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Ugo Bruzzo
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Alexander S. Tikhomirov

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Ugo Bruzzo
Dimitri Markushevich
Alexander S. Tikhomirov

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Scuola Internazionale Superiore
di Studi Avanzati (SISSA)
Via Bonomea 265
34136 Trieste
Italia

Istituto Nazionale di Fisica Nucleare
Sezione di Trieste

Mathématiques - bât. M2
Université Lille 1
59655 Villeneuve d'Ascq Cedex
France

Department of Mathematics
Yaroslavl State Pedagogical University
Respublikanskaya Str. 108
150 000 Yaroslavl
Russia

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MODULI OF SYMPLECTIC INSTANTON VECTOR BUNDLES OF HIGHER RANK ON PROJECTIVE SPACE $\mathbb{P}^3$. II

U. BRUZZO¶, D. MARKUSHEVICH§ AND A. S. TIKHOMIROV‡

¶ *Scuola Internazionale Superiore di Studi Avanzati (SISSA), Via Bonomea 265, 34136 Trieste, Italia and Istituto Nazionale di Fisica Nucleare, Sezione di Trieste*

§ *Mathématiques - bât. M2, Université Lille 1, F-59655 Villeneuve d'Ascq Cedex, France*

‡ *Department of Mathematics, Yaroslavl State Pedagogical University, Respublikanskaya Str. 108 150 000 Yaroslavl, Russia*

Abstract. Symplectic instanton vector bundles on the projective space $\mathbb{P}^3$ are a natural generalization of mathematical instantons of rank 2. We study the moduli space $I_{n,r}$ of rank-$2r$ symplectic instanton vector bundles on $\mathbb{P}^3$ with $r \geq 2$ and second Chern class $n \geq r+1,\ n-r \equiv 1(\mathrm{mod}2)$. We introduce the notion of tame symplectic instantons by excluding a kind of pathological monads and show that the locus $I^*_{n,r}$ of tame symplectic instantons is irreducible and has the expected dimension, equal to $4n(r+1)-r(2r+1)$.

1. Introduction

By a *symplectic instanton vector bundle* of rank $2r$ and charge n (shortly, a *symplectic (n,r)-instanton*) on the 3-dimensional projective space $\mathbb{P}^3$ we understand an algebraic vector bundle $E = E_{2r}$ of rank $2r$ on $\mathbb{P}^3$ with Chern classes

$$c_1(E) = c_3(E) = 0, \tag{1}$$

$$c_2(E) = n, \quad n \geq 1, \tag{2}$$

supplied with a symplectic structure and satisfying the vanishing conditions

$$h^0(E) = h^1(E \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = 0. \tag{3}$$

E-mail address: bruzzo@sissa.it, markushe@math.univ-lille1.fr, astikhomirov@mail.ru.

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By a symplectic structure we mean an anti-self-dual isomorphism

$$\phi: \ E \xrightarrow{\simeq} E^\vee, \quad \phi^\vee = -\phi, \tag{4}$$

considered modulo proportionality. The vanishing of the odd Chern classes (1) follows from the existence of a symplectic structure (4), and if $r = 1$, then the two conditions are equivalent. We will denote the moduli space of symplectic (n, r)-instantons by $I_{n,r}$.

Rank r symplectic instantons on $\mathbb{P}^3$ relate in a natural manner with "physical" $\mathbf{Sp}(r)$ instantons on the four-sphere S^4, i.e., connections on principal $\mathbf{Sp}(r)$-bundles on S^4 with self-dual curvature [1]; the moduli spaces of the former are in a sense a complexification of the moduli spaces of the latter. The relation is expressed by the so-called Atiyah-Ward correspondence [3, 1], which relies on the fact that the projective space $\mathbb{P}^3$ is the twistor space of the four-sphere S^4. The present paper, with its companion [7], are the first to study the geometry of the moduli spaces $I_{n,r}$. While [7] studied the case $n \equiv r$ (mod 2), with $n \geq r$, the present paper deals with the other case, $n \equiv r + 1$ (mod 2), with $n \geq r + 1$. We exploit as usual the monad method [8, 2, 4, 5, 11, 12], which allows one to study instantons by means of hyperwebs of quadrics. The main result of this paper is that a component $I^*_{n,r}$ of $I_{n,r}$ that is singled out by a certain open condition (which rules out some "badly behaved" monads) is irreducible.

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2. Notation and conventions

In many respects, we follow the exposition of [9], and stick to the notation there introduced. The base field $\mathbf{k}$ is assumed to be algebraically closed of characteristic 0. We identify vector bundles with locally free sheaves. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an algebraic variety or a scheme X, by $n\mathcal{F}$ we denote the direct sum of n copies of $\mathcal{F}$, while $H^i(\mathcal{F})$ denotes the i^{th} cohomology group of $\mathcal{F}$ and $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$, and $\mathcal{F}^\vee$ denotes the dual of $\mathcal{F}$, that is, $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If $X = \mathbb{P}^r$ and t is an integer, by $\mathcal{F}(t)$ we denote the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(t)$. $[\mathcal{F}]$ will denote the isomorphism class of a sheaf $\mathcal{F}$. For any morphism of $\mathcal{O}_X$-sheaves $f : \mathcal{F} \to \mathcal{F}'$ and any $\mathbf{k}$-vector space U (respectively, for any homomorphism $f : U \to U'$ of $\mathbf{k}$-vector spaces) we denote, for short, by the same letter

f the induced morphism of sheaves $id \otimes f : U \otimes \mathcal{F} \to U \otimes \mathcal{F}'$ (respectively, the induced morphism $f \otimes id : U \otimes \mathcal{F} \to U' \otimes \mathcal{F}$).

We fix an integer $n \geq 1$ and denote by H_n a fixed n-dimensional vector space over $\mathbf{k}$. Throughout this paper, V will be a fixed vector space of dimension 4 over $\mathbf{k}$, and we set $\mathbb{P}^3 := P(V)$. We reserve the letters u and v to denote the two morphisms in the Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} T_{\mathbb{P}^3}(-1) \to 0$. For any $\mathbf{k}$-vector spaces U and W and any vector $\phi \in \mathrm{Hom}(U, W \otimes \wedge^2 V^\vee) \subset \mathrm{Hom}(U \otimes V, W \otimes V^\vee)$ understood as a linear map $\phi : U \otimes V \to W \otimes V^\vee$ or, equivalently, as a map $^\sharp\phi : U \to W \otimes \wedge^2 V^\vee$, we will denote by $\widetilde{\phi}$ the composition $U \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{^\sharp\phi} W \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} W \otimes \Omega_{\mathbb{P}^3}(2)$, where ϵ is the induced morphism in the exact triple $0 \to \wedge^2 \Omega_{\mathbb{P}^3}(2) \xrightarrow{\wedge^2 v^\vee} \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} \Omega_{\mathbb{P}^3}(2) \to 0$ obtained by taking the second wedge power of the dual Euler exact sequence.

