Cubic Form Theorem for Affine Immersions

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An important theorem, due to Pick and Berwald, in classical affine differential geometry states that if a nondegenerate hypersurface M^n in the affine space $\mathbb{R}^{n+1}$ has vanishing cubic form, then it is a quadric. The main purpose of this paper is to prove a number of generalizations of this result to the case of more general affine immersions in the sense of our previous paper [7] including degenerate hypersurfaces.

In Section 1 we extend the notion of affine immersion in [7] to higher codimension and discuss basic formulas and examples. In Section 2 we prove some results on umbilical immersions and reduction of codimension. In Section 3 we discuss the condition that the cubic form is divisible by the second fundamental form and state a number of generalizations of the classical theorem of Pick and Berwald. The proofs of these results are given in Sections 4 and 5.

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1. Affine immersions for higher codimension

In this section we extend the notion of affine immersion in [7] to the case of higher codimension. Let (M,∇) and $(\tilde{M},\tilde{\nabla})$ be differentiable manifolds with torsion-free affine connections of dimension n and $\tilde{n} = n + p$, respectively.

An immersion $f: M \to \tilde{M}$ is called an affine immersion if around each point of M there is a field of transversal subspaces $x \mapsto N_x$:

(1) $T_{f(x)} = f_*(T_x(M)) + N_x$

such that for vector fields X and Y on M we have a decomposition

(2) $\tilde{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + \alpha(X,Y)$

where $\alpha(X,Y) \in N_x$ at each point x.

In the following we shall call N_x the normal space (rather than the transversal space) with the understanding that the choice in general is not unique. We have the normal bundle N with $x \mapsto N_x$. We call α the second fundamental form. Corresponding to Proposition 1 in [7] we have the following

Proposition 1. *Let* $f : (M,\nabla) \to (\tilde{M},\tilde{\nabla})$ *be an affine immersion and* $x \in M$. *Then a normal space* N_x *with the property that it is spanned by all* $\alpha(X,Y)$, *where* $X,Y \in T_x(M)$, *is uniquely determined*.

Proof. Let $N^1{}_x$ be another such normal space at x and α^1 the corresponding second fundamental form defined by the equation (2) using N^1. Write $\alpha(X,Y) = \tau(X,Y) + \beta(X,Y)$, where $\tau(X,Y) \in T_x(M)$ and $\beta(X,Y) \in N^1$. Then it follows that $\tau(X,Y) = 0$ and $\alpha(X,Y) = \beta(X,Y) = \alpha^1(X,Y)$. Since N_x (resp. $N^1{}_x$) is spanned by all $\alpha(X,Y)$ (resp. $\alpha^1(X,Y)$), we conclude that

$N_x = N^1{}_x$. □

In general, for each point $x \in M$ the subspace of $T_x(\tilde{M})$ spanned by $f_*(T_x(M))$ and all $\alpha(X,Y)$, $X,Y \in T_x(M)$, is called the <u>second osculating space</u> at x. It is determined uniquely, because it is also the span of all vectors $(\tilde{\nabla}_X f_*(Y))_x$, where X and Y are all vector fields on M. Its dimension is called the <u>second osculating dimension.</u>

If $\xi\colon x \mapsto \xi_x \in N_x$ is a normal vector field, then we write

$$\text{(3)} \qquad \tilde{\nabla}_X \xi = -f_*(A_\xi X) + \nabla^\perp{}_X \xi,$$

where $A_\xi X \in T_x(M)$ and $\nabla^\perp{}_X \xi \in N_x$ at each point. Just as in submanifold theory in Riemannian geometry, we have a bilinear mapping A, called the <u>shape tensor</u>:

$$(\xi, x) \in N_x \times T_x(M) \to A_\xi X \in T_x(M)$$

at each point x. We call A_ξ the <u>shape operator</u> for ξ. The mapping of the space of normal vector fields $\xi \to \nabla^\perp{}_X \xi$ is covariant differentiation relative to the normal connection.

Just as in submanifold theory we get several basic equations relating the curvature tensors $\tilde{R}$ for $(\tilde{M}, \tilde{\nabla})$ and R for (M, ∇), the second fundamental form form α, the shape tensor A, etc. in the usual way. Especially, the tangential component of $\tilde{R}(X,Y)Z$ is given by

$$\tan \tilde{R}(X,Y)Z = R(X,Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X$$

and the normal component by

$$\text{nor } \tilde{R}(X,Y)Z = (\nabla_X \alpha)(Y,Z) - (\nabla_Y \alpha)(X,Z),$$

where $\nabla_X \alpha$ is defined by

$(\nabla_X\alpha)(Y,Z) = \nabla^{\perp}_X \alpha(Y,Z) - \alpha(\nabla_X Y,Z) - \alpha(Y,\nabla_X Z).$

For a normal vector field ξ the tangential component of $\tilde{R}(X,Y)\xi$ is given by

$$\tan \tilde{R}(X,Y)\xi = (\nabla_Y A)_\xi(X) - (\nabla_X A)_\xi(Y),$$

where $\nabla_X A$ is defined by

$$(\nabla_X A)_\xi(Y) = \nabla_X(A_\xi Y) - A_\xi(\nabla_X Y) - (A_{\nabla^{\perp}_X \xi})(Y).$$

The normal component is given by

$$\text{nor } \tilde{R}(X,Y)\xi = \alpha(A_\xi X,Y) - \alpha(X,A_\xi Y) + R^{\perp}(X,Y)\xi,$$

where $R^{\perp}$ is the curvature tensor of the normal connection.

