AUTOMORPHISM GROUPS OF FIELDS, AND THEIR REPRESENTATIONS

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Abstract. This is an updated version of [41]. We study the automorphism group G of an extension $F|k$ of algebraically closed fields, especially in the case of countable transcendence degree and zero characteristic. In connection with their applications to algebraic geometry (birational geometry, algebraic cycles, motives, differential forms and sheaves in various topologies), we study the smooth linear and semi-linear representations of G.

Compared to [41], the principal new result is Theorem 1.1.5. It refines [35, Theorem 3.15]=[41, Theorem 1.1.10 1)]: the objects of $\mathcal{I}_G$ are characterized by their irreducible subquotients.

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1. Introduction

The study of field automorphism groups is an old subject. Without any attempt of describing its complicated history, let me just mention that many topological groups are field automorphism groups. E.g., automorphism groups of algebraic field extensions form usual Galois groups, and automorphism groups of function fields of algebraic varieties over topological fields contain groups of points of algebraic groups over that fields. Besides, groups of points of p-adic groups for $p < \infty$ (and also of finite adelic) arise also as automorphism groups of automorphic function fields. Some continuous automorphism groups of topological fields, e.g. of the Laurent series, have been also studied.

Let $F|k$ be a field extension of countable (this will be the principal case) or finite transcendence degree n, $0 \le n \le \infty$, and $G = G_{F|k}$ be its automorphism group.

Following the very general idea that a "sufficiently symmetric" (mathematical, physical or another) system is determined by a representation of its symmetry group, one tries to compare various "geometric categories over k", where F is interpreted as a "limit object", with various categories of representations of G.

To ensure that the representation theory of G is rich enough, F should be "big enough". For this reason, F will be usually algebraically closed. So F is "the function field of the universal tower of k-varieties of dimension $\le n$".

Some general notations, conventions and goals. Let $F|k$ be an extension of countable or finite transcendence degree n, $1 \le n \le \infty$. In order to avoid already complicated enough Galois theory, we assume by default that the fields F and k are algebraically closed fields of characteristic zero. The exceptions are §2.1, p.16, and §4.1, p.36. Following [14, 32, 44, 13] (and generalizing the case of algebraic extensions from [20]), we endow its automorphism group $G = G_{F|k}$ with the topology, whose base of open subgroups is given by the stabilizers of finite subsets in F. For a pair of field extensions K, L of k the set of all field embeddings of L into K over k is denoted by $\{L \overset{/k}{\hookrightarrow} K\}$. Let E be a field of characteristic zero. The E-vector space with the base given by a set S is denoted by $E[S]$.

We study the structure of G, its E-linear and F-semi-linear representations with open stabilizers, and their links with birational geometry, motives, differential forms and sheaves. In particular, we look for analogues of known results of representation theory of locally compact (especially, of p-adic) groups in the case of G.

1.1. How to translate geometric questions to the language of representation theory?

Depending on type of geometric questions we shall consider one of the following four categories of representations of G: $\Phi_G \subset \mathcal{S}m_G \supset \mathcal{I}_G \supset \mathcal{A}dm$, roughly corresponding to birational geometry over k and its more restrictive ("less functorial") version, birational motivic questions (such as those concerned with the structure of Chow groups of 0-cycles) and "finite-dimensional" birational motivic questions (such as description of "classical" motivic categories).

$\mathcal{S}m_G$**.** Usually an "algebro-geometric datum" D over F consists of a finite number of polynomial equations involving a finite number of coefficients $a_1, \dots, a_N \in F$, and the group G acts on the set of "similar" data. Then the stabilizer of D in G is open, since contains $G_{F|k(a_1,\dots,a_N)}$.

For a k-variety X, its F-subvarieties are examples of such data.

In particular, the $\mathbb{Q}$-vector space $\mathbb{Q}[X(F)]$ of 0-cycles on $X_F := X \times_k F$ is a G-module. Such representation is huge, but this is just a starting point.

Note that it is smooth, i.e. its stabilizers are open, so all representations we are going to consider will be smooth.

Conversely, any smooth representation of G with cyclic vector is isomorphic to a quotient of the G-module $\mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]$ of "generic" 0-cycles on X_F (equivalently, formal $\mathbb{Q}$-linear combinations of embeddings of the function field $k(X)$ into F over k), i.e., 0-cycles outside of the union of the divisors on X defined over k, for an appropriate irreducible variety X of dimension $\le n$ over k.

This follows from Lemma 3.1.3. If we fix a k-field embedding of the function field $k(X)$ into F then the module of generic 0-cycles on X_F becomes the representation $\mathbb{Q}[G/G_{F|k(X)}]$, coinduced by the trivial representation of $G_{F|k(X)}$. These G-modules are very complicated.

Remarks. 1. As $\mathbb{Q}[X(F)] = \bigoplus_{x\in X}\mathbb{Q}[\{k(x) \stackrel{/k}{\hookrightarrow} F\}]$, the representation $\mathbb{Q}[X(F)]$ reflects rather the class of X in the Grothendieck group $K_0(Var_k)$ of partitions of k-varieties, than X itself.

Moreover, suppose that $Z \subset X$ is such a closed subset that any subvariety of Z is birational to infinitely many subvarieties of $X \smallsetminus Z$, e.g. Z is a finite set of closed points of X. Then $\mathbb{Q}[X(F)] \cong \mathbb{Q}[(X \smallsetminus Z)(F)]$.

2. It is not clear, whether the birational type of X is determined by the representation $\mathbb{Q}[\{k(X) \stackrel{/k}{\hookrightarrow} F\}]$ of G of generic 0-cycles on X_F. There do exist pairs of non-birational varieties X and Y, whose G-modules of generic 0-cycles have the same irreducible subquotients, cf. §3.5. E.g., if a map $X \longrightarrow Y$ is generically finite then the pull-back induces an embedding $\mathbb{Q}[\{k(Y) \stackrel{/k}{\hookrightarrow} F\}] \hookrightarrow \mathbb{Q}[\{k(X) \stackrel{/k}{\hookrightarrow} F\}]$. On the other hand, if $X = Z \times \mathbb{P}^1$, $Y = Z' \times \mathbb{P}^1$ and Z' is a twofold cover of Z then there is also an embedding in the opposite direction $\mathbb{Q}[\{k(X) \stackrel{/k}{\hookrightarrow} F\}] \hookrightarrow \mathbb{Q}[\{k(Y) \stackrel{/k}{\hookrightarrow} F\}]$. What is in common between X and Y in this example, is that their primitive motives (see below) coincide (and vanish). But it seems that even this is not essential.

3. However, at least if $n = \infty$, one can extract dimension of X ($\dim X = \min\{q \ge 0 \mid W^{G_{F|\overline{L}}} \neq 0$, where $\mathrm{tr.deg}(L|k) = q\}$) and "birational motivic" invariants "modulo isogenies", such as $\mathrm{Alb}(X)$, $\Gamma(X, \Omega^{\bullet}_{X|k})$, out of $W = \mathbb{Q}[\{k(X) \stackrel{/k}{\hookrightarrow} F\}]$, cf. Theorem 1.1.6 (1), (3), (4), and Proposition 4.1.11.

4. If $W = \mathbb{Q}[X(F)]$ then $\mathbb{Q}[\{k(X) \stackrel{/k}{\hookrightarrow} F\}]$ is a nonzero quotient of W by the submodule generated by all $W^{G_{F|L}}$ with $\mathrm{tr.deg}(L|k) < q$ for a maximal possible $q (= \dim X)$.

5. Consider the category of smooth k-varieties with the morphisms, given by formal $\mathbb{Q}$-linear combinations of non-degenerate generically finite correspondences, i.e. irreducible subvarieties in the product of the source and the target, generically finite over a connected component of the source and dominant over a connected component of the target: $\mathrm{Hom}(X, Y) = Z^{\dim Y}(k(X) \otimes_k k(Y))_{\mathbb{Q}}$ ($:= \mathbb{Q}[\{$prime ideals of $k(X) \otimes_k k(Y)$ of depth equal to $\dim Y\}]$) for connected X and Y.

In the case $n = \infty$ there is a full embedding of this category into the category of smooth representations of G, given by $X \mapsto Z^{\dim X}(k(X) \otimes_k F)_{\mathbb{Q}} = \mathbb{Q}[\{k(X) \stackrel{/k}{\hookrightarrow} F\}]$.

Denote by $\mathcal{S}m_G(E)$ the category of smooth representations of G over a field E. This is a full abelian subcategory of the category of all representations of G over E.

It follows from the topological simplicity of G (Theorem 2.2.1) that in the case $n = \infty$ any finite-dimensional smooth representation of G is trivial.

$\mathcal{A}dm$. Now consider a more concrete geometric category, the category of motives.

An (effective) pure covariant motive is a pair (X, π), consisting of a smooth projective variety X over k with irreducible components X_j and a projector $\pi = \pi^2 \in \bigoplus_j B^{\dim X_j}(X_j \times_k X_j)$ in the algebra of correspondences on X modulo numerical equivalence. The morphisms are defined by $\mathrm{Hom}((X', \pi'), (X, \pi)) = \bigoplus_{i,j} \pi_j \cdot B^{\dim X_j}(X_j \times_k X'_i) \cdot \pi'_i$. The category of pure covariant motives carries an additive and a tensor structures:

$$(X', \pi') \bigoplus (X, \pi) := (X' \coprod X, \pi' \oplus \pi), \qquad (X', \pi') \otimes (X, \pi) := (X' \times_k X, \pi' \times_k \pi).$$

A *primitive q-motive* is a pair (X, π) as above, where $\dim X = q$ and $\mathrm{Hom}(Y \times \mathbb{P}^1, (X, \pi)) := \pi \cdot B^q(X \times_k Y \times \mathbb{P}^1) = 0$ for any smooth projective variety Y over k of dimension $< q$. E.g., it follows from Lefschetz's theorem on $(1,1)$-classes that the category of pure primitive 1-motives is equivalent to the category of abelian varieties over k with morphisms tensored with $\mathbb{Q}$. It is a result of Jannsen ([15]) that pure motives form an abelian semi-simple category. This implies that any pure motive admits a "primitive" decomposition $\bigoplus_{i,j} M_{ij} \otimes \mathbb{L}^{\otimes i}$, where M_{ij} is a primitive j-motive and $\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times \{0\})$ is the Lefschetz motive (cf. Remark on p.43).

Remark. Replacing the numerical equivalence by an arbitrary adequate equivalence relation we get a pseudoabelian tensor category of covariant Grothendieck motives.

DEFINITION. A representation W of a totally disconnected topological group is called *admissible* if it is smooth and the fixed subspaces W^U are finite-dimensional for all open subgroups U.

Denote by $\mathcal{A}dm = \mathcal{A}dm_G(\mathbb{Q})$ the category of admissible representations of G over $\mathbb{Q}$.

Note that there are infinite direct sums among admissible representations. For instance, let $\{A_\alpha\}$ be a collection of pairwise non-isogeneous simple abelian varieties over k. Then the representation $\bigoplus_\alpha (A_\alpha(F)/A_\alpha(k))$ of G is admissible.

Theorem 1.1.1 ([35]). *$\mathcal{A}dm$ is a Serre subcategory in $\mathcal{S}m_G := \mathcal{S}m_G(\mathbb{Q})$.*

In other words, $\mathcal{A}dm$ is abelian, stable under taking sub*quotients* (this is the point in the case $n = \infty$!) in the category of representations of G, and under taking extensions in $\mathcal{S}m_G$. This is shown in [35, Corollary 6.5] using an embedding of $\mathcal{A}dm$ into a bigger category $\mathcal{I}_G$.

Theorem 1.1.2 ([35]). *If $n = \infty$ then there exists a fully faithful functor $\mathbb{B}^\bullet$:*

$$\mathcal{M}_k := \left\{ \begin{array}{c} \text{pure covariant} \\ k\text{-motives} \end{array} \right\} \xrightarrow{\mathbb{B}^\bullet} \left\{ \begin{array}{c} \text{graded semi-simple admissible} \\ \text{representations of } G \text{ over } \mathbb{Q} \text{ of finite length} \end{array} \right\}.$$

The grading corresponds to the powers of the motive $\mathbb{L}$ in the above "primitive" decomposition.

Roughly speaking, the functor $\mathbb{B}^\bullet = \bigoplus_j^{\text{graded}} \mathbb{B}^{[j]}$ is defined as the space of 0-cycles over F modulo "numerical equivalence over k". More precisely, $\mathbb{B}^\bullet = \bigoplus_j^{\text{graded}} \varinjlim_L \operatorname{Hom}\left([L]^{\text{prim}} \otimes \mathbb{L}^{\otimes j}, -\right)$ is a graded direct sum of pro-representable functors. Here L runs over the set of all subfields in F of finite type over k, and $[L]^{\text{prim}}$ is the quotient of the motive of any smooth projective model of L over k by the sum of all submotives of type $M \otimes \mathbb{L}$ for all effective motives M.

Thus, the category $\mathcal{M}_k$ becomes a full subcategory of the category of graded semi-simple admissible representations of G over $\mathbb{Q}$ of finite length.

EXAMPLES. The motive of the point $\mathbf{Spec}(k)$ is mapped to the trivial representation $\mathbb{Q}$ in degree 0. The motive of a smooth proper curve C over k is sent to $\mathbb{Q} \oplus J_C(F)/J_C(F) \oplus \mathbb{Q}[1]$, where J_C is the Jacobian of C and $\mathbb{Q}[1]$ denotes the trivial representation in degree 1.

Conjecture 1.1.3. *The functor $\mathbb{B}^\bullet$ is an equivalence of categories.*

Of course, it would be more interesting to describe in a similar way the abelian category $\mathcal{MM}$ of mixed motives over k, whose semi-simple objects are pure. This is one more reason to study the category $\mathcal{A}dm$ of admissible representations of G.

Proposition 1.1.4 ([35]). *Assuming $n = \infty$, for any $W \in \mathcal{A}dm$, any abelian variety A over k and, conjecturally, for any effective motive M one has*

$$\begin{array}{ll} \operatorname{Ext}^{>0}_{\mathcal{A}dm}(\mathbb{Q}, W) = 0 & \operatorname{Ext}^{>0}_{\mathcal{MM}}(\mathbb{Q}, M) = 0 \\ \operatorname{Ext}^1_{\mathcal{A}dm}(\frac{A(F)}{A(k)}, W) = \frac{\operatorname{Hom}_{\mathbb{Z}}(A(k), W^G)}{\operatorname{Hom}_G(A(F)/A(k), W/W^G)} & \operatorname{Ext}^1_{\mathcal{MM}}(H^1(A), M) = \frac{A(k) \otimes W_0 M}{\operatorname{Hom}_{\mathcal{MM}}(H^1(A), M/W_0 M)} \\ \operatorname{Ext}^{\geq 2}_{\mathcal{A}dm}(A(F)/A(k), W) = 0 & \operatorname{Ext}^{\geq 2}_{\mathcal{MM}}(H^1(A), M) = 0 \end{array}$$

As $A(F)/A(k)$ is a canonical direct "H_1"-summand of $\mathbb{B}^\bullet(A)$, we see that the admissible representations of finite length should be related to effective motives. At least the Ext's between some irreducible objects are dual.

$\mathcal{I}_G$**.** The formal properties of $\mathcal{A}dm$ are not very nice. E.g., to prove Theorem 1.1.1 and Proposition 1.1.4 and also to give an evidence to Conjecture 1.1.3, one uses an inclusion of $\mathcal{A}dm$ into a bigger full subcategory in the category of smooth representations of G.

DEFINITION. An object $W \in \mathcal{S}m_G(E)$ is called "homotopy invariant" (in birational sense; the etymology comes from §3.4) if $W^{G_{F|L}} = W^{G_{F|L'}}$ for any purely transcendental subextension $L'|L$ in

$F|k$. The full subcategory in $\mathcal{S}m_G(E)$ of "homotopy invariant" objects is denoted by $\mathcal{I}_G(E)$. (The result will be the same, if we restrict ourselves to only those L', which are of finite type over k, cf. [35, Corollary 6.2].)

A typical object of $\mathcal{I}_G$ is the $\mathbb{Q}$-vector space $CH^q(X_F)_{\mathbb{Q}}$ of cycles of codimension $q \geq 0$ on the scheme $X \times_k F$ modulo rational equivalence, for any smooth variety X over k. (This follows from the descent property: $CH^*(X_F)_{\mathbb{Q}}^{G_{F|L}} = CH^*(X_L)_{\mathbb{Q}}$.)

On the other hand, if $\widetilde{F}$ is an algebraic closure of $F(t)$ in an algebraic extension of $F((t))$ then usually $CH^*(X_{\widetilde{F}})_{\mathbb{Q}} \notin \mathcal{I}_G$, since $\widetilde{F}^{G_{F|L}}(x) \subsetneq \widetilde{F}^{G_{F|L(x)}}$ if $x \notin \overline{L}$. For instance, if X is an elliptic curve $u^2 = P(v)$ then $k(x)((t)) \ni P(xt)^{1/2} \notin k((t))(x)$ and $[u \mapsto P(xt)^{1/2}, v \mapsto xt] \in CH_0(X_{k((xt))}) \subseteq CH_0(X_{k(x)((t))})$, but $\notin CH_0(X_{\overline{k(t)}})_{\mathbb{Q}}$.

Let us first prove the following characterization of the "homotopy invariant" representations.

Theorem 1.1.5 ($n = \infty$). *A smooth representation of G is "homotopy invariant" if and only if all its irreducible subquotients are. In particular, the category $\mathcal{I}_G(E)$ is a Serre subcategory in $\mathcal{S}m_G(E)$.*

Proof. Suppose that $W \notin \mathcal{I}_G$, but all its irreducible subquotients are "homotopy invariant". According to [35, Corollary 6.2], there exist a subfield L in $F|k$, an element $x \in F \smallsetminus \overline{L}$ and a vector $v \in W^{G_{F|L(x)}} \smallsetminus W^{G_{F|L}}$, i.e., there exists $\sigma \in G_{F|L}$ such that $\sigma v - v =: u \neq 0$.

Clearly, $G_{F|L} \not\subset \mathrm{Stab}_v \cup \mathrm{Stab}_{\overline{L(x)}}$. (Indeed, if a group H is a union of a pair of its subgroups, $H = H_1 \cup H_2$, and $H_1 \neq H$ then $hH_1 \subset H_2$ for any $h \in H \smallsetminus H_1$, so $H_1 \subseteq H_2$, and therefore, $H_2 = H$.) In other words, we may assume that $\sigma \in G_{F|L} \smallsetminus \mathrm{Stab}_{\overline{L(x)}}$.

We may replace W by the quotient by a maximal subrepresentation not containing u. Then the subrepresentation $\langle u \rangle$ generated by u becomes irreducible, and thus, an object of $\mathcal{I}_G$.

By its definition, $u \in W^{G_{F|L(x)}} + W^{G_{F|L(\sigma x)}} \subseteq W^{G_{F|L(x,\sigma x)}}$. As $\langle u \rangle \in \mathcal{I}_G$ and $x, \sigma x$ are algebraically independent over L, we conclude that $u \in W^{G_{F|L}}$. This implies that $\sigma v \in W^{G_{F|L(x)}}$. On the other hand, $\sigma v \in W^{G_{F|L(\sigma x)}}$, so $\sigma v \in W^{G_{F|L(x)}} \cap W^{G_{F|L(\sigma x)}}$. By [35, Lemma 2.16], the latter space is $W^{G_{F|L}}$, and finally, $v \in W^{G_{F|L}}$, contradicting our assumptions.

The converse is known from Lemma 4.1.1. □

Theorem 1.1.6 ([35], $n = \infty$).
1. *The inclusion functor $\mathcal{I}_G(E) \hookrightarrow \mathcal{S}m_G(E)$ admits a left and a right adjoints $\mathcal{I}, -^{(0)} : \mathcal{S}m_G(E) \longrightarrow \mathcal{I}_G(E)$, the universal quotient and subobject in $\mathcal{I}_G(E)$.*
2. *$\mathcal{A}dm(E) \subset \mathcal{I}_G(E)$, i.e. any admissible representation of G is "homotopy invariant".*
3. *The objects $C_{k(X)} := \mathcal{I}\mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]$ for all birational classes X of irreducible varieties over k form a system of projective generators of $\mathcal{I}_G$.*
4. *For any smooth proper k-variety X there is a canonical filtration $C_{k(X)} \supset \mathcal{F}^1 \supset \mathcal{F}^2 \supset \dots$, canonical isomorphisms $C_{k(X)}/\mathcal{F}^1 = \mathbb{Q}$ and $\mathcal{F}^1/\mathcal{F}^2 = \mathrm{Alb}(X_F)_{\mathbb{Q}}$, and a non-canonical splitting $C_{k(X)} \cong \mathbb{Q} \oplus \mathrm{Alb}(X_F)_{\mathbb{Q}} \oplus \mathcal{F}^2$. The term $\mathcal{F}^2$ is determined by these conditions together with $\mathrm{Hom}_G(\mathcal{F}^2, \mathbb{Q}) = \mathrm{Hom}_G(\mathcal{F}^2, A(F)/A(k)) = 0$ for any abelian variety A over k. Here* Alb *is the Albanese variety.* (Corollary 6.24)
5. *For any smooth proper k-variety X there is a canonical surjection $C_{k(X)} \longrightarrow CH_0(X_F)_{\mathbb{Q}}$, which is injective if X unirational over a curve (and in some other cases when $CH_0(X)$ is "known", e.g., if X is a quotient of even-dimensional Fermat hypersurface of degree* $\dim X + 2$ *or* $\dim X + 3$, *such that $CH_0(X)$ is cyclic).*
6. *There exist (co-) limits in $\mathcal{I}_G(E)$.*

Two filtrations. For a representation M of G define N_jM as the subspace, spanned by the invariants $M^{G_{F|F_j}}$ for all subfields $F_j \subseteq F$ of transcendence degree j over k. From the point of view of §3.4, on the smooth G-modules, N_j is "part coming from dimension $\leq j$". Clearly, the "level" filtration $N_\bullet$ is increasing and functorial.

Then the term $\mathcal{F}^jW$ of a functorial descending filtration $\mathcal{F}^\bullet$ on an object W of $\mathcal{I}_G$ is defined as the intersection of the kernels of all G-homomorphisms φ from W to the objects $W' \in \mathcal{I}_G$ of level j, i.e., such that $W' = N_jW'$. If $W = N_qW$ then $\mathcal{F}^{q+1}W = 0$, since $\ker(W \xrightarrow{id} W) = 0$.

It seems that in the case of G-modules of type $CH_0(X_F)_{\mathbb{Q}}$ for a smooth proper k-variety X this is the motivic filtration (cf. [5] and [7]), which agrees with Theorem 1.1.6 (4).

The following two conjectures link the Chow motives and the Kähler differentials (in fact, the holomorphic part of the de Rham cohomology, cf. Proposition 4.1.11 below) via the category $\mathcal{I}_G$.

Conjecture 1.1.7 ([35]). *If $n = \infty$ then the natural surjection $C_{k(X)} \longrightarrow CH_0(X \times_k F)_{\mathbb{Q}}$ is an isomorphism for any smooth proper irreducible variety X over k.*

The filtration $\mathcal{F}^\bullet$ on $C_{k(X)}$ coincides with the motivic filtration on the Chow groups of 0-cycles.

Remarks. 1. One deduces from Theorem 1.1.6 (5) a description of the category of abelian varieties over k with the groups of morphisms tensored with $\mathbb{Q}$ as a full subcategory of $\mathcal{A}dm_G \subset \mathcal{I}_G$ in terms of the "level" filtration $N_\bullet$ on smooth G-modules.

2. The conjecture of Bloch and Beilinson ([5] and [7]) on the "motivic" filtration on the Chow groups together with the semi-simplicity "standard" conjecture of Grothendieck (asserting that numerical and homological equivalences coincide on smooth proper varieties) imply that "numerical" and rational equivalences coincide on the cycles on the spectrum of the tensor product of a pair of fields over a common subfield. More precisely, for any smooth proper k-varieties X and Y the surjective localization homomorphism $CH^*(X \times_k Y) \longrightarrow CH^*(k(X) \otimes_k k(Y))_{\mathbb{Q}}$ "kills" the numerically trivial cycles (cf. [4] §1.4, or [38] Prop.1.1.1), or equivalently, that $CH^j(k(X) \otimes_k F)_{\mathbb{Q}}$ coincides with $\mathfrak{B}^j(M) := \mathbb{B}^{[0]}(M)$, where M is the maximal primitive j-submotive of (X, Δ_X).

If combined with the first part of Conjecture 1.1.7, this would give (when $n = \infty$) that
(a) $\mathbb{B}^\bullet$ is an equivalence of categories (Conjecture 1.1.3), cf. also "Corollary" 1.1.8.1 below;
(b) any irreducible object of $\mathcal{I}_G$ is admissible; and
(c) the G-modules $gr^N_j W$ are semi-simple for any $W \in \mathcal{I}_G$ (Conjecture 4.1.5), where $N_\bullet$ is defined above.

Indeed, for some collection of smooth projective j-dimensional k-varieties Y there is a surjective morphism $\bigoplus_Y \mathbb{Q}[\{k(Y) \overset{/k}{\hookrightarrow} F\}] \xrightarrow{\xi} gr^N_j W$, which factors through $\bigoplus_Y gr^N_j C_{k(Y)}$, cf. Proposition 4.1.3, p.36. If $C_{k(Y)} = CH_0(Y \times_k F)_{\mathbb{Q}}$ then $gr^N_j C_{k(Y)} = CH^j(k(Y) \otimes_k F)_{\mathbb{Q}}$, so ξ factors through $\bigoplus_Y CH^j(k(Y) \otimes_k F)_{\mathbb{Q}}$.

Finally, it follows from the semi-simplicity that there exist projectors π_Y and an isomorphism $\bigoplus_Y \mathfrak{B}^j((Y, \pi_Y)) \xrightarrow{\sim} gr^N_j W$. This proves (c), and taking an irreducible W (which coincides with $gr^N_j W$ for some j), we get also (a) and (b).

3. For any pair $W_1, W_2 \in \mathcal{S}m_G := \mathcal{S}m_G(\mathbb{Q})$ set $W_1 \otimes_{\mathcal{I}} W_2 := \mathcal{I}(W_1 \otimes W_2)$. As the example of $W_1 = W_2 = \mathbb{Q}[F \smallsetminus k]$ and $W_3 = \mathbb{Q}$ shows, this binary operation is *not* associative on $\mathcal{S}m_G$.

It follows from the first part of Conjecture 1.1.7 that there is a canonical isomorphism, the "Künneth formula": $C_{k(X \times_k Y)} \xrightarrow{\sim} C_{k(X)} \otimes_{\mathcal{I}} C_{k(Y)}$ for any pair of irreducible k-varieties X, Y. An evidence for this "corollary" (and an unconditional proof of the "Künneth formula" in the case when X is a curve) is given in §4.2, p.38.

It would follow from the "Künneth formula" that the restriction of $\otimes_{\mathcal{I}}$ to $\mathcal{I}_G$ is a commutative associative tensor structure on $\mathcal{I}_G$ (compatible with the inner $\mathcal{H}om$, cf. Remark on p.24 and Proposition 4.1.10), and that the class of projective objects is stable under $\otimes_{\mathcal{I}}$.

It would be interesting to find a "semi-simple graded" version of $\otimes_{\mathcal{I}}$ to make $\mathbb{B}^\bullet$ a tensor functor.

Conjecture 1.1.8 ([36]). *Any irreducible object of $\mathcal{I}_G$ is contained in the algebra $\Omega^\bullet_{F|k}$ if $n = \infty$.*

"Corollary" 1.1.8.1 ([36], $n = \infty$).
- *Any irreducible object of $\mathcal{I}_G$ is admissible. So "$\mathcal{I}_G \approx \mathcal{A}dm$".*
- *If numerical equivalence coincides with homological then $\mathbb{B}^\bullet$ is an equivalence of categories.*

Proof. Let W be an irreducible object of $\mathcal{I}_G$. There exists a smooth projective k-variety Y and a surjection $C_{k(Y)} \twoheadrightarrow W$. Assuming Conjecture, the representation W embeds into $\Omega^q_{F|k}$ for an integer $q \geq 0$. It follows from Proposition 4.1.11 that $\mathrm{Hom}_G(C_{k(Y)}, \Omega^q_{F|k}) = \Gamma(Y, \Omega^q_{Y|k})$, and thus, any homomorphism $C_{k(Y)} \longrightarrow \Omega^q_{F|k}$ factors through $A^{\dim Y}(Y_F)$, where A^* is the space of cycles "modulo homological (de Rham) equivalence over k". More precisely, $A^{\dim Y}(Y_F)$ is the image of $CH_0(Y_F)_{\mathbb{Q}}$ in $H^{2*}_{\mathrm{dR}/k}(Y_F)$. As the singular cohomologies of the smooth complex k-varieties are finite-dimensional, the representation $A^{\dim Y}(Y_F)$ is admissible, which implies the admissibility of its quotient W.

To establish that $\mathbb{B}^\bullet$ is an equivalence of categories, it suffices to show that any irreducible admissible representation W of G is the degree-zero component of $\mathbb{B}^\bullet(M)$ for some motive M. As W is a quotient of $A^{\dim Y}(Y_F)$, this follows from the fact that $A^{\dim Y}(Y_F)$ coincides with the degree-zero component of $\mathbb{B}^\bullet(Y)$, if numerical and homological equivalences coincide. $\square$

Conjecture 1.1.8 is one of the main motivations for the study of semi-linear representations of G, cf. §1.2. It has also the following geometric corollary, conjectured by S.Bloch.

"Corollary" 1.1.8.2 ([36]). *If $\Gamma(X, \Omega^{\geq 2}_{X|k}) = 0$ for a smooth proper variety X over k then the Albanese map induces an isomorphism $CH_0(X)^0 \xrightarrow{\sim} \mathrm{Alb}(X)$. In that case $C_{k(X)} = CH_0(X_F)_{\mathbb{Q}}$.* (The converse is well-known, cf. [30, 34].)

Proof. According to Theorem 1.1.6 (4), $\mathcal{F}^2 C_{k(X)}$ is a direct summand of the cyclic G-module $C_{k(X)}$: $C_{k(X)} \cong \mathbb{Q} \oplus \mathrm{Alb}X(F)_{\mathbb{Q}} \oplus \mathcal{F}^2 C_{k(X)}$, where Alb is the Albanese variety. Thus, if the G-module $\mathcal{F}^2 C_{k(X)}$ is non-zero then it is cyclic, and therefore, admits a non-zero irreducible quotient $W \in \mathcal{I}_G$. It follows from Conjecture 1.1.8 that there is an integer $q \geq 0$ and an embedding $W \hookrightarrow \Omega^q_{F|k}$.

However, it follows from Proposition 4.1.11 that $\mathrm{Hom}_G(C_{k(X)}, \Omega^\bullet_{F|k}) = \Gamma(X, \Omega^\bullet_{X|k})$, and therefore, $\mathrm{Hom}_G(C_{k(X)}, \Omega^q_{F|k}) = \mathrm{Hom}_G(\mathbb{Q} \oplus \mathrm{Alb}X(F)_{\mathbb{Q}}, \Omega^q_{F|k})$, if $q \leq 1$, so $q \geq 2$. This means that $\mathrm{Hom}_G(C_{k(X)}, \Omega^q_{F|k}) = \Gamma(X, \Omega^q_{X|k})$ is non-zero for some integer $q \geq 2$. $\square$

Φ_G and cohomology of smooth representations. As it is explained in §1.1, (at least some) irreducible admissible representations correspond to irreducible pure motives and the Ext-groups between certain irreducible admissible representations are dual to the expected values of the Ext-groups between the corresponding pure motives. Then there arise such problems as

- to find other groups $\mathrm{Ext}^*_{\mathcal{A}dm}$ and $\mathrm{Ext}^*_{\mathcal{S}m_G}$ and compare them with conjectural values of corresponding $\mathrm{Ext}^*_{\mathcal{MM}}$;
- to enlarge $\mathcal{A}dm$ (or $\mathcal{I}_G$) and relate this bigger category Ψ to the category of *effective* mixed motives, so that in particular, Ψ would contain such (non-admissible) objects as $F^\times/k^\times$ (playing the rôle of the Tate motive, since $\mathrm{Ext}^1_{\mathcal{S}m_G}(F^\times/k^\times, \mathbb{Q}) = \mathrm{Hom}(k^\times, \mathbb{Q})$, cf. Proposition 1.1.9) and $\mathbb{Q}$ was still its projective object.

If it is possible to describe the abelian category $\mathcal{MM}_k$ in a way similar to Conjecture 1.1.3 (or at least to Theorem 1.1.2), one should probably consider the category of smooth G-modules of finite length with no subquotients of certain type (e.g., isomorphic to F/k).

However, we shall see (after Proposition 1.1.11, p.9) that $\mathrm{Ext}^1_{\mathcal{S}m_G}(A(F)/A(k), F^\times/k^\times) \neq 0$ for any abelian k-variety A. Thus, the weight considerations show that in any case the relation between $\mathcal{MM}_k$ and $\mathcal{S}m_G$ cannot be very straightforward.

As an "upper bound" for Ψ, one can take the following full additive subcategory Φ_G of $\mathcal{S}m_G$. Let $\mathcal{G}l$ be the category of smooth k-varieties, whose morphisms are compositions of smooth ones and closed embeddings of type $X \hookrightarrow X \times Y$ defined by k-points of Y. The objects of Φ_G are limits $\mathcal{F}(F) := \lim\limits_{A \longrightarrow} \mathcal{F}(\mathbf{Spec}(A))$, where $\mathcal{F}$ is a functor on $\mathcal{G}l$, and A runs over the finitely generated smooth k-subalgebras in F. Examples of objects of Φ_G are $\bigotimes^\bullet_F \Omega^1_{F|k}$, or $A(F)_{\mathbb{Q}}$ for any commutative k-group A, and $\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]$ for any $L|k$ of finite type, but not $\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]^\circ := \ker[\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}] \xrightarrow{\deg} \mathbb{Q}]$.

Proposition 1.1.9 ([35], 5.1, 5.2). *Let $n = \infty$ and A be an irreducible commutative algebraic group over k. Then one has $\mathrm{Ext}^1_{\mathcal{S}m_G}(A(F)_{\mathbb{Q}}, \mathbb{Q}) = 0$ and therefore, $\mathrm{Ext}^1_{\mathcal{S}m_G}(A(F)/A(k), \mathbb{Q}) = \mathrm{Hom}(A(k), \mathbb{Q})$.*

One could guess that $\mathrm{Ext}^*_{\mathcal{S}m_G}(W_1, W_2) = \mathrm{Ext}^*_{\mathcal{I}_G}(W_1, W_2)$ for $W_1, W_2 \in \mathcal{I}_G$, if $n = \infty$. This follows from Theorem 1.1.5, when $* \le 1$. If $W_1, W_2 \in \mathcal{I}^q_G$ and W_1 is projective in $\mathcal{I}^q_G$ then $\mathrm{Ext}^1_{\mathcal{S}m_G}(W_1, W_2) = \mathrm{Ext}^1_{\mathcal{I}_G}(W_1, W_2) = \mathrm{Ext}^1_{\mathcal{I}^q_G}(W_1, W_2) = 0$ by Lemma 4.1.1.