Given an integer $n \geq 1$, we denote by $\mathbf{S}_n$ (resp. $\mathbf{\Sigma}_n$) the vector space $S^2 H_n^\vee \otimes \wedge^2 V^\vee$ (resp. $\mathrm{Hom}(H_n, H_n^\vee \otimes \wedge^2 V^\vee)$). By abuse of notation, denote by the same symbol a $\mathbf{k}$-vector space, say U, and the associated affine space $\mathbf{V}(U^\vee) = \mathrm{Spec}(\mathrm{Sym}^* U^\vee)$.

All the schemes considered in this paper are Noetherian. By a general point of an irreducible (but not necessarily reduced) scheme $\mathcal{X}$ we mean any closed point of some dense open subset of $\mathcal{X}$. An irreducible scheme is called generically reduced if it is reduced at any general point.

3. Generalities on symplectic instantons and definition of $MI_{n,r}$

In this section we enumerate some facts about symplectic instantons which are completely parallel to those for rank-2 usual instantons, see [9, Section 3].

For a given symplectic (n, r)-instanton E, the first condition (3) yields $h^0(E(-i)) = 0, i \geq 0$, which, together with the exact sequence $0 \to E(-j-1) \to E(-j) \to E(-j)|_{\mathbb{P}^2} \to 0$ for $j = 0$ and (3), implies that $h^0(E(-1)|_{\mathbb{P}^2}) = 0$, hence also $h^0(E(-i)|_{\mathbb{P}^2}) = 0,\ i \geq 1$. The last equality for $i = 2$, together with (3) and the above sequence for $j = 2$, gives $h^1(E(-3)) = 0$, hence also $h^1(E(-4)) = 0$. Then, from Serre duality and (4), we deduce

(5) $$h^i(E) = h^i(E(-1)) = h^{3-i}(E(-3)) = h^{3-i}(E(-4)) = 0, \quad i \neq 1,$$
$$h^i(E(-2)) = 0, \quad i \geq 0.$$

By Riemann-Roch and (3), (5), we have

(6) $$h^1(E(-1)) = h^2(E(-3)) = n, \quad h^1(E) = h^2(E(-4)) = 2n - 2r.$$

By tensoring the dual Euler sequence by E we also obtain

(7) $$h^1(E \otimes \Omega^1_{\mathbb{P}^3}) = h^2(E \otimes \Omega^2_{\mathbb{P}^3}) = 2n + 2r.$$

Consider a triple (E, f, ϕ) where E is an (n, r)-instanton, $f : H_n \xrightarrow{\simeq} H^2(E(-3))$ an isomorphism and $\phi : E \xrightarrow{\simeq} E^\vee$ a symplectic structure on E. Two triples (E, f, ϕ) and $(E'f', \phi')$ are considered to be equivalent if there is an isomorphism $g : E \xrightarrow{\simeq} E'$ such that $g_* \circ f = \lambda f'$ with $\lambda \in \{1, -1\}$ and $\phi = g^\vee \circ \phi' \circ g$, where $g_* : H^2(E(-3)) \xrightarrow{\simeq} H^2(E'(-3))$ is the induced isomorphism. We denote by $[E, f, \phi]$ the equivalence class of a triple (E, f, ϕ). It follows from this definition that the set $F_{[E]}$ of all equivalence classes $[E, f, \phi]$ with given $[E]$ is a homogeneous space of the group $GL(H_n)/\{\pm\mathrm{id}\}$.

Each class $[E, f, \phi]$ defines a point

$$A = A([E, f, \phi]) \in S^2 H_n^\vee \otimes \wedge^2 V^\vee \tag{8}$$

in the following way. Consider the exact sequences

$$\begin{gathered} 0 \to \Omega^1_{\mathbb{P}^3} \xrightarrow{i_1} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to 0, \\ 0 \to \Omega^2_{\mathbb{P}^3} \to \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \to \Omega^1_{\mathbb{P}^3} \to 0, \\ 0 \to \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4) \to \wedge^3 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{i_2} \Omega^2_{\mathbb{P}^3} \to 0, \end{gathered} \tag{9}$$

induced by the Koszul complex of $V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{ev}{\twoheadrightarrow} \mathcal{O}_{\mathbb{P}^3}$. Twisting these sequences by E and taking (3) and (5) into account, we obtain the vanishing

$$h^0(E \otimes \Omega_{\mathbb{P}^3}) = h^3(E \otimes \Omega^2_{\mathbb{P}^3}) = h^2(E \otimes \Omega_{\mathbb{P}^3}) = 0 \tag{10}$$

and the diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^2(E(-4)) \otimes \wedge^4 V^\vee & \longrightarrow & H^2(E(-3)) \otimes \wedge^3 V^\vee & \xrightarrow{i_2} & H^2(E \otimes \Omega^2_{\mathbb{P}^3}) & \longrightarrow 0 \\ & & & \downarrow A' & & \cong \uparrow \partial & \\ 0 \longleftarrow & H^1(E)) & \longleftarrow & H^1(E(-1)) \otimes V^\vee & \xleftarrow{i_1} & H^1(E \otimes \Omega_{\mathbb{P}^3}) & \longleftarrow 0, \end{array} \tag{11}$$

where $A' := i_1 \circ \partial^{-1} \circ i_2$. The Euler exact sequence (9) yields the canonical isomorphism $\omega_{\mathbb{P}^3} \xrightarrow{\simeq} \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4)$, and fixing an isomorphism $\tau : \mathbf{k} \xrightarrow{\simeq} \wedge^4 V^\vee$ we have the isomorphisms $\tilde{\tau} : V \xrightarrow{\simeq} \wedge^3 V^\vee$ and $\hat{\tau} : \omega_{\mathbb{P}^3} \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^3}(-4)$. We define A in (8) as the composition

$$\begin{gathered} A : H_n \otimes V \overset{\tilde{\tau}}{\xrightarrow{\simeq}} H_n \otimes \wedge^3 V^\vee \overset{f}{\xrightarrow{\simeq}} H^2(E(-3)) \otimes \wedge^3 V^\vee \xrightarrow{A'} H^1(E(-1)) \otimes V^\vee \overset{\phi}{\xrightarrow{\simeq}} \\ \overset{\phi}{\xrightarrow{\simeq}} H^1(E^\vee(-1)) \otimes V^\vee \overset{SD}{\xrightarrow{\simeq}} H^2(E(1) \otimes \omega_{\mathbb{P}^3})^\vee \otimes V^\vee \overset{\hat{\tau}}{\xrightarrow{\simeq}} H^2(E(-3))^\vee \otimes V^\vee \overset{f^\vee}{\xrightarrow{\simeq}} H_n^\vee \otimes V^\vee, \end{gathered} \tag{12}$$

where SD is the Serre duality isomorphism. One can verify that A is a skew-symmetric map which depends only on the class $[E, f, \phi]$, but does not depend on the choice of τ, and that $A \in \wedge^2(H_n^\vee \otimes V^\vee)$ lies in the direct summand $\mathbf{S}_n = S^2 H_n^\vee \otimes \wedge^2 V^\vee$ of the canonical decomposition

$$\wedge^2(H_n^\vee \otimes V^\vee) = S^2 H_n^\vee \otimes \wedge^2 V^\vee \oplus \wedge^2 H_n^\vee \otimes S^2 V^\vee. \tag{13}$$

Here $\mathbf{S}_n$ is the space of hyperwebs of quadrics in H_n. For this reason we call A the (n,r)-*instanton hyperweb of quadrics* corresponding to the data $[E,f,\phi]$.

