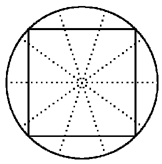
One can also derive from Lemma 12.1 analogous properties for the exterior cup-products of tracks.

13. The exterior tracks

For any two maps $f\colon \Sigma A \to \Sigma B$ and $g\colon \Sigma X \to \Sigma Y$ the suspended exterior cup-products $\Sigma(f\#g)$ and $\Sigma(f\underline{\#}g)$ are naturally homotopic to the composite

$$S^1\wedge S^1\wedge A\wedge X \overset{(2\;3)}{\cong} S^1\wedge A\wedge S^1\wedge X \xrightarrow{f\wedge g} S^1\wedge B\wedge S^1\wedge Y \overset{(2\;3)}{\cong} S^1\wedge S^1\wedge B\wedge Y.$$

In order to construct homotopies we only need to choose a track from the transposition map $(1\;2)\colon S^1\wedge S^1 \to S^1\wedge S^1$ to $\nu\wedge S^1$, where $\nu\colon S^1\to S^1$ is the co-H-inversion defined by $\nu(z)=z^{-1}$. Here we use the topological group structure of S^1. The set of all tracks $(1\;2)\Rightarrow \nu\wedge S^1$ and $1_{S^2}\Rightarrow 1_{S^2}$ is a group under horizontal composition. This group is an extension of $\mathbb{Z}/2$ by $\mathbb{Z}$ with the non-trivial action of $\mathbb{Z}/2$, compare [BM05b] 6.12. Up to isomorphism there is only one extension of this kind, the trivial extension, given by the infinite dyhedral group $\mathbb{Z}/2 * \mathbb{Z}/2$, hence this group of tracks is generated by two order 2 tracks $(1\;2)\Rightarrow\nu\wedge S^1$. One of these two generating tracks $\mathbb{H}\colon (1\;2)\Rightarrow \nu\wedge S^1$ can be constructed as follows. Since $\nu\wedge S^1$ is a homotopy equivalence it is enough to indicate which track is $\mathbb{H}^{\boxminus}(\nu\wedge S^1)\colon 1_{S^2}\Rightarrow (1\;2)(\nu\wedge S^1)$. The 2-sphere $S^2 = S^1\wedge S^1$ is a quotient of the square $[-1,1]^2$ by the map $[-1,1]^2\to S^1\wedge S^1\colon (x,y)\mapsto (\exp \pi i(1+x), \exp\pi i(1+y))$. The map $(1\;2)(\nu\wedge S^1)$ is induced by the 90° twist (counterclockwise) in the square, so we obtain $\mathbb{H}^{\boxminus}(\nu\wedge S^1)$ by using the homeomorphism from the square to the radius $\sqrt{2}$ circle projecting from the origin



and twisting continuously the circle 90° counterclockwise.

Definition 13.1. The *exterior tracks*

$$\mathbb{H}^{\#}_{f,g}\colon \Sigma(f\#g)\Rightarrow (2\;3)(f\wedge g)(2\;3),$$

$$\mathbb{H}^{\underline{\#}}_{f,g}\colon \Sigma(f\underline{\#}g)\Rightarrow (2\;3)(f\wedge g)(2\;3),$$

are given by the following diagram in $\mathbf{Top}^*$ where the **2**-cells without a track arrow $\Rightarrow$ are strictly commutative

$$
\begin{tikzcd}
S^1\wedge S^1\wedge A\wedge X \arrow[rrrrrrr, "\Sigma(f\#g)"] \arrow[d, equal] & & & & & & & S^1\wedge S^1\wedge B\wedge Y \arrow[d, equal] \\
S^1\wedge S^1\wedge A\wedge X \arrow[r, bend left, "1"] \arrow[r, bend right, "((\nu\wedge S^1)(1\ 2))\wedge A\wedge X"'] \arrow[r, phantom, "\Downarrow ((\nu\wedge S^1)\mathbb{H}^{\boxminus})\wedge A\wedge X"] \arrow[d, equal] & S^1\wedge S^1\wedge A\wedge X \arrow[r, "S^1\wedge f\wedge X"] & S^1\wedge S^1\wedge B\wedge X \arrow[r, bend left, "1"] \arrow[r, bend right, "((1\ 2)(\nu\wedge S^1))\wedge B\wedge X"'] \arrow[r, phantom, "\Downarrow (\mathbb{H}^{\boxminus}(\nu\wedge S^1))\wedge B\wedge X"] & S^1\wedge S^1\wedge B\wedge X \arrow[r, "(2\ 3)"] & S^1\wedge B\wedge S^1\wedge X \arrow[r, "S^1\wedge B\wedge g"] & S^1\wedge B\wedge S^1\wedge Y \arrow[rr, "(2\ 3)"] & & S^1\wedge S^1\wedge B\wedge Y \arrow[d, equal] \\
S^1\wedge S^1\wedge A\wedge X \arrow[r, "(2\ 3)"] \arrow[d, equal] & S^1\wedge A\wedge S^1\wedge X \arrow[rrrr, "f\wedge g"] & & & & S^1\wedge B\wedge S^1\wedge Y \arrow[rr, "(2\ 3)"] & & S^1\wedge S^1\wedge B\wedge Y \arrow[d, equal] \\
S^1\wedge S^1\wedge A\wedge X \arrow[r, "(2\ 3)"] \arrow[d, equal] & S^1\wedge A\wedge S^1\wedge X \arrow[r, "S^1\wedge A\wedge g"] & S^1\wedge A\wedge S^1\wedge Y \arrow[r, "(2\ 3)"] & S^1\wedge S^1\wedge A\wedge Y \arrow[r, bend left, "((\nu\wedge S^1)(1\ 2))\wedge A\wedge Y"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Uparrow ((\nu\wedge S^1)\mathbb{H}^{\boxminus})\wedge A\wedge Y"] & S^1\wedge S^1\wedge A\wedge Y \arrow[r, "S^1\wedge f\wedge Y"] & S^1\wedge S^1\wedge B\wedge Y \arrow[rr, bend left, "((1\ 2)(\nu\wedge S^1))\wedge B\wedge Y"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\Uparrow (\mathbb{H}^{\boxminus}(\nu\wedge S^1))\wedge B\wedge Y"] & & S^1\wedge S^1\wedge B\wedge Y \arrow[d, equal] \\
S^1\wedge S^1\wedge A\wedge X \arrow[rrrrrrr, "\Sigma(f\underline{\#}g)"] & & & & & & & S^1\wedge S^1\wedge B\wedge Y
\end{tikzcd}
$$

i. e. the tracks $\mathbb{H}^{\#}_{f,g}$ and $\mathbb{H}^{\underline{\#}}_{f,g}$ are the following composite tracks.

$$\begin{aligned}
\mathbb{H}^{\#}_{f,g} &= (2\ 3)(S^1\wedge B\wedge g)(2\ 3)((\mathbb{H}^{\boxminus}(\nu\wedge S^1))\wedge B\wedge X)(S^1\wedge f\wedge X)(((\nu\wedge S^1)\mathbb{H}^{\boxminus})\wedge A\wedge X)\\
&= (2\ 3)(S^1\wedge B\wedge g)(2\ 3)(\mathbb{H}^{\boxminus}\wedge B\wedge X)(S^1\wedge f\wedge X)(\mathbb{H}^{\boxminus}\wedge A\wedge X),\\
\mathbb{H}^{\underline{\#}}_{f,g} &= ((\mathbb{H}^{\boxminus}(\nu\wedge S^1))\wedge B\wedge Y)(S^1\wedge f\wedge Y)(((\nu\wedge S^1)\mathbb{H}^{\boxminus})\wedge A\wedge Y)(2\ 3)(S^1\wedge A\wedge g)(2\ 3)\\
&= (\mathbb{H}^{\boxminus}\wedge B\wedge Y)(S^1\wedge f\wedge Y)(\mathbb{H}^{\boxminus}\wedge A\wedge Y)(2\ 3)(S^1\wedge A\wedge g)(2\ 3).
\end{aligned}$$

In the next proposition we show elementary properties of the exterior tracks that are relevant for the definition of the smash product operation on secondary homotopy groups. They are analogous to the properties of exterior cup-products in Lemma 12.1.

Lemma 13.2. *The exterior tracks satisfy the following formulas for suspensions*

(1) $\mathbb{H}^{\#}_{\Sigma f',g} = 0^{\square}_{\Sigma((1\ 2)(f'\wedge g)(1\ 2))} = \mathbb{H}^{\underline{\#}}_{\Sigma f',g}$,
(2) $\mathbb{H}^{\#}_{f,\Sigma g'} = (\mathbb{H}^{\boxminus}\wedge B\wedge Y)(S^1\wedge f\wedge g')(\mathbb{H}^{\boxminus}\wedge A\wedge X) = \mathbb{H}^{\underline{\#}}_{f,\Sigma g'}$,

coproducts

(3) $\mathbb{H}^{\#}_{(f_1,f_2),g} = (\mathbb{H}^{\#}_{f_1,g}, \mathbb{H}^{\#}_{f_2,g})$,
(4) $\mathbb{H}^{\underline{\#}}_{(f_1,f_2),g} = (\mathbb{H}^{\underline{\#}}_{f_1,g}, \mathbb{H}^{\underline{\#}}_{f_2,g})$,
(5) $\mathbb{H}^{\#}_{f,(g_1,g_2)} = (\mathbb{H}^{\#}_{f,g_1}, \mathbb{H}^{\#}_{f,g_2})$,
(6) $\mathbb{H}^{\underline{\#}}_{f,(g_1,g_2)} = (\mathbb{H}^{\underline{\#}}_{f,g_1}, \mathbb{H}^{\underline{\#}}_{f,g_2})$,

and composition of maps

(7) $\mathbb{H}^{\#}_{f_1f_2,g_1(\Sigma g_2')} = \mathbb{H}^{\#}_{f_1,g_1}\mathbb{H}^{\#}_{f_2,\Sigma g_2'}$,
(8) $\mathbb{H}^{\underline{\#}}_{(\Sigma f_1')f_2,g_1g_2} = \mathbb{H}^{\underline{\#}}_{\Sigma f_1',g_1}\mathbb{H}^{\underline{\#}}_{f_2,g_2}$,
(9) $\mathbb{H}^{\underline{\#}}_{f_1(\Sigma f_2'),g_1g_2} = \mathbb{H}^{\underline{\#}}_{f_1,g_1}\mathbb{H}^{\underline{\#}}_{\Sigma f_2',g_2}$,
(10) $\mathbb{H}^{\underline{\#}}_{f_1f_2,(\Sigma g_1')g_2} = \mathbb{H}^{\underline{\#}}_{f_1,\Sigma g_1'}\mathbb{H}^{\underline{\#}}_{f_2,g_2}$.

They satisfy the following associativity rules.

(11) $(\mathbb{H}^{\#}_{f,g}\wedge h)(\Sigma\mathbb{H}^{\#}_{f\#g,h}) = (f\wedge\mathbb{H}^{\#}_{g,h})(\Sigma\mathbb{H}^{\#}_{f,g\#h})\colon \Sigma^2(f\#g\#h)\Rightarrow f\wedge g\wedge h$,
(12) $(\mathbb{H}^{\underline{\#}}_{f,g}\wedge h)(\Sigma\mathbb{H}^{\underline{\#}}_{f\underline{\#}g,h}) = (f\wedge\mathbb{H}^{\underline{\#}}_{g,h})(\Sigma\mathbb{H}^{\underline{\#}}_{f,g\underline{\#}h})\colon \Sigma^2(f\underline{\#}g\underline{\#}h)\Rightarrow f\wedge g\wedge h$.

These properties follow easily from the definition of exterior tracks above and from the fact that $\mathbb{H}\mathbb{H} = 0^{\square}_{1_{S^2}}$ is the trivial track. In the right hand side of the equalities (11) and(12) there are some permutations involved that we have omitted.

For suspended maps $\Sigma^{n-1}f\colon\Sigma^nA\to\Sigma^nB$, $\Sigma^{m-1}g\colon\Sigma^mX\to\Sigma^mY$ we define the tracks $\mathbb{H}^{\#}_{n,f,m,g}$, $\mathbb{H}^{\underline{\#}}_{n,f,m,g}$ from $\Sigma^{n+m-1}(f\#g)$, $\Sigma^{n+m-1}(f\underline{\#}g)$, respectively, to

(13.3)
$$S^n\wedge S^m\wedge A\wedge X\cong S^n\wedge A\wedge S^m\wedge X\stackrel{\Sigma^{n-1}f\wedge\Sigma^{m-1}g}{\longrightarrow}S^n\wedge B\wedge S^m\wedge Y\cong S^n\wedge S^m\wedge B\wedge Y$$

as

$$\begin{tikzcd}
S^{n-1}\wedge S^1\wedge S^{m-1}\wedge S^1\wedge A\wedge X \arrow[d, "\cong"', "(2\;3)"] \\
S^{n-1}\wedge S^{m-1}\wedge S^1\wedge S^1\wedge A\wedge X \arrow[d, bend right=60, "\Sigma^{n+m-1}(f\#g)"'] \arrow[d, bend left=60, "\Sigma^{n+m-2}((2\;3)(f\wedge g)(2\;3))"] \arrow[d, phantom, "\xRightarrow{\Sigma^{n+m-2}\mathbb{H}^{\#}_{f,g}}"] \\
S^{n-1}\wedge S^{m-1}\wedge S^1\wedge S^1\wedge B\wedge Y \arrow[d, "\cong"', "(2\;3)"] \\
S^{n-1}\wedge S^1\wedge S^{m-1}\wedge S^1\wedge B\wedge Y
\end{tikzcd}$$

and similarly for $\underline{\#}$. Notice that the last spherical coordinate in these smash products is always the same one. This is relevant in connection with Lemma 15.2 below. The tracks $\mathbb{H}^{\#}_{n,f,m,g}$, $\mathbb{H}^{\underline{\#}}_{n,f,m,g}$ satisfy properties analogous to Lemma 13.2 that we do not restate. They also satisfy the following further properties.