It is shown in [18] that the objects of the subcategories Φ_G and $\mathcal{I}_G$ of $\mathcal{S}m_G$ are acyclic. In the proof one interprets the smooth representations as sheaves in the dominant topology on $\mathrm{Spec}(k)$, and interprets their cohomology as Čech cohomology. Details are in §3.4. There are some applications of the acyclicity conditions to the irreducibility criteria of representations of G.

To look at a smooth representation of G "more geometrically", one would like to associate a "more geometric" sheaf to it, e.g. a sheaf in the smooth topology on $\mathrm{Spec}(k)$. This could be done, assuming some good properties of the functors $(-)_v$ from Proposition 1.2.2. (But, of course, the resulting sheaf can be zero.) This type of questions is discussed in §4.4.

Differential forms. To compare various cohomology theories H^*, one can associate with them some G-modules, such as $H^*(F) := \varinjlim H^*(U)$, where U runs over spectra of smooth subalgebras in F finitely generated over k, or the image $H^*_c(F)$ in $H^*(F)$ of $\varinjlim H^*(X)$, where X runs over smooth proper models of subfields in F of finite type over k.

Clearly, $H^*_c(F)$ is an admissible representation of G over $H^*(k)$. It would follow from the semi-simplicity standard conjecture that it is semi-simple. If $H^*_c(F)$ is semi-simple, one can omit the reference to the semi-simplicity standard conjecture in Remark 2 on p.6.

In the case $H^* = H^*_{\mathrm{dR}/k}$ of the de Rham cohomology the graded quotients of the (descending) Hodge filtration on $H^q_{\mathrm{dR}/k,c}(F)$ are $H^{p,q-p}_{F|k} = \varinjlim \mathrm{coker}[H^{p-1}(D, \Omega^{q-p-1}_{D|k}) \longrightarrow H^p(X, \Omega^{q-p}_{X|k})]$, where (X, D) runs over the set of pairs consisting of a smooth proper variety X with $k(X) \subset F$ and a normal crossing divisor D on X with smooth irreducible components. More particularly, $H^{q,0}_{F|k} = \Omega^q_{F|k,\mathrm{reg}} \subset H^q_{\mathrm{dR}/k,c}(F)$ is the G-submodule spanned by the spaces $\Gamma(X, \Omega^\bullet_{X|k})$ of regular differential forms on all smooth projective k-varieties X with the function fields embedded into F.

Another motivation for the study of differential forms $\Omega^\bullet_{F|k}$ is the calculation of integrals. To calculate an integral of a meromorphic differential form ω on an algebraic complex variety, one can transfer ω to other variety via a correspondence. In coordinates this looks as an algebraic change of variables. Assuming that all function fields are contained in a common field F, the problem of description of the properties of the (iterated) integrals of ω (of $\omega_1, \dots \omega_N$) becomes related to determining the structure of the G-submodule in the algebra of Kähler differentials $\Omega^\bullet_{F|k}$ (resp., in $\Omega^\bullet_{F|k} \otimes_k \cdots \otimes_k \Omega^\bullet_{F|k}$) generated by ω (resp., by $\omega_1 \otimes \cdots \otimes \omega_N$).

Let $H^q_{\mathrm{dR}/k,c}(F)$ be the image in $H^q_{\mathrm{dR}/k}(F)$ of $\varinjlim H^q_{\mathrm{dR}/k}(X)$, where X runs over smooth proper models of subfields in F of finite type over k. Clearly, this is an admissible representation over k. The Hodge filtration on $\Omega^\bullet_{X|k}$ induces a descending filtration on $H^q_{\mathrm{dR}/k,c}(F)$ with the graded quotients $H^{p,q-p}_{F|k} = \varinjlim \mathrm{coker}[H^{p-1}(D, \Omega^{q-p-1}_{D|k}) \longrightarrow H^p(X, \Omega^{q-p}_{X|k})]$, where (X, D) runs over the same pairs as above. More particularly, $H^{q,0}_{F|k} = \Omega^q_{F|k,\mathrm{reg}} \subset H^q_{\mathrm{dR}/k,c}(F)$.

Proposition 1.1.10 ([18]). *Suppose that the cardinality of k is at most continuum. Fix an embedding $\iota : k \hookrightarrow \mathbb{C}$ to the field of complex numbers. Then*

- *there exist a non-canonical $\mathbb{Q}$-linear isomorphism $H^{p,q}_{F|k} \cong H^{q,p}_{F|k}$, and a canonical $\mathbb{C}$-anti-linear isomorphism (depending on ι) $H^{p,q}_{F|k} \otimes_{k,\iota} \mathbb{C} \cong H^{q,p}_{F|k} \otimes_{k,\iota} \mathbb{C}$;*
- *the representation $H^n_{\mathrm{dR}/k,c}(F)$ (and thus, $\Omega^n_{F|k,\mathrm{reg}}$) is semi-simple for any $1 \le n < \infty$.*

It follows from Proposition 4.1.11 that the "homotopy invariant" part of $\bigotimes^\bullet_F \Omega^1_{F|k}$, i.e., its maximal subobject in $\mathcal{I}_G$, coincides with $\Omega^\bullet_{F|k,\mathrm{reg}}$, if $n = \infty$. This confirms once more the idea that the objects of $\mathcal{I}_G$ are of cohomological nature, since $\Omega^\bullet_{F|k,\mathrm{reg}} \subset H^q_{\mathrm{dR}/k,c}(F)$.

The above examples of G-modules come from certain (pro-)varieties over k by extending the base field to F. More generally, to each birationally invariant functor $\mathcal{F}$ on a category of k-varieties, or on a category of field extensions of k (as in Corollary 4.3.7), one can associate a G-module.

One gets two more examples of G-modules of this type from the birationally invariant functor $\mathrm{Div}_{\mathrm{alg}}$ of algebraically trivial divisors on the category of smooth proper k-varieties, and from the Picard functor: $\mathrm{Div}^{\circ}_{\mathbb{Q}} = \varinjlim_U \mathrm{Div}_{\mathrm{alg}}(Y_U)_{\mathbb{Q}}$, and $\mathrm{Pic}^{\circ}_{\mathbb{Q}} = \varinjlim_U \mathrm{Pic}^{\circ}(Y_U)_{\mathbb{Q}} = \mathrm{coker}[F^{\times}/k^{\times} \xrightarrow{\mathrm{div}} \mathrm{Div}^{\circ}_{\mathbb{Q}}]$, where U runs over the set of open subgroups of type $G_{F|L}$ and Y_U is a smooth projective model of $F^U = L$ over k. Clearly, $H^1_{\mathrm{dR}/k,c}(F) = \ker[H^1_{\mathrm{dR}/k}(F) \xrightarrow{\mathrm{Res}} k \otimes \mathrm{Div}^{\circ}_{\mathbb{Q}}]$.

Proposition 1.1.11 ([35], 3.11)**.** *Let $1 \le n \le \infty$ and $A^{\vee} := \mathrm{Pic}^{\circ} A$ be the dual abelian variety of an abelian variety A. Then $\mathrm{Pic}^{\circ}_{\mathbb{Q}} = \bigoplus_A A^{\vee}(k) \otimes_{\mathrm{End} A} (A(F)/A(k))$, where A runs over the set of isogeny classes of simple abelian varieties over k.*

Let us show that $\mathrm{Ext}^1_{\mathcal{S}m_G}(A(F)/A(k), F^{\times}/k^{\times}) \neq 0$ for any abelian variety A over k. The G-module $\mathrm{Div}^{\circ}_{\mathbb{Q}}$ fits into the exact sequence $0 \longrightarrow F^{\times}/k^{\times} \longrightarrow \mathrm{Div}^{\circ}_{\mathbb{Q}} \longrightarrow \mathrm{Pic}^{\circ}_{\mathbb{Q}} \longrightarrow 0$. According to Proposition 1.1.11, any non-zero element of $A^{\vee}(k)_{\mathbb{Q}}$ provides an embedding $A(F)/A(k)$ into $\mathrm{Pic}^{\circ}_{\mathbb{Q}}$, thus inducing an extension of $A(F)/A(k)$ by $F^{\times}/k^{\times}$ inside $\mathrm{Div}^{\circ}_{\mathbb{Q}}$. To see that this extension does not split, note that any generic F-point x of A, considered as an element of $A(F)_{\mathbb{Q}}$, identifies $\mathrm{Hom}_G(A(F)_{\mathbb{Q}}, \mathrm{Div}^{\circ}_{\mathbb{Q}})$ with a subspace in $(\mathrm{Div}^{\circ}_{\mathbb{Q}})^{\mathrm{Stab}_x}$, whose elements are the $\mathbb{Q}$-divisors on A, invariant under translations by torsion elements in $A(k)$. As the torsion subgroup in $A(k)$ is Zariski dense, any such divisor is zero, i.e., this subspace is zero.

It is not hard to deduce from Proposition 1.1.11 (modified in an evident way) the following description of the representation $\Omega^1_{F|k,\mathrm{closed}}$ for any $1 \le n \le \infty$.

Proposition 1.1.12. *Let $1 \le n \le \infty$.*

- *The maximal semi-simple subrepresentation of G in $\Omega^1_{F|k,\mathrm{closed}}$ is canonically isomorphic to $\bigoplus_A \Gamma(A, \Omega^1_{A|k})^{A(k)} \otimes_{\mathrm{End}(A)} (A(F)/A(k)) = (F/k) \oplus k \otimes (F^{\times}/k^{\times}) \oplus \Omega^1_{F|k,\mathrm{reg}}$, where A runs over the set of isogeny classes of simple commutative algebraic k-groups.*
- *The maximal semi-simple subrepresentation of G in $H^1_{\mathrm{dR}/k}(F)$ is canonically isomorphic to $\bigoplus_A H^1_{\mathrm{dR}/k}(A) \otimes_{\mathrm{End}(A)} (A(F)/A(k)) = k \otimes (F^{\times}/k^{\times}) \oplus H^1_{\mathrm{dR}/k,c}(F)$, where A runs over the set of isogeny classes of simple commutative algebraic k-groups (with the zero summand corresponding to $\mathbb{G}_a$).*
- *The representation $H^1_{\mathrm{dR}/k}(F)/(k \otimes (F^{\times}/k^{\times}))$ of G is canonically isomorphic to the direct sum $\bigoplus_A [H^1_{\mathrm{dR}/k}(k(A))/(k \otimes (k(A)^{\times}/k^{\times}))] \otimes_{\mathrm{End}(A)} (A(F)/A(k))$, where A runs over the set of isogeny classes of simple abelian k-varieties.*

This suggests that (i) the isomorphism classes of irreducible subquotients of $H^*_c(F)$ are the same as that of $\Omega^{\bullet}_{F|k,\mathrm{reg}}$, (ii) they can be naturally identified with the irreducible effective primitive motives, and (iii) the isomorphism classes of irreducible subquotients of $H^*(F)$ are related to more general irreducible effective motives, such as the Tate motive $\mathbb{Q}(-1)$ in the case of $H^1_{\mathrm{dR}/k}(F)$.

1.2. **From linear to semi-linear representations.** The representation $\Omega^{\bullet}_{F|k}$ of G is also an F-vector space endowed with a semi-linear G-action.

Definition. Let K be a field, H be a semigroup of endomorphisms of K and $k = K^H$.

A semi-linear representation of H over K is a K-vector space V endowed with a semi-linear H-action, i.e., with an additive H-action $H \times V \to V$ such that $\sigma(a \cdot v) = \sigma a \cdot \sigma v$ for any $\sigma \in H$, $v \in V$ and $a \in K$. This is the same as a module over the associative central k-algebra $K\langle H\rangle := K \otimes_{\mathbb{Z}} \mathbb{Z}[H]$ with the evident left K-action and the diagonal left H-action. We say that a semi-linear representations of H *non-degenerate* if the action of each element of H is injective. (If $\dim_K V < \infty$ this is equivalent to the condition $K \otimes_{\sigma(K)} \sigma(V) = V$.)

The semi-linear representations of H, finite-dimensional over K, form an abelian tensor k-linear category. This category is rigid if the elements of H are invertible. The set of isomorphism classes of non-degenerate semi-linear K-representations of H of degree r is canonically identified with the set $H^1(H, \mathrm{GL}_r K)$.

Denote by $\mathcal{C}$ the category of smooth semi-linear representations of G over F.

It is well-known after Hilbert, Tate, Sen, Fontaine... that the semi-linear representations is a powerful tool in the study of Galois representations. We try to use them in non-Galois context, namely, in the context of representations of G.

Once again, we are interested in linear representations of G, especially in irreducible ones, and more particularly, in irreducible "homotopy invariant" representations, i.e., objects of $\mathcal{I}_G$.

There are faithful forgetful functors $\mathcal{S}m_G \xleftarrow{\text{for}} \mathcal{C} \xrightarrow{\text{for}} \mathcal{S}m_G(k)$, admitting left adjoint functors of extending of coefficients to F: $\mathcal{S}m_G \xrightarrow{\otimes F} \mathcal{C} \xleftarrow{\otimes_k F} \mathcal{S}m_G(k)$, where $\mathcal{S}m_G(k)$ is the category of smooth representations of G over k, so $W \hookrightarrow \mathrm{for}(W \otimes F)$, or $W \hookrightarrow \mathrm{for}(W \otimes_k F)$.

The functor $F\otimes_k$ is faithful, but it is not full. E.g, if $U \subset G$ is an open subgroup and $\overline{f} \in (F^\times/k^\times)^U \smallsetminus \{1\}$ then $[\sigma] \mapsto \sigma f \cdot [\sigma]$ determines an element of $\mathrm{End}_{\mathcal{C}}(F[G/U])$, which is not in the subspace $\mathrm{End}_{\mathcal{S}m_G(k)}(k[G/U])$. Another example: $\mathrm{End}_{\mathcal{S}m_G(k)}(F) = k$, but $\mathrm{End}_{\mathcal{C}}(F \otimes_k F) = k \cdot id \oplus k \cdot (m \otimes 1)$, where $m : F \otimes_k F \longrightarrow F$ is the multiplication map.

On the other hand, the functor $\mathcal{I}_G(k) \xrightarrow{F\otimes_k} \mathcal{C}$ is fully faithful ([37, Lemma 0.1]), and any object $W \in \mathcal{I}_G(k)$ can be reconstructed from $W \otimes_k F \in \mathcal{C}$, cf. Lemma 1.2.3.

Though the functor $F\otimes_k$ does not respect the irreducibility, for any irreducible $W \in \mathcal{S}m_G$ the object $W \otimes F \in \mathcal{C}$ admits an irreducible semi-linear quotient V with an inclusion $W \subset V$.

Thus, any irreducible object $\mathcal{S}m_G$ is contained in an irreducible object of $\mathcal{C}$, and the problem of description of irreducible objects of $\mathcal{S}m_G$ splits into description of i) irreducible objects of $\mathcal{C}$ and ii) their linear submodules.

Here are two arguments, suggesting that in some respects $\mathcal{C}$ is simpler than $\mathcal{S}m_G$.

- All representations $A(F)/A(k)$ of G for all abelian k-varieties A (i.e. corresponding to all pure 1-motives) are contained in the irreducible object $\Omega^1_{F|k}$ of $\mathcal{C}$. Namely, any sufficiently general 1-form $\eta \in \Gamma(A, \Omega^1_{A|k})$ gives an embedding $A(F)/A(k) \hookrightarrow \Omega^1_{F|k}$, by sending a point $k(A) \overset{\sigma}{\hookrightarrow} F$ to $\sigma\eta \in \Omega^1_{F|k}$.
- It follows from Hilbert's Theorem 90 that the category $\mathcal{C}$ admits a countable system of cyclic generators: $P_m := F[\{K_m \overset{/k}{\hookrightarrow} F\}]$, where K_m is a purely transcendental field extension of k of transcendence degree m.

In general, for an arbitrary group, it may well happen that there are "much more" semi-linear representations than linear ones. E.g., if $\mu_\infty \cong \mathbb{Q}/\mathbb{Z}$ acts on $k(t)$ by $\zeta : t \mapsto \zeta t$, then the space of one-dimensional semi-linear representations $H^1(\mu_\infty, k(t)^\times) = \left(\varprojlim_n k(t)^\times/k(t^n)^\times \right) /k(t)^\times$ is enormous (and non-separated in the natural topology), though the space of one-dimensional E-linear representations $\mathrm{Hom}(\mu_\infty, E^\times) \subseteq \widehat{\mathbb{Z}}$ is relatively small.

If k is countable then the cardinality of the set of isomorphism classes of irreducible objects of $\mathcal{C}$ is continuum. (*Proof.* Let F' be an algebraically closed subfield of $F|k$ of a finite transcendence degree. For each one-dimensional representation $\varphi : G_{F'|k} \longrightarrow k^\times$ fix an irreducible quotient $V_\varphi \in \mathcal{C}$ of $F[G/G_{F|F'}] \otimes_{k[G_{F'|k}]} \varphi$. Note, that there is a subrepresentation of $G_{F'|k}$ in $V_\varphi^{G_{F|F'}}$ isomorphic to φ. Let us say that $\varphi \sim \psi$ if $V_\varphi \cong V_\psi$. If $\varphi \sim \psi$ then there is a subrepresentation of $G_{F'|k}$ in $V_\varphi^{G_{F|F'}}$, isomorphic to ψ. As $|V_\varphi^{G_{F|F'}}| = |k|$, the equivalence classes are of cardinality $\leq |k|$. Note, that there are $\geq 2^{|\mathbb{N}|}$ only those of φ, that factor through the modulus of $G_{F'|k}$. Therefore, the set of equivalence classes of φ is of cardinality $\geq 2^{|\mathbb{N}|}$, if k is countable, as we were going to show.

The upper bound $\leq 2^{|k|}$ for cardinality of the set of isomorphism classes of cyclic objects of $\mathcal{C}$ comes from the fact that there is a countable system $\{P_m\}_{m\in\mathbb{N}}$ of generators of $\mathcal{C}$, and the cardinality of each of P_m is $|k|$.)

On the other hand, I am not aware of a procedure, producing that many irreducible objects of $\mathcal{C}$, even conjecturally. It would be therefore natural to restrict oneself to a "relatively small" full subcategory in $\mathcal{C}$. (For instance, such that its objects are spanned as F-vector spaces by subrepresentations from $\mathcal{I}_G(k)$. Another, though a weaker, but a little bit more explicit condition, of "being globally generated", on the semi-linear quotients of $W\otimes F$ for $W\in\mathcal{I}_G$ is given below.)

However, the category $\mathcal{C}$ is "simple" in the sense that there are no non-trivial proper subcategories in it, closed under direct products and subquotients in $\mathcal{C}$. More precisely,

Lemma 1.2.1. *The annihilator in* $\lim\limits_{\longleftarrow U} F[G/U]$ *of any object* $0\neq V\in\mathcal{C}$ *is zero.*

For any integer $m\geq 0$ *there is an embedding of* P_m *into a direct product in* $\mathcal{C}$ *of copies of* V.

Proof. If $0\neq\alpha\in\lim\limits_{\longleftarrow U} F[G/U]$ annihilates V then for any L of finite type over k the projection $0\neq\alpha_L=\sum_{i=1}^N a_i\sigma_i\in F[G/G_{F|L}]$ annihilates $V^{G_{F|L}}$. Fix L, $0\neq v\in V^{G_{F|L}}$ and a functional $\varphi\in\mathrm{Hom}_F(V,F)$ such that $\varphi(\sigma_i v)\neq 0$ for all $1\leq i\leq N$. Then $\varphi(\alpha(fv))=\sum_{i=1}^N a_i\varphi(\sigma_i v)\cdot\sigma_i f$ vanishes for any $f\in L^\times$, contradicting to Artin's theorem on independence of characters of $L^\times$.

$\mathrm{Hom}_{\mathcal{C}}(P_m,V)$ is a non-zero K_m-vector space (by definition, $t\alpha:[id]\mapsto t\cdot\alpha[id]$ for any $t\in K_m$). Thus, $\bigcap_{\alpha\in\mathrm{Hom}_{\mathcal{C}}(P_m,V)}\ker\alpha\subseteq\bigcap_{t\in K_m^\times}\ker(P_m\xrightarrow{t\alpha}V)=0$ for any $\alpha\neq 0$ due to the linear independence of characters $\mathrm{Hom}(K_m^\times,F^\times)$. $\square$

Remark. This means that the central k-algebra $\lim\limits_{\longleftarrow U} F[G/U]$ is topologically simple, compare with [22, 23] in the case of finite Galois extensions.

Therefore, any "relatively small" subcategory of $\mathcal{C}$ cannot be "too nice".

Suppose from now on (until the end of this section) that $n=\infty$.

Valuations and associated functors ([39]). In order to associate a functor on a category of k-varieties to a representation of G one can try to "approximate" rings by their subfields. Evidently, this does not work literally, but apparently works in the case of discrete valuation rings of F.

Let $v:F^\times/k^\times\longrightarrow\mathbb{Q}$ be a discrete valuation of rank 1, and $\mathcal{O}_v$ be the valuation ring.

Set $G_v:=\{\sigma\in G\mid\sigma(\mathcal{O}_v)=\mathcal{O}_v\}$. This is a maximal closed non-open subgroup in G.

Proposition 1.2.2. *For any discrete valuation* $v:F^\times/k^\times\longrightarrow\mathbb{Q}$ *the additive functor* $(-)_v:\mathcal{S}m_G\longrightarrow\mathcal{S}m_{G_v}$, $W\mapsto W_v:=\sum_{F'\subset\mathcal{O}_v}W^{G_{F|F'}}\subseteq W$, *is fully faithful and preserves surjections and injections.*

Then the additive subfunctor $\Gamma:\mathcal{S}m_G\longrightarrow\mathcal{S}m_G$ of the identity functor, defined by $W\mapsto\Gamma(W):=\bigcap_v W_v$, where v runs over the set of discrete valuations of rank 1 trivial on k, preserves the injections.

EXAMPLE. $\Gamma(\Omega^1_{F|k})=\Omega^1_{F|k,\mathrm{reg}}\cong\bigoplus_A(A(F)/A(k))\otimes_{\mathrm{End}A}\Gamma(A,\Omega^1_{A|k})$, where A runs over the set of isogeny classes of simple abelian varieties over k, is the space of regular 1-forms.

Lemma 1.2.3. *If* $k=\overline{k}$ *then the functor* $\mathcal{I}_G(k)\xrightarrow{F\otimes_k}\mathcal{C}$ *is fully faithful. The compositions* $\mathcal{I}_G(k)\xrightarrow{F\otimes_k}\mathcal{C}\xrightarrow{\Gamma\circ\mathrm{for}}\mathcal{S}m_G(k)$ *and* $\mathcal{I}_G\hookrightarrow\mathcal{S}m_G\xrightarrow{\Gamma}\mathcal{S}m_G$ *are identical.*

The first part is [37, Lemma 0.1] and the second part follows from Lemmas 4.4.4 and 4.4.5.

Remarks. 1. To find a description of the Serre envelope of the abelian subcategory $\mathcal{I}_G\otimes F$ of $\mathcal{C}$ is a principal problem on the way to understand the structure of the category $\mathcal{I}_G(k)$. Conjecture 1.1.8 can serve as an indication in that direction.

2. It follows from Lemma 1.2.3 that any semi-linear quotient V of $W \otimes F$, with $W \in \mathcal{I}_G$, (in particular, any irreducible semi-linear representation V containing a "homotopy invariant" representation), is "globally generated", i.e., $\Gamma(V) \otimes F \longrightarrow\!\!\!\!\!\rightarrow V$ is surjective.

This is the condition one can impose on the class of "interesting" semi-linear representations. There are some reasons to expect that $(-)_v$ is exact, cf. [39]. This would imply some nice properties of the category of "globally generated" semi-linear representations.

Admissible semi-linear representations. As in the study of linear representations of any group, it is natural to start the study of semi-linear representations with the finite-dimensional representations. However, it will follow from Corollary 3.4.8 that they are trivial.

Theorem 1.2.4 ([35]). *Any finite-dimensional smooth semi-linear representation of G over F is trivial, if $n = \infty$.*

Definition. A smooth semi-linear representation V of G over F is called *admissible* if, for any open subgroup $U \subseteq G$, the fixed subspace V^U is finite-dimensional over the fixed subfield F^U (or equivalently, $\dim_L V^{G_{F|L}} < \infty$ for any subfield $L \subset F$ of finite type over k).

Theorem 1.2.5 ([36, 37]). *The admissible semi-linear representations of G over F form an abelian tensor (but not rigid) category, denoted by $\mathcal{A}$.*

The functor $H^0(G_{F|L}, -)$ is exact on $\mathcal{A}$ for any subfield $L \subseteq F$, so F is a projective object of $\mathcal{A}$.

The fact that $\mathcal{A}$ is a tensor category follows from Proposition 3.2.2.

As it is shown in [37, Lemma 3.1], one can take the quotients $Q_n := F[G/U_n]$ of the cyclic generators P_n of the category $\mathcal{C}$ for all $n \gg 0$ as cyclic generators of the category $\mathcal{A}$, since $\mathrm{Hom}_{\mathcal{C}}(Q_n, V) = V_n^{U_n}$. Here U_n denotes the preimage in $G_{\{F,K_n\}|k} \subset G$ of the subgroup in $G_{K_n|k}$ of translations by cyclotomic elements, i.e. consisting of transformations of type $x_j \mapsto x_j + b_j$ for all $1 \le j \le n$, where $b_j \in \mathbb{Q}^{\mathrm{ab}}$.

Example. Denote by $\mathfrak{m}$ the kernel of the multiplication homomorphism $F \otimes_{k_0} F \xrightarrow{\times} F$, where $k_0 = k \cap \overline{\mathbb{Q}}$ is the number subfield of k. This is an ideal in the algebra $F \otimes_{k_0} F$. We consider its powers $\mathfrak{m}^s \subseteq F \otimes_{k_0} F$ as objects of $\mathcal{C}$ for all $s \ge 0$ with the F-multiplication on the left, via $F \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}$.

Note that $\mathfrak{m}^s/\mathfrak{m}^{s+1} = \mathrm{Sym}^s_F \Omega^1_F$, so the semi-linear representations $\bigwedge^\bullet_F(\mathfrak{m}/\mathfrak{m}^s)$ and $\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^s)$ are admissible for any $q \ge 0$ and $s \ge 2$, if the transcendence degree of k is finite, and the object $\mathrm{Sym}^s_F \Omega^\bullet_{F|k}$ is admissible for any $s \ge 1$.

In the case of $k = \overline{\mathbb{Q}}$, the field of algebraic complex numbers, the category $\mathcal{A}$ admits the following explicit description.

For any $q \ge 0$ and $V \in \mathcal{A}$ let $W^q V$ be the sum of the images of the F-tensor powers $\bigotimes^{\ge q}_F \mathfrak{m}$ under all morphisms in $\mathcal{C}$ to V. Clearly, $W^\bullet$ is a functorial descending filtration on the objects of $\mathcal{A}$, and it is multiplicative: $(W^p V_1) \otimes_F (W^q V_2) \subseteq W^{p+q}(V_1 \otimes_F V_2)$ for any $p, q \ge 0$ and any $V_1, V_2 \in \mathcal{A}$.

(1) The graded quotients gr^q_W of the filtration $W^\bullet$ on the objects of $\mathcal{A}$ are finite direct sums of direct summands of $\bigotimes^q_F \Omega^1_F$, cf. [37, Theorem 4.10]. In particular, any object of $\mathcal{A}$ admits an irreducible quotient.
(2) The category $\mathcal{A}$ splits into the direct sum of its two full abelian subcategories, the first one equivalent to the category of finite-dimensional k-vector spaces, and the second one – $\mathcal{A}^\circ$ – consisting of objects V such that $V^G = 0$, cf. [37, Lemma 4.13].
(3) Any object V of $\mathcal{A}^\circ$ is a quotient of a direct sum of objects (of finite length) of type $\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^s)$ for some $q, s \ge 1$ ([37, Theorem 4.10]).
(4) If $V \in \mathcal{A}$ is finitely generated then it is of finite length and $\dim_k \mathrm{Ext}^j_{\mathcal{A}}(V, V') < \infty$ for any $j \ge 0$ and any $V' \in \mathcal{A}$; if $V \in \mathcal{A}$ is irreducible and $\mathrm{Ext}^1_{\mathcal{A}}(\mathfrak{m}/\mathfrak{m}^q, V) \ne 0$ for some $q \ge 2$ then $V \cong \mathrm{Sym}^q_F \Omega^1_F$ and $\mathrm{Ext}^1_{\mathcal{A}}(\mathfrak{m}/\mathfrak{m}^q, V) \cong k$ ([37, Corollary 4.17]).

(5) There are no non-zero projective objects in $\mathcal{A}^\circ$ ([37, Corollary 4.14]), but $\bigotimes_F^q \mathfrak{m}$ are its "projective pro-generators": the functor $\mathrm{Hom}_{\mathcal{C}}(\bigotimes_F^q \mathfrak{m}, -) = \varinjlim \mathrm{Hom}_{\mathcal{A}}(\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^N), -)$ is exact on $\mathcal{A}$ for any q ([37, Corollary 4.16]).

Moreover, (at least if $k = \overline{\mathbb{Q}}$) there is a functor $\mathcal{S} : V \longmapsto (Y \mapsto \mathcal{V}_Y(Y))$, providing an equivalence of $\mathcal{A}$ and the category of "coherent" sheaves in smooth topology $\mathfrak{Sm}_k$ on $\mathrm{Spec}(k)$, cf. [37, Corollary 5.2]. By definition, the underlying category of $\mathfrak{Sm}_k$ is the category of locally dominant morphisms of smooth k-varieties. We endow $\mathfrak{Sm}_k$ with the pretopology, where the coverings are the smooth surjective morphisms. Clearly, the base changes preserve the coverings.

A sheaf of $\mathcal{O}$-modules on $\mathfrak{Sm}_k$ is called "coherent" if its restriction to the small étale site (or equivalently, to the small Zariski site) of any smooth k-variety is coherent. Here $\mathcal{O}$ is the structure presheaf of the site $\mathfrak{Sm}_k$ (which associates to each $Y \in \mathfrak{Sm}_k$ its k-algebra of regular functions $\mathcal{O}(Y)$). Clearly, $\mathcal{O}$ is a sheaf on $\mathfrak{Sm}_k$.

First, one determines the restrictions of the sheaf $\mathcal{S}(V)$ to the projective spaces (they turn out to be sheaves of sections of homogeneous vector bundles). This part relies on some results on the "abstract" homomorphisms of algebraic groups, cf. [11, 26, 45]. Then the sheaves on the projective spaces can be locally extended to the arbitrary smooth k-varieties as pull-backs under étale morphisms to projective spaces. It is not hard to check that the sheaf $\mathcal{S}(V)$ is well-defined, when the object $V \in \mathcal{A}$ is described explicitly, cf. [37, Lemma 5.1].

The k-linear representations of G of particular interest are admissible ones, forming a full subcategory in $\mathcal{I}_G(k)$. Though tensoring with F does not transform them to admissible semi-linear representations,[1] there exists a similar functor in the opposite direction $\Gamma : \mathcal{A} \longrightarrow \mathcal{S}m_G(k)$, the "global sections" functor, faithful and left-exact, at least if $k = \overline{\mathbb{Q}}$. The functor Γ was already defined on p.11, even in a greater generality. However, it is sometimes useful to associate first to each smooth representation V of G a sheaf $\mathcal{V}$ on $\mathfrak{Sm}_k$, and only after that take the "global sections".

One has the following universal (though far from being unique) way of "globalization" of smooth representations of G. Let $V \in \mathcal{S}m_G$, Y be an irreducible smooth k-variety, $D \in Y^1$ be an irreducible divisor, and v_D be the corresponding discrete valuation of $k(Y)$. Choose an embedding $k(Y)$ into F over k and an extension v of v_D to a discrete valuation of rank 1 of F. Let $\widetilde{F} \subset F$ be a maximal subfield over k such that $v(\widetilde{F}^\times) = v(k(Y)^\times)$. Set $\mathcal{V}_{Y,D} = V^{G_{F|k(Y)}} \cap \bigcap_{\widetilde{F}} (\sum_{k \subset F' \subset \widetilde{F} \cap \mathcal{O}_v} V^{G_{F|F'}})$.

Note, that in the previous construction $\mathcal{S}(V)|_Y = \mathcal{V}_Y$ is a locally free coherent sheaf on Y with the generic fibre $V^{G_{F|k(Y)}}$ for each $V \in \mathcal{A}$. The functoriality follows from the fact that for any dominant morphism $X \xrightarrow{\pi} Y$ of smooth k-varieties the inclusion of generic fibres $k(X) \otimes_{k(Y)} V^{G_{F|k(Y)}} \subseteq V^{G_{F|k(X)}}$ induces an embedding of coherent sheaves $\pi^*\mathcal{V}_Y \hookrightarrow \mathcal{V}_X$ on X.

Slightly more generally, the "coherent" sheaves are contained in the category $\mathcal{F}l$ of flat (as $\mathcal{O}$-modules) "quasi-coherent" sheaves in the smooth topology. For any flat "quasi-coherent" sheaf $\mathcal{V}$ in the smooth topology the k-vector space $\Gamma(Y, \mathcal{V}_Y)$ is a birational invariant of a smooth proper Y. This follows from the Hartogs principle and the fact that any birational map is a composition of a birational morphism and of the inverse of a birational morphism, which is well-defined outside of codimension ≥ 2, cf. [37, Lemma 5.3].

Due to the birational invariance, one can define a left exact (but non-faithful) functor $\mathcal{F}l \xrightarrow{\mathit{\Gamma}} \mathcal{S}m_G(k)$ by $\mathcal{V} \mapsto \varinjlim \Gamma(Y, \mathcal{V}_Y)$, where Y runs over smooth projective models of subfields in F of finite type over k. Then $\Gamma(V) = \mathit{\Gamma}(\mathcal{V})$. The restriction of $\mathit{\Gamma}$ to the subcategory of the "coherent" sheaves is faithful, since $\Gamma(Y', \mathcal{V}_{Y'})$ generates the generic fibre of the sheaf $\mathcal{V}_{Y'}$ for appropriate finite covers Y' of Y, if $\mathcal{V}$ is "coherent".

[1] Suppose that $W \otimes F$ is admissible for a smooth representation W of G for $n \leq \infty$. Let us show that W is finite-dimensional. It follows from the inclusion $(W \otimes F)^{G_{F|L}} \supseteq W^{G_{F|L}} \otimes L$ that $\dim W^{G_{F|L}} < \infty$ for any L of finite transcendence degree over k. Therefore, $\dim W < \infty$ if $n < \infty$. If $n = \infty$ then W is trivial by Proposition 3.0.2, and everything is clear. □

If $\Gamma(Y,\mathcal{V}_Y)$ has the Galois descent property then $\Gamma(V)=\mathit{\Gamma}(\mathcal{V})$ is admissible. However, there is no Galois descent property in general, and $\Gamma(V)$ is not admissible.

