Quantized moduli spaces of the bundles on the elliptic curve and their applications.

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Introduction

Let $\mathfrak{b}$ be the Borel subalgebra of some Kac-Moody algebra $\mathfrak{g}$ and B the corresponding group. (If $\mathfrak{b}$ is infinite-dimensional, then B is a projective limit of finite-dimensional groups). Let $\mathcal{M}(\mathcal{E},\mathfrak{b})$ be the moduli space of B-bundles on the elliptic curve $\mathcal{E}$. There is the natural map $B\to H$, where H is a Cartan subgroup, so each B-bundle defines an H-bundle. Note, that each H-bundle on $\mathcal{E}$ has the characteristic class which lies in $H^2(\mathcal{E},L)\cong L$, where L is the root lattice of Kac-Moody algebra $\mathfrak{g}$. For each $n\in L$ we denote by $\mathcal{M}_n(\mathcal{E},\mathfrak{b})$ the corresponding connected component of $\mathcal{M}(\mathcal{E},\mathfrak{b})$.

In [3] we define the Hamiltonian structure on the manifold $\mathcal{M}(\mathcal{E},\mathfrak{b})$ and quantize the coordinate ring of the manifold $\mathcal{M}_n(\mathcal{E},\mathfrak{b})$ in the case when $n=\sum\limits_i n_i\delta_i$, $\{\delta_i\}$ are simple positive roots, $\{n_i\}$ are positive and big enough. Actually, our construction works in case of arbitrary $\{n_i\}$.

In this paper we consider the case when almost all n_i are zero and some of them may be negative. Discuss first the case when all $\{n_i\}$ are zero. If $\mathfrak{g}$ is finite-dimensional, then $\mathcal{M}_0(\mathcal{E},\mathfrak{b})$ is trivial. If $\mathfrak{g}$ is an affine Kac-Moody algebra, then $\mathcal{M}_0(\mathcal{E},\mathfrak{b})$ is infinite-dimensional with trivial Hamiltonian structure. Let $\mathfrak{g}=\widehat{\mathfrak{a}}$, where $\mathfrak{a}$ is a finite-dimensional semisimple Lie algebra and P be a moduli space of $\mathfrak{a}$-bundles on $\mathcal{E}$. Then $\mathcal{M}_0(\mathcal{E},\mathfrak{b})$ is a space of maps $\varphi: D\to P$, where D is a formal neighborhood of the origin in $\mathbb{C}$ such that $\varphi(0)$ is a trivial bundle. In §**5** we study the construction in this case. In the case $\mathfrak{g}=\widehat{sl_h}$ we write the quantization of $\mathcal{M}_0(\mathcal{E},\mathfrak{b})$ explicitly. It means that we construct an infinite set of commuting elements in some algebra.

In §**4.2** we consider the case when $\mathfrak{g}=sl_{h+1}$ and the set $(n_1,\dots,n_h)$ is $(m,0,0,\dots,0)$. The moduli space $\mathcal{M}_{(m,0,\dots,0)}(\mathcal{E},\mathfrak{b})$ has a projection on $\mathcal{E}^h$ and a typical fiber is $(\mathbb{C}P^{m-1})^h$. After quantization we get a new construction of elliptic R-matrix.

In the §**4.4** we consider again the case $\mathfrak{g}=sl_{h+1}$, when the set of $\{n_i\}$ is $(1,0,\dots,0,m)$. In this case our algebra contains the Sklyanin subalgebra $Q_{hm+1,h}(\mathcal{E},\tau)$ (see[2]).

In the §**4.5** again for $\mathfrak{g}=sl_{h+1}$ we consider the set $(0,\dots,0,1,0,\dots,0)$. In this case we get the generalized elliptic R-matrix (compare [2], formula (6)).

In the §**4.3** we consider the case $(-1,0,\dots,0,m)$ for $\mathfrak{g}=sl_{h+1}$. It gives us the quantization of the coordinate ring of the Grassmannian of h-dimensional planes in $m-h+1$-dimensional space.

Now we describe the contents of the paper.

In §**1** we introduce the notations and construct the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$. Here Δ is a root system of $\mathfrak{g}$, $\tau\in\mathbb{C}$. For $\tau=0$ this algebra is the coordinate ring of the manifold $\mathcal{M}_n(\mathcal{E},\mathfrak{b})$ and for generic τ it is a quantization of this coordinate ring.

In §**2** we construct some representations of the algebras $Q_{n,\Delta}(\mathcal{E},\tau)$ and in §**3** we study intertwining operators for these representations. Namely, we embed the algebra $Q_{n',\Delta}(\mathcal{E},-\tau)$ for some n' into the algebra of intertwining operators.

In §**4** we apply the construction of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ to produce some objects: the elliptic Belavin R-matrix, the quantization of the algebra of functions on the Grassmannian, the algebra $Q_{hm+1,h}(\mathcal{E},\tau)$ (see [2] for definition), some generalized elliptic R-matrix.

In §**5** we consider the case of affine $\mathfrak{g}$. We write down the explicit formula for commuting elements in the case $\mathfrak{g}=\widehat{sl_h}$.

In §**6** we construct the elliptic deformation of the Poison algebra and write down the explicit formula for commuting elements in this case (see also [5]).

§1. Algebra $Q_{n,\Delta}(\mathcal{E},\tau)$.

1.1. Notations. Let Δ be the root system of the Kac-Moody algebra $\mathfrak{g}$, Δ^+ are the positive roots, $\{\delta_1,\dots,\delta_h\}$ are the simple positive roots (so h is rank $\mathfrak{g}$). We suppose that the set of simple positive roots in Δ is fixed. Let $(a_{i,j}), 1\leqslant i,j\leqslant h$ be the Cartan matrix, $a_{i,j}=\frac{2(\delta_i,\delta_j)}{(\delta_i,\delta_i)}$. Actually, for our construction of $Q_{n,\Delta}(\mathcal{E},\tau)$, we need only a linear space with a fixed basis $\{\delta_1,\dots,\delta_h\}$ and a scalar product (δ_i,δ_j). But we don't know anything about the properties of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ if $\{\delta_1,\dots,\delta_h\}$ is not a set of simple positive roots of the finite dimensional or affine Kac-Moody algebra. Let L be a lattice with the basis $\{\delta_1,\dots,\delta_h\}$, $L\cong\mathbb{Z}^h$ and let $L^+=\{l_1\delta_1+\ldots+l_h\delta_h; l_1,\dots,l_h\geqslant 0\}\subset L$.

Let us $n: L\to\mathbb{Z}$ be homomorphism of the abelian groups, so $n_i=n(\delta_i)\in\mathbb{Z}$.

Let $\mathcal{E}=\mathbb{C}/\Gamma$ be an elliptic curve, where $\Gamma=\{m_1+m_2\eta; m_1,m_2\in\mathbb{Z}\}$ is a lattice, $\operatorname{Im}\eta>0$. For $m\in\mathbb{Z}$, $c\in\mathbb{C}$ we denote by $\xi_{m,c}$ a linear bundle on $\mathcal{E}$ such that its sections are realized as functions on $\mathbb{C}$ with the following properties:

$$f(z+1)=f(z); f(z+\eta)=e^{-2\pi i(mz+c)}f(z) \tag{1}$$

It is clear that $\xi_{m,c}\cong\xi_{m',c'}$ if and only if $m=m'$ and $c-c'\in\Gamma$. The space of holomorphic sections $H^0(\xi_{m,c})$ is realized as a space of holomorphic functions on $\mathbb{C}$ satisfying (1). We denote this space by $\Theta_{m,c}(\Gamma)$. It is easy to see that $\dim\Theta_{m,c}(\Gamma)=m$ if $m>0$, $\Theta_{m,c}(\Gamma)=0$ if $m<0$. If $m=0$ then $\Theta_{0,c}(\Gamma)=0$ for $c\notin\Gamma$. If $m=0$ and $c\in\Gamma$ then the bundle $\xi_{0,c}$ is trivial and $\dim H^0(\xi_{0,c})=1$. For $m>0$ the elements of $\Theta_{m,c}(\Gamma)$ are called θ-functions of order m. It is clear that for $m>0$ for different c the spaces $\Theta_{m,c}(\Gamma)$ are isomorphic and can be identified by translation of $\mathbb{C}$. It is easy to check that every θ-function of order m has exactly m zeros $\operatorname{mod}\Gamma$ and the sum of these zeros is equal to $c+\frac{1}{2}m \operatorname{mod}\Gamma$. Let $\theta(z)=\sum\limits_{\alpha\in\mathbb{Z}}(-1)^\alpha e^{2\pi i(\alpha z+\frac{\alpha(\alpha-1)}{2}\eta)}$. It is clear that $\theta(z)\in\Theta_{1,\frac{1}{2}}(\Gamma)$, $\theta(0)=0$, $\theta(-z)=-e^{-2\pi iz}\theta(z)$.

