New perspectives in Arakelov geometry

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Abstract

In this paper we use Connes' theory of spectral triples to provide a unified view, via non-commutative geometry, of the archimedean and the totally split degenerate fibers of an arithmetic surface.

1 Introduction.

The aim of this article is to report on some of the recent results obtained by the two authors on a non-commutative interpretation of the totally split degenerate fibers of an arithmetic surface. The results stated in the first part of the paper were announced in a talk given by the first author at the VII meeting of the CNTA in May 2002 at Montreal. Because of the survey nature of this article, we decided to include no proofs for the results stated here. For a detailed treatement of the material exposed in this note as well as for further results on this subject we refer the reader to the papers [6] and [7].

Let $\mathcal{X}$ be an arithmetic surface defined over $\mathrm{Spec}(\mathbb{Z})$ having the smooth algebraic curve $X_{/\mathbb{Q}}$ as its generic fiber. It is well known that, as a Riemann surface, X admits always a uniformization by means of a Schottky group Γ. In fact, the presence of this uniformization plays a fundamental role in the theory of the "fiber at infinity" of $\mathcal{X}$ described in [16] and [6]. In analogy to Mumford's p-adic uniformization of algebraic curves (*cf.* [17]), the Riemann surface $X_{/\mathbb{C}}$ can be interpreted as the boundary at infinity of a 3-manifold $\mathfrak{X}_\Gamma$ defined as the quotient of the real hyperbolic 3-space $\mathbb{H}^3$ by the action of the Schottky group Γ. The space $\mathfrak{X}_\Gamma$ contains in its interior an infinite link of bounded geodesics. Manin gave an expression for the Arakelov Green function on X in terms of configurations of geodesics in $\mathfrak{X}_\Gamma$, thus interpreting this tangle as the dual graph $\mathcal{G}$ of the "closed fiber at infinity" of $\mathcal{X}$.

In the first part of this paper we concentrate on Manin's description of such dual graph and we exhibit the suspension flow $\mathcal{S}_T$ of a dynamical system T on the limit set of the Schottky group Γ as a geometric model for $\mathcal{G}$. In particular, the first cohomology group of $\mathcal{S}_T$ determines a model of the first cohomology of the dual graph of the "fiber at infinity". Furthermore, the first (co)homology group of $\mathcal{S}_T$ carries a natural filtration, which we use to define a (co)homology theory associated to the dynamical system T.

The major breakthrough of this construction arises from the fact that this dynamical cohomology contains a subspace isomorphic to the archimedean cohomology of [5]: the group of invariants for the action of the local monodromy "at infinity". This result implies that the archimedean cohomology (a group retaining the correct arithmetic properties, in order to be interpreted as the cohomology of the "fiber at infinity" of $\mathcal{X}$) can be identified with the cohomology of a topological space constructed in terms of geodesics in $\mathfrak{X}_\Gamma$.

The Cuntz-Krieger algebra associated to T acts on a completion of $H^1(\mathcal{S}_T)$. This algebra carries a refined information on the action of the Schottky group Γ on its limit set and it induces an action, by bounded operators, on a Hilbert space completion $\mathcal{H}^1_{dyn}$ of the dynamical cohomology. This representation extends to a representation of the C*-algebra $\mathcal{A} = (\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$. As a non-commutative space, this algebra provides an analog, at arithmetic infinity, of the p-adic reduction maps of [16] and [17] which were realized in terms of certain finite graphs in a quotient of the Bruhat-Tits tree of $\mathrm{PGL}(2, K)$, when K is a finite extension of $\mathbb{Q}_p$ (see below).

Similarly, we find a representation, on a Hilbert space completion $\mathcal{H}_1^{dyn}$ of the dynamical homology, of a C*-algebra strongly Morita equivalent to $\mathcal{A}$, namely the algebra of sections of a flat $\mathrm{C}(\Lambda_\Gamma)$-bundle over the handlebody $\mathfrak{X}_\Gamma$.

We introduce a Dirac operator D on the Hilbert space $\mathcal{H}^1_{dyn} \oplus \mathcal{H}_1^{dyn}$, which agrees with the "logarithm of Frobenius" of [5] and [10] when restricted to the subspace identified with the archimedean cohomology. Using Connes' technique of transferring spectral geometry between Morita equivalent non-commutative spaces, we obtain a dynamical spectral triple, over the non-commutative space $\mathcal{A}$ acting on the Hilbert space $\mathcal{H}^1_{dyn} \oplus \mathcal{M} \otimes \mathcal{H}_1^{dyn}$, where $\mathcal{M}$ is the bimodule implementing the Morita equivalence.

This structure further enriches the geometric interpretation of the archimedean cohomology, giving it the meaning of spinors on the noncommutative manifold $\mathcal{A}$, with the logarithm of Frobenius introduced in [10] in the role of Dirac operator.

Let p be a finite prime where $\mathcal{X}$ has totally split degenerate reduction. Then, the completion $\hat{X}_p$ at p of the generic fiber of $\mathfrak{X}$ is a split-degenerate stable curve over $\mathbb{Q}_p$ (also called a Mumford curve) uniformized by the action of a p-adic Schottky group Γ. The dual graph of the reduction of $\hat{X}_p$ coincides with a finite graph obtained as the quotient of a tree Δ_Γ by the action of Γ.

The curve $\hat{X}_p$ is holomorphically isomorphic to a quotient of a subset of the ends of the Bruhat-Tits tree Δ of $\mathbb{Q}_p$ by the action of Γ. Thus, in this setting, the Bruhat-Tits tree at p replaces the hyperbolic space $\mathbb{H}^3$ "at infinity", and the analog of the tangle of bounded geodesics in $\mathfrak{X}_\Gamma$ is played by doubly infinite walks in Δ_Γ/Γ.

In analogy with the archimedean construction, we define the system $(\mathcal{W}(\Delta_\Gamma/\Gamma), T)$ where T is an invertible shift map on the set $\mathcal{W}(\Delta_\Gamma/\Gamma)$ of doubly-infinite walks on the graph Δ_Γ/Γ. The first cohomology group $H^1(\mathcal{W}(\Delta_\Gamma/\Gamma)_T, \mathbb{Z})$ of the mapping torus $\mathcal{W}(\Delta_\Gamma/\Gamma)_T$ of T inherits a natural filtration using which we introduce a dynamical cohomology group. The Cuntz-Krieger graph algebra $\mathrm{C}^*(\Delta_\Gamma/\Gamma)$ has a representation on a Hilbert space completion $\mathcal{H}^1_{dyn}(\Delta_\Gamma/\Gamma)$ of the dynamical cohomology on which a Dirac operator D acts. It follows that the data $(\mathrm{C}^*(\Delta_\Gamma/\Gamma), \mathcal{H}^1_{dyn}(\Delta_\Gamma/\Gamma) \oplus \mathcal{H}^1_{dyn}(\Delta_\Gamma/\Gamma), D)$

determine a spectral triple in the sense of Connes in exact analogy with the archimedean case. We refer to [7] for a description of the arithmetic significance of this construction.

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2 Notation.

Throughout this paper we will denote by K one among the following fields: (a) the complex numbers $\mathbb{C}$, (b) a finite extension of $\mathbb{Q}_p$. When (b) occurs, we write $\mathcal{O}_K$ for the ring of integers of K, $\mathfrak{m} \subset \mathcal{O}_K$ for the maximal ideal and $\pi \in \mathfrak{m}$ for a uniformizer (*i.e.* $\mathfrak{m} = (\pi)$). We also denote by k the residue classes field $k = \mathcal{O}/\mathfrak{m}$.

We denote by $\mathbb{H}^3$ the three-dimensional real hyperbolic space *i.e.* the quotient $\mathbb{H}^3 = \mathrm{SU}(2)\backslash\mathrm{PGL}(2,\mathbb{C})$. This space can also be described as the upper half space $\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}^+$ endowed with the hyperbolic metric. The group $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{H}^3$ by isometries. The complex projective line $\mathbb{P}^1(\mathbb{C})$ is identified with the conformal boundary at infinity of $\mathbb{H}^3$ and the action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{H}^3$ extends to an action on $\overline{\mathbb{H}}^3 := \mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$. The group $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations.

