Trajectory attractors for 2D Navier-Stokes system and some generalizations

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Introduction

We are dealing with the non-autonomous 2D Navier-Stokes system

$$\partial_t u + \nu L u + B(u) = g(x,t),\ (\nabla, u) = 0,\ u|_{\partial\Omega} = 0, \tag{1}$$

$x \in \Omega \subset\subset \mathbf{R}^2$, $t \geq 0, u = u(x,t) = (u^1, u^2) \equiv u(t)$, $g = g(x,t) = (g^1, g^2) \equiv g(t)$. Here $Lu = -P\Delta u$ is the Stokes operator, $\nu > 0$, $B(u) = P\sum_{i=1}^{2} u_i \partial_{x_i} u$; P is the orthogonal projector onto the space of divergence-free vector fields (see section 1).

Consider the autonomous case: $g(x,t) \equiv g(x)$, $g(x) \in H$ to begin with. Let for $t = 0$ we be given the initial condition:

$$u|_{t=0} = u_0(x),\ u_0(x) \in H. \tag{2}$$

The problem (1), (2) has a unique solution $u(t)$, $t \geq 0$, which can be represented in the form: $u(t) = S(t)u_0$. The family of mappings $\{S(t),\ t \geq 0\}$ forms *a semigroup*: $S(t_1)S(t_2) = S(t_1+t_2)\ \forall t_1, t_2 \geq 0$, $S(0) = Id$. A set $\mathbf{A} \subset H$ is said to be *an attractor* of this semigroup $\{S(t)\}$ (or *an attractor* of the equation (1)) if $\mathbf{A}$ is compact in H, $\mathbf{A}$ is strictly invariant with respect to $\{S(t)\} : S(t)\mathbf{A} = \mathbf{A}\ \forall t \geq 0$, and $\mathbf{A}$ attracts any bounded set B in H :

$$\mathbf{dist}_H(S(t)B, \mathbf{A}) \to 0\ (t \to +\infty)$$

(see, for example, [13], [20], [2], and the references cited there).

Non-autonomous equation (1) is less studied. Let an external force $g_0(x,t) \equiv g_0(t)$ in (1) depend on t, $t \geq 0$. Assume the function $g_0(t)$ is translation-compact in $L_2^{loc}(\mathbf{R}_+; H) \equiv L_2^{loc}$ (or $g_0(t)$ is translation-compact in $L_{2,w}^{loc}(\mathbf{R}_+; H) \equiv L_{2,w}^{loc}$). This means that the family of translations $\{g_0(t+h) \mid h \geq 0\}$ forms a precompact set in L_2^{loc} (respectively, in $L_{2,w}^{loc}$). It is easy to formulate the translation-compact criterions

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(see section 1). For example, a function $g_0(t)$ is translation-compact in $L^{loc}_{2,w}$ if and only if the following norm is bounded:

$$\|g_0\|_a^2 = \sup_{t \geq 0} \int_t^{t+1} |g_0(s)|^2 ds < +\infty. \tag{3}$$

By $\mathcal{H}_+(g_0)$ denote a hull of the function g_0 in the space $L^{loc}_{2,w}$, i.e.

$$\mathcal{H}_+(g_0) = [\{g(t+h) \mid h \geq 0\}]_{L^{loc}_{2,w}},$$

where $[\,.\,]_X$ means the closure in a topological space X.

Consider the family of equations (1) with external forces $g(t) \in \mathcal{H}_+(g_0) \equiv \Sigma$. Let $\{U_g(t,\tau) \mid t \geq \tau \geq 0\}$ be the family of operators (called *a process in* H) such that $U_g(t,\tau)u_\tau = u_g(t)$, $t \geq \tau \geq 0$, where $u_g(t)$ is a solution of equation (1) with the external force $g(t)$ and with the initial condition $u|_{t=\tau} = u_\tau(x)$, $u_\tau(x) \in H$. Evidently, $U_g(t,\tau) : E \to E$, $U_g(t,\theta)U_g(\theta,\tau) = U_g(t,\tau)$, $U_g(\tau,\tau) = Id\ \forall t \geq \theta \geq \tau \geq 0$. Consider the family of processes $\{U_g(t,\tau) \mid g \in \mathcal{H}_+(g_0)\}$ corresponding to the family of equations (1) with external forces $g \in \mathcal{H}_+(g_0)$. (To compare with autonomous case, $g_0(t) \equiv g_0$, $\mathcal{H}_+(g_0) = \{g_0\}$, $U_g(t,\tau) = S(t-\tau)$). It was proved that the family of processes $\{U_g(t,\tau) \mid g \in \mathcal{H}_+(g_0)\}$ possesses a uniform (w.r.t. $g \in \Sigma$) attractor $\mathbf{A}_\Sigma$ in H. More precisely, the set $\mathbf{A}_\Sigma$ is compact in H, $\mathbf{A}_\Sigma$ attracts any bounded set B in H uniformly w.r.t. $g \in \Sigma$:

$$\sup_{g \in \Sigma} \mathbf{dist}_H(U_g(t,\tau)B, \mathbf{A}_\Sigma) \to 0\ (t \to +\infty)\ \forall \tau \geq 0,$$

and $\mathbf{A}_\Sigma$ is a minimal compact, uniformly attracting set (see [9], [6], and [4] dealing with more restrictive case). In [6], [4] the structure and the properties of the uniform attractor for (1) was also studied.

In the present work we introduce and we study *a trajectory attractor* $\mathcal{A}_\Sigma$ for equation (1). We point out at once that a trajectory attractor $\mathcal{A}_\Sigma$ is a compact set in the corresponding trajectory space of equations (1) that consists of their solutions $u_g(t), t \geq 0$, considering as a whole as functions of t with values in H. In the previous considerations, an attractor $\mathbf{A}_\Sigma$ was a compact subset of points in H.

Consider as before a fixed external force $g_0(t)$ being a translation-compact function in L_2^{loc} (or in $L^{loc}_{2,w}$) and let $\mathcal{H}_+(g_0) \equiv \Sigma$ be a hull of g_0 in L_2^{loc}. (The case when $g_0(t)$ is translation-compact in $L^{loc}_{2,w}$ is studied in section 1). Let $H^{\mathbf{r}}(Q_{t_1,t_2}), \mathbf{r} = (2,2,1)$ be the Nicolskiy space in $Q_{t_1,t_2} = \Omega \times]t_1,t_2[$(see [3]) of functions $\varphi(x,t) = \varphi(t) = (\varphi^1\varphi^2) \in H$, $t \in]t_1,t_2[$ with a finite norm

$$\|\varphi\|^2_{H^{\mathbf{r}}(Q_{t_1,t_2})} = \int_{Q_{t_1,t_2}} \left(\sum_{|\alpha| \leq 2} |\partial_x^\alpha \varphi(x,t)|^2 + |\partial_t \varphi(x,t)|^2 \right) dxdt,\ \varphi|_{\partial\Omega} = 0.$$

To any external force $g(t) \in \mathcal{H}_+(g_0)$ there corresponds the trajectory space $\mathcal{K}_g^+$. The space $\mathcal{K}_g^+$ is the union of all solutions $u(t) = u_g(t), t \geq 0$, of equation (1) in the space $H^{\mathbf{r},loc}(Q_+) \equiv H^{\mathbf{r},loc}$, $Q_+ = \Omega \times]0,+\infty[$, (i.e. $u(t) \in H^{\mathbf{r}}(Q_{t_1,t_2})$ for any $]t_1,t_2[\subset \mathbf{R}_+$). Let $\mathcal{K}^+ = \bigcup_{g\in\mathcal{H}_+(g_0)} \mathcal{K}_g^+$ be a union of all $\mathcal{K}_g^+$. The translation semigroup $\{T(h) \mid h \geq 0\}$ acts on $H^{\mathbf{r},loc}$:

$$T(h)\varphi(t) = \varphi(t+h), \; h \geq 0.$$

Evidently, $T(h)u_g(t) = u_g(t+h) = u_{T(h)g}(t) \in \mathcal{K}^+_{T(h)g}$. Therefore,

$$T(h)\mathcal{K}^+ \subseteq \mathcal{K}^+ \; \forall h \geq 0 \tag{4}$$

(the inclusion may be strict, see section 1). It is proved that $\mathcal{K}^+$ is closed in $H^{\mathbf{r},loc}$. It is clear that the semigroup $\{T(h)\}$ is continuous on $H^{\mathbf{r},loc}$. Denote by $H^{\mathbf{r},a}(Q_+) \equiv H^{\mathbf{r},a}$ a subset of $H^{\mathbf{r},loc}$ of functions $\varphi(t)$, $t \geq 0$, having a finite norm

$$\|\varphi\|^2_{H^{\mathbf{r},\mathbf{a}}} = \sum_{|\alpha|\leq 2} \|\partial_x^\alpha \varphi\|_a^2 + \|\partial_t\varphi\|_a^2 < +\infty,$$

where $\|\,.\,\|_a^2$ is defined in (3).

A trajectory attractor of the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}^+$ is a set $\mathcal{A}_\Sigma \subseteq \mathcal{K}^+$ such that $\mathcal{A}_\Sigma$ is a set, compact in $H^{\mathbf{r},loc}$, bounded in $H^{\mathbf{r},a}$, invariant with respect to $\{T(h)\}$: $T(h)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma \; \forall h \geq 0$, and satisfying the following attracting property: for any set $B \subset \mathcal{K}^+$, bounded in $H^{\mathbf{r},a}$,and for any $[t_1,t_2] \subset \mathbf{R}_+$ the set $T(h)B$ tends to $\mathcal{A}_\Sigma$ in the strong topology of the space $H^{\mathbf{r}}(Q_{t_1,t_2})$ i.e.

$$\mathbf{dist}_{H^{\mathbf{r}}(Q_{t_1,t_2})}(T(h)B, \mathcal{A}_\Sigma) \to 0 \; (h \to +\infty). \tag{5}$$

In section 2, we construct the trajectory attractor $\mathcal{A}_\Sigma$ of the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}^+$. Section 1 deals with the trajectory attractor $\mathcal{A}_\Sigma$ in "weak" topology $H_w^{\mathbf{r},loc}(Q_+)$ under the assumption that $g_0(t)$ is translation-compact in $L^{loc}_{2,w}$ only. In this case $T(h)B$ tends to $\mathcal{A}_\Sigma$ in the weak topology of the space $H^{\mathbf{r}}(Q_{t_1,t_2})$ for any $[t_1,t_2] \subset \mathbf{R}_+$. In section 3, the structure of the trajectory attractor $\mathcal{A}_\Sigma$ is described.

Trajectory attractors have been constructed for various equations and systems of PDE for which the corresponding Cauchy problem has non-unique solution or for which the uniqueness theorem is not proved yet (see [7], [8], [9], [10], and [5]).

In section 4 we construct the trajectory attractor for 3D Navier-Stokes system; the structure and some properties of the trajectory attractor are given as well. In particular, the trajectory attractor $\mathcal{A}_\Sigma$ is stable with respect to a small perturbation of the external force $g_0(x,t)$; the trajectory attractor $\mathcal{A}_\Sigma^{(N)}$ of the Faedo-Galerkin approximation system of order N tends to $\mathcal{A}_\Sigma$ as $N \to +\infty$ in the corresponding topology. Some of other properties turns out to be unexpected.

1 Trajectory attractor for 2D N.-S. system with translation-compact external force in $L^{loc}_{2,w}$.

We consider the Navier-Stokes system in a bounded domain $\Omega \subset\subset \mathbf{R}^2$. Excluding the pressure, the system can be written in the form:

$$\partial_t u + \nu Lu + B(u) = g(x,t),\ (\nabla, u) = 0,\ u|_{\partial\Omega} = 0,\ x \in \Omega,\ t \geq 0, \tag{1.1}$$

where, $x = (x_1, x_2)$, $u = u(x,t) = (u^1, u^2)$, $g = g(x,t) = (g^1, g^2)$. L is the Stokes operator: $Lu = -P\Delta u$; $B(u) = B(u,u)$, $B(u,v) = P(u,\nabla)v = P\sum_{i=1}^2 u_i\partial_{x_i}v$, $\nu > 0$ (see [16], [15], [19], [21]). By H, V, and H_2 denote respectively the closure in $(L_2(\Omega))^2$, $(H^1(\Omega))^2$, and $(H^2(\Omega))^2$ of the set $\mathcal{V}_0 = \{v \mid v \in (C_0^\infty(\Omega))^2,\ (\nabla, v) = 0\}$. P denotes the orthogonal projector in $(L_2(\Omega))^2$ onto the Hilbert space H. The scalar products in H and in V are $(u,v) = \int_\Omega (u(x), v(x))dx$ and $((u,v)) = \langle Lu, v\rangle = \int_\Omega (\nabla u(x), \nabla v(x))dx$ and the norms are respectively $|u| = (u,u)^{1/2}$ and $\|u\| = \langle Lu, u\rangle^{1/2}$. The norm in H_2 is $\|\,.\,\|_2$.

To describe an external force $g(x,s)$ in (1.1) consider the topological space $L^{loc}_{2,w}(\mathbf{R}_+, H)$. By the definition, the space $L^{loc}_{2,w}(\mathbf{R}_+, H) = L^{loc}_{2,w}$ is $L^{loc}_2(\mathbf{R}_+, H) = L^{loc}_2$ endowed with the following local weak convergence topology. The sequence $\{g_n(s)\}$ converges to $g(s)$ as $n \to \infty$ in $L^{loc}_{2,w}$ whenever $\int_{t_1}^{t_2} (g_n(s) - g(s), v(s))\, ds \to 0$ $(n \to \infty)$ for any $[t_1, t_2] \subseteq \mathbf{R}_+$ and any $v(s) \in L_2(t_1, t_2; H)$.