1.2. Algebra $F_\Delta(\mathcal{E},\tau)$. Let $\tau\in\mathbb{C}$. We will construct an associative algebra $F_\Delta(\mathcal{E},\tau)$ that depends on 2 continues parameters: an elliptic curve $\mathcal{E}$ and a point $\tau\in\mathbb{C}$. The algebra $F_\Delta(\mathcal{E},\tau)$ is L^+-graded, so $F_\Delta(\mathcal{E},\tau)=\bigoplus\limits_{l\in L^+}F_l$ and $F_l*F_{l'}\subset F_{l+l'}$. Here $*$ is the product in the algebra $F_\Delta(\mathcal{E},\tau)$. By definition, the space F_l for $l=l_1\delta_1+\ldots+l_h\delta_h$ is a space of everywhere meromorphic functions $f(x_{1,1},\dots,x_{l_1,1};\dots;x_{1,h},\dots,x_{l_h,h};u_1,\dots,u_h)$ in $l_1+\ldots+l_h+h$ variables $\{x_{\alpha,i},u_i; 1\leqslant i\leqslant h, 1\leqslant\alpha\leqslant l_i\}$. We assume, that these functions are symmetric with respect to each group of variables $\{x_{1,i},\dots,x_{l_i,i}\}$ for all $1\leqslant i\leqslant h$. Let $f\in F_l, g\in F_{l'}$, where $l=l_1\delta_1+\ldots+l_h\delta_h, l'=l'_1\delta_1+\ldots+l'_h\delta_h$. By definition, for the product $f*g\in F_{l+l'}$ we have:

$$f*g(x_{1,1},\dots,x_{l_1+l'_1,1};\dots;x_{1,h},\dots,x_{l_h+l'_h,h};u_1,\dots,u_h)=$$

$$\frac{1}{l_1!l_1'!\dots l_h!l_h'!}\sum_{\substack{\sigma_1\in S_{l_1+l_1'}\\ \dots\dots\dots\\ \sigma_h\in S_{l_h+l_h'}}} f(x_{\sigma_1(1),1},\dots,x_{\sigma_1(l_1),1};\dots;x_{\sigma_h(1),h},\dots,x_{\sigma_h(l_h),h};u_1,\dots,u_h)\times$$

$$g(x_{\sigma_1(l_1+1),1},\dots,x_{\sigma_1(l_1+l_1'),1};\dots;x_{\sigma_h(l_h+1),h},\dots,x_{\sigma_h(l_h+l_h'),h};u_1^{(\tau)},\dots,u_h^{(\tau)})\times \quad (2)$$

$$\prod_{\substack{1\leqslant i,j\leqslant h,\\ 1\leqslant\alpha\leqslant l_i,\\ l_j+1\leqslant\beta\leqslant l_j+l_j'}} \frac{\theta(x_{\sigma_i(\alpha),i}-x_{\sigma_j(\beta),j}-(\delta_i,\delta_j)\tau)}{\theta(x_{\sigma_i(\alpha),i}-x_{\sigma_j(\beta),j})}$$

Here $u_i^{(\tau)}=u_i-2(l,\delta_i)\tau$ for $1\leqslant i\leqslant h$.

Proposition 1. *The formula (2) defines in the space* $F_\Delta(\mathcal{E},\tau)=\bigoplus_{l\in L^+}F_l$ *the structure of the associative algebra.*

Proof. By direct calculation.

Remarks. 1. Let $\tau=0$. In this case the algebra $F_\Delta(\mathcal{E},0)$ is commutative and does not depend on $\mathcal{E}$. The formula for $f*g$ takes such a form:

$$f*g(x_{1,1},\dots,x_{l_1+l_1',1};\dots;x_{1,h},\dots,x_{l_h+l_h',h};u_1,\dots,u_h)=$$

$$\frac{1}{l_1!l_1'!\dots l_h!l_h'!}\sum_{\substack{\sigma_1\in S_{l_1+l_1'}\\ \dots\dots\dots\\ \sigma_h\in S_{l_h+l_h'}}} f(x_{\sigma_1(1),1},\dots,x_{\sigma_1(l_1),1};\dots;x_{\sigma_h(1),h},\dots,x_{\sigma_h(l_h),h};u_1,\dots,u_h)\times$$

$$g(x_{\sigma_1(l_1+1),1},\dots,x_{\sigma_1(l_1+l_1'),1};\dots;x_{\sigma_h(l_h+1),h},\dots,x_{\sigma_h(l_h+l_h'),h};u_1,\dots,u_h) \quad (3)$$

Let $P_l\subset F_l$ be a space of functions which does not depend on $\{u_1,\dots,u_h\}$ and are polynomials in variables $\{x_{\alpha,i};1\leqslant i\leqslant h,1\leqslant\alpha\leqslant l_i\}$. It is clear that $P=\bigoplus_{l\in L^+}P_l$ is subalgebra in $F_\Delta(\mathcal{E},0)$. We have: $P_l=S^{l_1}P_{\delta_1}\otimes S^{l_2}P_{\delta_2}\otimes\dots\otimes S^{l_h}P_{\delta_h}$, so $P=S^*P_{\delta_1}\otimes S^*P_{\delta_2}\otimes\dots\otimes S^*P_{\delta_h}$. Here P_{δ_i} is a space of polynomials in one variable. The formula (3) defines the usual product in this tensor product of symmetric algebras.

2. By definition F_0 is a space of all meromorphic functions in variables $\{u_1, \ldots, u_h\}$. For $f, g \in F_0$ we have (see (2)) $f * g = fg$. It is the usual product of functions. So F_0 is a field. We have $F_0 * F_l \subset F_l$ and $F_l * F_0 \subset F_l$ so we have two structures of a F_0-vector space on F_l. These two structures are connected by the following: for $f \in F_0$ and $g \in F_l$ we have: $g * f(u_1, \ldots, u_h) = f(u_1 - 2(l, \delta_1)\tau, \ldots, u_h - 2(l, \delta_h)\tau) * g$ (see (2)). For $\tau = 0$, the algebra $F_\Delta(\mathcal{E}, 0)$ is a F_0-algebra.

1.3. Algebra $Q_{n,\Delta}(\mathcal{E}, \tau)$. For every linear form $n : L \to \mathbb{Z}$ we will construct a subalgebra $Q_{n,\Delta}(\mathcal{E}, \tau)$ in the algebra $F_\Delta(\mathcal{E}, \tau)$. By definition $Q_{n,\Delta}(\mathcal{E}, \tau) = \bigoplus_{l \in L^+} Q_l$, here $Q_l \subset F_l$ is a space of functions satisfied following conditions:

1. $f(x_{1,1}, \ldots, x_{l_h,h}; u_1, \ldots, u_h)$ as a function in variables $\{x_{1,1}, \ldots, x_{l_h,h}\}$ is holomorphic outside the divisors $\{x_{\alpha,i} - x_{\beta,j} = 0; i \neq j, (\delta_i, \delta_j) \neq 0\}$ and has a pole of order $\leqslant 1$ on these divisors.

2. For each $x_{\alpha,i}$, $1 \leqslant i \leqslant h$, $1 \leqslant \alpha \leqslant l_i$ f satisfies (1) for $m = n_i$, $c = u_i - (\delta_i, l)\tau$ as a function in $x_{\alpha,i}$. Here $n_i = n(\delta_i)$. So we have:

$$f(x_{1,1}, \ldots, x_{\alpha,i} + 1, \ldots, x_{l_h,h}; u_1, \ldots, u_h) = f(x_{1,1}, \ldots, x_{l_h,h}; u_1, \ldots, u_h);$$

$$f(x_{1,1}, \ldots, x_{\alpha,i} + \eta, \ldots, x_{l_h,h}; u_1, \ldots, u_h) =$$
$$e^{-2\pi i(n_i x_{\alpha,i} + u_i - (\delta_i, l)\tau)} f(x_{1,1}, \ldots, x_{l_h,h}; u_1, \ldots, u_h)$$

3. Let $1 \leqslant i, j \leqslant h$ such that $a_{i,j} < 0$. Let $1 \leqslant \alpha_1, \ldots, \alpha_{-a_{i,j}+1} \leqslant l_i$, $1 \leqslant \beta \leqslant l_j$. Then $f(x_{1,1}, \ldots, u_h) = 0$ on the affine subspace of codimension $-a_{i,j} + 1$ defined by the following relations:

$$x_{\alpha_1,i} - x_{\alpha_2,i} = (\delta_i, \delta_i)\tau, x_{\alpha_2,i} - x_{\alpha_3,i} = (\delta_i, \delta_i)\tau, \ldots, x_{\alpha_{-a_{i,j}},i} - x_{\alpha_{-a_{i,j}+1},i} = (\delta_i, \delta_i)\tau,$$

$$x_{\alpha_{-a_{i,j}+1},i} - x_{\beta,j} = (\delta_i, \delta_j)\tau, x_{\beta,j} - x_{\alpha_1,i} = (\delta_i, \delta_j)\tau.$$

4. Let $\{\delta_{i_1}, \ldots, \delta_{i_p}\}$ generates an irreducible component of the root system Δ; $\mu_1, \ldots, \mu_h; \lambda_1, \ldots, \lambda_h; \nu \in \mathbb{C}$ and $\mu_j = \lambda_j = 0$ if $j \notin \{i_1, \ldots, i_p\}$, $\mu_{i_\alpha} = -\nu$, $\lambda_{i_\alpha} = n_{i_\alpha}\nu$ for $1 \leqslant \alpha \leqslant p$. Then $f(x_{1,1} + \mu_1, \ldots, x_{l_1,1} + \mu_1; \ldots; x_{1,h} + \mu_h, \ldots, x_{l_h,h} + \mu_h; u_1 + \lambda_1, \ldots, u_h + \lambda_h) = f(x_{1,1}, \ldots, u_h)$.