Lemma 13.4. *Given tracks $F\colon \Sigma^{n-1}f_1 \Rightarrow \Sigma^{n-1}f_2$ and $G\colon \Sigma^{m-1}g_1 \Rightarrow \Sigma^{m-1}g_2$ between maps $\Sigma^{n-1}f_i\colon \Sigma^n A \to \Sigma^n B$ and $\Sigma^{m-1}g_i\colon \Sigma^m A \to \Sigma^m B$, $i = 1, 2$, the following equalities are satisfied*

(1) $(F\wedge G)\Box\mathbb{H}^{\#}_{n,f_1,m,g_1} = \mathbb{H}^{\#}_{n,f_2,m,g_2}\Box(\Sigma(F\#G))\colon$
$\Sigma^{n+m-1}(f_1\#g_1) \Rightarrow \Sigma^{n+m-2}(f_2\wedge g_2)$,

(2) $(F\wedge G)\Box\mathbb{H}^{\underline{\#}}_{n,f_1,m,g_1} = \mathbb{H}^{\underline{\#}}_{n,f_2,m,g_2}\Box(\Sigma(F\underline{\#}G))\colon$
$\Sigma^{n+m-1}(f_1\underline{\#}g_1) \Rightarrow \Sigma^{n+m-2}(f_2\wedge g_2)$.

Here the track $F\wedge G$ in (1) and (2) needs to be altered by permutations according to (13.3). Moreover $\Sigma(F\#G)$ and $\Sigma(F\underline{\#}G)$ need also to be altered by permutations as follows. The track $\Sigma(F\#G)$ should actually be

$$\begin{tikzcd}
S^{n-1}\wedge S^1\wedge S^{m-1}\wedge S^1\wedge A\wedge X \arrow[d, "\cong"', "(1\;2)"] \\
S^1\wedge S^{n-1}\wedge S^{m-1}\wedge S^1\wedge A\wedge X \arrow[d, bend right=60, "\Sigma^{n+m-1}(f_1\#g_1)"'] \arrow[d, bend left=60, "\Sigma^{n+m-1}(f_2\#g_2)"] \arrow[d, phantom, "\xRightarrow{\Sigma(F\#G)}"] \\
S^1\wedge S^{n-1}\wedge S^{m-1}\wedge S^1\wedge B\wedge Y \arrow[d, "\cong"', "(1\;2)"] \\
S^{n-1}\wedge S^1\wedge S^{m-1}\wedge S^1\wedge B\wedge Y
\end{tikzcd}$$

and similarly $\Sigma(F\underline{\#}G)$. Notice that the last spherical coordinate remains always in the same place in this diagram. This is again relevant in connection with Lemma 15.2 below.

14. The construction of the smash product operation

In this section we define the smash product morphism in **qpm** which appears in the statement of Theorem 7.1. In the last two sections we establish the properties which show that the definition given here is indeed consistent with the definition of the tensor product of quadratic pair modules.

The secondary homotopy groups $\pi_{n,*}X$, $n \geq 0$, of a pointed space X were introduced in [BM05a].

For $n = 0$, $\pi_{0,*}X$ is the fundamental pointed groupoid of X. We denote by $\pi_{0,0}X$ to the pointed set of objects, which can be regarded as the set of pointed maps $S^0 \to X$, and by $\pi_{0,1}X$ to the set of morphisms. Such a morphism $\alpha\colon x \to y$ is a track $\alpha\colon x \Rightarrow y$ between pointed maps $x, y\colon S^0 \to Y$.

For $n = 1$, $\pi_{1,*}X$ is a crossed module

$$\partial\colon \pi_{1,1}X \longrightarrow \pi_{1,0}X = \langle \Omega X\rangle.$$

In particular $\pi_{1,0}X$ acts on the right on $\pi_{1,1}X$.

For $n \geq 2$, $\pi_{n,*}X$ is a reduced quadratic module in the sense of [Bau91]

$$\otimes^2(\pi_{n,0}X)_{ab} \xrightarrow{\omega} \pi_{n,1}X \xrightarrow{\partial} \pi_{n,0}X = \langle \Omega^n X\rangle_{nil}, \tag{14.1}$$

which is stable for $n \geq 3$.

For all $n \geq 1$ the elements of $\pi_{n,1}X$ are equivalence classes $[f, F]$ represented by a map

$$f\colon S^1 \longrightarrow \vee_{\Omega^n X} S^1$$

and a track

$$\begin{tikzcd} S^n \arrow[r, "\Sigma^{n-1}f"'] \arrow[rr, bend left, "0"] & \vee_{\Omega^n X} S^n \arrow[r, "ev"'] \arrow[u, phantom, "\Uparrow F"] & X \end{tikzcd}$$

where ev is the obvious evaluation map. Recall from Section 6 the notation $S^n_X = \vee_{\Omega^n X} S^n$. Two elements $[f, F], [g, G] \in \pi_{n,1}X$ coincide provided there is a diagram like (6.7) parting to the trivial track with $\mathit{Hopf}(N) = 0$ for $n \geq 2$ and no conditions on N for $n = 1$. We refer the reader to [BM05a] for further details on the construction of the algebraic structure of $\pi_{n,*}X$.

According to the definition of additive secondary homotopy groups of a pointed space given in Remark 6.13 the quadratic pair module $\Pi_{n,*}X$ looks as follows.

(14.2)

$$\Pi_{n,*}X = \left(\begin{tikzcd} & \Pi_{n,ee}X = \otimes^2\mathbb{Z}[\Omega^n X] \arrow[dl, "P"'] & \\ \Pi_{n,1}X \arrow[rr, "\partial"'] & & \Pi_{n,0}X = \langle \Omega^n X\rangle_{nil} \arrow[ul, "H"'] \end{tikzcd} \right)$$

Here H is always defined as in (3.6).

For $n \geq 3$, $\Pi_{n,1}X = \pi_{n,1}X$ and $P = \omega\tau_\otimes$ and ∂ in (14.2) are given by the homomorphisms in (14.1).

For $n = 2$ the group $\Pi_{2,1}X$ is the quotient of $\pi_{2,1}X$ by the relations

$$P(a \otimes b + b \otimes a) = 0;\;\; a, b \in (\pi_{2,0}X)_{ab};$$

and P and ∂ in (14.2) are induced by $\omega\tau_\otimes$ and ∂ in (14.1) respectively.

For $n = 1$, $\Pi_{1,1}X$ is the quotient of the group

$$\pi_{1,1}X \times (\hat{\otimes}^2(\pi_{1,0}X)_{ab})$$

by the relations

$$(-[f, F] + [f, F]^x, 0) = (0, x\hat{\otimes}\partial[f, F]);\;\; [f, F] \in \pi_{1,1}X,\; x \in \pi_{1,0}X;$$

$P(a \otimes b) = (0, a\hat{\otimes}b)$ for $a, b \in (\pi_{1,0}X)_{ab}$; and $\partial([f, F], x\hat{\otimes}y) = \partial[f, F] - y - x + y + x$ in $\langle \Omega X\rangle_{nil}$ for $[f, F] \in \pi_{1,1}X$ and $x, y \in \pi_{1,0}X$.

Finally for $n = 0$ the group $\Pi_{0,1}X$ is the quotient of

$$\langle \pi_{0,1}X\rangle_{nil} \times (\hat{\otimes}^2 \mathbb{Z}[\pi_{0,0}X])$$

by the relations

$$([\alpha, \alpha'], 0) = (0, (-x' + y')\hat{\otimes}(-x + y))$$

for all morphisms $\alpha\colon x \to y$ and $\alpha'\colon x' \to y'$ in $\pi_{0,*}X$,

$$(\alpha\beta, 0) = (\beta + \alpha, 0),$$

for all composable morphisms $\bullet \stackrel{\beta}{\to} \bullet \stackrel{\alpha}{\to} \bullet$ in $\pi_{0,*}X$, $P(a \otimes b) = (0, a\hat{\otimes}b)$ for $a, b \in \pi_{0,0}X$; and $\partial(\alpha, a\hat{\otimes}b) = -x + y + [b, a]$ for $a, b \in \pi_{0,0}X$ and $\alpha\colon x \to y$ in $\pi_{0,1}X$.

We denote by $\Pi_{n,(0)}X$ and $\Pi_{n,(1)}X$ the square groups

$$\Pi_{n,(0)}X = (\Pi_{n,0}X \underset{\partial P}{\overset{H}{\rightleftarrows}} \Pi_{n,ee}X),$$

$$\Pi_{n,(1)}X = (\Pi_{n,1}X \underset{P}{\overset{H\partial}{\rightleftarrows}} \Pi_{n,ee}X),$$

defining the quadratic pair module $\Pi_{n,*}X$.

Definition 14.3. The *smash product operation* for the additive secondary homotopy groups of two pointed spaces X, Y is given by morphisms in **qpm**, $n, m \geq 0$,

$$\Pi_{n,*}X \odot \Pi_{m,*}Y \stackrel{\wedge}{\longrightarrow} \Pi_{n+m,*}(X \wedge Y). \tag{14.4}$$

These morphisms are induced by square group morphisms, $n, m \geq 0$, $0 \leq i, j, i+j \leq 1$,

$$\Pi_{n,(i)}X \odot \Pi_{m,(j)}Y \stackrel{\wedge}{\longrightarrow} \Pi_{n+m,(i+j)}(X \wedge Y). \tag{14.5}$$

defined as follows:

For $i = j = 0$ morphism (14.5) is the composition

$$\mathbb{Z}_{nil}[\Omega^n X] \odot \mathbb{Z}_{nil}[\Omega^m Y] \cong \mathbb{Z}_{nil}[(\Omega^n X) \wedge (\Omega^m Y)] \stackrel{\mathbb{Z}_{nil}[\wedge]}{\longrightarrow} \mathbb{Z}_{nil}[\Omega^{n+m}(X \wedge Y)]$$

of the isomorphism in Lemma 3.7 and the morphism induced by the map between discrete pointed sets

$$\wedge\colon (\Omega^n X) \wedge (\Omega^m Y) \longrightarrow \Omega^{m+n}(X \wedge Y), \quad n, m \geq 0, \tag{14.6}$$

defined by

$$(f\colon S^n \to X) \wedge (g\colon S^m \to Y) \mapsto (f \wedge g\colon S^{n+m} \to X \wedge Y).$$

All morphisms in (14.5) coincide in the ee-term.

Suppose now that $n, m \geq 1$.

For $i = 0$ and $j = 1$, an element $g \circledcirc [f, F]$ with $g \in \Pi_{n,0}X$ and $[f, F] \in \Pi_{m,1}Y$ is sent by (14.5) to the element $g \wedge [f, F] \in \Pi_{n+m,1}(X \wedge Y)$ represented by the map

$$S^1 \stackrel{\bar{g}\# f}{\longrightarrow} \vee_{(\Omega^n X)\wedge(\Omega^m Y)} S^1 \stackrel{\Sigma\wedge}{\longrightarrow} \vee_{\Omega^{n+m}(X\wedge Y)} S^1,$$

where $\bar{g}\colon S^1 \to \vee_{\Omega^n X} S^1$ is any map with $(\pi_1\bar{g})_{nil}(1) = g$ and the second arrow is the suspension of $\wedge$ in (14.6), and by the track

$$\begin{tikzcd}
S^{n+m} \arrow[r, "\Sigma^{n+m-1}(\bar g\# f)"'] \arrow[r, bend left, "\Sigma^{n-1}\bar g\wedge\Sigma^{m-1}f"] \arrow[rrr, bend left=40, "0"] \arrow[r, phantom, "\Uparrow \mathbb{H}^{\#}_{n,\bar g,m,f}"] & \vee_{\Omega^nX\wedge\Omega^mY}S^{n+m} \arrow[r, "\Sigma^{n+m}\wedge"'] \arrow[r, phantom, shift left=6, "\Uparrow \bar g_{ev}\wedge F"] & S^{n+m}_{X\wedge Y} \arrow[r, "ev"'] & X\wedge Y
\end{tikzcd}$$

i. e.

$$g \wedge [f,F] = [(\Sigma\wedge)(\bar{g}\# f), (\bar{g}_{ev}\wedge F)\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^{\#}_{n,\bar{g},m,f})].$$

Here, and in the following three cases, the smash products of maps and tracks need to be altered by permutations according to (13.3).

