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Contents

1 Introduction

This paper deals with Dirac type operators on compact planar domains. We consider such operators with self-adjoint locally elliptic local boundary conditions[1]. The paper is focused not on individual operators, but on paths in the space of such operators. We consider only paths connecting two operators conjugate by a scalar gauge transformation (so, they are loops up to a scalar gauge transformation). Such paths have a well known invariant, the spectral flow (which counts with signs the number of eigenvalues passing through zero from the start of the path to its end; the eigenvalues passing from negative values to positive one are counted with the plus sign, and egenvalues passing in the other direction are counted with the minus sign). The paper is devoted to the problem of computation of the spectral flow in the situation when all the operators along the path have the same symbol and the same boundary condition.

Because these results are potentially useful for the physics of condensed matter, the author attempted to avoid the advanced mathematical terminology and to explain the results and the ideas behind their proofs in a way accesible to non-mathematicians. By the same reason, we present the case of Dirac operators (Theorems 1 and 2) before dealing with the more general case of Dirac type operators on domains equipped with an arbitrary metric. Note that physicists sometimes use more general boundary conditions than the ones considered in this paper. For example, the so-called armchair boundary conditions for graphen are of the type considered in this paper, but the zigzag boundary conditions for graphen are not. While it is not completely obvious, boundary conditions considered in this paper are just another forms of local elliptic boundary conditions used in physics, as explained in Section 4.

We start with the following situation. Let X be a compact planar domain bounded by m smooth curves (topologically it is a disk with $m-1$ holes). Our operators act on spinor (i. e. spinor-valued) functions, which we identify with a column vectors of two complex-valued functions:

$$u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix},\ u^\pm \colon X \to \mathbb{C}.$$

A Dirac operator acting on spinor functions has the form

$$D = \mathbb{D} + \begin{pmatrix} 0 & \bar{q}(x) \\ q(x) & 0 \end{pmatrix},\ \mathbb{D} = -i \begin{pmatrix} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & 0 \end{pmatrix},$$

where q is a smooth function from X to $\mathbb{C}$, $x = (x^1, x^2) \in X$, $\partial_i = \partial/\partial x^i$. Our focus is on 1-parameter families D_t of such operators parametrized by $t \in [0,1]$. In such a family the first term $\mathbb{D}$ involving derivatives is always the same, but in the second term the function q is allowed to continuously change with t, i. e. $q = q_t$, where $t \in [0,1]$. In agreement with the above, we assume that $D_1 = \mu D_0 \mu^{-1}$ for some smooth scalar gauge transformation $\mu \colon X \to U(1)$.

[1]In particular, boundary conditions defined by general pseudo-differential operators are not allowed.

All operators D_t are considered with the same boundary condition of the form $i(n_1 + in_2)u^+ = B(x)u^-$, where $n = (n_1, n_2)$ is the outward conormal to the boundary, and B is a real-valued smooth function on the boundary of X without zeros. Our first main result, Theorem 1, asserts that the spectral flow of such a family of operators is equal to $c_m \sum_{j=1}^m b_j\mu_j$, where c_m is an integer constant depending on m only, μ_j is the degree of the restriction of μ on the j-th connected boundary component, $b_j = 1$ if B is negative on the j-th boundary component, and equal to 0 otherwise.

After considering this most special and very important situation, we turn our attention to the situation of Dirac operators acting on N-dimensional spinor functions

$$u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}, \; u^\pm \colon X \to \mathbb{C}^N,$$

where, as before, X is a compact planar domain bounded by m smooth curves. A Dirac operator acting on N-dimensional spinor functions has the form

$$D = \mathbb{D} + Q(x), \; \mathbb{D} = -i\left(\sigma_1\partial_1 + \sigma_2\partial_2\right),$$

where

$$\sigma_1 = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -iI_N \\ iI_N & 0 \end{pmatrix},$$

I_N is $N \times N$ unit matrix, and $Q(x)$ is complex self-adjoint $2N \times 2N$ matrix smoothly dependent on $x \in X$. Again, our focus is on 1-parameter families D_t of such operators parametrized by $t \in [0, 1]$. In such a family the first term $\mathbb{D}$ involving derivatives is always the same, but in the second term the matrix Q is allowed to continuously change with t, i. e. $Q = Q_t$, where $t \in [0, 1]$. We assume that $D_1 = \mu D_0 \mu^{-1}$ for some smooth scalar gauge transformation $\mu \colon X \to U(1)$, where $U(1)$ is considering as the subgroup of $U(2N)$ consisting of the diagonal matrices with equal diagonal elements.

All operators D_t are considered with the same boundary condition $i(n_1 + in_2)u^+ = B(x)u^-$, where B is a smooth map from the boundary to the space of complex self-adjoint invertible $N \times N$ matrices. Note that a local boundary condition is locally elliptic if and only if it can be written in such a form with $B(x)$ invertible at any x; this boundary condition is self-adjoint if and only if $B(x)$ is self-adjoint at any x.

Our second main result, Theorem 2, asserts that the spectral flow of such a family of operators is equal to $c_m \sum_{j=1}^m b_j\mu_j$, where c_m is the same constant as in Theorem 1 (in particular, c_m does not depend on the dimension N), μ_j is the degree of the restriction of μ on the j-th boundary component (this restriction give us the map from the circle to the circle because μ is a scalar gauge transformation), and b_j is the number of negative eigenvalues of $B(x)$ (counting with multiplicities) on the j-th boundary component.

Theorem 3 extends Theorem 2 to a still more general class of operators. These Dirac type operators involve in their definition an arbitrary (not necessarily flat) metric on X, and have the principal symbol defined by the Clifford multiplication which does not necessarily agree with this metric. While considering of arbitrary metric is important for some physical applications, considering of Clifford multiplication which does not agree with the metric on X does not seem neseccary. Nevertheless, we take care of this more general case because the proofs of our results crucially depend on its consideration. Moreover, we cannot prove Theorems 1 and 2 without proving Theorem 3 first.

Point out that *scalar* gauge transformations $\mu\colon X \to U(1)$ leave invariant every local boundary condition, as well as the first term $\mathbb{D}$ of the operator $\mathbb{D} + Q(x)$: $\mu(\mathbb{D} + Q)\mu^{-1} = \mathbb{D} + Q'$ for some function $Q'(x)$. So any operator D_0 can be connected with the conjugate operator $D_1 = \mu D_0 \mu^{-1}$ by the path $(\mathbb{D} + Q_t(x))$ with fixed boundary condition. On the contrary, non-scalar gauge transformations $\mu\colon X \to U(2N)$ do not have such properties, so the problem of the computing of the spectral flow can not be stated in such a form as described above. If we allow general non-scalar gauge transformations, then we have to allow the paths of the operators (D_t) and of the boundary conditions (B_t) with B_t and symbol of D_t being dependent on t. Some results about this more general case are outlined in Section 8.

Note that in this paper the spectral flow is computed only up to multiplication by an integer constant c_m depending only on m. For a disk with one hole ($m = 2$) the eigenvectors and hence the spectral flow are calculated explicitly in a special case; this is sufficient to determine c_2; it turns out that $c_2 = 1$ (see Theorem 4). For the case $m > 2$ this method fails because Fourier transform gives no help here. Nevertheless, the author expects that $c_m = 1$ for all m; some reasons in favour of this conjecture are provided after proving that $c_2 = 1$.

Part I

The spectral flow for Dirac operators

2 The spectral flow

Let H be a complex separable Hilbert space, (A_t), $t \in [0, 1]$ be a continuous 1-parameter family of bounded self-adjoint (or, what is the same, Hermitian) Fredholm operators in H. Near zero every A_t has the discrete real spectrum, which changes continuously with the variation of t. Hence one can count the net number of eigenvalues of A_t passing through zero in positive direction as t runs from 0 to 1, that is, the difference between the numbers of eigenvalues (counting multiplicities) crossing zero in positive and negative directions. This net number is called the spectral flow $\mathrm{sf}(A_t)$. The description of this notion can be found in [2, 3].

The case when A_0 or A_1 has zero eigenvalue require some agreement on the counting procedure; we use the following convention: take a small $\varepsilon > 0$ such that A_0, A_1 have no eigenvalues in the interval $[-\varepsilon, 0)$, and define the spectral flow as the net number of eigenvalues of $A_t + \varepsilon I$ which pass through zero.

Let now (A_t) be an 1-parameter family of (not necessarily bounded) self-adjoint Fredholm operators in H. For example, it can be a family of symmetric elliptic differential operators A_t on a sections of Hermitian bundle E over closed (that is, compact without boundary) manifold X. The definition of the spectral flow can be adjusted to this case, though more accurate consideration is needed, particularly due to the presence of various natural topologies on the space of such operators [4, 5, 6].

When a manifold has non-empty boundary, we have to consider the family (A_t, B_t), where A_t is a symmetric elliptic differential operator, and B_t is a "good" (self-adjoint elliptic) boundary condition for A_t at any t. The notion of self-adjoint elliptic boundary value problem for operators of Dirac type one can see in [3, 7], and for general first order elliptic operators – in [8].

Such differential operator A_t with boundary condition B_t defines the unbounded self-adjoint Fredholm operator on $L^2(X, E)$, which has the unbounded discrete real spectrum. Intuitively, the spectrum of (A_t, B_t) changes continuously at continuous change of (A_t, B_t), so the definition of the spectral flow works in this case as well [6]. However, the proofs of the correctness of the definition and of the standard properties of the spectral flow are considerably more difficult than in the case of closed manifolds. The crucial ingredient is the continuity (in t) of the family (A_t, B_t) in the space of unbounded self-adjoint Fredholm operators on $L^2(X, E)$ with an appropriate metric. This was proved in [8] (see [8], Theorem 7.16). This continuity property allows to use the theory developed in [4, 5] in full force. Our proof of Theorem 3 (see Part II) crucially depends on this theory, and, in particular, on Theorem 7.16 from [8]. The results of this theory needed for the proof of Theorem 3 are isolated in Section 10 as properties (P0-P4).

