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by

Boris Shoikhet

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Boris Shoikhet

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

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Monoidal cofibrant resolutions of dg algebras

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Abstract. Let k be a field of any characteristic. In this paper, we construct a functorial cofibrant resolution $\mathfrak{R}(A)$ for the $\mathbb{Z}_{\le 0}$-graded dg algebras A over k, such that the functor $A \rightsquigarrow \mathfrak{R}(A)$ is colax-monoidal with quasi-isomorphisms as the colax maps. More precisely, there are maps of bifunctors $\mathfrak{R}(A\otimes B)\to \mathfrak{R}(A)\otimes\mathfrak{R}(B)$, compatible with the projections to $A\otimes B$, and obeying the colax-monoidal axiom.
The main application of such resolution (which we consider in our next paper) is the existence of a colax-monoidal dg localization of pre-triangulated dg categories, such that the localization is a genuine dg category (not just an object of the homotopy category of dg categories), whose image in the homotopy category of dg categories is equivalent to the Toën's localization.

Introduction

0.1 The Main Theorem

Let k be a field of any characteristic.

It is known that any associative algebra A over k admits a *free resolution*, that is, a free associative dg algebra $\mathfrak{R}(A)$ endowed with a quasi-isomorphism of algebras $p_A\colon \mathfrak{R}(A)\to A$. Here we specify this claim as follows. Having two associative algebras A and B over k, $A\otimes B$ is an associative algebra again. We want to choose a functorial free resolution $\mathfrak{R}(A)$ for all algebras A over k, such that $\mathfrak{R}(A\otimes B)$ and $\mathfrak{R}(A)\otimes\mathfrak{R}(B)$ are related by a functorial quasi-isomorphism, defined over $A\otimes B$.

More precisely, we want to find a functor $A\rightsquigarrow\mathfrak{R}(A)$ with $\mathfrak{R}(A)$ a free associative dg algebra, such that there exist a morphism of bifunctors

$$\beta\colon \mathfrak{R}(A\otimes B)\to\mathfrak{R}(A)\otimes\mathfrak{R}(B)$$

such that for any two algebras A,B the map $\beta(A,B)$ is a *quasi-isomorphism*, the diagram

$$\begin{tikzcd}
\mathfrak{R}(A\otimes B) \arrow[rr, "\beta"] \arrow[dr, "p_{A\otimes B}"'] & & \mathfrak{R}(A)\otimes\mathfrak{R}(B) \arrow[dl, "p_A\otimes p_B"] \\
& A\otimes B &
\end{tikzcd} \tag{0.1}$$

is commuative, and such that for any three algebras A, B, C, the diagram

$$\begin{tikzcd} \mathfrak{R}(A\otimes B\otimes C) \arrow[r] \arrow[d] & \mathfrak{R}(A\otimes B)\otimes\mathfrak{R}(C) \arrow[d] \\ \mathfrak{R}(A)\otimes\mathfrak{R}(B\otimes C) \arrow[r] & \mathfrak{R}(A)\otimes\mathfrak{R}(B)\otimes\mathfrak{R}(C) \end{tikzcd} \tag{0.2}$$

is commutative. A functor $F\colon \mathcal{M}_1 \to \mathcal{M}_2$ between two monoidal categories, with a map of bifunctors $\beta\colon F(A\otimes B) \to F(A)\otimes F(B)$ for which the diagram (0.2) commutes, is called *colax-monoidal* (see Definition 5.1 below for the precise definition of a colax-monoidal functor).

We explain below in this Introduction (see Section 0.2 below) why the existence of such resolutions $\mathfrak{R}(A)$ is important for applications.

In fact, in this paper we allow for algebras themselves $A, B, \dots$ to be dg algebras (not just usual algebras, concentrated in degree 0). We use the technique of *closed model categories*, introduced and developed by Quillen in [Q].

Roughly speaking, the theory of closed model categories is a homological algebra in non-abelian (an non-additive) setting. A closed model category is a category $\mathcal{C}$ with some three classes of morphisms, called *weak equivalences, fibrations, and cofibrations.* The class of weak equivalences is the most essential. In abelian setting, the weak equivalences of complexes are the quasi-isomorphisms of complexes. The goal of the theory is to construct the *localization* (or homotopy category) $\mathrm{Ho}\mathcal{C} = \mathcal{C}[\mathcal{W}^{-1}]$ by the class of weak equivalences, and for some special "right or left exact" functors $F\colon \mathcal{C} \to \mathcal{D}$, to construct its *total derived functor* $\mathbb{F}\colon \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$.

In this paper, we assume the reader has some familiarity with the foundations of closed model categories. We use contemporary texts [Hir], [GS], and [DS] as references on closed model categories.

The closed model structure on the category of associative dg algebras $\mathcal{A}lg_k$ over a field k of any characteristic (and as well, for the category of dg algebras over any operad, over a field of characteristic 0) was constructed by Hinich [Hi]. For this structure, the weak equivalences are the quasi-isomorphisms of dg algebras, the fibrations are the component-wise surjective maps of dg algebras, and the cofibrations are uniquely defined from the weak equivalences and the fibrations, by the *left lifting property.*

Let us recall an explicit description of the cofibrant objects.

A dg associative algebra A is called *standard cofibrant* if there is an increasing filtration on A, $A = \mathrm{colim}\ A_i$,

$$A_0 \subset A_1 \subset A_2 \subset \dots$$

such that the underlying graded algebra A_i is obtained from A_{i-1} by adding the free generators graded space V_i,

$$A_i = A_{i-1} * T(V_i)$$

such that

$$d(V_i) \subset A_{i-1}$$

for $i \geq 1$, and A_0 has zero differential. Here $*$ is the free product of algebras, which is the same as the categorical coproduct of algebras, and $T(V_i)$ is the free graded algebra on generators V_i.

The description is: *any cofibrant dg associative algebra is a retract of a standard cofibrant dg algebra.*

As follows from this description, for $\mathbb{Z}_{\leq 0}$-graded dg algebras, the concepts of free dg algebras and of cofibrant dg algebras *coincide*. For $\mathbb{Z}$-graded dg algebras, the two concepts are not the same: for a cofibrant $\mathbb{Z}$-graded dg algebra, the underlying graded algebra is free, but the differential has special "triangular" form.

For the description of the closed model structure on the category of simplicial associative algebras, see (1.9) below.

Denote the category of $\mathbb{Z}_{\leq 0}$-graded associative algebras by $\mathcal{A}lg_k^{\leq 0}$.

Our main result is:

THEOREM 0.1. *Let k be a field of any characteristic. There is a functor $\mathfrak{R}\colon \mathcal{A}lg_k^{\leq 0} \to \mathcal{A}lg_k^{\leq 0}$ and a morphism of functors $w\colon \mathfrak{R} \to Id$ with the following properties:*

1. *$\mathfrak{R}(A)$ is cofibrant, and $w\colon \mathfrak{R}(A) \to A$ is a quasi-isomorphism, for any $A \in \mathcal{A}lg_k^{\leq 0}$,*

2. *there is a colax-monoidal structure on the functor $\mathfrak{R}$, such that all colax-maps $\beta_{A,B}\colon \mathfrak{R}(A \otimes B) \to \mathfrak{R}(A) \otimes \mathfrak{R}(B)$ are quasi-isomorphisms of dg algebras, and such that the diagram*

$$\begin{tikzcd} \mathfrak{R}(A\otimes B) \arrow[rr, "\beta_{A,B}"] \arrow[dr] & & \mathfrak{R}(A)\otimes\mathfrak{R}(B) \arrow[dl] \\ & A\otimes B & \end{tikzcd}$$

is commutative,

3. *the morphism $w(k^\bullet)\colon \mathfrak{R}(k^\bullet) \to k^\bullet$ coincides with $\alpha\colon \mathfrak{R}(k^\bullet) \to k^\bullet$, where α is a part of the colax-monoidal structure* (see Definition 5.1)*, and $k^\bullet = 1_{\mathcal{A}lg_k^{\leq 0}}$ is the dg algebra equal to the one-dimensional k-algebra in degree 0, and vanishing in other degrees.*

Note once again, that in the context of this Theorem, the words "cofibrant" and "free" are synonymous.

0.2 APPLICATIONS

Here we explain why Theorem 0.1 is interesting. The results mentioned here will be proven in our sequel paper(s).

Some of our intentions is to use the ideas of [KT] in the differential graded context. Kock and Toën (loc.cit.) work with simplicial algebras. They emphasize that their methods may not work for *non-cartesian monoidal categories.* The matter is that the standard definition of weak

Segal monoids, they essentially use, assumes that the monoidal product is isomorphic to the cartesian product.

There is, however, a definition due to Leinster [Le], of weak Segal monoids in non-cartesian monoidal categories. (Note that the category of vector spaces, the category of complexes of vector spaces, etc. are non-cartesian monoidal, as the monoidal product is the tensor product and the cartesian product is the direct sum).

Working with the Leinster's definition, one needs to know a finer monoidal property of the localization (see Theorem 0.3 below) than the one used in [KT].

Let $\mathcal{C}at^{\mathrm{dg}}_{\mathbb{U}}$ denotes the category of $\mathbb{U}$-small dg categories over a field k, let $\mathcal{CC}at^{\mathrm{dg}}_{\mathbb{U}}$ denotes the category of *colored* $\mathbb{U}$-small dg categories over k, and let, finally, $\mathcal{C}at^{\mathrm{pre-tr}}_{\mathbb{U}}$ (resp., $\mathcal{CC}at^{\mathrm{pre-tr}}_{\mathbb{U}}$) denotes the category of $\mathbb{U}$-small pre-triangulated dg categories (resp., the category of colored $\mathbb{U}$-small pre-triangulated dg categories) over k. The either of these three categories is symmetric monoidal, with $\otimes_k$ as the monoidal product.

Tabuada [Tab] constructed a closed model structure on the category of $\mathbb{U}$-small dg categories; this structure is assumed in the two following statements.

