LIE ALGEBRAS WITH CONTINUAL ROOT SYSTEM AND CARTAN SUBALGEBRA IS $\mathbb{C}[h]$

B. Shoikhet, A. Vershik

To the memory of M. Saveliev

Let $\mathfrak{G} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{G}_i$ be a $\mathbb{Z}$-graded Lie algebra with continual root system in sense of [SV1, SV2], $\mathfrak{G}_0$ is its Cartan subalgebra which is itself associative commutative algebra and operator $K : \mathfrak{G}_0 \hookleftarrow$ is a Cartan operator.

In [V1] the following problem was posed: to describe all $\mathbb{Z}$-graded Lie algebras of that type which have algebra of polynomials $\mathbb{C}[h]$ of one variable as Cartan subalgebra $\mathfrak{G}_0$.

The known examples are - algebras of differential operators with polynomial coefficients on $\mathbb{R}$ (see [VS2], [KR]). More deep example had been suggested by B. Feigin [F] and well studied in [Sh] - algebras $\mathfrak{G}\ell_\lambda$ - a quotient of universal enveloping algebra of $s\ell(2,\mathbb{R})$ with the given values of Casimir operator. It is not completely trivial how to include $\mathfrak{G}\ell_\lambda$ in the scheme of [SV1, 2] - we will do this below.

The main result of this note is the description of all algebras of this type of "series A". It happened that all nondegenerated algebras of this type are Lie algebras with bracket $[a,b] = ab - ba$ for some associative algebras. So we are describing associative $\mathbb{Z}$-graded algebras with zero component $\mathbb{C}[h]$. Such problems have been studied in the papers [D], [S], [J]. They have considered the algebras with the generators e, h, f with relations $he - eh = e, hf - fh = -f, ef - fe = p(h)$ where p is a polynomial.

There are two important things which consist the difference with our approach:

1) We are looking on *Lie algebras* and so we can find the limit cases which are not correspond to any associative algebras;
2) The natural construction of those algebras is *cross-product with 2-cocycle*. See [V2] about Lie algebras generated by the dynamical system.

In our case this dynamical system is the shift on $\mathbb{R}$ (nondegenerated case) or flow on $\mathbb{R}$ (degenerated case) and 2-cocycle gives various Cartan operators which are either second difference or second derivative.

Except of $\mathfrak{G}\ell_\lambda$ our examples are new, the degenerate cases are of the most interest - it gives some Poisson-type brackets on some surfaces.

Our plan is the following: we start from the definition of associative algebras as the cross-product with cocycle (§ 1) and then (§ 2) gives canonical description of Lie algebras with continual root system (real line in our case) and with Cartan operator.

The authours were partially supported by IHES (France) and RFBR, grants 96-15-96060

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

§ 1. Associative algebras

Let $\mathbb{C}[h]$ be algebras of polynomials. Each automorphism of its generated by the shift $(T_t g)(h) = q(h+t)$. Let us fix $t \neq 0$ we have an action of $\mathbb{Z}$ on $\mathbb{C}[h]$ and can define 2-cocycle $\alpha(n,m)$ with values in $\mathbb{C}[h]$ accordingly to the $\mathbb{Z}$-action. Then we can define associative algebras $\mathbb{C}[\mathbb{Z}] \otimes_{\alpha,T_t} \mathbb{C}[h]$ as a cross product with action T_t and 2-cocycle $\alpha : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}[h]$.

First of all cocycle α is uniquely defined by its value $\alpha_p(+1,-1) \equiv p$ because $\alpha_p(-1,1) = p \circ T_t^{-1}$, $\alpha_p(n,m) = p \circ T_t^{n+m}$ if $nm > 0$ etc. Algebra $\mathbb{C}[\mathbb{Z}] \otimes_{\alpha,T_t} \mathbb{C}[h]$ as linear space is a space of linear combinations of type

$$\sum_{n\in\mathbb{Z}} q_n \otimes T_t^n, \quad q_n \in \mathbb{C}[h]$$

and multiplication is the following

$$(q_n \otimes T_t^n) \cdot (q_m \otimes T_t^m) = \{q_n \cdot (q_m \circ T_t^n) \cdot \alpha(n,m)\} \otimes T_t^{n+1}$$