Denote $W_A := H_n \otimes V/\ker A$. Using the above chain of isomorphisms we can rewrite the diagram (11) as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker A & \longrightarrow & H_n \otimes V & \xrightarrow{c_A} & W_A & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle A} & & \cong \downarrow{\scriptstyle q_A} & & \\ 0 & \longleftarrow & \ker A^\vee & \longleftarrow & H_n^\vee \otimes V^\vee & \xleftarrow{c_A^\vee} & W_A^\vee & \longleftarrow & 0. \end{array} \tag{14}$$

In view of (7), $\dim W_A = 2n+2r$ and $q_A : W_A \overset{\simeq}{\to} W_A^\vee$ is a skew-symmetric isomorphism. An important property of $A = A([E,f,\phi])$ is that the induced morphism of sheaves

$$a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{c_A^\vee}{\to} H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{ev}{\to} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \tag{15}$$

is surjective and the composition $H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_A}{\to} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \overset{q_A}{\to} W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{a_A^\vee}{\to} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ is zero. Applying Beilinson spectral sequence [6] to $E(-1)$, we see that $E \simeq \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A$. Thus A defines a monad

$$\mathcal{M}_A : \quad 0 \to H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_A}{\to} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \overset{a_A^\vee \circ q_A}{\to} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \ , \tag{16}$$

whose cohomology sheaf

$$E_{2r}(A) := \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A. \tag{17}$$

is isomorphic to E. Twisting $\mathcal{M}_A$ by $\mathcal{O}_{\mathbb{P}^3}(-3)$ and using (17), we obtain the isomorphism $f : H_n \overset{\simeq}{\to} H^2(E(-3))$. Furthermore, the fact that q_A is symplectic implies that there is a canonical isomorphism of $\mathcal{M}_A$ with its dual which induces the symplectic isomorphism $\phi : E \overset{\simeq}{\to} E^\vee$. Thus, the data $[E,f,\phi]$ can be recovered from A. This leads to the following description of the moduli space $I_{n,r}$. Consider the *set of (n,r)-instanton hyperwebs of quadrics*

$$MI_{n,r} := \left\{ A \in \mathbf{S}_n \;\middle|\; \begin{array}{rl} \text{(i)} & rk(A : H_n \otimes V \to H_n^\vee \otimes V^\vee) = 2n+2r, \\ \text{(ii)} & \text{the morphism } a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \to H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \text{ defined} \\ & \text{by } A \text{ in (15) is surjective,} \\ \text{(iii)} & h^0(E_{2r}(A)) = 0, \text{ where } E_{2r}(A) = \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A \\ & \text{and } q_A : W_A \overset{\simeq}{\to} W_A^\vee \text{ is a symplectic isomorphism} \\ & \text{associated to } A \text{ by (14).} \end{array} \right\} \tag{18}$$

It is a locally closed subscheme of the affine space $\mathbf{S}_n$.

Theorem 3.1. *The natural morphism*

$$\pi_{n,r} : MI_{n,r} \to I_{n,r}, \ \ A \mapsto [E_{2r}(A)], \tag{19}$$

is a principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle in the étale topology. Hence $I_{n,r}$ is a quotient stack $MI_{n,r}/(GL(H_n)/\{\pm\mathrm{id}\})$, and is therefore an algebraic space.

Proof. See [9, Section 3]. □

Each fibre $F_{[E]} = \pi_n^{-1}([E])$ over an arbitrary point $[E] \in I_{n,r}$ is a principal homogeneous space of the group $GL(H_n)/\{\pm\mathrm{id}\}$. Hence the irreducibility of $(I_{n,r})_{red}$ is equivalent to the irreducibility of the scheme $(MI_{n,r})_{red}$.

We can also state:

Theorem 3.2. *For each $n \geq 1$, the space $MI_{n,r}$ of (n,r)-instanton nets of quadrics is a locally closed subscheme of the vector space $\mathbf{S}_n$ given locally at any point $A \in MI_{n,r}$ by*

$$\binom{2n-2r}{2} = 2n^2 - n(4r+1) + r(2r+1) \tag{20}$$

equations obtained as the rank condition (i) in (18).

Note that from (20) it follows that

$$\dim_A MI_{n,r} \geq \dim \mathbf{S}_n - (2n^2 - n(4r+1) + r(2r+1)) = n^2 + 4n(r+1) - r(2r+1) \tag{21}$$

at any point $A \in MI_{n,r}$. Hence,

$$\dim_{[E]} I_{n,r} \geq 4n(r+1) - r(2r+1) \tag{22}$$

at any point $[E] \in I_{n,r}$, since $MI_{n,r} \to I_{n,r}$ is a principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle in the étale topology.

4. Explicit construction of symplectic instantons

4.1. Example: symplectic $(n+1,n)$-instantons. We give a construction of symplectic $(n+1,n)$-instantons and describe their relation to usual rank-2 instantons with second Chern class $c_2 = 2n$. This relation is given at the level of spaces of hyperwebs of quadrics $MI_{n+1,n}$ and $MI_{2n,1}$, interpreted as spaces of monads.

Denote by $\mathrm{Isom}_{n+1,n-1}$ the set of all isomorphisms

$$\zeta : H_{n+1} \oplus H_{n-1} \xrightarrow{\simeq} H_{2n}. \tag{23}$$

This clearly coincides with the principal homogeneous space of the group $GL(2n)$. Besides, for any $\zeta \in \mathrm{Isom}_{n+1,n-1}$ let $p_\zeta : \mathbf{S}_{2n} \twoheadrightarrow \mathbf{S}_{n+1}$ be the induced epimorphism, and, for any monomorphism $i : H_n \hookrightarrow H_{n+1}$ let $pr_{(i)} : \mathbf{S}_{n+1} \to \mathbf{S}_n$ be the induced epimorphism.

Note that $MI_{2n,1}$ is irreducible [10, Theorem 1.1], and one has the following result [10, Theorem 3.1].