In a similar way the element $g\underline{\circledcirc}[f,F]$ is sent by (14.5) to $g\underline{\triangle}[f,F] \in \Pi_{n+m,1}(X\wedge Y)$, given by the map

$$S^1 \xrightarrow{\bar{g}\underline{\#}f} \vee_{(\Omega^n X)\wedge(\Omega^m Y)} S^1 \xrightarrow{\Sigma\wedge} \vee_{\Omega^{n+m}(X\wedge Y)} S^1,$$

and by the track

$$\begin{tikzcd}
S^{n+m} \arrow[r, "\Sigma^{n+m-1}(\bar g\underline{\#} f)"'] \arrow[r, bend left, "\Sigma^{n-1}\bar g\wedge\Sigma^{m-1}f"] \arrow[rrr, bend left=40, "0"] \arrow[r, phantom, "\Uparrow \mathbb{H}^{\underline{\#}}_{n,\bar g,m,f}"] & \vee_{\Omega^nX\wedge\Omega^mY}S^{n+m} \arrow[r, "\Sigma^{n+m}\wedge"'] \arrow[r, phantom, shift left=6, "\Uparrow \bar g_{ev}\wedge F"] & S^{n+m}_{X\wedge Y} \arrow[r, "ev"'] & X\wedge Y
\end{tikzcd}$$

i. e.

$$g\underline{\triangle}[f,F] = [(\Sigma\wedge)(\bar{g}\underline{\#}f), (\bar{g}_{ev}\wedge F)\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^{\underline{\#}}_{n,\bar{g},m,f})].$$

For $i = 1$ and $j = 0$ the generator $[f,F]\circledcirc g$ with $[f,F]\in\Pi_{n,1}X$ and $g\in\Pi_{m,0}Y$ is sent by (14.5) to the element $[f,F]\wedge g \in \Pi_{n+m,1}(X\wedge Y)$ represented by the map

$$S^1 \xrightarrow{f\#\bar{g}} \vee_{(\Omega^n X)\wedge(\Omega^m Y)} S^1 \xrightarrow{\Sigma\wedge} \vee_{\Omega^{n+m}(X\wedge Y)} S^1,$$

where $\bar{g}\colon S^1 \to \vee_{\Omega^m Y} S^1$ is any map with $(\pi_1\bar{g})_{nil}(1) = g$, and the track

$$\begin{tikzcd}
S^{n+m} \arrow[r, "\Sigma^{n+m-1}(f\#\bar g)"'] \arrow[r, bend left, "\Sigma^{n-1}f\wedge\Sigma^{m-1}\bar g"] \arrow[rrr, bend left=40, "0"] \arrow[r, phantom, "\Uparrow \mathbb{H}^{\#}_{n,f,m,\bar g}"] & \vee_{\Omega^nX\wedge\Omega^mY}S^{n+m} \arrow[r, "\Sigma^{n+m}\wedge"'] \arrow[r, phantom, shift left=6, "\Uparrow F\wedge\bar g_{ev}"] & S^{n+m}_{X\wedge Y} \arrow[r, "ev"'] & X\wedge Y
\end{tikzcd}$$

i. e.

$$[f,F]\wedge g = [(\Sigma\wedge)(f\#\bar{g}), (F\wedge\bar{g}_{ev})\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^{\#}_{n,f,m,\bar{g}})].$$

The element $[f,F]\underline{\circledcirc}g$ is sent by (14.5) to $[f,F]\underline{\triangle}g \in \Pi_{n+m,1}(X\wedge Y)$ given by the map

$$S^1 \xrightarrow{f\underline{\#}\bar{g}} \vee_{(\Omega^n X)\wedge(\Omega^m Y)} S^1 \xrightarrow{\Sigma\wedge} \vee_{\Omega^{n+m}(X\wedge Y)} S^1$$

and the track

$$\begin{tikzcd}[column sep=large]
S^{n+m} \arrow[r, "\Sigma^{n+m-1}(f\underline{\#}\bar{g})"'] \arrow[r, bend left, "\Sigma^{n-1}f\wedge\Sigma^{m-1}\bar{g}"] \arrow[r, phantom, shift left=2, "\Uparrow\mathbb{H}^{\#}_{n,f,m,\bar{g}}"] \arrow[rrr, bend left=40, "0"] & \bigvee_{\Omega^n X\wedge\Omega^m Y} S^{n+m} \arrow[r, "\Sigma^{n+m}\wedge"'] \arrow[r, phantom, shift left=6, "\Uparrow F\wedge\bar{g}_{ev}"] & S^{n+m}_{X\wedge Y} \arrow[r, "ev"'] & X\wedge Y
\end{tikzcd}$$

i. e.

$$[f,F]\underline{\triangle}g = [(\Sigma\wedge)(f\underline{\#}\bar{g}), (F\wedge\bar{g}_{ev})\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^{\#}_{n,f,m,\bar{g}})].$$

Suppose now that $n = 0$ and $m \geq 0$.

For $i = 0$ and $j = 1$, an element $g \circledcirc f$ with $g\colon S^0 \to X$ and $f \in \Pi_{m,1}Y$ is sent by (14.5) to $g \wedge f = (\Pi_{m,1}(g \wedge Y))(f) \in \Pi_{m,1}(X \wedge Y)$.

For $i = 1$ and $j = 0$, an element $F \circledcirc g$ with $F\colon f \Rightarrow f'$ a track between maps $f, f'\colon S^0 \to X$ and $g \in \Pi_{m,0}Y$ is sent by (14.5) to $F\wedge g = (\Pi_{m,*}(F\wedge Y))(g)\Pi_{m,1}(X\wedge Y)$.

Suppose now that $n \geq 0$ and $m = 0$.

For $i = 0$ and $j = 1$, an element $g\underline{\circledcirc}F$ with $g \in \Pi_{n,0}X$ and $F\colon f \Rightarrow f'$ a track between maps $f, f'\colon S^0 \to Y$ is sent by (14.5) to $g\underline{\triangle}F = (\Pi_{n,*}(X \wedge F))(g)\Pi_{n,1}(X \wedge Y)$.

For $i = 1$ and $j = 0$, an element $f\underline{\circledcirc}g$ with $f \in \Pi_{n,1}X$ and $g\colon S^0 \to Y$ is sent by (14.5) to $f\underline{\triangle}g = (\Pi_{m,1}(X \wedge g))(f)\Pi_{n,1}(X \wedge Y)$.

15. The Hopf invariant for tracks and smash products

In this section we prove two lemmas on the Hopf invariant for tracks defined in [BM05a] 3.3 which will be useful to check the properties of the smash product operation on secondary homotopy groups.

The first lemma computes the Hopf invariant of a track smashed with a discrete set.

Lemma 15.1. *Let $f, g\colon S^1 \wedge A \to S^1 \wedge B$ be maps between suspensions of discrete pointed sets A, B; let $F\colon S^1 \wedge f \Rightarrow S^1 \wedge g$ be a track; and let X be another discrete pointed set. Then the following equations hold*

(1) *$Hopf(F \wedge X) = (1 \otimes \tau_{\otimes} \otimes 1)(Hopf(F) \otimes \Delta)$,*
(2) *$Hopf(X \wedge F) = (1 \otimes \tau_{\otimes} \otimes 1)(\Delta \otimes Hopf(F))$.*

In particular the smash product of a track with trivial Hopf invariant and a discrete pointed set has always a trivial Hopf invariant.

This lemma follows easily from the elementary properties of the Hopf invariant for tracks in [BM05a] 3.

The second lemma computes the effect of conjugation by an automorphism of a sphere on certain tracks.

Lemma 15.2. *Let $f, g\colon S^1 \wedge A \to S^1 \wedge B$ be maps between suspensions of discrete pointed sets A, B; let $F\colon S^{n-1} \wedge f \Rightarrow S^{n-1} \wedge g$ be a track; and let $\alpha\colon S^{n-1} \cong S^{n-1}$ be a homeomorphism. Then $(\alpha\wedge S^1\wedge B)F(\alpha^{-1}\wedge S^1\wedge A)$ is a track with trivial Hopf invariant if and only if F has trivial Hopf invariant. Moreover, if α has degree 1 or $n \geq 3$ then*

$$F = (\alpha \wedge S^1 \wedge B)F(\alpha^{-1} \wedge S^1 \wedge A).$$

Proof. Let us denote also by $F\colon I_+\wedge S^{n-1}\wedge S^1\wedge A\to S^{n-1}\wedge S^1\wedge B$ to a map representing the track F. The adjoint map of pairs

$$\begin{gathered}ad((\alpha\wedge S^1\wedge B)F(I_+\wedge\alpha^{-1}\wedge S^1\wedge A))\colon\\ (I_+\wedge S^1\wedge A,(S^1\wedge A)\vee(S^1\wedge A))\longrightarrow(\Omega^{n-1}(S^{n-1}\wedge S^1\wedge B),S^1\wedge B)\end{gathered}$$

used to define the Hopf invariant coincides with the composite

$$\begin{tikzcd}(I_+\wedge S^1\wedge A,(S^1\wedge A)\vee(S^1\wedge A)) \arrow[d, "ad(F)"] \\ (\Omega^{n-1}(S^{n-1}\wedge S^1\wedge B),S^1\wedge B) \arrow[d, "\mathrm{map}_*(\alpha^{-1},\alpha\wedge S^1\wedge B)"] \\ (\Omega^{n-1}(S^{n-1}\wedge S^1\wedge B),S^1\wedge B)\end{tikzcd}$$

The map $\mathrm{map}_*(\alpha^{-1},\alpha\wedge S^1\wedge B)$ is a homeomorphism, hence the first part of the lemma follows from the very definition of the Hopf invariant for tracks.

The homeomorphism $\mathrm{map}_*(\alpha^{-1},\alpha\wedge S^1\wedge B)$ restricts to the identity on $S^1\wedge B$. Moreover, if α has degree 1 or $n\geq 3$ then this homeomorphism is compatible with the H-multiplication of the $(n-1)$-fold loop space up to homotopy, and therefore with the Pontrjagin product. In particular by the definition and elementary properties of the Hopf invariant for tracks

$$Hopf(F)=Hopf((\alpha\wedge S^1\wedge B)F(\alpha^{-1}\wedge S^1\wedge A))$$

and hence

$$F=(\alpha\wedge S^1\wedge B)F(\alpha^{-1}\wedge S^1\wedge A).$$

□

16. Hopf invariant computations related to exterior tracks

In this section we perform two Hopf invariant computations for tracks. The first computation is connected with axiom (7) in the definition of the tensor product of square groups, see Definition 2.1, and the second one is connected with the commutativity rule for the smash product operation on additive secondary homotopy groups, see Theorem 7.3. Both computations are crucial steps towards the proof of the main results of this paper stated in Section 7. They show that the algebraic structures described in Parts 1 and 2 are the right algebraic structures to describe the smash product operation.

First of all we define a concept which will be useful for computations.

Definition 16.1. Let **nil** be the category of free groups of nilpotency class 2 and let $\Phi\colon\mathbf{Ab}\to\mathbf{Ab}$ be a functor. The G-*group* of Φ is the class $\mathrm{G}(\Phi)$ of all functions χ sending two morphisms $f\colon\langle A\rangle_{nil}\to\langle B\rangle_{nil}$, $g\colon\langle X\rangle_{nil}\to\langle Y\rangle_{nil}$ in **nil** to a homomorphism

$$\chi(f,g)\colon\mathbb{Z}[A]\otimes\mathbb{Z}[X]\longrightarrow\Phi(\mathbb{Z}[B]\otimes\mathbb{Z}[Y])$$

in such a way that if f,f_i,g,g_i are morphisms in **nil** and f',f'_i,g',g'_i are maps between pointed sets $i=1,2$ then

(1) $\chi(\langle f'\rangle_{nil},g)=0$,
(2) $\chi(f,\langle g'\rangle_{nil})=0$,
(3) $\chi((f_1,f_2),g)=(\chi(f_1,g),\chi(f_2,g))$,

(4) $\chi(f,(g_1,g_2)) = (\chi(f,g_1),\chi(f,g_2))$,
(5) $\chi(f_1\langle f_2'\rangle_{nil}, g_1\langle g_2'\rangle_{nil}) = \chi(f_1,g_1)(\mathbb{Z}[f_2']\otimes\mathbb{Z}[g_2'])$,
(6) $\chi(\langle f_1'\rangle_{nil}f_2, \langle g_1'\rangle_{nil}g_2) = \Phi(\mathbb{Z}[f_1']\otimes\mathbb{Z}[g_1'])\chi(f_2,g_2)$.

A natural transformation $\zeta\colon \Phi \Rightarrow \Psi$ between functors $\Phi,\Psi\colon \mathbf{Ab}\to\mathbf{Ab}$ induces a function $\mathrm{G}(\zeta)\colon \mathrm{G}(\Phi)\to\mathrm{G}(\Psi)$ in the obvious way.

If $\mathrm{G}(\Phi)$ is a set then it is an abelian group by addition of abelian group homomorphisms. If $\mathrm{G}(\Psi)$ is also a set then $\mathrm{G}(\zeta)$ is an abelian group homomorphism.

Many functors have a G-group which is a set, see for example Lemma 16.8 below. Alternatively one can define G-groups by using a small subcategory of **nil** to obtain always sets. We therefore do not care about set theoretic subtleties in what follows.

The following lemma shows examples of non-trivial elements in the G-group of the reduced tensor square.

Lemma 16.2. *There are elements, $n,m\geq 0$,*

$$\bar\sigma\tau_\otimes(1\otimes\tau_\otimes\otimes 1)(H\otimes TH),\ \ -\bar\sigma\binom{(-1)^{nm}}{2}H(\underline{\#})\in \mathrm{G}(\hat\otimes^2)$$

which evaluated at $f\colon\langle A\rangle_{nil}\to\langle B\rangle_{nil}$, $g\colon\langle X\rangle_{nil}\to\langle Y\rangle_{nil}$ send an element $a\otimes x\in\mathbb{Z}[A]\otimes\mathbb{Z}[X]$ with $a\in A$ and $x\in X$ to

$$\bar\sigma\tau_\otimes(1\otimes\tau_\otimes\otimes 1)(H(f(a))\otimes TH(g(x))),$$
$$-\bar\sigma\binom{(-1)^{nm}}{2}H(f(a)\underline{\#}g(x)),$$

in $\hat\otimes^2(\mathbb{Z}[B]\otimes\mathbb{Z}[Y])$ respectively.