Theorem 0.2 (to be proven later). *Let k be a field of any characteristic. There is a functor $\mathfrak{R}\colon \mathcal{C}at^{\mathrm{pre-tr}}_{\mathbb{U}} \to \mathcal{C}^{\mathrm{dg}}_{\mathbb{U}}$ and a morphism of functors $w\colon \mathfrak{R} \to Id$ with the following properties:*

1. *$\mathfrak{R}(C)$ is cofibrant, and $w\colon \mathfrak{R}(C) \to C$ is a weak equivalence, for any $C \in \mathcal{C}at^{\mathrm{pre-tr}}_{\mathbb{U}}$,*

2. *there is a colax-monoidal structure on the functor $\mathfrak{R}$, such that all colax-maps $\beta_{C,D}\colon \mathfrak{R}(C\otimes D) \to \mathfrak{R}(C)\otimes\mathfrak{R}(D)$ are weak equivalences of dg categories, and such that the diagram*

$$\begin{tikzcd}
\mathfrak{R}(C\otimes D) \arrow[rr, "\beta_{C,D}"] \arrow[dr] & & \mathfrak{R}(C)\otimes\mathfrak{R}(D) \arrow[dl] \\
& C\otimes D &
\end{tikzcd}$$

is commutative.

To prove Theorem 0.2, we firstly extend our Main Theorem 0.1 from $\mathbb{Z}_{\le 0}$-graded dg algebras to $\mathbb{Z}_{\le 0}$-graded dg categories, this is rather straightforward. In the next step, we extend the result from $\mathbb{Z}_{\le 0}$-graded dg categories to pre-triangulated dg categories. It seems that the analogous statement is not true for general $\mathbb{Z}$-graded dg categories (not the pre-triangulated ones).

Theorem 0.3 (to be proven later). *There is a localization functor*

$$\mathbb{L}\colon \mathcal{CC}at^{\mathrm{pre-tr}}_{\mathbb{U}} \to \mathcal{C}at^{\mathrm{dg}}_{[\mathbb{U}]}$$

from colored pre-triangulated $\mathbb{U}$-small dg categories to dg $\mathbb{U}$-categories, with the following properties:

(i) $\mathbb{L}$ *is colax-monoidal, with weak equivalences of dg categories as the colax maps* $\beta(C, D)$*,*

(ii) $\beta(C, D)$ *is defined over* $C \otimes D$*, for any two pre-triangulated dg categories* C, D*,*

(iii) the image of $\mathbb{L}(C, S)$ *under the projection to the homotopy category* $\mathrm{Ho}\mathcal{C}at^{\mathrm{dg}}_{\mathbb{U}}$ *coincides with the Toën dg localization [To], Sect.8.2.*

Toën defines a localization of the dg categories with nice homotopy properties (see [To], Corollary 8.7) as the homotopy colimit of some push-out-angle diagram. It is possible to compute this homotopy colimit as the genuine colimit of a cofibrant replacement of the initial diagram. We use the cofibrant resolution $\mathfrak{R}(C)$, given in Theorem 0.2, for this replacement. (In fact, the push-out-angle diagrams and the pull-back-angle diagrams in a closed model category admit closed model structures, with component-wise weak equivalences as the weak equivalences, see [DS], Section 10).

We use Theorem 0.3 for a conceptual proof of the Deligne conjecture for abelian n-fold monoidal categories, as well as for proofs of other results in deformation theory. The details will appear elsewhere.

0.3 Plan of the paper

We start with proving a direct analogue of Theorem 0.1 for simplicial associative algebras over k, instead of $\mathbb{Z}_{\le 0}$-graded dg associative algebras (see Theorem 1.1 below). It turns that the Theorem becomes much simpler in the simplicial setting. We solve it by an explicit construction, which reminiscences the construction used by Dwyer-Kan in their first paper [DK1] on simplicial localization. This is done in Section 1.

The rough idea of the remaining Sections is to "transfer" the solution of Theorem 1.1 for simplicial algebras to a solution to the Main Theorem 0.1, using the Dold-Kan correspondence and passing to the categories of monoids. One should say, it is not very straightforward. The main techniques we use are, along with some well-known results on closed model categories, at first, some results on weak monoidal Quillen pairs, due to Schwede and Shipley [SchS03], and, at second, a result that the *bialgebra axiom* is fulfilled in the context of the Dold-Kan correspondence (proven independently in [Sh2] and, before, in [AM]).

Sections 2 and 3 are preparatory to the culminating Section 4, where the Main Theorem is proven (as Theorem 4.1). In Section 2 we recall those results on monoids, due to Schwede-Shipley [SchS03], which do not deal with homotopy theory, and discuss the role of the bialgebra axiom, following [Sh2]. In Section 3 we recall the Dold-Kan correspondence, emphasizing on its monoidal properties.

In Section 4 all different parts of the game play together, and lead us to the proof of the Main Theorem.

In Section 5 we collect the definitions expressed in some commutative diagrams, these are the precise definitions of a lax-monoidal and of a colax-monoidal functors, and formulate the bialgebra axiom. This Section is included to the paper to ease the reader's reference.

0.4 Notations

Throughout the paper, k denotes a field of any characteristic. An "algebra" always means an "associative algebra". All our algebras have unit.

All differentials in this paper have degree $+1$, as is common in the algebraic literature.

Let Δ be the category whose objects are $[0]$, $[1]$, $[2]$, $[3]$, and so on, where $[n]$ denotes the completely ordered sets with $n+1$ elements $0 < 1 < 2 < \dots < n$. A morphism $f\colon [m] \to [n]$ is any map obeying $f(i) \le f(j)$ when $i \le j$. A *simplicial* object in a category $\mathcal{C}$ is a functor $\Delta^{\mathrm{opp}} \to \mathcal{C}$, and a *cosimplicial* object in $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$. We denote by $\mathcal{C}^{\Delta}$ the category of simplicial objects in $\mathcal{C}$ and by $\mathcal{C}^{\Delta^{\mathrm{opp}}}$ the category of cosimplicial objects in $\mathcal{C}$. This notation is indeed confusing, but seemingly it is traditional now.

All categories we consider in this paper are small for some universe. We do not meet here any set-theoretical troubles related with the localization of categories, and we ignore the adjective "small" in all our formulations (except Section 0.2).

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1 The case of simplicial algebras

The category $\mathcal{A}lg_k^{\Delta}$ of simplicial algebras over field k is monoidal, with degree-wise $\otimes_k$ as the monoidal structure, and it admits a closed model structure. We recall this closed model structure in Section 1.2 below. We refer to this closed model structure in the following result.

Theorem 1.1 (Main Theorem for simplicial algebras). *Let k be a field of any characteristic. There is a functor $\mathfrak{R}\colon \mathcal{A}lg_k^{\Delta} \to \mathcal{A}lg_k^{\Delta}$ and a morphism of functors $w\colon \mathfrak{R} \to Id$ with the following properties:*

1. *$\mathfrak{R}(A)$ is cofibrant, and $w\colon \mathfrak{R}(A) \to A$ is a weak equivalence, for any $A \in \mathcal{A}lg_k^{\Delta}$,*

2. *there is a colax-monoidal structure on the functor $\mathfrak{R}$, such that all colax-maps $\beta_{A,B}\colon \mathfrak{R}(A\otimes B)\to\mathfrak{R}(A)\otimes\mathfrak{R}(B)$ are weak equivalences of simplicial algebras, and such that the diagram*

$$\begin{tikzcd} \mathfrak{R}(A\otimes B) \arrow[rr, "\beta_{A,B}"] \arrow[dr] & & \mathfrak{R}(A)\otimes\mathfrak{R}(B) \arrow[dl] \\ & A\otimes B & \end{tikzcd}$$

is commutative,

3. *the morphism $w(k_\bullet)\colon \mathfrak{R}(k_\bullet)\to k_\bullet$ coincides with $\alpha\colon R(k_\bullet)\to k_\bullet$, where α is a part of the colax-monoidal structure* (see Definition 5.1)*, and $k_\bullet = 1_{\mathcal{A}lg_k^\Delta}$ is the simplicial algebra equal to the one-dimensional k-algebra k in each degree.*

1.1 The construction

The idea is very easy. Let $A_\bullet$ be a simplicial algebra. There is the forgetful functor $\mathcal{A}lg_k^\Delta\to\mathcal{V}ect_k^\Delta$ to simplicial vector spaces, having a left adjoint functor of "free objects". This is the functor $A_\bullet\rightsquigarrow(TA)_\bullet$, with

$$(TA)_k = T(A_k) \tag{1.1}$$

where $T(A_k)$ is the free (tensor) algebra. We consider the cotriple, associated with the pair of adjoint functors (L is the left adjoint to R)

$$L\colon \mathcal{V}ect_k^\Delta \rightleftarrows \mathcal{A}lg_k^\Delta\colon R$$

(see [W], Section 8.6). Explicitly,

$$T = L\circ R \tag{1.2}$$

This implies that there are maps of functors $\epsilon\colon T\to\mathrm{id}$ and $\delta\colon T\to T^2$ obeying the cotriple axioms. These axioms guarantee that the following collection of algebras $(FA)_k$, $k\ge 0$, has a natural structure of a simplicial algebra $(FA)_\bullet$:

$$(FA)_k = T^{\circ(k+1)}A_k$$

(there is the $(k+1)$-st iterated tensor power in the r.h.s.), such that the natural map $FA_\bullet\to A_\bullet$, $T^{\circ(k+1)}A_k\xrightarrow{\epsilon^{k+1}}A_k$, is a weak equivalence of simplicial algebras.

Explicitly, having such a functor T with maps of functors $\epsilon\colon T\to\mathrm{id}$ and $\delta\colon T\to T^2$, the simplicial structure on $(FA)_\bullet$ is defined as follows.

When $A_k = A$ for any k, ($A_\bullet$ is a constant simplicial algebra), the formulas for simplicial algebra structure on $(FA)_\bullet$ are:

$$\begin{aligned} d_i &= T^{\circ i}\epsilon T^{\circ(n-i)}\colon T^{\circ(n+1)}A\to T^{\circ n}A \\ s_i &= T^{\circ i}\delta T^{\circ(n-i)}\colon T^{\circ(n+1)}A\to T^{\circ(n+2)}A \end{aligned} \tag{1.3}$$

The cotriple axioms then guarantee the simplicial identities (see [W], Section 8.6.4 for detail).