Proposition 2. *The space $Q_{n,\Delta}(\mathcal{E}, \tau)$ is closed with respect to the product $*$ defined by (2). So $Q_{n,\Delta}(\mathcal{E}, \tau)$ is a subalgebra of the algebra $F_\Delta(\mathcal{E}, \tau)$.*

Proof. By direct calculation it is easy to see that the product $*$ preserves each of the properties 1-4.

Proposition 3. *Let $\Delta_1 \subset \Delta$ is a root subsystem generated by some subset of $\{\delta_1, \ldots, \delta_h\}$. Then $F_{\Delta_1}(\mathcal{E}, \tau) \subset F_\Delta(\mathcal{E}, \tau)$, $Q_{m,\Delta_1}(\mathcal{E}, \tau) \subset Q_{n,\Delta}(\mathcal{E}, \tau)$ as graded subalgebras. Here m is a restriction of n to the sublattice generated by Δ_1.*

Proof. It is evident from the definitions.

Remarks. **1.** Q_0 is a field of meromorphic functions $f(u_1, \ldots, u_h)$ satisfying the property 4.

2. We have $Q_0 * Q_l \subset Q_l$ and $Q_l * Q_0 \subset Q_l$ so Q_l has two structures of a Q_0-vector space. These two structures are connected in the following way: for $f \in Q_0$ and $g \in Q_l$ we have; $g * f(u_1, \ldots, u_h) = f(u_1 - 2(l, \delta_1)\tau, \ldots, u_h - 2(l, \delta_h)\tau) * g$ (see (2)), so the dimension of these spaces is the same.

3. From the properties 1 and 2 of the elements from the space Q_l it follows that $\dim_{Q_0} Q_l$ is finite for each $l \in L^+$.

4. For $\tau = 0$ the algebra $Q_{n,\Delta}(\mathcal{E}, \tau)$ is a commutative Q_0-algebra.

5. Q_{δ_i} consists of the functions $f(x; u_1, \ldots, u_h)$ which are holomorphic with respect to x and satisfy the properties 2 and 4. So $\dim_{Q_0} Q_{\delta_i} = n_i$ if $n_i > 0$ and $Q_{\delta_i} = 0$ if $n_i \leqslant 0$.

6. $Q_{\alpha\delta_i}$ consists of the functions $f(x_1, \ldots, x_\alpha; u_1, \ldots, u_h)$ which are holomorphic and symmetric with respect to $\{x_1, \ldots, x_\alpha\}$ and satisfy the properties 2 and 4. It is clear that $\dim_{Q_0} Q_{\alpha\delta_i} = \frac{n_i(n_i+1)\ldots(n_i+\alpha-1)}{\alpha!}$ if $n_i > 0$ and $Q_{\alpha\delta_i} = 0$ if $n_i \leqslant 0$.

7. If $(\delta_i, \delta_j) = 0$ then $Q_{\alpha\delta_i+\beta\delta_j} \cong Q_{\alpha\delta_i} \otimes_{Q_0} Q_{\beta\delta_j}$.

8. If $(\delta_i, \delta_j) \neq 0$ (and $i \neq j$), than $Q_{\delta_i+\delta_j}$ becomes bigger then $Q_{\delta_i} \otimes_{Q_0} Q_{\delta_j}$ because of the pole part of the functions from $Q_{\delta_i+\delta_j}$. So we have: $\dim_{Q_0} Q_{\delta_i+\delta_j} = n_i n_j + n_i + n_j$, if $n_i, n_j \geqslant 0$. If n_i or $n_j < 0$, then $Q_{\delta_i+\delta_j} = 0$.

§2. Representations of the algebras $Q_{n,\Delta}(\mathcal{E},\tau)$.

Let $p \in L^+$, $p = p_1\delta_1 + \ldots + p_h\delta_h$. Let $A_{p,\Delta}(\mathcal{E},\tau)$ be the algebra generated by $\{e_{\alpha,i}, e_{\alpha,i}^{-1}; 1 \leqslant i \leqslant h, 1 \leqslant \alpha \leqslant p_i\}$ and $\{\varphi(y_{1,1},\ldots,y_{p_1,1};\ldots;y_{1,h},\ldots,y_{p_h,h};u_1,\ldots,u_h)\}$, where φ is any meromorphic function in variables $\{y_{\alpha,i}, u_i; 1 \leqslant i \leqslant h, 1 \leqslant \alpha \leqslant p_i\}$ such that as a function of $\{y_{\alpha,i}\}$ it has poles only on the divisors of the form $\{y_{\alpha,i} - y_{\beta,j} - (\delta_i,\delta_j)\tau + \mu(\delta_i,\delta_i)\tau - \nu(\delta_j,\delta_j)\tau = 0; \mu,\nu \in \mathbb{Z}, (\delta_i,\delta_j) \neq 0\}$. We assume such the following relations hold:

$$e_{\alpha,i}e_{\beta,j} = -e^{2\pi i(y_{\beta,j}-y_{\alpha,i})}\frac{\theta(y_{\alpha,i} - y_{\beta,j} - (\delta_i,\delta_j)\tau)}{\theta(y_{\beta,j} - y_{\alpha,i} - (\delta_i,\delta_j)\tau)}e_{\beta,j}e_{\alpha,i} \tag{4}$$

$e_{\alpha,i}y_{\beta,j} = y_{\beta,j}e_{\alpha,i}$, here $i \neq j$ or $\alpha \neq \beta$;

$$e_{\alpha,i}e_{\alpha,i}^{-1} = 1, e_{\alpha,i}y_{\alpha,i} = (y_{\alpha,i} + (\delta_i,\delta_i)\tau)e_{\alpha,i}, e_{\alpha,i}u_j = (u_j - 2(\delta_i,\delta_j)\tau)e_{\alpha,i};$$

$$[y_{\alpha,i}, y_{\beta,j}] = [y_{\alpha,i}, u_j] = [u_i, u_j] = 0.$$

Let us define the map $x : Q_{n,\Delta}(\mathcal{E},\tau) \to A_{p,\Delta}(\mathcal{E},\tau)$ by the following. For $f \in Q_0$, we put $x(f) = f$. Recall that $Q_0 \subset A_{p,\Delta}(\mathcal{E},\tau)$ by definition. For $f(x;u_1,\ldots,u_h) \in Q_{\delta_i}$ (if $n_i > 0$), we put $x(f) = \sum\limits_{1\leqslant\alpha\leqslant p_i} f(y_{\alpha,i};u_1,\ldots,u_h)e_{\alpha,i}$. In general case for $f(x_{1,1},\ldots,x_{l_h,h};u_1,\ldots,u_h) \in Q_l$, $l = l_1\delta_1 + \ldots + l_h\delta_h$, we define:

$$x(f) = \sum_{\substack{\varphi_{1,1},\ldots,\varphi_{p_h,h}\geqslant 0,\\ \varphi_{1,i}+\ldots+\varphi_{p_i,i}=l_i \text{ for each } 1\leqslant i\leqslant h}} B_{\varphi_{1,1},\ldots,\varphi_{p_h,h}} e_{1,1}^{\varphi_{1,1}}\ldots e_{p_1,1}^{\varphi_{p_1,1}}\ldots e_{1,h}^{\varphi_{1,h}}\ldots e_{p_h,h}^{\varphi_{p_h,h}} \tag{5}$$

Here $B_{\varphi_{1,1},\ldots,\varphi_{p_h,h}}$ is a function in variables $\{y_{\alpha,i}, u_i\}$,

$$B_{\varphi_{1,1},\ldots,\varphi_{p_h,h}} =$$

$$f(y_{1,1}, y_{1,1} + (\delta_1,\delta_1)\tau, \ldots, y_{1,1} + (\varphi_{1,1} - 1)(\delta_1,\delta_1)\tau, \ldots,$$

$$y_{p_h,h}, y_{p_h,h} + (\delta_h,\delta_h)\tau, \ldots, y_{p_h,h} + (\varphi_{p_h,h} - 1)(\delta_h,\delta_h)\tau; u_1,\ldots,u_h)\times$$

$$\prod_{\substack{1\leqslant i,j\leqslant h,\\ 1\leqslant\alpha\leqslant p_i, 1\leqslant\beta\leqslant p_j,\\ 0\leqslant\mu\leqslant\varphi_{\alpha,i}-1, 0\leqslant\nu\leqslant\varphi_{\beta,j}-1;\\ i\leqslant j;\text{ if } i=j, \text{ then } \alpha\leqslant\beta;\\ \text{if } i=j,\alpha=\beta, \text{ then } \mu<\nu}} \frac{\theta(y_{\alpha,i} + \mu(\delta_i,\delta_i)\tau - y_{\beta,j} - \nu(\delta_j,\delta_j)\tau)}{\theta(y_{\alpha,i} + \mu(\delta_i,\delta_i)\tau - y_{\beta,j} - \nu(\delta_j,\delta_j)\tau - (\delta_i,\delta_j)\tau)}$$

Proposition 4. *x is homomorphism of algebras.*

Proof. It is direct checking using the formulas (2) and (4).