If $\alpha \equiv \mathbf{1}$, then it is a skew product of $\mathbb{C}[\mathbb{Z}]$ on $\mathbb{C}[h]$. But we can define this algebra directly if we denote $1 \otimes T_t \equiv e, 1 \otimes T_t^{-1} \equiv f$ and h as its generators. Then we have the relations:

$$(q_n \otimes T_t^n) \cdot (q_m \otimes T_t^m) = \{q_n \cdot (q_m \circ T_t^n) \cdot \alpha(n,m)\} \otimes T_t^{n+1}$$

If $\alpha \equiv \mathbf{1}$, then it is a skew product of $\mathbb{C}[\mathbb{Z}]$ on $\mathbb{C}[h]$. But we can define this algebra directly if we denote $1 \otimes T_t \equiv e, 1 \otimes T^{-1} - t \equiv f$ and h as its generators. Then we have the relations:

$$e \cdot f = p(h), fe = p(T_t^{-1}h), he - eh = te, hf - fh = -tf$$

(see [J], [S]).

Lemma. Two cocycles α_p and $\alpha_{p'}$ are cohomologues iff $p'(h) = p(h+t')$ for some t'.

Proof. Direct calculations.

Definition. Let $\mathcal{M}_{p,t}$ denote the associative algebra $\mathbb{C}[Z] \otimes_{\alpha_p,T_t} \mathbb{C}[h]$ as defined above and consider the Lie algebra $Lie(\mathcal{M}_{p,t})$ with the brackets " $ab - ba$". Denote

$$\mathcal{L}(p,t) = [Lie(\mathcal{M}_{p,t}), Lie(\mathcal{M}_{p,t})]$$

the commutant of algerba $Lie(\mathcal{M}_{p,t}), p \in \mathbb{C}[h], t \in \mathbb{R}$. We will call $\mathcal{L}(p,t)$ *nondegenerated Lie algebra of type A with Cartan subalgebra* $\mathbb{C}[h]$ *and cocycle* $p \in \mathbb{C}[h]$.

Remark.

1. When $\deg p(h) = n > 0$, then codimension of the intersection of $\mathcal{L}(p,t)$ with $\mathbb{C}[h]$ in $\mathbb{C}[h]$ is equal to n-1.
2. If $t \neq 0$ we can put $t = 1$, because all algebras with the same p and $t \neq 0$ are isomorphic.
3. We can consider polynomial $p(h)$ of type

$$p(h) = a_1 h^n + a_2 h^{n-2} + \ldots + a_{n-1}h + a_n$$

because using the shift we can cancel the term of degree $n-1$.

We denote our algebra $\mathcal{L}(p,t) = \mathcal{L}(p)$. Another definition will be given in § 2.

§ 2 Lie algebras with continual root system

Define local Lie algebras as in the theory algebras Kac-Moody and as it defined for our case in [SV1, 2]. Local algebra is a sum

$$\mathfrak{G}_{-1} \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_1$$

where for our purpose each component as vector space is

$$\mathfrak{G}_i = \mathbb{C}[h], \qquad i = \pm 1, 0$$

and the brackets are defined as follows: we identify the elements of $\mathfrak{G}_i$ with $X_i(\phi), i = \pm 1, 0, \phi \in \mathbb{C}[h]$, then

$$[X_0(\phi), X_{\pm 1}(\psi)] = \pm X_{\pm 1}(K\phi \cdot \psi)$$
$$[X_{+1}(\phi), X_{-1}(\psi)] = X_0(\phi \cdot \psi)$$

(multiplication $\phi \cdot \psi$ etc. in $\mathbb{C}[h]$) where $K_{p,t} = \mathcal{D}_t \cdot L_p$ and

$$L_p : \mathbb{C}[h] \to \mathbb{C}[h], L_p q = pq$$
$$\mathcal{D}_t : \mathbb{C}[h] \to \mathbb{C}[h], (\mathcal{D}_t q)(h) = 2q(h) - q(h+t) - q(h-t)$$

$K_{p,t}$ is a Cartan operator (see [SV1, 2])
$\mathcal{D}_t$ - second difference operator
L_p - operator of multiplication on polynomial p.
We can put $t = 1$ as above. It is easy to check that Jacobi identity is true of our local algebra. Now define $\mathcal{F}_{p,t}$ - free Lie algebra generated by that local algebra and let $\mathfrak{M}$ - maximal nilpotent ideal in $\mathcal{F}_{p,t}$ which has zero intersection with $\mathfrak{G}_0$. The existence of that ideal could be proved in the same manner as in the theory of Kac-Moody algebras; this trick was used by Kac in the end of sixties, it allows to avoid a difficult probelm with the generalization of Serre's relations for our algebras. The Lie algebra $\mathcal{F}_{p,t}/\mathfrak{M}$ is called a *Lie algebra with root system* $\mathbb{R}$ *and a Cartan operator* $K_{p,t}$.