Properties (1)–(6) in Definition 16.1 are easy to check in these cases.

The following lemma is left as an exercise for the reader.

Lemma 16.3. *Given three functors $\Phi,\Psi,\Gamma\colon\mathbf{Ab}\to\mathbf{Ab}$ and a natural exact sequence*

$$\Phi\overset{\zeta}{\hookrightarrow}\Psi\overset{\xi}{\longrightarrow}\Gamma,$$

the sequence

$$\mathrm{G}(\Phi)\overset{\mathrm{G}(\zeta)}{\hookrightarrow}\mathrm{G}(\xi)\overset{\mathrm{G}(\xi)}{\longrightarrow}\mathrm{G}(\Gamma)$$

is exact.

This lemma can be applied to the natural exact sequence

$$A\otimes\mathbb{Z}/2\overset{\bar\tau}{\hookrightarrow}\hat\otimes^2A\overset{q}{\twoheadrightarrow}\wedge^2A,$$

where $\wedge^2A$ is the exterior square of A, $\bar\tau(a)=\bar\sigma(a\otimes a)$ and $q\bar\sigma(a\otimes b)=a\wedge b$.

Now we define elements in the G-group of the reduced tensor square by using the exterior tracks and the Hopf invriant for tracks.

In the rest of this section A,B,X,Y will always be pointed discrete sets. Given maps $f\colon\Sigma A\to\Sigma B$, $g\colon\Sigma X\to\Sigma Y$, we define the following abelian group homomorphism as a Hopf invariant for tracks

$$K(f,g)=\bar\sigma\mathit{Hopf}((\mathbb{H}^{\#}_{f,g})^{\boxminus}\Box\mathbb{H}^{\#}_{f,g})\colon\mathbb{Z}[A]\otimes\mathbb{Z}[X]\longrightarrow\hat\otimes^2(\mathbb{Z}[B]\otimes\mathbb{Z}[Y]).\tag{16.4}$$

Proposition 16.5. *The homomorphism $K(f,g)$ defined above only depends on $(\pi_1f)_{nil}$ and $(\pi_1g)_{nil}$. Moreover, $K\in\mathrm{G}(\hat\otimes^2)$. Furthermore,*

$$\mathrm{G}(q)(K)=\mathrm{G}(q)(\bar\sigma\tau_\otimes(1\otimes\tau_\otimes\otimes1)(H\otimes TH)).$$

Proof. By [BM05a] 3.6 (5) and Lemma 15.2

$$\text{(a)} \qquad K(f,g) = \mathit{Hopf}((\mathbb{H}^{\#}_{n,f,m,g})^{\boxminus}\square\mathbb{H}^{\#}_{n,f,m,g})$$

for $n, m \geq 1$ with $n+m > 2$. If $(\pi_1 f)_{nil} = (\pi_1\bar{f})_{nil}$ and $(\pi_1 g)_{nil} = (\pi_1\bar{g})_{nil}$ then there are tracks $F\colon \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}\bar{f}$, $G\colon \Sigma^{m-1}g \Rightarrow \Sigma^{m-1}\bar{g}$ with trivial Hopf invariant. Moreover, the tracks

$$\Sigma(F\#G)\colon \Sigma^{n+m-1}(f\#g) \Rightarrow \Sigma^{n+m-1}(\bar{f}\#\bar{g}),$$
$$\Sigma(F\underline{\#}G)\colon \Sigma^{n+m-1}(f\underline{\#}g) \Rightarrow \Sigma^{n+m-1}(\bar{f}\underline{\#}\bar{g}),$$

have trivial Hopf invariant by Lemmas 15.1 and 15.2, and [BM05a] 3.6.

By Lemma 13.4 we have that

$$\begin{aligned}\Sigma(F\#G)^{\boxminus}\square(\mathbb{H}^{\#}_{n,\bar{f},m,\bar{g}})^{\boxminus}\square\mathbb{H}^{\#}_{n,\bar{f},m,\bar{g}}\square\Sigma(F\underline{\#}G) &= (\mathbb{H}^{\#}_{n,f,m,g})^{\boxminus}\square(F\wedge G)^{\boxminus}\\ &\quad\ \square(F\wedge G)\square\mathbb{H}^{\#}_{n,f,m,g}\\ &= (\mathbb{H}^{\#}_{n,f,m,g})^{\boxminus}\square\mathbb{H}^{\#}_{n,f,m,g}.\end{aligned}$$

Hence the first part of the statement follows from the elementary properties of the Hopf invariant for tracks in [BM05a] 3.

The equation for the images by q follows from [BM05a] 3.6, equation (a), and the equality

$$\begin{aligned}(\pi_1(f\underline{\#}g))_{nil}(a\wedge x) &= (\pi_1(f\#g))_{nil}(a\wedge x)\\ &\quad +\partial\tau_{\otimes}(1\otimes\tau_{\otimes}\otimes 1)(H((\pi_1 f)_{nil}(a))\otimes TH((\pi_1 g)_{nil}(x))).\end{aligned}$$

This last equality is a consequence of Definition 2.1 (7) and Proposition 3.7.

Finally (1)–(6) in Definition 16.1 follow from Lemma 13.2 (1)–(10) and the elementary properties of the Hopf invariant for tracks in [BM05a] 3.6. □

Let $\hat{\tau}_{n,m}\colon \tau_{n,m} \Rightarrow (\cdot)^{(-1)^{nm}}_{n+m}$ be a track between maps

$$(\cdot)^{(-1)^{nm}}_{n+m}, \tau_{n,m}\colon S^{n+m}\to S^{n+m}.$$

This is a lift of the shuffle permutation in (6.5) to the symmetric track group $\mathrm{Sym}_{\square}(n+m)$. Given maps $f\colon \Sigma A\to\Sigma B$, $g\colon \Sigma X\to\Sigma Y$, we define the following Hopf invariants

(16.6)

$$\begin{aligned}L_{1,1}(f,g) &= \bar{\sigma}\mathit{Hopf}((\Sigma^2\tau_{\wedge})(\mathbb{H}^{\#}_{g,f})^{\boxminus}\hat{\tau}_{1,1})\square((\hat{\tau}^{\boxminus}_{1,1}\wedge B\wedge Y)\mathbb{H}^{\#}_{f,g})),\\ L_{n,m}(f,g) &= \mathit{Hopf}((\Sigma^{m+n}\tau_{\wedge})(\mathbb{H}^{\#}_{m,g,n,f})^{\boxminus}\hat{\tau}_{n,m})\square((\hat{\tau}^{\boxminus}_{n,m}\wedge B\wedge Y)\mathbb{H}^{\#}_{n,f,m,g})).\end{aligned}$$

Here $n,m\geq 1$ and $n>1$ or $m>1$, and $\tau_{\wedge}\colon Y\wedge B\to B\wedge Y$ is the symmetry map for the smash product. Notice that $L_{n,m}(f,g)$ is a homomorphism, $n,m\geq 1$,

$$L_{n,m}(f,g)\colon \mathbb{Z}[A]\otimes\mathbb{Z}[X]\longrightarrow \hat{\otimes}^2(\mathbb{Z}[B]\otimes\mathbb{Z}[Y]).$$

Proposition 16.7. *For any $n,m\geq 1$ the homomorphism $L_{n,m}(f,g)$ defined above does not depend on the choice of $\hat{\tau}_{n,m}$. Moreover, it only depends on $(\pi_1 f)_{nil}$ and $(\pi_1 g)_{nil}$. Furthermore, $L_{n,m}\in \mathrm{G}(\hat{\otimes}^2)$. In addition,*

$$\mathrm{G}(q)(L_{n,m}) = \mathrm{G}(q)\left(-\bar{\sigma}\binom{(-1)^{nm}}{2}H(\underline{\#})\right).$$

The proof of this proposition is analogous to Proposition 16.5.

We would like to omit $\mathrm{G}(q)$ in the equations of Propositions 16.5 and 16.7. For this we prove the following two lemmas.

Lemma 16.8. *If $\Phi\colon \mathbf{Ab} \to \mathbf{Ab}$ is an additive functor which preserves arbitrary filtered colimits then there is an isomorphism*

$$\mathrm{G}(\Phi) \cong \Phi(\mathbb{Z}).$$

Proof. We claim that the isomorphism sends χ to $\chi(-1,-1) \in \Phi(\mathbb{Z})$, where -1 is the homomorphism $-1\colon \mathbb{Z} \to \mathbb{Z}$. This will be a consequence of the following formula, that we claim to hold. We first notice that

$$\Phi(\mathbb{Z}[B] \otimes \mathbb{Z}[Y]) = \Phi(\mathbb{Z}) \otimes \mathbb{Z}[B] \otimes \mathbb{Z}[Y].$$

Given $f\colon \langle A\rangle_{nil} \to \langle B\rangle_{nil}$, $g\colon \langle X\rangle_{nil} \to \langle Y\rangle_{nil}$, $a \in A$ and $x \in X$, if $f_{ab}(a) = \sum_i n_i b_i$, $g_{ab}(x) = \sum_j m_j y_j$ are the linear expansions with $b_i \in B$, $y_j \in Y$, $n_i, m_j \in \mathbb{Z}$, then

$$\text{(a)} \qquad \chi(f,g)(a \otimes x) = \sum_{i,j} \epsilon(n_i, m_j)\, |n_i m_j|\, \chi(-1,-1) \otimes b_i \otimes y_j.$$

Here $\epsilon(n_i, m_j) = 1$ provided $n_i, m_j < 0$ and it is zero otherwise. Conversely any χ defined by formula (a) with $\chi(-1,-1)$ arbitrarily chosen out of $\Phi(\mathbb{Z})$ defines an element in $\mathrm{G}(\Phi)$. We will just prove the first part of the claim, the converse is easy.

By (3), (4), (5) and (6) in Definition 16.1 we have

$$\text{(b)} \qquad \chi(f,g)(a \otimes x) = \sum_{i,j} \chi(n_i, m_j) \otimes b_i \otimes y_j.$$

for $n_i, m_j\colon \mathbb{Z} \to \mathbb{Z}$. For any $n \in \mathbb{Z}$ we consider the homomorphism $\mu_n\colon \mathbb{Z} \to \langle c_1, \dots, c_{|n|}\rangle_{nil}$ defined by $\mu_n(1) = c_1 + \cdots + c_{|n|}$ if $n > 0$, $\mu_n(1) = -c_1 - \cdots - c_{|n|}$ if $n < 0$, and $\mu_n(1) = 0$ if $n = 0$. By (b) given $n, m \in \mathbb{Z}$ we have that

$$\text{(c)} \qquad \chi(\mu_n, \mu_m) = \sum_{i=1}^{|n|} \sum_{j=1}^{|m|} \chi(\epsilon(n), \epsilon(m)) \otimes c_i \otimes c_j.$$

Here ϵ sends a positive integer to 1, a negative integer to -1, and zero to itself. Now, again by Definition 16.1 (6), we have that $\chi(n,m)$ is the sum of all coefficients in (c), i. e.

$$\chi(n,m) = |nm|\, \chi(\epsilon(n), \epsilon(m)) \in \Phi(\mathbb{Z}).$$

But by Definition 16.1 (1) and (2) the element $\chi(\epsilon(n), \epsilon(m)) \in \Phi(\mathbb{Z})$ is zero unless $n, m < 0$, hence we are done. □

Lemma 16.9. *There is a commutative diagram*

$$\begin{tikzcd} \mathrm{G}(-\otimes \mathbb{Z}/2) \arrow[r, hook] \arrow[d, "\cong"'] & \mathrm{G}(\hat{\otimes}^2) \arrow[dl, bend left] \\ \mathbb{Z}/2 & \end{tikzcd}$$

Proof. Both vertical homomorphisms send an element χ in the corresponding G-group to

$$\chi(-1,-1) \in \mathbb{Z} \otimes \mathbb{Z}/2 = \hat{\otimes}^2 \mathbb{Z} = \mathbb{Z}/2.$$

The isomorphism is proved in the previous lemma. □

The following theorem is a key step towards the proof of Theorem 6.1. It is connected to relation (7) in the definition of the tensor product for square groups, see Definition 2.1.

Theorem 16.10. $K = \bar{\sigma}\tau_\otimes(1 \otimes \tau_\otimes \otimes 1)(H \otimes TH) \in \mathrm{G}(\hat{\otimes}^2)$.

Proof. By Proposition 16.5 and Lemma 16.9 we only have to check that

$$\bar{\sigma}\tau_\otimes(1 \otimes \tau_\otimes \otimes 1)(H(-1) \otimes TH(-1)) = K(\nu, \nu)$$

for $\nu\colon S^1 \to S^1\colon z \mapsto z^{-1}$ the complex inversion. It is easy to see from the very definition of H that

$$\bar{\sigma}\tau_\otimes(1 \otimes \tau_\otimes \otimes 1)(H(-1) \otimes TH(-1)) = 1 \in \mathbb{Z}/2,$$

hence this theorem follows from Lemma 17.3 below. $\square$

The next theorem is the main step in the proof of the commutativity rule for the smash product operation in Theorem 7.3.

Theorem 16.11. $L_{n,m} = -\bar{\sigma}\binom{(-1)^{nm}}{2} H(\underline{\#}) \in \mathrm{G}(\hat{\otimes}^2)$, $n, m \geq 1$.