In general case, when $A_\bullet$ is not constant, the simplicial algebra $(FA)_\bullet$ is defined as the diagonal of the bisimplicial set

$$(FA)_\bullet = \mathrm{diag}((FA_\bullet)_\bullet) \tag{1.4}$$

For two simplicial algebras $A_\bullet$ and $B_\bullet$, there is a canonical embedding

$$\beta_{A,B} \colon F(A \otimes B)_\bullet \to (FA)_\bullet \otimes (FB)_\bullet$$

defined *on the level of algebras* by iterations of the map

$$\begin{gathered} \alpha \colon T(A \otimes B) \to T(A) \otimes T(B) \\ (a_1 \otimes b_1) \otimes \cdots \otimes (a_k \otimes b_k) \xrightarrow{\alpha} (a_1 \otimes \cdots \otimes a_k) \otimes (b_1 \otimes \cdots \otimes b_k) \end{gathered} \tag{1.5}$$

The component $(\beta_{A,B})_k \colon (T^{\circ(k+1)} A_k) \otimes (T^{\circ(k+1)} B_k) \to T^{\circ(k+1)}(A_k \otimes B_k)$ is defined as the iterated power $\alpha^{\circ(k+1)}$.

LEMMA 1.2. *The collection of maps $\{\beta_\ell\}$, $\ell \geq 0$, defines a map of* simplicial algebras

$$\beta \colon (F(A \otimes B))_\bullet \to (FA)_\bullet \otimes (FB)_\bullet$$

Proof. Denote the product(s) in A_k by $\star$, the product in $T(V)$ by $\otimes$, and the product in $T(T(V))$ by $\bigotimes$. Then the formulas for $\epsilon \colon T \to \mathrm{id}$ and $\delta \colon T \to T^2$ are as follows:

$$\begin{aligned} \epsilon(a_1 \otimes \cdots \otimes a_k) &= a_1 \star \cdots \star a_k \\ \delta(a_1 \otimes \cdots \otimes a_k) &= a_1 \bigotimes \cdots \bigotimes a_k \end{aligned} \tag{1.6}$$

The statement of Lemma now follows directly from formulas (1.3), expressing the simplicial faces and degeneracies maps in ϵ and δ. $\square$

Our goal in this Subsection is to prove that $R(A_\bullet) = (FA)_\bullet$ solves Theorem 1.1. We need to prove

LEMMA 1.3. 1. *For any simplicial algebra $A_\bullet$, the simplicial algebra $FA_\bullet$ is cofibrant in the closed model structure on $\mathcal{A}lg_k^\Delta$,*

2. *the map $\beta_{A,B} \colon (F(A \otimes B))_\bullet \to (FA)_\bullet \otimes (FB)_\bullet$ is a weak equivalence.*

Before proving the above Lemma, we need to remind some results concerning the closed model structure on the category $\mathcal{A}lg_k^\Delta$, which goes back to Quillen [Q], Section 4.3.

1.2 The closed model category of simplicial algebras

Firstly recall the model structure on the category $\mathcal{V}ect_k^\Delta$ of simplicial vector spaces over field k.

$$\begin{aligned}
&\text{(i) A map } f\colon X \to Y \text{ in } \mathcal{V}ect_k^\Delta \text{ is a weak equivalence if it induces an isomorphism on homotopy groups } \pi_\bullet(X) \to \pi_\bullet(Y),\\
&\text{(ii) a map } f\colon X \to Y \text{ is a fibration if it induces a surjection } \pi_\bullet X \to \pi_0(X) \times_{\pi_0(Y)} \pi_\bullet(Y).\\
&\text{(iii) A map } f\colon X \to Y \text{ in } \mathcal{V}ect_k^\Delta \text{ is a cofibration if it has a form}
\end{aligned}\tag{1.7}$$

(i) A map $f\colon X \to Y$ in $\mathcal{V}ect_k^\Delta$ is a weak equivalence if it induces an isomorphism on homotopy groups $\pi_\bullet(X) \to \pi_\bullet(Y)$,

(ii) a map $f\colon X \to Y$ is a fibration if it induces a surjection $\pi_\bullet X \to \pi_0(X) \times_{\pi_0(Y)} \pi_\bullet(Y)$.

(iii) A map $f\colon X \to Y$ in $\mathcal{V}ect_k^\Delta$ is a cofibration if it has a form (1.7)

$$X_n \to Y_n = X_n \oplus V_n$$

for some collection of vector spaces $\{V_0, V_1, V_2, \dots\}$, such that each simplicial *degeneracy map* $s_i\colon [n+1] \to [n]$ maps V_n to V_{n+1}, $n \geq 0$.

The model category $\mathcal{V}ect_k^\Delta$ described above is *cofibrantly generated* (see [GS], Section 3, for a beautiful short survey of cofibrantly generated model categories). Recall that it means, in particular, that there are given sets I of generating cofibrations, and J of generating acyclic cofibrations, subject to the following two properties:

1. the source of any morphism in I obeys *the Quillen's small object argument* to the category of all cofibrations; the source of any morphism in J obeys the small object argument to the category of all acyclic cofibrations;

2. a morphism is a fibration if and only if it satisfies the left lifting property with respect to any morphism in J; a morphism is an acyclic fibration if and only if it satisfies the left lifting property with respect to any morphism in I.

Concerning *the Quillen's small object argument*, see [GS], Section 3.1, or [Hir], Section 10.5, for thorough treatment. The meaning of these two conditions is that they make possible to prove the last axiom (CM5) of a closed model category axioms, which is in a sense the hardest one (see loc.cit.).

See [GS], Examples 3.4, for explicit description of the sets I and J in the category $\mathcal{V}ect_k^\Delta$.

There is a pair of adjoint functors

$$L\colon \mathcal{V}ect_k^\Delta \rightleftarrows \mathcal{A}lg_k^\Delta\colon R \tag{1.8}$$

As the left-hand side category is a cofibrantly generated model category, the model structure can be "transferred" to the right-hand-side category, and this model category is again cofibrantly generated. This transfer principle is explained in [GS], Theorem 3.6, and [Hir], Theorem 11.3.2. As is explained in [GS], Sections 3,4, the assumptions of Theorem 3.6 are satisfied in (1.8).

In the situation when assumptions of Theorem 3.6 of [GS] are fulfilled, the sets $L(I)$ and $L(J)$ are generating cofibrations (resp., generating acyclic cofibrations) for the category in the right-hand side.

The obtained closed model structure on $\mathcal{A}lg_k^\Delta$ has the following explicit description, see [GS], Section 4.3.

(i) a map $f\colon X \to Y$ is a weak equivalence if $\pi_* f\colon \pi_* X \to \pi_* Y$ is an isomorphism,

(ii) a map $f\colon X \to Y$ is a fibration if the induced map $X \to \pi_0 X \times_{\pi_0 Y} Y$ is a surjection.

(iii) a map $f\colon X \to Y$ is a cofibration in $\mathcal{A}lg_k^\Delta$ if it is a retract of the following *free map*: (1.9)

$$X_n \to Y_n = X_n \sqcup T(V_n)$$

as algebras, for some collection $\{V_0, V_1, V_2, \dots\}$ of vector spaces, such that all degeneracy maps $s_i\colon [n+1] \to [n]$ maps V_n to V_{n+1}, $n \geq 0$.

See [Q], Section 4.3 and [GS], Proposition 4.21 for a proof.

1.3 Proof of Theorem 1.1

Firstly we prove

LEMMA 1.4. *For any simplicial algebra $A_\bullet$, the simplicial algebra $(FA)_\bullet$ is cofibrant, and the projection $p\colon (FA)_\bullet \to A_\bullet$ is an acyclic fibration.*

Proof. We need to find vector spaces V_i such that $(FA)_n = T(V_n)$ and such that all degeneracies maps $s_i\colon [n+1] \to [n]$ define maps of algebras $T(V_n) \to T(V_{n+1})$ induced by some maps of generators $V_n \to V_{n+1}$. We set $V_n = T^{\circ n}(V_n)$, it is clear that this choice satisfies the both conditions. The statement that the map $(FA)_\bullet \to A_\bullet$ is both a weak equivalence and a fibration, is clear. $\square$

Next follows

LEMMA 1.5. *For any two simplicial algebras $A_\bullet, B_\bullet$, the map $\beta_{A,B}\colon F(A \otimes B)_\bullet \to (FA)_\bullet \otimes (FB)_\bullet$ is a weak equivalence.*

Proof. It is a straightforward and simple check that the diagram

$$\begin{tikzcd} F(A\otimes B)_\bullet \arrow[rr, "\beta_{A,B}"] \arrow[dr, "p_{A\otimes B}"'] & & (FA)_\bullet \otimes (FB)_\bullet \arrow[dl, "p_A\otimes p_B"] \\ & A_\bullet \otimes B_\bullet & \end{tikzcd} \tag{1.10}$$

is commutative. The map $p_{A\otimes B}$ is a weak equivalence by Lemma 1.4, the product $p_A \otimes p_B$ is a weak equivalence by Lemma 1.4 again, and by Condition A (Lemma **??**). Then, the commutativity of the diagram (1.10) implies, by 2-out-of-3 axiom of closed model category, that the third arrow $\beta_{A,B}$ is also a weak equivalence. $\square$

2 Monoids and the Bialgebra axiom

2.1 The category of monoids

Let $\mathcal{M}$ be a symmetric monoidal category, $\mathbf{Mon}\mathcal{M}$ be the category of monoids in $\mathcal{M}$. There is the forgetful functor

$$f\colon \mathbf{Mon}\mathcal{M} \to \mathcal{M}$$

Under some conditions, the functor f has a left adjoint functor , "the free monoid functor". Recall the following result, from [ML], Chapter VII.3:

Lemma 2.1. *Let $\mathcal{M}$ be a monoidal category with all finite colimits, such that the functors $?\otimes a$ and $a\otimes ?$ (for fixed a) commute with finite colimits. Then the functor $\mathcal{M} \to \mathbf{Mon}\mathcal{M}$, $X \mapsto T(X)$, with*

$$T(X) = 1_\mathcal{M} \coprod X \coprod X \otimes X \coprod \dots \tag{2.1}$$

is left adjoint to the forgetful functor.

When the finite colimits exist, and there is the *inner Hom functor* (a right adjoint to the monoidal product), the functors $?\otimes a$ and $a\otimes ?$ commute with colimits by general categorical arguments.

We say that a monoidal category $\mathcal{M}$ is *good* when the assumptions of Lemma 2.1 hold.