Remarks. 1. If $\varphi_{\alpha,i} = 0$ for some α and i, then f does not contain variable $y_{\alpha,i}$ in corresponding part of the formula (5). For $\varphi_{\alpha,i} > 0$, f contains variables $y_{\alpha,i} + \mu(\delta_i, \delta_i)\tau$ for $0 \leqslant \mu \leqslant \varphi_{\alpha,i} - 1$.

2. It is possible to construct representations of the algebra $Q_{n,\Delta}(\mathcal{E}, \tau)$ using homomorphism x. For this let us construct the representation of the algebra $A_{p,\Delta}(\mathcal{E}, \tau)$ with diagonal action of the elements $\{y_{\alpha,i}, u_i\}$. The basis of this representation consists of monomials of the elements $\{e_{\alpha,i}\}$. The homomorphism x defines the representation of the algebra $Q_{n,\Delta}(\mathcal{E}, \tau)$ in this space.

§3. Intertwining operators.

In this paragraph we assume that Δ is a root system of the finite dimension Lie algebra. The intertwining operators for the representations of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ are constructed in the following way (see also [6], §3). Let σ be an automorphism of the algebra $A_{p,\Delta}(\mathcal{E},\tau)$. We say that an element $C \in A_{p,\Delta}(\mathcal{E},\tau)$ σ-commutes with the image $x(Q_{n,\Delta}(\mathcal{E},\tau))$, if for each element $f \in Q_{n,\Delta}(\mathcal{E},\tau)$ we have: $Cx(f) = (\sigma x(f))C$ in the algebra $A_{p,\Delta}(\mathcal{E},\tau)$. Let π be some representation of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ constructed by the homomorphism x (see remark 2 in §2). It is clear that C defines the intertwining operator between the representations π and π^σ of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$. Let $C_1 \in A_{p,\Delta}(\mathcal{E},\tau)$ σ_1-commutes and $C_2 \in A_{p,\Delta}(\mathcal{E},\tau)$ σ_2-commutes with $x(Q_{n,\Delta}(\mathcal{E},\tau))$. It is clear that $(\sigma_2 C_1)C_2$ $\sigma_2\sigma_1$-commutes with $x(Q_{n,\Delta}(\mathcal{E},\tau))$. This product coincides with the product of the intertwining operators.

Let $L_{\mathbb{C}} = L \otimes_{\mathbb{Z}} \mathbb{C}$ be a $\mathbb{C}$-vector space with a basis $\{\delta_1,\ldots,\delta_h\}$. Let $\mu \in L_{\mathbb{C}}$, $\mu = \mu_1\delta_1 + \ldots + \mu_h\delta_h$. Let us define an automorphism T_μ of the algebra $A_{p,\Delta}(\mathcal{E},\tau)$ by the following: $T_\mu(e_{\alpha,i}) = e_{\alpha,i}$, $T_\mu(y_{\alpha,i}) = y_{\alpha,i}$, $T_\mu(u_i) = u_i + \mu_i$. We will look for the elements of the algebra $A_{p,\Delta}(\mathcal{E},\tau)$ that T_μ-commutes with $x(Q_{n,\Delta}(\mathcal{E},\tau))$. The algebra $A_{p,\Delta}(\mathcal{E},\tau)$ is L-graded, if we assume $\deg y_{\alpha,i} = \deg u_i = 0, \deg e_{\alpha,i} = \delta_i$ for each $1 \leqslant i \leqslant h$, $1 \leqslant \alpha \leqslant p_i$. It is clear that $A_{p,\Delta}(\mathcal{E},\tau) = \bigoplus\limits_{l\in L} A_l$, here A_l is the space of elements of degree l and $A_l A_{l'} \subset A_{l+l'}$ in the algebra $A_{p,\Delta}(\mathcal{E},\tau)$. It is clear that the homomorphism x preserves the grading, so $x(Q_l) \subset A_l$. Let $\omega : L \to L_{\mathbb{C}}$ be a homomorphism of abelian groups. We denote by $A^\omega_{p,\Delta}(\mathcal{E},\tau)$ the associative algebra such that as a linear space it is isomorphic to $A_{p,\Delta}(\mathcal{E},\tau)$ and the product $\circ$ is defined by the following: $C_1 \circ C_2 = (T_{\omega(l_2)}C_1)C_2$. Here $C_1 \in A_{l_1}, C_2 \in A_{l_2}$ and $\circ$ is the product in the algebra $A^\omega_{p,\Delta}(\mathcal{E},\tau)$. We will construct a homomorphism

$$y : Q_{n',\Delta}(\mathcal{E},-\tau) \to A^\omega_{p,\Delta}(\mathcal{E},\tau)$$

for some n' and ω such that for each elements $f \in Q_{n,\Delta}(\mathcal{E},\tau)$, $g \in Q_{n',\Delta}(\mathcal{E},-\tau)$ we will have:

$$y(g)x(f) = (T_{-\omega(l)}x(f))y(g)$$

in the algebra $A_{p,\Delta}(\mathcal{E},\tau)$. Here $g \in Q'_l$, $Q_{n',\Delta}(\mathcal{E},-\tau) = \bigoplus\limits_{l\in L^+} Q'_l$ is the decomposition of $Q_{n',\Delta}(\mathcal{E},-\tau)$ for L^+-grading. We will have $y(Q'_l) \subset A_{-l}$.

Let $n' : L \to \mathbb{Z}$ be such homomorphism that

$$n'(\delta_i) = \sum_{1\leqslant j\leqslant h} a_{j,i}p_j - n(\delta_i)$$

We denote $n'_i = n'(\delta_i)$ for all $1 \leqslant i \leqslant h$.

Let $A'_{p,\Delta}(\mathcal{E},-\tau)$ be the algebra with generators $\{e'_{\alpha,i}, e'^{-1}_{\alpha,i}; 1 \leqslant i \leqslant h, 1 \leqslant \alpha \leqslant p_i\}$ and $\{\varphi(y'_{1,1},\ldots,y'_{p_1,1};\ldots;y'_{1,h},\ldots,y'_{p_h,h};u'_1,\ldots,u'_h)\}$, where φ are meromorphic functions. We assume the the relations are as in (4), with τ replaced by $-\tau$ and $e_{\alpha,i}, y_{\alpha,i}, u_i$ replaced by $e'_{\alpha,i}, y'_{\alpha,i}, u'_i$ respectively. It is clear, that there is a homomorphism $x' : Q_{n',\Delta}(\mathcal{E},-\tau) \to A'_{p,\Delta}(\mathcal{E},-\tau)$ defined by (5), where $Q_{n,\Delta}(\mathcal{E},\tau)$ is replaced by $Q_{n',\Delta}(\mathcal{E},-\tau)$ and $A_{p,\Delta}(\mathcal{E},\tau)$ is replaced by $A'_{p,\Delta}(\mathcal{E},-\tau)$. Here Q'_0 consists of meromorphic functions in variables $\{u'_1,\ldots,u'_h\}$.

Let $\omega : L \to L_{\mathbb{C}}$ be such homomorphism that

$$\omega(\delta_i) = -\sum_{1\leqslant j\leqslant h} 2(\delta_i,\delta_j)\tau\cdot\delta_j$$

Proposition 5. *Following formulas define the homomorphism of the algebras* $\varkappa : A'_{p,\Delta}(\mathcal{E},-\tau) \to A^{\omega}_{p,\Delta}(\mathcal{E},\tau)$*:*

$$\varkappa(y'_{\alpha,i}) = y_{\alpha,i}$$

$$\varkappa(u'_i) = -u_i - \sum_{\substack{1\leqslant j\leqslant h,\\ 1\leqslant\beta\leqslant p_j}} a_{j,i}y_{\beta,j} \tag{6}$$

$$\varkappa(e'_{\alpha,i}) = \prod_{\substack{j\neq i, 1\leqslant j\leqslant h,\\ 1\leqslant\beta\leqslant p_j, 0\leqslant\delta\leqslant -a_{j,i}-1}} (\theta(y_{\alpha,i}-y_{\beta,j}-((\delta_i,\delta_j)+(\delta_i,\delta_i)+\delta(\delta_j,\delta_j))\tau)e^{\pi i y_{\beta,j}})\times$$

$$\prod_{\substack{1\leqslant\beta\leqslant p_i\\ \beta\neq\alpha}} \frac{e^{-2\pi i y_{\beta,i}}}{\theta(y_{\alpha,i}-y_{\beta,i}-(\delta_i,\delta_i)\tau)\theta(y_{\alpha,i}-y_{\beta,i})}\cdot e^{-1}_{\alpha,i}$$

Proof. It is a direct checking of the relations (4).