Theorem 4.1. *There exists a dense open subset $MI^*_{2n,1}$ of $MI_{2n,1}$ such that, for any hyperweb $A \in MI^*_{2n,1}$ and a general $\zeta \in \mathrm{Isom}_{n+1,n-1}$ the rank of the homomorphism $B = p_\zeta(A) : H_{n+1} \otimes V \to H^\vee_{n+1} \otimes V^\vee$ coincides with the rank of $A : H_{2n} \otimes V \to H^\vee_{2n} \otimes V^\vee$:*

$$\mathrm{rk}B = \mathrm{rk}A = 4n + 2. \tag{24}$$

Set $W_{4n+2} := H_{2n} \otimes V/\ker A$ and let $c_A : H_{2n} \otimes V \twoheadrightarrow W_{4n+2}$ be the canonical epimorphism and $q_A : W_{4n+2} \xrightarrow{\sim} W^\vee_{4n+2}$ be the induced skew-symmetric isomorphism so that $A = c^\vee_A \circ q_A \circ c_A$. Now a morphism of sheaves

$$a_A : H_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} H_{2n} \otimes V \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{c_A} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \tag{25}$$

and its transpose

$${}^t a_A = a^\vee_A \circ q_A : W^\vee_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \to H^\vee_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$$

yield a monad

$$\mathcal{M}_A : \quad 0 \to H_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^t a_A} H^\vee_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{26}$$

with the cohomology sheaf $E(A)$, $[E(A)] \in I_{2n,1}$ (see (16) and (17)).

Let

$$i_\zeta : H_{n+1} \hookrightarrow H_{2n} \tag{27}$$

be the monomorphism defined by the isomorphism (23). The composition $a_B : H_{n+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{i_\zeta} H_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3}$ and its transpose ${}^t a_B = a^\vee_B \circ q_A$ yield a monad

$$\mathcal{M}_B : \quad 0 \to H_{n+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_B} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^t a_B} H^\vee_{n+1} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{28}$$

with the cohomology sheaf

$$E_{2n}(B) := \ker {}^t a_B / \mathrm{im} a_B, \quad c_2(E_{2n}(B)) = n + 1. \tag{29}$$

The symplectic isomorphism $q_A : W_{4n+2} \xrightarrow{\sim} W^\vee_{4n+2}$ induces a symplectic structure on $E_{2n}(B)$,

$$E_{2n}(B) \underset{\simeq}{\xrightarrow{\phi_B}} E_{2n}(B)^\vee. \tag{30}$$

Moreover, (24) implies an isomorphism $H_{n+1} \otimes V/\ker B \simeq W_{4n+2}$, hence a monomorphism of spases of sections $h^0({}^t a_B) : W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^t a_B} H^\vee_{n+1} V^\vee$ in (28). Hence (28) and (29) imply $h^0(E_{2n}(B)) = 0$. This together with (30) means that $E_{2n}(B)$ is a symplectic instanton:

$$[E_{2n}(B)] \in I_{n+1,n}. \tag{31}$$

Note that by construction the monads (26) and (28) fit in the commutative diagram
(32)

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_{n+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_B} & W_{4n+2}\otimes\mathcal{O}_{\mathbb{P}^3} & \underset{\simeq}{\xrightarrow{q_A}} & W_{4n+2}^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_B^\vee} & H_n^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle i_\zeta} & & \cong\Big\| & & {\scriptstyle w^\vee}\Big\|\cong & & {\scriptstyle i_\zeta^\vee}\Big\uparrow & & \\
0 & \longrightarrow & H_{2n}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_A} & W_{4n+2}\otimes\mathcal{O}_{\mathbb{P}^3} & \underset{\simeq}{\xrightarrow{q_A}} & W_{4n+2}^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_A^\vee} & H_{2n}^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0,
\end{array}$$

In view of (29) and (30) and the canonical isomorphism $H_{2n}/i_\zeta(H_{n+1}) \simeq H_{n-1}$, from this diagram we obtain the quotient monad

$$(33)\qquad \mathcal{M}_{A,B}:\quad 0\to H_{n-1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_{A,B}}{\to} E_{2n}(B) \underset{\simeq}{\overset{\phi_B}{\to}} E_{2n}(B)^\vee \overset{a_{A,B}^\vee}{\to} H_{n-1}^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1)\to 0$$

with the cohomology sheaf

$$(34)\qquad E_2(A) = \ker(a_{A,B}^\vee\circ\phi_B)/\mathrm{im}\ a_A.$$

4.2. Example: a special family of symplectic (n,r)-instantons. Now assume $n\geq 3$ and, for any integer $r,\ 2\leq r\leq n-1$, consider a monomorphism

$$(35)\qquad \tau: H_{2n-r+1}\hookrightarrow H_{2n}$$

such that, in the notation of (27),

$$(36)\qquad \tau(H_{2n-r+1})\supset i_\zeta(H_{n+1}).$$

We obtain a hyperweb of quadrics

$$A_\tau\in\mathbf{S}_{2n-r+1}$$

as the image of $A\in MI_{2r}$ under the projection $\mathbf{S}_{2n}\twoheadrightarrow\mathbf{S}_{2n-r+1}$ induced by τ. The corresponding monad

$$(37)\qquad \mathcal{M}_\tau:\quad 0\to H_{2n-r+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\overset{a_\tau}{\to} W_{4n+2}\otimes\mathcal{O}_{\mathbb{P}^3} \overset{a_\tau^\vee\circ q_A}{\to} H_{2n-r+1}^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1)\to 0,$$

has a rank-$2r$ cohomology bundle

$$(38)\qquad E_{2r}(A_\tau) = \ker(a_\tau^\vee\circ q_A)/\mathrm{im}\ a_\tau.$$

where $a_\tau := a_A\circ\tau$. By construction, $E_{2r}(A_\tau)$ inherits a natural symplectic structure

$$(39)\qquad \phi_r:\ E_{2r}(A_\tau)\overset{\simeq}{\to} E_{2r}(A_\tau)^\vee.$$

Besides, in view of (36), the monad (37) can be inserted as a midle row into the diagram (32), extending it to a three-row commutative anti-self-dual diagram. We obtain, in addition to the quotient monad (33), two more quotient monads:

$$(40)\qquad \mathcal{M}'_\tau:\quad 0\to H_{n-r}\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\overset{a'_\tau}{\to} E_{2n}(B)\underset{\simeq}{\overset{\phi}{\to}} E_{2n}(B)^\vee \overset{a'^\vee_\tau}{\to} H_{n-r}^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1)\to 0,$$

$$E_{2r}(A_\tau) = \ker({a'}_\tau^\vee \circ \phi)/\mathrm{im}\, a'_\tau,$$

(41) $$\mathcal{M}''_\tau: \quad 0 \to H_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a''_\tau}{\to} E_{2r}(A_\tau) \underset{\simeq}{\overset{\phi_\tau}{\to}} E_{2r}(A_\tau)^\vee \overset{{a''}_\tau^\vee}{\to} H_{r-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

$$E_2(A) = \ker({a''}_\tau^\vee \circ \phi_\tau)/\mathrm{im}\, a''_\tau.$$

Since $h^0(E_{2n}(B) = h^i(E_{2n}(B)(-2)) = 0$ by (31), from (40) we easily deduce:

(42) $$h^0(E_{2r}(A_\tau)) = h^i(E_{2r}(A_\tau)(-2)) = 0, \quad i \geq 0, \quad c_2(E_{2r}(A_\tau)) = 2n - r + 1.$$

By definition, this together with (39) means that

(43) $$[E_{2r}(A_\tau)] \in I_{2n-r+1,r}.$$

Remark 4.2. Observe that, in view of (35), the maps τ belong to the set

$$N_{n,r} := \{\tau \in \mathrm{Hom}(H_{2n-r+1}, H_{2n}) |\ \tau \text{ is injective and } \mathrm{im}\ \tau \supset \mathrm{im}\ i_\zeta\}.$$

When $A \in MI_{2n,1}(\zeta)$ is fixed, $N_{n,r}$ parametrizes some family of hyperwebs A_τ from $MI_{2n-r+1,r}$. Since $N_{n,r}$ is a principal $GL(H_{2n-r+1})$-bundle over an open subset of the Grassmannian $Gr(n-r, n-1)$, it is irreducible. Thus the family of the three-row extensions of the diagram (32) can be parametrized by the irreducible variety $MI_{2n,1}(\zeta) \times N_{n,r}$. Hence the family $D_{n,r}$ of isomorphism classes of symplectic rank-$2r$ bundles obtained from these diagrams by formula (38) is an irreducible locally closed subset of $I_{2n-r+1,r}$.

Note that it is a priori not clear whether the closure of $D_{n,r}$ in $I_{2n-r+1,r}$ is an irreducible component of $I_{2n-r+1,r}$.

Definition 4.3. Let $2 \leq r \leq n-1$. We say that $A \in MI_{2n-r+1,r}$ *satisfies property* (*) if there exists a monomorphism $i : H_n \hookrightarrow H_{2n-r+1}$ such that the image B of A under the projection $\mathbf{S}_{2n-r+1} \twoheadrightarrow \mathbf{S}_n$ induced by i is invertible as a homomorphism $B : H_n \otimes V \to H_n^\vee \otimes V^\vee$.

Property (*) is clearly an open condition on A. Moreover, since $\pi_{2n-r+1,r} : MI_{2n-r+1,r} \to I_{2n-r+1,r}$ is a principal bundle (Theorem 3.1), if an element $A \in \pi^{-1}_{2n-r+1,r}([E_{2r}])$ satisfies (*), then any other point $A' \in \pi^{-1}_{2n-r+1,r}([E_{2r}])$ satisfies (*). We thus say that a symplectic instanton E_{2r} from $I_{2n-r+1,r}$ *is tame* if some (hence any) $A \in \pi^{-1}_{2n-r+1,r}([E_{2r}])$ satisfies property (*). It is obviously an open condition on $[E_{2r}] \in I_{2n-r+1,r}$.

Remark 4.4. Using (36), we see that any $[E_{2r}] \in D_{n,r}$ is tame. We define

(44) $$I^*_{2n-r+1,r} := I_{(1)} \cup \ldots \cup I_{(k)},$$

where $I_{(1)}, \ldots, I_{(k)}$ are all the irreducible components of $I_{2n-r+1,r}$ whose general points are tame symplectic instantons. By definition, $D_{n,r} \subset I^*_{2n-r+1,r}$, hence $I^*_{2n-r+1,r}$ is nonempty. We also set $MI^*_{2n-r+1,r} = \pi^{-1}_{2n-r+1,r}(I^*_{2n-r+1,r})$, so that the map $\pi_{2n-r+1,r} : MI^*_{2n-r+1,r} \to I^*_{2n-r+1,r}$ is a principal bundle with structure group $GL(H_{2n-r+1})/\{\pm 1\}$.

5. Irreducibility of $I^*_{2n-r+1,r}$

5.1. A dense open subset $X_{n,r}$ of $MI^*_{2n-r+1,r}$. Reduction of the irreducibility of $I^*_{n,r}$ to that of $X_{n,r}$.. In this subsection we recall some known facts about usual rank-2 instantons considered as symplectic $(2n,1)$-instantons. Given an integer $n \geq 1$, set

$$\mathbf{S}^0_n := \{A \in \mathbf{S}_n \mid A : H_n \otimes V \to H_n^\vee \otimes V^\vee \text{ is an invertible map}\}. \tag{45}$$

This is a dense open subset of $\mathbf{S}_n$.

We need some more notation. Let $B \in \mathbf{S}^0_n$. By definition, B is an invertible anti-self-dual map $H_n \otimes V \to H_n^\vee \otimes V^\vee$. Then the inverse

$$B^{-1} : H_n^\vee \otimes V^\vee \to H_n \otimes V \tag{46}$$

is also anti-self-dual. Consider the vector space $\mathbf{\Sigma}_{n,r} := H^\vee_{n-r+1} \otimes H^\vee_n \otimes \wedge^2 V^\vee$. An element $C \in \mathbf{\Sigma}_{n,r}$ can be viewed as a linear map $C : H_{n-r+1} \otimes V \to H_n^\vee \otimes V^\vee$, and its dual $C^\vee : H_n \otimes V \to H^\vee_{n-r+1} \otimes V^\vee$. As the composition $C^\vee \circ B^{-1} \circ C$ is anti-self-dual, we can consider it as an element of $\wedge^2(H^\vee_{n-r+1} \otimes V^\vee) \simeq \mathbf{S}_{n-r+1} \oplus \wedge^2 H^\vee_{n-r+1} \otimes S^2 V^\vee$ (cf. (13)). Thus the condition

$$D - C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r+1}, \qquad D \in \wedge^2(H^\vee_{n-r+1} \otimes V^\vee) \tag{47}$$

makes sense.