For $g \geq 1$, $g \in \mathbb{Z}$, a Schottky group of rank g is a discrete subgroup $\Gamma \subset \mathrm{PSL}(2,K)$, which is purely loxodromic and isomorphic to a free group of rank g. The quotient space $\mathfrak{X}_\Gamma := \mathbb{H}^3/\Gamma$ is topologically a handlebody of genus g.

We denote by Λ_Γ, the limit set of the action of Γ. One sees that Λ_Γ is contained in $\mathbb{P}^1(K)$. This set can also be described as the closure of the set of the attractive and repelling fixed points $z^\pm(g)$ of the loxodromic elements $g \in \Gamma$. In the case $g = 1$ the limit set consists of two points, but for $g \geq 2$ the limit set is usually a fractal of some Hausdorff dimension $0 \leq \delta = \dim_H(\Lambda_\Gamma) < 2$.

We denote by $\Omega_\Gamma = \Omega_\Gamma(K)$ the domain of discontinuity of Γ, that is, the complement of Λ_Γ in $\mathbb{P}^1(K)$.

When $K = \mathbb{C}$, the quotient $X_{/\mathbb{C}} = \Omega_\Gamma/\Gamma$ is a Riemann surface of genus g and the covering $\Omega_\Gamma \to X_{/\mathbb{C}}$ is called a Schottky uniformization of $X_{/\mathbb{C}}$. Every complex Riemann surface $X_{/\mathbb{C}}$ admits a Schottky uniformization. The handlebody $\mathfrak{X}_\Gamma$ can be compactified by adding the conformal boundary at infinity $X_{/\mathbb{C}}$ to obtain $\overline{\mathfrak{X}}_\Gamma := \mathfrak{X}_\Gamma \cup X_{/\mathbb{C}} = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$.

A directed graph E consists of data $E = (E^0, E^1, E^1_+, r, s, \iota)$, where E^0 is the set of vertices, E^1 is the set of oriented edges $w = \{e, \epsilon\}$, where e is an edge of the graph and $\epsilon = \pm 1$ is a choice of orientation. The set E^1_+ consists of a choice of orientation for each edge, namely one element in each pair $\{e, \pm 1\}$. The maps $r, s : E^1 \to E^0$ are the range and source maps, and ι is the involution on E^1 defined by $\iota(w) = \{e, -\epsilon\}$.

A directed graph is finite if E^0 and E^1 are finite sets. It is locally finite if each vertex emits and receives at most finitely many oriented edges in E^1. A vertex v in a directed graph is a sink if there is no edge in E^1_+ with source v. A juxtaposition of oriented edges $w_1 w_2$ is said to be admissible if $w_2 \neq \iota(w_1)$ and $r(w_1) = s(w_2)$. A (finite, infinite, doubly infinite) walk in a directed graph E is an admissible (finite, infinite, doubly infinite) sequence of elements in E^1.

We denote by $\mathcal{W}^n(E)$ the set of walks of length n, by $\mathcal{W}^*(E) = \cup_n \mathcal{W}^n(E)$, by $\mathcal{W}^+(E)$ the set of infinite walks, and by $\mathcal{W}(E)$ the set of doubly infinite walks. A directed graph is a directed tree if, for any two vertices, there exists a unique walk in $\mathcal{W}^*(E)$ connecting them.

The edge matrix A_+ of a locally finite (or row finite) directed graph is an $(\#E^1_+) \times (\#E^1_+)$ (possibly infinite) matrix. The entries $A_+(w_i, w_j)$ satisfy $A_+(w_i, w_j) = 1$ if $w_i w_j$ is an admissible path, and $A_+(w_i, w_j) = 0$ otherwise. The directed edge matrix of E is a $\#E^1 \times \#E^1$ (possibly infinite) matrix with entries $A(w_i, w_j) = 1$ if $w_i w_j$ is an admissible walk and $A(w_i, w_j) = 0$ otherwise.

Even when not explicitly stated, all Hilbert spaces and algebras of operators we consider will be separable, *i.e.* they admit a dense (in the norm topology) countable subset.

3 A dynamical theory at infinity.

Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ be a Schottky group of rank $g \geq 2$. Given a choice of a set of generators $\{g_i\}_{i=1}^g$ for Γ, there is a bijection between the elements of Γ and the set of all admissible walks in the Cayley graph of Γ, namely reduced words in the $\{g_i\}_{i=1}^{2g}$, where we use the notation $g_{i+g} := g_i^{-1}$, for $i = 1, \ldots, g$.

In the following we consider the sets $\mathcal{S}^+$ and $\mathcal{S}$ of *resp.* right-infinite, doubly infinite admissible sequences in the $\{g_i\}_{i=1}^{2g}$:

$$\mathcal{S}^+ = \{a_0 a_1 \ldots a_\ell \ldots \ |a_i \in \{g_i\}_{i=1}^{2g}, \ a_{i+1} \neq a_i, \forall i \in \mathbb{N}\}, \tag{3.1}$$

$$\mathcal{S} = \{\ldots a_{-m} \ldots a_{-1} a_0 a_1 \ldots a_\ell \ldots \ |a_i \in \{g_i\}_{i=1}^{2g}, \ a_{i+1} \neq a_i, \forall i \in \mathbb{Z}\}. \tag{3.2}$$

On the space $\mathcal{S}$ we consider the topology generated by the sets $W^s(x, \ell) = \{y \in \mathcal{S} | x_k = y_k, k \geq \ell\}$, and $W^u(x, \ell) = \{y \in \mathcal{S} | x_k = y_k, k \leq \ell\}$ for $x \in \mathcal{S}$ and $\ell \in \mathbb{Z}$.

There is a two-sided shift operator T acting on $\mathcal{S}$ as the map

$$\begin{array}{cccccccccccc} T(& \ldots & a_{-m} & \ldots & a_{-1} & a_0 & a_1 & \ldots & a_\ell & \ldots &) = \\ & \ldots & a_{-m+1} & \ldots & a_0 & a_1 & a_2 & \ldots & a_{\ell+1} & \ldots & \end{array} \tag{3.3}$$

Definition 3.1 *A subshift of finite type $(\mathcal{S}_A, T)$ consists of all doubly infinite sequences in the elements of a given finite set W (alphabet) with the admissibility condition specified by a $\#W \times \#W$ elementary matrix,*

$$\mathcal{S}_A = \{(w_k)_{k \in \mathbb{Z}} : w_k \in W, A(w_k, w_{k+1}) = 1\},$$

and with the action of the invertible shift $(Tw)_k = w_{k+1}$.

Lemma 3.2 *The space $\mathcal{S}$ can be identified with the subshift of finite type $\mathcal{S}_A$ with the symmetric $2g \times 2g$ matrix A given by the directed edge matrix of the Cayley graph of Γ.*

The two-sided shift operator T on $\mathcal{S}$ of (3.3) decomposes $\mathcal{S}$ in a product of expanding and contracting directions, so that $(\mathcal{S}, T)$ is a Smale space.

The following topological space is defined in terms of the Smale space $(\mathcal{S}, T)$ and will be considered as a "dual graph" associated to the fiber at arithmetic infinity of the arithmetic surface $\mathcal{X}$.

Definition 3.3 *The mapping torus (suspension flow) of the dynamical system $(\mathcal{S}, T)$ is defined as*

$$\mathcal{S}_T := \mathcal{S} \times [0,1]/(x,0) \sim (Tx, 1) \tag{3.4}$$

3.1 (Co)homology of S_T.

In this paragraph we give an explicit description of the (co)homology $H^1(\mathcal{S}_T, \mathbb{Z})$.

The shift T acting on $\mathcal{S}$ induces an automorphism of the C^*–algebra of continuous functions $C(\mathcal{S})$. With an abuse of notation we still denote it by T. Consider the crossed product C^*–algebra $C(\mathcal{S}) \rtimes_T \mathbb{Z}$. This is a suitable norm completion of $C(\mathcal{S})[T, T^{-1}]$ with product $(V * W)_k = \sum_{r\in\mathbb{Z}} V_k \cdot (T^r W_{r+k})$, for $V = \sum_k V_k T^k$, $W = \sum_k W_k T^k$, and $V * W = \sum_k (V * W)_k T^k$.