Note that if the spectra of (A_0, B_0) and (A_1, B_1) are the same (isospectral operators), which is the case in this paper, there is another way to define the spectral flow of (A_t, B_t). The set

$$\{(t, \lambda) \colon \lambda \text{ is the eigenvalue of } (A_t, B_t)\}$$

can be uniquely represented as the union of the graphs of functions $\lambda_i(t)$ such that $\lambda_i(t) \leqslant \lambda_j(t)$ for $i \leqslant j$. These functions give us a bijection (one-to-one correspondence) of the spectrum of (A_0, B_0) to the spectrum of (A_1, B_1). If these spectra coincide as subsets of $\mathbb{R}$ then this correspondence give us the shift of the spectrum on the integer number of positions. This number is the spectral flow of (A_t, B_t). It is worth to note that for the isospectral case one can replace the level $\lambda = 0$ by any real number, and the difference between eigenvalues crossing the level in positive and negative directions will be the same [2].

3 Dirac operators: the simplest case

Suppose X is a compact planar domain bounded by m smooth curves (topologically it is a disk with $m-1$ holes). We will use the notations $x = (x^1, x^2) \in X$, $\partial_i = \partial/\partial x^i$.

Let us consider the Dirac operator on X

$$\mathbb{D} = -i \begin{pmatrix} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & 0 \end{pmatrix}, \tag{1}$$

acting on a spinor function $u\colon X \to \mathbb{C}^2$, $u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$.

A Dirac operator with non-zero vector potential has the form

$$D = \mathbb{D} + Q(x), \text{ where } Q(x) = \begin{pmatrix} 0 & \bar{q}(x) \\ q(x) & 0 \end{pmatrix},$$

q is a smooth function from X to $\mathbb{C}$.

Let $\mu\colon X \to U(1)$ be a gauge transformation; we suppose that $\mu(x) \in \mathbb{C}$, $|\mu(x)| \equiv 1$ for $x \in X$. Let us take a Dirac operator $D_0 = \mathbb{D} + Q_0(x)$ and connect it with the conjugate operator

$$D_1 = \mu D_0 \mu^{-1} = \mathbb{D} + Q_0 + \begin{pmatrix} 0 & i\mu^{-1}(\partial_1\mu - i\partial_2\mu) \\ i\mu^{-1}(\partial_1\mu + i\partial_2\mu) & 0 \end{pmatrix}$$

by an one-parameter family of Dirac operators

$$D_t = \mathbb{D} + Q_t, \text{ where } Q_t(x) = \begin{pmatrix} 0 & \overline{q}_t(x) \\ q_t(x) & 0 \end{pmatrix}, \tag{2}$$

q_t is a smooth function from X to $\mathbb{C}$ continuously depending on t, $t \in [0,1]$, $q_1 - q_0 = i\mu^{-1}(\partial_1\mu + i\partial_2\mu)$.

A self-adjoint local elliptic[2] local boundary condition for D_t has the form

$$in(x)u^+ = B(x)u^- \text{ on } \partial X, \tag{3}$$

where $B\colon \partial X \to \mathbb{R} \setminus \{0\}$ is a smooth function defining the boundary condition, $n = (n_1, n_2)$ is the outward conormal to the boundary ∂X of X at point x, and we identify n with the complex number $n_1 + in_2$ in (3).

Note that n_1, n_2 coincide with the components of the outward normal to ∂X for the case of Euclidean metric considered both here and in the next section. In Section 5 we consider a more general case of arbitrary metric on X, and the distinction between a normal and a conormal becomes essential there.

Remark. Boundary condition (3) coincides with the boundary condition of Berry and Mondragon for the "neutrino billiard" [9] up to replacement of B by B^{-1}.

Remark. $\mathbb{D} + Q(x)$ is the Dirac operator on the trivial 2-dimensional complex vector bundle over X with compatible unitary connection defined by the function $q(x)$. So the change of q_t with t is equivalent to the change of the connection.

Boundary condition (3) is gauge invariant with respect to the conjugation by μ, while D_0 and D_1 are conjugate by μ. So the operators D_0, D_1 with the same boundary condition (3) are isospectral, and the spectral flow of the family D_t give us the shift of the spectrum D_t when t runs from 0 to 1.

[2]Another name for "local elliptic boundary condition" is "Sapiro-Lopatinskii boundary condition"

Theorem 1. *The spectral flow of the family* (D_t), $t \in [0,1]$, *with boundary condition* (3) *is equal to*

$$c_m \sum_{j=1}^{m} b_j \mu_j,$$

where c_m *is the integer constant depending on* m *only,* μ_j *is the degree of the restriction of* μ *on* ∂X_j,

$$b_j = \begin{cases} 1, \text{ if } B < 0 \text{ on } \partial X_j \\ 0, \text{ if } B > 0 \text{ on } \partial X_j \end{cases}$$

Here ∂X_j *are the connected components of the boundary of* X*, equipped with the orientation in such a way that the pair (outward normal to* ∂X_j*, positive tangent vector to* ∂X_j*) has the positive orientation on the plane* (x, y).

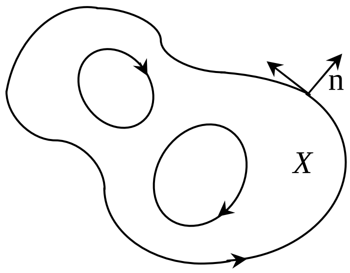

Figure 1: The case of two holes

Note that since $B \neq 0$, it has definite sign at each boundary component ∂X_j, so the constants b_j are correctly defined. The restriction of μ on j-th connected component of ∂X give us the map from the circle ∂X_j to the circle $U(1)$; μ_j is the degree of this map.

This theorem follows from more general result, which we formulate below. The generalization goes in two directions: (1) we admit arbitrary dimension of unknown complex functions u^-, u^+, (2) we replace Dirac operator by operators of more general form. The value of c_2 is computated is Section 6.

4 2N-dimensional Dirac operators

Let X be as in the previous section. The standard 2N-dimensional Dirac operator has the form

$$\mathbb{D} = -i\,(\sigma_1 \partial_1 + \sigma_2 \partial_2)\,, \text{ where } \sigma_1 = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -iI_N \\ iI_N & 0 \end{pmatrix}, \tag{4}$$

I_N is $N \times N$ unit matrix.

We will consider operators of the form $D = \mathbb{D} + Q(x)$ acting on spinor functions

$$u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}, \; u^\pm \colon X \to \mathbb{C}^N, \tag{5}$$

where Q is a smooth map from X to the space $\mathsf{H}(\mathbb{C}^{2N})$ of complex self-adjoint (or, what is the same, Hermitian) $2N \times 2N$ matrices.

A self-adjoint local elliptic boundary condition for the operator $\mathbb{D} + Q$ has the form

$$in(x)u^+ = B(x)u^- \text{ on } \partial X, \tag{6}$$

where B is a smooth map from ∂X to the space of complex self-adjoint invertible $N \times N$ matrices, $n = (n_1, n_2)$ is the outward conormal to ∂X at point x, and we identify n with the complex number $n_1 + in_2$.

The equivalent way of posing boundary condition (6) is

$$\left(i\,(n_1\sigma_1 + n_2\sigma_2) + \begin{pmatrix} B^{-1} & 0 \\ 0 & -B \end{pmatrix} \right) u = 0 \text{ on } \partial X. \tag{7}$$

Remark. Let us compare this boundary condition with the boundary condition $M\Psi = \Psi$ for the Dirac operator for 4-components spinors ($N = 2$) used in [10]. Note that the conditions on M in [10] (M is self-adjoint, unitary and anticommute with $n_1\sigma_1 + n_2\sigma_2$) mean nothing but the condition of self-adjointness of the boundary problem. The authors of [10] does not require local ellipticity from the boundary condition; however, in the absence of local ellipticity the spectrum of the operator is not expected to be discrete. The boundary condition $M\Psi = \Psi$ from [10] is both local elliptic and self-adjoint if and only if the matrix M can be represented by the formula

$$M = I_{2N} - 2 \begin{pmatrix} I_N + B^2 & 0 \\ 0 & I_N + B^2 \end{pmatrix}^{-1} \begin{pmatrix} I_N & i\bar{n}B \\ -inB & B^2 \end{pmatrix} \tag{8}$$

for some complex self-adjoint invertible $N \times N$ matrix function $B(x)$. For such M the boundary condition $M\Psi = \Psi$ is equivalent to our boundary condition (7) (with the replacement of Ψ by u).

Theorem 2. *Let $Q_t(x)$ be a continuous 1-parameter family of self-adjoint $2N \times 2N$ matrices smoothly dependent on $x \in X$ such that $\mathbb{D} + Q_1 = \mu\,(\mathbb{D} + Q_0)\,\mu^{-1}$ for some smooth gauge transformation $\mu \colon X \to U(1)$. Let B be a smooth map from ∂X to the space of complex self-adjoint invertible $N \times N$ matrices. Then the spectral flow of the family $(\mathbb{D} + Q_t)$ with the boundary condition (6) is equal to*

$$c_m \sum_{j=1}^{m} b_j \mu_j,$$

where c_m is the integer constant depending on m only, μ_j is the degree of the restriction of μ on ∂X_j, b_j is the number of negative eigenvalues of B (counting multiplicities) on the j-th connected component ∂X_j of the boundary (this number is correctly defined due to nondegeneracy of B).

This result is the corollary of Theorem 3 from the following section.

5 Dirac type operators

Let X be a compact planar domain bounded by m smooth curves and equipped with a Riemannian metric g (which is not necessarily flat).

We call a first order symmetric operator D over X a *Dirac type operator* if its symbol has the form

$$\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \Phi(x) \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \tag{9}$$

where Φ is a smooth map from X to the group $GL^+(2, \mathbb{R})$ of real invertible 2×2 matrices with positive determinant, and the matrices σ_1, σ_2 are defined by formula (4).

In other words, Dirac type operator is the operator acting on spinor functions (5) and having the following form:

$$D = D_{\Phi, Q} = -i\,(\rho_1(x)\partial_1 + \rho_2(x)\partial_2) + iR_\Phi(x) + Q(x), \tag{10}$$

where Q is a smooth map from X to the space $\mathsf{H}(\mathbb{C}^{2N})$ of complex self-adjoint $2N \times 2N$ matrices,

$$R_\Phi = -\frac{1}{2}\left[(\rho_1\partial_1 + \rho_2\partial_2) + (\rho_1\partial_1 + \rho_2\partial_2)^t\right]$$

(superscript t denotes the operation of taking the formal adjoint operator). More explicitely,

$$R_\Phi(x) = -\frac{1}{2}\left[\partial_1\left(\sqrt{g}\rho_1\right) + \partial_2\left(\sqrt{g}\rho_2\right)\right] \in \mathsf{H}(\mathbb{C}^{2N}),$$

where $\sqrt{g} = \sqrt{\det(g_{ij})}$, the matrix (g_{ij}) is inverse to the matrix $(g^{ij}) = (\langle dx^i, dx^j\rangle_g)$, $\sqrt{g}\, dx^1 dx^2$ is the volume element on X (of course, g_{ij}, g^{ij}, and $\sqrt{g}$ dependent on x).