Let we are given some fixed external force $g_0(s) \in L^{loc}_2$. Assume $g_0(s)$ is translation-compact (tr.-c.) in $L^{loc}_{2,w}$, i.e. the set $\{g_0(s+h) \mid h \in \mathbf{R}_+\}$ is precompact in $L^{loc}_{2,w}$. This condition is valid if and only if

$$\|g_0\|^2_{L_2^b(\mathbf{R}_+;H)} = \|g_0\|^2_a = \sup_{t\geq 0} \int_t^{t+1} |g_0(s)|^2 ds < +\infty \tag{1.2}$$

(see [6]). By $\mathcal{H}_+(g_0)$ denote the hull of a function $g_0(s)$ in $L^{loc}_{2,w}$: $\mathcal{H}_+(g_0) = [\{g_0(s+h) \mid h \in \mathbf{R}_+\}]_{L^{loc}_{2,w}}$. Here $[\,.\,]_{L^{loc}_{2,w}}$ means the closure in $L^{loc}_{2,w}$. It can be shown that the set $\mathcal{H}_+(g_0)$ being a topological subspace of $L^{loc}_{2,w}$ is metrizable and the corresponding metric space is complete. Moreower, any function $g(s) \in \mathcal{H}_+(g_0)$ is tr.-c. in $L^{loc}_{2,w}$, $\mathcal{H}_+(g) \subseteq \mathcal{H}_+(g_0)$, and $\|g\|_a \leq \|g_0\|_a$.

The translation semigroup $\{T(t) \mid t \geq 0\} = \{T(t)\}$ acts on $\mathcal{H}_+(g_0) : T(t)g(s) = g(s+t)$. Evidently, $T(t)$ is continuous in $L^{loc}_{2,w}$ and $T(t)\mathcal{H}_+(g_0) \subseteq \mathcal{H}_+(g_0)$ $\forall t \geq 0$.

We shall study the family of equations (1.1) with various external forces $g(\,.\,,s) \in \mathcal{H}_+(g_0)$

By Q_{t_1,t_2} denote the cylinder $\Omega \times [t_1, t_2]$, where $[t_1, t_2] \subset \mathbf{R}_+$.

Consider the space $H^{\mathbf{r}}(Q_{t_1,t_2})$, $\mathbf{r} = (2,2,1)$ (see [3]), $H^{\mathbf{r}}(Q_{t_1,t_2}) = L_2(t_1, t_2; H_2) \cap \{\partial_t v \in L_2(t_1, t_2; H)\}$. The norm in $H^{\mathbf{r}}(Q_{t_1,t_2})$ is

$$\|v\|^2_{H^{\mathbf{r}}(Q_{t_1,t_2})} = \int_{t_1}^{t_2} \left(\|v(s)\|_2^2 + |\partial_t v(s)|^2\right) ds. \tag{1.3}$$

Let us recall the existence and uniqueness theorem.

Theorem 1.1 *Let $g(s) \in L_2(t_1, t_2; H)$ and $u_0 \in V$. Then there exists a unique solution $u(s)$ of equation (1.1) belonging to the space $H^{\mathbf{r}}(Q_{t_1,t_2})$ such that $u(t_1) = u_0$. Moreover, $u(s) \in C([t_1, t_2], V)$.*

This theorem is a variant of the classical result (see [14], [15], [16], [19], [2]). The proof uses the Faedo-Galerkin approximations method.

We shall study equation (1.1) in the semicylinder $Q_+ = \Omega \times \mathbf{R}_+$, where $g(\,.\,, s) \in \mathcal{H}_+(g_0)$.

Consider the space $H^{\mathbf{r},loc}(Q_+) = L_2^{loc}(\mathbf{R}_+; H_2) \cap \{\partial_t v \in L_2^{loc}(\mathbf{R}_+; H)\}$, i.e. $v(s) \in H^{\mathbf{r},loc}(Q_+)$ if $\|\Pi_{t_1,t_2} v\|^2_{H^{\mathbf{r}}(Q_{t_1,t_2})} < +\infty$ for any $[t_1, t_2] \subset \mathbf{R}_+$, where Π_{t_1,t_2} is the restriction operator onto the segment $[t_1, t_2]$. We introduce two different topological spaces $H_s^{\mathbf{r},loc}(Q_+)$ and $H_w^{\mathbf{r},loc}(Q_+)$, ("strong" and "weak"). The space $H_s^{\mathbf{r},loc}(Q_+)$ ($H_w^{\mathbf{r},loc}(Q_+)$) is $H^{\mathbf{r},loc}(Q_+)$ with the following convergence topology. By the definition, $v_n(s) \to v(s)$ $(n \to \infty)$ in $H_s^{\mathbf{r},loc}(Q_+)$ (in $H_w^{\mathbf{r},loc}(Q_+)$) if $\Pi_{t_1,t_2} v_n(s) \to \Pi_{t_1,t_2} v(s)$ $(n \to \infty)$ strongly in $H^{\mathbf{r}}(Q_{t_1,t_2})$ (respectively, $\Pi_{t_1,t_2} v_n(s) \rightharpoonup \Pi_{t_1,t_2} v(s)$ $(n \to \infty)$ weakly in $H^{\mathbf{r}}(Q_{t_1,t_2})$) for any $[t_1, t_2] \subset \mathbf{R}_+$. It is easy to prove that the linear topological space $H_s^{\mathbf{r},loc}(Q_+)$ is metrizable, for example, by means of the Fréchet metric generated by the seminorms $\|\Pi_{n,n+1} v\|_{H^{\mathbf{r}}(Q_{n,n+1})}$, $n = 0, 1, 2, \ldots$. The space $H_w^{\mathbf{r},loc}(Q_+)$ is not metrizable, but it is a Hausdorff and Fréchet-Uryson space with a countable topology base.

We shall use also the space $H^{\mathbf{r},a}(Q_+)$ that is a subspace of $H^{\mathbf{r},loc}(Q_+)$. By the definition, $v(s) \in H^{\mathbf{r},a}(Q_+)$, if the following norm is finite

$$\|v\|^2_{H^{\mathbf{r},\mathbf{a}}(Q_+)} = \|v\|^2_{\mathbf{r},a} = \sup_{t \geq 0} \|\Pi_{t,t+1} v\|^2_{H^{\mathbf{r}}(Q_{t,t+1})}. \tag{1.4}$$

Evidently, $H^{\mathbf{r},a}(Q_+)$ with the norm (1.4) is a Banach space. We shall not use the topology generated by the norm (1.4). We need the Banach space $H^{\mathbf{r},a}(Q_+)$ to define bounded sets in $H^{\mathbf{r},loc}(Q_+)$ only.

We put into correspondence to any external force $g(\,.\,, s) \in \mathcal{H}_+(g_0)$ the trajectory space $\mathcal{K}_g^+$ that is the union of all solutions $u(s), s \geq 0$, of equation (1.1) in the space $H^{\mathbf{r},loc}(Q_+)$. Notice that $|B(v)| \leq C|v|^{1/2}\|v\|^2\|v\|_2^{1/2}$; therefore any solution $u(s) \in \mathcal{K}_g^+$ satisfies (1.1) in the strong sence of the space $L_2^{loc}(\mathbf{R}_+, H)$. By Theorem 1.1, the trajectory space $\mathcal{K}_g^+$ is wide enough for any $g \in \mathcal{H}_+(g_0)$. Denote, $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)} = \cup_{g \in \mathcal{H}_+(g_0)} \mathcal{K}_g^+$.

Lemma 1.1 *If $g_0(s) \in L_2^{loc}(\mathbf{R}_+, H)$ satisfies (1.2) then $\mathcal{K}^+ \subset H^{\mathbf{r},a}(Q_+)$.*

This lemma will be proved later on.

Consider the translation semigroup $\{T(t) \mid t \geq 0\}$ acting on $H^{\mathbf{r},loc}(Q_+)$ by the formula

$$T(t)v(s) = v(s+t),\ s \geq 0,\ v \in H^{\mathbf{r},loc}(Q_+).$$

Obviously, the family of trajectory spaces $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$ corresponding to the equation (1.1) satisfies the embedding:

$$T(t)\mathcal{K}_g^+ \subseteq \mathcal{K}^+_{T(t)g},\ \forall t \geq 0. \tag{1.5}$$

In other words, for any $t \geq 0$, a function $u(s+t)$, $s \geq 0$, is a solution of equation (1.1) with a shifted symbol $g(s+t) = T(t)g(s)$ for any solution $u(s) \in \mathcal{K}_g^+$ of equation (1.1) with a symbol $g(s) \in \mathcal{H}_+(g_0)$. Hence, the translation semigroup $\{T(t)\}$ takes $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ into itself: $T(t)\mathcal{K}^+ \subseteq \mathcal{K}^+$, $t \geq 0$.

In this section we study the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ attracts any set $T(t)B$ as $t \to +\infty$ in the topology of $\Theta_+^{loc} = H_w^{\mathbf{r},loc}(Q_+)$, where $B \subset \mathcal{K}^+$ and B is bounded in the Banach space $\mathcal{F}_+^a = H^{\mathbf{r},a}(Q_+)$.

Definition 1.1 *Let Σ be a complete metric space and let Θ be a topological space. Consider a family of sets $\{\mathcal{K}_\sigma,\ \sigma \in \Sigma\}$, $\mathcal{K}_\sigma \subset \Theta$, depending on a parameter $\sigma \in \Sigma$. The family $\{\mathcal{K}_\sigma,\ \sigma \in \Sigma\}$ is said to be* (Θ, Σ)-closed *if the graph set $\cup_{\sigma\in\Sigma}\mathcal{K}_\sigma \times \{\sigma\}$ is closed in the topological space $\Theta \times \Sigma$ with a usual product topology.*

Proposition 1.1 *Let Σ be a compact metric space and $\{\mathcal{K}_\sigma, \sigma \in \Sigma\}$ be (Θ, Σ)-closed; then the set $\mathcal{K}_\Sigma = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma$ is closed in Θ.*

Proof. We use the standard reasoning. Let $u \notin \mathcal{K}_\Sigma = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma$. Therefore, $(u, \sigma) \notin \cup_{\sigma'\in\Sigma}\mathcal{K}_{\sigma'} \times \{\sigma'\}$ for any $\sigma \in \Sigma$. The set $\cup_{\sigma'\in\Sigma}\mathcal{K}_{\sigma'} \times \{\sigma'\}$ is closed in $\Theta \times \Sigma$, so, there is a neighbourhood $\mathcal{W}_u \times \mathcal{O}_\sigma$ in $\Theta \times \Sigma$ such that $\mathcal{W}_\sigma \times \mathcal{O}_\sigma \cap (\cup_{\sigma'\in\Sigma}\mathcal{K}_{\sigma'} \times \{\sigma'\}) = \emptyset$, $u \in \mathcal{W}_\sigma$, $\sigma \in \mathcal{O}_\sigma$, where $\mathcal{W}_\sigma$ and $\mathcal{O}_\sigma$ are open sets in Θ and Σ respectively. The family $\{\mathcal{O}_\sigma \mid \sigma \in \Sigma\}$ forms an open covering of Σ. Since Σ is compact, there is a finite subcovering $\{\mathcal{O}_{\sigma_i} \mid i = 1, \dots, N\}$. Put $\mathcal{W}(u) = \bigcap_{i=1}^{N} \mathcal{W}_{\sigma_i}$. Evidently, $\mathcal{W}(u) \cap \mathcal{K}_\Sigma = \emptyset$. Hence, for any $u \notin \mathcal{K}_\Sigma$ there is a neighbourhood $\mathcal{W}(u) \cap \mathcal{K}_\Sigma = \emptyset$., i.e. $\mathcal{K}_\Sigma$ is closed in Σ. □

Lemma 1.2 *The family of trajectory spaces $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$ corresponding to the equation (1.1) is $(\Theta_+^{loc}, \mathcal{H}_+(g_0))$-closed and $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ is closed in Θ_+^{loc}.*

Proof. Assume that $u_n(s) \in \mathcal{K}_{g_n}$, $g_n \in \mathcal{H}_+(g_0)$, $u_n(s) \to u(s)$ $(n \to +\infty)$ in Θ_+^{loc} and $g_n(s) \to g(s)$ $(n \to +\infty)$ in $L_{2,w}^{loc}$. We claim that $u \in \mathcal{K}_g^+$. Indeed, for any fixed $[t_1, t_2] \subset \mathbf{R}_+$ we have: $u_n(s) \rightharpoonup u(s)$ $(n \to +\infty)$ weakly in $H^{\mathbf{r}}(Q_{t_1,t_2})$. Thus, $\partial_t u_n(s) \rightharpoonup \partial_t u(s)$ $(n \to +\infty)$ weakly in $L_2(t_1, t_2; H)$ and $\partial^\alpha u(s) \rightharpoonup \partial^\alpha u(s)$ $(n \to +\infty)$ weakly in $L_2(t_1, t_2; H)$ for any $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| \leq 2$. In particular, by refining, we may assume that $u_n(s) \to u(s)$ $(n \to +\infty)$ almost everywhere in Q_{t_1,t_2} and $B(u_n(s)) \rightharpoonup B(u(s))$ $(n \to +\infty)$ weakly in $L_2(t_1, t_2; H)$ (see compactness theorems in [16], [19]). Therefore we may, in the equation

$$\partial_t u_n + \nu L u_n + B(u_n) = g_n(x, t),$$

pass to the limit as $n \to \infty$ weakly in $L_2(t_1, t_2; H)$ and get

$$\partial_t u + \nu L u + B(u) = g(x, t),$$

so that $u(s) \in \mathcal{K}_g^+$. Finally, it follows from Proposition 1.1 that $\mathcal{K}^+_{\mathcal{H}_+(g_0)}$ is closed in Θ_+^{loc} since $\Sigma = \mathcal{H}_+(g_0)$ is a compact metric space. □

Consider the translation semigroup $\{T(t)\}$ acting on the metric space $\mathcal{H}_+(g_0)$. Evidently, the semigroup $\{T(t)\}$ is continuous in $\mathcal{H}_+(g_0)$.