Consider an arbitrary direct sum decomposition

$$\xi : H_n \oplus H_{n-r+1} \xrightarrow{\simeq} H_{2n-r+1}. \tag{48}$$

Under this decomposition, we can represent the hyperweb $A \in \mathbf{S}_{2n-r+1}$ considered as a homomorphism $A : H_n \otimes V \oplus H_{n-r+1} \otimes V \to H_n^\vee \otimes V^\vee \oplus H^\vee_{n-r+1} \otimes V^\vee$ by the $(8n-4r+4) \times (8n-4r+4)$-matrix of homomorphisms

$$A = \begin{pmatrix} A_1(\xi) & A_2(\xi) \\ -A_2(\xi)^\vee & A_3(\xi) \end{pmatrix}, \tag{49}$$

where

$$A_1(\xi) \in \mathbf{S}_n, \quad A_2(\xi) \in \mathbf{\Sigma}_{n,r} := \mathrm{Hom}(H_n, H^\vee_{n-r+1}) \otimes \wedge^2 V^\vee, \quad A_3(\xi) \in \mathbf{S}_{n-r+1}. \tag{50}$$

Under this notation, the decomposition (48) induces the isomorphism

$$\tilde{\xi} : \ \mathbf{S}_{2n-r+1} \xrightarrow{\sim} \mathbf{S}_n \oplus \mathbf{\Sigma}_{n,r} \oplus \mathbf{S}_{n-r+1}, \quad A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)). \tag{51}$$

Let $\mathrm{Isom}_{n,r}$ be the set of all isomorphisms ξ in (48). According to Definition 4.3, there exists $\xi \in \mathrm{Isom}_{n,r}$ such that the set

$MI^*_{2n-r+1,r}(\xi) := \{A \in MI_{2n-r+1,r} \mid A$ satisfies property $(*)$ for the monomorphism

$$i_\xi : H_n \hookrightarrow\ H_{2n-r+1} \text{ determined by } \xi\}$$

is a dense open subset of $MI^*_{2n-r+1,r}$. Now take $A \in MI^*_{2n-r+1,r}(\xi)$ and consider A as a matrix of homomorphisms (49). By definition, the submatrix $A_1(\xi)$ of this matrix is invertible. Hence by an appropriate elementary transformation we reduce the matrix A to an equivalent matrix $\tilde{A}$ of the form

$$\tilde{A} = \begin{pmatrix} \mathrm{id}_{H_n^\vee \otimes V^\vee} & A_1(\xi)^{-1} \circ A_2(\xi) \\ 0 & A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) + A_3(\xi) \end{pmatrix}. \tag{52}$$

Since $\mathrm{rk}\tilde{A} = \mathrm{rk}A = 2(2n-r+1)+2r = 4n+2$, we obtain the following relation between the matrices $A_1(\xi)$, $A_2(\xi)$ and $A_3(\xi)$:

$$\mathrm{rk}(A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) + A_3(\xi)) = 2. \tag{53}$$

Consider the embedding of the Grassmannian $G := Gr(2, H^\vee_{n-r+1} \otimes V^\vee) \hookrightarrow P(\wedge^2(H^\vee_{n-r+1} \otimes V^\vee))$, and let $KG \subset \wedge^2(H^\vee_{n-r+1} \otimes V^\vee)$ be the affine cone over G. Set $KG^* := KG \smallsetminus \{0\}$. We can now rewrite (53) as

$$A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) + A_3(\xi) \in KG^*, \tag{54}$$

where

$$A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) \in \wedge^2(H^\vee_{n-r+1} \otimes V^\vee), \qquad A_3(\xi) \in \mathbf{S}_{n-r+1}. \tag{55}$$

Now consider the set

$$\widetilde{X}_{n,r} := \{(B,C,D) \in \mathbf{S}^0_n \times \mathbf{\Sigma}_{n,r} \times KG^* \mid D - C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r+1}\}. \tag{56}$$

Since for an arbitrary point $y = (B,C,D) \in \tilde{X}_n$ the point $\tilde{\xi}^{-1}(B,C,D - C^\vee \circ B^{-1} \circ C)$ lies in $\mathbf{S}_{2n-r+1}$, hence may be considered as a homomorphism $A_y :\ H_{2n-r+1} \otimes V \to H^\vee_{2n-r+1} \otimes V^\vee$ of rank $4n+2$, we have a well-defined $(4n+2)$-dimensional vector space $W_{4n+2}(y) := H_{2n-r+1} \otimes V / \ker A_y$ together with a canonical epimorphism $c_y :\ H_{2n-r+1} \otimes V \twoheadrightarrow W_{4n+2}(y)$ and an induced skew-symmetric isomorphism $q_y :\ W_{4n+2}(y) \xrightarrow{\sim} W_{4n+2}(y)^\vee$ such that $A_y = c_y^\vee \circ q_y \circ c_y$. Now similarly to (25) a morphism of sheaves

$$a_y = c_y \circ u :\ H_{2n-r+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to W_{4n+2}(y) \otimes \mathcal{O}_{\mathbb{P}^3} \tag{57}$$

is defined, together with its transpose ${}^t a_y = a_y^\vee \circ q_y :\ W^\vee_{4n+2}(y) \otimes \mathcal{O}_{\mathbb{P}^3} \to H^\vee_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$, and we introduce an open subset $X_{n,r}$ of the set $\widetilde{X}_{n,r}$,

$$X_{n,r} := \left\{ y \in \widetilde{X}_{n,r} \;\middle|\; \begin{array}{l} (i)\ {}^t a_y \text{ is epimorphic,} \\ (ii)\ [\ker {}^t a_y / \mathrm{im} a_y] \in I^*_{2n-r+1,r} \end{array} \right\}. \tag{58}$$

Since the conditions (i) and (ii) on a point $y \in \widetilde{X}_{n,r}$ in (58) are open, from (54) and (55) we obtain the following result.

Proposition 5.1. *There exist a decomposition* $\xi \in \mathrm{Isom}_{n,r}$, *a dense open subset* $MI^*_{2n-r+1,r}(\xi)$ *of* $MI^*_{2n-r+1,r}$ *and an isomorphism of reduced schemes*

$$f_{n,r} : MI^*_{2n-r+1,r}(\xi) \xrightarrow{\simeq} X_{n,r},\ A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)). \tag{59}$$

The inverse isomorphism is given by the formula

$$f^{-1}_{n,r} :\ X_{n,r} \xrightarrow{\simeq} MI^*_{2n-r+1,r}(\xi) :\ (B, C, D) \mapsto\ \tilde{\xi}^{-1}(B,\ C,\ D - C^\vee \circ B^{-1} \circ C), \tag{60}$$

where $\widetilde{\xi}$ *is defined by (51).*

The proof of the following theorem will be given in Subsection 5.2.

Theorem 5.2. $X_{n,r}$ *is irreducible of dimension* $(2n-r+1)^2+4(2n-r+1)(r+1)-r(2r+1)$.

From Proposition 5.1 and Theorem 5.2 it follows that $MI^*_{2n-r,r}$ is irreducible of dimension $(2n-r+1)^2+4(2n-r+1)(r+1)-r(2r+1)$ for any $n \le 3$ and $2 \le r \le n-1$. Hence $I^*_{2n-r+1,r}$ is irreducible of dimension $4(2n-r+1)(r+1)-r(2r+1)$ for these values of n and r. Substituting $2n-r+1 \mapsto n$, we obtain the following main result of the paper.