We denote $\mathcal{D} = \mathcal{D}_{X, g, N}$ the space of all operators having the form (10) for fixed X, g, N. Note that Dirac type operator (that is an element of $\mathcal{D}_{X, g, N}$) is uniquely defined by the pair (Φ, Q).

A self-adjoint elliptic local boundary condition for Dirac type operator (10) has the form

$$in'(x)u^+ = B(x)u^- \ \text{ on } \partial X, \tag{11}$$

where B is a smooth map from ∂X to the space of complex self-adjoint invertible $N \times N$ matrices, the complex-valued function n' on ∂X is defined by the formula $n' = n_1' + in_2'$ with $(n_1', n_2') = (n_1, n_2)\Phi$ and $n = (n_1, n_2)$ being the outward conormal to ∂X at $x \in \partial X$.

Recall that $n_i = \sum g_{ij} n^j$ for the components (n^j) of the normal to the boundary.

Remark. Equation (11) is just another form of the equation

$$i\rho^+(x, n(x))u^+ = B(x)u^- \ \text{ on } \partial X, \tag{12}$$

where we denote by

$$\rho(x,\xi) = \begin{pmatrix} 0 & \rho^-(x,\xi) \\ \rho^+(x,\xi) & 0 \end{pmatrix}$$

the symbol $\xi_1\rho_1(x) + \xi_2\rho_2(x)$ of operator D in the direction of a covector $\xi = (\xi_1, \xi_2)$. Considering that in our case operator $\rho^+(x,\xi)$ is scalar, and $\rho^+(x, n(x)) = n'(x)I_N$, boundary condition (12) may be written in simplified form (11).

We denote $\mathcal{B} = \mathcal{B}_{X,N}$ the space of all smooth maps from ∂X to the space of complex self-adjoint invertible $N \times N$ matrices.

Suppose $D \in \mathcal{D}$, $B \in \mathcal{B}$. We will write (D, B) for operator (10) acting on the domain

$$\left\{u \in C^1(X, \mathbb{C}^{2N})\colon \text{ restriction of } u \text{ on } \partial X \text{ satisfies boundary condition (11)}\right\},$$

where $C^1(X, \mathbb{C}^{2N})$ is the space of continuously differentiable functions from X to $\mathbb{C}^{2N}$.

Such operators have the following properties:

1. For any $D \in \mathcal{D}$, $B \in \mathcal{B}$ the operator (D, B) is (unbounded) essentially self-adjoint Fredholm operator, which has the discrete real spectrum. All its eigenvectors are smooth functions. (Lemma 1, Section 9)

2. Suppose $Q_t(x)$ is continuous on (t, x), $D \in \mathcal{D}$, $B \in \mathcal{B}$. Then all the operators from the family $(D + Q_t, B)$ have the same domain, and this family is norm continuous in $L^2\left(X, g; \mathbb{C}^{2N}\right)$. Therefore the spectral flow of the operators family $(D + Q_t, B)$ is well defined ([5], Proposition 2.2).

Now we can formulate the main result of the present paper:

Theorem 3. *Let* $D \in \mathcal{D}$ *be a Dirac type operator* (10), $B \in \mathcal{B}$ *define boundary condition* (11) *for* D, $Q_t(x)$ *be a continuous 1-parameter family of self-adjoint* $2N \times 2N$ *matrices smoothly dependent on* $x \in X$ *such that* $D + Q_1 = \mu\,(D + Q_0)\,\mu^{-1}$ *for some smooth gauge transformation* $\mu\colon X \to U(1)$. *Then*

$$\mathsf{sf}\,(D + Q_t, B)_{t\in[0,1]} = c_m \sum_{j=1}^{m} b_j \mu_j,$$

where c_m *is the integer constant depending on* m *only,* b_j *is the number of negative eigenvalues of* B *(counting multiplicities) on* ∂X_j, μ_j *is the degree of the restriction of* μ *on* ∂X_j, ∂X *is oriented as described in the statement of Theorem 1.*

Note that constant c_m in all the Theorems 1-3 is the same.

Remark. Let S be a spinor bundle over X, $\langle\,\cdot\,,\,\cdot\,\rangle$ be an Hermitean metric on S compatible with its spinor structure, ∇ be a connection on S compatible with its spinor structure and the Levi-Civita connection on TX. Dirac operator on S in local coordinates has the form $D = \nu \cdot \nabla_\nu + w \cdot \nabla_w$, where (ν, w) is a positive oriented orthonormal basis in $T_x X$, and by the dot $\cdot$ we denote the action of tangent vectors on the spinors.

The unitary skew-adjoint isomorphism $J_x = (\nu \cdot)(w \cdot)$ of S_x does not dependent on the choice of a basis (ν, w) in $T_x X$ and define the bundle decomposition $S = S^+ \oplus S^-$, where $S^\pm$ are the subbundles of S such that $S^\pm_x$ are the eigenspaces of J_x corresponding to its eigenvalues $\mp i$. Due to the triviality of TX and of any complex bundle over X, we can fix some global positive oriented orthonormal basis field $(\nu(x), w(x))$ in TX and some trivialization of S^-. Extend the trivialization from S^- to S so that the action of the tangent vectors on the spinors in this trivialization has the form

$$\nu(x) \cdot u = -i \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix} u, \quad w(x) \cdot u = -i \begin{pmatrix} 0 & -iI_N \\ iI_N & 0 \end{pmatrix} u\,.$$

Then sections u of the spinor bundle S can be identified with the column vectors (5) of two functions $u^\pm \colon X \to \mathbb{C}^N$, and Dirac operator D acting on such a column vectors can be written in the form $D = -i\,(\rho_1 \nabla_1 + \rho_2 \nabla_2)$, where ρ_1, ρ_2 are defined by formula (9), $\Phi(x)$ is the transition matrix: $(\nu, w) = (e_1, e_2)\Phi(x)$, and by e_i we denote the vector (not differential operator) ∂_i to avoid misunderstanding.

So any Dirac operator over X has the form (10) with $\Phi(x)$ satisfying the condition $\Phi(x)\Phi^*(x) = (g^{ij}(x))$ and with the matrix $Q(x)$ of very special kind. While considering of arbitrary $Q(x)$ is important for some physical applications, considering of matrix function $\Phi(x)$ which does not satisfy the condition $\Phi(x)\Phi^*(x) = (g^{ij}(x))$ does not seem neseccary. Nevertheless, we take care of this more general case because the proofs of our results crucially depend on its consideration.

6 The case of one hole

Here we will compute the spectral flow for the case when X has just one hole ($m = 2$), and as a result will find c_2.

Theorem 4. $c_2 = 1$.

Proof. By Theorem 3, the spectral flow does not depend on the geometry of X and on the choice of $D \in \mathcal{D}$, so we can consider only the case when the computation is as simple as possible. Let us take the annulus $X = \{(r, \varphi) : 1 \leqslant r \leqslant 2\}$ in the polar coordinates (r, φ) on the plane, with the metric $ds^2 = dr^2 + d\varphi^2$, $N = 1$,

$$D = -i \begin{pmatrix} 0 & \partial_r - i\partial_\varphi \\ \partial_r + i\partial_\varphi & 0 \end{pmatrix}, \quad \mu = e^{i\varphi}, \quad Q_t = \begin{pmatrix} 0 & it \\ -it & 0 \end{pmatrix}, B = \begin{cases} +1 \text{ at } r = 1 \\ -1 \text{ at } r = 2 \end{cases}$$

We obtain the following system for the eigenvector u and the eigenvalue λ of $(D + Q_t, B)$:

$$\begin{cases} (-i\partial_r + \partial_\varphi - it)u^+ = \lambda u^- \\ (-i\partial_r - \partial_\varphi + it)u^- = \lambda u^+ \\ u^+ = iu^- \text{ at } r = 1, 2 \end{cases}$$

All the eigenvectors of $(D+Q_t, B)$ are smooth functions, so we can seek them in the form $u^{\pm}(r,\varphi)=\sum_{k\in\mathbb{Z}} u_k^{\pm}(r)e^{ik\varphi}$. Substituting it to the last system, we obtain

$$\begin{cases} \partial_r u_k^+ - (k-t)u_k^+ - i\lambda u_k^- = 0 \\ \partial_r u_k^- + (k-t)u_k^- - i\lambda u_k^+ = 0 \\ u_k^+ = iu_k^- \text{ at } r=1,2 \end{cases}$$

Equivalently,

$$\begin{cases} \partial_r\left(u_k^+ + iu_k^-\right) = (k-t-\lambda)\left(u_k^+ - iu_k^-\right) \\ \partial_r\left(u_k^+ - iu_k^-\right) = (k-t+\lambda)\left(u_k^+ + iu_k^-\right) \\ u_k^+ - iu_k^- = 0 \text{ at } r=1,2 \end{cases}$$

and $\partial_r^2\left(u_k^+ - iu_k^-\right) = \left((k-t)^2-\lambda^2\right)\left(u_k^+ - iu_k^-\right)$. So we have the following cases:

- either $u_k^+ = u_k^- \equiv 0$,
- or $k-t+\lambda=0$, $u_k^- = \text{const}$, $u_k^+ = iu_k^-$,
- or $(k-t)^2-\lambda^2 = -(\pi l)^2$, $l\in\mathbb{Z}\setminus\{0\}$, $u_k^+ - iu_k^- = \text{const}\cdot\left(e^{i\pi l r}-e^{-i\pi l r)}\right)$.

Therefore the set

$$\Lambda = \{(t,\lambda)\colon \lambda \text{ is the eigenvalue of } (D+Q_t,B)\}$$

is the union of the set $\Lambda_1=\{(t,\lambda)\colon \lambda - t\in\mathbb{Z}\}$ (with the multiplicities 1 of the eigenvalues) and of the set Λ_2 lying beyond the band $|\lambda|\geqslant\pi$.