Definition 1.2 *A set* $\mathbf{A}$ *is said to be a global attractor of a semigroup* $\{S(t)\}$ *acting on a complete metric space* X, *if* **(i)** $\mathbf{A}$ *is compact in* X *and* $\mathbf{A}$ *attracts any bounded set* $B : \mathbf{dist}_X(S(t)B, \mathbf{A}) \to 0$ $(t \to \infty)$; **(ii)** $S(t)\mathbf{A} = \mathbf{A}$ *for any* $t \geq 0$.

For the case $X = \Sigma = \mathcal{H}_+(g_0)$ we have

Proposition 1.2 *The translation semigroup* $\{T(t)\}$ *acting on the compact metric space* $\Sigma = \mathcal{H}_+(g_0)$ *possesses a global attractor* $\mathbf{A}$ *which coincides with the* ω*-limit set of* Σ*:*

$$\mathbf{A} = \omega(\Sigma) = \bigcap_{t\geq 0}\left[\bigcup_{h\geq t} T(h)\Sigma\right]_\Sigma, \quad \omega(\Sigma) \subseteq \Sigma,$$

where $[\,.\,]_\Sigma$ *means the closure in* Σ. *Moreower,* $T(t)\omega(\Sigma) = \omega(\Sigma)$ $\forall t \geq 0$.

This statement follows from well-known theorems from the theory of attractors of semigroups acting in metric spaces (see, for example, [2], [20], [13]).

Consider more general scheme. Let Σ be a complete metric space. Let also $\mathcal{F}$ be a Banach space. Assume $\mathcal{F} \subseteq \Theta$, where Θ is a Hausdorff topological space. Let a semigroup $\{T(t)\}$ acts on $\Theta : T(t)\Theta \subseteq \Theta$, $t \geq 0$. Let we be given a family of sets $\{\mathcal{K}_\sigma, \sigma \in \Sigma\}$, $\mathcal{K}_\sigma \subseteq \mathcal{F}$. Put $\mathcal{K}_\Sigma = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma$.

Definition 1.3 *A set* $P \subseteq \Theta$ *is said to be* a uniformly (w.r.t. $\sigma \in \Sigma$) attracting set *for the family* $\{\mathcal{K}_\sigma, \sigma \in \Sigma\}$ *in the topology* Θ *if for any bounded set* B *in* $\mathcal{F}$ *and* $B \subseteq \mathcal{K}_\Sigma$, *the set* P *attracts* $T(t)B$ *as* $t \to +\infty$ *in the topology* Θ, *i.e. for any neighbourhood* $\mathcal{O}(P)$ *in* Θ *there exists* $t_1 \geq 0$ *such that* $T(t)B \subseteq \mathcal{O}(P)$ *for any* $t \geq t_1$.

Definition 1.4 *A set* $\mathcal{A}_\Sigma \subseteq \Theta$ *is said to be* a uniform (w.r.t. $\sigma \in \Sigma$) attractor *of the semigroup* $\{T(t)\}$ *on* $\mathcal{K}_\Sigma$ *in the topology* Θ, *if* $\mathcal{A}_\Sigma$ *is compact in* Θ *and* $\mathcal{A}_\Sigma$ *is a minimal compact uniformly attracting set of* $\{\mathcal{K}_\sigma, \sigma \in \Sigma\}$, *i.e.* $\mathcal{A}_\Sigma$ *belongs to any compact uniformly attracting set* P *of* $\{\mathcal{K}_\sigma, \sigma \in \Sigma\} : \mathcal{A}_\Sigma \subseteq P$.

Let a semigroup acts on Σ, which we denote $\{T(t)\} : T(t)\Sigma \subseteq \Sigma$, $t \geq 0$.

Definition 1.5 *The family of trajectory spaces* $\{\mathcal{K}_\sigma, \sigma \in \Sigma\}$ *is said to be* translation-coordinated (tr.-coord.) *if for any* $\sigma \in \Sigma$ *and any* $u \in \mathcal{K}_\sigma$

$$T(t)u \in \mathcal{K}_{T(t)\sigma} \quad \forall t \geq 0.$$

It follows from (1.5) that the family $\{\mathcal{K}_g^+, g \in \mathcal{H}_+(g_0)\}$ is tr.-coord. with respect to the translation semigroup $\{T(t)\}$.

Proposition 1.3 *Let Σ be a compact metric space and let a continuous semigroup $\{T(t)\}$ act on Σ and on Θ : $T(t)\Sigma \subseteq \Sigma$, $T(t)\Theta \subseteq \Theta$, $t \geq 0$. Let we be given a family of sets $\{\mathcal{K}_\sigma, \sigma \in \Sigma\}$, $\mathcal{K}_\sigma \subseteq \mathcal{F}$. Assume, the family $\{\mathcal{K}_\sigma, \sigma \in \Sigma\}$ is (Θ, Σ)-closed and tr.-coord. Let there exist a uniformly (w.r.t. $\sigma \in \Sigma$) attracting set P for $\{\mathcal{K}_\sigma,\ \sigma \in \Sigma\}$ in Θ, such that P is compact in Θ and P is bounded in $\mathcal{F}$. Then the semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma$ possesses the uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathcal{A}_\Sigma \subseteq \mathcal{K}_\Sigma \cap P$ in the space Θ,*

$$T(t)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma \ \forall\, t \geq 0. \tag{1.6}$$

Moreover,

$$\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)},$$

where $\mathcal{A}_{\omega(\Sigma)}$ is the uniform (w.r.t. $\sigma \in \omega(\Sigma)$) attractor of the family $\{\mathcal{K}_\sigma, \sigma \in \omega(\Sigma)\}$, $\mathcal{A}_{\omega(\Sigma)} \subseteq \mathcal{K}_{\omega(\Sigma)}$. Here $\omega(\Sigma)$ is the attractor of the semigroup $\{T(t)\}$ on Σ, $T(t)\omega(\Sigma) = \omega(\Sigma)$. The set $\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}$ is compact in Θ and bounded in $\mathcal{F}$.

The proof of Proposition 1.3 is given in [5], (see also [10]).

In application to the Navier-Stokes system (1.1) in this section, $\Sigma = \mathcal{H}_+(g_0)$, $\mathcal{F} = \mathcal{F}_+^a = H^{\mathbf{r},a}(Q_+)$, $\Theta = \Theta_+^{loc} = H_w^{\mathbf{r},loc}(Q_+)$, $\{T(t)\}$ is the translation semigroup, and $\{\mathcal{K}_g^+, g \in \mathcal{H}_+(g_0)\}$ is the family of trajectory spaces of equation (1.1). In this case a uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ is called *a trajectory attractor* of the family $\{\mathcal{K}_g^+, g \in \mathcal{H}_+(g_0)\}$. In the next section we shall consider the "strong" space $\Theta = \Theta_+^{loc} = H_s^{\mathbf{r},loc}(Q_+)$.

Let us formulate the main result of this section.

Theorem 1.2 *Let $g_0(s)$ be tr.-c. in $L_{2,w}^{loc}(\mathbf{R}_+, H)$ then the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ possesses a trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ in $\Theta_+^{loc} = H_w^{\mathbf{r},loc}(Q_+)$; the set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ attracts any set $B \subseteq \mathcal{K}^+$, bounded in $\mathcal{F}_+^a = H^{\mathbf{r},a}(Q_+)$. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ is bounded in $\mathcal{F}^+$, compact in Θ_+^{loc}, and it is invariant with respect to the translation semigroup: $T(t)\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\mathcal{H}_+(g_0)}$ for any $t \geq 0$. Moreower,*

$$\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(g_0))}, \tag{1.7}$$

where $\mathcal{A}_{\omega(\mathcal{H}_+(g_0))}$ is the trajectory attractor of the family $\{\mathcal{K}_g, g \in \omega(\mathcal{H}_+(g_0))\}$, $\mathcal{A}_{\omega(\mathcal{H}_+(g_0))} \subseteq \mathcal{K}_{\omega(\mathcal{H}_+(g_0))}$. Any function $u(s) \in \mathcal{A}_{\mathcal{H}_+(g_0)}$ is tr.-c. in Θ_+^{loc}.

Notice that the topology of the space $H_w^{\mathbf{r}}(Q_{t_1,t_2})$ is stronger then the uniform convergence topology of the space $C([t_1,t_2]; H)$, $H_w^{\mathbf{r}}(Q_{t_1,t_2}) \subset C([t_1,t_2]; H)$. So, we have

Corollary 1.1 *For any set $B \subset \mathcal{K}^+$, bounded in $\mathcal{F}^+$, one has*

$$\mathbf{dist}_{C([t_1,t_2];H)}\left(\Pi_{0,\Gamma}T(t)B, \Pi_{0,\Gamma}\mathcal{A}_{\mathcal{H}_+(g_0)}\right) \to 0 \ (t \to \infty)\ \forall \Gamma \geq 0.$$

Similarly, from the embedding $H_w^{\mathbf{r}}(Q_{t_1,t_2}) \subset C_w([t_1,t_2]; V)$, it follows

Corollary 1.2 *For any set $B \subset \mathcal{K}^+$, bounded in $\mathcal{F}^+$, and for any $v \in V$, one has*

$$\mathbf{dist}_{C([0,\Gamma])}\left(\Pi_{0,\Gamma}J_vT(t)B, \Pi_{0,\Gamma}J_v\mathcal{A}_{\mathcal{H}_+(g_0)}\right) \to 0 \ (t \to \infty) \ \forall \Gamma \geq 0,$$

where J_v is the mapping from $H^{\mathbf{r}}_w(Q_{t_1,t_2})$ into $C([t_1,t_2]) : J_v(u(s)) = ((u(s),v))$, $((\,.\,,\,.\,))$ is the scalar product in V.

To prove theorem 1.2 we use Proposition 1.3. According to (1.5) and Lemma 1.2 we have only to check that the family of trajectory spaces $\{\mathcal{K}^+_g, g \in \mathcal{H}_+(g_0)\}$ corresponding to the equation (1.1) possesses a uniformly (w.r.t. $g \in \mathcal{H}_+(g_0)$) attracting set P compact in Θ^{loc}_+ and bounded in $\mathcal{F}^a_+$. This is the most difficult part of the proof. We separate the proof of this fact into a few lemmas.

Lemma 1.3 *For any $u \in \mathcal{K}^+_g$, $g \in \mathcal{H}_+(g_0)$, the following estimates are valid:*

$$|u(\tau+t)|^2 \leq e^{-\lambda t}|u(\tau)|^2 + C_1\|g\|^2_a, \ t,\tau \geq 0, \tag{1.8}$$

$$\|T(t)u\|^2_{L_\infty(\mathbb{R}_+;H)} \leq e^{-\lambda t}|u(0)|^2 + C_1\|g\|^2_a; \tag{1.9}$$

where λ is the first eigenvalue of the operator νL, $C_1 = \lambda^{-1}\left(1-e^{-\lambda}\right)^{-1}$;

$$\nu\int_t^{t+1}\|u(s)\|^2 ds \leq |u(t)|^2 + C_2\int_t^{t+1}|g(s)|^2 ds, \tag{1.10}$$

$$\nu\|T(t)u\|^2_{L^a_2(\mathbb{R}_+;V)} \leq e^{-\lambda t}|u(0)|^2 + C_3\|g\|^2_a, \tag{1.11}$$

where $C_2 = \lambda^{-1}$, $C_3 = C_1 + C_2$, $t \geq 0$.