The K–theory group $K_0(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z})$ is described by the co–invariants of the action of T (*cf.* [2],[18]).

Theorem 3.4 *The cohomology $H^1(\mathcal{S}_T, \mathbb{Z})$ satisfies the following properties*

1. *There is an identification of $H^1(\mathcal{S}_T, \mathbb{Z})$ with the K_0-group of the crossed product C^*-algebra for the action of T on $\mathcal{S}$,*

$$H^1(\mathcal{S}_T, \mathbb{Z}) \cong K_0(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z}). \tag{3.5}$$

2. *The identification (3.5) endows $H^1(\mathcal{S}_T, \mathbb{Z})$ with a filtration by free abelian groups $F_0 \hookrightarrow F_1 \hookrightarrow \cdots F_n \hookrightarrow \cdots$, with $\mathrm{rank} F_0 = 2g$ and $\mathrm{rank} F_n = 2g(2g-1)^{n-1}(2g-2)+1$, for $n \geq 1$, so that*

$$H^1(\mathcal{S}_T, \mathbb{Z}) = \varinjlim_n F_n.$$

The following result computes the first homology of $\mathcal{S}_T$.

Proposition 3.5 *The homology group $H_1(\mathcal{S}_T, \mathbb{Z})$ has a filtration by free abelian groups $\mathcal{K}_N$,*

$$H_1(\mathcal{S}_T, \mathbb{Z}) = \varinjlim_N \mathcal{K}_N, \tag{3.6}$$

with

$$K_N = \mathrm{rank}(\mathcal{K}_N) = \begin{cases} (2g-1)^N + 1 & N \text{ even} \\ (2g-1)^N + (2g-1) & N \text{ odd} \end{cases}$$

The group $H_1(\mathcal{S}_T, \mathbb{Z})$ can also be written as

$$H_1(\mathcal{S}_T, \mathbb{Z}) = \oplus_{N=0}^{\infty} \mathcal{R}_N$$

where $\mathcal{R}_n$ is a free abelian group of ranks $R_1 = 2g$ and

$$R_N = \mathrm{rank}(\mathcal{R}_N) = \frac{1}{N} \sum_{d|N} \mu(d)\, (2g-1)^{N/d},$$

for $N > 1$, with μ the Möbius function. This is isomorphic to a free abelian group on countably many generators.

The $\mathbb{Z}$-modules F_n are obtained as quotients $\mathcal{P}_n/\delta\mathcal{P}_{n-1}$, for $\delta(f) = f - f \circ T$, where $\mathcal{P} = \cup_n \mathcal{P}_n$ is the module of continuous $\mathbb{Z}$-valued functions on $\mathcal{S}$ that depend only on "future coordinates". The $\mathbb{Z}$-modules $\mathcal{K}_N$ are generated by all admissible words $a_0 \ldots a_N$ such that the word $a_N a_0$ is also admissible.

Combining Theorem 3.4 with Proposition 3.5, we can compute explicitly the pairing of homology and cohomology for $\mathcal{S}_T$.

Proposition 3.6 *Let F_n and $\mathcal{K}_N$ be the filtrations defined, respectively, in Theorem 3.4 and Proposition 3.5. There is a pairing*

$$\langle \cdot, \cdot \rangle : F_n \times \mathcal{K}_N \to \mathbb{Z} \qquad \langle [f], x \rangle = N \cdot f(\bar{x}), \tag{3.7}$$

with $x = a_0 \ldots a_N$. Here the representative $f \in [f]$ is a function that depends on the first $n+1$ terms $a_0 \ldots a_n$ of sequences in $\mathcal{S}$, and $\bar{x}$ is the truncation of the periodic sequence $\overline{a_0 \ldots a_N}$ after the first n terms. This pairing descends to the direct limits of the filtrations, where it agrees with the classical cohomology/homology pairing

$$\langle \cdot, \cdot \rangle : H^1(\mathcal{S}_T, \mathbb{Z}) \times H_1(\mathcal{S}_T, \mathbb{Z}) \to \mathbb{Z}. \tag{3.8}$$

3.2 Dynamical (co)homology.

We define the dynamical cohomology H^1_{dyn} as the graded vector space given by the sum of the graded pieces of the filtration of $H^1(\mathcal{S}_T)$, introduced in Theorem 3.4. These graded pieces Gr_p are considered with coefficients in the p-th Hodge–Tate twist $\mathbb{R}(p)$.

Similarly, we define the dynamical homology H_1^{dyn} as the graded vector space given by the sum of the terms in the filtration of $H_1(\mathcal{S}_T)$, introduced in Proposition 3.5. These vector spaces are again considered with twisted $\mathbb{R}(p)$-coefficients. The pair

$$H^1_{dyn} \oplus H_1^{dyn} \tag{3.9}$$

provides a geometric setting, defined in terms of the dynamics of the shift operator T, which contains a copy of the archimedean cohomology of [5] and of its dual.

Definition 3.7 *Let $H^1(\mathcal{S}_T, \mathbb{R}) = \lim_n F_n$, for a filtration F_n as in Theorem 3.4, with real coefficients. Let $\mathrm{Gr}_n = F_n/F_{n-1}$ be the corresponding graded pieces, with $\mathrm{Gr}_0 = F_0$.*

1. *We define the dynamical cohomology as*

$$H^1_{dyn} := \oplus_{p\leq 0} gr^{\Gamma}_{2p} H^1_{dyn}, \tag{3.10}$$

where we set

$$gr^{\Gamma}_{2p} H^1_{dyn} := \mathrm{Gr}_{-p} \otimes_{\mathbb{R}} \mathbb{R}(p) \tag{3.11}$$

with $\mathbb{R}(p) = (2\pi\sqrt{-1})^p\mathbb{R}$.

Furthermore, we define the graded subspace of H^1_{dyn}

$$\mathcal{V} := \oplus_{p\leq 0} gr^{\Gamma}_{2p} \mathcal{V}$$

where $gr^{\Gamma}_{2p}\mathcal{V}$ is generated by the elements $(2\pi\sqrt{-1})^p \chi_{-p+1,k}$, for $\chi_{n,k} := [\chi_{\mathcal{S}^+(w_{n,k})}] \in \mathrm{Gr}_{n-1}$.

Here $\chi_{\mathcal{S}^+(w_{n,k})}$ is the characteristic function of $\mathcal{S}^+(w_{n,k}) \subset \mathcal{S}^+$ with $w_{n,k} := a_0a_1\ldots a_{n-1} = \underbrace{g_k g_k \cdots g_k}_{n-times}$.

2. *The dynamical homology H_1^{dyn} is defined as*

$$H_1^{dyn} := \oplus_{p\geq 1} gr^{\Gamma}_{2p} H_1^{dyn}, \tag{3.12}$$

where we set

$$gr^{\Gamma}_{2p} H_1^{dyn} := \mathcal{K}_{p-1} \otimes_{\mathbb{R}} \mathbb{R}(p). \tag{3.13}$$

We also define $\mathcal{W} \subset H_1^{dyn}$ as the graded sub-space $\mathcal{W} = \oplus_{p\geq 1} gr^{\Gamma}_{2p}\mathcal{W}$, where $gr^{\Gamma}_{2p}\mathcal{W}$ is generated by the $2g$ elements $(2\pi\sqrt{-1})^p \; \underbrace{g_k g_k \cdots g_k}_{p-times}$.

In [6], we showed that the subspace $\mathcal{V}$ realizes a copy of the archimedean cohomology of [5] embedded in the dynamical cohomology.