If $\lambda_j(t)$ are the continuous functions from the interval $[0,1]$ to $\mathbb{R}$ such that $M\cap\{0\leqslant t\leqslant 1\}$ is the union of the graphs of functions $\lambda_j(t)$ and $\lambda_i(t)\leqslant\lambda_j(t)$ for $i\leqslant j$, then $\lambda_j(t)=j+t$ when $-3\leqslant j\leqslant 2$ (up to shift of the numeration). So

$$\mathsf{sf}\,(D+Q_t,B)_{t\in[0,1]} = 1.$$

On the other hand, by Theorem 3,

$$\mathsf{sf}\,(D+Q_t,B)_{t\in[0,1]} = c_2\left(b_1\mu_1+b_2\mu_2\right) = c_2\mu_2 = c_2$$

where we denote ∂X_1, ∂X_2 the inner and the outer boundary circles correspondingly. Therefore $c_2=1$.

7 The case of several holes

In this section we provide some evidence supporting the conjecture that $c_m = 1$ for all m.

Namely, let us realize $X = X^h$ as $(m-1)$ identical annuli arranged along the line and connected by the band of the width h, with the corners smoothed out, as on the Fig. 2.

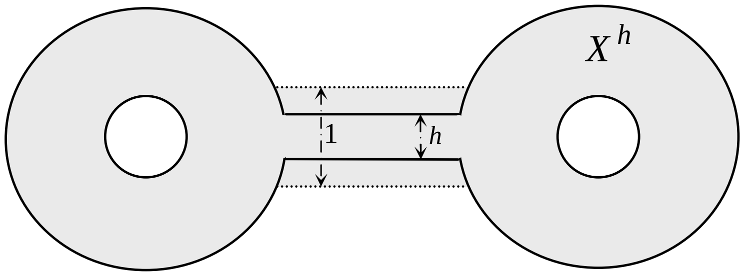

Figure 2: Contracting of the connecting band

Let us consider the process of continuous decreasing of the band's width from $h = 1$ to $h = 0$; we suppose that the annuli don't change in progress. Let us fix some function μ from X^1 to $U(1)$ and take $q_t = it\mu^{-1}(\partial_1\mu + i\partial_2\mu)$. Restricting μ and q_t on X^h, $0 < h \leqslant 1$, we obtain operator (2) over X^h. Let us define the boundary condition by $B^h = +1$ at the inner part $\cup_{j<m}\partial X_j^h$ of ∂X^h and $B^h = -1$ at the outer part ∂X_m^h of ∂X^h.

By Theorem 1, $\mathsf{sf}(\mathbb{D} + Q_t) = c_m \sum_{j<m} \mu_j$ does not depend on h. It is natural to suggest that the limit at $h \to +0$ of the (constant) spectral flow of the family $(\mathbb{D} + Q_t, B^h)$ for X^h is equal to the spectral flow of $(\mathbb{D} + Q_t, B^0)$ for the "limit" domain X^0, which is the disjoint union of $m-1$ annuli, and the "limit" boundary condition $B^0 = +1$ at the inner boundary and $B^0 = -1$ at the outer boundary of every annulus.

However, $\mathsf{sf}(\mathbb{D} + Q_t, B^0)$ for such union is the sum of $\mathsf{sf}(\mathbb{D} + Q_t, B^0)$ for the annuli, and hence is equal to $\sum_{j<m} c_2\mu_j = c_2 \sum_{j<m} \mu_j$.

Therefore, if the assumption on the limit behavior of the spectral flow is true, then $c_m = c_2$ at any $m > 2$.

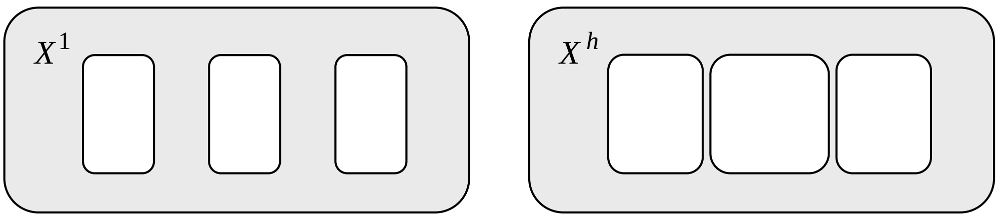

Figure 3: Increasing of the holes

Another way to have a look at the general case is to fix the outer boundary and to increase the holes up to their merging, as on Fig. 3. Here we obtain the single annulus in the limit of $h = 0$, and the same result $c_m = 1$ if the passage to the limit will be justified.

Alternatively, we can combine these two methods to obtain arbitrary number m', $1 \leqslant m' \leqslant m-1$ of annuli in the end of the limit process, with the same result for c_m.

8 General case: first order elliptic operators

The results of the present paper are concerned only with the case when X is a disk with holes. Light modification of the proof gives us analogue of this result for the case of smooth compact oriented surface X with nonempty boundary, with the only change of c_m to $c_{m,g}$, where $c_{m,g}$ is the integer constant depending on the number m of boundary components of X and on the genus g of X. However, this still remains within the very restricted framework: all the operators D_t are of Dirac type, both symbol of D_t and boundary condition don't depend on t, conjugating gauge transformation is scalar.

In fact, this result can be extended to much more general case.

Namely, let X be a smooth compact surface, (A_t) be an 1-parameter family of first order symmetric elliptic differential operators acting on sections of unitary vector bundle E over X, and subbundle L_t of $E|_{\partial X}$ defines a self-adjoint elliptic local boundary condition for A_t at any t. Suppose that (A_1, L_1) is conjugate to (A_0, L_0) by some gauge transformation μ (that is μ is unitary isomorphism of E, not necessarily scalar). Then operators (A_1, L_1), (A_0, L_0) are isospectral, and there arises the natural question about the spectral flow of the family (A_t, L_t). This question will be considered in a forthcoming paper by the author [1]. In that paper we will prove that $\operatorname{sf}(A_t, L_t)_{t\in[0,1]}$ has the form

$$c_{m,g} \sum_{j=1}^{m} \varphi_j,$$

where $c_{m,g}$ is the integer depending on the number m of boundary components of X and on the genus g of X, φ_j is the integer determined in a canonical way by the restrictions on j-th boundary component of the following data:

(1) family (σ_t), where σ_t is the symbol of A_t;

(2) family (L_t) of boundary conditions;

(3) gauge transformation μ.

In particular, the spectral flow of (A_t, L_t) does not depend on the choice of the operators in the interior of X but only on the symbol of the operators on the boundary.

Theorem 3 of present paper fits into this general result as follows: $c_m = c_{m,0}$, $\varphi_j = b_j\mu_j$. Recall that μ_j is invariant of the restriction of μ on j-th boundary component of X, b_j is defined from the restrictions of boundary condition and of the operator's symbol on j-th boundary component.

Part II
Proof of Theorem 3

Note that for $D' = D + Q_0$, $Q'_t = Q_t - Q_0$ we have $\mathsf{sf}(D + Q_t, B) = \mathsf{sf}(D' + Q'_t, B)$ with $Q'_0 = 0$. By this reason, in the proof we will restrict ourselves by the families Q_t with $Q_0 = 0$.

9 Two technical lemmas

First of all, we need to give some technical details. The reader interesting only in the ideas behind the proof can go directly to the next section.

Suppose $D \in \mathcal{D}$, $B \in \mathcal{B}$. We will write (D, B) for operator D acting on the domain

(13) $$\text{domain}(D,B) = \\ = \left\{ u \in L^2_1\left(X; \mathbb{C}^{2N}\right) : \text{ restriction of } u \text{ on } \partial X \text{ satisfies boundary condition (11)} \right\}.$$

Here $L^2_1\left(X; \mathbb{C}^{2N}\right)$ is the first Sobolev space; its elements are functions $u \in L^2\left(X; \mathbb{C}^{2N}\right)$ such that $\partial_1 u, \partial_2 u \in L^2\left(X; \mathbb{C}^{2N}\right)$. Strictly speaking, we use here not the restriction in the usual sense (trace map $u \mapsto u|_{\partial X}$) but the extension by continuity of the trace map $C^\infty\left(X; \mathbb{C}^{2N}\right) \to C^\infty\left(\partial X; \mathbb{C}^{2N}\right)$ to the bounded linear map from $L^2_1\left(X; \mathbb{C}^{2N}\right)$ to $L^2_{1/2}\left(\partial X; \mathbb{C}^{2N}\right)$ [8].

Note that operator (D, B) defined here is the closure of operator (D, B) defined in Section 5 (see [8], Proposition 2.9). Using of non-closed operators in the first part of the paper is explained by our intention to avoid the introduction of Sobolev spaces and of the extension of the trace map as long as possible. Due to the following Lemma, these two definitions give us the operators with the same eigenvectors, so this slight abuse of notation does not cause any troubles.

Lemma 1. *For any $D \in \mathcal{D}$, $B \in \mathcal{B}$ operator (D, B) is (unbounded) closed self-adjoint Fredholm operator on $L^2\left(X, g; \mathbb{C}^{2N}\right)$, which has the discrete real spectrum. Moreover, all its eigenvectors are smooth functions.*

Proof. Let B be a smooth function from ∂X to $GL(N, \mathbb{C})$. Then for any $D \in \mathcal{D}$, $\lambda \in \mathbb{C}$ boundary condition (11) satisfies the Sapiro-Lopatinskii condition for $D - \lambda$: the intersections of the subspace $\{u\colon in'(x)u^+ = B(x)u^-\} \subset \mathbb{C}^{2N}$ both with $\{u\colon u^- = 0\}$ and with $\{u\colon u^+ = 0\}$ are zero at any $x \in \partial X$. By Proposition 2.9 from [8], (11) is strongly regular boundary condition for D, so all eigenvectors of (D, B) in $L^2\left(X, g; \mathbb{C}^{2N}\right)$ are smooth functions. By the same Proposition, $(D - \lambda, B)$ is a closed Fredholm operator for any $\lambda \in \mathbb{C}$, so the spectrum of (D, B) is discrete.