Proof. Taking the scalar product in H of (1.1) with u, we get

$$\frac{d}{dt}|u(t)|^2 + \lambda|u(t)|^2 \leq \frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \leq \lambda^{-1}|g(t)|^2, \tag{1.12}$$

and we obtain after proper integrating from τ to $\tau + t$

$$|u(\tau+t)|^2 \leq e^{-\lambda t}|u(\tau)|^2 + \lambda^{-1}e^{-\lambda(\tau+t)}\int_\tau^{\tau+t}|g(s)|^2e^{\lambda s}ds.$$

Estimating the last expression, we get

$$\begin{aligned}
&\int_\tau^{\tau+t}|g(s)|^2e^{-\lambda(\tau+t-s)}ds \leq \\
&\int_{\tau+t-1}^{\tau+t}|g(s)|^2e^{-\lambda(\tau+t-s)}ds + \int_{\tau+t-2}^{\tau+t-1}|g(s)|^2e^{-\lambda(\tau+t-s)}ds + \ldots \leq \\
&\int_{\tau+t-1}^{\tau+t}|g(s)|^2ds + e^{-\lambda}\int_{\tau+t-2}^{\tau+t-1}|g(s)|^2ds + e^{-2\lambda}\int_{\tau+t-3}^{\tau+t-2}|g(s)|^2ds + \ldots \leq \\
&\|g\|^2_a\left(1 + e^{-\lambda} + e^{-2\lambda} + \ldots\right) = \|g\|^2_a\left(1-e^{-\lambda}\right)^{-1}.
\end{aligned}$$

So, inequality (1.8) is proved. Inequality (1.9) follows directly from (1.8). In the usual way, one derives (1.10) from (1.12). Combining (1.8) and (1.10), we get (1.11). □

Lemma 1.4 *For any $u \in \mathcal{K}_g^+$, $g \in \mathcal{H}_+(g_0)$,*

$$(1.13) \qquad \sup_{0\le t\le \Gamma} t\|u(\tau+t)\|^2 \le C_1\left(\Gamma, |u(\tau)|^2, \int_\tau^{\tau+\Gamma} |g(s)|^2 ds\right);\ \Gamma,\tau \ge 0,$$

where $C_1(\eta_1, \eta_2, \eta_3)$ is a continuous and increasing function with respect to each $\eta_i \ge 0$.

The proof is analogous to one given in [2]. We sketch the main points of it for convenience of readers. For brevity sake, we suppose without lose of generality that $\nu = 1$ and $\tau = 0$. Multiplying equation (1.1) by tLu we get:

$$(1.14) \qquad \frac{1}{2}\frac{d}{dt}\left(t\|u(t)\|^2\right) - \frac{1}{2}\|u(t)\|^2 + t\|u(t)\|_2^2 + t(B(u), Lu) \ \le \ t|g(t)|^2 + \frac{1}{4}t\|u(t)\|_2^2.$$

Recall that $(u, Lu) = \|u\|^2$, $(Lu, Lu) = \|u\|_2^2$. We have also that

$$(1.15) \qquad (B(u), Lu) \ \le \ |B(u)|\|u\|_2,$$

$$(1.16) \qquad |B(u)| \ \le \ c\left(\int_\Omega |u|^2|\nabla u|^2 dx\right)^{1/2} \le c\|u\|_{0,4}\|u\|_{1,4};$$

$$(1.17) \qquad \|u\|_{0,4} \ \le \ c_1\|u\|^{1/2}|u|^{1/2},\ \|u\|_{0,4} \le c_2\|u\|_2^{1/2}\|u\|^{1/2}$$

(see inequalities (1.17) in [15], [21]). It follows from (1.15), (1.16), and (1.17) that

$$(1.18) \qquad |B(u)| \ \le \ c_3\|u\|_2^{1/2}\|u\||u|^{1/2},$$

$$(1.19) \qquad t(B(u), Lu) \ \le \ tc_3\|u\|_2^{3/2}\|u\||u|^{1/2} \le \frac{t}{4}\|u\|_2^2 + \frac{tc_4}{2}\|u\|^4|u|^2.$$

Using (1.14) and (1.19) we obtain

$$(1.20) \qquad \frac{d}{dt}\left(t\|u(t)\|^2\right) + t\|u(t)\|_2^2 \le \|u(t)\|^2 + 2t|g(t)|^2 + tc_4\|u(t)\|^4|u(t)|^2,$$

Denote $z(t) = t\|u(t)\|^2$. Consequently,

$$z'(t) \le b(t) + \gamma(t)z(t),\ b(t) = \|u(t)\|^2 + 2t|g(t)|^2,\ \gamma(t) = c_4\|u(t)\|^2|u(t)|^2.$$

Applying Gronwall inequality, we get:

$$z(t) \le \int_0^t b(s)\exp\left(\int_s^t \gamma(\theta)d\theta\right)ds \le \left(\int_0^t b(s)ds\right)\exp\left(\int_0^t \gamma(s)ds\right).$$

Using (1.12), we have

$$(1.21) \qquad |u(t)|^2 + \int_0^t \|u(s)\|^2 ds \le |u(0)|^2 + \lambda^{-1}\int_0^t |g(s)|^2 ds.$$

Therefore

$$
\begin{aligned}
& t\|u(t)\|^2 \leq \\
\left(\int_0^t \left(\|u(s)\|^2 + 2s|g(s)|^2\right) ds\right) \exp\left(\int_0^t c_4\|u(s)\|^2|u(s)|^2 ds\right) & \leq \\
\left(|u(0)|^2 + (\lambda^{-1}+2t)\int_0^t |g(s)|^2 ds\right) \exp\left(c_4\left(|u(0)|^2 + \lambda^{-1}\int_0^t |g(s)|^2 ds\right)^2\right) &
\end{aligned}
$$

Finally,

$$
\sup_{0\leq t\leq \Gamma} t\|u(t)\|^2 \leq C_1\left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2 ds\right), \tag{1.22}
$$

where $C_1(\eta_1,\eta_2,\eta_3) = (\eta_2 + (\lambda^{-1}+2\eta_1)\eta_3)\exp\left(c_4(\eta_2+\lambda^{-1}\eta_3)^2\right)$. □

Inequality (1.13) implies that

$$
\|u(t+\tau+1)\|^2 \leq C_1\left(1, |u(t+\tau)|^2, \int_{t+\tau}^{t+\tau+1} |g(s)|^2 ds\right) \tag{1.23}
$$

Taking sup in (1.23) with respect to $\tau \geq 0$, we obtain according to (1.9) that

$$
\|T(t+1)u\|^2_{L_\infty(\mathbf{R}_+;V)} \leq C_1\left(1, \|T(t)u\|^2_{L_\infty(\mathbf{R}_+;H)}, \|T(t)g\|_a^2\right) \leq C_2\left(e^{-\lambda t}|u(0)|^2, \|g\|_a^2\right).
$$

Hence

Corollary 1.3 *For any $u \in \mathcal{K}_g^+$, $g \in \mathcal{H}_+(g_0)$,*

$$
\|T(t+1)u\|^2_{L_\infty(\mathbf{R}_+;V)} \leq C_2\left(e^{-\lambda t}|u(0)|^2, \|g\|_a^2\right),\ t \geq 0.
$$

Lemma 1.5 *For any $u \in \mathcal{K}_g^+$, $g \in \mathcal{H}_+(g_0)$,*

$$
\int_\tau^{\tau+\Gamma} (s-\tau)\left(\|u(s)\|_2^2 + |\partial_t u(s)|^2\right) ds \leq C_3\left(\Gamma, |u(\tau)|^2, \int_\tau^{\tau+\Gamma} |g(s)|^2 ds\right); \tag{1.24}
$$

$$
\begin{aligned}
\|T(t+1)u\|^2_{\Gamma,a} &= \sup_{\tau\geq t+1}\int_\tau^{\tau+1}\left(\|u(s)\|_2^2 + |\partial_t u(s)|^2\right) ds \leq \\
& C_4\left(e^{-\lambda t}|u(0)|^2, \|g\|_a^2\right),
\end{aligned} \tag{1.25}
$$

where τ, t, Γ are positive and any.

Proof. It is sufficient to prove (1.24) for $\tau = 0$ and $\nu = 1$. It follows from (1.20), (1.21), and (1.22) that

$$\int_0^\Gamma s\|u(s)\|_2^2 ds \leq \int_0^\Gamma \|u(s)\|^2 ds + 2\Gamma \int_0^\Gamma |g(s)|^2 ds \qquad +$$

$$c_4 \left(\sup_{0\leq t\leq\Gamma} |u(t)|^2 \right) \left(\sup_{0\leq t\leq\Gamma} t\|u(t)\|^2 \right) \int_0^\Gamma \|u(s)\|^2 ds \quad \leq$$

$$|u(0)|^2 + \lambda^{-1} \int_0^\Gamma |g(s)|^2 ds + 2\Gamma \int_0^\Gamma |g(s)|^2 ds \qquad +$$

$$c_4 \left(|u(0)|^2 + \lambda^{-1} \int_0^\Gamma |g(s)|^2 ds \right)^2 C_1 \left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2 ds \right) \quad =$$

$$C_3' \left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2 ds \right). \tag{1.26}$$

Now, equation (1.1) implies directly that

$$\left(\int_0^\Gamma s|\partial_t v(s)|^2 ds \right)^{1/2} \leq \left(\int_0^\Gamma s\|u(s)\|_2^2 ds \right)^{1/2} + \left(\int_0^\Gamma s|B(u)|^2 ds \right)^{1/2} +$$

$$\left(\int_0^\Gamma s|g(s)|^2 ds \right)^{1/2} \leq C_3' \left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2 ds \right) + \Gamma^{1/2} \left(\int_0^\Gamma |g(s)|^2 ds \right)^{1/2} +$$

$$c_3 \left(\int_0^\Gamma s\|u(s)\|_2 \|u(s)\|^2 |u(s)| ds \right)^{1/2} \tag{1.27}$$

We have used inequality (1.18). At the same time by (1.22) and (1.26), we get

$$\int_0^\Gamma s\|u(s)\|_2 \|u(s)\|^2 |u(s)| ds \leq \int_0^\Gamma s\|u(s)\|^4 |u(s)|^2 ds + \int_0^\Gamma s\|u(s)\|_2^2 ds \leq$$

$$\left(\sup_{0\leq t\leq\Gamma} |u(t)|^2 \right) \left(\sup_{0\leq t\leq\Gamma} t\|u(t)\|^2 \right) \int_\tau^\Gamma \|u(s)\|^2 ds + C_3' \left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2 ds \right) \quad \leq$$

$$\left(|u(0)|^2 + \lambda^{-1} \int_0^\Gamma |g(s)|^2 ds \right)^2 C_1 \left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2 ds \right) + C_3'(\,.\,). \tag{1.28}$$

Combining (1.27) and (1.28) we obtain

$$\int_0^\Gamma s|\partial_t u(s)|^2 ds \leq C_3'' \left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2 ds \right). \tag{1.29}$$

Summing (1.26) and (1.29), we derive (1.24). From (1.24) it follows for $\Gamma = 2$ that

$$\int_{t+\tau+1}^{t+\tau+2} \left(\|u(s)\|_2^2 + |\partial_t u(s)|^2 \right) ds \leq C_3 \left(2, |u(t+\tau)|^2, \int_{t+\tau}^{t+\tau+2} |g(s)|^2 ds \right). \tag{1.30}$$

Taking sup in (1.30) with respect to $\tau \geq 0$, we obtain according to (1.9) that

$$\|T(t+1)u\|_{\Gamma,a}^2 \leq C_3 \left(2, \|T(t)u\|_{L_\infty(\mathbb{R}_+;H)}^2, 2\|T(t)g\|_a^2 \right) \leq C_4 \left(e^{-\lambda t}|u(0)|^2, \|g\|_a^2 \right).$$

Lemma is proved. $\square$

Lemma 1.1 follows from more general

Lemma 1.6 *For any* $u \in \mathcal{K}_g^+$, $g \in \mathcal{H}_+(g_0)$,

$$(1.31) \qquad \int_\tau^{\tau+\Gamma} \left(\|v(s)\|_2^2 + |\partial_t v(s)|^2\right) ds \le C_5 \left(\|u(\tau)\|^2, \int_\tau^{\tau+\Gamma} |g(s)|^2 ds\right);$$

$$\|u\|_{\mathrm{r},a}^2 = \sup_{\tau\ge 0} \int_\tau^{\tau+1} \|u(s)\|_2^2 + |\partial_t u(s)|^2 ds \le C_6 \left(\|u(0)\|^2, \|g\|_a^2\right), \ \tau, \Gamma \ge 0.$$

Proof. Similarly to (1.20) we get

$$\begin{aligned}
\frac{d}{dt}\left(\|u(t)\|^2\right) + \|u(t)\|_2^2 &\le 2|g(t)|^2 + c_4\|u(t)\|^4|u(t)|^2, \\
z_1'(t) &\le b_1(t) + \gamma(t)z_1(t), \ \ z_1(t) = \|u(t)\|^2, \ b_1(t) = 2|g(t)|^2, \\
z_1(t) &\le \left(z_1(0) + \int_0^t b_1(s)ds\right)\exp\left(\int_0^t \gamma(s)ds\right).
\end{aligned}$$

So, using (1.21), we obtain,as above, (1.31). Finally, combinig (1.31) with $\tau \in [0,1]$ and (1.25) with $\tau \in]1, +\infty[$we get

$$\begin{aligned}
\|u\|_{\mathrm{r},a}^2 = \sup_{\tau\ge 0} \int_\tau^{\tau+1} \left(\|u(s)\|_2^2 + |\partial_t u(s)|^2\right) ds \ &\le \\
\max\left\{C_5\left(\|u(0)\|^2, \|g\|_a^2\right), C_4\left(|u(0)|^2, \|g\|_a^2\right)\right\} \ &= \ C_6'\left(\|u(0)\|^2, \|g\|_a^2\right).
\end{aligned}$$

Lemma is proved. $\square$

Coming back to the proof of Theorem 1.2, we construct the uniformly attracting set P in Θ_+^{loc} for the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+ = \mathcal{K}_{\mathcal{H}_+(g_0)}^+$. From (1.25) it follows that

$$(1.32) \qquad \|T(t+1)u\|_{\mathrm{r},a}^2 \le C_4\left(e^{-\lambda t}\|u\|_{\mathrm{r},a}^2, \|g\|_a^2\right) \le C_4\left(e^{-\lambda t}\|u\|_{\mathrm{r},a}^2, \|g_0\|_a^2\right)$$