Let now $\mathcal{M}_1, \mathcal{M}_2$ be two symmetric monoidal categories, and let $F\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a functor. Suppose a lax-monoidal structure ℓ_F is given. Then there is a functor $F^{\mathrm{mon}} = F^{\mathrm{mon}}(\ell_F)\colon \mathbf{Mon}\mathcal{M}_1 \to \mathbf{Mon}\mathcal{M}_2$, depending on ℓ_F. For a monoid X in $\mathcal{M}_1$, the underlying object of $F^{\mathrm{mon}}(X)$ is defined as $F(X)$, and the monoid structure is given as

$$F(X)\otimes F(X) \xrightarrow{\ell_F} F(X\otimes X) \xrightarrow{m_X} F(X) \tag{2.2}$$

We have immediately:

Lemma 2.2. *In the above notations, the following two diagram is commutative:*

$$\begin{tikzcd} \mathcal{M}_1 \arrow[r, "F"] & \mathcal{M}_2 \\ \mathbf{Mon}\mathcal{M}_1 \arrow[u] \arrow[r, "F^{\mathrm{mon}}"] & \mathbf{Mon}\mathcal{M}_2 \arrow[u] \end{tikzcd} \tag{2.3}$$

Here the vertical upward arrows are the forgetful functors.

2.2 The left adjoint functor on monoids

Suppose now that the functor F admits a left adjoint functor $L\colon \mathcal{M}_2 \to \mathcal{M}_1$. In this case, we want to construct a functor $L^{\mathrm{mon}}\colon \mathbf{Mon}\mathcal{M}_2 \to \mathbf{Mon}\mathcal{M}_1$, left adjoint to the functor F^{mon}.

The following Lemma (and the construction in its proof) is due to [SchS03], Section 3.3:

Lemma 2.3. *Suppose the monoidal categories $\mathcal{M}_1$ and $\mathcal{M}_2$ are good, and suppose that the functor L left adjoint to F exists. Then the left adjoint functor $L^{\mathrm{mon}}\colon \mathbf{Mon}\mathcal{M}_2 \to \mathbf{Mon}\mathcal{M}_1$ exists, and it makes the diagram*

$$\begin{tikzcd} \mathcal{M}_1 \arrow[d] & \mathcal{M}_2 \arrow[l, "L"'] \arrow[d] \\ \mathbf{Mon}\mathcal{M}_1 & \mathbf{Mon}\mathcal{M}_2 \arrow[l, "L^{\mathrm{mon}}"'] \end{tikzcd} \tag{2.4}$$

commutative. Here the downward vertical arrows are the free monoid functors.

Proof. The second claim is a formal consequence from the existence of L^{mon}, as the free monoid functors are left adjoint to the forgetful functors. The value $L^{\mathrm{mon}}(X)$ (for a monoid X in $\mathcal{M}_1$) is defined as the co-equalizer

$$T_{\mathcal{D}}(L(T_{\mathcal{M}_1}(X))) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} T_{\mathcal{M}_2}(LX) \tag{2.5}$$

where $T_{\mathcal{M}}$ denotes the free (tensor) monoid in a monoidal category $\mathcal{M}$, see (2.1). The map α in (2.5) comes from the map $T_{\mathcal{M}_1}(X) \to X$ defined from the monoid structure on X, and the map β in (2.5) is defined from the following map $L(T_{\mathcal{M}_1}(X)) \to T_{\mathcal{M}_2}(LX)$:

$$L(\underbrace{X \otimes X \otimes \dots \otimes X}_{n \text{ factors}}) \xrightarrow{{c_L}^{n-1}} \underbrace{L(X) \otimes L(X) \otimes \dots \otimes L(X)}_{n \text{ factors}}$$

where c_L is the colax-monoidal structure on L adjoint to the lax-monoidal structure ℓ_F on F.

The check that the functor L^{mon}, defined by (2.5), is left adjoint to the functor F^{mon}, is straightforward and is left to the reader. □

Now we pass to the situation when the functor F admits, besides the lax-monoidal structure ℓ_F, a colax-monoidal structure c_F, *compatible by the bialgebra axiom* (see Section 5.3 below).

2.3 The Bialgebra axiom

Fix some notations on adjoint functors.

Let $L\colon \mathcal{A} \rightleftarrows \mathcal{B}\colon R$ be two functors. They are called adjoint to each other, with L the left adjoint and R the right adjoint, when

$$\mathrm{Mor}_{\mathcal{B}}(LX, Y) \simeq \mathrm{Mor}_{\mathcal{A}}(X, RY) \tag{2.6}$$

where "$\simeq$" here means "isomorphic as bifunctors $\mathcal{A}^{\mathrm{opp}} \times \mathcal{B} \to \mathbf{Sets}$".

This gives rise to maps of functors $\epsilon\colon LR \to \mathrm{Id}_{\mathcal{B}}$ and $\eta\colon \mathrm{Id}_{\mathcal{A}} \to RL$ such that the compositions

$$\begin{aligned} &L \xrightarrow{L\circ\eta} LRL \xrightarrow{\epsilon\circ L} L \\ &R \xrightarrow{\eta\circ R} RLR \xrightarrow{R\circ\epsilon} R \end{aligned} \tag{2.7}$$

are identity maps of the functors.

The inverse is true: given maps of functors $\epsilon\colon LR \to \mathrm{Id}_{\mathcal{B}}$ and $\eta\colon \mathrm{Id}_{\mathcal{A}} \to RL$, obeying (2.7), gives rise to the isomorphism of bifunctors, that is, to adjoint equivalence (see [ML], Section IV.1, Theorems 1 and 2).

In particular, the case of *adjoint equivalence* is the case when $\epsilon\colon LR \to \mathrm{Id}_{\mathcal{B}}$ and $\eta\colon \mathrm{Id}_{\mathcal{A}} \to RL$ are *isomorphisms of functors*. In this case, setting $\epsilon_1 = \eta^{-1}$ and $\eta_1 = \epsilon^{-1}$, we obtain another adjunction, with L the *right* adjoint and R the *left adjoint*.

Let $\phi \in \mathrm{Mor}_{\mathcal{B}}(LX, Y)$. The following explicit formula for its adjoint $\psi \in \mathrm{Mor}_{\mathcal{A}}(X, RY)$ will be useful:

$$X \xrightarrow{\eta} RLX \xrightarrow{R(\phi)} RY \tag{2.8}$$

and analogously for the way back:

$$LX \xrightarrow{L(\psi)} LRY \xrightarrow{\epsilon} Y \tag{2.9}$$

(see [ML], Section IV.1).

Let now $\mathcal{C}$ and $\mathcal{D}$ be two symmetric monoidal categories, $F\colon \mathcal{C} \to \mathcal{D}$ a functor. Suppose a lax-monoidal structure ℓ_F and a colax-monoidal structure c_F on F are given. The *bialgebra axiom* is some compatibility condition on the pair (c_F, ℓ_F), see Section 5.3. Recall the following simple fact from [Sh2], Section 2:

Lemma 2.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be two strict symmetric monoidal categories, and let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be an adjoint equivalence of the underlying categories. Then given a pair (c_F, ℓ_F) where c_F is a colax-monoidal structure on F, ℓ_F is a lax-monoidal structure on F, one can assign to it a pair (c_G, ℓ_G) of analogous structures on G. If $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal, and if the pair (c_F, ℓ_F) satisfies the bialgebra axiom* (see Section 5.3)*, the pair (c_G, ℓ_G) satisfies the bialgebra axiom as well.*

Proof. Suppose (c_F, ℓ_F) are done. We firstly write down the formulas for ℓ_G and c_G.

Formula for ℓ_G:

$$GX \otimes GY \xrightarrow{\eta} GF(GX \otimes GY) \xrightarrow{c_F} G(FGX \otimes FGY) \xrightarrow{\epsilon\otimes\epsilon} G(X \otimes Y) \tag{2.10}$$

Formula for c_G:

$$G(X \otimes Y) \xrightarrow{\epsilon^{-1}\otimes\epsilon^{-1}} G(FGX \otimes FGY) \xrightarrow{\ell_F} GF(GX \otimes GY) \xrightarrow{\eta^{-1}} GX \otimes GY \tag{2.11}$$

When we now write down the bialgebra axiom diagram (see Section 5.3) for (c_G, ℓ_G) we see due to cancelations of ϵ with ϵ^{-1} and of η with η^{-1}, that the diagram is commutative as soon as the diagram for (c_F, ℓ_F) is. □

LEMMA 2.5. *Let $\mathcal{C}, \mathcal{D}$ be two symmetric monoidal categories, and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Suppose a lax-monoidal structure ℓ_F on F is given. Consider the functor*

$$F^{\mathrm{mon}} = F^{\mathrm{mon}}(\ell_F)\colon \mathbf{Mon}\mathcal{C} \to \mathbf{Mon}\mathcal{D}$$

defined in (2.2)*. Then the map $F^{\mathrm{mon}}(X) \otimes F^{\mathrm{mon}}(Y) \to F^{\mathrm{mon}}(X \otimes Y)$, defined on the underlying objects as ℓ_F, is a map of monoids, and, therefore, gives a lax-monoidal structure on F^{mon}. Let now c_F be a colax-monoidal structure on F. If (ℓ_F, c_F) satisfies the bialgebra axiom, the map $F^{\mathrm{mon}}(X \otimes Y) \to F^{\mathrm{mon}}(X) \otimes F^{\mathrm{mon}}(Y)$, defined on the underlying objects as c_F, is a map of monoids, and, therefore, gives a colax-monoidal structure on F^{mon}.*

The both claims are straightforward checks. The second claim was, in fact, our motivation for introduction of the bialgebra axiom in [Sh2].

3 THE DOLD-KAN CORRESPONDENCE

We use the following notations:

$\mathcal{C}(\mathbb{Z})$ is the category of unbounded complexes of abelian groups, $\mathcal{C}(\mathbb{Z})^+$ (resp., $\mathcal{C}(\mathbb{Z})^-$) are the full subcategories of $\mathbb{Z}_{\geq 0}$-graded (resp., $\mathbb{Z}_{\leq 0}$-graded) complexes. The category of abelian groups placed in degree 0 (with zero differential) is denoted by $\mathcal{M}od(\mathbb{Z})$, thus, $\mathcal{M}od(\mathbb{Z}) = \mathcal{C}(\mathbb{Z})^- \cap \mathcal{C}(\mathbb{Z})^+$.

3.1

The Dold-Kan correspondence is the following theorem:

THEOREM 3.1 (Dold-Kan correspondence). *There is an adjoint equivalence of categories*

$$N\colon \mathcal{M}od(\mathbb{Z})^{\Delta} \rightleftarrows \mathcal{C}(\mathbb{Z})^-\colon \Gamma$$

where N is the functor of normalized chain complex (which is isomorphic to the Moore complex).

We refer to [W], Section 8.4, and [SchS03], Section 2, which both contain excellent treatment of this Theorem.