Proof. By Proposition 16.7 and Lemma 16.9 we only have to check that

$$-\bar{\sigma}\binom{(-1)^{nm}}{2} H((-1)\underline{\#}(-1)) = L(\nu, \nu)$$

for $\nu\colon S^1 \to S^1\colon z \mapsto z^{-1}$ the complex inversion. But $(-1)\underline{\#}(-1) = 1$ and $H(1) = \binom{1}{2} = 0$, hence this theorem follows from Lemma 17.4 below. $\square$

17. Hopf invariants of tracks between orthogonal transformations

In this section we are concerned with the computation of Hopf invariants for tracks between self maps of spheres S^n, $n \geq 2$. More concretely, we are interested in tracks between maps $A\colon S^n \to S^n$ which are induced by the left action of the orthogonal group $O(n)$ on S^n, i. e. $A \in O(n)$. The pull-back of this action along the inclusion $\mathrm{Sym}(n) \subset O(n)$ induced by permutation of coordinates in $\mathbb{R}^n$ yields the action of S^n already considered in Section 6. Let $\det\colon O(n) \to \{\pm 1\}$ be the determinant homomorphism. We consider the group $\{\pm 1\}$ embedded in $O(n)$ as

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \pm 1 \end{pmatrix}.$$

In [BM05a] 6 we compute the group $\widetilde{O}(n)$ of tracks $A \Rightarrow \det A$, $A \in O(n)$, with multiplication given by horizontal composition. Notice that tracks $A \Rightarrow B$ between maps $A, B\colon S^n \to S^n$ with $A, B \in O(n)$ coincide with homotopy classes of paths from A to B in the Lie group $O(n)$ since the J-homomorphism $\pi_1 O(n) \cong \pi_{n+1} S$ is an isomorphism for $n \geq 2$. We can identify the group $\widetilde{O}(2)$ with the semidirect product $\{\pm 1\} \ltimes \mathbb{R}$. Here $\mathbb{R}$ is the additive group of real numbers and $\{\pm 1\}$ is the multiplicative group of order 2 acting on $\mathbb{R}$ by multiplication. See [BM05b] 6.12.

In order to describe $\widetilde{O}(n)$ for $n \geq 3$ we need to recall the definition of the positive Clifford algebra.

Definition 17.1. The *positive Clifford algebra* $C_+(n)$ is the unital $\mathbb{R}$-algebra generated by e_i, $1 \leq i \leq n$, with relations

(1) $e_i^2 = 1$ for $1 \leq i \leq n$,
(2) $e_i e_j = -e_j e_i$ for $1 \leq i < j \leq n$.

Clifford algebras are defined for arbitrary quadratic forms on finite-dimensional vector spaces, see for instance [BtD85] 6.1. The Clifford algebra defined above corresponds to the quadratic form of the standard positive-definite scalar product in $\mathbb{R}^n$. We identify the sphere S^{n-1} with the vectors of Euclidean norm 1 in the vector subspace $\mathbb{R}^n \subset C_+(n)$ spanned by the generators e_i. The vectors in S^{n-1} are units in $C_+(n)$. Indeed for any $v \in S^{n-1}$ the square $v^2 = 1$ is the unit element in $C_+(n)$, so that $v^{-1} = v$. The group $\widetilde{O}(n)$ can be identified with the subgroup of units in $C_+(n)$ generated by S^{n-1}. This group is also known as the *positive pin group.*

The group $\widetilde{O}(n)$ is a covering Lie group of $O(n)$ with simply connected components, and with kernel $\mathbb{Z} \subset \mathbb{R}$ if $n = 2$ and $\{\pm 1\}$ if $n \geq 3$. The covering homomorphism

$$q\colon \widetilde{O}(n) \longrightarrow O(n),$$

is defined for $n = 2$ as

$$q(x,y) = \left(\begin{array}{cc} \cos 2\pi y & -\sin 2\pi y \\ x\sin 2\pi y & x\cos 2\pi y \end{array}\right).$$

For $n \geq 3$ the homomorphism q sends an element $v \in S^{n-1} \subset \widetilde{O}(n) \subset C_+(n)$ to the matrix of the reflexion along the plane orthogonal to the unit vector v. An element $x \in \widetilde{O}(n)$ is identified with the track $q(x) \Rightarrow \det q(x)$ determined by the push-forward along q of the unique track in $\widetilde{O}(n)$ from the point x to the point $e_n^{\binom{\det q(x)}{2}}$ if $n \geq 3$, or from x to $(\det q(x), 0)$ if $n = 2$. With the track approach the covering map q sends a track to the source map. Moreover, a track $\alpha\colon 1_{S^n} \Rightarrow 1_{S^n}$ in the kernel of q is identified with $(1, -\mathit{Hopf}(\alpha))$ for $n = 2$, compare [BM05b] 3.4, and with $\mathit{Hopf}(\alpha) \in \mathbb{Z}/2 \cong \{\pm 1\}$ for $n \geq 3$.

The suspension of tracks defines group inclusions

$$\Sigma\colon \widetilde{O}(n) \hookrightarrow \widetilde{O}(n+1).$$

For $n \geq 3$ this is induced by the algebra inclusion $C_+(n) \hookrightarrow C_+(n+1)$ defined by $e_j \mapsto e_{j+1}$, $1 \leq j \leq n$. For $n = 2$ it is given by

$$(x,y) \mapsto e_3^{\binom{x+1}{2}}((\sin \pi y)e_2 + (\cos \pi y)e_3). \tag{17.2}$$

For K in (16.4) ane $\nu\colon S^1 \to S^1\colon z \mapsto z^{-1}$ we have the following result.

Lemma 17.3. $K(\nu,\nu) = 1$.

Proof. We are going to prove the following stronger equality:

$$\mathit{Hopf}((\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus} \square \mathbb{H}^{\#}_{\nu,\nu}) = 1. \tag{a}$$

From the very definition of $\mathbb{H}$ we obtain the following identity in $\widetilde{O}(2)$.

$$\mathbb{H}(\nu \wedge S^1) \quad = \quad (1, \frac{1}{4}).$$

Since $(\nu \wedge S^1)\mathbb{H}$ is the inverse of $\mathbb{H}(\nu \wedge S^1)$ then

$$(\nu \wedge S^1)\mathbb{H} \quad = \quad (1, -\frac{1}{4}).$$

Obviously the identity track $0^{\square}_{\Sigma\nu} \colon \Sigma\nu \Rightarrow \Sigma\nu$ is $(-1,0)$ in $\widetilde{O}(2)$. By using these equalities we obtain

$$\begin{aligned}
(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus} &= 0^{\square}_{\Sigma\nu}(\mathbb{H}(\nu \wedge S^1))0^{\square}_{\Sigma\nu}((\nu \wedge S^1)\mathbb{H}) \\
&= (-1,0)(1,\frac{1}{4})(-1,0)(1,-\frac{1}{4}) \\
&= (-1,\frac{1}{4})(-1,-\frac{1}{4}) \\
&= ((-1)(-1),-\frac{1}{4}-\frac{1}{4}) \\
&= (1,-\frac{1}{2}), \\
(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus} &= (\mathbb{H}(\nu \wedge S^1))0^{\square}_{\Sigma\nu}((\nu \wedge S^1)\mathbb{H})0^{\square}_{\Sigma\nu} \\
&= (1,\frac{1}{4})(-1,0)(1,-\frac{1}{4})(-1,0) \\
&= (-1,-\frac{1}{4})(-1,\frac{1}{4}) \\
&= ((-1)(-1),\frac{1}{4}+\frac{1}{4}) \\
&= (1,\frac{1}{2}).
\end{aligned}$$

Let $\beta \colon 1_{S^2} \Rightarrow 1_{S^2}$ be a track with $\mathit{Hopf}(\beta) = 1$, so that $\beta = (1,-1)$ in $\widetilde{O}(2)$. Equation (a) is equivalent to the following equation in $\widetilde{O}(2)$,

$$(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus} \quad = \quad (\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus}\beta,$$

which follows from the equalities above. $\square$

For $L_{n,m}$ in (16.6) we have the following result.

Lemma 17.4. *$L_{n,m}(\nu,\nu) = 0$, $n,m \geq 1$.*

Proof. We showed in the proof of Lemma 17.3 that in $\widetilde{O}(2)$

$$\begin{aligned}
(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus} &= (1,-\frac{1}{2}), \\
(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus} &= (1,\frac{1}{2}).
\end{aligned}$$

For $n=m=1$ we can take $\hat{\tau}_{1,1}=(-1,-\frac{1}{4})$. Then in $\widetilde{O}(2)$

$$\begin{aligned}
\hat{\tau}_{1,1}(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus} &= (-1,-\frac{1}{4})(1,-\frac{1}{2})\\
&= (-1,-\frac{1}{4}-\frac{1}{2})\\
&= (-1,-\frac{3}{4}),\\
(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus}\hat{\tau}_{1,1} &= (1,\frac{1}{2})(-1,-\frac{1}{4})\\
&= (-1,-\frac{1}{2}-\frac{1}{4})\\
&= (-1,-\frac{3}{4}).
\end{aligned}$$

This shows that $L_{1,1}(\nu,\nu)=0$.

Suppose now that $n>1$ or $m>1$. By using formula (17.2) one can easily check that for any $k\geq 1$

$$\begin{aligned}
\Sigma^k(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus} &= e_{k-1}e_k,\\
\Sigma^k(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus} &= e_k e_{k-1},
\end{aligned}$$

in $\widetilde{O}(k+2)$.

For any $k\geq 1$ the shuffle permutation $\tau_{1,k-1}$ can be decomposed as

$$\tau_{1,k-1} = (k\ k-1)\cdots(2\ 1),$$

hence for fixed $p,q\geq 0$ with $p+k+q>2$ we can lift $S^p\wedge\tau_{1,k-1}\wedge S^q$ to $\widetilde{O}(p+k+q)$ by

$$\hat{\tau}_{1,k-1} = \frac{1}{2^{\frac{k-1}{2}}}(e_{p+k}-e_{p+k-1})\cdots(e_{p+2}-e_{p+1}),$$

and its inverse in $\widetilde{O}(p+k+q)$ is

$$\hat{\tau}_{k-1,1} = \frac{1}{2^{\frac{k-1}{2}}}(e_{p+2}-e_{p+1})\cdots(e_{p+k}-e_{p+k-1}).$$

By using these equalities we obtain

$$\begin{aligned}
(\mathbb{H}^{\#}_{n,\nu,m,\nu})^{\boxminus} &= (S^{n-1}\wedge\tau_{m-1,1}\wedge S^1)(\Sigma^{m+n-2}(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus})(S^{n-1}\wedge\tau_{1,m-1}\wedge S^1)\\
&= \hat{\tau}_{m-1,1}(\Sigma^{m+n-2}(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus})\hat{\tau}_{1,m-1}\\
&= \frac{1}{2^{m-1}}(e_{n+1}-e_n)\cdots(e_{n+m-1}-e_{n+m-2})e_{n+m-1}e_{n+m}\\
&\quad\cdot(e_{n+m-1}-e_{n+m-2})\cdots(e_{n+1}-e_n),\\
(\mathbb{H}^{\#}_{m,\nu,n,\nu})^{\boxminus} &= (S^{m-1}\wedge\tau_{n-1,1}\wedge S^1)(\Sigma^{n+n-2}(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus})(S^{m-1}\wedge\tau_{1,n-1}\wedge S^1)\\
&= \hat{\tau}_{n-1,1}(\Sigma^{n+m-2}(\mathbb{H}^{\#}_{\nu,\nu})^{\boxminus})\hat{\tau}_{1,n-1}\\
&= \frac{1}{2^{n-1}}(e_{m+1}-e_m)\cdots(e_{n+m-1}-e_{n+m-2})e_{n+m}e_{n+m-1}\\
&\quad\cdot(e_{n+m-1}-e_{n+m-2})\cdots(e_{m+1}-e_m).
\end{aligned}$$

For $j>i$ we have the following useful identities.

(a) $$e_j(e_j-e_i)=-(e_j-e_i)e_i,$$

(b) $$e_i(e_j-e_i) = -(e_j-e_i)e_j,$$

(c) $$(e_j-e_i)^2 = 2.$$

By using (a), (b), (c), and the formulas above one can easily check that

$$\begin{aligned}(\mathbb{H}^{\#}_{n,\nu,m,\nu})^{\boxminus} &= e_n e_{n+m},\\ (\mathbb{H}^{\#}_{m,\nu,n,\nu})^{\boxminus} &= e_{n+m} e_m.\end{aligned}$$

The identity $\tau_{n,m} = \tau^n_{1,n+m-1}$ holds, therefore we can also take $\hat{\tau}_{n,m} = \hat{\tau}^n_{1,n+m-1}$. With this choice it is not difficult to compute that

$$\hat{\tau}_{n,m}(\mathbb{H}^{\#}_{n,\nu,m,\nu})^{\boxminus} = (\mathbb{H}^{\#}_{m,\nu,n,\nu})^{\boxminus}\hat{\tau}_{n,m}$$

in $\widetilde{O}(n+m)$. For this one uses (a) and (b). This last equation is equivalent to $L_{n,m}(\nu,\nu)=0$. □

18. Properties of the smash product in dimensions ≥ 1

Restricting to dimensions ≥ 1 in this section we show a series of properties of the smash product operation for secondary homotopy groups in Definition 14.3 which imply Theorems 7.1, 7.2, 7.3 and 8.13 within this range. The case of dimension 0 is a consequence of the fact that secondary homotopy groups are track functors and of the first technical lemma in the next section.