$\forall u \in \mathcal{K}^+$, since $\|g\|_a \le \|g_0\|_a$ for any $g \in \mathcal{H}_+(g_0)$. Consider the set

$$P_0 = \{v \in \mathcal{F}_+^a \mid \|v\|_{\mathrm{r},a}^2 \le C_4\left(1, \|g_0\|_a^2\right)\}.$$

Evidently, P_0 is a desired attracting set. Indeed, if $B \subseteq \mathcal{K}^+ \cap \mathcal{F}_+^a$ is a bounded set of trajectories then $e^{-\lambda t}\|u\|_{\mathrm{r},a}^2 \le 1$ for any $u \in B$ when $t \ge t' \gg 1$ and therefore, by (1.32) $T(t+1)B \subseteq P_0$. Hence P_0 is even a uniformly absorbing set. Notice that the set P_0 is bounded in $\mathcal{F}_+^a$ and compact in $\Theta_+^{loc} = H_w^{\mathrm{r},loc}(Q_+)$. The latter is true since the topology in $H_w^{\mathrm{r},loc}(Q_+)$ is generated by the weak topology of Banach spaces $H^{\mathrm{r}}(Q_{t_1,t_2}) = L_2(t_1,t_2;H_2) \cap \{\partial_t v \in L_2(t_1,t_2;H)\}$. Recall that $u_n(s) \rightharpoonup u(s)$ $(n \to +\infty)$ weakly in $H^{\mathrm{r}}(Q_{t_1,t_2})$ whenever $\partial_t u_n(s) \rightharpoonup \partial_t u(s)$ $(n \to +\infty)$ weakly in $L_2(t_1,t_2;H)$ and $\partial^\alpha u(s) \rightharpoonup \partial^\alpha u(s)$ $(n \to +\infty)$ weakly in $L_2(t_1,t_2;H)$ for any $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| \le 2$. That is, a bounded set in $H^{\mathrm{r}}(Q_{t_1,t_2})$ is weakly compact in $H^{\mathrm{r}}(Q_{t_1,t_2})$.

Remark 1.1 *The set P_0 being a compact subspace of $H_w^{r,loc}(Q_+)$ is a metrizable space and the corresponding metric space is compact. This proposition follows from the fact that a ball of a separable Banach space endowed with the weak topology of this space is metrizable and compact. The translation semigroup $\{T(t)\}$ is continuous on P_0 and $T(t)$ takes P_0 into itself: $T(t)P_0 \subseteq P_0$ for any $t \geq 0$. So the Proposition 1.2 is applicable. In particular the set $\mathbf{A} = \omega(P_0)$ is a global attractor of the semigroup $\{T(t)\}$ acting on P_0. Moreower, $\mathbf{A} = \mathcal{A}_{\mathcal{H}_+(g_0)}$ because P_0 is a uniformly absorbing set of the family of trajectory spaces $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$. This reasoning proves the first part of Theorem 1.2. To prove property (1.7) we have to use more subtle reasoning (See [5]).*

2 Trajectory attractor for 2D N.-S. system with translation-compact external force in L_2^{loc}.

Now consider the case when the external force $g(x,s)$ in (1.1) is a tr.-c. function in $L_2^{loc}(\mathbf{R}_+;H)$. The space $L_2^{loc}(\mathbf{R}_+,H) = L_2^{loc}$ is endowed with the following local strong convergence topology. The sequence $\{g_n(s)\}$ converges to $g(s)$ as $n \to \infty$ in L_2^{loc} whenever $\int_{t_1}^{t_2} |g_n(s) - g(s)|^2 ds \to 0$ $(n \to \infty)$ for any $[t_1,t_2] \subseteq \mathbf{R}_+$. The space L_2^{loc} is metrizable and complete. A function $g(s) \in L_2^{loc}$ is tr.-c. in L_2^{loc} whenever the set $\{g(s+h) \mid h \in \mathbf{R}_+\}$ is precompact in L_2^{loc}. The criterion of functions to be tr.-c. in L_2^{loc} is given in [6]. We recall: a function $g(s) \in L_2^{loc}$ is tr.-c. in L_2^{loc} if and only if

(i) for any $h \geq 0$ the set $\{\int_t^{t+h} g(s,x)ds \mid t \in \mathbf{R}_+\}$ is precompact in H;

(ii) there is a positive function $\beta(s) > 0, s > 0$, such that $\beta(s) \to 0+$ $(s \to 0+)$ and

$$\int_t^{t+1} |g(s) - g(s+l)|^2 ds \leq \beta(|l|)\ \forall t \geq 0.$$

Remark 2.1 *Let us give a simple sufficient condition. A function $g(s) \in L_2^{loc}$ is tr.-c. in L_2^{loc} if*

$$\|\Pi_{0,1} g(\,.\,, s+t)\|_{H^\delta(Q_{0,1})} \leq M\ \forall t \geq 0$$

for some $\delta > 0$. Here $H^\delta(Q_{0,1}) = H^\delta(\Omega \times [t_1,t_2])$ is the Sobolev space of order δ.

Let we be given a fixed tr.-c. in L_2^{loc} function $g_0(s)$. Evidently, $g_0(s)$ is tr.-c. $L_{2,w}^{loc}$ as well. Consider the set $\{g_0(s+h) \mid h \in \mathbf{R}_+\}$. Notice that $[\{g_0(s+h) \mid h \in \mathbf{R}_+\}]_{L_{2,w}^{loc}} \equiv [\{g_0(s+h) \mid h \in \mathbf{R}_+\}]_{L_2^{loc}}$ and the corresponding topological subspaces of $L_{2,w}^{loc}$ and L_2^{loc} are homeomorphic. Hence, $\mathcal{H}_+(g_0) = [\{g_0(s+h) \mid h \in \mathbf{R}_+\}]_\Xi$ does not depend on $\Xi = L_{2,w}^{loc}$ or $\Xi = L_2^{loc}$. As usually, the topological space $\mathcal{H}_+(g_0)$ is compact and any function $g(s) \in \mathcal{H}_+(g_0)$ is tr.-c. in L_2^{loc}, $\mathcal{H}_+(g) \subseteq \mathcal{H}_+(g_0)$, and $\|g\|_a \leq \|g_0\|_a$.

Now consider the "strong" space $H_s^{r,loc}(Q_+)$ introduced in section 1. Recall that $v_n(s) \to v(s)$ $(n \to \infty)$ in $H_s^{r,loc}(Q_+)$ if $\Pi_{t_1,t_2} v_n(s) \to \Pi_{t_1,t_2} v(s)$ $(n \to \infty)$ strongly in

$H^{\mathbf{r}}(Q_{t_1,t_2})$ with respect to the norm (1.3) for any $[t_1, t_2] \subseteq \mathbf{R}_+$. The linear topological space $H_s^{\mathbf{r},loc}(Q_+)$ is metrizable and complete.

To any $g \in \mathcal{H}_+(g_0)$ there corresponds the trajectory space $\mathcal{K}_g^+$ that is the union of all solutions $u(s), s \geq 0$, of equation (1.1) in the space $H^{\mathbf{r},loc}(Q_+)$. Consider the family $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$ and the union $\mathcal{K}_{\mathcal{H}_+(g_0)}^+ = \cup_{\mathcal{H}_+(g_0)} \mathcal{K}_g^+$.

In this section we study the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+ = \mathcal{K}_{\mathcal{H}_+(g_0)}^+$ in the "strong" topological space $H_s^{\mathbf{r},loc}(Q_+)$. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ attracts any set $T(t)B$ as $t \to +\infty$ in the topology of $\Theta_+^{loc} = H_s^{\mathbf{r},loc}(Q_+)$, where $B \subset \mathcal{K}^+$ and B is bounded in the Banach space $\mathcal{F}_+^a = H^{\mathbf{r},a}(Q_+)$.

Theorem 2.1 *Let $g_0(s)$ be tr.-c. in L_2^{loc}. Then the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ in $H_w^{\mathbf{r},loc}(Q_+)$ of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+$ from the Theorem 1.2 serves as the trajectory attractor in $H_s^{\mathbf{r},loc}(Q_+)$ of this semigroup. In particular, for any set $B \subset \mathcal{K}^+$*

$$\mathbf{dist}_{H^{\mathbf{r}}(Q_{0,\Gamma})}(T(t)B, \mathcal{A}_{\mathcal{H}_+(g_0)}) \to 0 \ (t \to \infty)\ \forall \Gamma > 0.$$

The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ is bounded in $\mathcal{F}^+$, compact in $H_s^{\mathbf{r},loc}(Q_+)$. Any function $u(s) \in \mathcal{A}_{\mathcal{H}_+(g_0)}$ is tr.-c. in $H_s^{\mathbf{r},loc}(Q_+)$.

Notice if the trajectory attractor in $H_s^{\mathbf{r},loc}(Q_+)$ exists then it coincides with the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ in $H_w^{\mathbf{r},loc}(Q_+)$ since the embedding $H_s^{\mathbf{r},loc}(Q_+) \subseteq H_w^{\mathbf{r},loc}(Q_+)$ is continuous and the trajectory attractor is the minimal attracting set. So to apply Proposition 1.3 we have to produce an attracting set P_1 that is compact in $H_s^{\mathbf{r},loc}(Q_+)$ and bounded in $\mathcal{F}_+^a = H^{\mathbf{r},a}(Q_+)$.

From the continuous embedding $H_s^{\mathbf{r}}(Q_{t_1,t_2}) \subset C([t_1, t_2]; V)$ and from Theorem 2.1, it follows

Corollary 2.1 *For any set $B \subset \mathcal{K}^+$, bounded in $\mathcal{F}^+$, one has*

$$\mathbf{dist}_{C([0,\Gamma];V)}\left(\Pi_{0,\Gamma} T(t)B, \Pi_{0,\Gamma}\mathcal{A}_{\mathcal{H}_+(g_0)}\right) \to 0 \ (t \to \infty)\ \forall \Gamma \geq 0.$$

Proof of Theorem 2.1. Consider the set $P_0' = P_0 \cap \mathcal{K}^+$, where P_0 is the absorbing set constructed in the section 1. Evidently, the set P_0' is uniformly absorbing for the family $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$. Put

$$P_1 = S(1)P_0' = \{\tilde{u}_g(s) \equiv u_g(s+1),\ s \geq 0 \mid u_g(s) \in \mathcal{K}_g^+ \cap P_0,\ g \in \mathcal{H}_+(g_0)\}.$$

The set P_1 is uniformly absorbing for the family $\{\mathcal{K}_g^+,\ g \in \mathcal{H}_+(g_0)\}$ as well. To complete the proof of Theorem 2.1 we have to establish the following

Lemma 2.1 *The set P_1 is compact in $H_s^{\mathbf{r},loc}(Q_+)$.*

It is easy to prove the following statement using the diagonal process.

Proposition 2.1 *The set B is compact in $H_s^{r,loc}(Q_+)$ if and only if the set $\Pi_{0,\Gamma}B$ is compact in $H^r(Q_{0,\Gamma})$ for any $\Gamma > 0$.*

Proof of Lemma 2.1. Fix any $\Gamma > 0$. Let $\tilde{u}^n(s) = T(1)u^n(s) = u^n(s+1)$ be any sequence from P_1, $u^n(s) \in \mathcal{K}_{g_n}^+ \cap P_0$, $g_n \in \mathcal{H}_+(g_0)$. Without lose of generality, we may assume that

$$\int_0^{\Gamma+1} |g_n(s) - g(s)|^2 ds \to 0 \; (n \to \infty) \tag{2.1}$$

for some $g \in \mathcal{H}_+(g_0)$. Let us show that the sequence $\{\tilde{u}^n(s)\}$ is precompact in $H^r(Q_{0,\Gamma})$. Since $u^n(s) \in P_0$, we have

$$\|u^n(\,.\,)\|_{H^r(Q_{0,\Gamma+1})} \le M(\Gamma+1) \; \forall n \in \mathbf{N}, \tag{2.2}$$

where the positive function $M(\theta)$ is not decreasing. We can represent the function $u^n(s)$ as a sum of two functions:

$$u^n(s) = u_1^n(s) + u_2^n(s), \; s \ge 0,$$

where $u_1^n(s)$ and $u_2^n(s)$ are solutions of the following problems:

$$\begin{aligned} \partial_t u_1^n(t) + L u_1^n(t) &= 0, \; t \ge 0, && (2.3)\\ u_1^n(0) &= u_1^n(0), \; u_1^n|_{\partial\Omega} = 0, \; \|u_1^n(0)\| \le M_1 && (2.4) \end{aligned}$$

$$\begin{aligned} \partial_t u_2^n(t) + L u_2^n(t) &= -B(u^n(t)) + g_n(t), \; t \ge 0, && (2.5)\\ u_2^n(0) &= 0, \; u_2^n|_{\partial\Omega} = 0. && (2.6) \end{aligned}$$

Respectively, $\tilde{u}^n(s) = \tilde{u}_1^n(s) + \tilde{u}_2^n(s)$.