For any $u, w \in L_1^2\left(X, g; \mathbb{C}^{2N}\right)$ we have

$$\langle Du, w\rangle_{L^2} - \langle u, Dw\rangle_{L^2} = \int_X \left(\langle Du, w\rangle - \langle u, Dw\rangle\right)\sqrt{g}\, dx^1 dx^2 =$$

$$= -i\int_X \left(\partial_1 \langle \sqrt{g}\rho_1 u, w\rangle + \partial_2 \langle \sqrt{g}\rho_2 u, w\rangle\right) dx^1 dx^2 =$$

$$= -i\int_X d\left(\langle \sqrt{g}\rho_1 u, w\rangle\, dx^2 - \langle \sqrt{g}\rho_2 u, w\rangle\, dx^1\right) = -i\int_{\partial X} \sqrt{g}\left(\langle \rho_1 u, w\rangle\, dx^2 - \langle \rho_2 u, w\rangle\, dx^1\right) =$$

$$= -i\int_{\partial X} \langle (n_1\rho_1 + n_2\rho_2)u, w\rangle \sqrt{g} ds = -\int_{\partial X} \left\langle \begin{pmatrix} i\bar{n}'u^- \\ in'u^+ \end{pmatrix}, \begin{pmatrix} w^+ \\ w^- \end{pmatrix} \right\rangle \sqrt{g} ds =$$

$$= \int_{\partial X} \left(\left\langle u^-, in'w^+\right\rangle - \left\langle in'u^+, w^-\right\rangle\right)\sqrt{g} ds,$$

where ds is the length element on ∂X. So for any $u, w \in \mathrm{domain}(D, B)$

$$\langle Du, w\rangle_{L^2} - \langle u, Dw\rangle_{L^2} = \int_{\partial X} \left\langle u^-, (B - B^*)w^-\right\rangle \sqrt{g} ds,$$

and operator (D, B) is symmetric on $L^2\left(X, g; \mathbb{C}^{2N}\right)$ if and only if $B(x)$ is self-adjoint at any x.

Let now $w \in \mathrm{domain}(D, B)^*$. By Proposition 2.9 from [8], $\mathrm{domain}(D, B)^*$ contains in $L_1^2\left(X, g; \mathbb{C}^{2N}\right)$, so we can use the computation above:

$$\langle Du, w\rangle_{L^2} - \langle u, Dw\rangle_{L^2} = \int_{\partial X} \left\langle u^-, (in'w^+ - Bw^-)\right\rangle \sqrt{g} ds$$

for any $u \in \mathrm{domain}(D, B)$. Therefore, $in'w^+ - Bw^-|_{\partial X} = 0$ for any $w \in \mathrm{domain}(D, B)^*$, $\mathrm{domain}(D, B)^* = \mathrm{domain}(D, B)$, and (D, B) is self-adjoint on $L^2\left(X, g; \mathbb{C}^{2N}\right)$. All eigenvalues of self-adjoint operator are real. This completes the proof.

In the statement of Theorem 3 we used only norm continuous paths of operators with fixed domain. But for the proof of Theorem 3 we have to deal with the paths of more general kind, when neither symbol of the operator nor boundary condition are fixed any more. The paths we need for the proof are not norm continuous but only graph continuous (note that by Proposition 2.2 from [5] any norm continuous path is graph continuous as well). So further we will use the graph topology on the space of closed densely defined self-adjoint operators on a separable Hilbert space H (in our case $H = L^2\left(X, g; \mathbb{C}^{2N}\right)$).

There are various definitions of the graph distance, all of which give the same graph topology [4]. One could take $d_G(A, A') = \left\|(A + iI)^{-1} - (A' + iI)^{-1}\right\|$, or alternatively $d_G(A, A') = \|P_A - P_{A'}\|$, where P_A, $P_{A'}$ are the orthogonal projectors of $H \times H$ onto the graphs of A, A' respectively.

Let us introduce the following metrics in $\mathcal{D}$ and $\mathcal{B}$:

$$d\left(D_{\Phi,Q}, D_{\Phi',Q'}\right) = \left\|Q - Q'\right\|_{C(X)} + \left\|\Phi - \Phi'\right\|_{C^1(X)} =$$
$$= \max_{x\in X} \left\|Q(x) - Q'(x)\right\| + \max_{x\in X} \left(\left\|\Phi(x) - \Phi'(x)\right\| + \left\|\partial_1\Phi(x) - \partial_1\Phi'(x)\right\| + \left\|\partial_2\Phi(x) - \partial_2\Phi'(x)\right\|\right),$$

$$d(B, B') = \left\|B - B'\right\|_{C^1(\partial X)} = \max_{x\in\partial X} \left(\left\|B(x) - B'(x)\right\| + \left\|\partial_s B(x) - \partial_s B'(x)\right\|\right),$$

where s is the length parameter on ∂X. Here we use any of the standard norms on the spaces $\mathsf{B}(\mathbb{C}^N)$ and $\mathsf{B}(\mathbb{C}^{2N})$ of complex $N \times N$ and $2N \times 2N$ matrices, and on the space $\mathsf{B}(\mathbb{R}^2)$ of real 2×2 matrices.

Note that $\left(D_{\Phi_t,Q_t}, B_t\right)$ is the continuous path in $\mathcal{D} \times \mathcal{B}$ if and only if $Q_t(x)$, $\Phi_t(x)$, $B_t(x)$, and the first partial derivatives of $\Phi_t(x)$, $B_t(x)$ with respect to x are continuous functions of (t, x).

We denote $\mathsf{HF}(H)$ the space of closed self-adjoint (or, what is the same, Hermitian) Fredholm operators in separable Hilbert space H. We fix graph metric on $\mathsf{HF}(H)$. Nevertheless we will write usually "graph continuous" instead of mere "continuous" for the maps to $\mathsf{HF}(H)$ to avoid a misunderstanding.

By Lemma 1, we have the natural inclusion $\mathcal{D} \times \mathcal{B} \hookrightarrow \mathsf{HF}\left(L^2\left(X, g; \mathbb{C}^{2N}\right)\right)$, which carries a pair $(D, B) \in \mathcal{D} \times \mathcal{B}$ to the operator D with the domain (13).

Lemma 2. *The natural inclusion $\mathcal{D} \times \mathcal{B} \hookrightarrow \mathsf{HF}\left(L^2\left(X, g; \mathbb{C}^{2N}\right)\right)$ is graph continuous.*

Therefore, if $t \mapsto (D_t, B_t)$ is the continuous path in $\mathcal{D} \times \mathcal{B}$, then (D_t, B_t) defines the graph continuous path in $\mathsf{HF}\left(L^2\left(X, g; \mathbb{C}^{2N}\right)\right)$, and the spectral flow of the operators family (D_t, B_t) is well defined.

Proof. Let us consider the smooth map

$$\psi\colon \mathsf{B}(\mathbb{C}^N) \to \mathsf{H}(\mathbb{C}^{2N}), \quad A \mapsto P = \begin{pmatrix} I_N & -A \\ -A^* & A^*A \end{pmatrix} \begin{pmatrix} I_N + AA^* & 0 \\ 0 & I_N + A^*A \end{pmatrix}^{-1},$$

which carries $A \in \mathsf{B}(\mathbb{C}^N)$ into the orthogonal projector P of $\mathbb{C}^{2N}$ with $\operatorname{Ker} P = \left\{u = (u^+, u^-) : u^+, u^- \in \mathbb{C}^N, u^+ = Au^-\right\}$. It induces the continuous map

$$\psi_*\colon C^1\left(\partial X, \mathsf{B}(\mathbb{C}^N)\right) \to C^1\left(\partial X, \mathsf{H}(\mathbb{C}^{2N})\right).$$

Composing ψ_* with the continuous map

$$\mathcal{D} \times \mathcal{B} \to C^1\left(\partial X, \mathsf{B}(\mathbb{C}^N)\right), \quad (D, B) \mapsto -i\rho^+(x, n(x))^{-1}B(x),$$

we obtain the continuous map

$$\Psi\colon \mathcal{D} \times \mathcal{B} \to C^1\left(\partial X, \mathsf{H}(\mathbb{C}^{2N})\right),$$

which carries (D, B) into the orthogonal projector P of $L^2\left(\partial X, g|_{\partial X}; \mathbb{C}^{2N}\right)$ with the kernel defined by boundary condition (11) [3].

By Proposition II.1.1 from [11], we have the continuous inclusion of the Banach spaces

$$\mathsf{B}\left(L^2_1\left(\partial X; \mathbb{C}^{2N}\right)\right) \subset \mathsf{B}\left(L^2_{1/2}\left(\partial X; \mathbb{C}^{2N}\right)\right),$$

where $\mathsf{B}(V)$ denote the space of bounded linear operators on a Banach space V, L^2_r is the (fractional) Sobolev space. Composing it with the natural continuous inclusion

$$C^1\left(\partial X, \mathsf{B}(\mathbb{C}^{2N})\right) \subset \mathsf{B}\left(L^2_1\left(\partial X; \mathbb{C}^{2N}\right)\right),$$

we obtain that the map $\Psi_* \colon \mathcal{D} \times \mathcal{B} \to \mathsf{B}\left(L^2_{1/2}\left(\partial X; \mathbb{C}^{2N}\right)\right)$ is continuous.

The natural map from $\mathcal{D}$ to the space of bounded linear operators from $L^2_1\left(X; \mathbb{C}^{2N}\right)$ to $L^2\left(X; \mathbb{C}^{2N}\right)$ is continuous too:

$$\left\|D_{\Phi,Q} - D_{\Phi',Q'}\right\|_{1,0} \leqslant \text{const}\left(\left\|\Phi - \Phi'\right\|_{C(X)} + \left\|R_\Phi - R_{\Phi'}\right\|_{C(X)} + \left\|Q - Q'\right\|_{C(X)}\right) \leqslant$$
$$\leqslant \text{const}\left(\left\|\Phi - \Phi'\right\|_{C^1(X)} + \left\|Q - Q'\right\|_{C(X)}\right).$$

By Theorem 7.16 from [8] and by Lemma 1, this imply that the inclusion $\mathcal{D} \times \mathcal{B} \hookrightarrow \mathsf{HF}\left(L^2\left(X, g; \mathbb{C}^2\right)\right)$ is graph continuous. This completes the proof.

10 The basic properties of the spectral flow

We will need the following properties of the spectral flow.