Example. Let Y' be a smooth projective hyperelliptic curve, considered as a two-fold cover of the projective line Y. Then for $\mathcal{V}=(\Omega^1_{\mathcal{O}|k})^{\otimes 2}$ the tensor square of any regular differential on Y' is a Galois-invariant element of $\Gamma(Y',\mathcal{V}_{Y'})$, which is not in $\Gamma(Y,\mathcal{V}_Y)=0$.

The functor $\mathit{\Gamma}$ coincides with the composition of the forgetful functor of the "generic fibre" to $\mathcal{S}m_G(k)$ with the functor Γ, defined on p.11. The functor $\Gamma=\mathit{\Gamma}\circ\mathcal{S}$ is faithful on $\mathcal{A}$. However, it is not full, and the objects in its image are highly reducible, cf. Example on p.11 (and it might be added that $\mathcal{S}(\Omega^1_{F|k})(Y)=\Gamma(Y,\Omega^1_{Y|k})$).

Conjecture 1.2.6. (1) *The functor* $\mathrm{Hom}_{\mathcal{C}}(\bigotimes^q_F\mathfrak{m},-)$ *is exact on* $\mathcal{A}$ *for any* $q\ge 0$.

(2) *Irreducible objects of* $\mathcal{A}$ *are direct summands of the tensor algebra* $\bigotimes^{\bullet}_F\Omega^1_{F|k}$.

(3) $\mathcal{A}$ *is equivalent to the category of "coherent" sheaves on* $\mathfrak{Sm}_k$.

It would follow from Conjecture 1.2.6 (1) that, e.g., $\mathrm{Ext}^1_{\mathcal{A}}(\Omega^1_{F|k},F)$ is isomorphic to the space of derivations of k: a non-zero $\eta\in\mathrm{Hom}_k(\Omega^1_k,k)$ is sent to the class of $0\longrightarrow F\xrightarrow{\cdot\eta^{-1}(1)}\Omega^1_F/\ker\eta\otimes_k F\longrightarrow\Omega^1_{F|k}\longrightarrow 0$ in $\mathcal{A}$. This is compatible with [37, Lemma 3.10].

As another evidence for Conjecture 1.2.6 (2), in addition to the case $k=\overline{\mathbb{Q}}$, it is shown in [37, Theorem 2.4] that for any $L\subset F$ purely transcendental of degree m over k and any $V\in\mathcal{A}$ any irreducible subquotient of the L-semi-linear representation $V^{G_{F|L}}$ of $\mathrm{PGL}_{m+1}k$ is a direct summand of $\bigotimes^{\bullet}_L\Omega^1_{L|k}$.

There exist too many smooth irreducible semi-linear representations. In particular, most of them are not admissible. For instance, neither of quotients (including the irreducible ones) of the cyclic object $F[\{L\subset F\mid L\cong\overline{K_q}\}]^{\circ}$, consisting of formal degree-zero F-linear combinations of algebraically closed subfields in F of transcendence degree q over k for some integer $q\ge 1$, belongs to $\mathcal{A}$, cf. [37, Corollary 3.5]. However, I do not even know, whether these objects are reducible. It is therefore unclear, how to describe the irreducible objects of $\mathcal{C}$ explicitly. Thus, one cannot replace the category $\mathcal{A}$ in the part (2) of Conjecture 1.2.6 by the whole category $\mathcal{C}$, and has to put some additional conditions, e.g., the one mentioned on p.11.

Remarks. 1. Assuming the part (2) of Conjecture 1.2.6, one can reformulate Conjecture 1.1.8 in the following linguistically more convincing form:

Any irreducible object of $\mathcal{A}dm$ *(and of* $\mathcal{I}_G$*) is contained in an irreducible object of* $\mathcal{A}$.

This reformulation is based on Proposition 4.1.11.

2. It does not always make sense to study the irreducible objects of an abelian category. E.g., there are no irreducible objects in the quotient of the category of vector spaces over a field by its subcategory of finite-dimensional vector spaces (as well as there are neither infinite sums, nor products in this quotient category).

On the other hand, any sheaf of vector spaces, or a quasi-coherent sheaf $\mathcal{F}$ on a scheme Y admits a quotient supported on a point p: $U\mapsto\varinjlim_{V\ni p}\mathcal{F}(V)$ if $p\in U$; $U\mapsto 0$ if $p\notin U$. Therefore, the irreducible objects correspond to the points of Y.

1.3. **Notations, conventions and terminology. Fields, their extensions, automorphisms, etc.** Let K be a field, and H be a group of its automorphisms. We consider H as a topological group with a base of open subgroups generated by the stabilizers of the elements of K. Then H becomes a Hausdorff totally disconnected group, i.e., any neighbourhood of its arbitrary element contains a closed subneighbourhood.

For any collection of subsets $K_0,(K_\alpha)_{\alpha\in I}$ of a field K the subgroup of H, consisting of the elements, leaving K_0 fixed and inducing automorphisms of each of K_α, is closed in H. In the case, when K_α are subfields, the natural homomorphism from this subgroup to the automorphism group of K_α is continuous. We denote by $G_{\{K,(K_\alpha)_{\alpha\in I}\}|K_0}$ the group of automorphisms of K, leaving K_0 fixed and preserving each of K_α. Set $G_{K|K_0}:=G_{\{K\}|K_0}$.

If K is a subfield in F then $\overline{K}$ denotes the algebraic closure of K in F, and $\mathrm{tr.deg}(F|K)$ is the transcendence degree of an extension $F|K$ (possibly infinite, but countable).

If the opposite is not stated explicitly, then $F|k$ is an extension of algebraically closed fields countable (by default) or finite transcendence degree $\mathrm{tr.deg}(F|k) = n \geq 1$, and $G := G_{F|k}$. Everywhere, unless stated otherwise, the characteristic of k is zero.

General notations. $\mathbb{Q}$ is the field of rational numbers, and a *module* is always a $\mathbb{Q}$-vector space (with a few exceptions, where the characteristic of k is allowed to be positive).

For an abelian group A set $A_{\mathbb{Q}} = A \otimes \mathbb{Q}$.

For a k-variety X and for any field extension $E|k$ we set $X_E := X \times_k E$.

$\mathbb{P}^M_K$ denotes M-dimensional projective space over a field K.

Let R be a ring. If U is a set then $R[U]$ denotes the free R-module with the basis U. If M is a R-module and $m \in M$ then $\langle m \rangle_R$ denotes the R-submodule in M, generated by m. If $R = \mathbb{Z}$ then $\langle m \rangle := \langle m \rangle_R$. Stab_w denotes the stabilizer of w.

If H is the group acting on the set S on the left (e.g., S is a group and $H \subseteq S$ is a subgroup) then $H \backslash S$ denotes the set of H-orbits (rights cosets), and if S_1 is a subset of a set S then $S \smallsetminus S_1$ is the complement in S to S_1. $N_H H_1$ denotes the normalizer of a subgroup H_1 in H.

If S is a set then $|S|$ denotes its cardinality. If $\alpha, \beta \in S$ then $\delta_{\alpha\beta}$ is the Kronecker symbol: $\delta_{\alpha\beta} = 1$, if $\alpha = \beta$, and $\delta_{\alpha\beta} = 0$, if $\alpha \neq \beta$.

Topological groups, their representations, measures, etc. If H is a totally disconnected topological group, denote by H° its subgroup, generated by all compact subgroups, and by H^{ab} the quotient of H by the closure of its commutant. Clearly, H° is a normal subgroup in H, which is open, at least if H is locally compact.

An H-set (or a representation of H in a vector space over a field, etc.) W is called *smooth*, if the stabilizers of all the elements of W are open. A smooth representation W is called *admissible*, if the fixed vectors of each open subgroup form a finite-dimensional subspace in W.

$\mathbb{Q}(\chi_H)$ is the H-module of right-invariant measures on H, and $\chi_H : H \longrightarrow \mathbb{Q}_+^\times$ is the modulus, if H is locally compact, cf. p.16. $\mathbb{Q}(\chi) := \mathbb{Q}(\chi_G)$ and $\chi := \chi_G$ if $n < \infty$.

$\mathcal{S}m_H(E)$ is the category of smooth representations of H over a field E of characteristic zero. $\mathcal{A}dm_H(E)$ is its full subcategory of admissible E-representations. $\mathcal{I}_G(E)$ is the full subcategory of $\mathcal{S}m_G(E)$, consisting of the representations W of G such that $W^{G_{F|L}} = W^{G_{F|L'}}$ for any extension $L|k$ in F and any purely transcendental extension $L'|L$ in F. When discussing $\mathcal{I}_G(E)$, the principal case will be $n = \infty$. Set $\mathcal{S}m_H = \mathcal{S}m_H(\mathbb{Q})$ and $\mathcal{I}_G = \mathcal{I}_G(\mathbb{Q})$.

The Hecke algebras $\mathcal{H}_E(H, U)$ for compact subgroups U in H are defined at the beginning of §3, p.25. The identity of $\mathcal{H}_E(H, U)$, the Haar measure on U, is denoted by h_U. There one can find also a definition of the action of the algebra of "measures" $\mathbb{D}_E(H) := \mathcal{H}_E(H, \{1\})$ on the objects of $\mathcal{S}m_H(E)$. Set $\mathbb{D}_E = \mathbb{D}_E(G)$ and $\mathcal{H}_E(U) = \mathcal{H}_E(G, U)$.

The *level* filtration $N_\bullet$ is defined on p.5.

2. The structure of G and Galois theories

In this section the principal results on the topological group G are described (in arbitrary characteristic, cf. Theorem 2.2.1), such as i) its simplicity in the case $n = \infty$; ii) simplicity of its (normal) subgroup G°, generated by all compact subgroups. In particular, G° is dense in G if $n = \infty$. Part i) complements a result of D.Lascar, [24]: G is simple as a discrete group, if the transcendence degree of $F|k$ is not countable.

Clearly, G° is open if $n < \infty$, so the projection $G \twoheadrightarrow G/G^\circ$ with the discrete topology on the target is continuous. If $n = 1$ Lemma 2.4.5 presents G/G° as a quotient of a certain, rather "structured" group. It is not known much on G/G°. This is why one is usually forced to work in the "stable" case $n = \infty$, and to pose questions in a way to be able to avoid the knowledge of the structure of this group.

If $n = \infty$ it follows from the simplicity of G ([35, Corollary 2.11]) that any non-trivial continuous homomorphism from G is injective; and if $n < \infty$ any non-injective continuous homomorphism from G factors through G/G°. One more consequence is that there are neither non-trivial smooth representations of G° (and of G, if $n = \infty$) of finite degree, nor proper closed subgroups of finite index.

Corollary ([35], 2.11)**.** *For any subgroup H of G, containing G°, and any continuous homomorphism π from H either π is injective, or the restriction of π to G° is trivial.*

Let H be a locally compact group, and $\mathbb{Q}(\chi_H)$ be the quotient of the free abelian group, generated by the set of compact open subgroups of H, by the relations $[U] = [U : U']\cdot[U']$ for all $U' \subset U$. As the intersection of any pair of compact open subgroups of H is of finite index in both of them, $\mathbb{Q}(\chi_H)$ is a one-dimensional $\mathbb{Q}$-vector space, oriented by the condition $[U] > 0$ for any U. In other words, $\mathbb{Q}(\chi_H)$ is the space of $\mathbb{Q}$-valued right-invariant measures on H. The group of bi-continuous automorphisms of H acts on it. In particular, the group H acts on it by conjugations. Let $\chi_H : H \longrightarrow \mathbb{Q}_+^\times$ be the character of this representation of H, the modulus of H. It is trivial in the restriction to the (open) subgroup H°, generated by all compact subgroups of H.

In §2.5 a locally compact group $\mathfrak{G}$ is introduced. If $n < \infty$ then $\mathfrak{G} := G$, while if $n = \infty$, there there is a continuous group embedding $\mathfrak{G} \hookrightarrow G$ with a dense image. It is clear from an explicit description of the modulus $\chi := \chi_{\mathfrak{G}}$ that χ is surjective for all $1 \le n \le \infty$.

However, I do not even know, whether the discrete group $\ker \chi/\mathfrak{G}^\circ$ is trivial. If it is trivial for $n = 1$ then it is trivial in general, cf. [35, Lemma 2.15].

It follows from Theorem 2.2.1 that $\mathfrak{G}^\circ$ is a topologically simple group.

2.1. **Galois theory for compact subgroups.** Let $F|k$ be an arbitrary extension of arbitrary fields, and $G = G_{F|k}$ be its automorphism group. The topology on G, described in Introduction, p.2, has been studied in [14, p.151, Exercise 5], [32], [44, Ch.6, §6.3], and [13, Ch.2, Part 1, §1]. It is shown there that G is a Hausdorff and totally disconnected group, and for any intermediate subfield K in $F|k$ the topology on $G_{F|K}$ coincides with the restriction of the topology on G. The subgroups $G_{\{F,(F_\alpha)_{\alpha\in I}\}|k}$ are closed in G, since if $\sigma(F_\alpha) \subseteq F_\alpha$, $\sigma(F \smallsetminus F_\alpha) \subseteq F \smallsetminus F_\alpha$ and $\sigma(F) = F$ then $\sigma(F_\alpha) = F_\alpha$.

The Galois–Krull theory associates intermediate subfields of a Galois extension to the closed (=compact) subgroups of the Galois group of this extension, and vice versa. Namely, one associates to a subgroup its fixed field, and associates to a subfield the group of automorphisms leaving it fixed.

These operations are mutually inverse to each other and admit the following direct generalization.

Definition. 1. A non-empty collection $\mathcal{L}$ of subfields in F is called a *sieve*, if it contains all extensions of its arbitrary element. We call the *saturation* of an arbitrary collection $\mathcal{L}$ of subfields in F the sieve $\overline{\mathcal{L}}$, whose elements are the extensions in F of the elements of $\mathcal{L}$.

2. A collection of subfields is called *directed*, if it contains the intersection of each pair of its elements.

Examples. a) The saturation of collections, consisting of a single subfield, defines an embedding of the set of subfields in $F|k$ to the set of sieves of subfields in $F|k$. b) For a descending sequence $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ of subfields K_j in $F|k$, over which F is algebraic, the collection of all the extensions of all K_j is a directed sieve. Clearly, the saturations of the collections $(K_j)_{j\ge1}$ and $(K_j)_{j\in S}$ coincide for any infinite subset S of the natural numbers.

Proposition 2.1.1. *There is a morphism of unitary monoids (transforming the compositum of subfields to the intersection of subgroups), inverting the inclusions, injective, if* $\mathrm{char}(k) = 0$ *and F is algebraically closed,*

$$\beta : \{\text{subfields in } F \text{ over } k\} \longrightarrow \{\text{closed subgroups in } G\}, \tag{1}$$

given by $K \mapsto \mathrm{Aut}(F|K) =: G_{F|K}$. It preserves the neutral elements: $k \mapsto G$. The image of β is stable under the passages to sup-/sub- groups with compact quotients; if F is algebraically closed *then the fibres of β consist of the subfields of F with the same sets of perfect subfields containing them; β induces compatible bijections*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{directed sieves consisting of such}\\ K \text{ in } F|k \text{ that } F|K \text{ are Galois extensions}\end{array}\right\} & \stackrel{A}{\leftrightarrow} & \left\{\begin{array}{c}\text{subgroups of } G\text{, exhausted by}\\ \text{compact subgroups}\end{array}\right\}\\
\uparrow \text{saturation} & & \bigcup\\
\left\{\begin{array}{c}\text{subfields } K \text{ in } F|k \text{ such that}\\ F|K \text{ is a Galois extension}\end{array}\right\} & \stackrel{B}{\leftrightarrow} & \{\text{compact subgroups in } G\}\\
\bigcup & & \bigcup\\
\left\{\begin{array}{c}\text{subfields } K \text{ in } F|k \text{ such that there is}\\ \text{a subfield } L \text{ in } K|k \text{ of finite type}\\ \text{such that } F|K' \text{ is a Galois extension only}\\ \text{for } K'=K \text{ if } K' \text{ is a subfield in } K|L\end{array}\right\} & \stackrel{C}{\leftrightarrow} & \left\{\begin{array}{c}\text{compact open}\\ \text{subgroups in } G\end{array}\right\}.
\end{array}$$

The latter set is non-empty if and only if G is locally compact. In particular, it is non-empty if transcendence degree of $F|F^G$ is finite (e.g., if $F=\overline{F}$ and $n<\infty$).

The inverse correspondences in the cases B and C are given by $G\supset H\longmapsto F^H$ (the fixed subfield in F of H). □

The correspondence B can be found in [32, §3, Lemma 1], or [44, Prop.6.11]; and C can be found in [14], or follows immediately from loc.cit., or [44, Prop.6.12]. In the case of an algebraically closed F there are the perfect closures K in F of subfields of finite type over k such that $F|K$ is a Galois extension on the left hand side of C. The correspondence A is induced by the maps

$$\left\{\begin{array}{c}\text{subgroups in } G\text{, generated}\\ \text{by compact subgroups}\end{array}\right\}_{\beta}\leftrightarrows^{\gamma}\left\{\begin{array}{c}\text{sieves consisting of subfields}\\ \text{in } F|k\text{, over which } F \text{ is algebraic}\end{array}\right\}$$

given by $\beta: G\supset H\mapsto (F^C\subseteq F)_C$, where C runs over the compact subgroups in H, and $\gamma:\mathcal{L}\mapsto\bigcup_{K\in\mathcal{L}}G_{F|K}$.

EXAMPLES. 1. If F is algebraically closed then those subfields of $F|k$, over which F is a Galois extension, are the perfect subfields containing k, over which F is algebraic. If, moreover, $n=1$ then the proper subgroups in the image of β are the compact subgroups.

2. Let $\mathrm{char}(k)=0$, $F=\overline{k(x)}$, and let H be generated by $G_{F|k((x-a)^2)}$ for all $a\in k$. Fix an embedding $F\hookrightarrow k((x^{-1/\infty}))$, and define the action of the translations by the elements of k on the formal Puiseux series in the evident way. This gives a group embedding $k\hookrightarrow G$. Let Γ be the image of $k\hookrightarrow G$. Then the multiplication $G_{F|k(x^2)}\times\Gamma\longrightarrow H$ is bijective. Clearly, the intersection of H with the union of the compact subgroups in G is not a group.

3. The group G may be locally compact, even if $\mathrm{tr.deg}(F|F^G)=\infty$. E.g., let a polynomial $P(X,Y,Z)$ over k define a surface with no birational automorphisms, and let x,y_i, where $i\in\mathbb{Z}$, be independent variables, and let z_i satisfy the conditions $P(\tau^i x,y_i,z_i)=0$ for all $i\in\mathbb{Z}$ and some element $\tau\in G_{k(x)|k}$ of infinite order (e.g., $\tau x=x+1$ if $\mathrm{char}(k)=0$, or $\tau x=qx$ if $q\in k^\times\smallsetminus\mu_\infty$). Set $F=k(x,y_i,z_i\mid i\in\mathbb{Z})$. Then $G_{F|k(x)}=\{1\}$, i.e., the group is discrete. On the other hand, G contains such an element σ that $\sigma x=\tau x$, $\sigma y_i=y_{i+1}$ and $\sigma z_i=z_{i+1}$ for all $i\in\mathbb{Z}$, and thus, $F^G=F^{\langle\sigma\rangle}=k$.

4. The map $H\mapsto F^H$ inverts the order, but in general it is not compatible with β. If F is algebraically closed then it is left inverse to the restriction of β to the perfect subfields, any perfect subfield is of type F^H and G contains the automorphism groups of all extensions in $F|k$ as its subquotients. But $H\mapsto F^H$ does not respect the monoid structure: $G_{\{F,k(x^2)\}|k},G_{\{F,k((x+1)^2)\}|k}\mapsto k$, but $G_{\{F,k(x^2)\}|k}\cap G_{\{F,k((x+1)^2)\}|k}=G_{F|k(x)}\mapsto k(x)\neq k$.

5. Further examples can be found in [33].

2.2. **Topological simplicity of G° and of G.** From now on the field F will be algebraically closed.

We say that a topological group is topologically simple, if its arbitrary closed normal proper subgroup is trivial.

Theorem 2.2.1 ([35], 2.9)**.** *If $n < \infty$ then any non-trivial subgroup in G normalized by G° is dense in G°. If $n = \infty$ then any non-trivial normal subgroup in G, e.g. G°, is everywhere dense.*

Here $F|k$ is an arbitrary extension of algebraically closed fields of an arbitrary characteristic. In particular, the subgroup G° of G, generated by the compact subgroups, is open and topologically simple, if $n < \infty$; if $n = \infty$ then G is topologically simple.

Remarks. 1. If $n = 1$ and $\mathrm{char}(k) \neq 0$ then the separable closure of $k(x)$ in F is generated by the G°-orbit of x for any $x \in F \smallsetminus k$, cf. Proposition 3.3.1.

2. An argument of [24] shows that G is simple as a discrete group provided that transcendence degree F over k is not countable.

It follows from the following lemma that G° is dense in G, if $n = \infty$.

Lemma 2.2.2. *Let L be a subfield of F such that* $\mathrm{tr.deg}(F|L) = \infty$. *Then $G_{F|L}$ is the closure of the set of products of all pairs of elements of all compact subgroups in $G_{F|L}$.*

Proof. Let $\sigma \in G_{F|L}$. We have to show that the restriction of σ to any finite subset $S \subset F$ coincides with the restriction of the product of a pair of elements of some compact subgroups in $G_{F|L}$. Let $T \subset F$ be a subset of order $|S|$ such that the elements of T are algebraically independent over $L(S, \sigma(S))$. Choose a subfield K in the algebraic closure of $L(T)$, isomorphic to $L(S)$ over L. Clearly, there are elements τ_1, τ_2 of some compact subgroups such that τ_1 interchanges $L(S)$ and K, τ_2 interchanges $L(\sigma(S))$ and K, and $\tau_2\tau_1|_S = \sigma|_S$. □

Proposition 2.2.3 ([35], 2.14, [39], 2.5)**.** *Let L_1 and L_2 be subfields of F such that $\overline{L_1} \bigcap \overline{L_2}$ is algebraic over $L_1 \bigcap L_2$ and* $\mathrm{tr.deg}(F|L_2) = \infty$, *or* $\mathrm{tr.deg}(L_1|L_1 \bigcap L_2) < \infty$. *Then the subgroup in G, generated by $G_{F|L_1}$ and $G_{F|L_2}$ is dense in $G_{F|L_1 \bigcap L_2}$.*

Remark. This (and [35, Lemma 2.16], cf. proof of Lemma 2.2.4) is an analogue of the following result from [3]: the Lie algebra of differentiations $\mathrm{Der}(F|L_1 \bigcap L_2)$ is topologically generated by its Lie subalgebras $\mathrm{Der}(F|L_1)$ and $\mathrm{Der}(F|L_2)$. (A base of open Lie subalgebras in $\mathrm{Der}(F|L)$ is given by the annihilators of finite subsets in F.)

Lemma 2.2.4. *Let $L_0 \subset L_1 \subset L_2$ be a pair of non-trivial purely transcendental extensions in F of finite type. Let $\mathfrak{S}$ be a transitive permutation group of a transcendence base S of L_2 over L_0, extending a transcendence base of L_1 over L_0. Let H be a subgroup in $G_{F|L_0}$, preserving L_2 and projecting onto a subgroup in $G_{L_2|L_0}$, containing $\mathfrak{S}$. Then the subgroup G' in G, generated by $G_{F|L_1}$ and H, coincides with $G_{F|L_0}$.*

Proof. G' contains the subgroups, conjugated to $G_{F|L_1}$ by all elements of H. In particular, G' contains the subgroups $G_{F|L_0(S \smallsetminus \{x\})}$ for all $x \in S$. According to [35, Lemma 2.16], for any subfield L in F, and any subset S in F consisting of elements, algebraically independent over L, the subgroup generated by the subgroups $G_{F|L(S \smallsetminus \{x\})}$ for all $x \in S$, is dense in $G_{F|L}$. Therefore, G' coincides with $G_{F|L_0}$. □

2.3. **Open and maximal proper subgroups; Galois theories,** [39]**.** The study of smooth representations of G and of stabilizers of their vectors leads to the study of open subgroups of G. For any $\mathrm{tr.deg}(F|k) = n \leq \infty$ there is a morphism of commutative associative monoids with the (minimal) unity, inverting inclusions, (transforming the intersection of subgroups to the algebraic closure of the compositum of subfields, and the unity G to the unity k)

$$\alpha : \{\text{open subgroups of } G\} \longrightarrow\!\!\!\!\!\rightarrow \left\{ \begin{array}{c} \text{algebraically closed subfields of } F \\ \text{of finite transcendence degree over } k \end{array} \right\} =: A\Pi.$$

It is determined uniquely by the following equivalent conditions:

- each open subgroup $H \subseteq G$ contains $G_{F|\alpha(H)}$ as a normal subgroup[2] and, if possible, $\alpha(H) \neq F$;
- $G_{F|\alpha(H)} \subseteq H$ and the transcendence degree of $\alpha(U)$ over k is minimal.

[2]i.e. H contains in the normalizer $G_{\{F,\alpha(H)\}|k}$ of $G_{F|\alpha(H)}$.

In particular, for any non-trivial algebraically closed extension $L \neq F$ of k of finite transcendence degree in F the normalizer $G_{\{F,L\}|k}$ in G of $G_{F|L}$ (which is evidently open) is maximal among the proper subgroups of G.

In the case $n = \infty$ any open proper subgroup of G is contained in a maximal proper open subgroup of G; and any maximal proper open subgroup of G is of type $G_{\{F,L\}|k}$ for some $L \in A\Pi$, $L \neq k$. Besides that, $\alpha(H) = \alpha(N_G H)$.

Questions. 1. The preimage of any proper subgroup of a prime (finite) index in $\mathbb{Q}_+^\times$ under the modulus character, if $n < \infty$, is one more type of maximal open proper subgroups, not encounted by Proposition 2.3. Any compact subset of G is contained in infinitely many subgroups of this type. Are there any other maximal proper open subgroups?

2. Do there exist closed subgroups not contained in maximal proper ones?

3. Can the maximal proper open subgroups be realized as stabilizers in *irreducible* (semi-)linear representations (if $n = \infty$) of G? The answer would be negative, if the representations $\mathbb{Q}[\{[\overline{L}] \mid k \subset L \subset F,\ \mathrm{tr.deg}(L|k) = m\}]^\circ$ (resp., $F[\{[\overline{L}] \mid k \subset L \subset F,\ \mathrm{tr.deg}(L|k) = m\}]^\circ \in \mathcal{C}$) turned out to be irreducible for all $m \geq 1$.

In the case of arbitrary transcendence degree the stabilizers of discrete valuations of rank one is another type of closed, but now not open maximal proper subgroups, cf. Proposition 2.4.3. Using them, one associates functors on categories of smooth k-varieties to the representations of G, cf. §1.2, p.11.

Remarks. 1. If $n < \infty$ and $H \subset G$ is contained in neither subgroup of type $G_{\{F,L\}|k}$, where $L \in A\Pi \smallsetminus \{k, F\}$, then F is algebraic over the subfield, generated over k by the H-orbit of x for any $x \in F \smallsetminus k$.

2. If $\mathrm{tr.deg}(L|k) = \mathrm{tr.deg}(F|L) = \infty$ then $G_{\{F,\overline{L}\}|k}$ is maximal among closed proper subgroups of G, i.e.., the subgroup H, generated by $G_{\{F,\overline{L}\}|k}$ and by any $\sigma \in G$ such that $\sigma(\overline{L}) \neq \overline{L}$, is dense in G. *Question.* Can one replace the condition "$\mathrm{tr.deg}(F|L) = \infty$" by the condition "$F \neq \overline{L}$"?

3. If subfields $\overline{K}$ and $\overline{L}$ are in general position then the subgroup $G_{\{F,\overline{K},\overline{L}\}|k}$ is contained in exactly three maximal proper open subgroups of G: $G_{\{F,\overline{K}\}|k}$, $G_{\{F,\overline{L}\}|k}$ and $G_{\{F,\overline{KL}\}|k}$, since if $\sigma \in G$ preserves neither of $\overline{K}$, $\overline{L}$ and $\overline{KL}$ then $\langle \sigma, G_{\{F,\overline{K},\overline{L}\}|k} \rangle = G$.

The union of proper open subgroups of G is characterized in the following way.

Corollary 2.3.1 ([39]). *The union of proper open subgroups of G is everywhere dense in G, and does not coincide with G if $n = \infty$. The following properties of an element $\sigma \in G$ are equivalent:*

1. *σ does not belong to the union of the proper open subgroups of G,*
2. *$W^{\langle \sigma \rangle} = W^G$ for any smooth G-set W,*
3. *there are no non-zero σ-invariant finite-dimensional F-vector subspaces in $\Omega^1_{F|k}$.*

Remark. If $n = \infty$ then any countable free group $H = *_{j \in S}\mathbb{Z}$ can be embedded into G in such a way that its intersection with any proper open subgroup in G is trivial. Namely, choose a transcendence base of $F|k$, and enumerate it by the elements of H: $\{x_h \mid h \in H\}$. Define an action of the generators $\{h_j \mid j \in S\}$ of H on the transcendence base by $h_j x_h = x_{h_j h}$. Clearly, this action extends, though not uniquely, to F.

Let $A = A_\sigma := F\langle \sigma, \sigma^{-1} \rangle$ be the algebra of endomorphisms of the additive group F generated by F and by $\sigma^{\pm 1}$ for some $\sigma \in G$. Clearly, A is a Euclidean simple central $F^{\langle \sigma \rangle}$-algebra, cf. [31]. The set of σ-invariant algebraically closed subfields in $F|k$ injects into the set of A-submodules in $\Omega^1_{F|k}$ by $L \mapsto F \otimes_L \Omega^1_{L|k}$.

Suppose now that $n = \infty$ and σ does not belong to the union of the proper open subgroups of G. In particular, $\Omega^1_{F|k}$ is a torsion-free A-module of at most countable rank. In a standard manner one checks that the finitely generated torsion-free A-modules are free.

An example of the A-module $\Omega^1_{F|k}$ of rank 1, which is not free, is given by $F=\overline{k(x_i,y_j \mid i\in\mathbb{Z},\ j\in\mathbb{N})}$, where x_i,y_j are algebraically independent, we set $y_0=x_0$ and $\sigma x_i=x_{i+1}$, $\sigma y_j=y_{j-1}+y_j$. The rank of the A-module $\Omega^1_{F|k}$ is an invariant of the conjugacy class of σ. What are the others?

In the case $F=\overline{k(x_i \mid i\in\mathbb{Z})}$, where x_i are algebraically independent and $\sigma x_i=x_{i+1}$, one has $A\xrightarrow{\sim}\Omega^1_{F|k}$, $\alpha\mapsto\alpha dx_0$, so the set of A-submodules in $\Omega^1_{F|k}$ is in bijection with the set of left ideals in A, i.e., with the set of monic (non-commutative) polynomials in σ with non-zero constant term. E.g., the polynomial $\sigma+1$ corresponds to $\overline{k(x_i+x_{i+1} \mid i\in\mathbb{Z})}\neq F$.

One more Galois theory, [39]**.** Now, as a corollary for $n=\infty$, we get a complete, though not very explicit, Galois theory of algebraically closed extensions of countable transcendence degree (a question of Krull, cf. [21]), i.e., a construction of all subgroups H of G, coincident with the automorphism groups of F over the fixed subfields F^H.

One can characterize

1. the normalizers $G_{\{F,L\}|k}$ of $G_{F|L}$ in G for all $L\in A\Pi\smallsetminus\{k\}$ as the maximal open proper subgroups of G;
2. the subgroups $G_{F|L}$ for all $L\in A\Pi\smallsetminus\{k\}$ as minimal closed non-trivial normal subgroups in $G_{\{F,L\}|k}$ (this follows from the topological simplicity of G);
3. subgroups $G_{F|L}$ of G for all non-trivial extensions $L|k$ in F of finite type as the open subgroups containing normal co-compact subgroups of type $G_{F|\overline{L}}$ from (2) (this follows from the classical Galois theory for $\overline{L}|k$);
4. the proper subgroups in the image of β as intersections of subgroups from (3).

Remark. The subgroups $G_{F|L}$ of G for all extensions $L|k$ in F of finite type and transcendence degree one are the subgroups from (3) with the only maximal proper subgroup of G containing them.

2.4. **Valuations and associated subgroups,** [39]**.** Let $\mathcal{O}_v$ be a valuation ring in F, $\mathfrak{m}_v=\mathcal{O}_v\smallsetminus\mathcal{O}_v^\times$ be the maximal ideal, and $\kappa(v)$ be the residue field of v. If $k\subseteq\mathcal{O}_v$, fix a subfield $k\subseteq F'\subseteq\mathcal{O}_v$ identified with $\kappa(v)$ by the reduction modulo $\mathfrak{m}_v$. In this case $\kappa(v)$ is of characteristic zero (and algebraically closed).

Set $G_v:=\{\sigma\in G \mid \sigma(\mathcal{O}_v)=\mathcal{O}_v\}$. This is a closed subgroup in G.

The valuation group $\Gamma:=F^\times/\mathcal{O}_v^\times\cong\mathbb{Q}^r$ is totally ordered: $v(x)\geq v(y)$ if and only if $xy^{-1}\in\mathcal{O}_v$, where $v:F^\times\longrightarrow\Gamma$ is the natural projection.

We call $r=\dim_{\mathbb{Q}}\Gamma$ the *rank* of v. We assume that it is finite.

Assume that the characteristics of a field L and of the residue field κ of a valuation w of L are equal. Then w is called *discrete*, if L is algebraic over the subfield generated by a lift of a transcendence base of κ and by a lift of a basis of the valuation group. In particular, v is discrete if and only if the transcendence degree of F over F' is equal to r.

Choose an arbitrary algebraically closed F' in F, over which F is of transcendence degree r, a transcendence base $x_1,\dots,x_r$ of $F|F'$, and embeddings $F\hookrightarrow\varinjlim_N F'((x_1^{1/N}))\dots((x_r^{1/N}))$ over $F'(x_1,\dots,x_r)$. In this case $v(x_1^{m_1})<\cdots<v(x_r^{m_r})$ for all $m_1,\dots,m_r>0$.

If $r<\infty$ and $\sigma(\mathcal{O}_v)\subseteq\mathcal{O}_v$ for some $\sigma\in G$ then $\sigma\in G_v$, since σ induces a surjective endomorphism of Γ, i.e. an automorphism.

It is well-known, [48], or [2, Chapter 5, Exercise 32], that for any valuation ring $\mathcal{O}_p$ with a valuation group Γ the map $\mathfrak{p}\mapsto\langle v(\mathcal{O}_p\smallsetminus\mathfrak{p})\rangle$ gives a bijection between the set $\mathbf{Spec}\mathcal{O}_p$ of prime ideals in $\mathcal{O}_p$ and the set of isolated subgroups in Γ. Moreover, $v(\mathcal{O}_p\smallsetminus\mathfrak{p})$ is the set of all non-negative elements of the corresponding isolated subgroup of Γ. Thus, there are exactly $r+1$ prime ideals in $\mathcal{O}_v$.