The both categories $\mathcal{M}od(\mathbb{Z})^\Delta$ and $\mathcal{C}(\mathbb{Z})^-$ are *symmetric monoidal* in natural way. However, neither of functors N and Γ is monoidal.

There is a colax-monoidal structure on N, called *the Alexander-Whitney map* $AW\colon N(A\otimes B)\to NA\otimes NB$ and a lax-monoidal structure on N, called *the shuffle map* $\nabla\colon N(A)\otimes N(B)\to N(A\otimes B)$.

Recall the explicit formulas for them.

The Alexander-Whitney colax-monoidal map $AW\colon N(A\otimes B)\to N(A)\otimes N(B)$ is defined as

$$AW(a^k\otimes b^k)=\sum_{i+j=k} d^i_{\mathrm{fin}}a^k\otimes d^j_0 b^k \tag{3.1}$$

where d_0 and d_{fin} are the first and the latest simplicial face maps.

The Eilenberg-MacLane shuffle lax-monoidal map $\nabla\colon N(A)\otimes N(B)\to N(A\otimes B)$ is defined as

$$\nabla(a^k\otimes b^\ell)=\sum_{(k,\ell)\text{-shuffles }(\alpha,\beta)}(-1)^{(\alpha,\beta)}S_\beta a^k\otimes S_\alpha b^\ell \tag{3.2}$$

where

$$S_\alpha=s_{\alpha_k}\dots s_{\alpha_1}$$

and

$$S_\beta=s_{\beta_\ell}\dots s_{\beta_1}$$

Here s_i are simplicial degeneracy maps, $\alpha=\{\alpha_1<\dots<\alpha_k\}$, $\beta=\{\beta_1<\dots<\beta_\ell\}$, $\alpha,\beta\subset[0,1,\dots,k+\ell-1]$, $\alpha\cap\beta=\varnothing$.

Let us summarize their properties in the following Proposition, see [SchS03], Section 2, and references therein, for a proof.

Proposition 3.2. *The colax-monoidal Alexander-Whitney and the lax-monoidal shuffle structures on the functor N enjoy the following properties:*

1. *the composition*
$$NA\otimes NB\xrightarrow{\nabla}N(A\otimes B)\xrightarrow{AW}NA\otimes NB$$
is the identity,

2. *the composition*
$$N(A\otimes B)\xrightarrow{AW}NA\otimes NB\xrightarrow{\nabla}N(A\otimes B)$$
is naturally chain homotopic to the identity,

3. *the shuffle map ∇ is symmetric,*

4. the Alexander-Whitney map AW is symmetric up to a natural chain homotopy.

□

Recall a Theorem proven independently in [AM], Sect. 5.4, and (later) in [Sh2], Sect. 2:

THEOREM 3.3. *The pair (∇, AW) of the lax-monoidal shuffle structure and the colax-monoidal Alexander-Whitney structure, defined on the normalized chain complex functor $N\colon \mathcal{M}od(\mathbb{Z})^\Delta \rightleftarrows \mathcal{C}(\mathbb{Z})^-$, obeys the bialgebra axiom.*

□

This Theorem, along with Proposition 3.2 (1.), play crucial role in the proof of Main Theorem in Section 4.

3.2 Monoidal properties

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors between monoidal categories.

DEFINITION 3.4. Suppose the functor F is colax-monoidal, with the colax-monoidal structure c_F, and G is lax-monoidal, with the lax-monoidal structure ℓ_G. A morphism of functors $\Phi\colon F \to G$ is called *colax-monoidal* if for any $X, Y \in \mathcal{C}$, the diagram

$$\begin{tikzcd} F(X\otimes Y) \arrow[r, "\Phi"] \arrow[d, "c_F"'] & G(X\otimes Y) \\ F(X)\otimes F(Y) \arrow[r, "\Phi\otimes\Phi"] & G(X)\otimes G(Y) \arrow[u, "\ell_G"'] \end{tikzcd} \tag{3.3}$$

As well, when F is lax-monoidal with the lax-monoidal structure ℓ_F, and G is colax-monoidal with the colax-monoidal structure c_G, a morphism $\Psi\colon F \to G$ is called *lax-monoidal*, if for any $X, Y \in \mathcal{C}$ the diagram

$$\begin{tikzcd} F(X)\otimes F(Y) \arrow[r, "\Psi\otimes\Psi"] \arrow[d, "\ell_F"'] & G(X)\otimes G(Y) \\ F(X\otimes Y) \arrow[r, "\Psi"] & G(X\otimes Y) \arrow[u, "c_G"'] \end{tikzcd} \tag{3.4}$$

Each of the functors N and Γ admits both lax-monoidal and colax monoidal structures. Therefore, the compositions $N \circ \Gamma$ and $\Gamma \circ N$ are both lax- and colax-monoidal.

Here are the main monoidal properties concerning the Dold-Kan correspondence, from which (ii) is used essentially in our proof of Main Theorem 4.1 below.

LEMMA 3.5. *(i) the adjunction map $\epsilon\colon N\circ\Gamma \to \mathrm{Id}$ is lax-monoidal,*

(ii) the adjunction map $\epsilon\colon N\circ\Gamma \to \mathrm{Id}$ is colax-monoidal.

Proof. The claim (i) is Lemma 2.11 of [SchS03]. The claim (ii) is proven analogously, we present here the proof for completeness. We need to prove the commutativity of the diagram

$$\begin{tikzcd} N\Gamma(X\otimes Y) \arrow[r, "\varphi"] \arrow[drr] & N(\Gamma(X)\otimes\Gamma(Y)) \arrow[r, "AW"] & N\Gamma(X)\otimes N\Gamma(Y) \arrow[d, "\epsilon\otimes\epsilon"] \\ & & X\otimes Y \end{tikzcd} \tag{3.5}$$

Here φ is the colax-monoidal structure on Γ adjoint to the shuffle lax-monoidal structure on N, see (2.11). The explicit formula for φ is:

$$\Gamma(X\otimes Y)\xrightarrow{\epsilon^{-1}\otimes\epsilon^{-1}}\Gamma(N\Gamma(X)\otimes N\Gamma(Y))\xrightarrow{\nabla}\Gamma N(\Gamma(X)\otimes\Gamma(Y))\xrightarrow{\eta}\Gamma(X)\otimes\Gamma(Y) \tag{3.6}$$

Now the horizontal composition map in (3.5) is:

$$N\Gamma(X\otimes Y)\xrightarrow{\epsilon^{-1}\otimes\epsilon^{-1}}N\Gamma(N\Gamma(X)\otimes N\Gamma(Y))\xrightarrow{\nabla}N\boxed{\Gamma N}(\Gamma(X)\otimes\Gamma(Y))\xrightarrow{\eta^{-1}}N(\Gamma(X)\otimes\Gamma(Y))$$
$$\xrightarrow{AW}N\Gamma(X)\otimes N\Gamma(Y) \tag{3.7}$$

where the map η^{-1} is applied to the "boxed" ΓN factor.

Now the idea is to use the identity $AW\circ\nabla=\mathrm{Id}$ (Lemma 3.2 (1.)) to "cancel" the second and the fourth arrows in (3.7). We have:

Lemma 3.6. *The following two compositions are equal:*

$$N\boxed{\Gamma N}(\Gamma(X)\otimes\Gamma(Y))\xrightarrow{\eta^{-1}}N(\Gamma(X)\otimes\Gamma(Y))\xrightarrow{AW}N\Gamma(X)\otimes N\Gamma(Y) \tag{3.8}$$

and

$$\boxed{N\Gamma}N(\Gamma(X)\otimes\Gamma(Y))\xrightarrow{AW}\boxed{N\Gamma}(N\Gamma(X)\otimes N\Gamma(Y))\xrightarrow{\epsilon}N\Gamma(X)\otimes N\Gamma(Y) \tag{3.9}$$

where in the first (corresp., second) equation the map η^{-1} (corresp., ϵ) is applied to the boxed factors.

Clearly Lemma 3.5 (ii) follows from Lemma 3.6 and Lemma 3.2 (1.).

Proof. For an adjoint *equivalence* $L\colon \mathcal{C}\rightleftarrows\mathcal{D}\colon R$ with the adjunction *isomorphisms* $\epsilon\colon LR\to\mathrm{Id}$ and $\eta\colon \mathrm{Id}\to RL$, the two arrows $\boxed{LR}L\xrightarrow{\epsilon}L$ and $L\boxed{RL}\xrightarrow{\eta^{-1}}L$ coincide. $\square$

$\square$

Remark 3.7. The adjunction map $\eta\colon \mathrm{Id}\to\Gamma\circ N$ is both lax-monoidal and colax-monoidal *only up to a homotopy*, see Remark 2.14 in [SchS03]. The reason is that the another order composition $\nabla\circ AW$ is equal to identity only up to a homotopy.

4 Main theorem for $\mathbb{Z}_{\le 0}$-graded dg algebras

4.1

Here we prove our main result:

Theorem 4.1. *Let k be a field of any characteristic. There is a functor $\mathfrak{R}\colon \mathcal{A}lg_k^{\le 0} \to \mathcal{A}lg_k^{\le 0}$ and a morphism of functors $w\colon \mathfrak{R} \to Id$ with the following properties:*

1. *$\mathfrak{R}(A)$ is cofibrant, and $w\colon \mathfrak{R}(A) \to A$ is a quasi-isomorphism, for any $A \in \mathcal{A}lg_k^{\le 0}$,*
2. *there is a colax-monoidal structure on the functor $\mathfrak{R}$, such that all colax-maps $\beta_{A,B}\colon \mathfrak{R}(A\otimes B) \to \mathfrak{R}(A)\otimes\mathfrak{R}(B)$ are quasi-isomorphisms of dg algebras, and such that the diagram*

$$\begin{tikzcd}
\mathfrak{R}(A\otimes B) \arrow[rr, "\beta_{A,B}"] \arrow[dr] & & \mathfrak{R}(A)\otimes\mathfrak{R}(B) \arrow[dl] \\
& A\otimes B &
\end{tikzcd}$$

   is commutative,

3. *the morphism $w(k^\bullet)\colon \mathfrak{R}(k^\bullet) \to k^\bullet$ coincides with $\alpha\colon \mathfrak{R}(k^\bullet) \to k^\bullet$, where α is a part of the colax-monoidal structure* (see Definition 5.1)*, and $k^\bullet = 1_{\mathcal{A}lg_k^{\le 0}}$ is the dg algebra equal to the one-dimensional k-algebra in degree 0, and vanishing in other degrees.*

The idea is to use the solution of the analogous problem for simplicial algebras (given in Theorem 1.1 in Section 1), and "transfer" it to dg algebras using the Dold-Kan correspondence.