In this section we will work with the track category $\mathbf{Top}^*$. This track category has a strict zero object 0 so that the zero morphism $0\colon X\to Y$ is always defined for a pair of pointed spaces. In this situation the *golden rule* says that

(GR) The composition of a trivial map 0 with a any track F is always a trivial track $0F = 0^{\square}, F0 = 0^{\square}$.

This is an obvious but crucial property that will be very useful for computations.

Let $n,m\geq 1$ and $0\leq i,j,i+j\leq 1$.

We need to show that the square group morphisms (14.5) are well defined. There is nothing to check in case $i=j=0$. For $i+j=1$ we define operations

(18.1) $$\wedge, \underline{\wedge}\colon \pi_{n,i}X\times\pi_{m,j}Y \longrightarrow \Pi_{n+m,1}(X\wedge Y)$$

as in Definition 14.3. For this we need to choose $\bar{g}$ more carefully in certain cases. For example, in order to define $g\wedge[f,F]$ and $g\underline{\wedge}[f,F]$ when $n=1$ and $i=0$ we need to take $\bar{g}$ in such a way that $(\pi_1\bar{g})(1) = g\in\langle\Omega X\rangle$.

It is not completely immediate that the operations $\wedge$ and $\underline{\wedge}$ do not depend on choices. We check here for instance that $g\wedge[f,F]$ does not depend on the choice of $\bar{g}$, f and F. Let $\bar{g}'$, f' and F' be another choice. Then there are tracks $N_1\colon \Sigma^{n-1}\bar{g}'\Rightarrow\Sigma^{n-1}\bar{g}$ and $N_2\colon\Sigma^{m-1}f'\Rightarrow\Sigma^{m-1}f$ with trivial Hopf invariant such that $F' = F\square(ev\,N_2)$. One can now use Lemmas 15.1, 13.4, and 15.2 together with the golden rule to show that, up to a permutation in the spherical coordinates, the track $\Sigma(N_1\# N_2)\colon \Sigma^{n+m-1}(\bar{g}'\# f')\Rightarrow\Sigma^{n+m-1}(\bar{g}\# f)$ is a track with trivial Hopf invariant which determines the desired equality.

In the next lemma we establish the fundamental properties of (18.1).

Lemma 18.2. (1) *The operation $\wedge$ is left linear*

$$\begin{aligned}(x+y)\wedge[f,F] &= x\wedge[f,F]+y\wedge[f,F],\\ ([f,F]+[g,G])\wedge x &= [f,F]\wedge x+[g,G]\wedge x.\end{aligned}$$

(2) *The operation* $\triangle$ *is right linear*

$$\begin{aligned}[f,F]\wedge(x+y) &= [f,F]\triangle x+[f,F]\triangle y,\\ x\triangle([f,F]+[g,G]) &= x\triangle[f,F]+x\triangle[g,G].\end{aligned}$$

(3) *For* $m\geq 2$, $i=0$ *and* $j=1$, *given* $a\in\otimes^2\mathbb{Z}[\Omega^m Y]$ *the equality* $x\wedge P(a)=P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(\Delta(x)\otimes a)$ *holds.*

(4) *For* $m\geq 2$, $i=1$ *and* $j=0$, *given* $a\in\otimes^2\mathbb{Z}[\Omega^m Y]$ *the equality* $[f,F]\wedge\partial P(a)=P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(\Delta\partial[f,F]\otimes a)$ *holds.*

(5) *For* $n\geq 2$, $i=1$ *and* $j=0$, *given* $a\in\otimes^2\mathbb{Z}[\Omega^n X]$ *the equality* $P(a)\triangle x=P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(a\otimes\Delta(x))$ *holds.*

(6) *For* $n\geq 2$, $i=0$ *and* $j=1$, *given* $a\in\otimes^2\mathbb{Z}[\Omega^n X]$ *the following equality holds*

$$\partial P(a)\triangle[f,F]=P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(a\otimes\Delta\partial[f,F]).$$

(7) *If* $m=1$, $i=0$ *and* $j=1$, *the following equality holds*

$$g\wedge(-[f,F]+[f,F]^x)=P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(\Delta(g)\otimes\{x\}\otimes\{\partial[f,F]\}).$$

(8) *For* $n=1$, $i=1$ *and* $j=0$, *the following equality holds*

$$(-[f,F]+[f,F]^x)\triangle g=P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(\{x\}\otimes\{\partial[f,F]\}\otimes\Delta(g)).$$

(9) *The equality* $g\triangle[f,F]-g\wedge[f,F]=P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(H(g)\otimes TH\partial[f,F])$ *holds.*

(10) *The equality* $[f,F]\triangle g-[f,F]\wedge g=P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(H\partial[f,F]\otimes TH(g))$ *holds.*

(11) *The equalities* $(\partial[f,F])\wedge[g,G]=[f,F]\wedge(\partial[g,G])$ *and* $(\partial[f,F])\triangle[g,G]=[f,F]\triangle(\partial[g,G])$ *hold.*

Proof. Let us check the first equation in (1).

$$\begin{aligned}
x\wedge[f,F]+y\wedge[f,F] &= [(\Sigma\wedge)(\bar{x}\#f,\bar{y}\#f)\mu,\\
&\quad ((\bar{x}_{ev}\wedge F)\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^\#_{n,\bar{x},m,f}),\\
&\quad (\bar{y}_{ev}\wedge F)\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^\#_{n,\bar{y},m,f}))\mu]\\
\text{12.1 (3) and (GR)} &= [(\Sigma\wedge)((\bar{x},\bar{y})\#f)\mu,\\
&\quad ((\bar{x}_{ev}\wedge F)\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^\#_{n,\bar{x},m,f}),\\
&\quad (\bar{y}_{ev}\wedge F)\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^\#_{n,\bar{y},m,f}))\mathbb{H}^\#_{n,\mu,m,1_{S^1}}]\\
\text{12.1 (7)} &= [(\Sigma\wedge)(((\bar{x},\bar{y})\mu)\#f),\\
&\quad ((\bar{x}_{ev}\wedge F,\bar{y}_{ev}\wedge F)(\Sigma^{n-1}\mu\wedge S^m))\square\\
&\quad (ev(\Sigma^{n+m}\wedge)(\mathbb{H}^\#_{n,\bar{x},m,f},\mathbb{H}^\#_{n,\bar{y},m,f})\mathbb{H}^\#_{n,\mu,m,1_{S^1}})]\\
\text{13.2 (3) and (7)} &= [(\Sigma\wedge)(((\bar{x},\bar{y})\mu)\#f),\\
&\quad (((\bar{x},\bar{y})\mu)_{ev}\wedge F)\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^\#_{n,(\bar{x},\bar{y})\mu,m,f})]\\
&= (x+y)\wedge[f,F].
\end{aligned}$$

The second equation in (1) and the equations in (2) are analogous.

In order to check equation (3) we notice that $x \wedge P(a)$ is represented by the following diagram

$$\begin{tikzcd}
S^{n+m} \arrow[r, "\Sigma^{n+m-1}(\bar{x}\#f)"'] \arrow[r, bend left=80, "0"] \arrow[r, bend left=30, "\Sigma^{n-1}\bar{x}\wedge\Sigma^{m-1}f"] \arrow[r, phantom, bend left=15, "\Uparrow\mathbb{H}^{\#}_{n,\bar{x},m,f}"] \arrow[r, phantom, bend left=55, "\Uparrow \Sigma^{n-1}\bar{x}\wedge(\Sigma^{m-2}F)"] & \bigvee_{\Omega^n X\wedge\Omega^m Y} S^{n+m} \arrow[r, "\Sigma^{n+m}\wedge"'] & S^{n+m}_{X\wedge Y} \arrow[r, "ev"'] & X\wedge Y
\end{tikzcd}$$

where $F\colon \Sigma f \Rightarrow 0$ is any track with $Hopf(F) = -\tau_{\otimes}(a)$. Here we use claim (*) in the proof of [BM05a] 4.9. By Lemmas 13.4 (1), 13.2 (2) and the golden rule this diagram coincides with

$$\begin{tikzcd}
S^{n+m} \arrow[r, "\Sigma^{n+m-1}(\bar{x}\#f)"'] \arrow[r, bend left=80, "0"] \arrow[r, phantom, bend left=40, "\Uparrow \Sigma((\Sigma^{n-1}\bar{x})\#(\Sigma^{m-2}F))"] & \bigvee_{\Omega^n X\wedge\Omega^m Y} S^{n+m} \arrow[r, "\Sigma^{n+m}\wedge"'] & S^{n+m}_{X\wedge Y} \arrow[r, "ev"'] & X\wedge Y
\end{tikzcd}$$

Now one can use Lemmas 15.1 (2) and 15.2 to check that

$$Hopf(\Sigma((\Sigma^{n-1}\bar{x})\#(\Sigma^{m-2}F))) = -\bar{\sigma}\tau_{\otimes}(1\otimes\tau_{\otimes}\otimes 1)(\Delta(x)\otimes a),$$

hence (3) follows form claim (*) in the proof of [BM05a] 4.9. Equation (5) is analogous. Equations (7) and (8) are unstable versions of (3) and (5). We leave them to the reader.

Both sides of the first equation in (11) are represented by

$$\begin{tikzcd}
S^{n+m} \arrow[r, "\Sigma^{n+m-1}(g\#f)"'] \arrow[r, bend left=30, "\Sigma^{n-1}g\wedge\Sigma^{m-1}f"] \arrow[r, phantom, bend left=15, "\Uparrow\mathbb{H}^{\#}_{n,g,m,f}"] \arrow[rrr, bend left=40, "0"] & \bigvee_{\Omega^n X\wedge\Omega^m Y} S^{n+m} \arrow[r, "\Sigma^{n+m}\wedge"'] \arrow[r, phantom, bend left=40, "\Uparrow G\wedge F"] & S^{n+m}_{X\wedge Y} \arrow[r, "ev"'] & X\wedge Y
\end{tikzcd}$$

In order to check this fact one only needs to use the golden rule. Similarly for the second equation in (11). Now (4) and (6) follow from (3), (5) and (11).

Finally (9) and (10) follow from Theorem 16.10, Lemma 15.2, and claim (*) in the proof of [BM05a] 4.9. $\square$

The equalities in Lemma 18.2 can be used to check the following properties of the operations (18.1).

Lemma 18.3. (1) *For $n = 1$ and $i = 0$ the elements $g\wedge[f,F]$ and $g\underline{\triangle}[f,F]$ only depend on $g\in\Pi_{1,0}X$.*

(2) *For $m = 1$ and $j = 0$ the elements $[f,F]\wedge g$ and $[f,F]\underline{\triangle}g$ only depend on $g\in\Pi_{1,0}Y$.*

(3) *For $m = 2$ and $j = 1$ the elements $g\wedge[f,F]$ and $g\underline{\triangle}[f,F]$ only depend on $[f,F]\in\Pi_{2,1}X$.*

(4) *For $n=2$ and $i=1$ the elements $[f,F]\wedge g$ and $[f,F]\underline{\triangle}g$ only depend on $[f,F]\in\Pi_{2,1}X$.*

Lemma 18.3 allows us to define the following operations when $m=j=1$ or $n=i=1$.

$$
\begin{aligned}
&\wedge,\underline{\triangle}\colon \Pi_{n,0}X\times\Big(\pi_{1,1}Y\times(\hat{\otimes}^2\mathbb{Z}[\Omega Y])\Big)\longrightarrow\Pi_{n+1,1}(X\wedge Y),\\
&\wedge,\underline{\triangle}\colon \Big(\pi_{1,1}X\times(\hat{\otimes}^2\mathbb{Z}[\Omega X])\Big)\times\Pi_{m,0}Y\longrightarrow\Pi_{1+m,1}(X\wedge Y),
\end{aligned}
\tag{18.4}
$$

by the formulas

$$
\begin{aligned}
g\wedge([f,F],a) &= g\wedge[f,F]+P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(\Delta(g)\otimes a),\\
g\underline{\triangle}([f,F],a) &= g\underline{\triangle}[f,F]+P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(\{g\}\otimes\{g\}\otimes a),\\
([f,F],a)\wedge g &= [f,F]\wedge g+P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(a\otimes\{g\}\otimes\{g\}),\\
([f,F],a)\underline{\triangle}g &= [f,F]\underline{\triangle}g+P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(a\otimes\Delta(g)).
\end{aligned}
$$

By using Lemma 18.2 one can check the following one.

Lemma 18.5. (1) *For $m=j=1$ the elements $g\wedge([f,F],a)$ and $g\underline{\triangle}([f,F],a)$ only depend on $([f,F],a)\in\Pi_{1,1}Y$*

(2) *For $n=i=1$ the elements $([f,F],a)\wedge g$ and $([f,F],a)\underline{\triangle}g$ only depend on $([f,F],a)\in\Pi_{1,1}X$.*

Moreover, one can extend all properties (1)–(6) and (9)–(11) in Lemma 18.2 to the operations in (18.4). The explicit statement is left to the reader.

In order to prove the following Lemma one uses Lemma 13.2 (11) and (12). We leave the details as an exercise.