Remarks. 1. If $\mathfrak{p}\neq 0$ is a non-maximal prime ideal of finite codimension in $\mathcal{O}_v$ and $\mathcal{O}_{v'}:=(\mathcal{O}_v)_{\mathfrak{p}}$ then $G_v\subseteq G_{v'}$ (since any element $\sigma\in G_v$ preserves $\mathfrak{p}$, thus also $\mathcal{O}_v\smallsetminus\mathfrak{p}$, i.e. induces an automorphism of $\mathcal{O}_{v'}$).

2. The inclusion $G_v\subset G_{\{F,\mathcal{O}_v[x_1^{-1}]\}|k}$ is proper for $r>1$, i.e. G_v is not maximal.

Let $\mathcal{P}^r_L$ be the set of discrete valuation rings of rank r in L, containing k, admitting also the following description.

Let $\mathcal{C}^r_X$ be the set of chains of irreducible normal subvarieties up to codimension r on an irreducible proper normal variety X over k. Any proper surjection with irreducible fibres, e.g. a birational morphism, $X' \xrightarrow{\pi} X$ induces an embedding $\mathcal{C}^r_X \hookrightarrow \mathcal{C}^r_{X'}$, $(Z^1 \supset \cdots \supset Z^r) \mapsto (W^1 \supset \cdots \supset W^r)$, where $W^0 := X'$ and W^j is the proper preimage of Z^j under the restriction of π to W^{j-1} for $1 \leq j \leq r$ (and $\pi|_{W_1} : W_1 \xrightarrow{\sim} Z_1$). If L is of finite type over k then $\mathcal{P}^r_L \cong \varinjlim_X \mathcal{C}^r_X$, where X runs over the models of $L|k$, and $\mathcal{P}^r_F = \varprojlim_L \mathcal{P}^r_L$. For instance, if C is a smooth projective curve over k then $\mathcal{P}^1_{k(C)}$ is the set of closed scheme points of C.

Any proper surjection $X' \xrightarrow{\pi} X$ induces embeddings $\mathbb{Z}[\mathcal{C}^r_X] \hookrightarrow \mathbb{Z}[\mathcal{C}^r_{X'}]$ and $\mathbb{Z}[\mathcal{C}^r_X]^\circ \hookrightarrow \mathbb{Z}[\mathcal{C}^r_{X'}]^\circ$, where $\mathbb{Z}[\mathcal{C}^r_X]^\circ := \bigcap_{j=0}^{r-1} \ker \left(\mathbb{Z}[\mathcal{C}^r_X] \longrightarrow \mathbb{Z}[\mathcal{C}^{r,j}_X]\right)$, $\mathcal{C}^{r,j}_X$ denotes the set of chains with no component of codimension j and $\mathcal{C}^r_X \longrightarrow \mathcal{C}^{r,j}_X$ is the omitting of such component.

Let $\amalg^r := \varinjlim_L \mathbb{Z}[\mathcal{P}^r_L]^\circ$, where L runs over the set of subfields in $F|k$ of finite type over k. Then one can define a morphism of smooth G-modules $gr^W_{2q} H^q_{\mathrm{dR}/k}(F) \xrightarrow{\mathrm{Res}} \amalg^q \otimes k$ by $\mathrm{Res}(\alpha \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_q}{t_q}) := \sum_{\sigma \in \mathfrak{S}_q} \mathrm{sgn}(\sigma) \alpha|_{D_{1\dots q}} \cdot (D_{\sigma(1)} \supset D_{\sigma(1)\sigma(2)} \supset \cdots \supset \bigcap_{j=1}^q D_j =: D_{1\dots q})$, where D_i is given locally by $t_i = 0$ and α is regular in a neighbourhood of $t_1 = \cdots = t_q = 0$.

Lemma 2.4.1. *If $0 \leq r < n+1 \leq \infty$ then the group G acts transitively on the set of pairs (v, Λ), where $v : F^\times/k^\times \twoheadrightarrow \Gamma \cong \mathbb{Q}^r$ is a discrete valuation of rank r and $\Lambda \cong \mathbb{Z}^r$ is a lattice in Γ. The stabilizer of (v, Λ) acts transitively on the set of maximal subfields $\widetilde{F}$ in $F|k$ such that $v(\widetilde{F}^\times) = \Lambda$. The residue field of $\widetilde{F}$ coincides with $\kappa(v)$ (in particular, it is algebraically closed).*

The G_v-action on $\kappa(v)$ induces a homomorphism $G_v \twoheadrightarrow G_{\kappa(v)|k}$. Let

$$G^\dagger_v := \{\sigma \in G_v \mid \sigma x - x \in \mathfrak{m}_v \text{ for any } x \in \mathcal{O}_v\} = \{\sigma \in G \mid \tfrac{\sigma x}{x} \in 1 + \mathfrak{m}_v \text{ for any } x \in \mathcal{O}^\times_v\}$$

be its kernel, the "inertia" subgroup.

Let L be the function field of a d-dimensional variety over k, $I \subseteq \{1, \dots, r\}$ be a subset and $\mathcal{O}_v \in \mathcal{P}^r_F$, $p \in \mathcal{P}^{|I|}_L$. Let $O_{p,v,I}$ be the set of all embeddings $\sigma : L \overset{/k}{\hookrightarrow} F$ such that $\sigma^{-1}(\mathcal{O}_v) = \mathcal{O}_p$ and $\sigma(L^\times) \cap \Gamma_i \neq \sigma(L^\times) \cap \Gamma_{i-1}$ if and only if $1 \leq i \leq r$ and $i \in I$.

Proposition 2.4.2. *If $m := \max(0, r+d-n) \leq |I| \leq M := \min(d, r)$ then $O_{p,v,I}$ is a non-empty G_v-orbit. The set $\{L \overset{/k}{\hookrightarrow} F\}$ of embeddings of L into F over k is a disjoint union of $O_{p,v,I}$. In particular, $\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]_{G^\dagger_v} = \bigoplus_{s=m}^M \bigoplus_{p \in \mathcal{P}^s_L} \mathbb{Q}[\{\kappa(p) \overset{/k}{\hookrightarrow} \kappa(v)\}]^{\binom{r}{s}}$ and $G_v \backslash \{L \overset{/k}{\hookrightarrow} F\} = \coprod_{s=m}^M (\mathcal{P}^s_L)^{\amalg \binom{r}{s}}$. If $|I| = r$ then the stabilizers of $O_{p,v,I}$ are isomorphic to $\widehat{\mathbb{Z}}^r \times G_{\kappa(v)|\kappa(p)}$ (and they are compact if $d = n$).*

Valuations and maximal subgroups. Example. If $n = 1$ and C is a smooth proper curve over k then to any valuation v the decomposition $\{k(C) \overset{/k}{\hookrightarrow} F\} = C(F) \smallsetminus C(k) = \coprod_{C(k)} (\mathfrak{m}_v \smallsetminus \{0\})$ is associated.

Proposition 2.4.3. *For any $\mathcal{O}_v \in \mathcal{P}^1_F$ the subgroup G_v is maximal among the closed subgroups of G.*

In the proof one checks that for any pair of distinct $\mathcal{O}_v, \mathcal{O}'_v \in \mathcal{P}^1_F$ the subgroup, generated by G_v and $G_{v'}$, is dense in G , i.e. it acts transitively on $\{L \overset{/k}{\hookrightarrow} F\}$ for any $L|k$ of finite type. The problem can be reduced to the case of $n = 1$, where v and v' are interpreted as compatible collections of points on the "universal tower" of curves over k. Then it remains to show that for any pair of distinct points $p, q \in C(k)$ on a smooth proper curve C over k there exist a level C_β of this tower and a morphism from C_β to C, sending the pair (v_β, v'_β) to (p, q).

Lemma 2.4.4. *The group $G_v^1 := \{\sigma \in G_v \mid \sigma x/x \in 1+\mathfrak{m}_v$ for any $x \in F^\times\} \subset G_v^\dagger$ is discrete if $n < \infty$.*

If $n = r = 1$ define $\varphi : G_v^1 \longrightarrow \Gamma \cup \{+\infty\}$ by $\varphi(\sigma) = v(\sigma x/x - 1)$ for any $x \in \mathfrak{m}_v \smallsetminus \{0\}$, or $x \in F \smallsetminus \mathcal{O}_v$. Clearly, φ is independent of x and determines a bounded non-archimedian bi-invariant distance on G_v^1. The logarithmic distance transforms the adjoint G_v-action on G_v^1 to the natural G_v-action on $\Gamma \cong \mathbb{Q}$. Using the Puiseux series, it is not hard to show that the self-map of G_v^1, $\sigma \mapsto \sigma^N$, is injective for any $N \in \mathbb{N}$.

Let $G_v^1(\beta) := \{\sigma \in G_v^1 \mid \varphi(\sigma) \geq \beta\}$, where $\beta \in \Gamma \otimes \mathbb{R}$. This is a normal subgroup in G_v°. Then $G_v^1 = G_v^1(0) = G_v^1(0)^+$, where $G_v^1(\beta)^+ := \bigcup_{\gamma>\beta} G_v^1(\gamma) = \{\sigma \in G_v^1 \mid \varphi(\sigma) > \beta\}$. Clearly, $G_v^1(\beta) \neq G_v^1(\gamma)$, if $\beta \neq \gamma$. The group $G_v^1(\beta)$ is "very unipotent". For instance, there is a canonical isomorphism $G_v^1(\beta)/G_v^1(\beta)^+ \xrightarrow{\sim} \mathrm{Hom}(\Gamma, \mathfrak{m}^{[\beta]})$, where

$$\mathfrak{m}^{[\beta]} = \{x \in \mathfrak{m} \mid v(x) \geq \beta\}/\{x \in \mathfrak{m} \mid v(x) > \beta\} \cong \begin{cases} k, & \text{if } \beta \in \Gamma \text{ and } \beta > 0 \\ 0, & \text{otherwise} \end{cases}$$

Lemma 2.4.5. *$G_v^1(\beta)$ is surjective over G/G° for any $\beta \in \Gamma \otimes \mathbb{R}$.*

2.5. **A "dense" locally compact "subgroup" $\mathfrak{G}$ of G.** It is well-known ([14, 32, 44, 13]) that in the case of algebraically closed F the group G is locally compact if and only if $n < \infty$.

Let $n = \infty$ and $\{x_1, x_2, \dots\}$ be a transcendence base of $F|k$. Set $L_m := k(x_m, x_{m+1}, \dots) \subset F$. Then $L_\bullet = (L_1 \supset L_2 \supset L_3 \supset \dots)$ is a descending sequence of subfields in F. Set $\mathfrak{G} = \mathfrak{G}_{L_\bullet} := \bigcup_{m\geq 1} G_{F|L_m}$. We take the set $\{G_{F|LL_1}\}$ of subgroups for all subfields L in $F|k$ of finite type as a base of open subgroups.

Geometrically (in a sense, analogous to §3.4), this corresponds to an inverse system of infinite-dimensional irreducible k-varieties given by finite systems of equations. They are related by dominant morphisms affecting only finitely many coordinates.

Then

- $\mathfrak{G}$ is locally compact (since F is algebraic over L_1), but is not unimodular;
- the inclusion $\mathfrak{G}$ into G is continuous with dense image (since $\bigcap_{m\geq 1} \overline{L_m} = k$).

To describe the modulus $\chi := \chi_{\mathfrak{G}} : \mathfrak{G} \longrightarrow \mathbb{Q}_+^\times$, for each $\sigma \in G_{F|L_m} \subseteq \mathfrak{G}$ choose a subfield L in $F|L_m$ of finite type, over which F is algebraic, e.g., generated over L_m by a transcendence base of $F|L_m$. Then $[G_{F|L}] = [L\sigma(L) : L^{\mathrm{insep}}] \cdot [G_{F|L\sigma(L)}]$ and $[G_{F|\sigma(L)}] = [L\sigma(L) : \sigma(L)^{\mathrm{insep}}] \cdot [G_{F|L\sigma(L)}]$ for any $\sigma \in G$, where $-^{\mathrm{insep}}$ is the purely inseparable closure in $L\sigma(L)$. Therefore, $\chi(\sigma) = \frac{[L\sigma(L):\sigma(L)^{\mathrm{insep}}]}{[L\sigma(L):L^{\mathrm{insep}}]}$.

For any integer $q > 1$ there is an element of $\mathfrak{G}$, leaving fixed all the elements of a transcendence base of $F|k$, except one, t, on which its acts as $t \mapsto t^q - t$. Thus, χ is surjective for any $1 \leq n \leq \infty$.

In particular, the group $\mathfrak{G}$ is compactly generated for neither $1 \leq n \leq \infty$, since otherwise the value group of the modulus was finitely generated, which is not the case for $\mathbb{Q}_+^\times$.

Examples of smooth semi-linear representations of $\mathfrak{G}$. We say that subsets I and J of $\mathbb{N}$ are commesurable if $I \smallsetminus (I \cap J)$ is finite and $|I \smallsetminus (I \cap J)| = |J \smallsetminus (I \cap J)|$. Denote by $[I]$ the class of subsets in $\mathbb{N}$, commesurable with the subset I. This is a countable set.

Define $\Omega_{F|k}^{[I]}$ as an F-vector space with the base $\{dx_{j_1} \wedge dx_{j_2} \wedge dx_{j_3} \wedge \dots \mid J \in [I]\}$, where $J = (j_1 < j_2 < j_3 < \dots)$. The group $\mathfrak{G}$ acts naturally on $\Omega_{F|k}^{[I]}$. If I is finite of cardinality q then we get the representation $\Omega_{F|k}^q$. If $I = \mathbb{N}$ then we get a representation of degree 1. If $J = \mathbb{N} \smallsetminus I$ then there is a non-degenerate pairing $\Omega_{F|k}^{[I]} \otimes_F \Omega_{F|k}^{[J]} \longrightarrow \Omega_{F|k}^{[\mathbb{N}]}$, natural if some $I \in [I]$ is fixed. It follows from [36, Lemma 7.7] that the semi-linear representation $\Omega_{F|k}^{[I]}$ is irreducible.

Let M be the set of all self-maps f of $\mathbb{N}$ such that $\lim_{m\to\infty} f(m) = \infty$. Define $\Omega_{F|k}^M$ as an F-vector space with the base $\{dx_{f(1)} \otimes dx_{f(2)} \otimes dx_{f(3)} \otimes \dots \mid f \in M\}$. The $\mathfrak{G}$-action is defined naturally. Define $\Omega_{F|k}^{[f]}$ as an F-vector subspace in $\Omega_{F|k}^M$, spanned by the $\mathfrak{G}$-orbit of $dx_{f(1)} \otimes dx_{f(2)} \otimes dx_{f(3)} \otimes \dots$

2.6. **Automorphisms of G.** The group G is quite rigid in the sense that the group of its continuous automorphisms is "of the same size" as G. Namely, it coincides with the group of field automorphisms of F preserving the algebraically closed subfield k. If $n > 1$ this follows from a stronger result of F.A.Bogomolov: any isomorphism between absolute Galois groups (and even between their maximal pro-p-quotients by the second term of the lower central series) of the function fields of k-varieties of dimension >1 is induced by an isomorphism between these function fields. The principal part of the proof in this case consists of checking that all abelian subgroups of rank >1 in the absolute Galois groups are contained in decomposition subgroups of various valuations, see [8, 9] and also [10].

If $n = 1$ this is shown in [40]. The general idea is as follows. The open compact subgroups of G° is the same as the absolute Galois group of function fields of curves over k with a marked F-rational generic point. This curves can be described functorially in terms of the topological group G° as projective schemes over $\mathbb{Q}$. E.g., the absolute Galois group of function fields of rational curves over k is the same as the open compact subgroups U such that $N_G(U)/U$ is infinite and has no abelian subgroups of finite index. In this case the decomposition subgroups in U^{ab} are parametrized by the set of parabolic subgroups P in $N_G(U)/U \cong \mathrm{PGL}_2 k$: the subgroup $D_P \cong \widehat{\mathbb{Z}}(1)$ consists of elements of U^{ab}, fixed under the adjoint P-action. For an arbitrary open compact subgroup $U = G_{F|L}$ the decomposition subgroups in U^{ab} can be described as the maximal subgroups in the closure of the additive envelope of the images of the transfers $G^{\mathrm{ab}}_{F|k(x)} \longrightarrow G^{\mathrm{ab}}_{F|L}$ for all $x \in L \smallsetminus k$, whose projections to $G^{\mathrm{ab}}_{F|k(x)}$ (with respect to the embedding $G^{\mathrm{ab}}_{F|L} \longrightarrow \prod_{x\in L\smallsetminus k} G^{\mathrm{ab}}_{F|k(x)}$, induced by inclusions $G_{F|L} \subseteq G_{F|k(x)}$) are subgroups of finite index in some decomposition subgroups in $G^{\mathrm{ab}}_{F|k(x)}$.

This implies that an automorphism of G° induces an automorphism of the whole collection of curves, i.e., an automorphism of F. A slightly more general statement looks as follows.

Theorem 2.6.1 ([40], 4.2)**.** *Let $n = 1$, and H be a subgroup of G, containing G°. Then $N_{G_{F|\mathbb{Q}}}(H)$ is contained in the group $N_{G_{F|\mathbb{Q}}}(G)$ of automorphisms of F, preserving k, and the adjoint action of $N_{G_{F|\mathbb{Q}}}(H)$ on H induces an isomorphism of $N_{G_{F|\mathbb{Q}}}(H)$ onto the group of continuous open automorphisms of H. If $H \supseteq \ker\chi$ then $N_{G_{F|\mathbb{Q}}}(H) = N_{G_{F|\mathbb{Q}}}(G^{\circ})$.*

3. General properties of smooth representations of G and their realizations

Before discussing the representations of G, let us make some general remarks on the category $\mathcal{S}m_H(E)$ of smooth E-representations of an arbitrary totally disconnected topological group H.

0. It is well-known (cf., e.g., [43, Exposé IV, §2.4–2.5] or [19, §8.1, Example 8.15 (iii)]), that the smooth H-sets and their H-equivariant maps form a topos.

Let $\mathfrak{T} = \mathfrak{T}(H, B)$ be a category, whose objects are the elements of some base B of open subgroups of H and $\mathrm{Hom}_{\mathfrak{T}}(U, V) = \{h \in H \mid hUh^{-1} \supseteq V\}/U$. The composition is defined in the natural way. We endow $\mathfrak{T}$ with the maximal topology, i.e., we assume that any sieve is covering. Then the sheaves of sets, groups, etc. are identified with the smooth H-sets, groups, etc.: $\mathcal{F} \mapsto \varinjlim_{U\in B} \mathcal{F}(U)$ (this is a smooth H-set, since its arbitrary element belongs to the image of some $\mathcal{F}(U)$, and, by definition, the U-action on it is trivial) and $W \mapsto (U \mapsto W^U)$.

E.g., if $B = \{1\}$ (and in particular, H is discrete) then there is a unique object $*$ in $\mathfrak{T}$, and $\mathrm{Hom}_{\mathfrak{T}}(*, *) = H$.

Let B be the set of open subgroups in G of type $G_{F|L}$ (where $L|k$ is an extension of finite type), and $\mathfrak{T} = \mathfrak{T}(G, B)$. Then $\mathrm{Hom}_{\mathfrak{T}}(G_{F|L}, G_{F|K}) = \{h \in G \mid h(L) \subseteq K\}/G_{F|L} = \{h : L \overset{/k}{\hookrightarrow} K\}$ is the set of field embeddings over k. When $n = \infty$, Lemma 3.4.1 describes the smooth G-sets as sheaves on a slightly different site, in the "dominant topology".

1. a) There are enough injectives in the category $\mathcal{S}m_H(E)$. Namely, the forgetful functor $\mathcal{S}m_H(E) \longrightarrow \mathrm{Vec}_E$ admits a right adjoint I: for any E-vector space V define $I(V)$ as the smooth part of the module of V-valued functions on H, i.e., $I(V) := \varinjlim_{U\in B} \mathrm{Maps}(H/U, V)$.

The group H acts on $I(V)$ by translations of the argument. It follows from the semi-simplicity of the category of E-vector spaces Vec_E that $I(V)$ is injective. If V is a smooth H-module then there is an H-equivariant embedding $V \longrightarrow I(V)$, $v \mapsto (h \mapsto hv)$.

b) The objects $E[H/U]$, where $U \in B$, form a generating system of $\mathcal{S}m_H(E)$, i.e., any smooth cyclic E-representation of H is a quotient of $E[H/U]$ for some $U \in B$. There are $\le 2^{\max(|H/U|,|E|)}$ quotients of the representation $E[H/U]$. Thus, there are $\le \max(|B|, \sup\limits_U 2^{\max(|H/U|,|E|)})$ cyclic E-representations of H. In the case $H = G$ we get the bound $\le 2^{\max(|k|,|E|)}$. If H is locally compact, but not unimodular, then there are $\ge \max(2^{\mathrm{rk}(\chi_H(H))}, |E|)$ irreducible representations. E.g., if $n < \infty$ then in the case of $H = G$ we get the bound $\ge \max(2^{|\mathbb{N}|}, |E|)$. Lower bounds in the case of $H = G$ can be found in Proposition 3.5.2.

c) There are direct sums, direct products, tensor products and the inner $\mathcal{H}om$ functor in the category $\mathcal{S}m_H(E)$. They are the smooth parts of the corresponding functors on Vec_E. Namely, the direct product of a family in $\mathcal{S}m_H(E)$ is the smooth part of its set-theoretic direct product, and $\mathcal{H}om(W_1, W_2) := \lim\limits_{U \longrightarrow} \mathrm{Hom}_{E[U]}(W_1, W_2)$, where U runs over open subgroups in H. The functor $\mathcal{H}om(W, -)$ is right adjoint to the functor $- \otimes_E W$:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{S}m_H(E)}(W_1 \otimes_E W, W_2) &= \mathrm{Hom}_{E[H]}(W_1 \otimes_E W, W_2) = \mathrm{Hom}_{E[H]}(W_1, \mathrm{Hom}_E(W, W_2)) = \\ &= \mathrm{Hom}_{E[H]}(W_1, \mathcal{H}om(W, W_2)) = \mathrm{Hom}_{\mathcal{S}m_H(E)}(W_1, \mathcal{H}om(W, W_2))\end{aligned}$$

for any $W_1, W_2, W \in \mathcal{S}m_H(E)$.

d) If $\varphi : H_2 \longrightarrow H_1$ is a homomorphism with a dense image then the pull-back functor $\varphi^{-1} : \mathcal{S}m_{H_1} \longrightarrow H_2$-mod is fully faithful. (*Proof.* Let $W_1, W_2 \in \mathcal{S}m_{H_1}$, $\alpha \in \mathrm{Hom}_{H_2}(\varphi^{-1}W_1, \varphi^{-1}W_2)$, $v \in W_1$ and $\sigma \in H_1$. Let S be the common stabilizer of the elements v and $\alpha(v)$. Choose some element $\sigma' \in \varphi^{-1}(\sigma S) \subseteq H_2$. Then $\alpha(\sigma v) = \alpha(\sigma' v) = \sigma' \alpha(v) = \sigma\alpha(v)$. $\Box$)

If φ is continuous then φ^{-1} factors through $\varphi^* : \mathcal{S}m_{H_1} \longrightarrow \mathcal{S}m_{H_2}$.

If the homomorphism φ is continuous and with dense image then the functor φ^* admits a right adjoint $\varphi_* : W \mapsto \bigcup_U W^{U \times_{H_1} H_2}$, where U runs over open subgroups of H_1. In particular, φ^* preserves the irreducibility. (*Proof.* The H_1-action on $\varphi_* W$ is defined as follows. If $w \in W^{\varphi^{-1}(U)}$ and $\sigma \in H_1$ then $\sigma w := \sigma' w$, where $\sigma' \in H_2$ and $\varphi(\sigma') \in \sigma U$, which is independent of σ'. $\Box$)

EXAMPLE. The forgetful functor $\mathcal{S}m_G \longrightarrow \mathfrak{G}$-mod is fully faithful, preserves the irreducibility, factors through $r : \mathcal{S}m_G \longrightarrow \mathcal{S}m_{\mathfrak{G}}$, and r admits a right adjoint: $W \mapsto \bigcup_L \bigcap_{m \ge 1} W^{G_{F|LL_m}}$, where L runs over the set of all subfields of finite type in $F|k$.

2. If any open subgroup of H contains an open subgroup of infinite index (e.g., $H = G$ and $n = \infty$) then there are no non-zero projective objects in the category of smooth representations of H.

(*Proof.* Let W be a projective object in the category of smooth E-representations of H. Choose a generating system $\{e_j\}_{j \in J}$ of the representation W. This gives rise to a surjective homomorphism $\bigoplus_{j \in J} E[H/\mathrm{Stab}_{e_j}] \xrightarrow{\pi} W$. Fix an element $i_0 \in J$ and for each $j \in J$ fix an open subgroup U_j of infinite index in $\mathrm{Stab}_{e_j} \cap \mathrm{Stab}_{e_{i_0}}$. As W is projective, the composition of π with the surjection $\bigoplus_{j \in J} E[H/U_j] \longrightarrow \bigoplus_{j \in J} E[H/\mathrm{Stab}_{e_j}]$ splits, and therefore, there exists an element in $\bigoplus_{j \in J} E[H/U_j]$ with the same stabilizer as e_{i_0}. However, as $E[H/U_j]^{\mathrm{Stab}_{e_{i_0}}} = 0$, this would imply that $e_{i_0} = 0$, and thus, $W = 0$. $\Box$)

3. a) If H is locally compact (e.g., $H = \mathfrak{G}$, cf. §2.5, for any $n \le \infty$) then the category $\mathcal{S}m_H(E)$ has enough projectives. Namely, any smooth H-module is a quotient of a direct sum of objects of type $E[H/U]$ for some open compact subgroups U of H.

However, the G-modules $E[G/U]$ are too complicated, cf. §3.5. Besides that (Proposition 3.5.2), there are "too many" ($\ge \max(2^{|\mathbb{N}|}, |k|, |E|)$) smooth irreducible representations of G for any $1 \le n \le \infty$. (This is one of the reasons to study rather $\mathcal{I}_G$ from the following §4, where the objects are supposed to be more controllable than in $\mathcal{S}m_G(E)$, since it is expected that they are of "cohomological nature", cf. §1.1.)

b) The category $\mathcal{A}dm_H(E)$ of admissible representations of any totally disconnected group H is closed under extensions and under the passages to subobjects in $\mathcal{S}m_H(E)$.

If H is locally compact then $\mathcal{A}dm_H(E)$ is a Serre subcategory in $\mathcal{S}m_H(E)$.

4. Representation theory of locally compact groups is largely determined by representation theory of Hecke algebras. Though, let H be arbitrary.

Define $\mathbb{D}_E(H) := \lim\limits_{\longleftarrow U} E[H/U]$, where E is a characteristic-zero field and the projective system is formed with respect to the projection $E[H/V] \xrightarrow{r_{VU}} E[H/U]$ and $H/V \longrightarrow H/U$, induced by the inclusions $V \subset U$ of open subgroups of H. For any $\nu \in \mathbb{D}_E(H)$, $\sigma \in H$ and an open subgroup U let $\nu(\sigma U)$ be the $[\sigma U]$-coefficient in the image of ν in $E[H/U]$. Clearly, any continuous homomorphism $H \longrightarrow H'$ induces a homomorphism of algebras $\mathbb{D}_E(H) \longrightarrow \mathbb{D}_E(H')$.

For each smooth E-representation W of H define a pairing $\mathbb{D}_E(H) \times W \longrightarrow W$ by $(\nu, w) \longmapsto \sum_{\sigma \in H/U} \nu(\sigma U) \cdot \sigma w$, where U is an arbitrary open subgroup in the stabilizer of w, e.g., $U = \mathrm{Stab}_w$. Clearly, the result is independent of the choice of U. This determines a $\mathbb{D}_E(H)$-module structure on W. When $W = E[H/U]$, this pairing is compatible with the projections r_{VU}, so it gives rise to a pairing $\mathbb{D}_E(H) \times \lim\limits_{\longleftarrow U} E[H/U] \longrightarrow \lim\limits_{\longleftarrow U} E[H/U] = \mathbb{D}_E(H)$, and thus, an associative multiplication $\mathbb{D}_E(H) \times \mathbb{D}_E(H) \xrightarrow{*} \mathbb{D}_E(H)$, extending the convolution of the compactly supported measures. (The *support* of ν is the minimal closed subset S in the semi-group $\lim\limits_{\longleftarrow U} H/U$ such that $\nu(\sigma U) = 0$ for σU, that does not meet S.)

The Hecke algebra of a pair (H, U), where U is a compact subgroup in H, is the subalgebra $\mathcal{H}_E(H, U) := h_U * \mathbb{D}_E(H) * h_U$ in $\mathbb{D}_E(H)$ of U-biinvariant measures. Here h_U is the Haar measure on U, defined by the system $(h_U)_V = [U : U \bigcap V]^{-1} \sum_{\sigma \in U/U \bigcap V} [\sigma V] \in \mathbb{Q}[H/V]$ for all open subgroups $V \subset H$. h_U is the unity of the algebra $\mathcal{H}_E(H, U)$ and $h_U h_{U'} = h_U$ for a closed subgroup $U' \subseteq U$. For any smooth E–representation W of H the Hecke algebra $\mathcal{H}_E(H, U)$ act on W^U, since $W^U = h_U(W)$.

When H is locally compact, and U is open and compact, this definition of the Hecke algebra is equivalent to the usual one, and for each smooth E-representation W of H the Hecke algebra act on W^U in the usual way, cf. [6].

In the case of $H = G$ and $n < \infty$ the Hecke algebras become the algebras of non-degenerate correspondences on some n-dimensional k-varieties, cf. §3.1, p.27.

5. The smooth representations of any compact group are semi-simple. Let $U \subseteq H = G$ be a compact subgroup, ρ be a non-zero smooth irreducible representation of U over $\mathbb{Q}$, and $\sigma \in H$ be an element such that $\sigma^{-1} H \sigma \subseteq H$. Define the representation ρ^σ of U by $(\tau, u) \mapsto (\sigma^{-1}\tau\sigma)u$ for all $\tau \in U$ and $u \in \rho$. Let W be a smooth representation of G. Then the multiplicity $m_W(\rho) \geq 0$ of ρ in W is equal to $\frac{\dim_{\mathbb{Q}} \mathrm{Hom}_U(\rho, W)}{\dim_{\mathbb{Q}} \mathrm{End}_U(\rho)}$. It is finite, if W is admissible. Define an embedding $\mathrm{Hom}_U(\rho, W) \hookrightarrow \mathrm{Hom}_U(\rho^\sigma, W)$ by $\lambda \mapsto \sigma\lambda$: $\begin{array}{ccc} u & \stackrel{\sigma\lambda}{\longmapsto} & \sigma\lambda(u) \\ \downarrow \tau & & \downarrow \tau \\ (\sigma^{-1}\tau\sigma)u & \stackrel{\sigma\lambda}{\longmapsto} & \tau\sigma\lambda(u) \end{array}$. In particular, if $m_W(\rho) \neq 0$, then also $m_W(\rho^\sigma) \neq 0$. Clearly, $\mathrm{End}_U(\rho) \subseteq \mathrm{End}_U(\rho^\sigma)$. Therefore, $m_W(\rho) = m_W(\rho^\sigma)$, if $\sigma^{-1} H \sigma = H$.

Note, that for any pair of compact subgroups $U_1, U_2 \subset G$ any pair of their smooth representations intertwine in the usual sense, i.e., there exist $g \in G$ and a non-zero $\varphi \in \mathrm{Hom}(\rho_1, \rho_2)$ such that if $gk_1 = k_2 g$ and $k_i \in U_i$, $i \in \{1, 2\}$ then $\varphi\rho_1(k_1) = \rho_2(k_2)\varphi$. Namely, there exists $g \in G$ such that $\rho_1|_{U_1 \cap g^{-1} U_2 g}$ and $\rho_2|_{U_2 \cap g U_1 g^{-1}}$ are trivial (if ρ_i are of finite length). This can be explained by the fact that the representation F (and therefore, the irreducible representation F/k) of G contains all smooth irreducible (and thus, finite-dimensional) representations of U_i (Hilbert's Satz 90). It is easy to show that the same holds for $F^\times/k^\times$.

It is shown in [35, Appendix A, Theorem A.4] that for any compact subgroup K in G the centres of the Hecke algebras $\mathcal{H}_E(K)$ and $\mathcal{H}_E(G^\circ, K)$ of the pairs (G, K) and (G°, K) (the definition of G° is in §1.3 of the introduction) coincide with $E \cdot h_K$, if $n < \infty$, i.e., consist of scalars. This is a negative result, especially, compared to the analogous questions for p-adic groups.

In some cases the morphism groups between geometric objects can be identified with the morphism groups between the corresponding G-modules (cf. Propositions 3.1.5 and 3.3.2, and Corollary 3.1.6).

We establish (in Corollary 3.4.9) that the cohomological dimensions of $\mathcal{S}m_G$ and of $\mathcal{C}$ (of smooth semi-linear representations of G) are infinite when $n = \infty$.

By analogy with the Langlands correspondences, one can call the irreducible representations of G in the image of the functor $\mathfrak{B}^n$ *cuspidal*, where $\mathfrak{B}^n$ is a functor on the category of primitive n-motives, defined (in a greater generality) in §4. For groups GL over local non-archimedian fields there are several equivalent definitions of quasi-cuspidal representations. One of them (finiteness): the supports of all *matrix coefficients* of a smooth representation W of a topological group (i.e., the functions on this group of type $\langle \sigma w, \widetilde{w}\rangle$ for some vector $w \in W$ and a vector with an open stabilizer $\widetilde{w}$ in the dual representation) are compact modulo centre. However, it is shown in [35, Proposition 4.6] that for $n < \infty$ any such representation of any subgroup of G, containing G°, is zero. This is deduced from the irreducibility of smooth representations F/k and/or $F^\times/k^\times$ of the subgroup G° of G, and their faithfulness as modules over the corresponding algebras of measures on G: for any $1 \le n \le \infty$ the annihilator of F/k in $\mathbb{D}_k$ as well as the annihilator of $F^\times/k^\times$ in $\mathbb{D}_{\mathbb{Q}}$ are trivial ([35, Proposition 4.2]).

In the case $n = \infty$ one can establish some analogues of Hilbert's Satz 90. In particular, as it follows from Corollary 3.4.8, any smooth G-torsor under a smooth G-group $B(F)$ is trivial for any algebraic k-group B. However, there are interesting examples of torsors in the case $n < \infty$.