More precisely, consider the Dold-Kan correspondence

$$N\colon \mathcal{M}od(\mathbb{Z})^{\Delta} \rightleftarrows \mathcal{C}(\mathbb{Z})^{-}\colon \Gamma$$

The functors N and Γ form an adjoint equivalence of categories; therefore, we have some freedom which of these two functors to consider as the left (right) adjoint. We consider Γ as the right adjoint. From now on, we use the notations: $\Gamma = R$, $N = L$.

The functor L comes with the colax-monoidal (Alexander-Whitney) structure AW and with the lax-monoidal (shuffle) structure ∇. They induce a lax-monoidal structure ℓ_R and a colax-monoidal structure c_R on the functor R by the adjunction, as is explained in (2.10) and (**??**).

Consider the functor $R^{\mathrm{mon}}\colon \mathcal{A}lg_k^{\le 0} \to \mathcal{A}lg_k^{\Delta}$, induced by the functor R and from its lax-monoidal structure ℓ_R, on the categories of monoids (see Section 2.1). It admits a left adjoint functor L^{mon}, defined in Section 2.2.

From now on, we use the notation $\mathfrak{F}(A)$ for the solution of Theorem 1.1 (instead of former notation $\mathfrak{R}(A)$, which we now reserve for solution to Theorem 4.1).

Define

$$\boxed{\mathfrak{R}(A) = L^{\mathrm{mon}}(\mathfrak{F}(R^{\mathrm{mon}}(A)))} \tag{4.1}$$

where $A \in \mathcal{A}lg_k^{\le 0}$.

There is a projection $p_{\mathfrak{F}}\colon \mathfrak{F}(R^{\mathrm{mon}}(A)) \to R^{\mathrm{mon}}(A)$. Define the projection

$$p_A\colon \mathfrak{R}(A) \to A \tag{4.2}$$

as the composition of the projection $p_{\mathfrak{F}}$ with the adjunction map $L^{\mathrm{mon}} \circ R^{\mathrm{mon}} \to \mathrm{Id}$.

We claim that this functor $\mathfrak{R}$ gives a solution to Theorem 4.1. We need to prove the following statements:

Proposition 4.2. *(i) the functor $\mathfrak{R}\colon \mathcal{A}lg_k^{\le 0} \to \mathcal{A}lg_k^{\le 0}$ has a natural colax-monoidal structure β,*

(ii) $\mathfrak{R}(A)$ is cofibrant, and the projection $p_A\colon \mathfrak{R}(A) \to A$ is a weak equivalence, for any $A \in \mathcal{A}lg_k^{\le 0}$,

(iii) the diagram

$$\begin{tikzcd} \mathfrak{R}(A\otimes B) \arrow[rr, "\beta_{A,B}"] \arrow[dr, "p_{A\otimes B}"'] & & \mathfrak{R}(A)\otimes\mathfrak{R}(B) \arrow[dl, "p_A\otimes p_B"] \\ & A\otimes B & \end{tikzcd} \tag{4.3}$$

commutes; consequently, it follows from (ii) and from the 2-out-of-3 axiom that $\beta_{A,B}$ is a weak equivalence for any $A, B \in \mathcal{A}lg_k^{\le 0}$.

The three items of this Proposition rely on three different theories: they are the bialgebra axiom for (i), the Schwede-Shipley theory of weak monoidal Quillen pairs for (ii), and the monoidal property of the Dold-Kan correspondence (Lemma 3.5) for (iii). We give the detailed proof in the rest of this Section.

4.2 Proof of Proposition 4.2, (i)

The functor $\mathfrak{R} = L^{\mathrm{mon}} \circ \mathfrak{F} \circ R^{\mathrm{mon}}$ is a composition of three functors. The functor L^{mon} comes with its colax-monoidal structure (adjoint to the lax-monoidal structure on R^{mon}), and the functor $\mathfrak{F}$ has the colax-monoidal structure (1.5). It remains to define a colax-monoidal structure on R^{mon} (a priori R^{mon} has only a lax-monoidal structure).

Recall that the functor $L = N$ has the Alexander-Whiteney colax-monoidal structure AW and a lax-monoidal structure ∇, *compatible by the bialgebra axiom* (see Theorem 3.3). Then it follows from Lemma 2.4 that the adjoint lax-monoidal and colax-monoidal structures on R obey the bialgebra axiom as well. In general, the lax-monoidal structure on R induces a lax-monoidal structure on $R^{\mathrm{mon}}\colon \mathbf{Mon}\mathcal{D} \to \mathbf{Mon}\mathcal{C}$ (the same as the lax-monoidal structure on R

for the underlying objects), *but that is not true for the colax-monoidal structure.* When the both structures are compatible by the bialgebra axiom, Lemma 2.5 says the colax-monoidal structure, defined on the underlying objects as the one on R, defines a colax-monoidal structure on R^{mon}.

Thus, $\mathfrak{R}$ is a composition of three functors, each of which comes with natural colax-monoidal structure. Therefore, $\mathfrak{R}$ is colax-monoidal. We always assume this structure when refer to a colax-monoidal structure on $\mathfrak{R}$. □

4.3 Quillen pairs and weak monoidal Quillen pairs

4.3.1 Quillen pairs and Quillen equivalences

To prove the statement (ii) of Proposition, we need firstly to recall some definitions on Quileen pairs of functors between two closed model categories, and to recall some results of Schwede-Schipley [SchS03] on weak monoidal Quillen pairs.

In classical homological algebra one can derive left exact or right exact functor. When we work with closed model categories, we try to extend the classical homological algebra to non-abelian (and non-additive) context. A typical examples are the category of topological spaces and the category of dg associative algebras. How we can define the notions of a left (right) exact functor (i.e., of those functors we can derive) in such generality? The answer is given (by Quillen) in the concept of *a Quillen pair of functors.*

To motivate the definition below, recall the following simple fact:

Lemma 4.3. *Suppose $\mathcal{A}, \mathcal{B}$ are two abelian categories, and let*

$$L\colon \mathcal{A} \rightleftarrows \mathcal{B}\colon R$$

be a pair of adjoint functors, with L left adjoint. Then L is right exact and R is left exact.

Prove as an exercise, or see the proof in [W], Theorem 2.6.1. □

Morally, we can not say in abstract situation what is the right (left) exactness, be we now what adjoint functors. These are (among other assumptions) the functors we can derive. Therefore, they come in pairs.

Definition 4.4. Let $\mathcal{C}, \mathcal{D}$ be two closed model categories, and let

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$$

is a pair of adjoint functors, with L the left adjoint. The pair (L, R) is called *a Quillen pair of functors* if

(1) L preserves cofibrations and trivial cofibrations,

(2) R preserves fibrations and trivial fibrations.

It is proven (see, e.g., [Hir], Prop. 8.5.7) that, under these conditions, L takes weak equivalences between cofibrant objects to weak equivalences, and R takes weak equivalences between fibrant object to weak equivalences.

It is proven that a Quillen pair of functors defines an adjoint pair of functors between the homotopy categories,

$$\mathbb{L}\colon \mathrm{Ho}\mathcal{C} \rightleftarrows \mathrm{Ho}\mathcal{D}\colon \mathbb{R}$$

The next step is to find conditions on (L, R) under which the pair $(\mathbb{L}, \mathbb{R})$ is an adjoint equivalence. This is the case when (L, R) is *a Quillen equivalence.*

DEFINITION 4.5. A Quillen pair

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$$

is called *a Quiilen equivalence* if for any cofibrant object X in $\mathcal{C}$ and for every fibrant object Y in $\mathcal{D}$, a map $f\colon X \to RY$ is a weak equivalence if and only if the adjoint map $f^\sharp\colon LX \to Y$ is a weak equivalence.

It is proven (see e.g. [Hir], Theorem 8.5.23) that if (L, R) is a Quillen equivalence, the functors

$$\mathbb{L}\colon \mathrm{Ho}\mathcal{C} \rightleftarrows \mathrm{Ho}\mathcal{D}\colon \mathbb{R}$$

form an adjoint *equivalence* of categories.

All these results are due to D.Quillen [Q].

4.3.2 Weak monoidal Quillen pairs

Here we recall a result on weak monoidal Quillen pairs which is essential for our proof of Proposition 4.2, (ii) below. Our intention here is not to give a throughout treatment (as it would be just a copy of published papers), but rather to recall very briefly the definitions and results, for convenient reference in the next Subsection.

The categories we consider here are at once closed model and monoidal. There is some reasonable compatibility between these two structures on a category $\mathcal{C}$, which guarantee, in particular, that $\mathrm{Ho}\mathcal{C}$ is a monoidal category. The concept is called *a monoidal model category.* We do not reproduce this definition here as we do not use it practically, all our categories in this paper fulfill this definition. The interested reader is referred to [SchS00].

The following definition is due to Schwede-Shipley [SchS03] (Definition 3.6).

DEFINITION 4.6. Let $\mathcal{C}, \mathcal{D}$ be *monoidal model categories*, and let

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$$

be a Quillen pair of the underlying closed model categories. Suppose there is a lax-monoidal structure ℓ on the functor R, denote by φ the adjoint colax-monoidal structure on L.

The triple (L, R, ℓ) is called *a weak monoidal Quillen pair* if

(i) for all cofibrant objects X, Y in $\mathcal{C}$, the colax-monoidal map

$$\varphi_{X,Y}\colon L(X\otimes Y)\to L(X)\otimes L(Y)$$

is a weak equivalence,

(ii) for some cofibrant replacement $q\colon \mathbb{I}^c\to\mathbb{I}$ of the unit object in $\mathcal{C}$, the composition

$$L(\mathbb{I}^c_{\mathcal{C}})\xrightarrow{L(q)} L(\mathbb{I}_{\mathcal{C}})\xrightarrow{\mu}\mathbb{I}_{\mathcal{D}}$$

is a weak equivalence (where μ is a part of colax-monoidal structure, see Definition 5.1).

A triple (L, R, ℓ) is called *a weak monoidal Quillen equivalence*, if is a weak monoidal Quillen pair, such that the underlying Quillen pair (L, R) is a Quillen equivalence.

We use essentially the following result from [SchS03] (Theorem 3.12 (3)).