Since u_1^n is a solution of the Stokes problem (2.3), (2.4), we obtain

$$\int_0^{t+1} \left(\|u_1^n(s)\|_2^2 + |\partial_t u_1^n(s)|^2\right) ds \le M_2(t+1, \|u_1^n(0)\|) = M_3(t+1), \tag{2.7}$$

$0 \le t \le \Gamma$. Let $\psi(t)$ be the cutoff function:

$$\psi(t) \equiv 1, \; t \ge 1; \; \psi(t) \equiv 0, \; 0 \le t \le 1/2; \; \psi \in C_0^\infty(\mathbf{R}), \; \psi(t) \ge 0.$$

It follows from (2.3) that

$$\partial_t\left(\psi(t)u_1^n(t)\right) + L\left(\psi(t)u_1^n(t)\right) = \psi'(t)u_1^n(t),$$

Differentiating this equation in t and denoting $\partial_t\left(\psi(t)u_1^n(t)\right) = p^n$, we get

$$\begin{aligned} \partial_t p^n + L p^n &= \psi''(t)u_1^n(t) + \psi'(t)\partial_t u_1^n(t),\\ p^n(0) &= 0, \; p^n|_{\partial\Omega} = 0. \end{aligned}$$

So,

$$\int_0^{t+1} \left(|L\partial_t(\psi u_1^n)|^2 + |\partial_t^2(\psi u_1^n)|^2\right) ds = \int_0^{t+1} \left(\|p^n(s)\|_2^2 + |\partial_t p^n(s)|^2\right) ds \le$$

$$C\int_0^{t+1} \left(|u_1^n|^2 + |\partial_t u_1^n|^2\right) ds \le M_4(t+1). \tag{2.8}$$

Combining (2.8) and (2.7), we obtain

$$\int_0^{t+1} \psi^2(s)\left(\|\partial_t u_1^n(s)\|_2^2 + |\partial_t^2 u_1^n(s)|^2\right) ds \le M_5(t+1). \tag{2.9}$$

Now we apply the operator L to both sides of equation (2.3) and get:

$$\begin{aligned} L^2 u_1^n(t) &= -\partial_t L u_1^n(t), \\ L u_1^n|_{\partial\Omega} &= -\partial_t u_1^n(t)|_{\partial\Omega} = 0. \end{aligned}$$

Therefore

$$\int_0^{t+1} \psi^2(s)|L^2 u_1^n(s)|^2 ds = \int_0^{t+1} \psi^2(s)|\partial_t\left(L u_1^n(s)\right)|^2 ds \le M_6(t+1). \tag{2.10}$$

Finally, by virtue of (2.7), (2.9), and (2.10), we conclude:

$$\int_1^{t+1} \left(\|\partial_t u_1^n(s)\|_2^2 + |\partial_t^2 u_1^n(s)|^2 + \|u_1^n(s)\|_2^2 + \|u_1^n(s)\|_4^2\right) ds \le M_7(t+1).$$

In particular, the sequence $\{u_1^n(s)\}$ is compact in $H^{\mathbf{r}}(Q_{1,\Gamma+1})$ and $\{\tilde{u}_1^n(s)\}$ is compact in $H^{\mathbf{r}}(Q_{0,\Gamma})$.

Now we shall prove that the sequence $\{u_2^n(s)\}$ is compact in $H^{\mathbf{r}}(Q_{0,\Gamma+1})$ as well. According to (2.5) it is sufficient to prove that the sequence $B(u^n(s)) = B(u^n(s), u^n(s))$ is precompact in $L_2(0, \Gamma+1; H)$. (From (2.1) it follows that the sequence $\{g_n(s)\}$ is precompact in $L_2(0, \Gamma+1; H)$). The sequence $\{u^n(s)\}$ is bounded in $H^{\mathbf{r}}(Q_{0,\Gamma+1})$, hence, by refining, we may assume that $u_n(s) \rightharpoonup u(s)$ $(n \to +\infty)$ weakly in $H^{\mathbf{r}}(Q_{0,\Gamma+1})$. Thus, $\partial_t u_n(s) \rightharpoonup \partial_t u(s)$ $(n \to +\infty)$ weakly in $L_2(0, \Gamma+1; H)$ and $\partial^\alpha u(s) \rightharpoonup \partial^\alpha u(s)$ $(n \to +\infty)$ weakly in $L_2(0, \Gamma+1; H)$ for any $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| \le 2$. Let us prove that

$$B(u^n(s)) \to B(u(s)) \ (n \to +\infty) \text{ strongly in } L_2(0, \Gamma+1; H). \tag{2.11}$$

By the Nicolskiy theorem (see [3])

$$H^{\mathbf{r}}(Q_{0,\Gamma+1}) \subset H_q^\rho(Q_{0,\Gamma+1}), \ \mathbf{r} = (r_1, r_2, r_3), \ \rho = (\rho_1, \rho_2, \rho_3), \ q \ge 2, \tag{2.12}$$

whenever

$$\rho_j/r_j \le 1 - (1/2 - 1/q)(1/r_1 + 1/r_2 + 1/r_3), \ j = 1, 2, 3. \tag{2.13}$$

Moreower, the embedding (2.12) is compact if inequalities in (2.13) are strict.

The values $\mathbf{r} = (r_1, r_2, r_3) = (2,2,1)$, $\rho = (\rho_1, \rho_2, \rho_3) = (1,1,0)$, $q \leq 4$ suit the conditions (2.13), since $\rho_j/r_j \leq 1/2 \leq 1 - (1/2 - 1/q)2$,. So we conclude that

$$\| \frac{\partial v}{\partial x_i} \|_{L_q(Q_{0,\Gamma+1})} \leq C_q \|v\|_{H^{\mathbf{r}}(Q_{0,\Gamma+1})}, \ i = 1,2; \ 2 \leq q \leq 4. \tag{2.14}$$

For $q < 4$ the embedding $H^{\mathbf{r}}(Q_{0,\Gamma+1}) \subset\subset H_q^{(1,1,0)}(Q_{0,\Gamma+1})$ is compact. Similarly, taking $\rho = (\rho_1, \rho_2, \rho_3) = (0,0,0)$, $\rho_j/r_j = 0 < 1 - (1/2 - 1/q_1)2$ for any $q_1 \geq 2$, we obtain

$$\|v\|_{L_{q_1}(Q_{0,\Gamma+1})} \leq C'_{q_1} \|v\|_{H^{\mathbf{r}}(Q_{0,\Gamma+1})}$$

and the embedding $H^{\mathbf{r}}(Q_{0,\Gamma+1}) \subset\subset L_{q_1}(Q_{0,\Gamma+1})$ is compact.

Finally we get

$$\begin{aligned}
\|B(u^n) - B(u)\|_{L_2(0,\Gamma+1;H)} &\equiv \\
\|B(u^n) - B(u)\| \leq \|B(u^n - u, u^n)\| + \|B(u, u^n - u)\| &\leq \\
C\left(\int_{Q_{0,\Gamma+1}} |u^n - u|^2 |\nabla u^n|^2 dxds\right)^{1/2} + C\left(\int_{Q_{0,\Gamma+1}} |u|^2 |\nabla (u^n - u)|^2 dxds\right)^{1/2} &\leq \\
C_1 \left(\int_{Q_{0,\Gamma+1}} |\nabla u^n|^3 dxds\right)^{1/3} \left(\int_{Q_{0,\Gamma+1}} |u^n - u|^6 dxds\right)^{1/6} &+ \\
C_1 \left(\int_{Q_{0,\Gamma+1}} |u|^6 dxds\right)^{1/6} \left(\int_{Q_{0,\Gamma+1}} |\nabla (u^n - u)|^3 dxds\right)^{1/6} &
\end{aligned} \tag{2.15}$$

Since $H^{\mathbf{r}}(Q_{0,\Gamma+1}) \subset\subset H_3^{(1,1,0)}(Q_{0,\Gamma+1})$ and $H^{\mathbf{r}}(Q_{0,\Gamma+1}) \subset\subset L_6(Q_{0,\Gamma+1})$, we get

$$\int_{Q_{0,\Gamma+1}} |\nabla (u^n - u)|^3 dxds \to 0, \ \int_{Q_{0,\Gamma+1}} |u^n - u|^6 dxds \to 0 \ (n \to \infty)$$

and, by (2.14) and (2.2),

$$\int_{Q_{0,\Gamma+1}} |\nabla u^n|^3 dxds \leq M'.$$

Therefore, the right-hand side of (2.15) tends to zero as $n \to \infty$ and (2.11) is proved.

Thus, the right-hand sides of (2.5) forms a precompact set in $L_2(0, \Gamma + 1; H)$ and, hence, the set of solutions $\{u_2^n(s)\}$ is precompact in $H^{\mathbf{r}}(Q_{0,\Gamma+1})$. Consequently, $\{\tilde{u}_2^n(s)\}$ is precompact in $H^{\mathbf{r}}(Q_{0,\Gamma})$. The sum $\{\tilde{u}^n(s)\}$ of two precompact sequences $\{\tilde{u}_1^n(s)\}$ and $\{\tilde{u}_2^n(s)\}$ is precompact in $H^{\mathbf{r}}(Q_{0,\Gamma})$. Lemma 2.1 is proved. □

3 On the structure of trajectory attractors

In this section we shall describe the structure of trajectory attractors from Theorems 1.2 and 2.1 in terms of complete trajectories of equation 1.1, i.e. when solutions $u(s), s \in \mathbf{R}$, are determined on the whole time axis $\mathbf{R}$.

Let the function $g_0(x,s)$ satisfies (1.2) and let $\mathcal{H}_+(g_0)$ be the hull of g_0 in $L^{loc}_{2,w}(\mathbf{R}_+, H)$. As usually, $\mathcal{H}_+(g_0)$ is a complete metric space and the translation semigroup $\{T(t)\}$ acts on $\mathcal{H}_+(g_0)$, $T(t)\mathcal{H}_+(g_0) \subseteq \mathcal{H}_+(g_0)$, $T(t)$ is continuous for any $t \geq 0$. Consider the attractor $\omega(\mathcal{H}_+(g_0))$ of the semigroup $\{T(t)\}$ on $\mathcal{H}_+(g_0)$,

$$T(t)\omega(\mathcal{H}_+(g_0)) = \omega(\mathcal{H}_+(g_0)) \ \forall t \geq 0, \tag{3.1}$$

(See Proposition 1.2).

Similarly to $L^{loc}_2(\mathbf{R}_+, H)$ and $L^a_2(\mathbf{R}_+, H)$ we consider spaces $L^{loc}_2(\mathbf{R}, H)$ and $L^a_2(\mathbf{R}, H)$ of functions on the whole axis. The space $L^a_2(\mathbf{R}, H)$ has a norm:

$$\|\zeta\|^2_{L^a_2(\mathbf{R};H)} = \sup_{t\in\mathbf{R}} \int_t^{t+1} |\zeta(s)|^2 ds < +\infty.$$

Consider any external force $g \in \omega(\mathcal{H}_+(g_0))$. The invariance property (3.1) implies that there is a function $g_1(s), g_1 \in \omega(\mathcal{H}_+(g_0))$ such that $T(1)g_1 = g$. Consider the function $\zeta(s), s \geq -1$, $\zeta(s) = g_1(s+1)$. Obviously, $\zeta(s) \equiv g(s)$ for $s \geq 0$, hence, $\zeta(s)$ is a prolongation of $g(s)$ on the semiaxis $[-1, +\infty[$. In such doing, there is $g_2 \in \omega(\mathcal{H}_+(g_0))$ such that $T(1)g_2 = g_1, T(2)g_2 = g$. Put $\zeta(s) = g_2(s+2)$ for $s \geq -2$. Evidently, the function $\zeta(s)$ is well defined, since $g_2(s+2) = g_1(s+1)$ for $s \geq -1$. Continuing this process, we define $\zeta(s) = g_n(s+n)$ for $s \in [-n, +\infty[$, where $g_n \in \omega(\mathcal{H}_+(g_0))$ and $n \in \mathbf{N}$. We have defined a function $\zeta(s), s \in \mathbf{R}$, which is a prolongation of the initial external force $\zeta(s), s \in \mathbf{R}_+$. Moreover, the function $\zeta(s)$ satisfies the following property: $\Pi_+\zeta_t(s) \in \omega(\mathcal{H}_+(g_0))$ for any $t \in \mathbf{R}$, where $\zeta_t(s) = \zeta(t+s)$. Here $\Pi_+ = \Pi_{0,\infty}$ is the restriction operator to the semiaxis $\mathbf{R}_+$. Evidently, $\zeta(s) \in L^a_2(\mathbf{R}, H)$ and $\|\zeta\|^2_{L^a_2(\mathbf{R};H)} \leq \|g_0\|^2_{L^a_2(\mathbf{R}_+;H)}$.

Definition 3.1 **(i)** *A function $\zeta(s) \in L^a_2(\mathbf{R}, H)$ is said to be a complete external force in $\omega(\mathcal{H}_+(g_0))$ if $\Pi_+\zeta_t(s) = \Pi_+\zeta(t+s) \in \omega(\mathcal{H}_+(g_0))$, $s \in \mathbf{R}_+$, for any $t \in \mathbf{R}$.*

(ii) *Let $Z(g_0)$ be the set of all complete external forces in $\mathcal{H}_+(g_0)$.*

As it was showed above, for any symbol $g \in \omega(\mathcal{H}_+(g_0))$ there exist at least one complete external force $\zeta(s)$ which is the prolongation of g for negative s. Notice at once, that, in general, this prolongation need not be unique.