According to Proposition 1.1.9, $\mathrm{Ext}^1_{\mathcal{S}m_G(\mathbb{Q})}(A(F)/A(k), \mathbb{Q}) = \mathrm{Hom}(A(k), \mathbb{Q})$ if $n = \infty$ and A is an irreducible commutative algebraic k-group. If A is an abelian variety then $A(F)/A(k) = \mathfrak{B}^1(A^\vee)$ (here $A^\vee := \mathrm{Pic}^\circ A$ is the dual abelian variety), where $\mathfrak{B}^1$ is a functor on the category of primitive 1-motives, defined (in a greater generality) in §4. Therefore, it is natural to compare this equality with the identity $\mathrm{Ext}^1_{\mathcal{MM}_k}(\mathbb{Q}(0), H_1(A)) = A(k)_{\mathbb{Q}}$ in the category of mixed motives over k.

If $A = \mathbb{G}_m$ then the identity $\mathrm{Ext}^1_{\mathcal{MM}_k}(\mathbb{Q}(0), \mathbb{Q}(1)) = k^\times \otimes \mathbb{Q}$ suggests that the smooth representation $F^\times/k^\times$ of G may admit a motivic interpretation, analogous to $\mathbb{Q}(1)$, though it is not admissible.

The last section 3.5 contains examples of pairs of distinct extensions of finite type L_1 and L_2 of k with the same collections $\mathrm{JH}(L_1)$ and $\mathrm{JH}(L_2)$ of irreducible subquotients of representations $\mathbb{Q}[\{L_1 \overset{/k}{\hookrightarrow} F\}]$ and $\mathbb{Q}[\{L_2 \overset{/k}{\hookrightarrow} F\}]$. In two of these examples the primitive motives of maximal level of models of L_1 and of L_2 are trivial. In one more example $L_1 = k(X)$ and $L_2 = k(\mathbb{P}^{\dim X})$, where X is a product of generically twofold covers of projective spaces (e.g., hyperelliptic curves) over k, at least one of which is a curve of genus ≤ 1. Thus, it is not excluded that $\mathrm{JH}(L)$ depends only on $\mathrm{tr.deg}(L|k)$.

Let us first generalize the fact that there are no finite-dimensional non-trivial smooth representations of G if $n = \infty$.

Proposition 3.0.2. *Let $W \in \mathcal{S}m_G(E)$, and for some subfields $\overline{L_1} \subsetneqq \overline{L_2}$ in $F|k$ the subspaces $W^{G_{F|\overline{L_1}}}$ and $W^{G_{F|\overline{L_2}}}$ be finite-dimensional and non-zero. Then $W^G = W^{G_{F|\overline{L_1}}} \neq 0$. In particular, if $n = \infty$ and $\dim_E W^{G_{F|\overline{L}}} < \infty$ for any L of sufficiently big finite transcendence degree over k then W is trivial.*

Proof. The representation $W^{G_{F|\overline{L_1}}}$ of $G^\circ_{\overline{L_1}|k}$ and the representation $W^{G_{F|\overline{L_2}}}$ of $G^\circ_{\overline{L_2}|k}$ are trivial. Therefore, any vector $w \in W^{G_{F|\overline{L_1}}}$ is fixed under the action of the subgroup H in G, generated by the subgroups $G_{F|\overline{L_1}}$ (in $G_{\{F,\overline{L_1}\}|k} \times_{G_{\overline{L_1}|k}} G^\circ_{\overline{L_1}|k}$) and $G_{\{F,\overline{L_2}\}|k} \times_{G_{\overline{L_2}|k}} G^\circ_{\overline{L_2}|k}$. It follows from Lemma 2.2.4 that this subgroup H coincides with G. □

3.1. **Hecke algebras and correspondences.** Let $\mathcal{H}_E(U) := \mathcal{H}_E(G, U) = h_U * \mathbb{D}_E * h_U$.

Proposition 3.1.1. (1) *Let H be a totally disconnected topological group and T be a filtering family of its compact subgroups, i.e. such that any open subgroup contains an element of T.*

Then a smooth E-representation W of H is irreducible, resp. semi-simple, if and only if the $\mathcal{H}_E(H,U)$-module W^U is irreducible, resp. semi-simple, for each compact subgroup $U \in T$.

EXAMPLE. Let $H = G$ and T consists of compact subgroups U with F^U purely inseparable over a purely transcendental extension of an extension of k of finite type.

(2) *Let W_j for $j = 1, 2$ be smooth irreducible E-representations of H and $W_1^U \neq 0$ for some compact subgroup U. Then W_1 is equivalent to W_2 if and only if the $\mathcal{H}_E(H,U)$-modules W_1^U and W_2^U are equivalent.*

(3) *For each open compact subgroup $U \subset H$ and each irreducible E-representation τ of the algebra $\mathcal{H}_E(H,U)$ there is a smooth irreducible representation W of H with $\tau \cong W^U$.*

Proofs from [6, Proposition 2.10] and [35, Lemma 3.2] go through with almost no modifications.

Remark. It would be natural to replace the semi-simplicity or irreducibility conditions for representations of Hecke algebras from Proposition 3.1.1 by the corresponding conditions on the representations of *groups* $G_{F'|k}$ for algebraically closed extensions $F'|k$ in F of finite transcendence degree. For a certain small (but important) class of representations this is done in Lemma 4.3.1. The following lemma is a very preliminary step in the general direction.

Lemma 3.1.2 ([18]). *Let $n = \infty$, $\mathcal{H}$ be a subcategory in $\mathcal{S}m_G$, closed under passages to subobjects, and $F'|k$ be an algebraically closed extension in F of finite transcendence degree. The following conditions on the subcategory $\mathcal{H}$ and F' are equivalent.*

(1) *For any $W \in \mathcal{H}$ any $G_{F'|k}$-submodule $U \subseteq W^{G_{F|F'}}$ coincides with the $G_{F'|k}$-submodule of $G_{F|F'}$-invariants in G-submodule, spanned by U: $U = \langle U \rangle_G^{G_{F|F'}}$.*

(2) *For any $W \in \mathcal{H}$ any surjection $\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]^N \longrightarrow W$ in $\mathcal{S}m_G$ induces a surjection of $G_{F|F'}$-invariants $\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F'\}]^N \longrightarrow W^{G_{F|F'}}$ in $\mathcal{S}m_{F'|k}$, where $F' = \overline{L}$.*

(3) *For any extension $L|k$ in F of finite type, where $F' = \overline{L}$, and any $Q \subseteq \mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]^N$ such that the quotient belongs to $\mathcal{H}$, one has $H^1_{\mathcal{S}m_G}(G_{F|F'}, Q) = 0$.*

For any irreducible variety Y over k with the function field $k(Y) = F^U$ for a compact open subgroup U in G one can identify the Hecke algebra $\mathcal{H}_{\mathbb{Q}}(U)$ with the $\mathbb{Q}$-algebra of non-degenerate correspondences on Y (i.e., of formal linear combinations of n-subvarieties in $Y \times_k Y$ dominant over both factors Y). This follows from the following Lemma and the facts that

- the set of double classes $U \backslash G/U$ can be identified with a basis of $\mathcal{H}_{\mathbb{Q}}(U)$ as a $\mathbb{Q}$-space via $[\sigma] \longmapsto h_U * \sigma * h_U$;
- that irreducible n-subvarieties in $Y \times_k Y$ dominant over both factors Y are in a natural bijection with the set of maximal ideals of the algebra $F^U \otimes_k F^U$.

Lemma 3.1.3 ([35], 3.3). *Let $L, K \subseteq F$ be field subextension of k with* $\mathrm{tr.deg}(L|k) = q < \infty$. *Then the set of double classes $G_{F|K} \backslash G/G_{F|L}$ is canonically identified with the set of all points in $\mathbf{Spec}(L \otimes_k K)$ of codimension $\geq q - \mathrm{tr.deg}(F|K)$ (so $G_{F|K} \backslash G/G_{F|L} = \mathbf{Max}(L \otimes_k K)$, if $F = \overline{K}$). Here $G/G_{F|L}$ is the set of all embeddings of L into F over k.*

Let $A^q(Y)$ be the quotient of the $\mathbb{Q}$-vector space $Z^q(Y)$ of cycles on a smooth proper variety Y over k of codimension q by the $\mathbb{Q}$-vector subspace $Z^q_\sim(Y)$ of cycles $\sim$-equivalent to zero for an adequate equivalence relation $\sim$. According to Hironaka theorem on resolution of singularities, each smooth variety X admits an open embedding i into a smooth proper variety $\overline{X}$ over k. Then $A^q(-)$ can be extended to arbitrary smooth variety X as the cokernel of the map $Z^q_\sim(\overline{X}) \xrightarrow{i^*} Z^q(X)$ induced by restriction of cycles. This is independent of the choice of variety $\overline{X}$.[3]

[3]since for any pair of smooth compactifications $(\overline{X}, \overline{X}')$ of X there is their common refinement $\overline{X} \xleftarrow{\beta} \overline{X}'' \xrightarrow{\beta'} \overline{X}'$, i^* factors through $Z^q_\sim(\overline{X}) \xrightarrow{\beta^*} Z^q_\sim(\overline{X}'') \xrightarrow{(i'')^*} Z^q(X)$ and $i^* Z^q_\sim(\overline{X}) = (i'')^* Z^q_\sim(\overline{X}'')$.

In the standard way one extends the contravariant functors $A^q(\ )$ and $Z^q(\ )$ to contravariant functors on the category of smooth pro-varieties over k. Namely, if for a set of indices I, an inverse system $(X_j)_{j\in I}$ of smooth varieties over k is formed with respect to flat morphisms and X is the limit, then $Z^q(X) = \lim\limits_{\overrightarrow{j\in I}} Z^q(X_j)$, where the direct system is formed with respect to the pull-backs, and similarly for $A^q(\ )$. Then $A^q(X)$ is the cokernel of $\bigoplus_{j\in I} Z^q_{\sim}(\overline{X_j}) \longrightarrow Z^q(X)$. This is independent of the choice of the projective system defining X.

In particular, as for any commutative k-algebra R the scheme $\mathbf{Spec}(R)$ is an inverse limit of a system of k-varieties, $A^q(R) := A^q(\mathbf{Spec}(R))$ is defined. Any automorphism α of the k-algebra R induces a morphism of a system $(X_j)_{j\in I}$ defining $\mathbf{Spec}(R)$ to a system $(\alpha^*(X_j))_{j\in I}$ canonically equivalent to $(X_j)_{j\in I}$, and therefore, induces an automorphism of $A^q(Y_R)$ for any k-scheme Y. This gives a contravariant functor from a category of varieties over k to the category of $\mathrm{Aut}(R|k)$-modules.

In what follows X will be of type Y_F for a k-subscheme Y in a variety over k.

The homomorphism of algebras $\mathcal{H}_{\mathbb{Q}}(U)(U) \longrightarrow A^{\dim Y}(Y\times_k Y)$ is surjective for any smooth projective Y, where $k(Y)\subset F$ and $U = G_{F|k(Y)}$. This can be seen from the following "moving lemma", applied in the case $X_1 = X_2 = Y$ and $Z = X_1\times_k X_2$. (Its present form is suggested by the referee of [35].)

Lemma 3.1.4 ([35], 3.4). *Let $Z, X_1, \dots, X_r$ be irreducible projective varieties over k, and let $Z \xrightarrow{p_j} X_j$ be surjective maps. Let $\alpha\subset Z$ be an irreducible subvariety of dimension at least $\max\limits_{1\le j\le r} \dim X_j$. Then α is rationally equivalent to a linear combination of some irreducible subvarieties in Z surjective (under the maps p_j) over all X_j's.*

In particular, the natural projection $Z^q(k(X)\otimes_k k(Y)) \longrightarrow CH^q(X\times_k Y)$ is surjective for $q\le \dim X\le\dim Y$; and $Z^q(k(Y)\otimes_k F) \longrightarrow CH^q(Y_F)$ is surjective for $q\le\dim Y\le n$.

Proposition 3.1.5 ([35], 3.6 $+\varepsilon$). *Let Y be a smooth irreducible proper variety over k and $\dim Y\le n$. Let X be a smooth variety over k, and W be a quotient G-representation of $A^q(X_F)$ for some $q\ge 0$. Then there are canonical isomorphisms $A^q(X_{k(Y)}) \xrightarrow{\sim} \mathrm{Hom}_G(A^{\dim Y}(Y_F), A^q(X_F))$ and $\mathrm{Hom}_G(A^{\dim Y}(Y_F), W) \xrightarrow{\sim} \mathrm{Hom}_G(Z^{\dim Y}(k(Y)\otimes_k F), W)$*

Proof uses Lemmas 3.1.4, 3.1.3 and elementary intersection theory.

Corollary 3.1.6 ([35], 3.7). *For any field L' of finite type and of transcendence degree $m\le n$ over k, any field L of finite type over k and any integer $q\ge m$ there is a canonical isomorphism $A^q(L\otimes_k L') \xrightarrow{\sim} \mathrm{Hom}_G(A^m(L'\otimes_k F), A^q(L\otimes_k F))$, where the both groups are zero if $q>m$.*

3.2. Invariants of subgroups and tensor products.

Lemma 3.2.1. *Let W be a smooth G-set, and L be an extension of k in F. Then $W^{G_{F|L}} = \bigcup_{L_0\subseteq L} W^{G_{F|L_0}}$, where L_0 runs over extensions of k of finite type.*

The proof is identical to the proof of [35, Lemma 6.1].

Proposition 3.2.2. *Let E be either F, or any characteristic zero field with the trivial G-action, and let W_1, W_2 be smooth semi-linear representations of G over E. Assume that either a subgroup $H\subseteq G$ admits no non-trivial smooth finite-dimensional semi-linear representations over E, or $\mathrm{tr.deg}(F|k)=\infty$ and $H = G_{F|\overline{L}}$ for a field extension L of k in F. Then one has $(W_1\otimes_E W_2)^H = W_1^H\otimes_{E^H} W_2^H$.*

(This is not true if $\mathrm{tr.deg}(F|k)<\infty$. Namely, if W_1 and W_2 are non-trivial mutually dual representations of G of degree one then $(W_1\otimes_E W_2)^G = E^G$, but $W_1^G = W_2^G = 0$.)

"Conversely", if for a subgroup $H\subset G$ one has $(W_1\otimes_E W_2)^H = W_1^H\otimes_{E^H} W_2^H$ then the field F^H is algebraically closed.

(This is a tiny generalization of [36, Lemma 7.5], the proof is similar.)

3.3. G- **and** G°**-modules of type** $A(F)/A(k)$**, where** A **is a commutative** k**-group, morphisms between them and a separable closure of a one-dimensional extension of** k**.**

Proposition 3.3.1. *If $n=1$ then the G°-orbit of x generates the separable closure K_x of $k(x)$ in F for any $x \in F \smallsetminus k$. More precisely, $K_x^\times/k^\times$ is an irreducible G°-module, and K_x/k is an irreducible G°-module if* $\mathrm{char}(k) \neq 2$.

The G°-modules F/k and $F^\times/k^\times$ are irreducible if either $\mathrm{char}(k)=0$*, or* $2 \le n \le \infty$.

Proof. Let A be the additive subgroup of F generated by the G°-orbit of some $x \in F \smallsetminus k$. It is shown in [35, Prop. 4.1] that if $\mathrm{char}(k) \neq 2$ then A is a subfield of F. Besides that, if M is the multiplicative subgroup of $F^\times$ generated by the G°-orbit of some $x \in F \smallsetminus k$ then $M \bigcup \{0\}$ is a G°-invariant subfield of F.

Clearly, $A = M\bigcup\{0\} = F$ if $n \ge 2$. If $n=1$ then $\mathrm{Gal}(F|\mathbb{Q}(G^\circ x))$ is a compact normal subgroup in G°, i.e., it is trivial by Theorem 2.2.1. Thus, the extension $F|\mathbb{Q}(G^\circ x)$ is purely inseparable.

Let us show that $k(G^\circ x)$ is a separable extension of $k(x)$. Equivalently, that if $\sigma^N x = x$ for some $N \ge 1$ then $k(x, \sigma x)$ is a separable extension of $k(x)$. Let $P(x, \sigma x)$ be a minimal polynomial. Then $P_I dx + P_{II} d(\sigma x) = 0 \in \Omega^1_{k(x,\sigma x)|k}$, where either $P_I \neq 0$, or $P_{II} \neq 0$ as otherwise $P = Q^p$ for another polynomial Q. If $P_{II} \neq 0$ then $k(x, \sigma x)$ is a separable extension of $k(x)$. If $P_I \neq 0$ then $k(x,\sigma x)$ is a separable extension of $k(\sigma x)$, and thus, $k(x, \sigma^{-1}x)$ is a separable extension of $k(x)$. Then the subfield, generated over k by $x, \sigma^{-1}x, \dots, \sigma^{-(N-1)}x = \sigma x$, is a separable extension of $k(x)$. ☐

Proposition 3.3.2 ([35], 3.6, 4.3)**.** *Let $1 \le n \le \infty$. Let A and B be reduced irreducible group schemes over k. Then the natural map* $\mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}_G(A(F), B(F))$*, where* $\mathrm{Hom}(A,B) := \mathrm{Hom}_{\text{group schemes}/k}(A,B)$ *and* Hom_G *is the set of G-homomorphisms, is bijective.*

Suppose that the k-groups A and B commutative and simple. Then

$$\mathrm{Hom}(A,B)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Hom}_G(A(F)/A(k), B(F)/B(k)) \xrightarrow{\sim} \mathrm{Hom}_{G^\circ}(A(F)/A(k), B(F)/B(k)).$$

Unfortunately, the proof of the second part consists of checking individual cases: A and B are simple abelian varieties, $\mathbb{G}_a$, or $\mathbb{G}_m$.

3.4. **The dominant topology, acyclicity of certain smooth representations of** G **and cohomological dimension (**[18]**).** In this section we are going to identify the smooth representations with the abelian sheaves on a "small" site, and interpret the smooth cohomology (i.e. $\mathrm{Ext}^*_{\mathcal{S}m_G}(\mathbb{Q}, -)$) as Čech cohomology of sheaves.

We shall assume that $F|k$ is an extension of algebraically closed fields of characteristic zero of countable transcendence degree.

Let $\mathfrak{D}m_k$ be the category of smooth morphisms of smooth k-schemes. We endow $\mathfrak{D}m_k$ with the pre-topology, where the coverings are the dominant morphisms.

Lemma 3.4.1 ([18], 1.1)**.** *The category of sheaves on $\mathfrak{D}m_k$ is equivalent to the category of smooth G-sets.*

To a sheaf one associates its "generic fibre", i.e., $\varinjlim_U \mathcal{F}(U)$, where U runs over the smooth integral k-varieties with the function field embedded into F over k.

Proposition 3.4.2. *Let the transcendence degree of an extension $F'|k$ in F be infinite. Then $H^{>0}_{\mathcal{S}m_G}(G_{F|\overline{F'}}, W) = H^{>0}_{\mathcal{S}m_G}(G_{F|F'}, W \otimes \mathbb{Q}) = 0$ for any smooth G-module W.*

Čech cohomology.

Lemma 3.4.3. *The complex* $(\cdots \to \mathbb{Q}[\{L(Y^2) \overset{/k}{\hookrightarrow} F\}] \to \mathbb{Q}[\{L(Y) \overset{/k}{\hookrightarrow} F\}] \to \mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}] \to 0)$ *is acyclic for any L-variety Y. If $L = k$ and Y is smooth and proper then the complexes* $\cdots \to C_{k(Y^2)} \to C_{k(Y)} \to \mathbb{Q} \to 0$ *and* $\cdots \to CH_0(Y^2)_{\mathbb{Q}} \to CH_0(Y)_{\mathbb{Q}} \to \mathbb{Q} \to 0$ *are also acyclic.*

Corollary 3.4.4. *For any $W \in \mathcal{I}_G$ the complex* $0 \to W^G \to W^{G_{F|k(Y)}} \to W^{G_{F|k(Y^2)}} \to \dots$ *is exact. In particular,* $0 \to CH^q(X)_{\mathbb{Q}} \to CH^q(X_{k(Y)})_{\mathbb{Q}} \to CH^q(X_{k(Y^2)})_{\mathbb{Q}} \to \dots$ *is exact.*

As $\mathcal{I}$ is right exact, we get from Lemma 3.4.3 the following

Corollary 3.4.5. *Sending the function field L of a k-variety to $\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}] \in \mathcal{S}m_G$, resp. to $C_L \in \mathcal{I}_G$, defines a sheaf on $\mathfrak{D}m_k$ with values in $\mathcal{S}m_G^{\mathrm{op}}$, resp. in $\mathcal{I}_G^{\mathrm{op}}$.* □

Denote by $\check{H}^*$ the Čech cohomology.

Corollary 3.4.6 ([29], Ch.III, Corollary 2.5)**.** *$\check{H}^*$ coincides with H^* for any sheaf if and only if $\check{H}^*$ transforms any short exact sequence of sheaves to a long exact sequence of Čech cohomologies.*

For any extension K of k in F with F of infinite transcendence degree over K fix a transcendence basis $\{x_1, x_2, x_3, \dots\} = \{x_1^{(K)}, x_2^{(K)}, x_3^{(K)}, \dots\}$ of F over K. For each $m \geq 0$ set $F_m = \overline{K(x_{2^m}, x_{2^m\cdot 3}, x_{2^m\cdot 5}, \dots)}$, and for each $j \geq 0$ fix a self-embedding σ_j of F over $\overline{K}$ such that $\sigma_j|_{F_s} = id$ if $j > s$ and $\sigma_j|_{F_s} = \sigma_s|_{F_s} : F_s \xrightarrow{\sim} F_{s+1}$ if $j \leq s$. For any extension L of K we fix $L_0 \subseteq F_0$ isomorphic to L over K and set $L_s := \sigma_0^s(L_0)$.

For a G-module W set ${}^L_K W^\bullet := (W^{G_{F|L_0}} \xrightarrow{\sigma_0-\sigma_1} W^{G_{F|L_0L_1}} \xrightarrow{\sigma_0-\sigma_1+\sigma_2} W^{G_{F|L_0L_1L_2}} \longrightarrow \dots)$,

Proposition 3.4.7. *$H^q_{\mathcal{S}m_G}(G_{F|K}, W) = H^q({}^F_K W^\bullet)$ for any smooth representation W of G and any algebraically closed extension $K \supseteq k$ in F.*

Acyclicity of "geometric" G-modules and cohomological dimension of $\mathcal{S}m_G$. As in the proof of Lemma 3.4.1, we associate to a presheaf $\mathcal{F}$ on $\mathfrak{D}m_k$ and a filtered union $\mathcal{O} = \varinjlim_U A$ of finitely generated smooth k-subalgebras A a smooth $\mathrm{Aut}(\mathcal{O}|k)$-set $\mathcal{F}(\mathcal{O}) := \varinjlim_A \mathcal{F}(\mathbf{Spec}(A))$. Our main examples of $\mathcal{O}$ will be F and $\mathcal{O}_v$.

Let us show that $\mathcal{F}(F)^{G_{F|F'}} = \mathcal{F}(F')$ for any $F' = \overline{F'} \subseteq F$ with $\mathrm{tr.deg}(F'|k) = \infty$.

(As it shows the example of the presheaf $\mathcal{F} : U \mapsto \Gamma(\overline{U}, \bigotimes^2_{\mathcal{O}} \Omega^1_{\overline{U}/k})$, the condition $F' = \overline{F'}$ is essential.)

Fix an isomorphism $\alpha : F \xrightarrow{\sim} F'$ over k. Then $\alpha^* : \varinjlim_U \mathcal{F}(U) \xrightarrow{\sim} \varinjlim_V \mathcal{F}(V)$, where $\mathcal{O}(U) \subset F$ and $\mathcal{O}(V) \subset F'$ are smooth. For any U there is $\sigma \in G$ such that $\sigma|_{\mathcal{O}(U)} = \alpha|_{\mathcal{O}(U)}$, so $\varinjlim_V \mathcal{F}(V) = \varinjlim_{(U,\sigma)} \sigma\mathcal{F}(U) = \mathcal{F}(F)^{G_{F|F'}}$, cf. Lemma 4.1.2.

Assume that $\mathcal{F}$ is endowed with transformations $i^x_{X,Y} : \mathcal{F}(X \times_k Y) \longrightarrow \mathcal{F}(Y)$ for any smooth X, Y and any $x \in X(k)$ such that $i^x_{X,Y} \circ {}_X\mathrm{pr}^*_Y = id_{\mathcal{F}(Y)}$ and $i^x_{X,Y\times_k Z} \circ {}_Z\mathrm{pr}^*_{X\times_k Y} = {}_Z\mathrm{pr}^*_Y \circ i^x_{X,Y}$, where ${}_X\mathrm{pr}_Y : X \times_k Y \longrightarrow Y$ is the projection.

Let $\mathcal{R}$ be a presheaf of commutative rings, and $\mathcal{F}$ be an $\mathcal{R}$-module. Then the representation $\mathcal{F}(F)$ is an $\mathcal{R}(F)$-module, and such representations (for a fixed $\mathcal{R}$ and for all $\mathcal{F}$) form a tensor category with respect to the operation $\otimes_{\mathcal{R}(F)}$. In particular, the category of representations of type $\mathcal{F}(F)$ for $\mathcal{F}$ taking values in commutative groups is tensor.

Corollary 3.4.8 ([18])**.** *$H^{>0}_{\mathcal{S}m}(G, W) = 0$ for any $W \in \mathcal{I}_G$; $H^1_{\mathcal{S}m}(G, \mathcal{F}(F)) = \{*\}$ for any group-valued $\mathcal{F}$; and $H^{>0}_{\mathcal{S}m}(G, \mathcal{F}(F)) = 0$ for any $\mathcal{F}$ with values in commutative groups.*

Examples of representations of type $\mathcal{F}(F)$ as in Corollary 3.4.8 are $A(F)$ for any group k-variety A, or $A(F)/A(k)$, if A is commutative, $CH_0(X_F)_{\mathbb{Q}}$, $\Omega^\bullet_{F|k,\mathrm{reg}}$, $\bigotimes^\bullet_F \Omega^1_{F|k_0}$, $\Omega^\bullet_{F|k_0,\mathrm{closed}}$, $\Omega^\bullet_{F|k_0,\mathrm{exact}}$, and $H^\bullet_{\mathrm{dR}/k_0}(F)$ for any $k_0 \subseteq k$. (Corresponding functors are: $A(\mathcal{O})$, or $A(\mathcal{O})/A(k)$, $CH_0(X \times_k (-))_{\mathbb{Q}}$, $\Gamma(\overline{-}, \Omega^\bullet_{\mathcal{O}|k})$, $\bigotimes^\bullet_{\mathcal{O}} \Omega^1_{\mathcal{O}|k_0}$, $\Omega^\bullet_{\mathcal{O}|k_0,\mathrm{closed}}$, $\Omega^\bullet_{\mathcal{O}|k_0,\mathrm{exact}}$, and $H^\bullet_{\mathrm{dR}/k_0}(\mathcal{O})$.)

Corollary 3.4.9 ([18])**.**
(1) *The categories $\mathcal{S}m_G$ and $\mathcal{C}$ (cf. §1.2) admit systems of acyclic generators $\{\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]\}_L$ and $\{F[\{L \overset{/k}{\hookrightarrow} F\}]\}_L$, where L runs over the subfields of finite type over k, containing any given extension of k of finite type.*
(2) *Cohomological dimensions of $\mathcal{S}m_G$ and $\mathcal{C}$ are infinite.*

$\mathbb{A}^1$-invariance of some presheaves. Let $\mathcal{V}_k$ be a category of k-varieties, containing all smooth varieties. Let $\mathcal{L}$ be a category, where all self-embeddings are isomorphisms (e.g., an abelian category such that for any object the multiplicities of its irreducible subquotients are finite[4]).

An $\mathcal{L}$-valued presheaf $\mathcal{F}$ on $\mathcal{V}_k$ is $\mathbb{A}^1$-*invariant*, if $\mathcal{F}(U) = \mathcal{F}(U \times \mathbb{A}^1)$ for any $U \in \mathcal{V}_k$.

Consider any pretopology on $\mathcal{V}_k$ such that $\mathbb{A}^1_k \longrightarrow \mathbf{Spec}k$, $(\mathbb{A}^1_k \smallsetminus \{0\}) \coprod (\mathbb{A}^1_k \smallsetminus \{1\}) \longrightarrow \mathbb{A}^1_k$ and $\mathbb{G}_{m,k} \longrightarrow \mathbb{G}_{m,k}$, $x \mapsto x^2$ are coverings (in particular, $\mathcal{F}(\mathbb{A}^1_X) \longrightarrow \mathcal{F}(\mathbb{G}_{m,X})$ is injective for any $X \in \mathcal{V}_k$ and any sheaf on $\mathcal{V}_k$).

Proposition 3.4.10 ([18]). *Any sheaf $\mathcal{F}$ on $\mathcal{V}_k$ with values in $\mathcal{L}$ is $\mathbb{A}^1$-invariant.*

For a proof one has to note that $\mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta_{\mathbb{A}^1}$ is isomorphic to its quotient by the permutation σ of the two multiples $\mathbb{A}^1$, and therefore, as open dense embeddings are covers, $\mathcal{F}(U \times \mathbb{A}^1 \times \mathbb{A}^1)$ embeds into the $\mathfrak{S}_2$-invariant part of $\mathcal{F}(U \times \mathbb{A}^1 \times \mathbb{A}^1)$, i.e., σ acts trivially. As $U \times \mathbb{A}^1 \longrightarrow U$ is a cover, $\mathrm{pr}_1 = \mathrm{pr}_2 \circ \sigma$, and $\mathcal{F}(U)$ is the equalizer of the injections $\mathrm{pr}_1^*, \mathrm{pr}_2^* : \mathcal{F}(U \times \mathbb{A}^1) \rightrightarrows \mathcal{F}(U \times \mathbb{A}^1 \times \mathbb{A}^1)$, induced by the projections, we get that $\mathcal{F}(U) \longrightarrow \mathcal{F}(U \times \mathbb{A}^1)$ is an isomorphism.

Proposition 3.4.11.
- *Any sheaf in dominant topology $\mathcal{F}$ is birationally invariant and has the Galois descent property, i.e. for any Galois covering $Y \to X$ one has $\mathcal{F}(X) = \mathcal{F}(Y)^{\mathrm{Aut}(Y|X)}$.*
- *Any $\mathbb{A}^1$- and birationally invariant presheaf $\mathcal{F}$ with the Galois descent property is a sheaf.*

Proof.

- This is clear, since étale morphisms with dense images are covering and $Y \times_X Y \cong Y \coprod \cdots \coprod Y$ for (copies correspond to the elements of $\mathrm{Aut}(Y|X)$, the first projection is identical on each copy and the second projection on the copy corresponding to $g \in \mathrm{Aut}(Y|X)$ is given by g).
- Clearly, any birationally invariant $\mathbb{A}^1$-invariant Galois-separable presheaf is separable: if $Y \to X$ is a cover, i.e. a smooth dominant morphism, then for any sufficiently general dominant map $\varphi : Y \dashrightarrow \mathbb{P}^\delta$ (where $\delta = \dim Y - \dim X$) we can choose a dominant étale morphism $\widetilde{Y} \to Y$ so that the composition $\widetilde{Y} \to Y \dashrightarrow X \times \mathbb{P}^\delta$ is Galois with the group H, and therefore, the composition $\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(X \times \mathbb{P}^\delta) \longrightarrow \mathcal{F}(Y) \longrightarrow \mathcal{F}(\widetilde{Y})$ is injective. In the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(X) & \to & \mathcal{F}(\widetilde{Y}) & \rightrightarrows & \mathcal{F}(\widetilde{Y} \times_X \widetilde{Y}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{F}(X) & \to & \mathcal{F}(Y) & \rightrightarrows & \mathcal{F}(Y \times_X Y) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{F}(X) & \to & \mathcal{F}(X \times \mathbb{P}^\delta) & \rightrightarrows & \mathcal{F}(X \times \mathbb{P}^\delta \times \mathbb{P}^\delta) \end{array}$$

the vertical arrows are injective, so it suffices to show the exactness of the upper row. Let f be an element of the equalizer of $\mathcal{F}(\widetilde{Y}) \rightrightarrows \mathcal{F}(\widetilde{Y} \times_X \widetilde{Y})$, considered as an element of $\mathcal{F}(\widetilde{Y} \times_X \widetilde{Y})$. Then $f \in \mathcal{F}(\widetilde{Y} \times_X \widetilde{Y})^{H \times \{1\}}$ and $f \in \mathcal{F}(\widetilde{Y} \times_X \widetilde{Y})^{\{1\} \times H}$, so $f \in \mathcal{F}(\widetilde{Y} \times_X \widetilde{Y})^{H \times H} = \mathcal{F}(X \times \mathbb{P}^\delta \times \mathbb{P}^\delta) = \mathcal{F}(X)$. □

3.5. **Coinduced representations.** Let K, L, M be a triple of algebraically closed field extensions of k. Denote by i_K^L the coinduction functor $W \mapsto \mathbb{Z}[\{K \overset{/k}{\hookrightarrow} L\}] \otimes_{\mathbb{Z}[G_{K|k}]} W$ from the category of smooth E-representations of $G_{K|k}$ to the category of smooth E-representations of $G_{L|k}$. Clearly, if K and L are isomorphic then i_K^L is an equivalence of categories. If K is a subfield of L then the functor i_K^L coincides with the functor $W \mapsto \mathbb{Z}[G_{L|k}] \otimes_{\mathbb{Z}[G_{\{L,K\}|k}]} W$, and i_K^K is the identity functor.

Proposition 3.5.1.
- *There is a natural isomorphism of functors $H^0(G_{M|L}, -) \circ i_K^M = i_K^L$.*
- *If K is embeddable into L and L is embeddable into M then the functors $i_L^M \circ i_K^L$ and i_K^M are naturally isomorphic.*

[4]The multiplicity of an irreducible object X in W is defined inductively: it is 0 if for any filtration $W \supseteq Y \supseteq Z$ the quotient Y/Z is not isomorphic to X; it is $N > 0$ if there is a filtration $W \supseteq Y \supseteq Z$ such that $Y/Z \cong X$ and the sum the multiplicities of X in W/Y and in Z is $N - 1$. By Jordan–Hölder theorem, the multiplicity is well-defined.

- *For any smooth E-representation W_1 of $G_{L|k}$ and any smooth E-representation W_2 of $G_{K|k}$ one has* $\mathrm{Hom}_{E[G_{M|k}]}(i_L^M(W_1), i_K^M(W_2)) = \mathrm{Hom}_{E[G_{L|k}]}(W_1, i_K^L(W_2))$.