(P0) Zero crossing. In the absence of zero crossing the spectral flow vanish. More precisely, suppose $\gamma \colon [0,1] \to \mathcal{D} \times \mathcal{B}$ is the continuous path such that 0 is not the eigenvalue of $\gamma(t)$ for any $t \in [0,1]$. Then $\mathsf{sf}(\gamma) = 0$.

(P1) Homotopy invariance. The spectral flow along the continuous path $\gamma \colon [0,1] \to \mathcal{D} \times \mathcal{B}$ does not change when γ changes continuously in the space of paths in $\mathcal{D} \times \mathcal{B}$ with the fixed endpoints (the same as the endpoints of γ).

In other words, for the continuous map $h \colon [0,1] \times [0,1] \to \mathcal{D} \times \mathcal{B}$ such that $h_s(0) \equiv (D_0, B_0)$, $h_s(1) \equiv (D_1, B_1)$, we have $\mathsf{sf}\left(h_0(t)\right)_{t\in[0,1]} = \mathsf{sf}\left(h_1(t)\right)_{t\in[0,1]}$.

(P2) Path additivity. Suppose $\gamma \colon [a, c] \to \mathcal{D} \times \mathcal{B}$ is a continuous path, $a \leqslant b \leqslant c$. Then $\mathsf{sf}\left(\gamma(t)\right)_{t\in[a,c]} = \mathsf{sf}\left(\gamma(t)\right)_{t\in[a,b]} + \mathsf{sf}\left(\gamma(t)\right)_{t\in[b,c]}$.

(P3) Additivity with respect to direct sum. Let N_1, N_2 be natural numbers, $\left(D^i_t, B^i_t\right)$ be continuous paths in $\mathcal{D}_{N_i} \times \mathcal{B}_{N_i}$. Then the spectral flow along the path $\left(D^1_t \oplus D^2_t, B^1_t \oplus B^2_t\right)$ is equal to the sum of the spectral flows along the paths $\left(D^1_t, B^1_t\right)$ and $\left(D^2_t, B^2_t\right)$.

[3] We use here the general formula for the orthogonal projector P with the kernel $\{u^+ = Au^-\}$ for arbitrary matrix A. Actually, in our case $A = (in')^{-1}B$ is normal, $A = A^*$.

(P4) Conjugacy invariance. Let $J_\pm \colon X \to U(N)$ be unitary $N \times N$ matrices smoothly dependent on $x \in X$, $J = \begin{pmatrix} J_+ & 0 \\ 0 & J_- \end{pmatrix} : X \to U(2N)$, (D_t, B_t) be a smooth path in $\mathcal{D} \times \mathcal{B}$. Then $\mathsf{sf}\,(D_t, B_t) = \mathsf{sf}\,(JD_tJ^{-1}, J_-B_tJ_-^{-1})$.

More generally, if H is a separable complex Hilbert space, J is an unitary isomorphism of H, (A_t) is an 1-parameter graph continuous family of closed self-adjoint Fredholm operators, then $\mathsf{sf}\,(A_t) = \mathsf{sf}\,(JA_tJ^{-1})$.

Remark. Properties (P1) and (P2) imply that the spectral flow along the path is opposite to the spectral flow along the same path passing in the opposite direction.

Proof. By Lemmas 1-2, the inclusion of $\mathcal{D} \times \mathcal{B}$ into $\mathsf{HF}\left(L^2\left(X, g; \mathbb{C}^{2N}\right)\right)$ is graph continuous. So it is sufficient to prove Properties (P0-P4) for graph continuous paths in the space $\mathsf{HF}(H)$ of all closed self-adjoint Fredholm operators in separable Hilbert space H; this will implies properties (P0-P4) for the paths in $\mathcal{D} \times \mathcal{B}$.

First three properties of the spectral flow for graph continuous paths in $\mathsf{HF}(H)$ are proved in [4] (Proposition 2.2), taking into account the convention from Section 2 for the case when $\gamma(0)$ or $\gamma(1)$ are non-invertible.

Conjugacy invariance of the spectral flow for graph continuous paths in $\mathsf{HF}(H)$ follows from the uniqueness property of the spectral flow, which is proved in [5] (Theorem 5.9).

To prove (P4), consider graph continuous paths (A_t), (A'_t) in $\mathsf{HF}(H)$, $\mathsf{HF}(H')$ respectively. The path $(A_t \oplus A'_t)_{t\in[0,1]}$ is homotopic to the concatenation of paths $(A_t \oplus A'_0)_{t\in[0,1]}$ and $(A_1 \oplus A'_t)_{t\in[0,1]}$ in $\mathsf{HF}(H \oplus H')$. The spectral flow of the path $(A_t \oplus A'_0)$ in $\mathsf{HF}(H \oplus H')$ considering as the function of (A_t) satisfies the properties of the spectral flow for paths in $\mathsf{HF}(H)$, so by the uniqueness property of the spectral flow we have $\mathsf{sf}\,(A_t \oplus A'_0) = \mathsf{sf}\,(A_t)$. Similarly, $\mathsf{sf}\,(A_1 \oplus A'_t) = \mathsf{sf}\,(A'_t)$. Therefore, $\mathsf{sf}\,(A_t \oplus A'_t) = \mathsf{sf}\,(A_t) + \mathsf{sf}\,(A'_t)$. This completes the proof.

11 Independence of the choice of family (Q_t)

Let us prove that the spectral flow along $(D + Q_t, B)$ does not depend on the choice of (Q_t) when D, B, μ are fixed.

Let Q_t, Q'_t be two 1-parameter families of smooth maps from X to $\mathsf{H}(\mathbb{C}^{2N})$ such that $Q_0 = Q'_0 = 0$, $Q_1 = Q'_1 = \mu D \mu^{-1} - D$.

The path $D + Q_t$ could be continuously changed to the path $D + Q'_t$ in the class of paths in $\mathcal{D}$ with the fixed endpoints. For example, we could take the homotopy $h(s,t) = D + (1-s)Q_t + sQ'_t$. By the homotopy invariance property (P1) of the spectral flow, $\mathsf{sf}\,(D + Q_t, B)_{t\in[0,1]} = \mathsf{sf}\,(D + Q'_t, B)_{t\in[0,1]}$.

Therefore, if $Q_0 = 0$, $Q_1 = \mu D \mu^{-1} - D$ then $\mathsf{sf}\,(D + Q_t, B)_{t\in[0,1]} = F(X, g, N, D, B, \mu)$ for some integer-valued function F. Now we will investigate the properties of this function.

12 Independence of the choice of operator D

1. Suppose that D_0 is homotopic to D_1 in $\mathcal{D}$, that is there exist a continuous 1-parameter family of the Dirac type operators D_s connecting D_0 with D_1. We will show now that $F(X, g, N, D_0, B, \mu) = F(X, g, N, D_1, B, \mu)$.

Let us consider 2-parameter family of the Dirac type operators $D_{s,t} = (1-t)D_s + t\mu D_s \mu^{-1}$. Note that $D_{s,0} = D_s$, $D_{s,1} = \mu D_s \mu^{-1}$, $D_{s,t} - D_{s,0} = tQ_s$, where $Q_s = \mu D_s \mu^{-1} - D_s$ is the 1-parameter family of $2N \times 2N$ self-adjoint complex matrices smoothly dependent on $x \in X$.

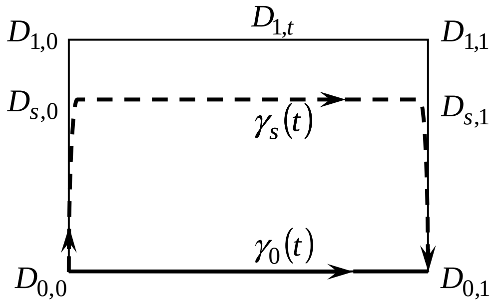

Figure 4: Homotopy from $\gamma_0(t)$ to $\gamma_1(t)$

Let us define the path $\gamma_1\colon [0,3] \to \mathcal{D}$ by the formula

$$\gamma_1(t) = \begin{cases} D_{t,0}, & t \in [0,1] \\ D_{1,t-1}, & t \in [1,2] \\ D_{3-t,1}, & t \in [2,3] \end{cases}$$

In other words, we consequently go around the left, top and right sides of the rectangle on Fig. 4 in clockwise direction. The path γ_1 could be continuously deformed to the path $D_{0,t} = D_0 + tQ_0$ within the rectangle. For example, we can take as such a deformation the family

$$\gamma_s(t) = \begin{cases} D_{st,0}, & t \in [0,1] \\ D_{s,t-1}, & t \in [1,2] \\ D_{s(3-t),1}, & t \in [2,3] \end{cases}$$

Then $\gamma_0(t)$ is the path $(D_0 + (t-1)Q_0)_{t\in[1,2]}$ concatenated with two steady paths, the spectral flows along which are zero by property (P0).

By the homotopic invariance property of the spectral flow,

$$\mathsf{sf}\,(\gamma_1(t), B)_{t\in[0,3]} = \mathsf{sf}\,(\gamma_0(t), B)_{t\in[0,3]} = \mathsf{sf}\,(D_0 + tQ_0, B)_{t\in[0,1]}.$$

On the other hand, the spectral flows along the first and the third parts of γ_1 are mutually reduced by (P4):

$$\mathsf{sf}\,(\gamma(t), B)_{t\in[0,1]} + \mathsf{sf}\,(\gamma(t), B)_{t\in[2,3]} = \mathsf{sf}\,(D_s, B)_{s\in[0,1]} - \mathsf{sf}\,(\mu D_s \mu^{-1}, B)_{s\in[0,1]} = 0.$$

Therefore, $\mathsf{sf}\,(D_0 + tQ_0, B)_{t\in[0,1]} = \mathsf{sf}\,(\gamma_1(t), B)_{t\in[1,2]} = \mathsf{sf}\,(D_1 + tQ_1, B)_{t\in[0,1]}$, and $F(X, g, N, D_0, B, \mu) = F(X, g, N, D_1, B, \mu)$.

2. Now we will simplify D step by step.

At first, we can continuously change $D = D_{\Phi,Q} = -i\,(\rho_1\partial_1 + \rho_2\partial_2) + iR_\Phi(x) + Q(x)$ to the operator $D_{\Phi,0}$, for example, along the path $D_{\Phi,(1-s)Q}$.