Lemma 18.6. *The following associativity rules for the operations (18.1) hold.*

(1) $g\wedge(g'\wedge[f,F])=(g\#g')\wedge[f,F]$,
(2) $g\wedge([f,F]\wedge g')=(g\wedge[f,F])\wedge g'$,
(3) $[f,F]\wedge(g\#g')=([f,F]\wedge g)\wedge g'$,
(4) $g\underline{\triangle}(g'\underline{\triangle}[f,F])=(g\underline{\#}g')\underline{\triangle}[f,F]$,
(5) $g\underline{\triangle}([f,F]\underline{\triangle}g')=(g\underline{\triangle}[f,F])\underline{\triangle}g'$,
(6) $[f,F]\underline{\triangle}(g\underline{\#}g')=([f,F]\underline{\triangle}g)\underline{\triangle}g'$.

One can accordingly obtain associativity properties involving also the operations in (18.4).

Recall that $\hat{\tau}_{n,m}\colon\tau_{n,m}\Rightarrow(\cdot)^{(-1)^{nm}}_{n+m}\in\mathrm{Sym}_\square(n+m)$ is a track from the shuffle permutation in (6.5) to its sign, and $\tau_\wedge$ is the symmetry isomorphism for the smash product of pointed spaces. In the following lemma we establish the commutativity rule for the operations (18.1). One can similarly state the commutativity rules for (18.4).

Lemma 18.7. *Given two pointed spaces X, Y, $[f,F]\in\Pi_{n,1}X$, and $g\in\Pi_{m,0}Y$ we have the equalities*

(1) $\langle g\underline{\triangle}\partial[f,F],\hat{\tau}_{n,m}\rangle=-(\tau_\wedge)_*([f,F]\wedge g)+((-1)^{nm})^*(g\underline{\triangle}[f,F])$,
(2) $\langle g\wedge\partial[f,F],\hat{\tau}_{n,m}\rangle=-(\tau_\wedge)_*([f,F]\underline{\triangle}g)+((-1)^{nm})^*(g\wedge[f,F])$.

Moreover if $g\in\Pi_{n,0}X$ and $[f,F]\in\Pi_{m,1}Y$

(3) $\langle\partial[f,F]\underline{\triangle}g,\hat{\tau}_{n,m}\rangle=-(\tau_\wedge)_*(g\wedge[f,F])+((-1)^{nm})^*([f,F]\underline{\triangle}g)$,
(4) $\langle\partial[f,F]\wedge g,\hat{\tau}_{n,m}\rangle=-(\tau_\wedge)_*(g\underline{\triangle}[f,F])+((-1)^{nm})^*([f,F]\wedge g)$.

Proof. Here we prove (1). The other equations are analogous.

The following diagram of pointed sets is commutative.

$$\begin{tikzcd}
\Omega^n X \wedge \Omega^m Y \arrow[rr, "\wedge"] \arrow[d, "\tau_\wedge"'] & & \Omega^{n+m}(X \wedge Y) \arrow[d, "\Omega^{n+m}\tau_\wedge"] \\
\Omega^m Y \wedge \Omega^n X \arrow[r, "\wedge"] & \Omega^{m+n}(Y \wedge X) \arrow[r, "\tau^*_{n,m}"] & \Omega^{n+m}(Y \wedge X)
\end{tikzcd}$$

Therefore using the definition of induced morphisms for secondary homotopy groups in [BM05a] 4.2 we have

$$\begin{aligned}
(\tau_\wedge)_*([f,F] \wedge g) &= [(\Sigma\Omega^{n+m}\tau_\wedge)(\Sigma\wedge)(f\#\bar{g}),\\
&\quad \tau_\wedge((F \wedge \bar{g}_{ev})\square(ev(\Sigma^{n+m}\wedge)\mathbb{H}^{\#}_{n,f,m,g}))]\\
&= [(\Sigma\tau^*_{n,m})(\Sigma\wedge)(\Sigma\tau_\wedge)(f\#\bar{g}),\\
&\quad (\tau_\wedge(F \wedge \bar{g}_{ev}))\square(\tau_\wedge ev(\Sigma^{n+m}\wedge)\mathbb{H}^{\#}_{n,f,m,g}))]\\
(3.4) \quad &= [(\Sigma\tau^*_{n,m})(\Sigma\wedge)(\bar{g}\underline{\#}f),\\
&\quad (\tau_\wedge(F \wedge \bar{g}_{ev}))\square(ev(\Sigma^{n+m}\wedge)(\tau_{n,m} \wedge \tau_\wedge)\mathbb{H}^{\#}_{n,f,m,g}))]\\
&= \text{(a)}.
\end{aligned}$$

On the other hand by using the definition of the $(\mathbb{Z},\cdot)$-action, see Definition 1.4, together with claim (*) in the proof of [BM05a] 4.9, if

$$Q\colon (\Sigma^{n+m}\wedge)((\cdot)^{(-1)^{nm}}_{n+m} \wedge \tau_\wedge)(\Sigma^{n+m-1}(f\#\bar{g})) \Rightarrow (\Sigma^{n+m}\wedge)(\Sigma^{n+m-1}(\bar{g}\underline{\#}f))(\cdot)^{(-1)^{nm}}_{n+m}$$

is a track with

$$\overline{Hopf}(Q) = -\bar{\sigma}\tau_\otimes \binom{(-1)^{nm}}{2} H(g\underline{\triangle}\partial[f,F])$$

then

$$\begin{aligned}
((-1)^{nm})^*(g\underline{\triangle}[f,F]) &= [((\cdot)^{(-1)^{nm}} \wedge \Omega^{m+n}(Y \wedge X))(\Sigma\wedge)(\bar{g}\underline{\#}f),\\
&\quad ((\bar{g}_{ev} \wedge F)(\cdot)^{(-1)^{nm}}_{m+n})\\
&\quad \square(ev(\Sigma^{m+n}\wedge)\mathbb{H}^{\underline{\#}}_{m,\bar{g},n,f}(\cdot)^{(-1)^{nm}}_{m+n})\square(ev\ Q)]\\
\text{(GR)} \quad &= [((\cdot)^{(-1)^{nm}} \wedge \Omega^{m+n}(Y \wedge X))(\Sigma\wedge)(\bar{g}\underline{\#}f),\\
&\quad ((\bar{g}_{ev} \wedge F)\tau_{n,m})\\
&\quad \square(ev(\Sigma^{m+n}\wedge)\mathbb{H}^{\underline{\#}}_{m,\bar{g},n,f}\hat{\tau}^{\boxminus}_{n,m})\square(ev\ Q)]\\
\text{(Theorem 16.11)} \quad &= [((\cdot)^{(-1)^{nm}} \wedge \Omega^{m+n}(Y \wedge X))(\Sigma\wedge)(\bar{g}\underline{\#}f),\\
&\quad (\tau_\wedge(F \wedge \bar{g}_{ev}))\square(ev(\Sigma^{n+m}\wedge)(\hat{\tau}^{\boxminus}_{n,m} \wedge \tau_\wedge)\mathbb{H}^{\#}_{n,f,m,\bar{g}})]\\
&= \text{(b)}.
\end{aligned}$$

Finally given a map $\varepsilon\colon S^1 \to S^1 \vee S^1$ with $(\pi_1\varepsilon)_{nil}(1) = -a + b \in \langle a, b\rangle_{nil}$ and a track $N\colon (1,1)(\Sigma^{n+m-1}\varepsilon) \Rightarrow 0$ with $\overline{Hopf}(N) = 0$

$$
\begin{aligned}
-(\mathrm{a}) + (\mathrm{b}) \quad = \quad & [((\Sigma\tau^*_{n,m})(\Sigma\wedge)(\bar g\underline{\#}f), ((\cdot)^{(-1)^{nm}} \wedge \Omega^{m+n}(Y\wedge X)(\Sigma\wedge)(\bar g\underline{\#}f))\varepsilon,\\
& (\tau_\wedge(F\wedge \bar g_{ev}, F\wedge \bar g_{ev})(\Sigma^{n+m}\varepsilon))\\
& \square(ev(\Sigma^{n+m}\wedge)((\tau_{n,m}\wedge\tau_\wedge)\mathbb{H}^{\#}_{n,f,m,g},\\
& (\hat\tau^{\boxminus}_{n,m}\wedge\tau_\wedge)\mathbb{H}^{\#}_{n,f,m,g})(\Sigma^{n+m}\varepsilon))]\\
= \quad & [((\Sigma\tau^*_{n,m})(\Sigma\wedge)(\bar g\underline{\#}f), ((\cdot)^{(-1)^{nm}} \wedge \Omega^{m+n}(Y\wedge X)(\Sigma\wedge)(\bar g\underline{\#}f))\varepsilon,\\
& (\tau_\wedge(F\wedge \bar g_{ev})(1,1)(\Sigma^{n+m}\varepsilon))\\
& \square(ev(\Sigma^{n+m}\wedge)(\hat\tau^{\boxminus}_{n,m}\wedge\tau_\wedge)\mathbb{H}^{\#}_{n,f,m,g})(1,1)(\Sigma^{n+m}\varepsilon))\\
& \square(ev(\Sigma^{n+m}\wedge)((\hat\tau_{n,m}\wedge\tau_\wedge)(\Sigma^{n+m-1}(f\#\bar g)),\\
& ((\cdot)^{(-1)^{nm}}_{n+m}\wedge\tau_\wedge)(\Sigma^{n+m-1}(f\#\bar g)))(\Sigma^{n+m}\varepsilon))]\\
(3.4) \quad = \quad & [((\Sigma\tau^*_{n,m})(\Sigma\wedge)(\bar g\underline{\#}f), ((\cdot)^{(-1)^{nm}} \wedge \Omega^{m+n}(Y\wedge X)(\Sigma\wedge)(\bar g\underline{\#}f))\varepsilon,\\
& (((\tau_\wedge(F\wedge \bar g_{ev}))\\
& \square(ev(\Sigma^{n+m}\wedge)(\hat\tau^{\boxminus}_{n,m}\wedge\tau_\wedge)\mathbb{H}^{\#}_{n,f,m,g}))(1,1)(\Sigma^{n+m}\varepsilon))\\
& \square(ev(\Sigma^{n+m}\wedge)((\hat\tau_{n,m}\wedge\Omega^mY\wedge\Omega^nX)(\Sigma^{n+m-1}(\bar g\underline{\#}f)),\\
& ((\cdot)^{(-1)^{nm}}_{n+m}\wedge\Omega^mY\wedge\Omega^nX)(\Sigma^{n+m-1}(\bar g\underline{\#}f)))(\Sigma^{n+m}\varepsilon))]\\
(\mathrm{GR}) \quad = \quad & [((\Sigma\tau^*_{n,m})(\Sigma\wedge)(\bar g\underline{\#}f), ((\cdot)^{(-1)^{nm}} \wedge \Omega^{m+n}(Y\wedge X))(\Sigma\wedge)(\bar g\underline{\#}f))\varepsilon,\\
& (((\tau_\wedge(F\wedge \bar g_{ev}))\\
& \square(ev(\Sigma^{n+m}\wedge)(\hat\tau^{\boxminus}_{n,m}\wedge\tau_\wedge)\mathbb{H}^{\#}_{n,f,m,g}))N)\\
& \square(ev(\Sigma^{n+m}\wedge)((\hat\tau_{n,m}\wedge\Omega^mY\wedge\Omega^nX)(\Sigma^{n+m-1}(\bar g\underline{\#}f)),\\
& ((\cdot)^{(-1)^{nm}}_{n+m}\wedge\Omega^mY\wedge\Omega^nX)(\Sigma^{n+m-1}(\bar g\underline{\#}f)))(\Sigma^{n+m}\varepsilon))]\\
(\mathrm{GR}), (3.4) \quad = \quad & [((\Sigma\tau^*_{n,m})(\Sigma\wedge)(\bar g\underline{\#}f), ((\cdot)^{(-1)^{nm}} \wedge \Omega^{m+n}(Y\wedge X))(\Sigma\wedge)(\bar g\underline{\#}f))\varepsilon,\\
& (ev(\Sigma^{n+m}\wedge)(S^{n+m-1}\wedge(\cdot)^{(-1)^{nm}}\wedge\Omega^mY\wedge\Omega^nX)\\
& (\Sigma^{n+m-1}(\bar g\underline{\#}f))N)\\
& \square(ev(\Sigma^{n+m}\wedge)((\hat\tau_{n,m}\wedge\Omega^mY\wedge\Omega^nX)(\Sigma^{n+m-1}(\bar g\underline{\#}f)),\\
& ((\cdot)^{(-1)^{nm}}_{n+m}\wedge\Omega^mY\wedge\Omega^nX)(\Sigma^{n+m-1}(\bar g\underline{\#}f)))(\Sigma^{n+m}\varepsilon))]\\
= \quad & \langle g\underline{\triangle}\partial[f,F],\hat\tau_{n,m}\rangle
\end{aligned}
$$

Here we use the definition of the bracket operation $\langle -,-\rangle$ in [BM05b] 4.5. $\square$

Finally we prove the compatibility of the smash product operation with the action of the symmetric track group.

Lemma 18.8. *Let $f \in \Pi_{n,0}X$ and $g \in \Pi_{m,0}Y$. Given $\hat\sigma \in \mathrm{Sym}_\square(m)$ and $\hat\gamma \in \mathrm{Sym}_\square(n)$ with $\delta(\hat\sigma) = \sigma$ and $\delta(\hat\gamma) = \gamma$ the following equalities hold.*

(1) $\langle f\wedge g, S^n\wedge\hat\sigma\rangle = f\wedge\langle g,\hat\sigma\rangle$
$+P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)(H(f)\otimes(-\sigma^*g + (\mathrm{sign}\,\sigma)^*g|\sigma^*g)_H)$,

(2) $\langle f\underline{\triangle}g, \hat\tau^{-1}_{n,m}(S^m\wedge\hat\gamma)\hat\tau_{n,m}\rangle = \langle f,\hat\gamma\rangle\underline{\triangle}g$
$+P(\otimes^2\mathbb{Z}[\wedge])(1\otimes\tau_\otimes\otimes 1)((-\sigma^*f + (\mathrm{sign}\,\sigma)^*f|\sigma^*f)_H\otimes H(g))$.