In particular, if K is embedded into L then

- *the coinduction functor i_K^L is fully faithful,*
- *the functor $H^0(G_{L|K}, -)$ is its left quasi-inverse and*
- *the representation $i_K^L W_2$ is "homotopy invariant", cf. §1.1, p.4, if and only if either $W_2 = 0$ or $K = L$ (therefore, the natural morphism $i_K^F W^{G_{F|K}} \longrightarrow W$ is never injective for non-zero $W \in \mathcal{I}_G$).*

Proof. One has

$$\mathrm{Hom}_{E[G_{M|k}]}(i_L^M(W_1), i_K^M(W_2)) = \mathrm{Hom}_{E[G_{\{M,L\}|k}]}(W_1, i_K^M(W_2)) = \mathrm{Hom}_{E[G_{L|k}]}(W_1, (i_K^M(W_2))^{G_{M|L}}).$$

Suppose that $\sum_i a_i\sigma_i \otimes w_i$ is a shortest presentation of an element of the module $i_K^M(W_2) = \mathbb{Z}[\{K \overset{/k}{\hookrightarrow} M\}] \otimes_{\mathbb{Z}[G_{K|k}]} W_2$, which is fixed by the group $G_{M|L}$. Then, for each i, the orbit $G_{M|L}\sigma_i$, considered as a subset of the set of subfields in $M|k$ isomorphic to K, should be finite. This can happen if and only if $\sigma_i(K) \subseteq L$, i.e., $(i_K^M(W_2))^{G_{M|L}} = \mathbb{Z}[\{K \overset{/k}{\hookrightarrow} L\}] \otimes_{\mathbb{Z}[G_{K|k}]} W_2$.

Assume that K is properly embedded into L. Let K' be a non-trivial purely transcendental extension of K. There is such a field embedding $\xi : K \overset{/k}{\hookrightarrow} \overline{K'}$ that its $G_{\overline{K'}|K'}$-orbit consists of the same amount of embeddings as the $G_{\overline{K'}|K'}$-orbit of its image. Therefore, the sum of elements of the $G_{\overline{K'}|K'}$-orbit of $\xi \otimes v$ is non-zero in $i_K^L(W_2)$ for any non-zero $v \in W_2$ and does not belong to $(i_K^L(W_2))^{G_{L|K}}$. $\square$

In this section we give an example (in Proposition 3.5.3) of a pair of essentially distinct open compact subgroups U and U' in G such that there are embeddings of E-representations $E[G/U] \hookrightarrow E[G/U']$ and $E[G/U'] \hookrightarrow E[G/U]$ of G. This implies that the irreducible subquotients of $E[G/U]$ and of $E[G/U']$ are the same. In this example the primitive motives of the maximal level of models of the fields F^U and $F^{U'}$ coincide (and vanish). However, as it shows the example of Proposition 3.5.4, the coincidence of the collections of irreducible subquotients is a rather general phenomenon.

But let us start with some general remarks.

Remarks. 0. If $U \subset U' \subset H$ are subgroups, and index of U in U' is finite then there is a natural embedding $E[H/U'] \hookrightarrow E[H/U]$, $[u] \mapsto \sum_{h\in H/U,\ hU'=uU'}[h]$.

1. **Representations of G/G°.** Let E be an arbitrary field, and U be a compact open subgroup of G. Then any irreducible E-representation of G, that factors through G/G°, is a quotient of the E-representation $E[G/U]$ of G.

Proposition 3.5.2. *Let $1 \le n \le \infty$, $F'|k$ be an algebraically closed extension in F of a finite transcendence degree, and φ be a smooth irreducible E-representation of $G_{F'|k}$. Let W be an irreducible quotient of the (cyclic, generated by any non-zero element of $V^{G_{F|F'}} = [id] \otimes \varphi$) representation $V := E[G/G_{F|F'}] \otimes_{E[G_{F'|k}]} \varphi$. Then there are $\le \max(|k|, |E|)$ irreducible subrepresentations of $G_{F'|k}$, one of which is φ, in $W^{G_{F|F'}}$.*

There are $\ge |k|$ smooth irreducible E-representations of G. There are exactly $2^{|\mathbb{N}|}$ smooth irreducible E-representations of G, if k and E are countable.

Proof. Note, that there are $|\mathrm{Hom}(\mathbb{Q}_+^\times, E^\times)| = \max(2^{|\mathbb{N}|}, |E|)$ one-dimensional representations of $G_{F'|k} \longrightarrow E^\times$, factorizing through the modulus of $G_{F'|k}$. For each φ choose $W =: W_\varphi$. We say that $\varphi \sim \psi$ if $W_\varphi \cong W_\psi$. As $|W^{G_{F|F'}}| \le |W| = \max(|k|, |E|)$, the cardinalities of the equivalence classes are $\le \max(|k|, |E|)$. Therefore, $|\{\varphi\}/\sim| \ge \max(2^{|\mathbb{N}|}, |E|)$, if it is $> \max(|k|, |E|)$, i.e. if $2^{|\mathbb{N}|} > |k|$ and $2^{|\mathbb{N}|} > |E|$. In any case, there are $|k|$ smooth irreducible representations of G of type $(A(F)/A(k)) \otimes E$, where A is an elliptic curve over k without complex multiplication. $\square$

2. **Twists by one-dimensional representations.** Let $n < \infty$, and φ be a homomorphism from G/G° to $E^\times$. Consider $E[G/U](\varphi)$ as the same E-vector space as $E[G/U]$, but with the

G-action by $[\sigma] \xmapsto{\tau} \varphi(\tau) \cdot [\tau\sigma]$. Then $\lambda_\varphi([\sigma]) := \varphi(\sigma) \cdot [\sigma]$ gives an isomorphism of representations $E[G/U] \xrightarrow{\lambda_\varphi} E[G/U](\varphi)$ of G.

This implies that for any irreducible E-representation W of G the multiplicities of W and of $W(\varphi)$ in $E[G/U]$ coincide. It is likely, however, that these multiplicities are infinite.

E.g., let L be an extension of k of finite type and of transcendence degree q in F. Then, at least if certain conjectures hold, any motivic G-module of level $< q$ is a subquotient of $\mathbb{Q}[G/G_{F|L}]$ of infinite multiplicity. To see this, fix a transcendence base $x_1, \dots, x_q$ of L over k. Then there is a surjection $\mathbb{Q}[G/G_{F|L}] \longrightarrow \Omega^s_{F|k}$, given by $[1] \longmapsto x_{s+1}dx_1 \wedge \dots \wedge dx_s$ for any $s < q$. Any motivic G-module of level s is a submodule of $\Omega^s_{F|k}$ of infinite multiplicity. (In the case $s = 1$ this is shown in Proposition 1.1.12.)

Purely transcendental extensions of quadratic extensions.

Proposition 3.5.3 ([35], Corollary 7.3)**.** *Let $L'' \subset F$ be a subfield, finitely generated over k, and $F \neq \overline{L''}$. For some $u \in \sqrt{(L'')^\times} \smallsetminus (L'')^\times$ and some $t \in F$, transcendental over L'', set $L = L''(u, T)$, where $T = (2t - u)^2$, and $L' = L''(t)$. Then for $U = G_{F|L}$ and $U' = G_{F|L'}$ there are embeddings $E[G/U'] \hookrightarrow E[G/U]$ and $E[G/U] \hookrightarrow E[G/U']$.*

This results from the following combinatorial claim ([35, Lemma 7.2]).

Let H be a group and U and U' be subgroups of H such that $U \bigcap U'$ is of index two in U: $U = (U \bigcap U') \bigcup \sigma(U \bigcap U')$. Suppose that $\tau_1 \cdots \tau_N \neq 1$ for any integer $N \geq 1$ and for any collection $\tau_1, \dots, \tau_N \in U'\sigma \smallsetminus U$. Then the morphism of E-representations $E[H/U] \xrightarrow{[\xi] \mapsto [\xi\sigma] + [\xi]} E[H/U']$ of H is injective.

Proposition 3.5.4 ([35], 7.4)**.** *Fix an odd integer $m \geq 1$, and let $m - 1 \leq n \leq \infty$. Fix a collection $x_1, \dots, x_m$ of elements in F with the only relation $\sum_{j=1}^m x_j^d = 1$ over k, where $d \in \{m+1, m+2\}$. Set $L'' = k(x_1, \dots, x_m)$ and $L = (L'')^{\langle e_1 e_2^2 \cdots e_m^m \rangle}$, where $e_i x_j = \zeta^{\delta_{ij}} \cdot x_j$ for a primitive d-th root of unity ζ. Let L' be a maximal purely transcendental extension of k in L. Then if $U = G_{F|L}$ and $U' = G_{F|L'}$, the E-representations $E[G/U]$ and $E[G/U']$ of G have the same irreducible subquotients.*

Proposition 3.5.5. *Let $g_1, \dots, g_N$ be rational involutions of a k-variety X, generating an infinite group. Then the natural map of E-representations $r : E[\{k(X) \overset{/k}{\hookrightarrow} F\}] \longrightarrow \bigoplus_{j=1}^N E[\{k(X)^{\langle g_j \rangle} \overset{/k}{\hookrightarrow} F\}]$ of G is injective.*

Proof. If a non-zero 0-cycle α is in the kernel of r, and P is a point in the support of α, then the support of α contains the $\langle g_1, \dots, g_N \rangle$-orbit of the point P. As this orbit is infinite, but the support of α is finite, we get the contradiction, i.e., $\alpha = 0$. $\square$

EXAMPLES. 1. Let X be an algebraic k-group, $g_1 : x \mapsto x^{-1}$ and $g_2 : x \mapsto h \cdot x^{-1}$, where $h \in X(k)$ is a point of infinite order. Then the E-representations $E[\{k(X) \overset{/k}{\hookrightarrow} F\}]$ and $E[\{k(K(X)) \overset{/k}{\hookrightarrow} F\}]$ of G have the same irreducible subquotients, where $K(X)$ is the quotient of X by the involution g_1 (the Kummer variety).

2. If Y_j are generically twofold covers of projective spaces over k, at least one of which, for example Y_1, is a curve of genus ≤ 1 then there are embeddings of G-representations $E[\{k(\prod_{j=1}^N Y_j) \overset{/k}{\hookrightarrow} F\}] \hookrightarrow \bigoplus_{i=1}^N E[\{k(\prod_{1 \leq j \leq N,\ j \neq i} Y_j)(\mathbb{P}^{d_i}) \overset{/k}{\hookrightarrow} F\}]^{1+\delta_{1i}} \hookrightarrow E[\{k(\mathbb{P}^d) \overset{/k}{\hookrightarrow} F\}]^{N+1}$, where $d_i = \dim Y_i$ and $d = \sum_{j=1}^N d_j$.

4. Homotopy invariant representations of G

In this section we continue describing the abelian category $\mathcal{I}_G$ (cf. §1.1, p.4 and further).

The category $\mathcal{I}_G$ is closed under taking subquotients in $\mathcal{S}m_G$ (Lemma 4.1.1). If $n = \infty$ then a smooth representation of G is "homotopy invariant" if and only if all its irreducible subquotients are (Theorem 1.1.5).[5] If $n = \infty$ then the subcategory $\mathcal{I}_G$ is closed under the inner $\mathcal{H}om$ functor on $\mathcal{S}m_G$ (Proposition 4.1.10). It follows from Lemma 3.4.1 and Proposition 3.4.10 that $\mathcal{I}_G(E)$ contains the category $\mathcal{A}dm_G(E)$ of admissible representations of G (Theorem 1.1.6(2)), if $n = \infty$. A direct (but essentially the same) proof can be found in [35, Proposition 6.4].

The inclusion functor $\mathcal{I}_G \hookrightarrow \mathcal{S}m_G$ admits a left adjoint $\mathcal{I} = (\varprojlim_L C_L) \otimes_{\mathbb{D}_{\mathbb{Q}}} : \mathcal{S}m_G \longrightarrow \mathcal{I}_G$, cf. §4.6, so any morphism from $W \in \mathcal{S}m_G$ to an object of $\mathcal{I}_G$ factors through $\mathcal{I}W \in \mathcal{I}_G$ (Proposition 4.1.3).

In the case $n = \infty$ there are no non-zero projective objects in $\mathcal{S}m_G$ (Remark on p.24). Unlike $\mathcal{S}m_G$, there are enough projective objects in $\mathcal{I}_G$ (Theorem 1.1.6 (3)). Namely, the objects $C_L := \mathcal{I}\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]$ for all field extensions of finite type $L|k$ form a system of projective generators of $\mathcal{I}_G$. The sheaf $X \mapsto C_{k(X)}$ on $\mathfrak{D}m_k$ with values in $\mathcal{I}_G^{\mathrm{op}}$ (Corollary 3.4.5) in $\mathbb{A}^1$-invariant (Lemma 4.1.6).

For any smooth irreducible proper k-variety X there is a natural surjection $C_{k(X)} \longrightarrow CH_0(X \times_k F)_{\mathbb{Q}}$. The first part of Conjecture 1.1.7 asserts that this is an isomorphism if $n = \infty$. If X is a curve then this is verified in Corollary 4.1.7. It is explained in Remark 3 on p.6 that this conjecture implies the existence of a commutative associative tensor structure on $\mathcal{I}_G$.

There are some reasons to expect that the category of mixed motives can be linked to a full subcategory of the category of smooth G-modules, whose objects have "motivic" irreducible subquotients. In particular, by analogy with the Hodge theory, it is conjectured (Conjecture 4.1.5) that the adjoint quotients of the level filtration $N_\bullet$ are semi-simple for any object of $\mathcal{I}_G$. This implies easily ("Corollary" 4.1.5.1) that the level filtration $N_\bullet$ is strictly compatible with morphisms in $\mathcal{I}_G$. In particular, extensions of G-modules from $\mathcal{I}_G$ of lower level by irreducible G-modules from $\mathcal{I}_G$ of higher level are (canonically) split. In turn, Conjecture 4.1.5 follows from the first part of Conjecture 1.1.7 and from the "motivic" conjectures, cf. Remark 2 on p.6.

It is desirable to extend the category $\mathcal{I}_G$ in order to be able to consider such G-modules as $F^\times/k^\times$, and to extend the filtration $N_\bullet$ on $\mathcal{I}_G$ to a "weight filtration" in such a way that it was still strictly compatible with the morphisms.

Note, that this is not true for the filtration $N_\bullet$ on the arbitrary smooth G-modules. E.g., any irreducible admissible G-module of level 1 (corresponding to an abelian variety) admits a non-trivial extension by the irreducible G-module $F^\times/k^\times$ (which is also of level 1), cf. p.9.

Usually, the weight of an irreducible object W_1 is greater than the weight of an irreducible object W_2, if $\mathrm{Ext}^1(W_1, W_2) \neq 0$, so weight($\mathbb{Q}$) $<$weight($F^\times/k^\times$) $<$weight($A(F)/A(k)$) for any abelian variety A over k, which is not good, if $A(F)/A(k)$ corresponds to the motive $H_1(A)$ of weight 1. To resolve this "contradiction" one could try to use a grading of rank > 1.

This bigger category should admit a duality functor, which is absent in the case of the category $\mathcal{I}_G$.

We summarize the principal results of §4.3, except those mentioned in §1.1, in the following way.

Theorem 4.0.6. (1) *For any $1 \leq n \leq \infty$ and $q \geq 0$ there is a functor $\mathfrak{B}^q$:*

$$\{\text{pure primitive } q\text{-motives over } k\} \xrightarrow{\mathfrak{B}^q} \left\{ \begin{array}{c} \text{semi-simple admissible } G\text{-modules} \\ \text{of finite type and of level } q \end{array} \right\},$$

fully faithful for $q \leq n$. (The level of a G-module W is an integer q such that $N_q W = W$ and $N_{q-1}W = 0$ for the filtration $N_\bullet$, defined on p.5.)

(2) *If $n < \infty$ then there is a bilinear symmetric non-degenerate G-equivariant form on the G-module $\mathfrak{B}^n(M)$ with values in the oriented G-module $\mathbb{Q}(\chi)$ of degree 1, where $M = (X, \Delta_{k(X)})$ is the maximal primitive n-submotive of the motive (X, Δ_X) and $\dim X = n$.*

This form is definite, if for the $(n-1)$-cycles on the $2n$-dimensional complex varieties numerical and homological equivalences coincide (e.g., if $n \leq 2$), and therefore, $\mathfrak{B}^n$ factors through the subcategory of "polarizable" G-modules (i.e., admitting a positive form as above).

[5] If $n < \infty$ then there exist non-trivial extensions of $\mathbb{Q}$ by $\mathbb{Q}$, i.e., the category $\mathcal{I}_G$ is *not* closed under extensions in $\mathcal{S}m_G$.

This follows directly from Corollary 3.1.6 and Propositions 4.3.2, 4.3.11, 4.3.13. Roughly speaking, the functor $\mathfrak{B}^q$ is defined as the space of 0-cycles over F modulo “numerical equivalence over k”. Details are in §4.3, p.42, where it is shown that it is pro-representable. It follows from Proposition 4.3.9 that $\mathfrak{B}^q((X,\pi))$ depends only on the birational class of X. Moreover, it follows from Proposition 3.3.2 that the composition of the functor $\mathfrak{B}^1$ with the foretful functor to the category of G°-modules is also fully faithful, and the functor $\mathfrak{B}^1$ from Theorem 4.0.6(1) is an equivalence of categories if $n=\infty$, cf. §4.1.

Conjecture 1.1.3 admits the following form, “convenient for checking in particular cases”.

Conjecture 4.0.7. *For any $q \geq 0$ the functor $\mathfrak{B}^q$ is an equivalence of categories if $n=\infty$.*

One can show that if $U \neq 0$ is a quotient of $F^\times$ then the functor $U\otimes : \mathcal{I}_G \longrightarrow \mathcal{S}m_G$ is fully faithful. Therefore, there exist other fully faithful functors from the category of pure motives to the category of smooth graded representations of G, besides $\mathbb{B}^\bullet$, cf. also Lemma 1.2.3. However, these functors do not preserve the irreducibility.[6]

There are some indications that the category of primitive n-motives “is not too far” from the category of polarizable (in the sense of Theorem 4.0.6(2)) G-modules (at least if $n \leq 2$). In particular, the vanishing of the subspaces in the polarizable G-representations fixed by the compact subgroup $G_{F|\overline{L(x)}}$ (here $L|k$ is a subextension in F, and x is an element of F, transcendental over L, such that $F = \overline{L(x)}$), corresponds to the triviallity of the primitive n-submotives of the motive $(Y \times \mathbb{P}^1, \pi)$, where $\dim Y < n$. However, one has to impose some extra conditions, since it can be easily deduced from Proposition 3.3.2 and the full faithfulness of the functor $\mathfrak{B}^1$ that for any G-module W there is at most one character ψ such that $W(\psi) \cong \mathfrak{B}^1(M)$ for a pure 1-motive M, cf. [35, Corollary 4.5], but the twists of polarizable G-modules by the characters of G of order 2 are also polarizable.

Possible links to mixed motives. There are several cohomology theories of algebraic varieties are related by comparison isomorphisms and behave in a parallel way. This led A.Grothendieck, P.Deligne, A.A.Beilinson et al to a conjecture on existence of a universal cohomology theory – with values in an abelian category of mixed motives – and on identities between the extension groups between these cohomological objects and K-groups.

The references for this circle of ideas are, e.g., [5], [16].

For smooth projective varieties this theory is given by the Grothendieck motives, but only under assumption that numerical and homological equivalences coincide.

V.A.Voevodsky, M.Levine and M.Hanamura (cf. [46, 25, 12]) have defined triangulated categories, supposed to be equivalent to the derived category of mixed motives. The principal difficulty consists of constructing of a t-structure, whose core was the desired abelian category of mixed motives. This would be possible, if the “standard” (including Beilinson's) conjectures were proved, cf. [4]. (It should follow from Conjecture 1.1.7 that $\mathcal{I}_G$ is equivalent to a localization of the homotopy t-structure on the Voevodsky triangulated category of motives.)

Another approach, due to Deligne and Jannsen, cf. [17], consists of considering of compatible collections of “realizations”. Here the difficulties are related to the Hodge and Tate conjectures.

As it is mentioned in §1.1, to a given cohomology theory H^* one can associate the G-module $H^*_c(F) := \varinjlim H^*(Y)/N^1H^*(Y)$, where Y runs over all smooth proper irreducible varieties over k with function fields embedded into F, and $N^\bullet$ is the coniveau filtration. Clearly, $H^*_c(F) \in \mathcal{I}_G$ and if $H^*(k)$ is finite-dimensional over a field E then $H^*_c(F) \in \mathcal{A}dm_G(E)$.

Then (assuming that numerical and homological equivalences coincide) $H^*_c(F)$ is a semi-simple G-module, admitting a decomposition $H^*_c(F) \cong \bigoplus_M H^*(M) \otimes_{\mathrm{End}(M)} \mathbb{B}(M)$, where M runs over the isomorphism classes of irreducible primitive motives (as in Proposition 1.1.11). This implies, in

[6] $W \otimes U$ is irreducible only if U is irreducible and $W \not\cong U$. Therefore, if $U \neq 0$ is a quotient of $A(F)$ for a commutative k-group A then one may suppose that A is simple and it is not an abelian variety, i.e. either $\mathbb{G}_m$, or $\mathbb{G}_a$, and that $U = A(F)/A(k)$. If $W = E(F)/E(k)$ for an elliptic curve E over k then the morphism $W \otimes U \xrightarrow{\omega\wedge\eta} \Omega^2_{F|k}$ is non-zero and it is not injective, where ω is a non-zero regular 1-form on E, and η is a non-zero invariant 1-form on A, i.e., $W \otimes U$ is reducible.

notations of §4.3, that $H^*(N) \cong \bigoplus_i \mathrm{Hom}_G(\mathbb{B}^{[i]}(N), H_c^*(F))$ for any motive N. Thus, the realization functor (on the category of pure motives over k), corresponding to a theory H^*, can be decomposed into a composition of the functor $\mathbb{B}^\bullet$ and a (contravariant) functor $\mathrm{Hom}_G(-, H_c^*(F))$ (on the category of G-modules).

4.1. The category $\mathcal{I}_G$, level filtration, differential forms...

Lemma 4.1.1 ([35], 6.6). *The functor $H^0(G_{F|L}, -) : \mathcal{I}_G \longrightarrow \mathcal{V}ect_{\mathbb{Q}}$ is exact for any field extension L of k in F. $\mathcal{I}_G$ is closed under taking subquotients in $\mathcal{S}m_G$.*

For each integer $q \geq 0$ let $\mathcal{I}_G^q$ be the full subcategory in $\mathcal{I}_G$ with the objects W such that $W^{G_{F|F'}} = 0$ for any algebraically closed F' with $\mathrm{tr.deg}(F'|k) = q-1$. Then $\{\mathcal{I}_G^q\}_{q\geq 0}$ is a descending filtration of $\mathcal{I}_G$ by Serre subcategories.[7]

This implies that $\mathcal{A}dm_G(E)$ is an abelian Serre subcategory in $\mathcal{S}m_G(E)$, [35, Corollary 6.5].

Lemma 4.1.2 ([35], 6.7). *If $F'|k$ is an extension in F of infinite transcendence degree then the functor $H^0(G_{F|\overline{F'}}, -)$ from $\mathcal{S}m_G$ to $\mathcal{S}m_{G_{\overline{F'}|k}}$ is an equivalence of categories (inducing an equivalence of $\mathcal{I}_G$ and $\mathcal{I}_{G_{\overline{F'}|k}}$). The functor $H^0(G_{F|K}, -)$ from $\mathcal{S}m_G$ to $\mathcal{V}ect_{\mathbb{Q}}$ is exact if and only if* $\mathrm{tr.deg}(K|k) = \mathrm{tr.deg}(F|k)(\leq \infty)$.

The proof makes use of a field isomorphism $F \xrightarrow{\sim} \overline{F'}$, identical on k.

The functor $\mathcal{I}$. The level filtration $N_\bullet$ on a G-module M is defined on p.5. Equivalently, N_jM is the minimal subrepresentation of G in M, containing $M^{G_{F|F_j}}$ for some algebraically closed $F_j \subseteq F$. Clearly, $N_\bullet$ is a functorial (restriction to N_jM of any G-homomorphism $M \longrightarrow M'$ factors through N_jM') non-negative increasing ($N_jM \subseteq N_{j+1}M$) multiplicative (with respect to the tensor products: $N_{i+j}(M_1 \otimes M_2) \supseteq N_iM_1 \otimes N_jM_2$) filtration, which is exhausting on the smooth representations ($M = \bigcup_{j\geq 0} N_jM$, if M is smooth).

Proposition 4.1.3 ([35], 6.8). *For any integer $q \geq 0$ any $W \in \mathcal{S}m_G$ admits a quotient $\mathcal{I}^qW \in \mathcal{I}_G^q$ such that any G-homomorphism from W to any object of $\mathcal{I}_G^q$ factors through $\mathcal{I}^qW$. The functor $\mathcal{S}m_G \xrightarrow{\mathcal{I}^q} \mathcal{I}_G^q$, given by $W \longmapsto \mathcal{I}^qW$, is right exact and $\mathcal{I}^qW = \mathcal{I}W/N_{q-1}\mathcal{I}W$.*

One can deduce the existence of the functors $\mathcal{I}^q$ from general categorical facts, cf. [27, §5.8]. However, they are constructed in [35] "explicitly", which makes a link from the generators of $\mathcal{I}_G$ to the Chow groups of 0-cycles rather transparent, cf. [35, Proposition 6.17]:

Proposition 4.1.4. *If $n = \infty$ then for any irreducible variety X over k the kernel of the natural projection $\mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}] \longrightarrow C_{k(X)}$ is the sum over all curves $y \in (k(X) \otimes_k F)_1$ of the subspaces spanned by those linear combinations of generic (with respect to some field of definition of the curves y) F-points of the curves $\overline{\{y\}}$ that are linearly equivalent to zero on any compactification of the normalization of $\overline{\{y\}}$.*

Example. Let A be a one-dimensional group scheme over k, $m \geq 1$ be an integer, and $W = N_qW$ be a smooth representation of G, where $q \leq n-1$. Then $\mathcal{I}(S^mA(F) \otimes W) = 0$, if either m is even, or $A = \mathbb{G}_a$, or $A = \mathbb{G}_m$. In particlar, the natural projection $A(F)_{\mathbb{Q}}^{\otimes N} \longrightarrow \bigwedge^N_{\mathrm{End}_{\mathbb{Q}}A} A(F)_{\mathbb{Q}}$ induces an isomorphism $\mathcal{I}(A(F)_{\mathbb{Q}}^{\otimes N}) \xrightarrow{\sim} \mathcal{I}(\bigwedge^N_{\mathrm{End}_{\mathbb{Q}}A} A(F)_{\mathbb{Q}})$, if $n \geq N-1$. (This is shown in [35, p.204] when $\mathrm{End}A = \mathbb{Z}$; the general case is similar.)

Remark. $\mathcal{I}$ is not left exact. E.g., it transforms embedding $k \hookrightarrow F$ to $k \longrightarrow 0$.

Conjecture 4.1.5. *If $n = \infty$ then for any $j \geq 0$ and any $W \in \mathcal{I}_G$ the representation $gr_j^N W$ of G is semi-simple.*

[7] A full subcategory of an abelian category $\mathcal{B}$ is called a Serre subcategory, if it is closed under taking subquotients and extensions in $\mathcal{B}$.

This is clear, when $j = 0$, and can be easily deduced from Corollary 4.1.8, when $j = 1$.

"Corollary" 4.1.5.1. *Suppose that $n = \infty$ and Conjecture 4.1.5 holds for any $0 \le j \le q-1$. Let $L|k$ be an extension of finite type and* $\mathrm{tr.deg}(L|k) < q$. *Then*

- *the functor $\mathcal{I}^q$ is exact on $\mathcal{I}_G$ (which is equivalent to the strict compatibility of the filtration $N_0 \subseteq \cdots \subseteq N_{q-1}$ with morphisms in $\mathcal{I}_G$);*
- *the algebra $A = CH^s\left(\mathbf{Spec}(L \otimes_k L)\right)_{\mathbb{Q}}$ of correspondences modulo rational equivalence is semi-simple and its length is finite,[8] where* $s = \mathrm{tr.deg}(L|k)$.

The first part of this "Corollary" is proved in [35, Corollary 6.10], and the second one is evident from the formula $A = \mathrm{End}_G(W)$, where $W := CH^s\left(\mathbf{Spec}(L \otimes_k F)\right)_{\mathbb{Q}} = gr_s^N W$ is semi-simple and cyclic.

Remark. The inclusion $\mathbb{Q}[G/G_{F|L}]^\circ \hookrightarrow \mathbb{Q}[G/G_{F|L}]$ is an example of a morphism of smooth G-modules, which is *not* strictly compatible with the filtration $N_\bullet$, since $N_{\mathrm{tr.deg}(L|k)}\mathbb{Q}[G/G_{F|L}]^\circ$ coincides with

$$\left\{ \sum_{[\sigma]\in G/G_{F|L}} a_\sigma[\sigma] \;\middle|\; \sum_{\sigma(L)\subset \overline{F'}} a_\sigma = 0 \text{ for any } F' \text{ with } \mathrm{tr.deg}(F'|k) = \mathrm{tr.deg}(L|k) \right\},$$

which is different from $\mathbb{Q}[G/G_{F|L}]^\circ$, whereas $\mathbb{Q}[G/G_{F|L}] = N_{\mathrm{tr.deg}(L|k)}\mathbb{Q}[G/G_{F|L}]$.

Lemma 4.1.6 ([35], 6.12). *For any $1 \le n \le \infty$, any subfield $L_1 \subset F$ of finite type over k, and any unirational extension L_2 of L_1 in F of finite type there is a natural isomorphism $C_{L_2} \xrightarrow{\sim} C_{L_1}$.*

The objects of $\mathcal{I}_G$ of level 1. For any $W \in \mathcal{S}m_G$ there is a surjection $\bigoplus_{e\in W^{G_{F|F'}}} \langle e\rangle_G \longrightarrow N_1 W$, where $F'|k$ is an algebraically closed extension in F with $\mathrm{tr.deg}(F'|k) = 1$. This means that to describe the objects of $\mathcal{I}_G$ of level 1, it suffices to treat the case of $W = \langle e\rangle_G$, where $\mathrm{Stab}_e \supseteq G_{F|L}$ with $L \cong k(X)$ for a smooth proper curve X over k of genus $g \ge 0$. Then W is dominated by C_L. Let $\mathrm{Pic}^j X$ be the Picard variety of the linear equivalence classes of divisors on X of degree j.

Proposition 4.1.7 ([35], 6.20 + 6.21). *Let X be a smooth projective curve over k, $k(X)$ be its function field, $Z_0^{\mathrm{rat}}(k(X) \otimes_k F)$ be the kernel of the natural projection $\mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}] \longrightarrow \mathrm{Pic}(X_F)_{\mathbb{Q}}$ and $\mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]^\circ$ be the group of generic degree-zero 0-cycles over F. If $n = \infty$ then $\mathcal{I}Z_0^{\mathrm{rat}}(k(X) \otimes_k F) = 0$, $\mathcal{I}\mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]^\circ = \mathrm{Pic}^\circ(X_F)_{\mathbb{Q}}$ and $C_{k(X)} = \mathrm{Pic}(X_F)_{\mathbb{Q}}$.*

The proof is based on the facts that i) sufficiently big symmetric powers of a smooth projective curve are projective bundles over its Jacobian; ii) [35, Lemma 6.18]: the G-module $Z_0^{\mathrm{rat}}(k(X) \otimes_k F)$ is generated by $w_N = \sum_{j=1}^N \sigma_j - \sum_{j=1}^N \tau_j$ for all $N \gg 0$, where $(\sigma_1, \dots, \sigma_N; \tau_1, \dots, \tau_N)$ is the generic F-point of the fibre over zero of the morphism $X^N \times_k X^N \xrightarrow{p_N} \mathrm{Pic}^\circ X$, sending a point $(x_1, \dots, x_N; y_1, \dots, y_N)$ to the class of $\sum_{j=1}^N (x_j - y_j)$.

Corollary 4.1.8 ([35], 6.22 + 6.23). *If $n = \infty$ then $A(F)_{\mathbb{Q}}$ is a projective object of $\mathcal{I}_G$ for any abelian k-variety A, and any object of $\mathcal{I}_G$ of level 1 is a direct sum of a trivial module and a quotient of a direct sum of modules $A(F)_{\mathbb{Q}}$ for some A by a trivial submodule.*

The inner $\mathcal{H}om$.

Corollary 4.1.9 ([35], 6.25). *The inclusion $\mathbb{Q}[\{L(X) \overset{/L}{\hookrightarrow} F\}] \subseteq \mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]$ induces a surjection of $G_{F|L}$-modules $\mathbb{Q}[\{L(X) \overset{/L}{\hookrightarrow} F\}] \longrightarrow C_{k(X)}$ for any extension L of k in F with* $\mathrm{tr.deg}(F|L) = \infty$ *and any irreducible k-variety X.*

[8] Together with "Corollary" 1.1.8.1 this would imply that in particular the algebra A is finite-dimensional over $\mathbb{Q}$.

Theorem 1.1.5 and the following Proposition indicate a connection between $\mathcal{I}_G$ and the category of *effective* homological motives. In §4.3 one discusses also non-effective motives.

Proposition 4.1.10 ([35], 6.26)**.** *The inner $\mathcal{H}om$ functor on $\mathcal{S}m_G$ (cf. §3, p.24) induces an inner $\mathcal{H}om$ functor on $\mathcal{I}_G$, if $n=\infty$. The level of $\mathcal{H}om(W_1, W_2)$ is $\le q$, if $W_1, W_2 = N_qW_2 \in \mathcal{I}_G$ and $q \le 1$.*

Example. Representing G as the cokernel of a closed embedding $G_{\overline{F(X)}|F(X)} \hookrightarrow G_{\{\overline{F(X)},F\}|k(X)}$ of topological groups, one gets a G-action on the set of orbits $\{k(Y) \overset{/k}{\hookrightarrow} \overline{F(X)}\}/\mathrm{Gal}(\overline{F(X)}|F(X))$. Then $\mathcal{H}om(C_{k(X)}, CH_0(Y_F)_{\mathbb{Q}}) = CH_0(Y_{F(X)})_{\mathbb{Q}}$.

Remark. Unlike the objects of $\mathcal{I}_G$ (in the case $n=\infty$), for any totally disconnected topological group H there are many smooth representations H with non-trivial contragredients. Namely, the H-equivariant pairing $\mathbb{Q}[H/U]\otimes\mathbb{Q}[H/U] \longrightarrow \mathbb{Q}$, given by $[\sigma]\otimes[\tau] \longmapsto 0$, if $[\sigma] \ne [\tau]$, and $[\sigma]\otimes[\sigma] \longmapsto 1$, defines an embedding of $\mathbb{Q}[H/U]$ into its contragredient. Here U is any open subgroup of H.

Proposition 4.1.11 ([36], 7.6)**.** *Let $W \in \mathcal{I}_G$ and $q \ge 0$ be an integer. Then*

- *any G-homomorphism $W \xrightarrow{\varphi} \bigotimes_F^q \Omega^1_{F|k}$ factors through $W \longrightarrow \Omega^q_{F|k} \subseteq \bigotimes_F^q \Omega^1_{F|k}$;*
- *for any smooth proper k-variety Y a field embedding $k(Y) \overset{\iota}{\hookrightarrow} F$ over k induces an injection $\varphi(W) \bigcap \iota_*\Omega^q_{k(Y)|k} \hookrightarrow \Gamma(Y, \Omega^q_{Y|k})$, and there are the following canonical isomorphisms*

$$\mathrm{Hom}_G(C_{k(Y)}, \bigotimes\nolimits_F^{\bullet} \Omega^1_{F|k}) \xleftarrow{\sim} \Gamma(Y, \Omega^{\bullet}_{Y|k}) \xrightarrow{\sim} \mathrm{Hom}_G(CH_0(Y_F), \bigotimes\nolimits_F^{\bullet} \Omega^1_{F|k}). \tag{2}$$

The first isomorphism is functorial with respect to the dominant morphisms $Y \longrightarrow Y'$, the second one is functorial with respect to arbitrary morphisms $Y \longrightarrow Y'$.