Let T'_μ be an automorphism of the algebra $A'_{p,\Delta}(\mathcal{E},-\tau)$ such that $T'_\mu e'_{\alpha,i} = e'_{\alpha,i}$, $T'_\mu y'_{\alpha,i} = y'_{\alpha,i}$, $T'_\mu u'_i = u'_i + \mu_i$. Here $\mu \in L_{\mathbb{C}}$, $\mu = \mu_1\delta_1 + \ldots + \mu_h\delta_h$. We put:

$$\mu_i = (-(n'_i+3)(\delta_i,\delta_i)+(\delta_i,p))\tau + \frac{1}{2}\sum_{1\leqslant j\leqslant h} a_{j,i}p_j$$

Proposition 6. *Let* $y : Q_{n',\Delta}(\mathcal{E},-\tau) \to A^{\omega}_{p,\Delta}(\mathcal{E},\tau)$ *be the composition of the homomorphisms* x', T'_μ *and* $\varkappa$, *so* $y(g) = \varkappa T'_\mu x'(g)$ *for* $g \in Q_{n',\Delta}(\mathcal{E},-\tau)$. *Then for each* $g \in Q_{n',\Delta}(\mathcal{E},-\tau)$, $f \in Q_{n,\Delta}(\mathcal{E},\tau)$ *there is a relation* $y(g)x(f) = (T_{-\omega(l)}x(f))y(g)$ *in the algebra* $A_{p,\Delta}(\mathcal{E},\tau)$. *Here* $g \in Q'_l$.

Proof. See [6], §3 for the case of strict dominant n and n' (this means $n(\delta_i) > 0$ and $n'(\delta_i) > 0$). In general case the proof is similar.

Remarks. **1.** This construction for the case of strict dominant n and n' was done in [6], §3, propositions 3 and 4. We have extended this construction for general case above.

2. In [6], §3, Proposition 5 we have constructed another elements $K_d \in A_{p,\Delta}(\mathcal{E},\tau)$ that T_ν-commute with $x(Q_{n,\Delta}(\mathcal{E},\tau))$. Here d is such that $\frac{d}{(\delta_i,\delta_i)} \in \mathbb{N}$ for each $1 \leqslant i \leqslant h$, $\nu = -d\tau(n'_1\delta_1 + \ldots + n'_h\delta_h)$. In [6] we considered only the case of strict dominant n and n', but it is clear that the Proposition 5 holds for general case.

3. In [6] we used slightly different notations (compare [6], §1 and §1 of this paper). In notations of [6] we had to divide by n_i (see for example [6], formula (1)) but it is not good if we want to consider the case $n_i = 0$ for some i. So we have to change the notations.

§4. Examples and applications.

4.1. Case of finite root system Δ and dominant n.

This case was studied in [6]. We remind the main result about the structure of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$.

Let $\mathfrak{g}=\mathfrak{g}^-\oplus\mathfrak{h}\oplus\mathfrak{g}^+$ be the Cartan decomposition, $\{g_\gamma;\gamma\in\Delta^+\}$ be a basis in $\mathfrak{g}^+$, $\varphi_{\gamma_1,\gamma_2}\in\mathbb{C}$ be the structure constants, so $[g_{\gamma_1},g_{\gamma_2}]=\varphi_{\gamma_1,\gamma_2}g_{\gamma_1+\gamma_2}$. We denote by $\widehat{\mathfrak{g}^+}(\mathcal{E},n)$ the following L^+-graded Lie algebra over field Q_0:

$$\widehat{\mathfrak{g}^+}(\mathcal{E},n)=\bigoplus_{\gamma\in\Delta^+}\Theta_{n(\gamma),u(\gamma)}(\Gamma)$$

as a linear space. Here $u(\gamma)=u_1\gamma_1+\ldots+u_h\gamma_h$ for $\gamma=\gamma_1\delta_1+\ldots+\gamma_h\delta_h$. The commutator for $\mu\in\Theta_{n(\gamma),u(\gamma)}(\Gamma)$, $\lambda\in\Theta_{n(\gamma'),u(\gamma')}(\Gamma)$ is by definition $[\mu,\lambda]=\varphi_{\gamma,\gamma'}\mu\cdot\lambda$. Here $\mu\cdot\lambda$ is the usual product of functions. We remind that by definition $\Theta_{n(\gamma),u(\gamma)}(\Gamma)=0$ if $n(\gamma)=0$ (see §1.1). It is clear that $\widehat{\mathfrak{g}^+}(\mathcal{E},n)$ is a subalgebra of the current algebra of $\mathfrak{g}^+$ over field Q_0.

Proposition 7. *The algebra $Q_{n,\Delta}(\mathcal{E},\tau)$, defined for finite root system Δ and dominant n is a deformation of the universal enveloping algebra $U(\widehat{\mathfrak{g}^+}(\mathcal{E},n))$ in the class of L^+-graded associative algebras.*

Remark. From this proposition it follows that the Hilbert function of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ for finite Δ and dominant n is

$$\sum_{l\in L^+}\dim_{Q_0}Q_l w^l=\prod_{\delta\in\Delta^+}(1-w^\delta)^{-n(\delta)}$$

Here $w^l=w_1^{l_1}\ldots w_h^{l_h}$ for $l=l_1\delta_1+\ldots+l_h\delta_h$; $w_1,\ldots,w_h$ are formal parameters.

4.2. The Zamolodchikove algebra for the elliptic Belavin R-matrix.

Let Δ be root system A_h, so $(\delta_i,\delta_i)=2$, $1\leqslant i\leqslant h$; $(\delta_i,\delta_{i+1})=-1$, $1\leqslant i<h$; $(\delta_i,\delta_j)=0$ if $|i-j|>1$. Let $n(\delta_1)=m>0$, $n(\delta_2)=\ldots=n(\delta_h)=0$. The Hilbert function of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ has a form (see §4.1): $\prod\limits_{\gamma\in\Delta^+}(1-w^\gamma)^{-n(\gamma)}=(1-w_1)^{-m}(1-w_1w_2)^{-m}\ldots(1-w_1\ldots w_h)^{-m}$. So the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ is a deformation of the polynomial ring in hm variables: m variables of degree $\delta_1+\ldots+\delta_i$ for each $1\leqslant i\leqslant h$. It is easy to check that the space $Q_{\delta_1+\ldots+\delta_i}$ consists of functions of the form:

$$f(x_{1,1};\ldots;x_{1,i};u_1,\ldots,u_h)=\frac{\prod\limits_{1\leqslant\alpha<i}\theta(x_{1,\alpha}-x_{1,\alpha+1}-\sum\limits_{\alpha<\nu\leqslant i}u_\nu+\tau)}{\prod\limits_{1\leqslant\alpha<i}\theta(x_{1,\alpha}-x_{1,\alpha+1})}\varphi(x_{1,1}),$$

where $\varphi(x)\in\Theta_{m,u_1+\ldots+u_i}(\Gamma)$.

Proposition 8. *The space $Q_{\delta_1+\ldots+\delta_i}$ generates the algebra $Q_m(\mathcal{E},\frac{2}{m}\tau)$ for each $1\leqslant i\leqslant h$. The commutation relations between $Q_{\delta_1+\ldots+\delta_i}$ and $Q_{\delta_1+\ldots+\delta_j}$, $i\neq j$, $1\leqslant i,j\leqslant h$ are defined by the elliptic R-matrix $R_m(\mathcal{E},\frac{2}{m}\tau)$ (for the definitions of $Q_m(\mathcal{E},\tau)$ and $R_m(\mathcal{E},\tau)$ see [1]).*

Proof. It is enough to proof the proposition for the image of the homomorphism $x: Q_{n,\Delta}(\mathcal{E},\tau)\to A_{p,\Delta}(\mathcal{E},\tau)$, because for large $\{p_i\}$ this homomorphism is injection. Let us denote by $z_{\varphi,i}$ the image of the element from the space $Q_{\delta_1+\ldots+\delta_i}$ for a function $\varphi(x)\in\Theta_{m,u_1+\ldots+u_i}(\Gamma)$. We denote by $v_i=u_1+\ldots+u_i$ for $1\leqslant i\leqslant h$. The formula (5) gives us the following expression for $z_{\varphi,i}\in A_{p,\Delta}(\mathcal{E},\tau)$:

$$z_{\varphi,i}=\sum_{\substack{1\leqslant\alpha_1\leqslant p_1\\ \cdots\cdots\cdots\\ 1\leqslant\alpha_i\leqslant p_i}}\frac{\prod\limits_{1\leqslant\nu<i}\theta(y_{\alpha_\nu,\nu}-y_{\alpha_{\nu+1},\nu+1}+v_\nu-v_i+\tau)}{\prod\limits_{1\leqslant\nu<i}\theta(y_{\alpha_\nu,\nu}-y_{\alpha_{\nu+1},\nu+1}+\tau)}\varphi(y_{\alpha_1,1})e_{\alpha_1,1}e_{\alpha_2,2}\ldots e_{\alpha_i,i} \tag{7}$$