Proof. Equation (2) follows from (1), Lemma 18.7, and the laws of a sign group action. Let us prove (1). By Lemma 13.2 (1), (2), (5), (6), (8) and (9), given $\varepsilon\colon S^1 \to S^1 \vee S^1$ with $(\pi_1\varepsilon)_{nil}(1) = -a + b \in \langle a, b\rangle_{nil}$

$$\begin{array}{ll}\mathbb{H}^{\#}_{n,\bar f,m,(\Sigma\sigma^*,(\cdot)^{\operatorname{sign}\sigma}\wedge\Omega^m Y)(\bar g\vee\bar g)\varepsilon} & = \\ (\Sigma^{n+m}(\Omega^n X\wedge\sigma^*),(\cdot)^{\operatorname{sign}\sigma}_{n+m}\wedge\Omega^n X\wedge\Omega^m Y)\mathbb{H}^{\#}_{n,\bar f,m,(\bar g\vee\bar g)\varepsilon}, & \\ \mathbb{H}^{\#}_{n,\bar f,m,(\bar g\vee\bar g)\varepsilon} = (\mathbb{H}^{\#}_{n,\bar f,m,\bar g}\vee\mathbb{H}^{\#}_{n,\bar f,m,\bar g})(\Sigma^{n+m-1}\varepsilon). &\end{array}$$

Now one can use Theorem 16.10, Lemma 15.2 and the elementary properties of the Hopf invariant for tracks in [BM05a] 3.6 to check that any track Q from

$$(\Sigma^{n+m}(\Omega^n X\wedge\sigma^*),(\cdot)^{\operatorname{sign}\sigma}_{n+m}\wedge\Omega^n X\wedge\Omega^m Y)((\Sigma^{n+m-1}\bar f\#\bar g)\vee(\Sigma^{n+m-1}\bar f\#\bar g))(\Sigma^{n+m-1}\varepsilon)$$

to

$$\Sigma^{n+m-1}(\bar f\#((\Sigma\sigma^*,(\cdot)^{\operatorname{sign}\sigma}\wedge\Omega^m Y)(\bar g\vee\bar g)\varepsilon))$$

with

$$\overline{Hopf(Q)} = -\bar\sigma\tau_\otimes(1\otimes\tau_\otimes\otimes 1)(H(f)\otimes(-\sigma^* g + (\operatorname{sign}\sigma)^* g|\sigma^* g)_H),$$

satisfies

$$\begin{array}{l}\mathbb{H}^{\#}_{n,\bar f,m,(\Sigma\sigma^*,(\cdot)^{\operatorname{sign}\sigma}\wedge\Omega^m Y)(\bar g\vee\bar g)\varepsilon}\square Q \quad = \\ (\Sigma^{n+m}(\Omega^n X\wedge\sigma^*),(\cdot)^{\operatorname{sign}\sigma}_{n+m}\wedge\Omega^n X\wedge\Omega^m Y)(\mathbb{H}^{\#}_{n,\bar f,m,\bar g}\vee\mathbb{H}^{\#}_{n,\bar f,m,\bar g})(\Sigma^{n+m-1}\varepsilon).\end{array}$$

Then given $N\colon (1,1)(\Sigma^{m-1}\varepsilon)\Rightarrow 0$ with $\overline{Hopf}(N) = 0$, by using claim (*) in the proof of [BM05a] 4.9 and the definition of the bracket $\langle -,-\rangle$ in [BM05b] 4.5, the right hand side of equation (1) is

$$
\begin{aligned}
&[(\Sigma\wedge)(\Sigma(\Omega^nX\wedge\sigma^*),(\cdot)^{\mathrm{sign}\,\sigma}\wedge\Omega^nX\wedge\Omega^mY)((\bar f\#\bar g)\vee(\bar f\#\bar g))\varepsilon,\\
&(\bar f_{ev}\wedge(ev((\cdot)^{\mathrm{sign}\,\sigma}_n\wedge\Omega^mY)(\Sigma^{m-1}\bar g)N))\\
&\square(\bar f_{ev}\wedge(ev(\hat\sigma\wedge\Omega^mY,0^{\square}_{(\cdot)^{\mathrm{sign}\,\sigma}_n\wedge\Omega^mY})((\Sigma^{m-1}\bar g)\vee(\Sigma^{m-1}\bar g))(\Sigma^{m-1}\varepsilon)))\\
&\square(ev(\Sigma^{n+m}\wedge)(\Sigma^{n+m}(\Omega^nX\wedge\sigma^*),(\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY)\\
&(\mathbb{H}^{\#}_{n,\bar f,m,\bar g}\vee\mathbb{H}^{\#}_{n,\bar f,m,\bar g})(\Sigma^{n+m-1}\varepsilon))]=\\
&[(\Sigma\wedge)(\Sigma(\Omega^nX\wedge\sigma^*),(\cdot)^{\mathrm{sign}\,\sigma}\wedge\Omega^nX\wedge\Omega^mY)((\bar f\#\bar g)\vee(\bar f\#\bar g))\varepsilon,\\
&(ev(\Sigma^{n+m}\wedge)((\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY)(\Sigma^{n-1}\bar f\wedge\Sigma^{m-1}\bar g)(\Sigma^nN))\\
&\square(ev(\Sigma^{n+m}\wedge)(S^n\wedge\hat\sigma\wedge\Omega^nX\wedge\Omega^mY,0^{\square}_{(\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY})\\
&((\Sigma^{n-1}\bar f\wedge\Sigma^{m-1}\bar g)\vee(\Sigma^{n-1}\bar f\wedge\Sigma^{m-1}\bar g))(\Sigma^{n+m-1}\varepsilon))\\
&\square(ev(\Sigma^{n+m}\wedge)(\Sigma^{n+m}(\Omega^nX\wedge\sigma^*),(\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY)\\
&(\mathbb{H}^{\#}_{n,\bar f,m,\bar g}\vee\mathbb{H}^{\#}_{n,\bar f,m,\bar g})(\Sigma^{n+m-1}\varepsilon))]=\\
&[(\Sigma\wedge)(\Sigma(\Omega^nX\wedge\sigma^*),(\cdot)^{\mathrm{sign}\,\sigma}\wedge\Omega^nX\wedge\Omega^mY)((\bar f\#\bar g)\vee(\bar f\#\bar g))\varepsilon,\\
&(ev(\Sigma^{n+m}\wedge)((\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY)(\Sigma^{n-1}\bar f\wedge\Sigma^{m-1}\bar g)(\Sigma^nN))\\
&\square(ev(\Sigma^{n+m}\wedge)(S^n\wedge\hat\sigma\wedge\Omega^nX\wedge\Omega^mY,0^{\square}_{(\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY})\\
&(\mathbb{H}^{\#}_{n,\bar f,m,\bar g}\vee\mathbb{H}^{\#}_{n,\bar f,m,\bar g})(\Sigma^{n+m-1}\varepsilon))]=\\
&[(\Sigma\wedge)(\Sigma(\Omega^nX\wedge\sigma^*),(\cdot)^{\mathrm{sign}\,\sigma}\wedge\Omega^nX\wedge\Omega^mY)((\bar f\#\bar g)\vee(\bar f\#\bar g))\varepsilon,\\
&(ev(\Sigma^{n+m}\wedge)((\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY)(\mathbb{H}^{\#}_{n,\bar f,m,\bar g})(\Sigma^nN))\\
&\square(ev(\Sigma^{n+m}\wedge)(S^n\wedge\hat\sigma\wedge\Omega^nX\wedge\Omega^mY,0^{\square}_{(\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY})\\
&((\Sigma^{n+m-1}(\bar f\#\bar g))\vee(\Sigma^{n+m-1}(\bar f\#\bar g)))(\Sigma^{n+m-1}\varepsilon))]=\\
&[(\Sigma\wedge)(\Sigma(\Omega^nX\wedge\sigma^*),(\cdot)^{\mathrm{sign}\,\sigma}\wedge\Omega^nX\wedge\Omega^mY)((\bar f\#\bar g)\vee(\bar f\#\bar g))\varepsilon,\\
&(ev(\Sigma^{n+m}\wedge)((\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY)(\Sigma^{n+m-1}(\bar f\#\bar g))(\Sigma^nN))\\
&\square(ev(\Sigma^{n+m}\wedge)(S^n\wedge\hat\sigma\wedge\Omega^nX\wedge\Omega^mY,0^{\square}_{(\cdot)^{\mathrm{sign}\,\sigma}_{n+m}\wedge\Omega^nX\wedge\Omega^mY})\\
&((\Sigma^{n+m-1}(\bar f\#\bar g))\vee(\Sigma^{n+m-1}(\bar f\#\bar g)))(\Sigma^{n+m-1}\varepsilon))]=\\
&\langle f\wedge g,S^n\wedge\hat\sigma\rangle
\end{aligned}
$$

Here we essentially use the golden rule, concretely for the fourth equation. □

19. The smash product in dimension 0

The first lemma in this section implies that the smash product operation in Definition 14.3 is well defined when dimension 0 is involved and Theorems 7.1, 7.2, 7.3, and 8.13 are satisfied also in this case.

Recall that **qpm** is a track category. Therefore the morphism set $\mathrm{Hom}_{\mathbf{qpm}}(C,D)$ in **qpm** is indeed a groupoid. This groupoid is pointed by the zero morphism. If $G_\square$ is a sign group acting on C and D we can consider the full pointed subgroupoid of $G_\square$-equivariant morphisms

$$\mathrm{Hom}_{G_\square}(C,D)\subset\mathrm{Hom}_{\mathbf{qpm}}(C,D).$$

If $G_\square$ is the trivial sign group then this inclusion is always an equality.

A pointed groupoid $\mathbf{G}$ gives rise to a stable quadratic module $\mathsf{Ad}_3\mathsf{Ad}_2\mathsf{Ad}_1\mathbf{G}$, compare Remark 6.13. The low-dimensional group of this stable quadratic module is the free group of nilpotency class 2 on the pointed set of objects, therefore if we define H as in (3.6) we obtain a quadratic pair module $\tilde{\mathbf{G}}$ which corresponds to $\mathsf{Ad}_3\mathsf{Ad}_2\mathsf{Ad}_1\mathbf{G}$ by the forgetful functor in Remark 1.5. Compare [BM05b] 1.15.

Lemma 19.1. *Let C, D be quadratic pair modules endowed with an action of the sign group $G_\square$, let $\mathbf{G}$ be a groupoid, and let*

$$\varphi\colon \mathbf{G} \longrightarrow \mathrm{Hom}_{G_\square}(C, D)$$

be a pointed groupoid morphism. Suppose that $(-|-)_H\colon \otimes^2 (C_0)_{ab} \cong C_{ee}$ is an isomorphism. Then there is a well-defined quadratic pair module morphism

$$\tilde{\varphi}\colon \tilde{\mathbf{G}} \odot C \longrightarrow D$$

given by

$$\tilde{\varphi}_i(g \circledcirc x) = \varphi(g)(x),$$

where g is an object in $\mathbf{G}$ and $x \in C_i$ for some $i \in \{0,1\}$, or g is a morphism in $\mathbf{G}$, $x \in C_0$ and $i = 1$; and by

$$\tilde{\varphi}_{ee}((g|g')_H \otimes (x|x')_H) = (\varphi(g)(x)|\varphi(g')(x'))_H,$$

for g, g' objects in $\mathbf{G}$ and $x, x' \in C_0$. This morphism is $G_\square$-equivariant with respect to the action of $G_\square$ on the tensor product given by Lemma 8.9

The proof of the lemma is technical but straightforward. The reader can also check that he construction is natural in $\mathbf{G}$, $G_\square$, C and D.

Finally we show how Lemma 5.3 can be used to obtain the tracks in **qpm** induced by the additive secondary homotopy groups from the smash product operation. For this we notice that there is a unique morphism in **qpm** from the interval quadratic pair module $\mathbb{I}$ in Section 5 to $\Pi_{0,*}$ of the interval I_+,

$$v\colon \mathbb{I} \longrightarrow \Pi_{0,*}I_+,$$

sending $i_k \in \mathbb{I}_0$, $k = 0, 1$, to the map $i_k\colon S^0 \to I_+ \in \Pi_{0,0}I_+$ corresponding to the inclusion of $k \in I = [0, 1]$.

Lemma 19.2. *Given a track $F\colon f \Rightarrow g$ between maps $f, g\colon X \to Y$ represented by a homotopy $F\colon I_+ \wedge X \to Y$ the composite*

$$\mathbb{I} \odot \Pi_{n,*}X \xrightarrow{v \odot 1} \Pi_{0,*}I_+ \odot \Pi_{n,*}X \xrightarrow{\wedge} \Pi_{n,*}(I_+ \wedge X) \xrightarrow{\Pi_{n,*}F} \Pi_{n,*}Y$$

corresponds by Lemma 5.3 to the track $\Pi_{n,*}F\colon \Pi_{n,*}f \Rightarrow \Pi_{n,*}g$ in **qpm**, *$n \geq 0$.*

This lemma follows easily from the definition of the smash product operation for secondary homotopy groups.

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