Sketch of the proof. $\omega \in \bigotimes_F^q \Omega^1_{F|k}$ is interpreted as a rational section of the coherent sheaf $\Omega^q_{Y^q|k}|_{\Delta_Y}$ on a smooth projective k-variety Y. The principal idea: if ω is not in the span of the images of $\Gamma(Y, \Omega^{\bullet}_{Y|k})$ for smooth proper k-varieties Y then the direct image $f_*\omega = \mathrm{tr}_{/k(\mathbb{P}^M)}(\omega)$ of ω is a fixed non-zero element of the G-module, generated by ω, for an appropriate finite morphism $f : Y \longrightarrow \mathbb{P}^M_k$. To ensure that $f_*\omega \ne 0$, one uses the poles. Even if ω has no poles, but does not belong to $\Omega^{\bullet}_{F|k}$, its direct image under an appropriate finite ramified morphism has poles.

4.2. **The "Künneth formula" and tensor structure. A tensor structure on $\mathcal{I}_G$.** As it shows Example after Proposition 4.1.3 on p.36, $\mathcal{I}_G$ is not closed under tensor products in $\mathcal{S}m_G$. Define $W_1 \otimes_{\mathcal{I}} W_2$ by $\mathcal{I}(W_1 \otimes W_2)$.

It can be seen from the following example that this operation is not associative on $\mathcal{S}m_G$. Let $W_j = \mathbb{Q}[\{k(X_j) \overset{/k}{\hookrightarrow} F\}]$ for some irreducible k-varieties X_j, $1 \le j \le N$, $N \ge 2$. Then $W_1 \otimes \cdots \otimes W_N = \bigoplus_{x \in \mathbf{Spec}(k(X_1)\otimes_k \cdots \otimes_k k(X_N))} \mathbb{Q}[\{k(x) \overset{/k}{\hookrightarrow} F\}]$, so $\mathcal{I}(W_1 \otimes \cdots \otimes W_N)$ is isomorphic to the direct sum over all $x \in \mathbf{Spec}(k(X_1) \otimes_k \cdots \otimes_k k(X_N))$ of the representations $C_{k(x)}$. If $X_1 = X_2 = \mathbb{A}^1_k$ then $\mathcal{I}W_1 = \mathcal{I}W_2 = \mathbb{Q}$, and therefore, $W_1 \otimes_{\mathcal{I}} (W_2 \otimes_{\mathcal{I}} \mathbb{Q}) = W_1 \otimes_{\mathcal{I}} \mathcal{I}W_2 = \mathcal{I}W_1 = \mathbb{Q}$.

On the other hand, by Noether's Normalization Lemma, $(W_1 \otimes_{\mathcal{I}} W_2) \otimes_{\mathcal{I}} \mathbb{Q} = \mathcal{I}(W_1 \otimes W_2)$ contains submodules, isomorphic to $C_{k(X)}$ for any curve X over k.

Lemma 4.2.1 ([35], 6.27 + 6.28)**.** *Let $n=\infty$. Then for any finite collection of smooth irreducible proper k-varieties $X_1, \dots, X_N$ there is a canonical surjective morphism $\mathcal{I}(\alpha) : C_{k(X_1 \times_k \cdots \times_k X_N)} \longrightarrow \mathcal{I}\left(C_{k(X_1)} \otimes \cdots \otimes C_{k(X_N)}\right)$ of G-modules.*

If $C_{k(X_1 \times_k \cdots \times_k X_N)} = CH_0((X_1 \times_k \cdots \times_k X_N)_F)_{\mathbb{Q}}$ then $\mathcal{I}(\alpha)$ is an isomorphism.

If $\mathcal{I}(\alpha)$ is an isomorphism then $\otimes_{\mathcal{I}}$ is associative, the class of projective objects of $\mathcal{I}_G$ is closed under $\otimes_{\mathcal{I}}$, and $W_1 \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} W_N = \mathcal{I}(W_1 \otimes \cdots \otimes W_N)$.

The "Künneth formula" for products with curves. The restriction map $\tau \mapsto \tau|_{k(X)} \otimes \tau|_{k(Y)}$ defines a G-homomorphism $\mathbb{Q}[\{k(X) \otimes_k k(Y) \overset{/k}{\hookrightarrow} F\}] \xrightarrow{\alpha} C_{k(X)} \otimes C_{k(Y)}$. It follows from Lemma 4.2.1 that α is surjective, which gives a surjection $C_{k(X \times_k Y)} \longrightarrow C_{k(X)} \otimes_{\mathcal{I}} C_{k(Y)}$.

For arbitrary $A \in C_{k(X)}$ and $B \in C_{k(Y)}$ choose some liftings $\widetilde{A} \in \mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]$ and $\widetilde{B} \in \mathbb{Q}[\{k(Y) \overset{/k}{\hookrightarrow} F\}]$ such that all embeddings from $\widetilde{A}$ and from $\widetilde{B}$ are pairwise in general position.[9]

One has to check that the class of $\widetilde{A} \times \widetilde{B} \in \mathbb{Q}[\{k(X \times_k Y) \overset{/k}{\hookrightarrow} F\}]$ in $C_{k(X \times_k Y)}$ is independent of the choice of $\widetilde{A}$ and $\widetilde{B}$. If some other liftings $\widetilde{A}' \in \mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]$ and $\widetilde{B}' \in \mathbb{Q}[\{k(Y) \overset{/k}{\hookrightarrow} F\}]$ are defined similarly, choose some lifting $\widetilde{B}'' \in \mathbb{Q}[\{k(Y) \overset{/k}{\hookrightarrow} F\}]$ of B such that all embeddings from $\widetilde{A}$ and from $\widetilde{B}''$, as well as from $\widetilde{A}'$ and from $\widetilde{B}''$, are pairwise in general position. Then $\widetilde{A} \times \widetilde{B} - \widetilde{A}' \times \widetilde{B}' = (\widetilde{A} - \widetilde{A}') \times \widetilde{B}'' + \widetilde{A} \times (\widetilde{B} - \widetilde{B}'') + \widetilde{A}' \times (\widetilde{B}'' - \widetilde{B}')$.

Thus, one has to check the following condition $\star_{X,Y}$: if the class of $\sum_{i=1}^{N} a_i\tau_i \in \mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]$ in $C_{k(X)}$ is zero and all τ_i are in general position with respect to $\sigma : k(Y) \overset{/k}{\hookrightarrow} F$ then the class of $\gamma := \sum_{i=1}^{N} a_i(\tau_i, \sigma) \in \mathbb{Q}[\{k(X \times_k Y) \overset{/k}{\hookrightarrow} F\}]$ in $C_{k(X \times_k Y)}$ is zero. Also, one has to check the condition $\star_{Y,X}$.

By definition of the functor $\mathcal{I}$, there exist purely transcendental extensions $L'_j|L_j$, elements $\alpha_j \in \mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]^{G_{F|L'_j}}$ and $\xi_j \in G_{F|L_j}$ such that $\sum_{i=1}^{N} a_i\tau_i = \sum_j(\xi_j\alpha_j - \alpha_j)$.

If σ is in general position with respect to the compositum L of all $\tau_i(k(X))$ then there exists $\kappa \in G_{F|L}$ such that $\kappa\sigma =: \sigma'$ is in general position with respect to the compositum of all L'_j. Then $\gamma' := \kappa\gamma = \sum_i a_i(\tau_i, \sigma') = \sum_j(\xi_j\alpha_j - \alpha_j) \otimes \sigma'$. Set $K_j := L_j\sigma'(k(Y))$ and $K'_j := L'_j\sigma'(k(Y))$. Then $\alpha_j \otimes \sigma' \in \mathbb{Q}[\{k(X \times_k Y) \overset{/k}{\hookrightarrow} F\}]^{G_{F|K'_j}}$, K'_j is a purely transcendental extension of K_j, and there exist $\xi'_j \in G_{F|\sigma'(k(Y))}$ such that $\xi'_j|_{L'_j} = \xi_j|_{L'_j}$. This implies that $\gamma' = \sum_j(\xi'_j(\alpha_j \otimes \sigma') - \alpha_j \otimes \sigma')$ belongs, by definition of the functor $\mathcal{I}$, to the kernel of the projection $\mathbb{Q}[\{k(X \times_k Y) \overset{/k}{\hookrightarrow} F\}] \longrightarrow C_{k(X \times_k Y)}$, and therefore, the same is true for γ.

Let us check that the conditions $\star_{X,Y}$ and $\star_{Y,X}$ are equivalent. Let a generic curve C on Y, passing through σ, be defined over a field containing all $\tau_i(k(X))$. Then σ is linearly equivalent to a linear combination β of generic points of C (which are therefore generic points of Y). Then the image of γ in $C_{k(X \times_k Y)}$ coincides with the image of $\sum_i a_i\tau_i \times (\sigma - \beta)$, which shows the implication $\star_{Y,X} \Rightarrow \star_{X,Y}$.

Example. Let us check the condition $\star_{X,Y}$ in the case, when X is a smooth proper curve. Let $K = \overline{\sigma(k(Y))}$. Then $\sum_i a_i\tau_i$ is a generic divisor on the curve X_K over K, linearly equivalent to zero. According to [35, Lemma 6.18], the $G_{F|K}$-module of generic divisors on X_K over K, linearly equivalent to zero, is generated by the elements $w_M = \sum_{j=1}^{M}(\sigma_j - \sigma'_j)$ for all $M \gg 0$, where $(\sigma_1, \dots, \sigma_M; \sigma'_1, \dots, \sigma'_M)$ is a generic F-point of the fibre over 0 of the morphism $X_K^M \times_K X_K^M \longrightarrow \mathrm{Pic}^\circ X_K$, sending $(x_1, \dots, x_M; y_1, \dots, y_M)$ to the class of $\sum_{j=1}^{M}(x_j - y_j)$. Clearly, the compositum of all $\sigma_j(k(X))\sigma'_j(k(X))$ is in general position with respect to K. The same is true for any other element in the $G_{F|K}$-orbit of w_M. Therefore, as we have already seen above, the image of $\sum_i a_i(\tau_i, \sigma)$ in $C_{k(X \times_k Y)}$ is zero.

[9]First, choose arbitrary $\widetilde{A}$ and $\widetilde{B}$. For each point P of the support of $\widetilde{B}$ choose a generic curve C_P passing through P, on which P is a generic point with respect to a field of definition of C_P. Replace P by a linearly equivalent linear combination of points of C_P in general position with respect to $\widetilde{A}$. Then we get the desired $\widetilde{B}$.

Thus, one has a canonical G-module surjection $C_{k(X)} \otimes C_{k(Y)} \twoheadrightarrow C_{k(X \times_k Y)}$, at least if X is a curve, and the composition $C_{k(X\times_k Y)} \longrightarrow C_{k(X)} \otimes_{\mathcal{I}} C_{k(Y)} \longrightarrow C_{k(X\times_k Y)}$ is identical.

Corollary 4.2.2. *If $n = \infty$, X and Y are irreducible k-varieties, and X is a curve then $C_{k(X\times_k Y)} = C_{k(X)} \otimes_{\mathcal{I}} C_{k(Y)}$.* $\square$

Example. Let X_j, $1 \le j \le N$, and Y be irreducible k-varieties. Assume that all X_j, possibly except one of them, are curves. Let $Y \dashrightarrow X_j$, $1 \le j \le N$, be dominant maps. Then there is a natural morphism $C_{k(Y)} \longrightarrow C_{k(\prod_{j=1}^N X_j)}$. Indeed, in this case there is a natural morphism $\bigotimes_{j=1}^N C_{k(X_j)} \twoheadrightarrow C_{k(\prod_{j=1}^N X_j)}$, such that its composition with $\mathbb{Q}[\{k(Y) \overset{/k}{\hookrightarrow} F\}] \longrightarrow \mathbb{Q}[\prod_{j=1}^N \{k(X_j) \overset{/k}{\hookrightarrow} F\}] = \bigotimes_{j=1}^N \mathbb{Q}[\{k(X_j) \overset{/k}{\hookrightarrow} F\}] \twoheadrightarrow \bigotimes_{j=1}^N C_{k(X_j)}$ factors through $C_{k(Y)}$. Assuming that the functor $(-)_v$ from Proposition 1.2.2 is exact (cf. also p.44), one can construct a natural morphism $C_{k(D)} \longrightarrow C_{k(X)}$ for any irreducible divisor D on any irreducible k-variety X, cf. Corollary 4.4.8.

4.3. **Geometric construction of admissible representations.** Now we turn to constructing of a supply of semi-simple admissible representations of G. Conjecturally, in the case $n = \infty$ all semi-simple admissible representations of G are obtained in this way.

Set $B^q(X) = A^q(X)$, if $\sim$ is numerical equivalence (over k!). As before, $X_E := X \times_k E$ for any k-variety X and any field extension $E|k$.

Recall, that $B^q(X)$ is a limit of certain quotients of $\mathbb{Q}$-vector spaces of numerical equivalence classes of cycles of codimension q on the smooth proper k-varieties, but not over F, even if $X = Y_F$, cf. p.27 before Lemma 3.1.4.

Lemma 4.3.1. *Let $W \in \mathcal{S}m_G$. If $\mathrm{Hom}_G(Z^{\dim X}(k(X) \otimes_k F), W) = \mathrm{Hom}_G(CH_0(X_F), W)$ for any smooth proper k-variety X then W is semi-simple if and only if the $G_{F'|k}$-modules $W^{G_{F|F'}}$ are semi-simple for all algebraically closed F' of finite transcendence degree over k.*

Proof. Clearly, $W \in \mathcal{I}_G$. By Proposition 3.1.1, G-module W is semi-simple if and only if for any $L \subset F$ of finite type over k and any purely transcendental extension $L'|L$ in F with $\overline{L'} = F$ the module $W^{G_{F|L'}} = W^{G_{F|L}}$ over the Hecke algebra $\mathcal{H}_{G_{F|L'}} = h_{L'}\mathbb{D}h_{L'} \supseteq \langle h_{L'}\sigma h_{L'} \mid \sigma \in G\rangle_{\mathbb{Q}}$ is semi-simple. Here $h_{L'}$ is the Haar measure on $G_{F|L'}$. As W is a quotient of a direct sum of objects of type $CH_0(X_F)_{\mathbb{Q}}$, the action $\mathbb{D} \otimes W^{G_{F|L}} \twoheadrightarrow \mathbb{Q}[G/G_{F|L}] \otimes W^{G_{F|L}} \longrightarrow W$ factors through $CH_0(Y_F)_{\mathbb{Q}} \otimes W^{G_{F|L}} \longrightarrow W$, where Y is a smooth proper model of $L|k$, cf. Proposition 3.1.5. Then the action $\mathcal{H}_{G_{F|L'}}$ factors through $CH_0(Y_{k(Y)})_{\mathbb{Q}} = h_{L'}CH_0(Y_F)_{\mathbb{Q}}$.

In other words, the action of $\mathcal{H}_{G_{F|L'}}(G)$ on $W^{G_{F|L}}$ is determined by the action of $\mathcal{H}_{G_{\overline{L}|L}}(G_{\overline{L}|k})$, so the semi-simplicity of the $\mathcal{H}_{G_{F|L'}}(G)$-module $W^{G_{F|L}}$ is equivalent to its semi-simplicity as a $\mathcal{H}_{G_{\overline{L}|L}}(G_{\overline{L}|k})$-module. $\square$

Proposition 4.3.2. *G-module $\mathbf{B}^q_X := B^q(X_F)$ is admissible and semi-simple for any smooth proper k-variety X and any $q \ge 0$. If $q \in \{0, 1\}$, or $q = \dim X \le n$ then $\mathbf{B}^q_X$ is of finite length.*

Proof. By the standard argument, we may assume that k is embedded into the field $\mathbb{C}$ of complex numbers, and thus, for any smooth irreducible proper d-dimensional k-variety Y with $k(Y) =: L \subset F$, the space $B^q(X_F)^{G_{F|L}}$ is a quotient of a finite-dimensional space $Z^q(X \times_k Y)/\sim_{\mathrm{hom}} \subseteq H^{2q}((X \times_k Y)(\mathbb{C}), \mathbb{Q}(q))$, so the representation $B^q(X_F)$ is admissible.

By Proposition 3.1.5 and Lemma 4.3.1 (with $W = B^q(X_F)$, $W^{G_{F|L'}} = W^{G_{F|L}} = B^q(X_L)$), the semi-simplicity of the G-module $B^q(X_F)$ is equivalent to the semi-simplicity of the $\mathcal{H}_{G_{\overline{L}|L}}(G_{\overline{L}|k})$-modules $B^q(X_L)$ for all $L \subset F$ as above.

The kernel of $A^q(X \times_k Y) \longrightarrow A^q(X_{k(Y)})$ is a $A^d(Y \times_k Y)$-submodule in $A^q(X \times_k Y)$, since $\alpha \circ \beta = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*\alpha \cdot \mathrm{pr}_{23}^*\beta)$ for its arbitrary element α and for any element $\beta \in A^d(Y \times_k Y)$, so the projection to Y of the support of $\alpha \circ \beta$ is contained in $\mathrm{pr}_2((D \times_k Y) \bigcap \mathrm{supp}(\beta))$ for some divisor

D on Y. As its dimension is equal to $d-1$, the divisor D cannot dominate Y. This implies that $A^q(X_{k(Y)})$ carries a natural $A^d(Y \times_k Y)$-module structure.

According to [15], the algebra $B^d(Y \times_k Y)$ is semi-simple, so the $B^d(Y \times_k Y)$-module $B^q(X_{k(Y)})$ is also semi-simple. By the moving Lemma 3.1.4, the ring homomorphism $\mathcal{H}_{G_{\overline{L}|L}}(G_{\overline{L}|k}) \longrightarrow B^d(Y \times_k Y)$, induced by the identification of the Hecke algebra $\mathcal{H}_{G_{\overline{L}|L}}(G_{\overline{L}|k})$ with the algebra of non-degenerate correspondences on Y (cf. p.27), is surjective. This endows any $B^d(Y \times_k Y)$-module with a structure of a (semi-simple) $\mathcal{H}_{G_{\overline{L}|L}}(G_{\overline{L}|k})$-module.

The length of any cyclic semi-simple G-module, in particular of $\mathbf{B}_X^{\dim X}$ (Lemma 3.1.4), is finite.

It follows from Lefschetz' theorems on hyperplane section and on (1,1)-classes that $\mathbf{B}_X^q$ is a subquotient of $\mathbf{B}_H^q$ for any smooth q-dimensional plane section H of X, so the length of $\mathbf{B}_X^q$ is also finite. □

Corollary 4.3.3. $\mathrm{Hom}_G(B^q(L' \otimes_k F), B^p(L \otimes_k F)) = 0$ *for any pair of fields L, L' of finite type over k with* $\mathrm{tr.deg}(L|k) = p$, $\mathrm{tr.deg}(L'|k) = q$ *and* $p \neq q$.

Proof. If either $p > n$, or $q > n$, then at least one of the modules $B^q(L' \otimes_k F)$ and $B^p(L \otimes_k F)$ is zero, so we may assume that $\max(p, q) \le n$. By Proposition 4.3.2, the G-modules $B^q(L' \otimes_k F)$ and $B^p(L \otimes_k F)$ are semi-simple, so $\mathrm{Hom}_G(B^q(L' \otimes_k F), B^p(L \otimes_k F))$ is isomorphic to $\mathrm{Hom}_G(B^p(L \otimes_k F), B^q(L' \otimes_k F))$, so we may assume that $p > q$. Then, by Corollary 3.1.6, $\mathrm{Hom}_G(B^q(L' \otimes_k F), B^p(L \otimes_k F)) = B^p(L \otimes_k L') = 0$. □

Corollary 4.3.4 ([35], 3.12)**.** *For any smooth irreducible k-variety X of dimension $\le n+1$ and for $q \in \{0, 1, 2, \dim X\}$ there exists a unique G-submodule in $\mathbf{B}_X^q = B^q(X \times_k F)$, isomorphic to $B^q(k(X) \otimes_k F)$.*

For each open compact subgroup $U \subset G$ and a smooth irreducible k-variety Y with $k(Y) = F^U$ define a semi-simple G-module (of finite length) $\mathbf{B}_{Z,Y}^q$ as the minimal one among such that the $\mathcal{H}_{\mathbb{Q}}(U)$-module $(\mathbf{B}_{Z,Y}^q)^U$ is isomorphic to $B^q(Z \times_k Y)$. By Proposition 3.1.1, it exists and it is unique.

Lemma 4.3.5 ([35], 3.13)**.** *Let X, Y and Z be smooth irreducible k-varieties,* $\dim X = \dim Y = n \ge \dim Z$, *and $p, q \ge 0$ be integers.*

Then $\mathrm{Hom}_{\mathcal{H}_{\mathbb{Q}}(U)}(B^q(Z \times_k Y), B^p(k(X) \otimes_k k(Y))) = 0$, *if either $q = \dim Z < p$, or $q = n$ and $\dim Z < p$, or $q > n$ and $p + q > \dim Z + n$, or $q < p$ and $q \in \{0, 1\}$.*

Proposition 4.3.6 ([35], 3.14)**.** *Let X and Y be smooth irreducible k-varieties, and either $q \in \{0, 1, 2\}$, or $q = \dim X = \dim Y$. Then there is a unique submodule in $B^q(X \times_k Y)$ over the algebra $\left(B^{\dim X}(X \times_k X) \otimes B^{\dim Y}(Y \times_k Y)^{\mathrm{op}}\right)$, isomorphic to its quotient $B^q(k(X) \otimes_k k(Y))$.*

The existence of such submodule follows from the semi-simplicity of the module $B^q(X \times_k Y)$ ([15]). The uniqueness follows from Lemma 4.3.5.

Representations contragredient to the motivic ones. For each open compact subgroup U in $\mathfrak{G}$ (cf. §2.5) fix a smooth proper irreducible k-variety Y_U and an embedding of its function field into F such that $F^U = k(Y_U)L_m$ for an integer $m \ge 1$, where $k(Y_U)$ and L_m are algebraically independent over k, i.e. $\dim_k Y_U = m - 1$. Let $n_U := \dim_k Y_U$. If Y_U' is another variety with the same properties then there is a canonical isomorphism $B^{n_U'}((Y_U')_L) = B^{n_U}((Y_U)_L)$, induced by direct image isomorphisms of type $B^{n_U}((Y_U)_L) = B^{n_U+q}((Y_U \times \mathbb{P}^q)_L)$.

For any ordered pair of open compact subgroups $U \supseteq U'$ in $\mathfrak{G}$ one can choose smooth proper irreducible k-varieties $Y_U, Y_{U'}$ and embeddings of their function fields into F such that $F^U = k(Y_U)L_m$ and $F^{U'} = k(Y_{U'})L_m$ for an integer $m \ge 1$, and $k(Y_U)k(Y_{U'})$ and L_m are algebraically independent over k. Then the direct image homomorphism induces a canonical embedding $B^{n_U}((Y_U)_L) \hookrightarrow B^{n_{U'}}((Y_{U'})_L)$.

This enables us to form an inductive system $(B^{n_U}((Y_U)_{k(X)}))_U$, where U runs over the set of open compact subgroups in $\mathfrak{G}$.

Corollary 4.3.7. *Let X and Y be smooth irreducible proper k-varieties. then the $\mathbb{Q}$-vector spaces $B^{\dim X}(X_{k(Y)})$ and $B^{\dim Y}(Y_{k(X)})$ are naturally dual to each other. If $\dim X \le n$ then this duality induces a non-degenerate $\mathfrak{G}$-equivariant pairing of admissible $\mathfrak{G}$-modules*

$$B^{\dim X}(X_F) \otimes \lim_{U \longrightarrow} B^{n_U}((Y_U)_{k(X)}) \longrightarrow \mathbb{Q}(\chi).$$

Proof. Let $n \ge \dim Y \ge \dim X$. By Proposition 3.1.5,

$$B^{\dim X}(X_{k(Y)}) = \mathrm{Hom}_G(B^{\dim Y}(Y_F), B^{\dim X}(X_F))$$

and $B^{\dim Y}(Y_{k(X)}) = \mathrm{Hom}_G(B^{\dim X}(X_F), B^{\dim Y}(Y_F))$. By Proposition 4.3.2, the representations $B^{\dim Y}(Y_F)$ and $B^{\dim X}(X_F)$ of G are semi-simple, and their lengths are finite. For any $\alpha \in B^{\dim X}(X_{k(Y)})$ and $\beta \in B^{\dim Y}(Y_{k(X)})$ set $\langle \alpha \cdot \beta \rangle = \mathrm{tr}(\alpha \circ \beta)$ $(= \mathrm{tr}(\beta \circ \alpha))$. Here α and β are considered as G-homomorphisms. If $\alpha \ne 0$ then there is an element $\gamma \in B^{\dim Y}(Y_{k(X)})$ such that $\alpha \circ \gamma$ is a non-zero projector in $\mathrm{End}_G B^{\dim X}(X_F)$, so the form $\langle \quad \cdot \quad \rangle$ is non-degenerate. Define a form $B^{\dim X}(X_F) \otimes \lim_{U \longrightarrow} B^{n_U}((Y_U)_{k(X)}) \longrightarrow \mathbb{Q}(\chi)$ by $\alpha \otimes \beta \longmapsto \langle \alpha \cdot \beta \rangle \cdot [U]$ for any $\alpha \in B^{\dim X}(X_F)^U$ and $\beta \in B^{n_U}((Y_U)_{k(X)})$. This is well-defined by the projection formula. □

The projector $\Delta_{k(X)}$. For any pair of varieties X, Y let ^{t} be the transposing of cycles, induced by $X \times Y \xrightarrow{\sim} Y \times X$. Denote by $\Delta_{k(X)} = {}^t\Delta_{k(X)}$ the identity (diagonal) element in $B^{\dim X}(k(X) \otimes_k k(X))$, considered as an element of the ring $B^{\dim X}(X \times_k X)$.

Lemma 4.3.8 ([35], 3.15, 3.16)**.** *For any irreducible smooth proper k-variety X of dimension n the element $\Delta_{k(X)}$ is a central projector of the algebra $B^n(X \times_k X)$. The left (equivalently, the right) ideal, generated by $\Delta_{k(X)}$, coincides with (the image of the ring) $B^n(k(X) \otimes_k k(X))$. $\Delta_{k(X)} B^{\dim X}(X_L) = B^{\dim X}(k(X) \otimes_k L)$ for any field extension $L|k$.*

Proposition 4.3.9 ([35], 3.17)**.** *$(X, \Delta_{k(X)})$ is the maximal primitive n-submotive of the motive (X, Δ_X) for any irreducible smooth proper n-dimensional k-variety X. The motive $(X, \Delta_{k(X)})$ is a birational invariant of X.*

The birational invariantness of $(X, \Delta_{k(X)})$ follows from the birational invariantness of Y^{prim} for any smooth projective k-variety Y, cf. p.42 below.

Corollary 4.3.10 ([35], 3.18)**.** *Let X and Y be smooth irreducible proper k-varieties, and $\dim X = \dim Y = n$. Then $\Delta_{k(X)} \cdot B^n(X \times_k Y) = B^n(X \times_k Y) \cdot \Delta_{k(Y)} = \Delta_{k(X)} \cdot B^n(X \times_k Y) \cdot \Delta_{k(Y)}$ is a unique $(B^n(X \times_k X) \otimes B^n(Y \times_k Y)^{\mathrm{op}})$-submodule in $B^n(X \times_k Y)$, isomorphic to its quotient $B^n(k(X) \otimes_k k(Y))$. Similarly, $B^q(k(X) \otimes_k k(Y)) = \Delta_{k(X)} \cdot B^q(X \times_k Y) \cdot \Delta_{k(Y)}$ for $q = 0, 1$.*

The functors $\mathbb{B}^\bullet$ and $\mathfrak{B}^q$. For a smooth projective k-variety Y let the motive Y^{prim} be defined as the intersection of the kernels of all morphisms $\varphi : Y \longrightarrow M \otimes \mathbb{L}$ for all possible *effective* motives M, or equivalently, Y^{prim} is the cokernel of the morphism $\sum \varphi : \bigoplus_{M \otimes \mathbb{L} \xrightarrow{\varphi} Y} M \otimes \mathbb{L} \longrightarrow Y$ (with the same M). Clearly, $Y \longmapsto Y^{\mathrm{prim}}$ is a functor from the category of smooth projective varieties to the category of pure motives. Any birational map is a composition of a blow-up and a blow-down with smooth centres ([1, 47]). As a blow-up does not affect Y^{prim} (cf. [28]), this implies that Y^{prim} is an invariant of the function field $k(Y)$. According to the Hironaka theorem, for any extension $L|k$ in F of finite type there is a smooth projective k-variety $Y_{[L]}$ with the function field L, and therefore, we get a canonical projective system of motives $\{Y_{[L]}^{\mathrm{prim}}\}_L$, indexed by the subfields L in F of finite type over k.

Now define the functor $\mathbb{B}^\bullet = \bigoplus \mathbb{B}^{[i]}$ in Theorem 1.1.2 from the category of pure motives to the category of graded $\mathbb{Q}$-vector spaces, by setting $\mathbb{B}^{[i]} = \lim_{L \longrightarrow} \mathrm{Hom}\left(Y_{[L]}^{\mathrm{prim}} \otimes \mathbb{L}^{\otimes i}, -\right)$ for its degree i component. Let also $\mathfrak{B}^q$ denote the restriction of $\mathbb{B}^{[0]}$ to the subcategory of primitive q-motives.

G acts on the projective system $\{Y^{\mathrm{prim}}_{[L]}\}_L$ by $Y_{[L]} \xrightarrow{\sigma} Y_{[\sigma(L)]}$, $\sigma(L) \xrightarrow{\sigma^{-1}} L$, so G acts on the limits $\mathfrak{B}^q(M)$ and $\mathbb{B}^{\bullet}(M)$.

Remark. Any pure motive $M = (X, \pi)$ is isomorphic to $\bigoplus\limits_{0\le i,j,i+j\le \dim X} M_{ij} \otimes \mathbb{L}^{\otimes i}$, where M_{ij} is a primitive j-motive and $\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times \{0\})$, so $\mathbb{B}^{[i]}(M) \cong \bigoplus_j \mathfrak{B}^j(M_{ij})$. This is proved by induction on dimension d of X as follows. Let $M_{0d} = \bigcap_{\varphi} \ker(\varphi)$, where φ runs over the morphisms from M to the motives of type $(Y \times \mathbb{P}^1, \Delta)$ for all Y with $\dim Y < d$. (By Proposition 4.3.9, $M_{0d} = (X, \pi \circ \Delta_{k(X)})$.) As the length of M is $\le \dim_{\mathbb{Q}} \mathrm{End}(M) < \infty$, the motive M/M_{0d} can be embedded into a finite direct sum of motives $(Y_j \times \mathbb{P}^1, \Delta)$ with $\dim Y_j < d$. As $(Y_j \times \mathbb{P}^1, \Delta) = (Y_j, \Delta) \oplus (Y_j, \Delta) \otimes \mathbb{L}$, the induction is completed. In fact, the decomposition $M = \bigoplus\limits_{0\le i,j,i+j\le \dim X} \widetilde{M}_{ij}$, where $\widetilde{M}_{ij}$ is isomorphic to $M_{ij} \otimes \mathbb{L}^{\otimes i}$, is canonical, since $\widetilde{M}_{ij}$ is the sum of the images of all morphisms $\varphi : N \otimes \mathbb{L}^{\otimes i} \longrightarrow M$ for all possible *primitive* j-motives N.

Proposition 4.3.11 ([35], 3.19). *If $\dim X = q \le n$ and $M = (X, \pi)$ is a primitive q-motive then $\mathfrak{B}^q(M) = \mathbb{B}^{[0]}(M) = \pi B^q(X_F)$.*

The proof uses an equivariant version of resoluiton of singularities and [15].

It follows from Proposition 4.3.11 and Lemma 4.3.8 that

Corollary 4.3.12 ([35], 3.20). *$\Delta_{k(X)} B^d(X \times_k Y) = B^d(k(X) \otimes_k k(Y))$, and $\Delta_{k(X)} B^q(X \times_k Y)$ vanishes for any $q < d := \dim X$, any irreducible smooth proper k-variety X and any irreducible smooth k-variety Y.*

"Polarization" on $B^n(k(X) \otimes_k F)$ and polarizable G-modules.

Proposition 4.3.13 ([35], 3.21). *For any irreducible k-variety X of dimension n there exists a symmetric G-equivariant non-degenerate pairing*

$$B^n(k(X) \otimes_k F) \otimes B^n(k(X) \otimes_k F) \xrightarrow{\langle\ ,\ \rangle} \mathbb{Q}(\chi)$$

such that $\langle p^*(\cdot),\ \cdot\ \rangle = \langle\ \cdot\ , p_*(\cdot)\rangle$ for any generically finite rational map p. In particular, $\langle\ ,\ \rangle$ induces a non-degenerate pairing between the submodules $W := \pi B^n(k(X) \otimes_k F)$ and ${}^tW := {}^t\pi B^n(k(X) \otimes_k F)$ for all projectors $\pi \in B^n(k(X) \otimes_k k(X))$.

If for the $(n-1)$-cycles on the $2n$-dimensional complex varieties numerical equivalence coincides with homological one then $\langle\ ,\ \rangle$ is $(-1)^n$-definite. E.g., this is true for $n \le 2$.

The form $\langle \alpha, \gamma\rangle \in \mathbb{Q}(\chi)$ is defined as $\langle \widehat{\alpha} \cdot \widehat{\gamma}\rangle \cdot [U]$, where $\alpha, \gamma \in B^n(k(X) \otimes_k F)$ are fixed by a compact open subgroup $U \subset G$, $\widehat{\alpha}, \widehat{\gamma}$ are the images of $\alpha, \gamma \in B^n(k(X) \otimes_k k(Y_U))$ in $B^n(X \times_k Y_U)$ in the sense of Proposition 4.3.6. Here Y_U is a smooth proper k-variety with the function field $k(Y_U)$, identified with F^U, and $\langle\ \cdot\ \rangle$ is the intersection form on $B^n(X \times_k Y_U)$. By the projection formula, $\langle \alpha, \gamma\rangle$ is independent of the choices made, and $\langle p^*(\cdot),\ \cdot\ \rangle = \langle\ \cdot\ , p_*(\cdot)\rangle$.

The rest of the proof uses the standard intersection theory, Lemma 4.3.5, and Hodge index theorem.

4.4. **Valuations and associated functors,** [39]. In this section to each smooth representation of G we associate a sheaf in the smooth topology on $\mathrm{Spec}(k)$. For that to each smooth k-variety X, its scheme-theoretic point $p \in X$ and an embedding $k(X_p) \overset{/k}{\hookrightarrow} F$ we associate a collection $J_{X,p}$ of subfields in F, and define the stalks by $\mathcal{W}_{X,p} := W^{G_{F|k(X_p)}} \cap \left(\sum_{F'' \in J_{X,p}} W^{G_{F|F''}}\right)$.