Theorem 5.3. *For any integer* $r \ge 2$ *and for any integer* $n \ge r-1$ *such that* $n \equiv r-1(\mathrm{mod} 2)$, *the moduli space* $I^*_{n,r}$ *of tame symplectic instantons is an open subset of an irreducible component of* $I_{n,r}$ *of dimension* $4n(r+1)-r(2r+1)$.

5.2. Proof of the irreducibility of $X_{n,r}$.

In this subsection we give the proof of Theorem 5.2. Consider the set $\widetilde{X}_{n,r}$ defined in (56). Since $X_{n,r}$ is an open subset of $\widetilde{X}_{n,r}$, it is enough to prove the irreducibility of $\widetilde{X}_{n,r}$. In view of the isomorphism $\mathbf{S}^0_n \xrightarrow{\simeq} (\mathbf{S}^\vee_n)^0 :\ B \mapsto B^{-1}$, we rewrite $\widetilde{X}_{n,r}$ as

$$\widetilde{X}_{n,r} := \{(B, C, D) \in (\mathbf{S}^\vee_n)^0 \times \mathbf{\Sigma}_{n,r} \times KG^* \mid D - C^\vee \circ B \circ C \in \mathbf{S}_{n-r+1}\}. \tag{61}$$

Fix a direct sum decomposition

$$H_n \xrightarrow{\simeq} H_{n-r+1} \oplus H_{r-1}.$$

Then any linear map

$$C \in \mathbf{\Sigma}_{n,r} = \mathrm{Hom}(H_{n-r+1}, H^\vee_n \otimes \wedge^2 V^\vee), \quad C : H_{n-r+1} \otimes V \to H^\vee_n \otimes V^\vee, \tag{62}$$

can be represented as a map

$$C : H_{n-r+1} \otimes V \to H^\vee_{n-r+1} \otimes V^\vee\ \oplus\ H^\vee_{r-1} \otimes V^\vee, \tag{63}$$

or else as a block matrix

$$C = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{64}$$

where
(65)
$\phi \in \mathrm{Hom}(H_{n-r+1}, H^\vee_{n-r+1}) \otimes \wedge^2 V^\vee = \mathbf{\Phi}_{n-r+1}, \quad \psi \in \mathbf{\Psi}_{n,r} := \mathrm{Hom}(H_{n-r+1}, H^\vee_{r-1}) \otimes \wedge^2 V^\vee.$

Similarly, any $D \in (\mathbf{S}^\vee_n)^0 \subset \mathbf{S}^\vee_n = S^2 H_n \otimes \wedge^2 V \subset \mathrm{Hom}(H^\vee_n \otimes V^\vee, H_n \otimes V)$ can be represented in the form

$$B = \begin{pmatrix} B_1 & \lambda \\ -\lambda^\vee & \mu \end{pmatrix}, \tag{66}$$

where

$$B_1 \in \mathbf{S}^\vee_{n-r+1} \subset \mathrm{Hom}(H^\vee_{n-r+1} \otimes V^\vee, H_{n-r+1} \otimes V), \tag{67}$$

$$\lambda \in \mathbf{L}_{n,r} := \mathrm{Hom}(H^\vee_r, H_{n-r+1}) \otimes \wedge^2 V, \quad \mu \in \mathbf{M}_{r-1} := S^2 H_{r-1} \otimes \wedge^2 V.$$

By (64) and (66) the composition

$$C^\vee \circ B \circ C : H_{n-r+1} \otimes V \to H^\vee_{n-r+1} \otimes V^\vee \quad (C^\vee \circ B \circ C \in \wedge^2(H^\vee_{n-r+1} \otimes V^\vee))$$

can be written in the form

$$C^\vee \circ B \circ C = \phi^\vee \circ B_1 \circ \phi + \phi^\vee \circ \lambda \circ \psi - \psi^\vee \circ \lambda^\vee \circ \phi + \psi^\vee \circ \mu \circ \psi. \tag{68}$$

By (64)-(67) we have

$$\mathbf{S}^\vee_n \times \mathbf{\Sigma}_{n,r} = \mathbf{S}^\vee_{n-r+1} \times \mathbf{\Phi}_{n-r+1} \times \mathbf{\Psi}_{n,r} \times \mathbf{L}_{n,r} \times \mathbf{M}_{r-1},$$

and there are well defined morphisms

$$\tilde{p} : \widetilde{X}_{n,r} \to \mathbf{L}_{n,r} \times \mathbf{M}_r \times KG, \ (B_1, \phi, \psi, \lambda, \mu, D) \mapsto (\lambda, \mu, D).$$

and

$$p := \tilde{p} | \overline{X}_{n,r} : \overline{X}_{n,r} \to \mathbf{L}_{n,r} \times \mathbf{M}_{r-1} \times KG,$$

where $\overline{X}_{n,r}$ is the closure of $\widetilde{X}_{n,r}$ in $(\mathbf{S}^\vee_n)^0 \times \mathbf{\Sigma}_{n,r} \times KG$. We now invoke the following result from [9]:

Proposition 5.4. *Let $n \geq 2$. For any $B \in (\mathbf{S}^\vee_n)^0$ and for a general choice of the decomposition $H_n \simeq\!\!\to H_{n-r+1} \oplus H_{r-1}$, the block B_1 of B in (66) is nondegenerate.*

Proof. See [9, Proposition 7.3]. By applying this proposition r times, we can find a decomposition $H_n \xrightarrow{\sim} H_{n-r+1} \oplus H_{r-1}$ such that $B_1 : H^\vee_{n-r+1} \otimes V^\vee \to H_{n-r+1} \otimes V$ in (66) is nondegenerate, i.e., $B_1 \in (\mathbf{S}^\vee_{n-r+1})^0$. □

Let $\mathcal{X}$ be any irreducible component of $X_{n,r}$ considered as a reduced scheme and let $\overline{\mathcal{X}}$ be its closure in $\overline{X}_{n,r}$. Fix a point $z=(B_1,\phi,\psi,\lambda,\mu,D)\in\mathcal{X}$ not lying in the components of $X_{n,r}$ different from $\mathcal{X}$. Consider the morphism

$$f:\ \mathbb{A}^1\to\overline{\mathcal{X}},\ t\mapsto(B_1,t^2\phi,t\psi,t\lambda,t^2\mu,t^4D),\quad f(1)=z, \tag{69}$$

which is well defined by (68). By definition, the point $f(0)=(B_1,0,0,0,0,0)$ lies in the fibre $p^{-1}(0,0,0)$. Hence, $p^{-1}(0,0,0)\cap\overline{\mathcal{X}}\neq\varnothing$. In other words,

$$\rho^{-1}(0,0,0)\neq\varnothing,\qquad\text{where}\quad\rho:=p|\overline{\mathcal{X}}. \tag{70}$$