By analogy to section 1, we introduce in the cylinder $Q = \Omega \times \mathbf{R}$ the space $H^{\mathbf{r},loc}(Q) = L^{loc}_2(\mathbf{R}; H_2) \cap \{\partial_t v \in L^{loc}_2(\mathbf{R}; H)\}$, i.e. $v(s) \in H^{\mathbf{r},loc}(Q)$ if

$$\|\Pi_{t_1,t_2} v\|^2_{H^{\mathbf{r}}(Q_{t_1,t_2})} < +\infty \ \ \forall [t_1, t_2] \subseteq \mathbf{R}.$$

We shall use the topological spaces $H_s^{\mathbf{r},loc}(Q)$, $H_w^{\mathbf{r},loc}(Q)$, and the Banach space $H^{\mathbf{r},a}(Q)$ with the norm:

$$\|v\|^2_{H^{\mathbf{r},\mathbf{a}}(Q)} = \|v\|^2_{\mathbf{r},a} = \sup_{t\in\mathbf{R}} \|\Pi_{t,t+1}v\|^2_{H^{\mathbf{r}}(Q_{t,t+1})}.$$

Let we be given some complete external force $\zeta(s), s \in \mathbf{R}$, in $\omega(\mathcal{H}_+(g_0))$. Consider the equation

$$\partial_t u + \nu L u + B(u) = \zeta(x,t),\ (\nabla, u) = 0,\ u|_{\partial\Omega} = 0,\ x \in \Omega,\ t \in \mathbf{R}, \tag{3.2}$$

Definition 3.2 *The kernel $\mathcal{K}_\zeta$ of equation (3.2) with the complete external force $\zeta(s) \in Z(g_0)$ is the set of all solutions $u(s), s \in \mathbf{R}$, of the equation (3.2) that are bounded in the space $H^{\mathbf{r},a}(Q)$.*

The following Theorem specifies the structure of the trajectory attractor from Theorems 1.2 and 2.1.

Theorem 3.1 **(i)** *Let $g_0(s)$ be tr.-c. in $L^{loc}_{2,w}(\mathbf{R}_+, H)$ then the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ in $H_w^{\mathbf{r},loc}(Q_+)$ of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ can be represented in the form:*

$$\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(g_0))} = \Pi_+ \bigcup_{\zeta\in Z(g_0)} \mathcal{K}_\zeta = \Pi_+\mathcal{K}_{Z(g_0)}. \tag{3.3}$$

The set $\mathcal{K}_{Z(g_0)}$ is compact in $H_w^{\mathbf{r},loc}(Q)$ and it is bounded in $H^{\mathbf{r},a}(Q)$. For any $\zeta \in Z(g_0)$ the kernel $\mathcal{K}_\zeta$ is not empty. Any function $u(s) \in \mathcal{K}_\zeta$ is tr.-c. in $H_w^{\mathbf{r},loc}(Q)$.

(ii) *Let $g_0(s)$ be tr.-c. in $L_2^{loc}(\mathbf{R}_+, H)$ then the set $\mathcal{K}_{Z(g_0)}$ is compact in $H_s^{\mathbf{r},loc}(Q)$ and any function $u(s) \in \mathcal{K}_\zeta$ is tr.-c. in $H_s^{\mathbf{r},loc}(Q)$.*

The proof of Theorem 3.1 is given in [5] and it uses the invariance property 1.6 of the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)} : T(t)\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\mathcal{H}_+(g_0)}\ \forall t \geq 0$.

Remark 3.1 *It was mentioned above that, in general, the prolongation $\zeta(s)$ of an external force $g(s) \in \omega(\mathcal{H}_+(g_0))$ for $s < 0$ need not be unique. Let us describe an important case when it is unique. Let $g_0(s)$ be tr.-c. function in $L_2^a(\mathbf{R}_+; H)$, i.e. the set $\{g_0(s+h) \mid h \in \mathbf{R}_+\}$ is precompact in the Banach space $L_2^a(\mathbf{R}_+; H)$ with the uniform norm (1.2) and, hence, the hull $\mathcal{H}_+(g_0)$ is compact in $L_2^a(\mathbf{R}_+; H)$. It can be proved that there exists a unique function $\tilde{g}_0(s), s \in \mathbf{R}$, such that $\tilde{g}_0(s)$ is tr.-c. in $L_2^a(\mathbf{R}; H)$ and*

$$\int_t^{t+1} |g_0(s) - \tilde{g}_0(s)|^2 ds \to 0\ (t \to \infty).$$

Therefore, $\omega(\mathcal{H}_+(g_0)) = \mathcal{H}_+(\tilde{g}_0)$. Tr.-c. functions in $L_2^a(\mathbf{R}; H)$ are also called almost periodic functions in the Stepanov sence. These functions meet all the main properties of usual almost periodic function (in the Bohr or Bochner-Amerio sence, *see [1]). In particular, the translation semigroup* $\{T(t)\}$ *is invertible on* $\mathcal{H}_+(\tilde{g}_0)$ *and* $\mathcal{H}_+(\tilde{g}_0) = \Pi_+\mathcal{H}(\tilde{g}_0)$, *where* $\mathcal{H}(\tilde{g}_0) = [\{\tilde{g}_0(s+h) \mid h \in \mathbf{R}\}]_{L_2^a(\mathbf{R};H)}$ *is a hull of the almost periodic function* $\tilde{g}_0$. *Finally, in (3.3)* $Z(g_0) = \mathcal{H}(\tilde{g}_0)$ *and every external force* $g(s) \in \omega(\mathcal{H}_+(g_0))$ *possesses a unique prolongation for* $s < 0$ *as an almost periodic function.*

To conclude the section we describe the uniform (w.r.t. $g \in \mathcal{H}_+(g_0)$) attractor $\mathbf{A}_{\mathcal{H}_+(g_0)}$ for the family of processes $\{U_g(t,\tau) \mid t \geq \tau \geq 0\}$, $g \in \mathcal{H}_+(g_0)$, corresponding to the equation (1.1). By Theorem 1.1, for any $g \in \mathcal{H}_+(g_0)$, one defines a process $\{U_g(t,\tau) \mid t \geq \tau \geq 0\}$ acting on $V : U_g(t,\tau)u_\tau = u_g(t)$, where $u_g(t)$ is a solution of (1.1) with the initial condition $u|_{t=\tau} = u_\tau$, $\tau \geq 0$. Now consider the set $Z(g_0)$. By the similar way, to any $\zeta \in Z(g_0)$ there corresponds a complete process $\{U_\zeta(t,\tau) \mid t \geq \tau,\ \tau \in \mathbf{R}\}$, $U_g(t,\tau)u_\tau = u_\zeta(t)$, where $u_\zeta(t)$ is a solution of (3.2) with the initial condition $u|_{t=\tau} = u_\tau$, $\tau \in \mathbf{R}$. Consider the kernel $\mathcal{K}_\zeta$ corresponding to ζ.

By $\mathcal{K}_\zeta(t)$ we denote *a kernel section at time* $t \in \mathbf{R}$: $\mathcal{K}_\zeta(t) = \{u(t) \mid u(.) \in \mathcal{K}_\zeta\} \subset V$. It is clear that

$$U_\zeta(t,\tau)\mathcal{K}_\zeta(\tau) = \mathcal{K}_\zeta(t)\ \forall t \geq \tau,\ \tau \in \mathbf{R}.$$

Using Theorem 3.1, Corollary 2.1, and Corollary 1.2 we get

Corollary 3.1 (i) *If* $g_0(s)$ *is tr.-c. in* $L_2^{loc}(\mathbf{R}_+, H)$ *then the set*

$$\mathbf{A}_{\mathcal{H}_+(g_0)} = \bigcup_{\zeta \in Z(g_0)} \mathcal{K}_\zeta(0) \tag{3.4}$$

is the uniform (w.r.t. $g \in \mathcal{H}_+(g_0)$*) attractor* $\mathbf{A}_{\mathcal{H}_+(g_0)}$ *in* V *of the family of processes* $\{U_g(t,\tau) \mid t \geq \tau \geq 0\}$, $g \in \mathcal{H}_+(g_0)$, *the set* $\mathbf{A}_{\mathcal{H}_+(g_0)}$ *is compact in* V.

(ii) *If* $g_0(s)$ *is tr.-c. in* $L_{2,w}^{loc}(\mathbf{R}_+, H)$ *then the set* $\mathbf{A}_{\mathcal{H}_+(g_0)}$ *defined in (3.4) serves as the uniform (w.r.t.* $g \in \mathcal{H}_+(g_0)$*) attractor in* V_w *(with a weak topology of* V*) and it is bounded in* V. *In particular,* $\mathbf{A}_{\mathcal{H}_+(g_0)}$ *is the uniform attractor in* $H_{1-\delta}$, $\mathbf{A}_{\mathcal{H}_+(g_0)} \subset\subset H_{1-\delta}$, $0 < \delta \leq 1$.

4 Trajectory attractors for 3D N.-S. system.

In this section we shall construct a trajectory attractor for the non-autonomous Navier-Stokes system in a 3D domain $\Omega \subset\subset \mathbf{R}^3$. The structure of the trajectory attractor will be described and some properties of the attractor will be given. Only the brief general scheme will be sketched, without proofs and detailed explanations. This part will be expounded in more detail in another publication (see also [7], [10], [18]).

Consider 3D Navier-Stokes system in the semicylinder $Q_+ = \Omega \times \mathbf{R}_+$:

(4.1) $\partial_t u + \nu L u + B(u) = g(x,t)$, $(\nabla, u) = 0$, $u|_{\partial\Omega} = 0$, $x \in \Omega \subset\subset \mathbf{R}^3$, $t \geq 0$, where, $x = (x_1, x_2, x_3)$, $u = u(x,t) = (u^1, u^2, u^3)$, $g = g(x,t) = (g^1, g^2, g^3)$. L is the 3D Stokes operator: $Lu = -P\Delta u$; $B(u) = B(u,u)$, $B(u,v) = P(u,\nabla)v = P\sum_{i=1}^{3} u_i \partial_{x_i} v$. The spaces H and V are determined similar to the 2D case. Suppose $g(x,t) \in L_2^{loc}(\mathbf{R}_+, H)$.

Let we are given an initial external force $g_0(x,t) \in L_2^{loc}(\mathbf{R}_+, H)$ in (4.1). Assume that g_0 is tr.-c. in $L_{2,w}^{loc}(\mathbf{R}_+, H) \equiv L_{2,w}^{loc}$, i.e.

$$(4.2) \qquad \|g_0\|^2_{L_2^a(\mathbf{R}_+;V')} = \|g_0\|_a^2 = \sup_{t\in\mathbf{R}_+} \int_t^{t+1} |g_0(s)|^2 ds < +\infty.$$

Let $\Sigma = \mathcal{H}_+(g_0) \equiv [\{g_0(s+t) \mid t \geq 0\}]_{L_{2,w}^{loc}(\mathbf{R}_+,V')}$ be a hull of the function $g_0(s)$ in the space $L_{2,w}^{loc}(\mathbf{R}_+, H)$. It can be proved that $\mathcal{H}_+(g_0)$ is a complete metric space. The translation semigroup $\{T(t)\}$ is continuous on $\mathcal{H}_+(g_0)$ and $T(t)\mathcal{H}_+(g_0) \subseteq \mathcal{H}_+(g_0)$ for any $t \geq 0$, moreower, for any $g \in \mathcal{H}_+(g_0)$ one has: $\|g\|_a^2 \leq \|g_0\|_a^2$.

To study the trajectory attractor of the equation (4.1) we consider the family of these equations with various external forces $g \in \mathcal{H}_+(g_0)$.

To describe a trajectory space $\mathcal{K}_g^+$ of equation (4.1) with the external force g we shall consider weak solutions of equation (4.1) in the space $L_2^{loc}(\mathbf{R}_+; V) \cap L_\infty^{loc}(\mathbf{R}_+; H)$. If $u(s) \in L_2^{loc}(\mathbf{R}_+; V) \cap L_\infty^{loc}(\mathbf{R}_+; H)$ then equation (4.1) makes sence in the distribution space $D'(\mathbf{R}_+; V')$, where V' is the dual space of V. This is a usual way to define weak solutions of equation (4.1) (see [16]).

Definition 4.1 *The trajectory space $\mathcal{K}_g^+$ is the union of all weak solutions $u(s) \in L_2^{loc}(\mathbf{R}_+; V) \cap L_\infty^{loc}(\mathbf{R}_+; H)$ of equation (4.1) with the external force g that satisfy the following inequality:*

$$(4.3) \qquad \frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \leq (g(t), u(t)), \ t \in \mathbf{R}_+.$$

The equality (4.3) should be read as follows: for any function $\psi(s) \in C_0^\infty(]0,+\infty[)$, $\psi \geq 0$,

$$(4.4) - \frac{1}{2}\int_0^{+\infty} |u(s)|^2 \psi'(s) ds + \nu \int_0^{+\infty} \|u(s)\|^2 \psi(s) ds \leq \int_0^{+\infty} (g(s), u(s))\, \psi(s) ds.$$

Let us formulate the existence

Theorem 4.1 *Let $g \in L_2^{loc}(\mathbf{R}_+; H)$ and $u_0 \in H$; then there exists a weak solution $u(s)$ of equation (4.1) belonging to the space $L_2^{loc}(\mathbf{R}_+; V) \cap L_\infty^{loc}(\mathbf{R}_+; H)$ such that $u(0) = u_0$ and $u(s)$ satisfies the inequality (4.4).*

The existence theorem is a classical result (see [14], [15], [16], [19]). The proof uses the Faedo-Galerkin approximations method. To get inequality (4.4) one has to pass to the limit in the corresponding a priori equality involving the sequence $\{u_m\}$ of the Faedo-Galerkin approximations.