An explicit geometric description for the space $\mathcal{V}$ in terms of geodesics in $\mathfrak{X}_\Gamma$ is obtained by interpreting the characteristic function $\chi_{\mathcal{S}^+(w_{n,k})}$ as the "best approximation" within $\mathcal{P}_n$ to a distribution supported on the periodic sequence of period g_k,

$$f(g_k g_k g_k g_k g_k \ldots) = 1 \quad \text{and} \quad f(a_0 a_1 a_2 a_3 \ldots) = 0 \ \text{ otherwise.}$$

Such periodic sequence $g_k g_k g_k g_k g_k \ldots$ describes the closed geodesic in $\mathfrak{X}_\Gamma$ that is the oriented core of one of the handles in the handlebody. Thus, the subspace $gr^\Gamma_{2p}\mathcal{V} \subset gr^\Gamma_{2p} H^1_{dyn}$ is spanned by the "best approximations" within $gr^\Gamma_{2p} H^1_{dyn}$ to cohomology classes supported on the core handles of the handlebody. In other words, this interpretation views the index $p \leq 0$ of the graded structure $\mathcal{V} = \oplus_p gr^\Gamma_{2p}\mathcal{V}$ as a measure of "zooming in", with increasing precision for larger $|p|$, on the core handles of the handlebody $\mathfrak{X}_\Gamma$.

3.3 A dynamical spectral triple.

The Cuntz–Krieger algebra $\mathcal{O}_A$ (*cf.* [8] [9]) is defined as the universal C*–algebra generated by partial isometries $S_1, \ldots, S_{2g}$, satisfying the relations

$$\sum_j S_j S_j^* = I \tag{3.14}$$

$$S_i^* S_i = \sum_j A_{ij}\, S_j S_j^*, \tag{3.15}$$

where $A = (A_{ij})$ is the $2g \times 2g$ transition matrix of the subshift of finite type $(\mathcal{S}, T)$, namely the matrix whose entries are $A_{ij} = 1$ whenever $|i - j| \neq g$, and $A_{ij} = 0$ otherwise.

The algebra $\mathcal{O}_A$ can be also described in terms of the action of the free group Γ on its limit set Λ_Γ (*cf.* [19], [20]), so that we can regard $\mathcal{O}_A$ as a noncommutative space replacing the classical quotient Λ_Γ/Γ.

The action of Γ on $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ determines a unitary representation

$$\Gamma \to \mathrm{Aut}(\mathrm{C}(\Lambda_\Gamma)) \qquad (T_\gamma f)(x) = f(\gamma^{-1}x), \tag{3.16}$$

where $\mathrm{C}(\Lambda_\Gamma)$ is the C*–algebra of continuous functions on Λ_Γ. Thus, we can form the (reduced) crossed product C*–algebra $\mathrm{C}(\Lambda_\Gamma) \rtimes \Gamma$. It follows that $\mathcal{O}_A \cong C(\Lambda_\Gamma) \rtimes \Gamma$. The algebra $\mathcal{O}_A$ acts on H^1_{dyn} in the following way

Proposition 3.8 *Let $\mathcal{O}_A^{alg}$ be the $*$–algebra over $\mathbb{R}$ generated algebraically by the S_i satisfying* (3.14) (3.15). *The action $S_i : F_n \mapsto F_n$*

$$(S_i h)(a_0, \ldots a_n) = h(g_i^{-1} a_0 \ldots a_{n-1}) \cdot \chi_{a_0 \neq g_i}(a_0, \ldots a_n).$$

induces a representation

$$\rho : \mathcal{O}_A^{alg} \to End(H^1_{dyn}).$$

This extends to a representation

$$\rho : \mathcal{O}_A \to \mathcal{B}(\mathcal{H}^1_{dyn}) \tag{3.17}$$

as bounded operators on the real Hilbert space completion $\mathcal{H}^1_{dyn}$*, where* $\mathcal{O}_A$ *is considered here as a real* C^**-algebra.*

On the space $\mathcal{H}^1_{dyn}$ one can also define an action of the (real) C^*-algebra $(\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$. Here $\mathrm{C}_0(\mathbb{H}^3)$ denotes the C^*-algebra of continuous functions that vanish on the sphere at infinity $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{H}^3$.

Proposition 3.9 *Let* $x_0 \in \mathbb{H}^3$ *be a fixed base point. For* $f \in \mathrm{C}_0(\mathbb{H}^3)$*, we define an operator* $\tilde{\rho}(f) : F_n \to F_n$ *as follows. For* $h \in F_n$*, we set*

$$(\tilde{\rho}(f)h)(a_0 \ldots a_n) = f((a_0 \ldots a_n) \cdot x_0) \cdot h(a_0 \ldots a_n), \tag{3.18}$$

where we identify the reduced word $a_0 \ldots a_n$ *with an element of* Γ*. This extends the representation of* $\mathcal{O}_A$ *on* $\mathcal{H}^1_{dyn}$ *given in Proposition 3.8 to a representation*

$$\tilde{\rho} : (\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma \to \mathcal{B}(\mathcal{H}^1_{dyn}). \tag{3.19}$$

On the space $\mathcal{H}_1^{dyn}$ we construct an action of the C^*-algebra $\mathrm{C}(\Lambda_\Gamma)$ in the following way.

Lemma 3.10 *For* f *a real valued continuous functions on* Λ_Γ*, consider the operator*

$$\rho(f) : (2\pi\sqrt{-1})^p \, a_0 \ldots a_p \mapsto f(\overline{a_0 \ldots a_p}) \; (2\pi\sqrt{-1})^p \, a_0 \ldots a_p. \tag{3.20}$$

These operators define a representation of the algebra of real valued continuous functions $\mathrm{C}(\Lambda_\Gamma, \mathbb{R})$ *as bounded operators on* $\mathcal{H}_1^{dyn}$*,*

$$\rho : \mathrm{C}(\Lambda_\Gamma, \mathbb{R}) \to \mathcal{B}(\mathcal{H}_1^{dyn}).$$

For a C^*-algebra $\mathcal{A}$, we denote by $\mathrm{C}_b(\mathbb{H}^3, \mathcal{A})$ the C^*-algebra of bounded continuous functions $f : \mathbb{H}^3 \to \mathcal{A}$. We extend the representation of $\mathrm{C}(\Lambda_\Gamma)$ on the space $\mathcal{H}_1^{dyn}$, given in Lemma 3.10, to a representation of the algebra $\mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma))$.

An element $f \in \mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma))$ is a collection f_u of elements in $\mathrm{C}(\Lambda_\Gamma)$, which vary continuously in the parameter $u \in \mathbb{H}^3$, and converge to zero in the norm of $\mathrm{C}(\Lambda_\Gamma)$ as u approaches the boundary at infinity $\mathbb{P}^1(\mathbb{C})$ in $\mathbb{H}^3$.

Lemma 3.11 *Let* $x_0 \in \mathbb{H}^3$ *be a fixed base point. For* $f \in \mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma))$ *we set:*

$$\tilde{\rho}(f) \, a_0 \ldots a_p = f_{(a_0 \ldots a_p) \, x_0}(\overline{a_0 \ldots a_p}) \cdot a_0 \ldots a_p. \tag{3.21}$$

This extends the representation (3.20) *to a representation*

$$\tilde{\rho} : \mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma)) \to \mathcal{B}(\mathcal{H}_1^{dyn}).$$

Let $\mathfrak{A}$ be a C*-algebra with an action of Γ by automorphisms. Let $\mathcal{E}$ be the flat $\mathfrak{A}$-bundle

$$\mathcal{E} = (\mathfrak{A} \times \mathbb{H}^3)/\Gamma \longrightarrow \mathfrak{X}_\Gamma, \tag{3.22}$$

where we consider the diagonal action of Γ on $\mathfrak{A} \times \mathbb{H}^3$. We consider the C*-algebra $\mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$ of continuous sections of $\mathcal{E}$ that vanish on the boundary at infinity $X_{/\mathbb{C}}$ of the handlebody $\mathfrak{X}_\Gamma$.

Proposition 3.12 *The representation of* $\mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma))$ *defined by* (3.21) *induces a representation*

$$\tilde{\rho} : \mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E}) \to \mathcal{B}(\mathcal{H}_1^{dyn}),$$

where $\mathcal{E}$ *is the bundle* (3.22) *for* $\mathfrak{A} = \mathrm{C}(\Lambda_\Gamma)$.