Further, take a smooth map $h\colon [0,1] \times GL^+(2,\mathbb{R}) \to GL^+(2,\mathbb{R})$ such that $h(0,\cdot)$ is the identity map, while the image of $h(1,\cdot)$ is the group $SO(2,\mathbb{R})$ of 2×2 orthogonal real matrices with determinant equal to one (existing of such a family is well-known fact for any dimension; it can be easily obtained using Gram–Schmidt process of orthonormalising, for example). The operator $D_{\Phi,0}$ can be continuously changed in $\mathcal{D}$ along the path $D_{h(t,\Phi),0}$ to the operator $D_{\Phi',0}$, where

$$\Phi'(x) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \in SO(2,\mathbb{R})$$

for some smooth function φ from X to S^1. So we have

$$F(X, g, N, D_{\Phi,Q}, B, \mu) = F(X, g, N, D_{\Phi',0}, B, \mu).$$

On the other hand, $D_{\Phi',0}$ can be represented as $J^{-1}D_{I,0}J$, where $I = I_2$ is the identity 2×2 matrix,

$$J(x) = \begin{pmatrix} J_+ & 0 \\ 0 & J_- \end{pmatrix} = \begin{pmatrix} e^{i\varphi}I_N & 0 \\ 0 & I_N \end{pmatrix} \in U(2N).$$

Let Q_t be an 1-parameter family of self-adjoint $2N\times 2N$ complex matrices such that $Q_0 = 0$, $Q_1 = \mu D_{\Phi',0}\mu^{-1} - D_{\Phi',0}$. Applying property (P4) of the spectral flow, we obtain

$$\begin{aligned} F(X, g, N, D_{\Phi',0}, B, \mu) = \mathsf{sf}\,(D_{\Phi',0} + Q_t, B) = \mathsf{sf}\,(J\left(D_{\Phi',0} + Q_t\right)J^{-1}, J_-BJ_-^{-1}) = \\ = \mathsf{sf}\,(D_{I,0} + JQ_tJ^{-1}, B) = F(X, g, N, D_{I,0}, B, \mu), \end{aligned}$$

because $JQ_1J^{-1} = \mu\left(JD_{\Phi',0}J^{-1}\right)\mu^{-1} - JD_{\Phi',0}J^{-1} = \mu D_{I,0}\mu^{-1} - D_{I,0}$.

Therefore, $F(X, g, N, D, B, \mu)$ does not depend on the choice of $D \in \mathcal{D}$, so from now on we will write $F(X, g, N, B, \mu)$ instead of $F(X, g, N, D, B, \mu)$.

13 Independence of the metric and invariance under the change of variables

We prove here that $F(X, g, N, B, \mu)$ is independent from the choice of the metric g on X, invariant under the change of variables, and does not depend on the geometry of X, using the fact that the number of holes is the only topological invariant of the disk with holes, and that the spectral flow is conjugacy invariant and does not depend on the choice of the operator.

Let X, X' be compact planar domains, each bounded by m smooth curves, and g, g' be Riemannian metrics on X, X' correspondingly.

As well known, there exists an orientation-preserving diffeomorphism $f\colon X' \to X$. [4]

We define θ as the smooth function from X' to $\mathbb{R}^+$ such that $f^*\,\mathrm{dvol} = \theta\,\mathrm{dvol}'$, where dvol, dvol' are volume elements on X, X' correspondingly. [5]

Diffeomorphism f defines the unitary isomorphism J from the Hilbert space $L^2\left(X, g; \mathbb{C}^{2N}\right)$ to the Hilbert space $L^2\left(X', g'; \mathbb{C}^{2N}\right)$, $u \mapsto \sqrt{\theta} f^* u$. [6]

Isomorphism J transforms operator $D \in \mathcal{D}_{X,g,N}$ with symbol ρ to the symmetric operator $D' = JDJ^{-1}$ on X' with symbol ρ'. For any $x' \in X'$, $x = f(x')$, any cotangent vector $\xi \in T^*_x X$, $\xi' = f^*\xi$, we have $\rho'(x', \xi') = \rho(x, \xi)$, that is $\rho'(x') = \left(\frac{\partial x'}{\partial x}\right)\rho(x) = \left(\frac{\partial x'}{\partial x}\right)\Phi\sigma$ in coordinate representation. The matrix $\left(\frac{\partial x'}{\partial x}\right)\Phi(x)$ contains in $GL^+(2, \mathbb{R})$ for any $x \in X$, so $D' \in \mathcal{D}_{X',g',N}$.

Let μ be a smooth function from X to $U(1)$. Taking the map $\mu' = f^*\mu$ from X' to $U(1)$ and the map $B' = \|f^* n\|^{-1}_{g'} f^* B$ from $\partial X'$ to $H(\mathbb{C}^N)$, we obtain

$$\mu' D' \mu'^{-1} - D' = \mu' \left(JDJ^{-1}\right) \mu'^{-1} - JDJ^{-1} = J\left(\mu D \mu^{-1} - D\right) J^{-1}.$$

So if Q_t connects $Q_0 = 0$ with $Q_1 = \mu D \mu^{-1} - D$, then $Q'_t = JQ_tJ^{-1}$ connects $Q'_0 = 0$ with $Q'_1 = \mu' D' \mu'^{-1} - D'$, and by the conjugacy invariance of the spectral flow (P4), we have

$$\mathsf{sf}\,(D + Q_t, B) = \mathsf{sf}\left(J(D + Q_t)J^{-1}, B'\right) = \mathsf{sf}\,(D' + Q'_t, B').$$

However, B' is homotopic to f^*B in $\mathcal{B}_{X',N}$, while the spectral flow of $(D' + Q'_t, \widetilde{B})$ is invariant under the continuous change of $\widetilde{B}$ in $\mathcal{B}_{X',N}$ (this is verified in a similar way as in the proof in Section 12). Therefore $\mathsf{sf}\,(D' + Q'_t, B') = \mathsf{sf}\,(D' + Q'_t, f^*B)$, and finally we obtain

$$F(X, g, N, B, \mu) = F(X', g', N, f^*B, f^*\mu). \tag{14}$$

This completes the proof.

In particular, for any two metrics g, g' on the same X, using the identity diffeomorphism f, we have

$$F(X, g, N, B, \mu) = F(X, g', N, B, \mu).$$

Further we will write $F(N, B, \mu)$ instead of $F(X, g, N, B, \mu)$.

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[4] In other words, there exist a smooth one-to-one change of variables $(x'^1, x'^2) = x' \xrightarrow{f} x = (x^1, x^2)$ with the smooth inverse and with positive Jacobian determinant $\det(\partial x/\partial x')$, which transforms X' onto X.

[5] As usually, we denote f^* the homomorphism from the differential forms (in particular, functions) on X to the differential forms on X', which is induced by f. In coordinate form, $\theta(x') = \frac{\sqrt{g(f(x'))}}{\sqrt{g'(x')}} \det(\frac{\partial x}{\partial x'})$.

[6] That is $(Ju)(x') = \sqrt{\theta(x')}\,u(f(x'))$.

14 Boundary conditions

Let us investigate the dependence of $F(N, B, \mu)$ on B.

$F(N, B, \mu)$ does not change when B continuously changes in $\mathcal{B}$; this is verified in a similar way as the proof in Section 12.

Let b_j be the number of negative eigenvalues of B (counting multiplicities) on ∂X_j. We prove that the ordered set $\hat{b} = \left(b_j\right)_{j=1}^{m}$ uniquely determines B up to continuous variation of B in $\mathcal{B}$.

Obviously, $\hat{b}$ is invariant with respect to such variations, so we have only to prove that any two B, B' with the same $\hat{b}$ are homotopic. It is sufficient to prove that any smooth map A from the circle S^1 to the space of complex self-adjoint invertible $N \times N$ matrices is homotopic (in the space of all such maps with C^1-metric) to the steady map sending S^1 to the point $(-I_k) \oplus I_{N-k} \in H(\mathbb{C}^N)$, where k is the number of negative eigenvalues of $A(x)$, $x \in S^1$.

1. Let us consider the continuous 1-parameter family A_s of smooth maps from S^1 to the space of complex self-adjoint invertible $N \times N$ matrices defined by the formula $A_s = A \cdot \left((1-s)I_N + sA^2\right)^{-1/2}$. This expression is correct because $(1-s)I_N + sA^2$ is self-adjoint and positive definite for any $s \in [0, 1]$. The family A_s give us the deformation from $A = A_0$ to smooth map A_1 from S^1 to the space of self-adjoint *unitary* $N \times N$ matrices.

2. The connected component of $A_1(x)$ in the space of self-adjoint unitary $N \times N$ matrices is diffeomorphic to the space $Gr_{\mathbb{C}}(k, N)$ of all k-dimensional linear subspaces of $\mathbb{C}^N$. This diffeomorphism is defined by the correspondence $U \mapsto \operatorname{Ker}(I_N + U)$, which associate with U the invariant subspace $V \subseteq \mathbb{C}^N$ of U corresponding to eigenvalue -1 of U. The inverse diffeomorphism is defined by the formula $V \mapsto U = (-I)_V \oplus I_{V^\perp}$.

The complex Grassmanian $Gr_{\mathbb{C}}(k, N)$ is known to be simply connected, so any two continuous maps from the circle to $Gr_{\mathbb{C}}(k, N)$ are homotopic. Taking into account that $Gr_{\mathbb{C}}(k, N)$ is the smooth manifold, we obtain that the space of smooth maps from the circle to $Gr_{\mathbb{C}}(k, N)$ (with C^1-metric) is path-connected. The same is true for the connected component of the space of self-adjoint unitary $N \times N$ matrices which is diffeomorphic to $Gr_{\mathbb{C}}(k, N)$, so A can be continuously changed in the class of smooth maps to the steady map $x \mapsto (-I_k) \oplus I_{N-k}$. This completes the proof.

15 Gauge transformations

1. We will prove that F is linear by μ, that is $F(N, B, \mu_1\mu_2) = F(N, B, \mu_1) + F(N, B, \mu_2)$ for any smooth functions $\mu_1, \mu_2 \colon X \to U(1)$.