THEOREM 4.7. *Let (L, R, ℓ) be a weak monoidal Quillen equivalence, and let*

$$L\colon \mathcal{C}\rightleftarrows\mathcal{D}\colon R$$

be the underlying Quillen pair. Suppose that the unit objects in $\mathcal{C}$ and $\mathcal{D}$ are cofibrant, and suppose that the forgetful functors $\mathbf{Mon}\mathcal{C}\to\mathcal{C}$ and $\mathbf{Mon}\mathcal{D}\to\mathcal{D}$ create model structures in $\mathbf{Mon}\mathcal{C}$ and $\mathbf{Mon}\mathcal{D}$ (see the explanation just below). Then

$$L^{\mathrm{mon}}\colon \mathbf{Mon}\mathcal{C}\rightleftarrows\mathbf{Mon}\mathcal{D}\colon R^{\mathrm{mon}} \tag{4.4}$$

is a Quillen equivalence.

REMARK 4.8. 1. See [GS], Section 3, or [Hir], Chapter 11, for detailed explanation of the meaning of "the forgetful functors generate closed model structures". This concept refers to the transfer of closed model structures for *cofibrantly generated model categories*. See loc.cit. for all these concepts, as well as for a proof that our categories $\mathcal{C}=\mathcal{M}od(\mathbb{Z})^{\Delta}$ and $\mathcal{D}=\mathcal{C}(\mathbb{Z})^-$ satisfy this assumptions.

2. The Quillen equivalence (4.4) is *not* a weak monoidal Quillen equivalence. In fact, the natural monoidal structure on $\mathbf{Mon}\mathcal{M}$ for a monoidal model category $\mathcal{M}$, is *not* a monoidal model category in general. For instance, the monoidal bifunctor does not commute with the coproducts as a functor of one argument, for fixed another one.

4.4 Proof of Proposition 4.2, (ii)

We need to prove that, for any dg algebra $A \in \mathcal{A}lg_k^{\le 0}$, the dg algebra $\mathfrak{R}(A)$ is cofibrant, and the projection

$$p_A\colon \mathfrak{R}(A) \to A \tag{4.5}$$

is a quasi-isomorphism of dg algebras.

Consider the Dold-Kan correspondence. In [SchS03], Section 3.4, there is given a criterium for when a triple (L, R, ℓ) is a weak Quillen equivalence. It is proven as well, that this criterium works in the following two cases. The first case is the case of the Dold-Kan correspondence, with $\Gamma = R$ the right adjoint, with the lax-monoidal structure being the adjoint to the colax-monoidal structure AW on $N = L$. The second case is the case when $N = R$ is the right adjoint, with the lax-monoidal shuffle structure ∇. In our applications, we need only the first possibility. Let us summarize.

Lemma 4.9. *Consider the Dold-Kan correspondence*

$$N\colon \mathcal{M}od(Z)^{\Delta} \rightleftarrows \mathcal{C}(\mathbb{Z})^{-}\colon \Gamma$$

Use the notations $L = N$, $R = \Gamma$, and let ℓ be the lax-monoidal structure on R, adjoint to the colax-monoidal structure AW on L. Then (L, R, ℓ) is a weak Quillen equivalence.

□

Now we apply Theorem 4.7.

Corollary 4.10. *In the above notations, the adjoint pair of functors*

$$L^{\mathrm{mon}}\colon \mathbf{Mon}(\mathcal{M}od(\mathbb{Z})^{\Delta}) \rightleftarrows \mathbf{Mon}(\mathcal{C}(\mathbb{Z})^{-})\colon R^{\mathrm{mon}} \tag{4.6}$$

is a weak Quillen equivalence.

Proof. It follows immediately from Lemma 4.9 and from Theorem 4.7. □

Now we pass to a proof of the claims of Proposition 4.2, (ii).

Proof. Prove that $\mathfrak{R}(A)$ is cofibrant dg algebra for any $A \in \mathcal{A}lg_k^{\le 0}$.

Indeed,

$$\mathfrak{R}(A) = L^{\mathrm{mon}} \circ \mathfrak{F} \circ R^{\mathrm{mon}}(A)$$

We know that $\mathfrak{F}(?)$ is cofibrant (Lemma 1.4), and that $(L^{\mathrm{mon}}, R^{\mathrm{mon}})$ is a Quillen equivalence, in particular, a Quillen pair (Corollary 4.10). By Definition of a Quillen pair, L^{mon} maps cofibrations to cofibrations. As L^{mon} maps the initial object to the initial object, it maps the cofibrant objections to cofibrant objects. Therefore, $\mathfrak{R}(A)$ is cofibrant, as $\mathfrak{F}(R^{\mathrm{mon}}(A))$ is cofibrant.

Prove that, for any $A \in \mathcal{A}lg_k^{\le 0}$, the projection $p_A\colon \mathfrak{R}(A) \to A$ is a weak equivalence.

The projection
$$p_A\colon L^{\mathrm{mon}}\circ\mathfrak{F}\circ R^{\mathrm{mon}}(A)\to A \tag{4.7}$$
is adjoint to
$$p_A^\sharp\colon \mathfrak{F}\circ R^{\mathrm{mon}}(A)\to R^{\mathrm{mon}}(A) \tag{4.8}$$
According to Corollary 4.10, the pair $(L^{\mathrm{mon}}, R^{\mathrm{mon}})$ is a Quillen equivalence. We are going to apply the defining property of Quillen equivalences, see Definition 4.5. Moreover, $\mathfrak{F}\circ R^{\mathrm{mon}}(A)$ is cofibrant by Lemma 1.4 (as $\mathfrak{F}(?)$ is cofibrant by this Lemma), and $R^{\mathrm{mon}}(A)$ is fibrant by the description in Section 1.2. Moreover, the map $p_A^\sharp$ is a weak equivalence, again by Lemma 1.4. Therefore, p_A is a weak equivalence as well, by Definition 4.5.

□

4.5 Proof of Proposition 4.2, (iii)

The claim of Proposition 4.2, (iii) follows from the two Lemmas below. The first one is straightforward, while the second one is very essential, and is based, by its own, on the Key-Lemma 4.13 formulated and proven below in this Subsection.

Lemma 4.11. *For any X, Y, the following diagram is commutative:*
$$\begin{tikzcd} L^{\mathrm{mon}}(\mathfrak{F}(R^{\mathrm{mon}}(X\otimes Y))) \arrow[r] \arrow[d] & L^{\mathrm{mon}}(\mathfrak{F}(R^{\mathrm{mon}}(X)))\otimes L^{\mathrm{mon}}(\mathfrak{F}(R^{\mathrm{mon}}(Y))) \arrow[d] \\ L^{\mathrm{mon}}R^{\mathrm{mon}}(X\otimes Y) \arrow[r] & L^{\mathrm{mon}}R^{\mathrm{mon}}(X)\otimes L^{\mathrm{mon}}R^{\mathrm{mon}}(Y) \end{tikzcd} \tag{4.9}$$

Lemma 4.12. *For any X, Y, the following diagram is commutative:*
$$\begin{tikzcd} L^{\mathrm{mon}}R^{\mathrm{mon}}(X\otimes Y) \arrow[rr] \arrow[dr, "p_{X\otimes Y}"'] & & L^{\mathrm{mon}}R^{\mathrm{mon}}(X)\otimes L^{\mathrm{mon}}R^{\mathrm{mon}}(Y) \arrow[dl, "p_X\otimes p_Y"] \\ & X\otimes Y & \end{tikzcd} \tag{4.10}$$

Proof of Lemma (4.11)*:* We have the commutative diagram:
$$\begin{tikzcd} L^{\mathrm{mon}}(\mathfrak{F}(R^{\mathrm{mon}}(X\otimes Y))) \arrow[r, "c_R"] \arrow[d] & L^{\mathrm{mon}}(\mathfrak{F}(R^{\mathrm{mon}}(X)\otimes R^{\mathrm{mon}}(Y))) \arrow[d] \\ L^{\mathrm{mon}}(R^{\mathrm{mon}}(X\otimes Y)) \arrow[r, "c_R"] & L^{\mathrm{mon}}(R^{\mathrm{mon}}(X)\otimes R^{\mathrm{mon}}(Y)) \end{tikzcd} \tag{4.11}$$
by tautological reasons. Now consider the diagram
$$\begin{tikzcd} \mathfrak{F}(R^{\mathrm{mon}}(X)\otimes R^{\mathrm{mon}}(Y)) \arrow[rr] \arrow[dr] & & \mathfrak{F}(R^{\mathrm{mon}}(X))\otimes \mathfrak{F}(R^{\mathrm{mon}}(Y)) \arrow[dl] \\ & R^{\mathrm{mon}}(X)\otimes R^{\mathrm{mon}}(Y) & \end{tikzcd} \tag{4.12}$$

from Lemma 1.10. The composition of the diagram (4.11) with the application of L^{mon} to the diagram (4.12) gives the diagram in Lemma 4.11. $\square$

Proof of Lemma (4.12)*:* We firstly describe the functor L^{mon} explicitly, with its colax-monoidal structure.

Recall that the functor $L^{\mathrm{mon}}\colon \mathbf{Mon}\mathcal{C} \to \mathbf{Mon}\mathcal{D}$ is constructed as the left adjoint to the functor $\mathbf{Mon}\mathcal{C} \leftarrow \mathbf{Mon}\mathcal{D}\colon R^{\mathrm{mon}}$. The functor R^{mon} coincides with the functor R on the underlying objects, and the monoid structure is defined from a lax-monoidal structure ℓ_R on R (see (2.2)). Then ℓ_R defines a lax-monoidal structure on R^{mon} as well, and the colax-monoidal structure $c_{L^{\mathrm{mon}}}$ is defined from ℓ_R by the adjunction (see (2.11)).

Here we give a more explicit description of this colax-monoidal structure on L^{mon}.

Recall (see ???) that the value $L^{\mathrm{mon}}(X)$ (for a monoid X in $\mathcal{C}$) is defined as the co-equalizer

$$T_{\mathcal{D}}(L(T_{\mathcal{C}}(X))) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} T_{\mathcal{D}}(LX) \tag{4.13}$$

where $T_{\mathcal{M}}$ denotes the free (tensor) monoid in a monoidal category $\mathcal{M}$, see (2.1). In particular, it is some quotient of the free monoid $T_{\mathcal{D}}(LX)$.