We introduce the notation $\mathcal{A} := (\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$ and $\hat{\mathcal{A}} := \mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$. In Propositions 3.9 and 3.12 we have obtained representations

$$\begin{aligned} \tilde{\rho} : \mathcal{A} &\to \mathcal{B}(\mathcal{H}^1_{dyn}) \\ \tilde{\rho} : \hat{\mathcal{A}} &\to \mathcal{B}(\mathcal{H}_1^{dyn}). \end{aligned} \tag{3.23}$$

The important observation is that the algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ represent the same non-commutative space. To understand this point, we recall the following result (*cf.* [12]).

Proposition 3.13 *Let* G *be a connected Lie group and* K *a maximal compact subgroup. For* $\Gamma \subset G$ *a lattice and* $\mathfrak{A}$ *a* C**-algebra with an action of* Γ *by automorphisms, the algebras* $(\mathfrak{A} \otimes \mathrm{C}_0(G/K)) \rtimes \Gamma$ *and* $\mathrm{C}_0(\Gamma\backslash G/K, \mathcal{E})$ *are strongly Morita equivalent. Here* $\mathcal{E}$ *is the flat* $\mathfrak{A}$*-bundle over* $\Gamma\backslash G/K$,

$$\mathcal{E} = (\mathfrak{A} \times G/K)/\Gamma \longrightarrow \Gamma\backslash G/K,$$

where Γ *acts diagonally on* $\mathfrak{A} \times G/K$.

In our case, the result of Proposition 3.13 implies that the algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ in (3.23) are strongly Morita equivalent.

We consider the linear operator D on $H^1_{dyn} \oplus H_1^{dyn}$ defined by setting, for $p \leq 0$,

$$D : gr^\Gamma_{2p} H^1_{dyn} \to gr^\Gamma_{2p} H^1_{dyn} \qquad D : x \mapsto p \cdot x \tag{3.24}$$

and, for $p \geq 1$,

$$D : gr^\Gamma_{2p} H_1^{dyn} \to gr^\Gamma_{2p} H_1^{dyn} \qquad D : x \mapsto p \cdot x. \tag{3.25}$$

This defines an unbounded linear operator

$$D : \mathcal{H}^1_{dyn} \oplus \mathcal{H}_1^{dyn} \to \mathcal{H}^1_{dyn} \oplus \mathcal{H}_1^{dyn}. \tag{3.26}$$

This operator has the correct properties of a Dirac operator.

Theorem 3.14 *Let* $\mathcal{H} = \mathcal{H}^1_{dyn} \oplus \mathcal{M} \otimes_{\hat{\mathcal{A}}} \mathcal{H}_1^{dyn}$*, with* $\mathcal{M}$ *a bimodule implementing the Morita equivalence between the* C**-algebras* $\hat{\mathcal{A}}$ *and* $\mathcal{A}$ *of* (3.23)*. Then* $(\mathcal{A}, \mathcal{H}, \tilde{D})$ *is a spectral triple, where* $\tilde{D}|_{\mathcal{H}^1_{dyn}} = D$ *as defined in* (3.24)*, and* $\tilde{D}|_{\mathcal{M}\otimes\mathcal{H}_1^{dyn}} = 1 \otimes D$*, with* D *as defined in* (3.25).

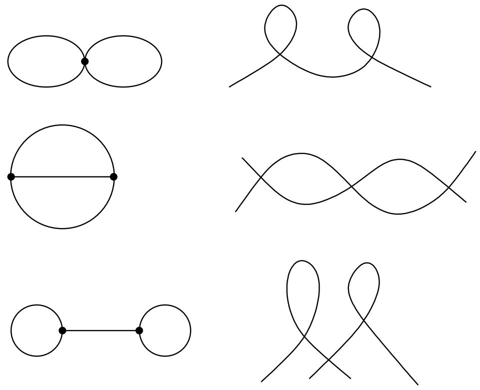

Figure 1: The graphs Δ_Γ/Γ for genus $g = 2$, and the corresponding fibers.

4 A Dynamical theory for Mumford curves.

Throughtout this chapter K will denote a finite extension of $\mathbb{Q}_p$ and Δ_K the Bruhat-Tits tree associated to $G = \mathrm{PGL}(2, K)$. In the following we recall few results about the action of a Schottky group on a Bruhat-Tits tree and on C*-algebras of graphs. Detailed explanations are contained in [15], [17] and [1], [13], [14], [20].

For a Schottky group $\Gamma \subset \mathrm{PGL}(2, K)$ there is a smallest subtree $\Delta'_\Gamma \subset \Delta_K$ containing the axes of all elements of Γ. The set of ends of Δ'_Γ in $\mathbb{P}^1(K)$ is Λ_Γ, the limit set of Γ. The group Γ carries Δ'_Γ into itself so that the quotient Δ'_Γ/Γ is a finite graph that coincides with the dual graph of the closed fibre of the minimal smooth model of the algebraic curve C/K holomorphically isomorphic to $X_\Gamma := \Omega_\Gamma/\Gamma$ (*cf.* [17] p. 163). There is a smallest tree Δ_Γ on which Γ acts and such that Δ_Γ/Γ is the (finite) graph of the specialization of C. The curve C is a k-split degenerate, stable curve. When the genus of the fibers is at least 2 - *i.e.* when the Schottky group has at least $g \geq 2$ generators - the curve X_Γ is called a Schottky–Mumford curve.

The possible graphs Δ_Γ/Γ and the corresponding fiber for the case of genus 2 are illustrated in Figure 1.

To a locally finite directed graph one associates a C*-algebra in the following way.

A Cuntz–Krieger family consists of a collection $\{P_v\}_{v \in E^0}$ of mutually orthogonal projections and a $\{S_w\}_{w \in E^1_+}$ of partial isometries, satisfying the conditions: $S^*_w S_w = P_{r(w)}$ and, for all $v \in s(E^1_+)$, $P_v = \sum_{w : s(w) = v} S_w S^*_w$.

The Cuntz–Krieger elements $\{P_v, S_w\}$ satisfy the relation $S^*_w S_w = \sum A_+(w, \tilde{w}) S_{\tilde{w}} S^*_{\tilde{w}}$, with A_+ the edge matrix of the graph.

One defines a universal C^*-algebra $C^*(E)$ generated by a Cuntz–Krieger family. If E is a finite graph with no sinks, we have $C^*(E) \simeq \mathcal{O}_{A_+}$, where $\mathcal{O}_{A_+}$ is the Cuntz-Krieger algebra of the edge matrix A_+. If the directed graph is a tree Δ, then $C^*(\Delta)$ is an AF algebra strongly Morita equivalent to the commutative C^*-algebra $C_0(\partial\Delta)$. A monomorphism of directed trees induces an injective $*$-homomorphism of the corresponding C^*-algebras.

If $G \subset Aut(E)$ is a group acting freely on the directed graph E, with quotient graph E/G, then the crossed product C^*-algebra $C^*(E) \rtimes G$ is strongly Morita equivalent to $C^*(E/G)$. In particular, if Δ is the universal covering tree of a directed graph E and Γ is the fundamental group, then the algebra $C^*(E)$ is strongly Morita equivalent to $C_0(\partial\Delta) \rtimes \Gamma$.

In the following we consider the Bruhat–Tits tree Δ_K for a fixed finite extension K of $\mathbb{Q}_p$, and the corresponding C^*-algebra $C^*(\Delta_K)$, which is strongly Morita equivalent to the abelian C^*-algebra of complex valued functions $C(\mathbb{P}^1(K))$.

Notice that the commutative C^*-algebra $C^*(X_K)$ of complex valued continuous functions on the Mumford curve $X_K = \Omega_\Gamma/\Gamma$ is strongly Morita equivalent to the crossed product $C_0(\Omega_\Gamma) \rtimes \Gamma$, with $C_0(\Omega_\Gamma)$ a Γ-invariant ideal of $C(\mathbb{P}^1(K))$ with quotient algebra $C(\Lambda_\Gamma)$. The algebra $C^*(\Delta_\Gamma/\Gamma)$, in turn, is strongly Morita equivalent to $C^*(\Delta_\Gamma) \rtimes \Gamma$ and to $C(\Lambda_\Gamma) \rtimes \Gamma$, where $\partial\Delta_\Gamma = \Lambda_\Gamma$. Similarly, one sees that the algebra $C^*(\Delta_K/\Gamma)$ is strongly Morita equivalent to the crossed product algebra $C^*(\Delta_K) \rtimes \Gamma$, which in turn is strongly Morita equivalent to $C^*(\mathbb{P}^1(K)) \rtimes \Gamma$.