Let

$$f_{\alpha_1,i}=\sum_{\substack{1\leqslant\alpha_2\leqslant p_2\\ \cdots\cdots\cdots\\ 1\leqslant\alpha_i\leqslant p_i}}\frac{\prod\limits_{1\leqslant\nu<i}\theta(y_{\alpha_\nu,\nu}-y_{\alpha_{\nu+1},\nu+1}+v_\nu-v_i+\tau)}{\prod\limits_{1\leqslant\nu<i}\theta(y_{\alpha_\nu,\nu}-y_{\alpha_{\nu+1},\nu+1}+\tau)}e_{\alpha_1,1}e_{\alpha_2,2}\ldots e_{\alpha_i,i} \tag{8}$$

From (7) and (8) we have:

$$z_{\varphi,i}=\sum_{1\leqslant\alpha\leqslant p_1}\varphi(y_{\alpha,1})f_{\alpha,i} \tag{9}$$

Using (4) it is easy to check the following relations between $\{f_{\alpha,i},y_{\alpha,1},v_i;1\leqslant\alpha\leqslant p_1;1\leqslant i\leqslant h\}$:

$$f_{\alpha,i}f_{\beta,j}=\frac{e^{2\pi i\tau}\theta(v_i-v_j)\theta(y_{\alpha,1}-y_{\beta,1}-2\tau)}{\theta(v_i-v_j+2\tau)\theta(y_{\alpha,1}-y_{\beta,1})}f_{\beta,j}f_{\alpha,i}+$$

$$\frac{e^{2\pi i(v_i-v_j-\tau)}\theta(2\tau)\theta(y_{\alpha,1}-y_{\beta,1}+v_j-v_i)}{\theta(v_i-v_j+2\tau)\theta(y_{\alpha,1}-y_{\beta,1})}f_{\alpha,j}f_{\beta,i},\ \text{here } \alpha\neq\beta, i<j; \tag{10}$$

$$f_{\alpha,i}f_{\beta,i}=-e^{2\pi i(y_{\beta,1}-y_{\alpha,1})}\frac{\theta(y_{\alpha,1}-y_{\beta,1}-2\tau)}{\theta(y_{\beta,1}-y_{\alpha,1}-2\tau)}f_{\beta,i}f_{\alpha,i},\ \text{here } \alpha\neq\beta;$$

$$f_{\alpha,j}f_{\alpha,i}=e^{2\pi i\tau}f_{\alpha,i}f_{\alpha,j},\ \text{here } i<j;$$

$$f_{\alpha,i}y_{\beta,1}=y_{\beta,1}f_{\alpha,i},\ \text{here } \alpha\neq\beta;\ f_{\alpha,i}y_{\alpha,1}=(y_{\alpha,1}+2\tau)f_{\alpha,i};$$

$$f_{\alpha,i}v_j=(v_j-2\tau)f_{\alpha,i},\ \text{here } i\neq j;\ f_{\alpha,i}v_i=(v_i-4\tau)f_{\alpha,i};$$

$$[y_{\alpha,1},y_{\beta,1}]=[y_{\alpha,1},v_i]=[v_i,v_j]=0.$$

The proposition follows from the relations (10), the formula (9) and the results of [2].

4.3. The elliptic deformation of the algebra functions on Grassmannian.

Let us set $\Delta=A_h$, $(\delta_i,\delta_i)=2$, $1\leqslant i\leqslant h$; $(\delta_i,\delta_{i+1})=-1$, $1\leqslant i<h$; $(\delta_i,\delta_j)=0$ if $|i-j|>1$ as above. Let $n(\delta_1)=-1$, $n(\delta_i)=0$ for $2\leqslant i\leqslant h-1$, $n(\delta_h)=m>0$. The space $Q_{\delta_1+2\delta_2+\ldots+h\delta_h}$ consists of the following functions:

$$f(x_{1,1};x_{1,2},x_{2,2};\ldots;x_{1,h},\ldots,x_{h,h};u_1,\ldots,u_h)=$$

$$\frac{\prod\limits_{1\leqslant i<h}\theta(\sum\limits_{1\leqslant\alpha\leqslant i}x_{\alpha,i}-\sum\limits_{1\leqslant\alpha\leqslant i+1}x_{\alpha,i+1}+\sum\limits_{1\leqslant\alpha\leqslant i}u_\alpha)\prod\limits_{\substack{1\leqslant i\leqslant h,\\ 1\leqslant\alpha\neq\beta\leqslant i}}\theta(x_{\alpha,i}-x_{\beta,i}-\tau)}{\prod\limits_{\substack{1\leqslant i<h,\\ 1\leqslant\alpha\leqslant i,1\leqslant\beta\leqslant i+1}}\theta(x_{\alpha,i}-x_{\beta,i+1})}\times \tag{11}$$

$$\varphi(x_{1,h},\ldots,x_{h,h};u_1,\ldots,u_h)$$

Here $\varphi(x_1,\ldots,x_h;u_1,\ldots,u_h)$ is holomorphic and symmetric in the variables $\{x_1,\ldots,x_h\}$ and satisfies relations:

$$\varphi(x_1+1,x_2,\ldots,x_h;u_1,\ldots,u_h)=\varphi(x_1,\ldots,x_h;u_1,\ldots,u_h)$$

$$\varphi(x_1+\eta,x_2,\ldots,x_h;u_1,\ldots,u_h)=$$
$$e^{-2\pi i((m-2h+2)x_1+x_2+\ldots+x_h+u_1+\ldots+u_h)}\varphi(x_1,\ldots,x_h;u_1,\ldots,u_h)$$

It is easy to check that the space of such functions is isomorphic to $\Lambda^h\Theta_{m-h+1,u_1+\ldots+u_h}(\Gamma)$.

Proposition 9. *For $\tau=0$ the subalgebra $\bigoplus\limits_{\alpha\geqslant1}Q_{\alpha(\delta_1+2\delta_2+\ldots+h\delta_h)}$ of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ is an algebra of functions on the manifold of h-dimensional subspaces in $m-h+1$-dimensional vector space. This means that the algebra is commutative and generated by $Q_{\delta_1+2\delta_2+\ldots+h\delta_h}\cong\Lambda^h\Theta_{m-h+1,u_1+\ldots+u_h}(\Gamma)$ and the Plucker relations. For generic τ the algebra $\bigoplus\limits_{\alpha\geqslant1}Q_{\alpha(\delta_1+2\delta_2+\ldots+h\delta_h)}$ is a flat deformation of the algebra of functions on the Grassmannian $Gr(h,m-h+1)$.*

Proof. For $\tau=0$ the proof is similar to the proof of proposition 8. From our definitions it follows that $\dim_{Q_0}Q_{\alpha(\delta_1+2\delta_2+\ldots+h\delta_h)}$ does not depend on τ. So this dimension is the same as for $\tau=0$.

4.4. Algebra $Q_{hm+1,h}(\mathcal{E},\tau)$.

Let $\Delta = A_h$, $(\delta_i,\delta_i) = 2$, $1 \leqslant i \leqslant h$; $(\delta_i,\delta_{i+1}) = -1$, $1 \leqslant i < h$; $(\delta_i,\delta_j) = 0$ if $|i-j| > 1$ as above. Let $n(\delta_1) = 1$, $n(\delta_i) = 0$ for $2 \leqslant i \leqslant h-1$, $n(\delta_h) = m > 0$. From the proposition 7 it follows that the algebra $Q_{n,\Delta}(\mathcal{E},0)$ is the algebra of polynomials in $h(m+1)$ variables: one variable of degree $\delta_1 + \ldots + \delta_i$ for each $1 \leqslant i < h$, m variables of degree $\delta_i + \ldots + \delta_h$ for each $1 < i \leqslant h$ and $m+1$ variables of degree $\delta_1 + \ldots + \delta_h$. It is easy to see from this that the subalgebra $\bigoplus\limits_{\alpha\geqslant 1} Q_{\alpha(\delta_1+\ldots+\delta_h)}$ of the algebra $Q_{n,\Delta}(\mathcal{E},0)$ is the algebra of polynomials in $hm+1$ variables of degree $\delta_1 + \ldots + \delta_h$. So the subalgebra $\bigoplus\limits_{\alpha\geqslant 1} Q_{\alpha(\delta_1+\ldots+\delta_h)}$ of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ is a flat deformation of the algebra of polynomials in $hm+1$ variables. The Hilbert function is $1 + \sum\limits_{\alpha\geqslant 1} \dim_{Q_0} Q_{\alpha(\delta_1+\ldots+\delta_h)} w^\alpha = (1-w)^{-(hm+1)}$.

Proposition 10. *The subalgebra $\bigoplus\limits_{\alpha\geqslant 1} Q_{\alpha(\delta_1+\ldots+\delta_h)}$ of the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ is isomorphic to $Q_{hm+1,h}(\mathcal{E},\frac{2}{hm+1}\tau)$.*

Proof. The proof is similar to the proof of proposition 8, using the construction of the algebras $Q_{n,k}(\mathcal{E},\tau)$ (see [2]).