Theorem. *Lie algebras $\mathfrak{G}(\mathbb{C}[h], K_{p,t}) = \mathcal{F}_{p,t}/\mathfrak{M}$ and $\mathcal{L}(p,t)$ which was defined in § 1 are canonically isomorphic.*

■

It is a $\mathbb{Z}$-graded Lie algebra of "constant" growth (all graded components are isomorphic as $\mathfrak{G}_0$-modules). We preserve denotation $\mathfrak{G}(K_{p,t})$, $p \in \mathbb{C}[h], t \in \mathbb{R})$ for this Lie algebra. The presentation of it in the form $\mathcal{F}_{p,t}/\mathfrak{M}$ is called *canonical form of the Lie algebra with continual root system*, (see [SV1,SV2,V2]. These algebras play the role of $s\ell$ and so we called them of type A. It is possible to include classical $s\ell\ (n, \mathbb{C})$ and $\widetilde{s\ell}(n, \mathbb{C})$ in this scheme. (See [SV1, 2]). It is natural analog of Kac-Moody Lie algebras. Previous algebras can be called as nondegenerated. Now we define the limit (degenerated) case which is may be most of interest. In the case of rotation Lie algebra (=sine algebra=quantum torus, etc) this degenerated case gave us algebra of Poisson brackets on the 2-torus. The scheme of definition in this case is the same. It is clear that if $t \to 0$ then $\frac{1}{t^2}\mathcal{D}_t \to \frac{d^2}{dh^2}$, and

$$K_{p,1}(\cdot) \to \frac{d^2}{dh^2}(p\cdot) \equiv K_p.$$

This is defined as Lie algebra $\mathfrak{G}(K_p)$ as above. More exactly: canonical form of this is the following

$$[X_0(\phi), X_{\pm 1}(\psi)] = \pm X_{\pm 1}(\frac{d^2}{dh^2}(p\phi)\cdot\psi)$$
$$[X_{+1}(\phi), X_{-1}(\psi)] = X_0(\phi\cdot\psi).$$

The algebra $\mathfrak{G}(K_p)$ has no associative conterpart.

Main theorem. *Algebras $\mathfrak{G}(K_{p,1})$ and $\mathfrak{G}(K_p)$ when p runs over $\mathbb{C}[h]$ exhaust all Lie algebras of type A with Cartan subalgebras $\mathbb{C}[h]$.*

Examples.

1) If $p \equiv 1$, then $\mathfrak{G}(K_{p,1})$ is a product $\mathbb{C}(S^1)$ and $Lie(\mathbb{C}[\mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z}])$; the last one is the A_∞-algebra of infinite matrices.
2) If $p(h) = h$, the $\mathfrak{G}(K_{p,1})$ is the algebra of the differential operators with polynomial coefficients (see [KR]).
3) If $p(h) =$ quadratic polynomial then this algebra becomes $\mathfrak{G}\ell_\lambda$, in the sense of Feigin (see [F], [Sh]). In this case K is invertible. The algebra $\mathfrak{G}\ell_\lambda$ and its degenerated case are very important; the degenerated algebra is an algebra of Poisson brackets on the hyperboloid.
4) If $\deg p \geq 3$ we obtain completely a new class of algebras which will be considered in the other paper.

Literature

[SV1] M. Saveliev, A. Vershik **126** (1989), Comm.Math.P., 367-378.
[SV2] M. Saveliev, A. Vershik (Jan. 1990), Phys. Lett. A 143(3), 121-128.
[V1] A. Vershik **1510** (1992), SLNM, 397.
[KR] V. Kac, H. Radul **157** (1993), Comm. Math. Phys., 429-457.
[F] B. Feigin **43 (2)** (1988), Russian Math. Surv., 169-170.
[Sh] B. Shoikhet **92 (2)** (1998), Journal of Math. Sci., 3764-3806.
[D] J. Dixmier **24** (1973), J. of Algebra, 551-564.
[J] A. Joseph **28(3)** (1977), Isr. J. Math., 177-192.
[S] P. Smith **322(1)** (1990), TAMS, 285-314.
[V2] A. Vershik **4(6)** (1992), St. P. Math. J., 103-113.