Thus, the graph algebra $C^*(\Delta_K/\Gamma)$ is, up to Morita equivalence, an extension of the commutative C^*-algebra $C^*(X_K)$ by the Cuntz-Krieger algebra $C^*(\Delta_\Gamma/\Gamma)$ associated to the edge matrix of the finite graph Δ_Γ/Γ.

In this paragraph we introduce a dynamical system associated to the space $\mathcal{W}(\Delta/\Gamma)$ of walks on the directed tree Δ on which Γ acts. In particular, we are interested in the cases when $\Delta = \Delta_K,\ \Delta_\Gamma$.

For $\Delta = \Delta_\Gamma$, we obtain a subshift of finite type associated to the action of the Schottky group Γ on the limit set Λ_Γ, of the type that was considered in [6].

Let $\bar{V} \subset \Delta_\Gamma$ be a finite subtree whose set of edges consists of one representative for each Γ-class. This is a fundamental domain for Γ in the weak sense (following the notation of [15]), since some vertices may be identified under the action of Γ. Correspondingly, $V \subset \mathbb{P}^1(K)$ is the set of ends of all infinite paths starting at points in $\bar{V}$.

Consider the set $\mathcal{W}(\Delta_\Gamma/\Gamma)$ of doubly infinite walks on the finite graph Δ_Γ/Γ. These are doubly infinite admissible sequences in the finite alphabet given by the edges of $\bar{V}$ with both possible orientations. On $\mathcal{W}(\Delta_\Gamma/\Gamma)$ we consider the topology generated by the sets $\mathcal{W}^s(\omega,\ell) = \{\tilde{\omega} \in \mathcal{W}(\Delta_\Gamma/\Gamma) : \tilde{\omega}_k = \omega_k, k \geq \ell\}$ and $\mathcal{W}^u(\omega,\ell) = \{\tilde{\omega} \in \mathcal{W}(\Delta_\Gamma/\Gamma) : \tilde{\omega}_k = \omega_k, k \leq \ell\}$, for $\omega \in \mathcal{W}(\Delta_\Gamma/\Gamma)$ and $\ell \in \mathbb{Z}$. With this topology, the space $\mathcal{W}(\Delta_\Gamma/\Gamma)$ is a totally disconnected compact Hausdorff space.

The invertible shift map T, given by $(T\omega)_k = \omega_{k+1}$, is a homeomorphism of $\mathcal{W}(\Delta_\Gamma/\Gamma)$. We can describe again the dynamical system $(\mathcal{W}(\Delta_\Gamma/\Gamma), T)$ in terms of subshifts of finite type.

Lemma 4.1 *The space $\mathcal{W}(\Delta_\Gamma/\Gamma)$ with the action of the invertible shift T is a subshift of finite type, where $\mathcal{W}(\Delta_\Gamma/\Gamma) = \mathcal{S}_A$ with A the directed edge matrix of the finite graph Δ_Γ/Γ.*

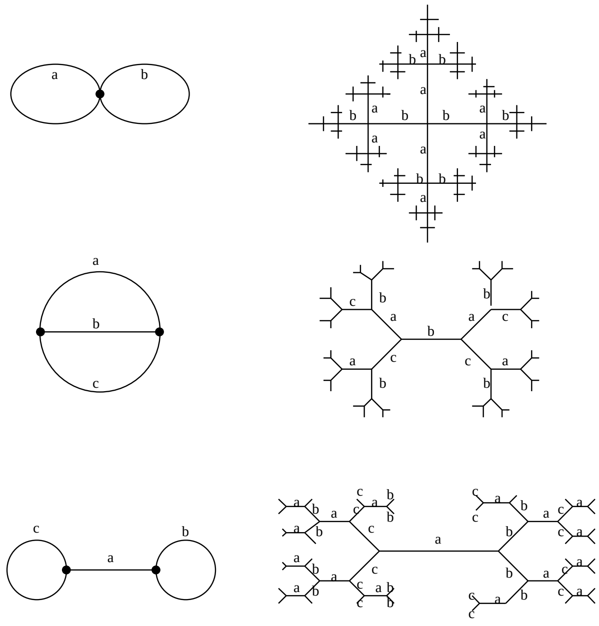

Figure 2: The graphs Δ_Γ/Γ for genus $g = 2$, and the corresponding trees Δ_Γ.

We consider the mapping torus of T:

$$\mathcal{W}(\Delta_\Gamma/\Gamma)_T := \mathcal{W}(\Delta_\Gamma/\Gamma) \times [0,1]/(Tx, 0) \sim (x, 1). \tag{4.1}$$

4.1 Genus two example

In the example of Mumford–Schottky curves of genus $g = 2$, the tree Δ_Γ is illustrated in Figure 2.

In the first case in the Figure, the tree Δ_Γ is just a copy of the Cayley graph of the free group Γ on two generators, hence we can identify doubly infinite walks in Δ_Γ with doubly infinite reduced words in the generators of Γ and their inverses. The directed edge matrix is given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

In the second case in Figure 2, we label by $a = e_1$, $b = e_2$ and $c = e_3$ the oriented edges in the graph Δ_Γ/Γ, so that we have a corresponding set of labels $E = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$ for the edges in the covering Δ_Γ. A choice of generators for the group $\Gamma \simeq \mathbb{Z} * \mathbb{Z}$ acting on Δ_Γ is obtained by identifying the generators g_1 and g_2 of Γ

with the chains of edges $a\bar{b}$ and $a\bar{c}$. Doubly infinite walks in the tree Δ_Γ are admissible doubly infinite sequences of such labels, where admissibility is determined by the directed edge matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The third case in Figure 2 is analogous. A choice of generators for the group $\Gamma \simeq \mathbb{Z} * \mathbb{Z}$ acting on Δ_Γ is given by $ab\bar{a}$ and c. Doubly infinite walks in the tree Δ_Γ are admissible doubly infinite sequences in the alphabet $E = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$, with admissibility determined by the directed edge matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The construction is analogous for genus $g > 2$, for the various possible finite graphs Δ_Γ/Γ. The directed edge matrix can then be written in block form as

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

where each block α_{ij} is a $\#(\Delta_\Gamma/\Gamma)^1_+ \times \#(\Delta_\Gamma/\Gamma)^1_+$–matrix with $\alpha_{12} = \alpha_{12}^t$, $\alpha_{21} = \alpha_{21}^t$, and $\alpha_{11} = \alpha_{22}^t$.

4.2 Cohomology of $\mathcal{W}(\Delta/\Gamma)_T$.

Let $\Delta = \Delta_\Gamma$. We identify the first cohomology group $H^1(\mathcal{W}(\Delta_\Gamma/\Gamma)_T, \mathbb{Z})$ with the group of homotopy classes of continuous maps of $\mathcal{W}(\Delta_\Gamma/\Gamma)_T$ to the circle. Let $\mathrm{C}(\mathcal{W}(\Delta_\Gamma/\Gamma), \mathbb{Z})$ be the $\mathbb{Z}$-module of integer valued continuous functions on $\mathcal{W}(\Delta_\Gamma/\Gamma)$, and let

$$\mathrm{C}(\mathcal{W}(\Delta_\Gamma/\Gamma), \mathbb{Z})_T := Coker(\delta),$$

for $\delta(f) = f - f \circ T$. The analog of Theorem 3.4 holds:

Proposition 4.2 *The map $f \mapsto [\exp(2\pi i t f(x))]$, which associates to an element $f \in \mathrm{C}(\mathcal{W}(\Delta_\Gamma/\Gamma), \mathbb{Z})$ a homotopy class of maps from $\mathcal{W}(\Delta_\Gamma/\Gamma)_T$ to the circle, gives an isomorphism $\mathrm{C}(\mathcal{W}(\Delta_\Gamma/\Gamma), \mathbb{Z})_T \simeq H^1(\mathcal{W}(\Delta_\Gamma/\Gamma)_T, \mathbb{Z})$. Moreover, there is a filtration of*

$C(\mathcal{W}(\Delta_\Gamma/\Gamma),\mathbb{Z})_T$ by free $\mathbb{Z}$-modules $F_0 \subset F_1 \subset \cdots F_n \cdots$, of rank $\theta_n - \theta_{n-1} + 1$, where θ_n is the number of admissible words of length $n+1$ in the alphabet, so that we have

$$H^1(\mathcal{W}(\Delta_\Gamma/\Gamma)_T, \mathbb{Z}) = \varinjlim_n F_n.$$

The quotients F_{n+1}/F_n are also torsion free.