4.5. Generalized elliptic R-matrix.

Let $\Delta = A_h$, $(\delta_i,\delta_i) = 2$, $1 \leqslant i \leqslant h$; $(\delta_i,\delta_{i+1}) = -1$, $1 \leqslant i < h$; $(\delta_i,\delta_j) = 0$ if $|i-j| > 1$ as above. We fix ν, $1 < \nu < h$. Let $n(\delta_\nu) = 1$, $n(\delta_i) = 0$ for $1 \leqslant i \leqslant h$, $i \neq \nu$. According to the proposition 7 the algebra $Q_{n,\Delta}(\mathcal{E},0)$ is the algebra of polynomials in $\nu(h-\nu+1)$ variables: one variable of degree $\delta_{i_1} + \delta_{i_1+1} + \ldots + \delta_{i_2}$ for each i_1, i_2 such that $1 \leqslant i_1 \leqslant \nu \leqslant i_2 \leqslant h$. The algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ is a flat deformation of this polynomial ring. It is clear that in the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ we have $\dim_{Q_0} Q_{\delta_{i_1}+\ldots+\delta_{i_2}} = 1$ for $1 \leqslant i_1 \leqslant \nu \leqslant i_2 \leqslant h$, $Q_{\delta_{i_1}+\ldots+\delta_{i_2}} = 0$ for $\nu < i_1$ or $i_2 < \nu$. Let e_{i_1,i_2} be a nonzero element from $Q_{\delta_{i_1}+\delta_{i_1+1}+\ldots+\delta_{i_2}}$ (for $1 \leqslant i_1 \leqslant \nu \leqslant i_2 \leqslant h$). It is easy to check that we can choose e_{i_1,i_2} in such a form:

$$e_{i_1,i_2} =$$

$$\frac{\prod\limits_{i_1\leqslant\varphi<\nu} \theta(x_{1,\varphi} - x_{1,\varphi+1} + \sum\limits_{i_1\leqslant\alpha\leqslant\varphi} u_\alpha - \tau) \prod\limits_{\nu\leqslant\psi<i_2} \theta(x_{1,\psi} - x_{1,\psi+1} - \sum\limits_{\psi<\alpha\leqslant i_2} u_\alpha + \tau)}{\prod\limits_{i_1\leqslant p<i_2} \theta(x_{1,p} - x_{1,p+1})} \times \tag{12}$$

$$\theta(x_{1,\nu} + \sum_{i_1\leqslant\alpha\leqslant i_2} u_\alpha - 2\tau)$$

Let us denote $v_i = u_i + u_{i+1} + \ldots + u_\nu$ for $1 \leqslant i \leqslant \nu$ and $w_i = u_\nu + u_{\nu+1} + \ldots + u_i$ for $\nu \leqslant i \leqslant h$. We denote $q(j,j') = e^{2\pi i\tau}$ for $j < j'$, $q(j,j') = e^{-2\pi i\tau}$ for $j > j'$, $q(j,j) = 1$.

Proposition 11. *The elements* $\{e_{i_1,i_2}, v_{i_1}, w_{i_2}; 1 \leqslant i_1 \leqslant \nu \leqslant i_2 \leqslant h\}$ *satisfy following commutation relations:*

$$e_{i_1,i_2}e_{i'_1,i'_2} = \frac{\theta(v_{i_1}-v_{i'_1}-2\tau)\theta(w_{i'_2}-w_{i_2})q(i'_1,i_1)q(i'_2,i_2)}{\theta(v_{i_1}-v_{i'_1})\theta(w_{i'_2}-w_{i_2}-2\tau)}e_{i'_1,i'_2}e_{i_1,i_2}+ \tag{13}$$

$$\frac{\theta(-2\tau)q(i'_2,i_2)\theta(w_{i'_2}-w_{i_2}-v_{i_1}+v_{i'_1})}{\theta(v_{i'_1}-v_{i_1})\theta(w_{i'_2}-w_{i_2}-2\tau)}e_{i_1,i'_2}e_{i'_1,i_2},\ \textit{for } i_1 \neq i'_1, i_2 \neq i'_2;$$

$$e_{i_1,i_2}e_{i'_1,i_2} = q(i'_1,i_1)e_{i'_1,i_2}e_{i_1,i_2};\, e_{i_1,i_2}e_{i_1,i'_2} = q(i'_2,i_2)e_{i_1,i'_2}e_{i_1,i_2};$$

$$e_{i_1,i_2}v_{i'_1} = v_{i'_1}e_{i_1,i_2}\ \textit{for } i_1 \neq i'_1;\, e_{i_1,i_2}w_{i'_2} = w_{i'_2}e_{i_1,i_2}\ \textit{for } i_2 \neq i'_2;$$

$$e_{i_1,i_2}v_{i_1} = (v_{i_1}-2\tau)e_{i_1,i_2};\, e_{i_1,i_2}w_{i_2} = (w_{i_2}-2\tau)e_{i_1,i_2};$$

$$[v_{i_1}, v_{i'_1}] = [w_{i_2}, w_{i'_2}] = [v_{i_1}, w_{i_2}] = 0$$

Proof. It is clear that

$$e_{i_1,i_2}e_{i'_1,i'_2},\ e_{i'_1,i'_2}e_{i_1,i_2},\ e_{i_1,i'_2}e_{i'_1,i_2} \in Q_{\delta_{i_1}+\delta_{i_1+1}+\ldots+\delta_{i_2}+\delta_{i'_1}+\delta_{i'_1+1}+\ldots+\delta_{i'_2}}$$

but $\dim_{Q_0} Q_{\delta_{i_1}+\ldots+\delta_{i_2}+\delta_{i'_1}+\ldots+\delta_{i'_2}} = 2$, so there is a linear relation between these elements. It is easy to find the coefficients of this relation using (2). Another relations in (13) are easy to check directly.

Remark. If we put $n(\delta_\nu) = m > 0$ instead of $n(\delta_\nu) = 1$, then we will have an algebra that is a deformation of the polynomial ring in $m\nu(h-\nu+1)$ variables: m variables of degree $\delta_{i_1}+\delta_{i_1+1}+\ldots+\delta_{i_2}$ for each $1 \leqslant i_1 \leqslant \nu \leqslant i_2 \leqslant h$. It is clear that $\dim_{Q_0} Q_{\delta_{i_1}+\ldots+\delta_{i_2}} = m$ for $1 \leqslant i_1 \leqslant \nu \leqslant i_2 \leqslant h$, $Q_{\delta_{i_1}+\ldots+\delta_{i_2}} = 0$ for $i_1 > \nu$ or $i_2 < \nu$ and $\dim_{Q_0} Q_{\delta_{i_1}+\ldots+\delta_{i_2}+\delta_{i'_1}+\ldots+\delta_{i'_2}} = 2m^2$ for $1 \leqslant i_1, i'_1 \leqslant \nu \leqslant i_2, i'_2 \leqslant h$. So the elements from the product $Q_{\delta_{i_1}+\ldots+\delta_{i_2}} * Q_{\delta_{i'_1}+\ldots+\delta_{i'_2}}$ are uniquely expressed as linear combination of the elements from the space $Q_{\delta_{i'_1}+\ldots+\delta_{i'_2}} * Q_{\delta_{i_1}+\ldots+\delta_{i_2}} \oplus Q_{\delta_{i_1}+\ldots+\delta_{i'_2}} * Q_{\delta_{i'_1}+\ldots+\delta_{i_2}}$. So we have a map

$$Q_{\delta_{i_1}+\ldots+\delta_{i_2}} \otimes Q_{\delta_{i'_1}+\ldots+\delta_{i'_2}} \to Q_{\delta_{i'_1}+\ldots+\delta_{i'_2}} \otimes Q_{\delta_{i_1}+\ldots+\delta_{i_2}} \oplus Q_{\delta_{i_1}+\ldots+\delta_{i'_2}} \otimes Q_{\delta_{i'_1}+\ldots+\delta_{i_2}}$$

It is clear that this map satisfies the Yang-Baxter equations.

§5. Case of affine root system. A family of commuting elements.

Let Δ be the root system of the Lie algebra $\widehat{sl_h}$. So $(\delta_i,\delta_i)=2$, $(\delta_i,\delta_{i+1})=-1$ and $(\delta_i,\delta_j)=0$ otherwise. Here we assume that $i\in\mathbb{Z}_h$. It is possible to prove that for dominant n the Hilbert function of $Q_{n,\Delta}(\mathcal{E},\tau)$ is equal to

$$\prod_{\gamma\in\Delta^+}(1-w^\gamma)^{-n(\gamma)k(\gamma)}\prod_{\alpha\geqslant\mu}(1-w^{\alpha(\delta_1+\ldots+\delta_h)})^{-\alpha n(\delta_1+\ldots+\delta_h)}$$

Here $k(\gamma)$ is a multiplicity of the root γ in the algebra $\widehat{sl_h}$, $\mu=2$ for $h=2$ and $\mu=1$ for $h>2$. So the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ is bigger then in the finite dimensional case. Note that the element $u_1+\ldots+u_h$ is central in the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$, because the root $\delta_1+\ldots+\delta_h$ is imaginary. We denote by $Q_{n,\Delta,c}(\mathcal{E},\tau)$ (for $c\in\mathbb{C}$) the factor algebra of $Q_{n,\Delta}(\mathcal{E},\tau)$ by the relation $u_1+\ldots+u_h=c$. If n is dominant and $n\neq 0$ then the Hilbert function of the algebra $Q_{n,\Delta,c}(\mathcal{E},\tau)$ is the same as for the algebra $Q_{n,\Delta}(\mathcal{E},\tau)$ and does not depend on c. Let us $n=0$. The Hilbert function of the algebra $Q_{0,\Delta,c}(\mathcal{E},\tau)$ is equal to 1 if $c\notin\Gamma$ and is equal to $\prod\limits_{\alpha\geqslant1}(1-w^{\alpha(\delta_1+\ldots+\delta_h)})^{-h}$ for $h>2$ and $(1-w_1w_2)^{-1}\prod\limits_{\alpha\geqslant2}(1-w_1^\alpha w_2^\alpha)^{-2}$ for $h=2$ if $c\in\Gamma$.