In the case where $(\tilde{M},\tilde{\nabla})$ is projectively flat (with symmetric Ricci tensor, see [6]), we have

$$\tilde{R}(X,Y)Z = \tilde{\gamma}(Y,Z)X - \tilde{\gamma}(X,Z)Y,$$

where $\tilde{\gamma}$ is the normalized Ricci tensor for $(\tilde{M},\tilde{\nabla})$:

$$\tilde{\gamma}(X,Y) = \text{Ric}(X,Y)/(\tilde{n}-1).$$

In this case, all the formulas above become simpler. Thus we have

(4) $R(X,Y) = \tilde{\gamma}(Y,Z)X - \tilde{\gamma}(X,Z)Y + A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y$

- equation of Gauss-

(5) $(\nabla_X\alpha)(Y,Z) = (\nabla_Y\alpha)(X,Z)$

-equation of Codazzi for α-

(6) $(\nabla_X A)_\xi Y + \tilde{\gamma}(Y,\xi)X = (\nabla_Y A)_\xi X + \tilde{\gamma}(X,\xi)Y$

-equation of Codazzi for A-

(7) $R^{\perp}(X,Y)\xi = \alpha(X,A_\xi Y) - \alpha(A_\xi X,Y)$

- equation of Ricci-

When the ambiant affine connection $\tilde{\nabla}$ is flat, equations (4) an (6) get further simplified:

(4a) $R(X,Y) = A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y$

(6a) $(\nabla_X A)_\xi Y = (\nabla_Y A)_\xi X$.

If $\alpha = 0$ at a point x, we say that f is totally geodesic at x. If $\alpha = 0$ at every point $x \in M$, we say that f is totally geodesic.

An affine immersion f: $(M,\nabla) \to (\tilde{M},\tilde{\nabla})$ is said to be umbilical at $x \in M$ if there is a 1-form ρ on N_x such that

(8) $A_\xi = \rho(\xi) I$ for every $\xi \in N_x$,

where I denotes the identity transformation. If f is umbilical at every point, we say that f is umbilical. If f is umbilical and the ambiant connection $\tilde{\nabla}$ is projectively flat, then the normal connection is flat (i.e. $R^\perp = 0$) as follows from (7).

We now discuss a few examples.

Example 1. Let (M,g) and $(\tilde{M},\tilde{g})$ be Riemannian or pseudo-Riemannian manifolds with Levi-Civita connections ∇ and $\tilde{\nabla}$, respectively. An isometric immersion f: $(M,g) \to (\tilde{M},\tilde{g})$ gives rise to an affine immersion $(M,\nabla) \to (\tilde{M},\tilde{\nabla})$. Here, of course, there is a natural choice of normal space N_x as the orthogonal component of $T_x(M)$ relative to $\tilde{g}$.

Example 2. Curves in affine space $\mathbb{R}^3$ are studied in [1], Chapter 3. Also see [5] for surfaces in $\mathbb{R}^4$.

Example 3. Graph immersion. Let F: $\mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function and consider the graph immersion f: $M = \mathbb{R}^n \to \tilde{M} = \mathbb{R}^{n+p}$ given by

(9) $f(x) = (x,F(x)) \in \mathbb{R}^n \times \mathbb{R}^p = \mathbb{R}^{n+p}, \quad x \in \mathbb{R}^n.$

For each $x \in M$, let N_x be the subspace of $T_x(\mathbb{R}^{n+p})$ that is parallel to the affine p-space $\mathbb{R}^p$ of $\mathbb{R}^{n+p}$. We get an affine immersion f: $(M,\nabla) \to (\tilde{M},\tilde{\nabla})$, both spaces $M = \mathbb{R}^n$ and $\tilde{M} = \mathbb{R}^{n+p}$ with the usual flat affine connections. As in Example 3 in [7], the second fundamental form α is given essentially as the Hessian of the function F with values in $\mathbb{R}^p$ identified with each N_x. We have also $A = 0$. Thus f is umbilical but not totally geodesic.