Remark 4.1 *For 3D case, the uniqueness problem is still open. Also, it is not known whether any weak solution of (4.1) satisfies inequality (4.3).*

It can be showed that any weak solution $u(s) \in L_2^{loc}(\mathbf{R}_+; V) \cap L_\infty^{loc}(\mathbf{R}_+; H)$ of the equation (4.1) satisfies

$$\partial_t^{1/4-\varepsilon} u \in L_2^{loc}(\mathbf{R}_+; H) \ \ \forall \varepsilon\ 0 < \varepsilon < 1/4$$

(see [16]), and

$$\partial_t u \in L_{4/3}^{loc}(\mathbf{R}_+; V')$$

(see [20]). Consider the following space:

$$\begin{aligned} \mathcal{F}_+^{loc} &= L_2^{loc}(\mathbf{R}_+; V) \cap L_\infty^{loc}(\mathbf{R}_+; H) \cap \\ \{v \mid \partial_t^{1/4-\varepsilon} v &\in L_2^{loc}(\mathbf{R}_+; V')\} \cap \{v \mid \partial_t v \in L_{4/3}^{loc}(\mathbf{R}_+; V')\}, \end{aligned}$$

where ε is fixed, $0 < \varepsilon < 1/4$. The space $\mathcal{F}_+^{loc}$ is endowed with the following "weak" convergence topology.

Definition 4.2 *A sequence $\{v_n\} \subset \mathcal{F}_+^{loc}$ converges (in a weak sense) to $v \in \mathcal{F}_+^{loc}$ as $t \to \infty$ if $v_n(s) \to v(s)$ $(n \to \infty)$ weakly in $L_2(t_1, t_2; V)$, $*$-weakly in $L_\infty(t_1, t_2; H)$, $\partial_t^{1/4-\varepsilon} v_n(s) \to \partial_t^{1/4-\varepsilon} v(s)$ $(n \to \infty)$ weakly in $L_2(t_1, t_2; H)\}$, and $\partial_t v_n(s) \to \partial_t v(s)$ $(n \to \infty)$ weakly in $L_{4/3}(t_1, t_2; V')\}$ for any $[t_1, t_2] \subset \mathbf{R}_+$.*

The space $\mathcal{F}_+^{loc}$ with the above weak topology is denoted by Θ_+^{loc}. We shall use also the space

$$\begin{aligned} \mathcal{F}_+^{a} &= L_2^{a}(\mathbf{R}_+; V) \cap L_\infty^{a}(\mathbf{R}_+; H) \cap \\ \{v \mid \partial_t^{1/4-\varepsilon} v &\in L_2^{a}(\mathbf{R}_+; V')\} \cap \{v \mid \partial_t v \in L_{4/3}^{a}(\mathbf{R}_+; V')\}, \end{aligned}$$

that is a subspace of $\mathcal{F}_+^{loc}$. If X is a Banach space then $L_p^a(\mathbf{R}_+; X)$ means the subspace of $L_p^{loc}(\mathbf{R}_+; X)$ having the finite norm

$$\|v\|_{L_p^a(\mathbf{R}_+;X)}^p = \sup_{t \geq 0} \int_t^{t+1} \|v(s)\|_X^p ds.$$

Similarly, the space $L_p^a(\mathbf{R}; X)$ has the norm

$$\|v\|_{L_p^a(\mathbf{R};X)}^p = \sup_{t \in \mathbf{R}} \int_t^{t+1} \|v(s)\|_X^p ds.$$

Lemma 4.1 **(i)** *$\mathcal{K}_g^+ \in \mathcal{F}_+^a$ for any $g \in \mathcal{H}_+(g_0)$;*
(ii) *for any $u(s) \in \mathcal{K}_g^+$*

$$\|T(t)u(\,.\,)\|_{\mathcal{F}_+^a} \leq C \|u(\,.\,)\|_{L_\infty(0,1;H)}^2 \exp(-\lambda t) + R_0 \ \forall t \geq 0, \tag{4.5}$$

where λ is the first eigenvalue of the operator νL; C depends on λ and R_0 depends on λ and $\|g_0\|_{L_2^a(\mathbf{R}_+;V')}^2$.

Put

$$\mathcal{K}_\Sigma^+ = \bigcup_{g \in \mathcal{H}_+(g_0)} \mathcal{K}_g^+, \ \Sigma = \mathcal{H}_+(g_0).$$

The translation semigroup $\{T(t) \mid t \geq 0\}$ acts on $\mathcal{K}_\Sigma^+$:

$$T(t)u(s) = u(t+s), \ s \geq 0.$$

Evidently

$$T(t)u(s) \in \mathcal{K}_{T(t)g}^+ \ \forall u \in \mathcal{K}_g^+, \ t \geq 0,$$

so, the family $\{\mathcal{K}_g^+, \ g \in \mathcal{H}_+(g_0)\}$ is translation-coordinated. Therefore

$$T(t)\mathcal{K}_\Sigma^+ \subseteq \mathcal{K}_\Sigma^+ \ \forall t \geq 0.$$

It is clear that every mapping $T(t)$ is continuous in Θ_+^{loc}.

It follows from (4.5) that the ball $B_0 = \|v\|_{\mathcal{F}_+^a} \leq 2R_0$ serves as a uniformly absorbing set of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma^+$. The set B_0 is bounded in $\mathcal{F}_+^a$ and it is compact in Θ_+^{loc}.

Lemma 4.2 *The family $\{\mathcal{K}_g^+, \ g \in \Sigma\}$ is $(\Theta_+^{loc}, \mathcal{H}_+(g_0))$-closed and $\mathcal{K}_\Sigma^+$ is closed in Θ_+^{loc}.*

In such a way, by Lemmas 4.1 and 4.2, Proposition 1.2 is applicable.

Let $\omega(\mathcal{H}_+(g_0))$ denote the global attractor of the semigroup $\{T(t)\}$ on $\mathcal{H}_+(g_0)$. Here

$$\omega(\mathcal{H}_+(g_0)) = \bigcap_{\tau \geq 0} \left[\bigcup_{t \geq \tau} T(t)\mathcal{H}_+(g_0) \right]_{L_{2,w}^{loc}}$$

is an ω-limit set of $\mathcal{H}_+(g_0)$.

Let $Z(g_0)$ be the set of all complete external forces in $\mathcal{H}_+(g_0)$, i.e. the set of all functions $\zeta(s), s \in \mathbf{R}$, $\zeta(s) \in L_2^{loc}(\mathbf{R}, H)$ such that $\zeta_t \in \omega(\mathcal{H}_+(g_0))$ for any $t \in \mathbf{R}$, where $\zeta_t(s) = \Pi_+\zeta(s+t), s \geq 0$. Evidently, for any $g(s) \in \omega(\mathcal{H}_+(g_0))$ there is at least one function $\zeta \in Z(g_0)$ such that $\zeta(s)$ is a prolongation $g(s)$ for negative s. To any complete external force $\zeta \in Z(g_0)$ there corresponds the kernel $\mathcal{K}_\zeta$ of equation (4.1). The kernel $\mathcal{K}_\zeta$ consists of all weak solutions $u(s), s \in \mathbf{R}$, of the equation

$$\partial_t u + \nu L u + B(u) = \zeta(x,t), \ t \in \mathbf{R},$$

that satisfy inequality (4.4) and that are bounded in the space

$$\begin{aligned} \mathcal{F}^a &= L_2^a(\mathbf{R}; V) \cap L_\infty^a(\mathbf{R}; H) \cap \\ \{v \mid \partial_t^{1/4-\epsilon} v &\in L_2^a(\mathbf{R}; V')\} \cap \{v \mid \partial_t v \in L_{4/3}^a(\mathbf{R}; V')\}. \end{aligned}$$

Let us formulate the main

Theorem 4.2 *Let $g_0(s)$ be tr.-c. in $L^{loc}_{2,w}(\mathbf{R}_+;H)$ then the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+_\Sigma$ ($\Sigma = \mathcal{H}_+(g_0)$) possesses a trajectory attractor $\mathcal{A}_\Sigma = \mathcal{A}_{\mathcal{H}_+(g_0)}$ in Θ^{loc}_+. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ is bounded in $\mathcal{F}^a_+$ and compact in Θ^{loc}_+. Moreower,*

$$\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(g_0))} = \Pi_+ \bigcup_{\zeta \in Z(g_0)} \mathcal{K}_\zeta = \Pi_+ \mathcal{K}_{Z(g_0)}.$$

The kernel $\mathcal{K}_\zeta$ is not empty for any $\zeta \in Z(g_0)$; the set $\mathcal{K}_{Z(g_0)}$ is bounded in $\mathcal{F}^a$ and compact in Θ^{loc}.

The detailed proof of Lemma 4.1, Lemma 4.2, and Theorem 4.2 will be given in [5].

Notice that the following embedding is continuous: $\Theta^{loc}_+ \subset L^{loc}_2(\mathbf{R}_+;H_{1-\delta})$, $0 < \delta \le 1$, so, we get

Corollary 4.1 *For any set $B \subset \mathcal{K}^+$ bounded in $\mathcal{F}^a_+$*

$$\mathbf{dist}_{L_2(0,\Gamma;H_{1-\delta})}\left(\Pi_{0,\Gamma}T(t)B, \Pi_{0,\Gamma}\mathcal{K}_{Z(g_0)}\right) \to 0 \ (t \to \infty),$$

where Γ is fixed and any.

In conclusion, we shall formulate some properties of trajectory attractors of the Navier-Stokes system.

I) Let in (4.1) $g_0(x,s) = g_1(x,s) + a(x,s)$, where $g_1(x,s)$ and $a(x,s)$ are tr.-c. functions in $L^{loc}_{2,w}(\mathbf{R}_+;H)$. Assume that $T(t)a \to 0$ $(t \to +\infty)$ in $L^{loc}_{2,w}(\mathbf{R}_+;H)$, i.e.

$$\int_0^1 (a(s+t), \psi(s))\, ds \to 0 \ (t \to +\infty) \tag{4.6}$$

for any $\psi(s) \in L_2(0,1;H)$. Then the trajectory attractors corresponding to $\Sigma = \mathcal{H}_+(g_1 + a)$ and to $\Sigma_1 = \mathcal{H}_+(g_1)$ coincide:

$$\mathcal{A}_{\mathcal{H}_+(g_1+a)} = \mathcal{A}_{\mathcal{H}_+(g_1)}. \tag{4.7}$$

In particular, if $g_1 \equiv 0$ then $\mathcal{A}_{\mathcal{H}_+(a)} = \mathcal{A}_{\mathcal{H}_+(0)} = \{0\}$.

For example, the function $a(x,s) = \varphi(x)\sin(t^2)$ satisfies (4.6) for any $\varphi \in H$. Thus more and more rapidly oscillating additional term $a(s)$ does not effect on the trajectory attractor. The equality (4.7) is valid as for 3D as for 2D N.-S. systems.

II) Let in (4.1) $g_0(x,s) = g_{0\varepsilon}(x,s) = g_1(x,s) + \varepsilon g_2(x,s)$, where $g_i(x,s)$ are tr.-c. functions in $L^{loc}_{2,w}(\mathbf{R}_+;H)$ and $|\varepsilon| \le 1$. Put $\mathcal{A}(\varepsilon) = \mathcal{A}_{\mathcal{H}_+(g_{0\varepsilon})}$. Then $\mathcal{A}(\varepsilon)$ is lower semi-continuous with respect to ε. More precisely. It can be proved that the ball $B_0 = \|v\|_{\mathcal{F}^a_+} \le R_1$ being a topological subspace of Θ^{loc}_+ is metrizable and in this metric:

$$\mathbf{dist}_{\Theta^{loc}_+}(\mathcal{A}(\varepsilon), \mathcal{A}(0)) \to 0 \ (t \to \infty). \tag{4.8}$$

The radius R_1 is big enough to provide the inclusion: $\mathcal{A}(\varepsilon) \subseteq B_1$ for any $\varepsilon, |\varepsilon| \leq 1$. For the 2D N.-S. system (1.1) the property (4.8) is also valid with $\mathbf{dist}_{\Theta_+^{loc}}$ being replaced by $\mathbf{dist}_{\mathcal{H}^{r,loc}}$ or by $\mathbf{dist}_{\mathcal{H}_w^{r,loc}}$ depending on the tr.-c. class the external force belongs to.

III) Let $\mathcal{A}^{(N)}_{\mathcal{H}_+(P_N g_0)} \equiv \mathcal{A}^{(N)}$ be the trajectory attractor of the Faedo-Galerkin approximation system of order N for the equation (4.1), where P_N is the projection onto the finite-dimensional subspace of H spanned by the first N eigenfunctions of the Stokes operator. Then

$$\mathbf{dist}_{\Theta_+^{loc}} \left(\mathcal{A}^{(N)}, \mathcal{A}_{\mathcal{H}_+(g_0)}\right) \to 0 \ (t \to \infty).$$

In other words, for any neighbourhood $\mathcal{O}(\mathcal{A}_{\mathcal{H}_+(g_0)})$ in Θ_+^{loc} there is N_1 such that $\mathcal{A}^{(N)} \subseteq \mathcal{O}(\mathcal{A}_{\mathcal{H}_+(g_0)})$ for any $N \geq N_1$.

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