On a conjecture of Fontaine and Mazur

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In this talk we discuss the following theorem, which provides some evidence for a very influential conjecture of Fontaine and Mazur (which for many years circulated only as the abstract of Fontaine's talk at the 1988 Arbeitstagung - MPIM preprints 1988 number 30).

Theorem A *Let $l > 3$ be a prime and fix $i : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. Let $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_l^{ac})$ be a continuous irreducible representation such that*

- *ρ is ramifies at only finitely many primes,*
- *$\rho|_{G_l}$ is crystalline with Hodge-Tate numbers n_1, n_2 where $1 \leq |n_1 - n_2| \leq (l-1)/2$,*
- *$\det \rho(c) = -1$.*

Then the following assertions hold.

1. *There is a Galois totally real field $F/\mathbb{Q}$, a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ and an embedding λ of the field of rationality of π into $\mathbb{Q}_l^{ac}$ such that $\rho|_{G_F} \sim \rho_{\pi,\lambda}$, the usual l-adic representation associated to π (and λ).*
2. *If ρ is unramified at a prime p and if α is an eigenvalue of $\rho(\mathrm{Frob}_p)$ then $\alpha \in \mathbb{Q}^{ac}$ and for any embedding $\mathbb{Q}^{ac} \hookrightarrow \mathbb{C}$ we have*
$$|\alpha|^2 = p^{n_1+n_2}.$$
3. *There is a rational function $L_l(X) \in \mathbb{C}(X)$ such that the product*
$$L(i\rho, s) = L_l(l^{-s})^{-1} \prod_{p \neq l} i \det(1 - \rho^{I_p}(\mathrm{Frob}_p)p^{-s})^{-1}$$
converges in $\mathrm{Re}\, s > 1 + (n_1+n_2)/2$ and extends to a meromorphic function on the entire complex plane which satisfies a functional equation
$$(2\pi)^{n_2-s}\Gamma(s-n_2)L(i\rho,s) = \epsilon(i\rho,s)(2\pi)^{s-n_1-1}\Gamma(1+n_1-s)L(i\rho^\vee, 1-s),$$
where $\epsilon(i\rho, s)$ denotes the usual product of local ϵ-factors and so has the form N^sW for some constants N and W.
4. *If $k = 2$ further assume that for some prime $p \neq l$ we have*
$$\rho|_{G_p} \sim \begin{pmatrix} \epsilon\chi & * \\ 0 & \chi \end{pmatrix}.$$
Then ρ occurs in the l-adic cohomology (with coefficients in some Tate twist of the constant sheaf) of some smooth projective variety over $\mathbb{Q}$.

As a consequence we see that the L-function of any regular (i.e. distinct Hodge numbers), rank two motive over $\mathbb{Q}$ has meromorphic continuation to the entire complex plane and satisfies the expected functional equation. This applies for instance to the L-function of H^1 of any abelian variety of GL_2-type, and to the zeta function of any rigid (i.e. $h^{2,1} = 0$) Calabi-Yau 3-fold.

Because it is not entirely clear what is meant by motive we can make this statement precise as follows. By a *regular, rank 2 weakly compatible system of l-adic representations* $\mathcal{R}$ over $\mathbb{Q}$ we shall mean a 5-tuple
$$(M, S, \{Q_p(X)\}, \{\rho_\lambda\}, \{n_1, n_2\})$$
where

- M is a number field;
- S is a finite set of rational primes;
- for each prime $p \notin S$ of $\mathbb{Q}$, $Q_p(X)$ is a monic degree 2 polynomial in $M[X]$;
- for each prime λ of M (with residue characteristic l say)

$$\rho_\lambda : G_\mathbb{Q} \longrightarrow GL_2(M_\lambda)$$

  is a continuous representation such that $\det \rho_\lambda(c) = -1$ and such that for each rational prime $p \notin S \cup \{l\}$, ρ_λ is unramified at p and $\rho_\lambda(\mathrm{Frob}_p)$ has characteristic polynomial $Q_p(X)$; and

- n_1, n_2 are distinct integers such that for all primes λ of M lying above a rational prime $l \notin S$ the representation $\rho_\lambda|_{G_l}$ is crystalline with Hodge-Tate numbers n_1 and n_2.

(We remark that if $\mathcal{R}$ arises from l-adic realisations of a regular motive one could use the Hodge realisation to check that $\det \rho_\lambda(c) = -1$ for all λ. Thus we consider this oddness condition part of regularity.) It is easy to see that either all the ρ_λ are absolutely reducible or all but finitely many are absolutely irreducible. In the former (resp. latter) case we call $\mathcal{R}$ reducible (resp. irreducible).

We will call $\mathcal{R}$ *strongly compatible* if for each rational prime p there is a representation $\mathrm{WD}_p(\mathcal{R})$ of the Weil-Deligne group of $\mathbb{Q}_p$ such that for primes λ of M not dividing p, $\mathrm{WD}_p(\mathcal{R})$ is equivalent to the Frobenius semi-simplification of the representation of the Weil-Deligne group associated to $\rho_\lambda|_{G_p}$. If $\mathcal{R}$ is strongly compatible and if $i : M \hookrightarrow \mathbb{C}$ then we define an L-function $L(i\mathcal{R}, s)$ as the infinite product

$$L(i\mathcal{R}, s) = \prod_p L_p(i\mathrm{WD}_p(\mathcal{R}), s)$$

which may or may not converge. Fix an additive character $\Psi = \prod \Psi_p$ of $\mathbb{A}/\mathbb{Q}$ with $\Psi_\infty(x) = e^{2\pi i x}$, and a Haar measure $dx = \prod dx_p$ on $\mathbb{A}$ with dx_∞ the usual measure on $\mathbb{R}$ and with $dx(\mathbb{A}/\mathbb{Q}) = 1$. Then we can also also define an ϵ-factor $\epsilon(i\mathcal{R}, s)$ by the formula

$$\epsilon(i\mathcal{R}, s) = i^{1+n_1-n_2} \prod_p \epsilon(i\mathrm{WD}_p(\mathrm{RS}), \Psi_p, dx_p, s).$$

Theorem B *Suppose that* $\mathcal{R} = (M, S, \{Q_x(X)\}, \{\rho_\lambda\}, \{n_1, n_2\})/\mathbb{Q}$ *is a regular, irreducible, rank* 2 *weakly compatible system of* l*-adic representations with* $n_1 > n_2$ *and suppose* $i : M \hookrightarrow \mathbb{C}$*. Then the following assertions hold.*

1. *For all rational primes* $p \notin S$ *and for all* $i : M \hookrightarrow \mathbb{C}$ *the roots of* $i(Q_p(X))$ *have absolute value* $p^{(n_1+n_2)/2}$.
2. $\mathcal{R}$ *is strongly compatible.*
3. *The* L*-function* $L(i\mathcal{R}, s)$ *converges to a meromorphic function in* $\mathrm{Re}\, s > 1+(n_1+n_2)/2$*, has meromorphic continuation to the entire complex plane, and satisfies a functional equation*

$$(2\pi)^{n_2-s}\Gamma(s-n_2)L(i\mathcal{R}, s) = \epsilon(i\mathcal{R}, s)(2\pi)^{s-n_1-1}\Gamma(1+n_1-s)L(i\mathcal{R}^\vee, 1-s).$$