From now on, we restrict ourselves with the case of the Dold-Kan correspondence, $\mathcal{C} = \mathcal{M}od(\mathbb{Z})^{\Delta}$, $\mathcal{D} = \mathcal{C}(\mathbb{Z})^{-}$, $L = N$, $R = \Gamma$.

Consider the shuffle lax-monoidal structure ∇ on L. This structure defines a functor $\tilde{L}^{\mathrm{mon}}\colon \mathbf{Mon}\mathcal{C} \to \mathbf{Mon}\mathcal{D}$, as in (2.2). We claim that the functors L^{mon} and $\tilde{L}^{\mathrm{mon}}$ coincide.

Key-lemma 4.13. *For the Dold-Kan correspondence and the above notations, the two functors*

$$L^{\mathrm{mon}}, \tilde{L}^{\mathrm{mon}}\colon \mathbf{Mon}\mathcal{C} \to \mathbf{Mon}\mathcal{D}$$

coincide. Moreover, the colax-monoidal structure $c_{L^{\mathrm{mon}}}$ on L^{mon} (a priori defined by adjunction from the lax-monoidal structure on R^{mon}) coincides on the underlying objects with the Alexander-Whiteney colax-monoidal structure AW on L (the latter defines a colax-monoidal structure on monoids as the pair (∇, AW) obeys the bialgebra axiom, see Lemma 2.5).

Proof. Consider the definition (4.13) of the functor L^{mon}, in the special case of the Dold-Kan correspondence. It gives that $L^{\mathrm{mon}}(X)$ is the quotient of the free dg algebra $T(LX)$ by the *two-sided ideal*, generated by

$$AW(L(x \otimes y)) - L(x \star y), \quad x, y \in X \tag{4.14}$$

where $\star$ is the product in X.

Now the crucial idea, special for the case of Dold-Kan correspondence, is that *the subspace in $L(X) \otimes L(X)$ generated by the elements $AW(L(x \otimes y))$ coincides with the entire space $L(X) \otimes L(X)$.*

Indeed, we know from Proposition 3.2, (1.), that the composition

$$L(X)\otimes L(X)\xrightarrow{\nabla} L(X\otimes X)\xrightarrow{AW} L(X)\otimes L(X)$$

is the identity map of $L(X)\otimes L(X)$. Then, if we have any element $\omega\in L(X)\otimes L(X)$, we can write it as

$$\omega = AW(\nabla(\omega))$$

This claim simplifies the ideal (4.14). Namely, we have now that this ideal is generated by

$$L(a)\otimes L(b) - m_\star(\nabla(L(a)\otimes L(b))) \tag{4.15}$$

where the notation $m_\star$ in the second summand assumes the composition

$$L(X)\otimes L(X)\xrightarrow{\nabla} L(X\otimes X)\xrightarrow{L\circ\star} L(X)$$

The latter formula is precisely the equation (2.2), which defines the monoid structure on $F(X)$ from a lax-monoidal structure on F and from a monoid structure on X.

All other claims of Key-Lemma follow from this argument as well. $\square$

We continue to prove Lemma 4.12. The diagram (4.10) may be now rewritten as

$$\begin{tikzcd} LR(X\otimes Y) \arrow[rr] \arrow[dr, "p_{X\otimes Y}"'] & & LR(X)\otimes LR(Y) \arrow[dl, "p_X\otimes p_Y"] \\ & X\otimes Y & \end{tikzcd} \tag{4.16}$$

where the horizontal map comes from the colax-monoidal structure on R (adjoint to the lax-monoidal structure ∇ on L), and from the colax-monoidal structure AW on L. The projections comes from the adjunction maps.

Now the commutativity of diagram (4.16) is precisely the statement of Lemma 3.5 (ii). We are done.

$\square$

Theorem 4.1 is proven. $\square$

5 Diagrams

5.1 Colax-monoidal structure on a functor

Definition 5.1 (Colax-monoidal functor). Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two strict associative monoidal categories. A functor $F\colon \mathcal{M}_1\to\mathcal{M}_2$ is called *colax-monoidal* if there is a map of bifunctors $\beta_{X,Y}\colon F(X\otimes Y)\to F(X)\otimes F(Y)$ and a morphism $\alpha\colon F(1_{\mathcal{M}_1})\to 1_{\mathcal{M}_2}$ such that:

(1): for any three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{M}_1)$, the diagram

$$\begin{tikzcd}
& F(X \otimes Y) \otimes F(Z) \arrow[dr, "\beta_{X,Y} \otimes \mathrm{id}_{F(Z)}"] & \\
F(X \otimes Y \otimes Z) \arrow[ur, "\beta_{X \otimes Y, Z}"] \arrow[dr, "\beta_{X, Y \otimes Z}"'] & & F(X) \otimes F(Y) \otimes F(Z) \\
& F(X) \otimes F(Y \otimes Z) \arrow[ur, "\mathrm{id}_{F(X)} \otimes \beta_{Y,Z}"'] &
\end{tikzcd} \tag{5.1}$$

is commutative. The functors $\beta_{X,Y}$ are called the *colax-monoidal maps.*

(2): for any $X \in \mathrm{Ob}\mathcal{M}_1$ the following two diagrams are commutative

$$\begin{tikzcd}
F(1_{\mathcal{M}_1} \otimes X) \arrow[r, "\beta_{1,X}"] \arrow[d] & F(1_{\mathcal{M}_1}) \otimes F(X) \arrow[d, "\alpha \otimes \mathrm{id}"] & F(X \otimes 1_{\mathcal{M}_1}) \arrow[r, "\beta_{X,1}"] \arrow[d] & F(X) \otimes F(1_{\mathcal{M}_1}) \arrow[d, "\mathrm{id} \otimes \alpha"] \\
F(X) & 1_{\mathcal{M}_2} \otimes F(X) \arrow[l] & F(X) & F(X) \otimes 1_{\mathcal{M}_2} \arrow[l]
\end{tikzcd} \tag{5.2}$$

5.2 Lax-monoidal structure on a functor

Definition 5.2 (Lax-monoidal functor). Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two strict associative monoidal categories. A functor $G\colon \mathcal{M}_1 \to \mathcal{M}_2$ is called *lax-monoidal* if there is a map of bifunctors $\gamma_{X,Y}\colon G(X) \otimes G(Y) \to G(X \otimes Y)$ and a morphism $\kappa\colon 1_{\mathcal{M}_2} \to G(1_{\mathcal{M}_1})$ such that:

(1): for any three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{M}_1)$, the diagram

$$\begin{tikzcd}
& G(X \otimes Y) \otimes G(Z) \arrow[dl, "\gamma_{X \otimes Y, Z}"'] & \\
G(X \otimes Y \otimes Z) & & G(X) \otimes G(Y) \otimes G(Z) \arrow[ul, "\gamma_{X,Y} \otimes \mathrm{id}_{G(Z)}"'] \arrow[dl, "\mathrm{id}_{G(X)} \otimes \gamma_{Y,Z}"] \\
& G(X) \otimes G(Y \otimes Z) \arrow[ul, "\gamma_{X, Y \otimes Z}"] &
\end{tikzcd} \tag{5.3}$$

is commutative. The functors $\gamma_{X,Y}$ are called the *lax-monoidal maps.*

(2): for any $X \in \mathrm{Ob}\mathcal{M}_1$ the following two diagrams are commutative

$$\begin{tikzcd}
F(1_{\mathcal{M}_1} \otimes X) \arrow[d] & F(1_{\mathcal{M}_1}) \otimes F(X) \arrow[l, "\gamma_{1,X}"'] & F(X \otimes 1_{\mathcal{M}_1}) \arrow[d] & F(X) \otimes F(1_{\mathcal{M}_1}) \arrow[l, "\gamma_{X,1}"'] \\
F(X) & 1_{\mathcal{M}_2} \otimes F(X) \arrow[l] \arrow[u, "\kappa \otimes \mathrm{id}"'] & F(X) & F(X) \otimes 1_{\mathcal{M}_2} \arrow[l] \arrow[u, "\mathrm{id} \otimes \kappa"']
\end{tikzcd} \tag{5.4}$$

5.3 Bialgebra axiom

This axiom, expressing a compatibility between the lax-monoidal and colax-monoidal structures on a functor between *symmetric* monoidal categories, seems to be new.

Definition 5.3 (Bialgebra axiom). Suppose there are given both colax-monoidal and lax-monoidal structures on a functor $F\colon \mathcal{C}\to\mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are strict *symmetric* monoidal categories. Denote these structures by $c_F(X,Y)\colon F(X\otimes Y)\to F(X)\otimes F(Y)$, and $l_F\colon F(X)\otimes F(Y)\to F(X\otimes Y)$. We say that the pair (l_F,c_F) satisfies the bialgebra axiom, if for any for objects $X,Y,Z,W\in \mathrm{Ob}\mathcal{C}$, the following two morphisms $F(X\otimes Y)\otimes F(Z\otimes W)\to F(X\otimes Z)\otimes F(Y\otimes W)$ coincide:

$$\begin{gathered}F(X\otimes Y)\otimes F(Z\otimes W)\xrightarrow{l_F} F(X\otimes Y\otimes Z\otimes W)\xrightarrow{F(\mathrm{id}\otimes\sigma\otimes\mathrm{id})}\\ F(X\otimes Z\otimes Y\otimes W)\xrightarrow{c_F} F(X\otimes Z)\otimes F(Y\otimes W)\end{gathered}\tag{5.5}$$

and

$$\begin{gathered}F(X\otimes Y)\otimes F(Z\otimes W)\xrightarrow{c_F\otimes c_F} F(X)\otimes F(Y)\otimes F(Z)\otimes F(W)\xrightarrow{\mathrm{id}\otimes\sigma\otimes\mathrm{id}}\\ F(X)\otimes F(Z)\otimes F(Y)\otimes F(W)\xrightarrow{l_F\otimes l_F} F(X\otimes Z)\otimes F(Y\otimes W)\end{gathered}\tag{5.6}$$

where σ denotes the symmetry morphisms in $\mathcal{C}$ and in $\mathcal{D}$.

Thus, the commutative diagram, expressing the bialgebra axiom, is

$$\begin{tikzcd} F(X\otimes Y)\otimes F(Z\otimes W) \arrow[r, bend left, "(5.5)"] \arrow[r, bend right, "(5.6)"'] & F(X\otimes Z)\otimes F(Y\otimes W)\end{tikzcd}\tag{5.7}$$

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Max-Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, GERMANY

E-mail address: borya_port@yahoo.com