The space $\mathcal{W}(\Delta_\Gamma/\Gamma)_T$ corresponds to a space of "bounded geodesics" on the graph Δ_K/Γ, where geodesics, in this setting, are just doubly infinite walks in Δ_K/Γ. In particular, a closed geodesic is the image under the quotient map $\pi_\Gamma : \Delta_K \to \Delta_K/\Gamma$ of a doubly infinite walk in the Bruhat-Tits tree Δ_K with ends given by the pair $z^+(\gamma), z^-(\gamma)$ of fixed points of some element $\gamma \in \Gamma$. Similarly, a bounded geodesic is an element $\omega \in \mathcal{W}(\Delta_K/\Gamma)$ which is the image, under the quotient map, of a doubly infinite walk in Δ_K with both ends on $\Lambda_\Gamma \subset \mathbb{P}^1(K)$. This implies that a bounded geodesic is a walk of the form $\omega = \pi_\Gamma(\tilde\omega)$, for some $\tilde\omega \in \mathcal{W}(\Delta_\Gamma/\Gamma)$. By construction, any such walk is an axis of Δ_Γ.

Orbits of $\mathcal{W}(\Delta_\Gamma/\Gamma)$ under the action of the invertible shift T correspond bijectively to orbits of the complement of the diagonal in $\Lambda_\Gamma \times \Lambda_\Gamma$ under the action of Γ. Thus, we see that $\mathcal{W}(\Delta_\Gamma/\Gamma)_T$ gives a geometric realization of the space of "bounded geodesics" on the graph Δ_K/Γ, much as, in the case of the geometry at arithmetic infinity, we used the mapping torus of the shift T as a model of the tangle of bounded geodesics in a hyperbolic handlebody.

In [6] §4, we showed how the mapping torus $\mathcal{S}_T$ of the subshift of finite type $(\mathcal{S}, T)$, associated to the limit set of the Schottky group, maps surjectively to the tangle of bounded geodesics inside the hyperbolic handlebody, through a map that resolves all the points of intersection of different geodesics. In the case of the Mumford curve, where we replace the real hyperbolic 3-space by the Bruhat–Tits building Δ_K, the analog of the surjective map from $\mathcal{S}_T$ to the tangle of bounded geodesics is a map from $\mathcal{W}(\Delta_\Gamma/\Gamma)_T$ to the dual graph Δ_Γ/Γ. Here is a description of this map.

As before, we write elements of $\mathcal{W}(\Delta_\Gamma/\Gamma)$ as admissible doubly infinite sequences

$$\omega = \ldots w_{i_{-m}} \ldots w_{i_{-1}} w_{i_0} w_{i_1} \ldots w_{i_n} \ldots,$$

with the $w_{i_k} = \{e_{i_k}, \epsilon_{i_k}\}$ oriented edges on the graph Δ_Γ/Γ. We consider each oriented edge w of normalized length one, so that it can be parameterized as $w(t) = \{e(t), \epsilon\}$, for $0 \le t \le 1$, with $\bar w(t) = \{e(1-t), -\epsilon\}$. Since $\omega \in \mathcal{S}_A$ is an admissible sequence of oriented edges we have $w_{i_k}(1) = w_{i_{k+1}}(0) \in \Delta_\Gamma^{(0)}$.

We consider a map of the covering space $\mathcal{W}(\Delta_\Gamma/\Gamma) \times \mathbb{R}$ of $\mathcal{W}(\Delta_\Gamma/\Gamma)_T$ to $|\Delta_\Gamma|$ of the form

$$\tilde E(\omega, \tau) = w_{i_{[\tau]}}(\tau - [\tau]). \tag{4.2}$$

Here $|\Delta_\Gamma|$ denotes the geometric realization of the graph. By construction, the map $\tilde E$ satisfies $\tilde E(T\omega, \tau) = \tilde E(\omega, \tau + 1)$, hence it descends to a map E of the quotient

$$E : \mathcal{W}(\Delta_\Gamma)_T \to |\Delta_\Gamma|. \tag{4.3}$$

We then obtain a map to $|\Delta_\Gamma/\Gamma|$, by composing with the quotient map of the Γ action, $\pi_\Gamma : \Delta_\Gamma \to \Delta_\Gamma/\Gamma$, that is,

$$\bar{E} := \pi_\Gamma \circ E : \mathcal{W}(\Delta_\Gamma)_T \to |\Delta_\Gamma/\Gamma|. \tag{4.4}$$

Thus, we obtain the following.

Proposition 4.3 *The map $\bar{E}$ of (4.4) is a continuous surjection from the mapping torus $\mathcal{W}(\Delta_\Gamma)_T$ to the geometric realization $|\Delta_\Gamma/\Gamma|$ of the finite graph Δ_Γ/Γ.*

The fibers of the map (4.4) are explicitly described as

$$\bar{E}^{-1}(w_i(t)) = \mathcal{W}(\Delta_\Gamma)(w_i) \times \{t\} \cup \mathcal{W}(\Delta_\Gamma)(\bar{w}_i) \times \{1-t\}, \tag{4.5}$$

where $w_i(t)$, for $t \in [0,1]$ is a parameterized oriented edge in the graph Δ_Γ/Γ, and $\mathcal{W}(\Delta_\Gamma)(w_i) \subset \mathcal{W}(\Delta_\Gamma)$ consists of

$$\mathcal{W}(\Delta_\Gamma)(w_i) = \{\omega \in \mathcal{W}(\Delta_\Gamma) : \ w_{i_0} = w_i\},$$

for $\omega = \ldots w_{i_{-m}} \ldots w_{i_{-1}} w_{i_0} w_{i_1} \ldots w_{i_n} \ldots$.

We now consider the case where Δ is the Bruhat-Tits tree Δ_K. The set of doubly infinite walks $\mathcal{W}(\Delta_K/\Gamma)$ can be identified with the set of admissible doubly infinite sequences in the oriented edges of a fundamental domain for the action of Γ. Therefore, we obtain the following identification

$$\mathcal{W}(\Delta_K/\Gamma) \simeq V \times \mathcal{W}^*(\Delta_\Gamma/\Gamma) \times V \cup V \times \mathcal{W}^+(\Delta_\Gamma/\Gamma) \cup \mathcal{W}^+(\Delta_\Gamma/\Gamma) \times V \cup \mathcal{W}(\Delta_\Gamma/\Gamma), \tag{4.6}$$

where we distinguish between walks that wander off from the finite graph Δ_Γ/Γ along one of the paths leading to an end in V, in one or in both directions, and those that stay confined within the finite graph Δ_Γ/Γ. Here we include in $\mathcal{W}^*(\Delta_\Gamma/\Gamma) = \cup_n \mathcal{W}^n(\Delta_\Gamma/\Gamma)$ also the case $\mathcal{W}^0(\Delta_\Gamma/\Gamma) = \emptyset$, where the walk in Δ_K/Γ does not intersect the finite graph Δ_Γ/Γ at all.

In the topology induced by the p-adic norm, $\mathbb{P}^1(K)$ is a totally disconnected compact Hausdorff space, and so is $\mathcal{W}(\Delta_K/\Gamma)$ by (4.6). Again, we consider the invertible shift map T, which is a homeomorphism of $\mathcal{W}(\Delta_K/\Gamma)$, and we form the mapping torus

$$\mathcal{W}(\Delta_K/\Gamma)_T := \mathcal{W}(\Delta_K/\Gamma) \times [0,1]/(Tx, 0) \sim (x, 1). \tag{4.7}$$

We obtain the analog of Proposition 4.2, though in the case of Δ_K the $\mathbb{Z}$-modules F_n will not be finitely generated.