Proposition 12. *Let $c\in\Gamma$. The algebra $Q_{0,\Delta,c}(\mathcal{E},\tau)$ is commutative for each $\tau\in\mathcal{E}$ and isomorphic to polynomials in infinite number of variables. Namely, for $h>2$ it is the algebra of polynomials in h variables of degree $\alpha(\delta_1+\ldots+\delta_h)$ for each $\alpha\geqslant1$ and for $h=2$ it is the algebra of polynomials in one variable of degree $\delta_1+\delta_2$ and two variables of degree $\alpha(\delta_1+\delta_2)$ for each $\alpha\geqslant2$.*

Proof. Let $c=0$. Firstly we consider the case $h>2$. Let $g(z_1,\ldots,z_h)$ be the holomorphic function in variables $\{z_j;j\in\mathbb{Z}_h\}$ that satisfies the relations:

$g(z_1,\ldots,z_j+1,\ldots,z_h)=g(z_1,\ldots,z_h)$,

$g(z_1,\ldots,z_j+\eta,\ldots,z_h)=e^{-2\pi i(2z_j-z_{j-1}-z_{j+1}-\eta+u_j)}g(z_1,\ldots,z_h)$ for each $j\in\mathbb{Z}_h$.

It is easy to see that the space of such functions is h-dimensional (over field of functions in variables $\{u_j\}$) and $g(z_1+p,\ldots,z_h+p)=g(z_1,\ldots,z_h)$ for each $p\in\mathbb{C}$. Let $K_{\alpha,g}$ be the following element of the space $Q_{\alpha(\delta_1+\ldots+\delta_h)}$:

$$K_{\alpha,g}=\frac{\prod\limits_{\substack{i\in\mathbb{Z}_h,\\1\leqslant\mu,\nu\leqslant\alpha}}\theta(x_{\mu,i}-x_{\nu,i}-2\tau)\cdot g(x_{1,1}+\ldots+x_{\alpha,1},\ldots,x_{1,h}+\ldots+x_{\alpha,h})}{\prod\limits_{\substack{i\in\mathbb{Z}_h,\\1\leqslant\mu,\nu\leqslant\alpha}}\theta(x_{\mu,i}-x_{\nu,i+1})}\tag{14}$$

One can check by direct calculation that $K_{\alpha_1,g_1}*K_{\alpha_2,g_2}=K_{\alpha_2,g_2}*K_{\alpha_1,g_1}$ for each α_1,α_2; g_1,g_2.

For $h = 2$ the elements $K_{\alpha,g}$ defined by (14) commute also, but in this case $K_{\alpha,g} \notin Q_{\alpha(\delta_1+\delta_2)}$ because $K_{\alpha,g}$ has a pole of order 2. In this case the algebra $Q_{0,\Delta,0}(\mathcal{E},\tau)$ is a subalgebra of the algebra generated by $\{K_{\alpha,g}\}$.

§6. Elliptic deformation of the Poison algebra. Another family of commuting elements.

Let $\tau_1, \tau_2, \tau_3 \in \mathcal{E}$, $\tau_1+\tau_2+\tau_3 = 0$ and $m \in \mathbb{N}$. We define an associative $\mathbb{N}$-graded algebra $H_m(\mathcal{E}, \tau_1, \tau_2, \tau_3)$ by the following: as a linear space $H_m(\mathcal{E}, \tau_1, \tau_2, \tau_3) = F_1 \oplus F_2 \oplus F_3 \oplus \ldots$, where $F_\alpha * F_\beta \subset F_{\alpha+\beta}$ and F_α is a space of functions $f(x_1, \ldots, x_\alpha)$ on $\mathbb{C}^\alpha$ that satisfies the properties:

1. $f(x_1, \ldots, x_\alpha)$ is symmetric by $x_1, \ldots, x_\alpha$.

2. $f(x_1, \ldots, x_\alpha)$ is holomorphic outside the divisors $\{x_\mu - x_\nu = 0\}$ and has a pole of order $\leqslant 2$ on these divisors.

3. For $\alpha \geqslant 3$ we have:

$$f(x, x+\tau_1, x+\tau_1+\tau_2, x_4, \ldots, x_\alpha) = 0$$

$$f(x, x+\tau_2, x+\tau_2+\tau_1, x_4, \ldots, x_\alpha) = 0$$

4. $f(x_1+1, x_2, \ldots, x_\alpha) = f(x_1, \ldots, x_\alpha)$,
$f(x_1+\eta, x_2, \ldots, x_\alpha) = e^{-2\pi i(mx_1+c)} f(x_1, \ldots, x_\alpha)$, here $c \in \mathbb{C}$ is fixed.

We define the product $*$ in the algebra $H_m(\mathcal{E}, \tau_1, \tau_2, \tau_3)$ by the following rule: for $f \in F_\alpha$, $g \in F_\beta$ we set:

$$f * g(x_1, \ldots, x_{\alpha+\beta}) =$$

$$\frac{1}{\alpha!\beta!} \sum_{\sigma \in S_{\alpha+\beta}} f(x_{\sigma_1}, \ldots, x_{\sigma_\alpha}) g(x_{\sigma_{\alpha+1}}, \ldots, x_{\sigma_{\alpha+\beta}}) \prod_{\substack{1 \leqslant \mu \leqslant \alpha, \\ \alpha+1 \leqslant \nu \leqslant \alpha+\beta}} \lambda(x_{\sigma_\mu}, x_{\sigma_\nu}),$$

here $\lambda(x, y) = \frac{\theta(x-y-\tau_1)\theta(x-y-\tau_2)\theta(x-y-\tau_3)}{\theta(x-y)^3}$.

It is clear that for $m > 0$ the algebra $H_m(\mathcal{E}, \tau_1, \tau_2, \tau_3)$ does not depend on c. It is possible to proof that for $m > 0$ the algebra $H_m(\mathcal{E}, \tau_1, \tau_2, \tau_3)$ is a deformation in class of $\mathbb{N}$-graded associative algebras of the universal enveloping algebra of the following $\mathbb{N}$-graded Lie algebra $h_{m,c}(\mathcal{E})$: as a vector space, $h_{m,c}(\mathcal{E}) = \Theta_{m,c}(\Gamma) \oplus \Theta_{2m,2c}(\Gamma) \oplus \ldots$ and for $\varphi \in \Theta_{\alpha m, \alpha c}(\Gamma)$, $\psi \in \Theta_{\beta m, \beta c}(\Gamma)$, the commutator $[\varphi, \psi] = \beta\varphi'\psi - \alpha\psi'\varphi \in \Theta_{(\alpha+\beta)m, (\alpha+\beta)c}(\Gamma)$. The Hilbert function of the algebra $H_m(\mathcal{E}, \tau_1, \tau_2, \tau_3)$ is $1 + \sum\limits_{\alpha \geqslant 1} \dim F_\alpha w^\alpha = \prod\limits_{\alpha \geqslant 1} (1-w^\alpha)^{-m\alpha}$ (see also [5]).

Now let us apply this construction to the case $m = 0$. For $c \notin \Gamma$ we have $H_0(\mathcal{E}, \tau_1, \tau_2, \tau_3) = 0$. For $c \in \Gamma$ the Hilbert function of the algebra $H_0(\mathcal{E}, \tau_1, \tau_2, \tau_3)$ is equal to $\prod\limits_{\alpha \geqslant 1} (1-w^\alpha)^{-1}$. In this case the algebra $H_0(\mathcal{E}, \tau_1, \tau_2, \tau_3)$ is commutative for each $\tau_1, \tau_2, \tau_3 \in \mathbb{C}$, $\tau_1+\tau_2+\tau_3 = 0$ and isomorphic to the polynomial algebra in infinite number of variables: one variable of degree α for each $\alpha \geqslant 1$. Let us $c = 0$. We define the elements $K_\alpha \in F_\alpha$ by the formula:

$$K_\alpha(x_1, \ldots, x_\alpha) = \prod_{1 \leqslant \mu < \nu \leqslant \alpha} \frac{\theta(x_\mu - x_\nu - \tau_1)\theta(x_\mu - x_\nu + \tau_1)}{\theta(x_\mu - x_\nu)^2} \text{ for } \alpha > 1,$$

$$K_1(x_1) = 1$$

It is easy to check by direct calculation that $K_\alpha * K_\beta = K_\beta * K_\alpha$ for each α, $\beta \in \mathbb{N}$ (see for details [5]).

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