Let $Q_i = \mu_i D \mu_i^{-1} - D$. Then $Q_1 + Q_2 = (\mu_1\mu_2)\, D\, (\mu_1\mu_2)^{-1} - D$, so by definition $F(N, B, \mu_1\mu_2)$ is equal to the spectral flow along the path $(D + P_t, B)_{t\in[0,2]}$, where $P_0 = 0$, $P_2 = Q_1 + Q_2$. We could take P_t composed from two parts: from 0 to Q_1 and then from

Q_1 to $Q_1 + Q_2$, for example,

$$P_t = \begin{cases} tQ_1, & t \in [0,1] \\ Q_1 + (t-1)Q_2, & t \in [1,2] \end{cases}$$

Using the property (P2) of the spectral flow, we obtain

$$\begin{aligned} F(N, B, \mu_1\mu_2) = \mathsf{sf}\,(D + P_t, B)_{t\in[0,1]} + \mathsf{sf}\,(D + P_t, B)_{t\in[1,2]} = \\ = \mathsf{sf}\,(D + tQ_1, B)_{t\in[0,1]} + \mathsf{sf}\,((D + Q_1) + tQ_2, B)_{t\in[0,1]} = \\ = F(N, B, \mu_1) + F(N, B, \mu_2), \end{aligned}$$

so F is linear by μ.

2. Let us denote M the set of equivalence classes of smooth functions $\mu\colon X \to U(1)$, where two functions are equivalent if one of them could be continuously changed to another in the space of smooth functions from X to $U(1)$ (with C^1-metric). We will consider M as the Abelian group, where the group structure on M is induced by the group structure on $U(1)$. It is well known that

$$M = \left\{(\mu_1, .., \mu_m) \in \mathbb{Z}^m\colon \sum \mu_j = 0\right\},$$

with the group structure induced from $\mathbb{Z}^m$, and the class of μ in M is defined by the m-tuple $\hat{\mu} = (\mu_j)$, where μ_j is the degree of the restriction of μ on ∂X_j.

Let us prove that $F(N, B, \mu)$ depends only on the class of μ in M.

Suppose that μ_t is a continuous path in the space of smooth functions from X to $U(1)$ such that $\mu_0(x) \equiv 1$. By the previous clause, it is sufficient to prove that $F(N, B, \mu_1) = 0$. Let us take $Q_t = \mu_t D \mu_t{}^{-1} - D$. Taking into account that $Q_1 = \mu_1 D \mu_1^{-1} - D$, we obtain $F(N, B, \mu_1) = \mathsf{sf}\,(D + Q_t, B)$. But all the operators $(D + Q_t, B)$ are conjugate to (D, B) by μ_t and therefore are isospectral. Let $\varepsilon > 0$ be such that (D, B) has no zero eigenvalues in the interval $[-\varepsilon, 0)$. Then $\mathsf{sf}\,(D + Q_t, B) = \mathsf{sf}\,(D + Q_t + \varepsilon I_{2N}, B) = 0$ by (P0) because all the operators $(D + Q_t + \varepsilon I_{2N}, B)$ have no zero eigenvalues. This completes the proof.

16 Bilinearity

In the previous sections we have proven that F depends only on the integer numbers N, $b_1, \ldots, b_m, \mu_1, \ldots, \mu_m$. Here we look at this dependence more closely.

Let us denote S the set of all possible $(m+1)$-tuples $(N, b_1, \ldots, b_m)$:

$$S = \left\{(N, b_1, \ldots, b_m) \in \mathbb{Z}^{m+1}\colon N \geqslant 1, 0 \leqslant b_j \leqslant N\right\}.$$

F defines the map from $S \times M$ to $\mathbb{Z}$ (which we will denote by the same letter F for simplicity) satisfying the following conditions:

$$F\left(N, \hat{b}, \hat{\mu} \oplus \hat{\mu}'\right) = F\left(N, \hat{b}, \hat{\mu}\right) + F\left(N, \hat{b}, \hat{\mu}'\right)$$
$$F\left(N + N', \hat{b} \oplus \hat{b}', \hat{\mu}\right) = F\left(N, \hat{b}, \hat{\mu}\right) + F\left(N', \hat{b}', \hat{\mu}\right)$$

where $\hat{\mu} = (\mu_j)_{j=1\ldots m}$, $\hat{b} = (b_j)_{j=1\ldots m}$, symbol $\oplus$ denote the componentwise addition. Indeed, the first equality has been proven in section 15, while the second equality is by the property (P3) of the spectral flow.

Hence F is a bilinear function, and therefore there is a homomorphism from $\mathbb{Z}^{m+1} \otimes M$ to $\mathbb{Z}$ such that F can be represented as the composition

$$S \times M \hookrightarrow \mathbb{Z}^{m+1} \times M \to \mathbb{Z}^{m+1} \otimes M \to \mathbb{Z}, \tag{15}$$

where the first arrow is induced by the natural embedding of S into $\mathbb{Z}^{m+1}$, and the second arrow is the canonical map of the direct product to the tensor product.

Let us consider operator (2) with boundary condition (3). If $(\mathbb{D} + Q_t)\, u = 0$ and $i\,(n_1 + in_2)\, u^+ = Bu^-$ on ∂X, then

$$\int_{\partial X} \left\langle B(x)u^-, u^- \right\rangle ds = \int_{\partial X} \left\langle i\,(n_1 + in_2)\, u^+, u^- \right\rangle ds =$$
$$= \int_X \left\langle (-i\,(\partial_1 + i\partial_2) + q_t)\, u^+, u^- \right\rangle dx^1 dx^2 - \int_X \left\langle u^+, (-i\,(\partial_1 - i\partial_2) + \overline{q}_t)\, u^- \right\rangle dx^1 dx^2 = 0,$$

where ds is the length element on ∂X.

Suppose now that the sign of B is the same on all boundary components. Then from the last equality we have $u^- \equiv 0$ on ∂X, $u^+ = -i\,(n_1 - in_2)\, Bu^- \equiv 0$ on ∂X. Thus $u \equiv 0$ on X by the weak inner unique continuation property of Dirac operator [3]. So $(\mathbb{D} + Q_t, B)$ has no zero eigenvalues at any t for such B, and by Property (P0) $\mathrm{sf}\,(\mathbb{D} + Q_t, B) = 0$. Finally we obtain $F(1, \hat{0}, \hat{\mu}) = F(1, \hat{1}, \hat{\mu}) = 0$ at any $\hat{\mu}$, where we denote $\hat{0} = (0, \ldots, 0)$, $\hat{1} = (1, \ldots, 1) \in \mathbb{Z}^m$.

Let us consider the group M' which is quotient of $\mathbb{Z}^{m+1}$ by subgroup spanned on elements $(1, \hat{0})\,, (1, \hat{1}) \in \mathbb{Z}^{m+1}$. Note that M' coincide with the quotient group $\mathbb{Z}^m / \langle \hat{1} \rangle$, so it is naturally isomorphic to the Abelian group $\mathrm{Hom}(M, \mathbb{Z})$ of all homomorphisms of M to $\mathbb{Z}$.

By previous arguments, there exists such homomorphism $\widetilde{F}\colon M' \otimes M \to \mathbb{Z}$ that F is the composition of the following homomorphisms:

$$S \times M \hookrightarrow \mathbb{Z}^{m+1} \times M \to \mathbb{Z}^{m+1} \otimes M \to M' \otimes M \xrightarrow{\widetilde{F}} \mathbb{Z}, \tag{16}$$

where the first two arrows are the same as in (15), and the third arrow is induced by the natural projection $\mathbb{Z}^{m+1} \to M'$.

17 Invariance under the action of symmetric group

Let $\mathrm{Diff}^+(X)$ be the group of all diffeomorphisms of X preserving orientation, $f \in \mathrm{Diff}^+(X)$. As was shown in Section 13, $F\,(N, f^*B, f^*\mu) = F(N, B, \mu)$, and hence

$$F\left(N, f^*\hat{b}, f^*\hat{\mu}\right) = F(N, \hat{b}, \hat{\mu}),$$

where f^* acts on $\hat{b}$ and $\hat{\mu}$ by the permutation of the coordinates, corresponding to the permutation of the boundary components of X by f. It is well known that any permutation of the boundary components of X is realized by some element of $\mathrm{Diff}^+(X)$. Thus $F(N, \hat{b}, \hat{\mu})$ is invariant under the action of symmetric group S_m (the group of the permutations of m elements) on $(\hat{b}, \hat{\mu})$ by the permutations of the coordinates.

All permutations of the coordinates leave invariant the element $\hat{1}$ of $\mathbb{Z}^m$, so S_m acts on $M' = \mathbb{Z}^m / \langle \hat{1} \rangle$ in exactly the same way, and $\widetilde{F}$ is invariant under the action of S_m, too.

Extending $\widetilde{F}$ by the linearity from $M' \otimes M$ to $V' \otimes V$, $V' = M' \otimes \mathbb{C} = \mathbb{C}^m / \langle \hat{1} \rangle$, $V = M \otimes \mathbb{C} = \{ v \in \mathbb{C}^m \colon \sum v_j = 0 \}$, we obtain homomorphism $\widetilde{F}_{\mathbb{C}} \colon V' \otimes V \to \mathbb{C}$, coinciding with $\widetilde{F}$ on the lattice $M' \otimes M \subset V' \otimes V$. Obviously, $\widetilde{F}_{\mathbb{C}}$ is invariant with respect to the action of S_m on $V' \otimes V$ as well.

V' and $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ coincide as the representations of S_m, so the vector space of all invariant homomorphisms from $V' \otimes V$ to $\mathbb{C}$ is isomorphic to the vector space of all equivariant homomorphisms $V \to V$. But the last space is 1-dimensional by Schur's lemma, because V is the irreducible representation of S_m [12]. So $\widetilde{F}_{\mathbb{C}}(v' \otimes v) = c \sum_j v'_j v_j$ for some constant $c \in \mathbb{C}$, and $F(N, \hat{b}, \hat{\mu}) = c \sum b_j \mu_j$, where c dependent only on m.

On the other hand, F is integer-valued and, in particular, $c = F(1, (0,1), (-1,1)) \in \mathbb{Z}$.

Finally, we obtain $\mathrm{sf}(D + Q_t, B)_{t \in [0,1]} = c_m \sum_{j=1}^m b_j \mu_j$, where c_m is the integer constant depending on m only, and Theorem 3 is proved.

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