4.3 A dynamical theory.

On $\mathbb{P}^1(K)$ we consider the measure $d\mu$ as in [3]. This induces a measure on $\mathcal{W}(\Delta_K/\Gamma)$ and on $\mathcal{W}(\Delta_\Gamma/\Gamma)$, with respect to which the shift map T is measure preserving.

Consider the submodule $\mathcal{P} \subset \mathrm{C}(\mathcal{W}(\Delta_\Gamma/\Gamma), \mathbb{Z})$ of functions of future coordinates where $\mathcal{P} \simeq \mathrm{C}(\mathcal{W}^+(\Delta_\Gamma/\Gamma), \mathbb{Z})$. This gives a filtration $\mathcal{P} = \cup_n \mathcal{P}_n$, where $\mathcal{P}_n$ is identified with the submodule of $\mathrm{C}(\mathcal{W}^+(\Delta_\Gamma/\Gamma), \mathbb{Z})$ generated by characteristic functions of $\mathcal{W}^+(\Delta_\Gamma/\Gamma, \rho) \subset \mathcal{W}^+(\Delta_\Gamma/\Gamma)$, where $\rho \in \mathcal{W}^*(\Delta_\Gamma/\Gamma)$ is a finite walk $\rho = w_0 \cdots w_n$ of length $n+1$, and $\mathcal{W}^+(\Delta_\Gamma/\Gamma, \rho)$ is the set of infinite paths $\omega \in \mathcal{W}^+(\Delta_\Gamma/\Gamma)$, with $\omega_k = w_k$ for $0 \leq k \leq n+1$. The free $\mathbb{Z}$-module $\mathcal{P}_n$ of functions of at most $n+1$ future coordinates determines a vector subspace $\mathcal{P}_n \otimes_{\mathbb{Z}} \mathbb{C}$ of $L^2(\mathbb{P}^1(K), d\mu)$. The operator δ of §4.2 is bounded in norm, and the $F_n = \mathcal{P}_n/\delta\mathcal{P}_{n-1}$ have induced Hilbert norms and bounded inclusions $F_n \otimes \mathbb{C} \hookrightarrow F_{n+1} \otimes \mathbb{C}$, where the F_n are the torsion free $\mathbb{Z}$-modules of Proposition 4.2. Using the inner product induced from $L^2(\mathbb{P}^1(K), d\mu)$, we can split $F_n \cong F_{n-1} \oplus (F_{n-1}^\perp \cap F_n)$.

Definition 4.4 *The dynamical cohomology $\mathcal{H}^1_{dyn}(\Delta_\Gamma/\Gamma)$ is the norm completion of the graded complex vector space*

$$H^1_{dyn}(\Delta_\Gamma/\Gamma) := F_0 \oplus (F_1 \cap F_0^\perp) \oplus (F_2 \cap F_1^\perp) \oplus \cdots (F_n \cap F_{n-1}^\perp) \cdots .$$

Proposition 4.5 *There is a representation of the algebra $\mathrm{C}^*(\Delta_\Gamma/\Gamma)$ by bounded linear operators on the Hilbert space $\mathcal{H}^1_{dyn}(\Delta_\Gamma/\Gamma)$.*

4.4 A dynamical spectral triple for a Mumford curve.

Using the representation of Proposition 4.5, we construct a spectral triple associated to a Mumford curve, where, in defining the Dirac operator we only use the graded structure of the space $H^1_{dyn}(\Delta_\Gamma/\Gamma)$.

Let $\mathcal{H}$ be the Hilbert space

$$\mathcal{H} = \mathcal{H}^1_{dyn}(\Delta_\Gamma/\Gamma) \oplus \mathcal{H}^1_{dyn}(\Delta_\Gamma/\Gamma).$$

We consider the algebra $\mathrm{C}^*(\Delta_\Gamma/\Gamma)$ acting on $\mathcal{H}$ diagonally.

We define an unbounded self-adjoint operator $|D|$ on $\mathcal{H}$, with the property that $(\lambda - |D|)^{-1}$ is a compact operator for all $\lambda \notin \mathbb{R}$.

Let $Gr_{+,n} = 0 \oplus Gr_n$ and $Gr_{n,-} = Gr_n \oplus 0$. We define the Dirac operator D by setting

$$D|_{Gr_{+,n}} = n \qquad D|_{Gr_{-,n}} = -n-1. \tag{4.8}$$

Proposition 4.6 *The data $(\mathrm{C}^*(\Delta_\Gamma/\Gamma), \mathcal{H}, D)$ determine a spectral triple in the sense of Connes [4].*

The proof is completely analogous to the argument given in [6].

In [7] we recover arithmetic information such as the local L–factors of [11] from the dynamical cohomology and the spectral triple of Proposition 4.6.

References

[1] T. Bates, D. Pask, I. Raeburn, W. Szymański, *The C^*-algebras of row finite graphs*, New York J. Math. 6 (2000) 307–324.

[2] M. Boyle, D. Handelman, *Orbit equivalence, flow equivalence, and ordered cohomology*, Israel J. Math. 95 (1996) 169–210.

[3] L. Chekhov, A. Mironov, A. Zabrodin, *Multiloop calculations in p-adic string theory and Bruhat-Tits trees.* Comm. Math. Phys. 125 (1989), no. 4, 675–711.

[4] A. Connes, *Geometry from the spectral point of view.* Lett. Math. Phys. 34 (1995), no. 3, 203–238.

[5] C. Consani, *Double complexes and Euler L–factors*, Compositio Math. 111 (1998) 323–358.

[6] C. Consani, M. Marcolli, *Noncommutative geometry, dynamics and ∞-adic Arakelov geometry*, preprint arXiv:math.AG/0205306.

[7] C. Consani, M. Marcolli, *Spectral triples from Mumford curves*, Preprint (2002).

[8] J. Cuntz, *A class of C^*–algebras and topological Markov chains II: reducible chains and the Ext–functor for C^*–algebras*, Invent. Math. 63 (1981) 25–40.

[9] J. Cuntz, W. Krieger, *A class of C^*–algebras and topological Markov chains*, Invent. Math. 56 (1980) 251–268.

[10] C. Deninger, *On the Γ–factors attached to motives*, Invent. Math. 104 (1991) 245–261.

[11] C. Deninger, *Local L-factors of motives and regularized determinants.* Invent. Math. 107 (1992), no. 1, 135–150.

[12] G.G. Kasparov, *K–theory, group C^*–algebras, and higher signatures (Conspectus).* Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), 101–146, London Math. Soc. Lecture Note Ser., 226, Cambridge Univ. Press, 1995.

[13] A. Kumjian, D. Pask, *C^*-algebras of directed graphs and group actions*, Ergod. Th. & Dynam. Sys. (1999) 19 1503–1519.

[14] A. Kumjian, D. Pask, I. Raeburn, *Cuntz–Krieger algebras of directed graphs*, Pacific J. Math. 184 (1998) N.1, 161–174.

[15] Yu.I. Manin, *p-adic automorphic functions.* Journ. of Soviet Math., 5 (1976) 279-333.

[16] Yu.I. Manin, *Three–dimensional hyperbolic geometry as ∞–adic Arakelov geometry*, Invent. Math. 104 (1991) 223–244.

[17] D. Mumford, *An analytic construction of degenerating curves over complete local rings*, Compositio Math. 24 (1972) 129–174.

[18] W. Parry, S. Tuncel, *Classification problems in ergodic theory*, London Math. Soc. Lecture Notes Series 67, 1982.

[19] G. Robertson, *Boundary actions for affine buildings and higher rank Cuntz–Krieger algebras*, in J. Cuntz, S. Echterhoff (eds.) "C^*–algebras", Springer Verlag 2001, pp. 182–202.

[20] J. Spielberg, *Free–product groups, Cuntz–Krieger algebras, and covariant maps.* Internat. J. Math. 2 (1991), no. 4, 457–476.

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