Naturally, for each $q \ge 0$ we would like to obtain the sheaf $\Omega^q_{\mathcal{O}|k}$ from the smooth representation $\Omega^q_{F|k}$ of G. There are two options for the representation $\bigotimes^q_F \Omega^1_{F|k}$: one "homotopy invariant" (and more natural) – $\bigotimes^q_{\mathcal{O}} \Omega^1_{\mathcal{O}|k}$, and another with the Galois descent property. The "homotopy invariance" means that for any projective bundle $X \longrightarrow Y$ over a smooth proper base Y the induced map of sections is an isomorphism.

The ''globalization" functor. For any collection J of subfields $F'' \subset F$ the additive functor $\Phi_J : W \mapsto \sum_{F'' \in J} W^{G_{F|F''}}$ on $\mathcal{S}m_G$ preserves the surjections, if any element of J is contained in an element of J of an arbitrary big finite, or countable transcendence degree over k, and the injections in general.

Remark. If a collection J consists of all purely transcendental extensions of k then $\Phi_J(\Omega^q_{F|k}) = \Omega^q_{F|k}$ if $n > q$, and $\Phi_J(\Omega^q_{F|k,\mathrm{reg}}) = 0$ for any $q \geq 1$. Therefore, in general Φ_J is not exact, even if $n = \infty$.

In particular, to any discrete valuation ring $\mathcal{O}_v \in \mathcal{P}^r_F$ one can associate the set J of all its subfields. We consider the following functor Φ_J: $(-)_v : \mathcal{S}m_G \longrightarrow \mathcal{S}m_{G_v}$, $W \mapsto W_v := \sum_{\sigma \in G_v} W^{G_{F|\sigma(F')}} = \sum_{\sigma \in G_v^\dagger} W^{G_{F|\sigma(F')}} \subseteq W$. Set $\Gamma_r(W) := \bigcap_{\mathcal{O}_v \in \mathcal{P}^r_F} W_v$ and $\Gamma := \Gamma_1$, i.e., Γ_r are additive functors on $\mathcal{S}m_G$ to itself.

Example. $\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]_v = \mathbb{Q}[\{L \overset{/k}{\hookrightarrow} \mathcal{O}_v\}]$ and $(F[\{L \overset{/k}{\hookrightarrow} F\}])_v = \mathcal{O}_v[\{L \overset{/k}{\hookrightarrow} \mathcal{O}_v\}]$ (and all these modules are zero, if $\mathrm{tr.deg}(L|k) > n - r$); $\Gamma(\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]) = \Gamma(F[\{L \overset{/k}{\hookrightarrow} F\}]) = 0$, if $L \neq k$.

Lemma 4.4.1. *If $n = \infty$ then there are canonical isomorphisms*

$$\mathrm{Hom}_G(W, W') \xrightarrow{\sim} \mathrm{Hom}_{G_v}(W_v, W') = \mathrm{Hom}_{G_v}(W_v, W'_v)$$

for any $W, W' \in \mathcal{S}m_G$. In particular, the functor $(-)_v : \mathcal{S}m_G \longrightarrow \mathcal{S}m_{G_v}$ is fully faithful.

Remark. Clearly, the functor $(-)_v$ does not preserve the irreducibility: usually the surjection $W_v \longrightarrow H_0(G_v^\dagger, W_v)$ is non-trivial and non-injective. E.g., the length of the G_v-module (F/k) is $r+1$: $(F/k)_v = \mathcal{O}_v/k \supsetneqq \mathfrak{m}_v = \mathfrak{p}_1 \supsetneqq \mathfrak{p}_2 \supsetneqq \cdots \supsetneqq \mathfrak{p}_r$. However, $(-)_v$ preserves the existence of a cyclic vector: if $W \in \mathcal{S}m_G$ is cyclic then the $G_{F'/k}$-module $W^{G_{F/F'}}$ admits some cyclic vector w (as $H^0(G_{F/F'}, -) : \mathcal{S}m_G \longrightarrow \mathcal{S}m_{G_{F'/k}}$ is an equivalence of categories), and thus, w generates the G_v-module W_v. It follows from Lemma 4.4.1 that if W is irreducible and W_v is semi-simple then W_v is irreducible.

Lemma 4.4.2. *Let J be a collection of algebraically closed subfields $F'' \subset F$ of countable transcendence degree over k. Then the functor Φ_J is exact if and only if one of the following equivalent conditions on J holds:*

- *for any integer $N \geq 1$, any extension L of k of finite type, any collection of embeddings $\xi_j : L \overset{/k}{\hookrightarrow} F_j$ such that $F_j \in J$ for all $1 \leq j \leq N$, and any $\sigma : L \overset{/k}{\hookrightarrow} F$ there is an element $\alpha \in \mathbb{Q}[G]$ such that $\alpha\xi_j = 0$ for all $1 \leq j \leq N$ and $\alpha\sigma - \sigma \in \mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F'' \mid F'' \in J\}]$;*
- *for any irreducible k-variety X, any integer $N \geq 1$, any collection of dominant k-morphisms $f_j : X \longrightarrow Y_j$ such that $\dim Y_j < \dim X$ for all $0 \leq j \leq N$ and f_0 factors through f_j for neither of $1 \leq j \leq N$, and any generic point $\sigma : k(Y_0) \overset{/k}{\hookrightarrow} F$ there is a generic 0-cycle $\alpha \in \mathbb{Q}[X(F)]$ such that $(f_j)_*\alpha = 0$ for all $1 \leq j \leq N$, and $(f_0)_*\alpha - \sigma \in \mathbb{Q}[\{k(Y_0) \overset{/k}{\hookrightarrow} F'' \mid F'' \in J\}]$.*

The conditions of Lemma 4.4.2 are satisfied for σ in general position with respect to the compositum of all $\xi_j(L)$. This and the following fact suggest that the functor $(-)_v$ can be exact.

Proposition 4.4.3. *Let H be an algebraic k-group, $N \geq 1$ be an integer, and H_i be a k-subgroup for each $0 \leq i \leq N$. Suppose that H_j normalizes H_i for each pair $0 \leq i < j \leq N$, and H_i is contained in H_0 for neither $1 \leq i \leq N$. Denote by $f_i : H \longrightarrow H/H_i$ the corresponding projections. Then there exits a 0-cycle $\alpha \in \mathbb{Q}[H(F)]$ such that $(f_i)_*\alpha = 0$ for any $1 \leq i \leq N$, and $(f_0)_*\alpha \neq 0$. More explicitly, almost all 0-cycles of type $(h_1 - 1) \cdots (h_N - 1)$, where $h_i \in H_i$ for all $1 \leq i \leq N$, satisfy these conditions.*

Denote by $\mathcal{I} = \mathcal{I}_{/k} = \mathcal{I}_{F|k} : \mathcal{S}m_G \longrightarrow \mathcal{I}_G$ the left adjoint of the inclusion functor $\mathcal{I}_G \hookrightarrow \mathcal{S}m_G$, and set $C_L := \mathcal{I}_{F|k}\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]$ for any finitely generated extension $L|k$, cf. Theorem 1.1.6 (3).

Lemma 4.4.4. *If $n=\infty$ and $r=1$ the projection $\mathbb{Q}[\{k(X)\stackrel{/k}{\hookrightarrow}\mathcal{O}_v\}]\longrightarrow C_{k(X)}$ is surjective for any irreducible variety X over k. In particular, $(C_{k(X)})_v=C_{k(X)}$, and $W_v=W$ (and thus, $\Gamma(W)=W_v=W$) for any $W\in\mathcal{I}_G$.*

Lemma 4.4.5. *Suppose that $n=\infty$. Then $(W_1\otimes W_2)_v\subseteq(W_1)_v\otimes(W_2)_v$ and $\Gamma(W_1\otimes W_2)\subseteq\Gamma(W_1)\otimes\Gamma(W_2)$ for any $W_1,W_2\in\mathcal{S}m_G$. However, $(W\otimes W)_v\neq W_v\otimes W_v$, if $W=\mathbb{Q}[F\smallsetminus k]$. If either W_1 is a quotient of $A(F)$ for some commutative algebraic k-group A, or $W_1\in\mathcal{I}_G$, then $(W_1\otimes W_2)_v=(W_1)_v\otimes(W_2)_v$ for any $W_2\in\mathcal{S}m_G$.*

Remark. 1. If W carries an F-vector space structure $F\otimes W\longrightarrow W$ then, by Lemma 4.4.5, W_v carries an $\mathcal{O}_v$-module structure: $(F\otimes W)_v=\mathcal{O}_v\otimes W_v\longrightarrow W_v$. Clearly, the morphism $F\otimes_{\mathcal{O}_v}W_v\longrightarrow W$ is injective, but not surjective, as it shows the example of $W=F[\{L\stackrel{/k}{\hookrightarrow}F\}]$.

2. Clearly, Γ_r preserves injection, but not surjections. Namely, let $W:=\bigotimes_k^N F\longrightarrow\Omega^{N-1}_{F|k}$ be given by $a_1\otimes\cdots\otimes a_N\mapsto a_1da_2\wedge\cdots\wedge da_N$. Then $W_v=\bigotimes_k^N\mathcal{O}_v$, if $n\geq 2N$, so $(\bigotimes_F^{N-1}\Omega^1_{F|k_0})_v=\bigotimes_F^{N-1}\Omega^1_{\mathcal{O}_v|k_0}$ for any $k_0\subseteq k$; and $\Gamma(\bigotimes_k^N F)=k$, but $\Gamma_r(\Omega^\bullet_{F|k})=\Omega^\bullet_{F|k,\mathrm{reg}}$ for any $r\geq 1$, cf. [39]. In the case $n=\infty$ one can also use Lemma 4.4.5.

For an integral normal k-variety X with $k(X)\subset F$ let $\mathfrak{V}(X)$ be the set of all discrete valuations of F of rank one, trivial on k such that their restrictions to $k(X)$ are either trivial, or correspond to divisors on X.

Set $\mathcal{W}(X):=W^{G_{F|k(X)}}\cap\bigcap_{v\in\mathfrak{V}(X)}W_v\subseteq\mathcal{W}$.

Clearly, if a dominant morphism $U\longrightarrow X$ transforms the divisors on U to divisors on X then $\mathfrak{V}(U)\subseteq\mathfrak{V}(X)$, so $\mathcal{W}(X)\subseteq\mathcal{W}(U)$.

If $X=U_1\cup U_2$ then $\mathfrak{V}(X)=\mathfrak{V}(U_1)\cup\mathfrak{V}(U_2)$, since $X^1=U_1^1\cup U_2^1$, so $\mathcal{W}(X)=\mathcal{W}(U_1)\cap\mathcal{W}(U_2)$, i.e., $U\mapsto\mathcal{W}(U)$ for open $U\subseteq X$ is a Zariski sheaf on X.

Remark. $W^{G_{F|k(X)}}\cap W_v$ depends only on the restriction of v to $k(X)$, since the set of $G_{F|k(X)}$-orbits $G_{F|k(X)}\backslash G/G_v$ of valuations of F coincides, by Proposition 2.4.2, with the set of discrete valuations of $k(X)$ of rank $\leq r$. E.g., if the restriction of v to $k(X)$ is trivial then $W^{G_{F|k(X)}}\subseteq W_v$.

Examples. 1. If $V=\mathbb{Q}[\{L\stackrel{/k}{\hookrightarrow}F\}]$, or $V=F[\{L\stackrel{/k}{\hookrightarrow}F\}]$ then $\mathcal{V}(U)=0$ for any non-trivial field extension $L|k$ of finite type and any smooth U over k.

2. If $V=\Omega^\bullet_{F|k}$ then $\mathcal{V}(U)=\Omega^\bullet_{\mathcal{O}(U)|k}$ for any smooth U over k.

3. If $V=\mathrm{Sym}^s_F\Omega^1_{F|k}$ then $\mathcal{V}(U)\subset\mathrm{Sym}^s_{k(U)}\Omega^1_{k(U)|k}$ consists of elements with poles (with respect to the lattice $\mathrm{Sym}^s_{\mathcal{O}(U)}\Omega^1_{\mathcal{O}(U)|k}$) of order $<s$ for any smooth curve U over k.

Note, that $\mathcal{V}$ is functorial with respect to all morphisms of smooth k-varieties; $\Gamma(V)$ is "homotopy invariant" if and only if $s=1$.

4. If $V=W\otimes F$ for some $W\in\mathcal{I}_G$ then $\mathcal{V}(U)=(W^{G_{F|\overline{k(U)}}}\otimes\overline{\mathcal{O}(U)})^{G_{\overline{k(U)}|k(U)}}$ for any irreducible smooth affine U over k, where $\overline{\mathcal{O}(U)}$ is the integral closure of $\mathcal{O}(U)$ in F.

Consider the following site $\mathfrak{H}$. The objects of $\mathfrak{H}$ are the smooth k-varieties. The morphisms in $\mathfrak{H}$ are the locally dominant morphisms, transforming the non-dominant divisors to divisors. The coverings are smooth morphisms, surjective over the generic point of each divisor downstairs. Denote by $\mathrm{Shv}(\mathfrak{H})$ the category of sheaves on $\mathfrak{H}$. Consider the functor $\Phi:\mathrm{Shv}(\mathfrak{H})\longrightarrow\mathcal{S}m_G$, given by $\mathcal{F}\mapsto\mathcal{F}(F):=\lim\limits_{A\longrightarrow}\mathcal{F}(\mathbf{Spec}(A))\in\mathcal{S}m_G$. Here A runs over the smooth k-subalgebras of F.

Example. If $j\leq 1$ then $\mathcal{F}:X\mapsto Z^j(X_L)$ is a sheaf on $\mathfrak{H}$, and $\mathcal{F}(F)=Z^j(L\otimes_k F)$. In particular, Φ is not faithful, since $\mathcal{F}(F)=0$ if $j=1$ and $L=k$.

Proposition 4.4.6. *A choice of embeddings into F over k of the function fields of all irreducible k-varieties determines a functor $\mathcal{S}m_G\longrightarrow\mathrm{Shv}(\mathfrak{H})$, $V\mapsto\mathcal{V}$.*

Question. Is it right adjoint to Φ?

The "specialization" functor.

Lemma 4.4.7. *If $r = 1$ and $n = \infty$ then $H_0(G_v^\dagger, -_v)$ gives functors $\mathcal{S}m_G \longrightarrow \mathcal{S}m_{G_{\kappa(v)|k}}$ and $\mathcal{I}_G \longrightarrow \mathcal{I}_{G_{\kappa(v)|k}}$. The natural homomorphisms of $G_{\kappa(v)|k}$-modules $W^{G_{F|F'}} \longrightarrow\!\!\!\!\rightarrow H_0(G_v^\dagger, W_v)$ are surjective for all $W \in \mathcal{S}m_G$. They are isomorphisms, if the functor $(-)_v$ is exact.*

Corollary 4.4.8. *For any smooth irreducible divisor D on any smooth proper irreducible k-variety X there is a natural morphism $C_{k(D)} \longrightarrow C_{k(X)}$, if $(-)_v$ is exact for $r = 1$, making commutative the diagram*
$$\begin{array}{ccc} C_{k(D)} & \longrightarrow\!\!\!\!\rightarrow & CH_0(D_F)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ C_{k(X)} & \longrightarrow\!\!\!\!\rightarrow & CH_0(X_F)_{\mathbb{Q}} \end{array}.$$

This would be evident if the first part of Conjecture 1.1.7 hold true.

Lemma 4.4.9. *Let $\mathcal{F}$ be a functor on the category of smooth k-varieties (and of all their morphisms). Suppose that $\mathcal{F}(\mathcal{O}_v) = \mathcal{F}(F)_v$, cf. p.30. Then $H_0(G_v^\dagger, \mathcal{F}(F)_v) = \mathcal{F}(\kappa(v))$.*

Examples. 1. For a smooth proper k-variety X and $q \geq 0$ the functor $\mathcal{F} : Y \mapsto CH^q(X \times_k Y)$ satisfies the assumptions of Lemma 4.4.9, and $\mathcal{F}(F) = CH^q(X_F)$.

The isomorphism $H_0(G_v^\dagger, CH^q(X_F)) = CH^q(X_{\kappa(v)})$ is nothing but the specialization homomorphism $CH^q(X_F) \longrightarrow\!\!\!\!\rightarrow CH^q(X_{\kappa(v)})$ (cf. [42]), which is $G_v^\dagger$-invariant, and thus, factors through the coinvariants $H_0(G_v^\dagger, CH^q(X_F))$.

2. The functor $\mathcal{F} : Y \mapsto \Gamma(\overline{Y}, \Omega^\bullet_{\overline{Y}|k})$, where $\overline{Y}$ is a smooth compactification of Y, also satisfies the assumptions of Lemma 4.4.9, and $\mathcal{F}(F) = \mathcal{F}(F)_v = \Omega^\bullet_{F|k,\mathrm{reg}}$.

The reduction modulo the maximal ideal induces a surjection $\Omega^\bullet_{\mathcal{O}_v|k} \longrightarrow\!\!\!\!\rightarrow \Omega^\bullet_{\kappa(v)|k}$ and an isomorphism $H_0(G_v^\dagger, \Omega^\bullet_{F|k,\mathrm{reg}}) = \Omega^\bullet_{\kappa(v)|k,\mathrm{reg}}$. $\square$

3. $\mathcal{F} : Y \mapsto \mathbb{Q}[\mathcal{O}(Y)]$ is an example of a functor with $\mathcal{F}(\mathcal{O}_v) = \mathbb{Q}[\mathcal{O}_v] \neq \mathcal{F}(F)_v = \mathbb{Q}[\mathcal{O}_v \smallsetminus (k + \mathfrak{m}_v)] \oplus \mathbb{Q}[k]$. However, even in this case one has $H_0(G_v^\dagger, \mathcal{F}(F)_v) = \mathcal{F}(\kappa(v)) = \mathbb{Q}[\kappa(v)]$.

Corollary 4.4.10. *Let X be an irreducible variety over k with the function field embedded into F, and $Y \subset X$ be an irreducible divisor. The discrete valuations $v : F^\times/k^\times \longrightarrow\!\!\!\!\rightarrow \Gamma$ of F of rank 1 such that $k(X) \cap \mathcal{O}_v = \mathcal{O}_{X,Y}$ (so $\kappa(v|_{k(X)}) = k(Y)$) form a single $G_{F|k(X)}$-orbit. Then any embedding $k(Y) \overset{/k}{\hookrightarrow} F$ induces a canonical isomorphism $W^{G_{F|k(Y)}} \xrightarrow{\sim} H_0(G_v^\dagger, W_v)^{G_{F|k(X)} \cap G_v}$, if $(-)_v$ is exact.*

Proof. By Lemma 4.4.7, $W^{G_{F|F'}} \xrightarrow{\sim} H_0(G_v^\dagger, W_v)$. One can show that the sequence $1 \longrightarrow G_{F|L} \cap G_v^\dagger \longrightarrow G_{F|L} \cap G_v \longrightarrow G_{\kappa(v)|\kappa(v|_L)} \longrightarrow 1$ is exact, which implies that
$$\left(W^{G_{F|F'}}\right)^{G_{\kappa(v)|\kappa(v|_L)}} \xrightarrow{\sim} H_0(G_v^\dagger, W_v)^{G_{F|L} \cap G_v} = H_0(G_v^\dagger, W_v)^{G_{\kappa(v)|\kappa(v|_L)}}. \qquad \square$$

Restrictions on the objects of $\mathcal{I}_G$ and on the quotients of objects of $\mathcal{I}_G \otimes F$. Suppose that $n = \infty$.

One gets from Lemma 1.2.3 the conditions $V_v \otimes_{\mathcal{O}_v} F = V$ and $\Gamma(V) \otimes_k F \longrightarrow\!\!\!\!\rightarrow V$ for any $W \in \mathcal{I}_G$ and any semi-linear quotient V of $W \otimes F$ ("the interesting objects of the category $\mathcal{C}$ of smooth semi-linear representations of G are globally generated"). However, it remains to check that these conditions are non-empty on the set of irreducible objects.

Corollary 4.4.11. *Let $\mathcal{I}'_G$ be the maximal full subcategory of $\mathcal{I}_G$, such that for its objects W the map $W^{G_{F|F'}} \longrightarrow H_0(G_v^\dagger, W)$ is an isomorphism, where v is a discrete valuation of rank 1 trivial on k. If $n = \infty$ then $\mathcal{I}'_G$ is an abelian subcategory, closed under passages to the subquotients in $\mathcal{I}_G$.*

The proof uses Lemmas 4.1.2 and 4.4.7. It follows from Conjecture 1.1.7 that $\mathcal{I}'_G = \mathcal{I}_G$.

Let $\mathcal{I}_G^+$ (resp., $\mathcal{C}_-$) be the (maximal) full subcategory of $\mathcal{S}m_G$ (resp., of $\mathcal{C}$), whose objects W satisfy $W = W_v$ (resp., $W = F \otimes_{\mathcal{O}_v} W_v$). Clearly, these subcategories are closed under taking the quotients and contain $\mathcal{I}_G$ (resp., $\mathcal{I}_G \otimes F$).

Lemma 4.4.12. *Assume that $(-)_v$ is exact. Then $\mathcal{I}_G^+$ (resp., $\mathcal{C}_-$) is a Serre subcategory in $\mathcal{S}m_G$ (resp., in $\mathcal{C}$). Moreover, $\mathcal{I}_G^+ \neq \mathcal{I}_G$.*

The inclusion functors $\mathcal{I}_G^+ \hookrightarrow \mathcal{S}m_G$ and $\mathcal{C}_- \hookrightarrow \mathcal{C}$ admit right adjoints $W \mapsto \Gamma(W)$ and $V \mapsto \bigcap_v (F \otimes_{\mathcal{O}_v} V_v)$, respectively, but do not admit left adjoints.

Remark. Assuming that Corollary 4.4.10 holds, the following construction should give a fully faithful functor from $\mathcal{I}_G^+$ to a category of (birationally invariant) functors on the smooth k-varieties with all, not necessarily smooth, morphisms, which is a right quasi-inverse of $\Phi : \mathcal{F} \mapsto \mathcal{F}(F)$, cf. p.45.

As usually, we assume that the function fields of irreducible k-varieties $Y \subset X$ are embedded into F. For any $W \in \mathcal{I}_G^+$ the natural homomorphism $W^{G_{F|k(X)}} \longrightarrow H_0(G_v^\dagger, W)$ factors through $W^{G_{F|k(X)}} \longrightarrow H_0(G_v^\dagger, W_v)^{G_v \cap G_{F|k(X)}}$. By Corollary 4.4.10, the space $H_0(G_v^\dagger, W_v)^{G_v \cap G_{F|k(X)}}$ is canonically isomorphic to $W^{G_{F|k(Y)}}$ if $k(X) \cap \mathcal{O}_v = \mathcal{O}_{X,Y}$. □

4.5. **Restriction of the objects of $\mathcal{I}_G$ to some special Galois subgroups, and $\mathcal{I}$-induction.** In the spirit of Howe, Bushnell–Kutzko et al., one can study the smooth representations of a locally compact group, restricting them to open compact subgroups.

In the case of group G and $n = \infty$, if one restricts oneself to the subcategory $\mathcal{I}_G$ then a natural replacement of open compact subgroups is the open compact subgroups of $\mathfrak{G}$, cf. Proposition 4.6.1. Fix a subfield K in F, purely transcendental over k, over which F is algebraic.

Let Π be the set of isomorphism classes of all (non-zero) smooth irreducible representations ρ of $U := G_{F|K}$ over $\mathbb{Q}$, and W be a smooth representation of G. Then, as a U-module, W is isomorphic to a direct sum of all representations $\rho \in \Pi$ with some multiplicities $m(\rho) \geq 0$. Let $\sigma \in G$ be an element such that $\sigma(K) \subseteq K$. The twist ρ^σ of a representation ρ of U, and an embedding $\mathrm{Hom}_U(\rho, W) \hookrightarrow \mathrm{Hom}_U(\rho^\sigma, W)$ were defined in §3, p.25. It was mentioned there that if $m(\rho) \neq 0$ then $m(\rho^\sigma) \neq 0$. Besides, $m(\rho) = m(\rho^\sigma)$, if $\sigma(K) = K$.

Remark. It was mentioned at the beginning of §3, p.25 that any pair $\rho, \rho' \in \Pi$ intertwine.

The restriction of the $\mathcal{I}$-induction functor $\mathcal{S}m_U \longrightarrow \mathcal{I}_G$ to the finite-dimensional ρ is defined by $\rho \mapsto W_\rho := \mathcal{I}(\mathbb{Q}[G] \otimes_{\mathbb{Q}[G_{F|L}U]} \rho)$, where $F^{\ker \rho}$ is unirational over L, $L|k$ finitely generated and U-invariant. In general, the functor of the $\mathcal{I}$-induction is defined by the additivity.

Conjecture 4.5.1.
(1) *There are finitely many (or there are no) isomorphism classes of irreducible objects of $\mathcal{I}_G$ containing a given irreducible smooth representation of $G_{F|K}$, where K is purely transcendental over k and F is algebraic over K.*
(2) *Any irreducible object of $\mathcal{I}_G$ contains an irreducible (smooth) representation of $G_{F|K}$ that does not enter in any other irreducible object of $\mathcal{I}_G$.*

Remarks. 1. There are many irreducible smooth representation of $G_{F|K}$, entering in neither of objects of $\mathcal{I}_G$. Any non-trivial $\rho \in \Pi$ such that $F^{\ker \rho} \supseteq K$ is unirational (e.g., purely transcendental) over k is an example of such representation.

2. Examples of representation of $G_{F|K}$, entering into a unique irreducible object of $\mathcal{I}_G$, are the trivial one-dimensional and such non-trivial irreducible $\rho \in \mathcal{S}m_{G_{F|K}}$, entering into at least one object of $\mathcal{I}_G$, that $F^{\ker \rho}$ is a purely transcendental extension of the function field of a smooth projective curve over k with a simple (non-zero!) Jacobian.

3. Let us deduce the part (1) of this Conjecture from the motivic conjectures and Conjecture 1.1.7.

Let ρ be an irreducible smooth representation of $G_{F|K}$, and $F^{\ker \rho}$ be a finite extension of K, i.e. a purely transcendental extension of L, which is of finite type over k. If W contains ρ then $G_{F|L}$ has a fixed vector. If W is irreducible then there is a surjection $C_L \longrightarrow W$. Let us show that, assuming "all the conjectures", C_L has only a finitely many irreducible quotients. The quotients of level j are the quotients of $gr^j_{\mathcal{F}} C_L$. If C_L coincides with the Chow group $CH_0(X_F)_{\mathbb{Q}}$ for any smooth projective model X of an extension $L|k$, and the filtration $\mathcal{F}^\bullet$ coincides with the motivic one then $gr^j_{\mathcal{F}} C_L$ is determined by the motive $H^{2\dim X - j}(X, \mathbb{Q}(\dim X)) \cong H^j(X, \mathbb{Q}(j))$. We may thus assume that X is j-dimensional. In that case the quotients of level j are the summands of the semi-simple representation $gr_j^N CH_0(X_F)_{\mathbb{Q}} = CH_0(k(X) \otimes_k F)_{\mathbb{Q}}$ of finite length.

4.6. **Alternative descriptions of $\mathcal{I}_G$.** There are (at least) three more ways to describe the category $\mathcal{I}_G$: 1) as the category of non-degenerate modules over an associative idempotented algebra; 2) as a full subcategory in $\mathcal{S}m_{\mathfrak{G}}$; 3) as a category of homotopy invariant sheaves of vector spaces in the dominant topology on Spec(k), cf. §3.4.

"Homotopy invariant" representations as non-degenerate modules. If $n = \infty$ then the action of the associative algebra $\mathbb{D}_E := \lim\limits_{\longleftarrow_U} E[G/U]$ of the "oscillating" measures on G (for which all open subgroups and their translates are measurable), cf. the beginning of §3, p.25, on any object of $\mathcal{I}_G(E)$ factors through the action of its quotient $\mathbf{C}_E := \lim\limits_{\longleftarrow_L} C_L \otimes E$, since the morphism $E[G/G_{F|L}] \otimes_E W^{G_{F|L}} \longrightarrow W$ of representations of G factors through $\mathcal{I}(E[G/G_{F|L}] \otimes_E W^{G_{F|L}}) = C_L \otimes W^{G_{F|L}} \longrightarrow W$.

For any compact subgroup U in G the action of its Hecke algebra $\mathcal{H}_E(U) := h_U * \mathbf{D}_E * h_U$ on W^U factors through the action of its quotient $\mathcal{C}_E(U) := h_U * \mathbf{C}_E * h_U$ in $\mathbf{C}_E$ for any $W \in \mathcal{I}_G(E)$. E.g., if F^U is purely transcendental over L and L is of finite type k then $\mathcal{C}_E(U) = C_L^U \otimes E = \mathrm{End}_{\mathcal{I}_G(E)}(C_L \otimes E)$.

Let $\mathcal{H}_{\mathcal{I}} := \lim\limits_{K\longrightarrow} \lim\limits_{\longleftarrow L} C_L^{G_{F|K}}$ be the associative idempotented algebra without unity. The images h_K of the Haar measures on $G_{F|K}$ for purely transcendental extensions K of subfields of finite type over k in F over which F is algebraic, are projectors in the algebra $\mathcal{H}_{\mathcal{I}}$.

Then the category $\mathcal{I}_G$ is equivalent to the category of non-degenerate modules over $\mathcal{H}_{\mathcal{I}}$, i.e. such modules W that $W = \mathcal{H}_{\mathcal{I}} W$.

The algebra $\mathcal{H}_{\mathcal{I}}$ is isomorphic to the Hecke algebra (of locally invariant measures with compact support) of neither locally compact group, since any, e.g. finite-dimensional, subspace in $\lim\limits_{\longleftarrow L} C_L^G$ is a left ideal in $\mathcal{H}_{\mathcal{I}}$, which never happens in the Hecke algebras. Indeed, if there is a non-zero finite-dimensional left ideal $\mathfrak{a}$ in the Hecke algebra of a locally compact group H then the union of the supports of the measures in $\mathfrak{a}$ is compact and left-invariant, and therefore, the group H is compact. Then the smooth representations of H are semi-simple. It follows from the Mittag-Leffler property of the system $(C_L^G)_L$ and from Theorem 1.1.6 (4) that $\lim\limits_{\longleftarrow L} C_L^G \neq 0$.

The categories $\mathcal{I}_{\mathfrak{G}}$ and $\mathcal{A}dm_{\mathfrak{G}}$. The category $\mathcal{I}_G$ admits also a description in terms of the locally compact group $\mathfrak{G}$ from §2.5, if $n = \infty$. Namely, define $\mathcal{I}_{\mathfrak{G}}$ as the full subcategory in $\mathcal{S}m_{\mathfrak{G}}$ with "homotopy invariant" objects W: $W^{G_{F|LL_m}} = W^{G_{F|LL_m(S)}}$ for any $m \geq 1$, any extension $L|k$ in F of finite type and any transcendence base S of F over LL_m.

Proposition 4.6.1. *If $n = \infty$ then the forgetful functor to the category of $\mathfrak{G}$-modules induces the following equivalences of categories: $\mathcal{I}_G \xrightarrow{\sim} \mathcal{I}_{\mathfrak{G}}$ and $\mathcal{A}dm_G \xrightarrow{\sim} \mathcal{I}_{\mathfrak{G}} \cap \mathcal{A}dm_{\mathfrak{G}}$.*

Proof. To construct a quasi-inverse functor $\mathcal{I}_{\mathfrak{G}} \longrightarrow \mathcal{I}_G$ we have to define the value of σv for some given $W \in \mathcal{I}_{\mathfrak{G}}$, $v \in W$ and $\sigma \in G$. There exist a subfield $L \subset F$ of finite type over k and an integer $m \geq 1$ such that the stabilizer of v contains $G_{F|LL_m}$. Let $LL_m = L'L_{m'}$, where $L' \subset F$ is of finite type over k, and let L' and $L_{m'}$ be algebraically independent over k.

Let $N > m'$ be an integer such that $L'\sigma(L')$ and L_N are algebraically independent over k. Take any $\sigma' \in G_{F|L_N}$ such that $\sigma'|_{L'} = \sigma|_{L'}$ and set $\sigma v := \sigma' v$. One has $v \in W^{G_{F|L'L_{m'}}} = W^{G_{F|L'L_N}}$, so σv is independent of particular choices of N and of σ'.

Now we check independence of L'. Suppose that $v \in W^{G_{F|L'L_{m'}}} \cap W^{G_{F|L''L_{m''}}}$. Since $v \in W^{G_{F|L'L''L_{m'+m''}}}$, it suffices to treat the case $L' \subseteq L''$. As above, we choose an integer $N > m''$ such that $L''\sigma(L'')$ and L_N are algebraically independent over k, and some $\sigma'' \in G_{F|L_N}$ such that $\sigma''|_{L''} = \sigma|_{L''}$. Then σ'' can also serve as a σ', i.e., $\sigma'' v = \sigma' v$.

This gives us a map $G \times W \longrightarrow W$. Clearly, this is a linear action, and the stabilizer of v contains the open subgroup $G_{F|L'}$, and thus, W becomes an object of $\mathcal{I}_G$.

As L_j is purely transcendental over L_{j+1} for any $j \gg 1$, and the admissible representations of G are "homotopy invariant", the forgetful functor induces $\mathcal{A}dm_G \longrightarrow \mathcal{A}dm_{\mathfrak{G}}$, thus giving the second equivalence. $\square$

Remarks. 1. There exist admissible representations of $\mathfrak{G}$ outside of $\mathcal{I}_{\mathfrak{G}}$, e.g. $\mathbb{Q}(\rho) \notin \mathcal{I}_{\mathfrak{G}}$ for any non-trivial character ρ of $\mathfrak{G}$.

2. $\mathcal{I}_{\mathfrak{G}}$ is closed under taking subquotients and direct products (cf. §3, p.24), but not under extensions in $\mathcal{S}m_{\mathfrak{G}}$. As any morphism from $W \in \mathcal{S}m_{\mathfrak{G}}$ to an object of $\mathcal{I}_{\mathfrak{G}}$ factors through the canonical map to the direct product over all morphisms from W to representatives of all isomorphism classes in $\mathcal{I}_{\mathfrak{G}}$, there is a functor $\mathcal{I} : \mathcal{S}m_{\mathfrak{G}} \longrightarrow \mathcal{I}_{\mathfrak{G}}$ left adjoint to the inclusion functor $\mathcal{I}_{\mathfrak{G}} \hookrightarrow \mathcal{S}m_{\mathfrak{G}}$.

The $\mathcal{I}$-induction functor from §4.5 is the composition of the coinducing $\mathcal{S}m_U \longrightarrow \mathcal{S}m_{\mathfrak{G}}$, of $\mathcal{I}$ and of the equivalence from Proposition 4.6.1.

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