Now, it follows from (68) and the definition of $\widetilde{X}_{n,r}$ that

$$\tilde{p}^{-1}(0,0,0)=\{(B_1,\phi,\psi)\in(\mathbf{S}^\vee_{n-r+1})^0\times\mathbf{\Phi}_{n-r+1}\times\mathbf{\Psi}_{n,r}\mid\phi^\vee\circ B_1\circ\phi\in\mathbf{S}_{n-r+1}\}. \tag{71}$$

Consider the set

$$Z_{n-r+1}=\{(B,\phi)\in(\mathbf{S}^\vee_{n-r+1})^0\times\mathbf{\Phi}_{n-r+1}\mid\phi^\vee\circ B\circ\phi\in\mathbf{S}_{n-r+1}\}.$$

It carries a natural structure of a closed subscheme of $(\mathbf{S}^\vee_{n-r+1})^0\times\mathbf{\Phi}_{n-r+1}$. Comparing the definition of Z_{n-r+1} with (71) we see there are scheme-theoretic inclusions of schemes

$$\rho^{-1}(0,0,0)\subset p^{-1}(0,0,0)\subset\tilde{p}^{-1}(0,0,0)=Z_{n-r+1}\times\mathbf{\Psi}_{n,r}. \tag{72}$$

By [9, Theorem 7.2], Z_{n-r+1} is an integral scheme of dimension $4(n-r+1)(n-r+3)$. This together with (72) implies that

$$\begin{aligned}\dim\rho^{-1}(0,0,0)\le\dim p^{-1}(0,0,0)\le\dim Z_{n-r+1}+\dim\mathbf{\Psi}_{n,r}=4(n-r+1)(n-r+3)\\ +6(r-1)(n-r+1)=(n-r+1)(4n+2r+6).\end{aligned} \tag{73}$$

Hence in view of (70)

$$\begin{aligned}\dim\overline{\mathcal{X}}&\le\dim\rho^{-1}(0,0,0)+\dim\mathbf{L}_{n,r}+\dim\mathbf{M}_{r-1}+\dim KG\\ &\le(n-r+1)(4n+2r+6)+6(r-1)(n-r+1)+3(r-1)r+(8n-8r+5)\\ &=(2n-r+1)^2+4(2n-r+1)(r+1)-r(2r+1).\end{aligned} \tag{74}$$

On the other hand, formula (21) — with n replaced by $2n-r+1$ — and Proposition 5.1 show that, for any point $x\in\mathcal{X}$ such that $A:=f_{n,r}^{-1}(x)\in MI^*_{2n-r+1,r}(\xi)$,

$$(2n-r+1)^2+4(2n-r+1)(r+1)-r(2r+1)\le\dim_A MI^*_{2n-r+1,r}(\xi)=\dim\overline{\mathcal{X}}. \tag{75}$$

Comparing (74) with (75), we see that all the inequalities in (73)-(75) are equalities. In particular,

$$\dim\rho^{-1}(0,0)=\dim(Z_{n-r+1}\times\mathbf{\Psi}_{n,r})=\dim\overline{\mathcal{X}}-\dim(\mathbf{L}_{n,r}\times\mathbf{M}_{r-1}\times KG). \tag{76}$$

Since by Theorem [9, Theorem 7.2] the scheme Z_{n-r+1} is integral and so $Z_{n-r+1} \times \mathbf{\Psi}_{n,r}$ is integral as well, (72) and (76) yield the equalities of integral schemes

$$\rho^{-1}(0,0,0) = p^{-1}(0,0,0) = \tilde{p}^{-1}(0,0,0) = Z_{n-r+1} \times \mathbf{\Psi}_{n,r}. \tag{77}$$

Now we invoke the following easy lemma which is a slight generalization of Lemma 7.4 from [9]. The proof of this lemma is left to the reader.

Lemma 5.5. *Let $f : X \to Y$ be a morphism of reduced schemes, where Y is an integral scheme. Assume that there exists a closed point $y \in Y$ such that for any irreducible component X' of X the following conditions are satisfied:*

(a) $\dim f^{-1}(y) = \dim X' - \dim Y$*,*

(b) the scheme-theoretic inclusion of fibres $(f|_{X'})^{-1}(y) \subset f^{-1}(y)$ is an isomorphism of integral schemes.

Then

(i) there exists an open subset U of Y containing y such that the morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is flat and

(ii) X is integral.

Applying assertions (i)-(ii) of this lemma to $X = X_{n,r}$, $X' = \mathcal{X}$, $Y = \mathbf{L}_{n,r} \times \mathbf{M}_{r-1} \times KG$, $y = (0,0)$, $f = p$, and using (76) and (77), we obtain that $X_{n,r}$ is integral of dimension $(2n-r+1)^2 + 4(2n-r+1)(r+1) - r(2r+1)$. Theorem 5.2 is thus proved.

References

[1] Atiyah, M. F., *Geometry of Yang-Mills fields*, Scuola Normale Superiore, Pisa, 1979, 99 pp.

[2] Atiyah, M. F., Drinfeld, V. G., Hitchin, N. J., Manin, Yu. I., *Construction of instantons*, Phys. Lett. **A 65** (1978), 185–187.

[3] Atiyah, M. F., and Ward, R. S., *Instantons and algebraic geometry*, Comm. Math. Phys. **55** (1977), 117-124.

[4] Barth, W., *Lectures on mathematical instanton bundles,* in: Gauge Theories: Fundamental Interactions and Rigorous Results, P. Dita, V. Georgescu, and R. Purice, eds., Birkhäuser, Boston, 1982, pp. 177–206.

[5] Barth, W., Hulek K., *Monads and moduli of vector bundles*, Manuscripta Math. **25** (1978), 323–347.

[6] Beilinson, A., *Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra,* Functional Anal. Appl. **12** (1978), 212–214.

[7] Bruzzo, U., Markushevich, D., Tikhomirov, A. S, *Moduli of symplectic instanton vector bundles of higher rank on projective space $\mathbb{P}^3$*. Cent. Eur. J. Math. **10** (2012), 1232-1245.

[8] Horrocks, G., *Vector bundles on the punctured spectrum of a local ring*, Proc. Lond. Math. Soc. 14 (1964), 684–713.

[9] Tikhomirov, A. S., *Moduli of mathematical instanton vector bundles with odd c_2 on projective space*, Izvestiya: Mathematics **76**:5 (2012), 143–224.

[10] Tikhomirov, A. S., *Moduli of mathematical instanton vector bundles with even c_2 on projective space*, Izvestiya RAN: Ser. Mat. **77**:6 (2013), 139–168.

[11] Tyurin, A. N., *On the superposition of mathematical instantons II*, In: Arithmetic and Geometry, Progress in Mathematics 36, Birkhäuser 1983.

[12] Tyurin, A. N., *The structure of the variety of pairs of commuting pencils of symmetric matrices*, Math. USSR Izvestiya, **20(2)** (1982), 391–410.