Example 4. Centro-affine immersion. Supppose M is an n-dimensional submanifold immersed in $\tilde{M} = \mathbb{R}^{n+p}$. Assume that there exists an affine (p-1)-subspace $V = \mathbb{R}^{p-1}$ in $\mathbb{R}^{n+p}$ such that for each point x of M the affine p-subspace spanned by x and V is transversal to M. Choosing N_x to be the tangent space at x of this transversal affine p-space, we write equation (2) and define an affine connection ∇ on M. The resulting affine immersion f: $(M, \nabla) \to \mathbb{R}^{n+p}$ is a generalization of centro-affine hypersurface in [7]. We show that f is umbilical and that ∇ is projectively flat. To see this, let $x_0 \in M$ and let $\xi_0 = \lambda_0 x_0 + U_0$ be a normal vector at x_0, where x_0 is also considered as a position vector for the point x_0 from a fixed point of $\mathbb{R}^{n+p}$. To compute A_ξ we extend ξ_0 to a normal vector field $\xi = \lambda_0 x + U_0$ and find $\tilde{\nabla}_X \xi = \lambda_0 X$. Thus $A_\xi = -\lambda_0 I$. This shows that f is umbilical. Next we consider another submanifold transversal to the family of normal affine p-spaces to M. It is given by a mapping of the form

(10) $\quad x \in M \mapsto \varphi(x) = \lambda x + F(x),$

where $\lambda: M \to \mathbb{R}^+$ and $F: M \to \mathbb{R}^{p-1}$. The connection induced by φ on M is

$$\nabla'_X Y = \nabla_X Y + \mu(X)Y + \mu(Y)X, \quad \text{where } \mu = d(\log \lambda).$$

By taking an affine n-space as $\varphi(M)$, we can get ∇' to be a flat affine connection. This means that ∇ is projectively flat.

2. Umbilical immersions and reduction of codimension

First we prove the following result on umbilical immersions.

Theorem 2. Let $f: (M^n, \nabla) \to (\mathbb{R}^{n+p}, \tilde{\nabla})$ be an umbilical affine immersion, where $n \geq 2$. Then it is affinely equivalent to a graph immersion or a centro-affine submanifold immersion.

Proof. Let ρ be the 1-form on the normal bundle such that $A_\xi = \rho(\xi)I$. From Codazzi's equation (6a) and from $(\nabla_X A)_\xi = (\nabla_X \rho)(\xi)I$, we get $(\nabla_X \rho)(\xi)Y = (\nabla_Y \rho)(\xi)X$ for any two vectors X and Y. Thus $\nabla_X \rho = 0$ for any X. Thus $\operatorname{Ker} \rho_x = \{\xi \in N_x;\ \rho(\xi) = 0\}$ has constant dimension. Now we show that the distribution $x \in M^n \mapsto \operatorname{Ker} \rho_x \subset T_x(\mathbb{R}^{n+p})$ along the immersion f is parallel in $\mathbb{R}^{n+p}$. This is obvious, however, because if ξ_t is parallel along a curve x_t in M^n relative to the normal connection, then $\rho(\xi_t)$ is constant since $\nabla\rho = 0$.

i) Case where $\rho \neq 0$. Take a normal vector field $\xi \notin \operatorname{Ker} \rho$, and consider the mapping $x \in M^n \mapsto y = x + \xi/\rho(\xi) \in \mathbb{R}^{n+p}$. Then for any tangent vector X we get

$$\tilde{\nabla}_X y = X + [-X(\rho(\xi))\xi]/\rho(\xi)^2 + (-\rho(\xi)X + \nabla^\perp{}_X \xi)/\rho(\xi)$$

$$= -[X(\rho(\xi))/\rho(\xi)^2]\xi + (\nabla^\perp{}_X \xi)/\rho(\xi)$$

and

$$\rho(\tilde{\nabla}_X y) = 0$$

so that $\tilde{\nabla}_X(y) \in \operatorname{Ker} \rho$. This means that all points y lie in the (p-1)-dimensional affine subspace, say V, through one point y_0 and parallel to the parallel distribution $\operatorname{Ker} \rho$. It now follows that for each $x \in M^n$ the normal space N_x coincides with the tangent space at x to the p-dimensional affine subspace generated by x and V. We conclude that M^n is a centro-affine submanifold immersed in $\mathbb{R}^{n+p}$.

Finally, consider the case where $\rho = 0$, thus $A = 0$. For any normal vector field ξ, we see that $\tilde{\nabla}_X\xi = \nabla^{\perp}{}_X\xi$ belongs to N_x. This means that the normal spaces $N_x \subset T_x(\mathbb{R}^{n+p})$ are parallel in $\mathbb{R}^{n+p}$. Since M^n is transversal to this family of parallel p-dimensional affine subspaces N, it is a graph. □

We now prove two results concerning reduction of codimension for affine immersions.

The first is a variation of Erbacher's result in Riemannian geometry [3].

<u>Proposition 3. Let</u> $f: (M^n, \nabla) \to (\mathbb{R}^{n+p}, \tilde{\nabla})$ <u>be an affine immersion. Suppose</u> N_1 <u>is a subbundle of the normal bundle</u> N <u>such that</u>

i) $N_1(x)$ <u>contains the range of</u> α_x <u>for every</u> $x \in M^n$;

ii) N_1 <u>is parallel relative to the normal connection.</u>

<u>Then</u> $f(M^n)$ <u>is contained in a certain</u> $(n+q)$-<u>dimensional affine subspace of</u> $\mathbb{R}^{n+p}$, <u>where</u> $q = \dim N_1(x)$.

Proof. We can easily check that the distribution $x \to \Delta(x) = T_x(M) + N_1(x)$ along the mapping f is parallel in $\mathbb{R}^{n+p}$. Thus we have a parallel distribution Δ of dimenison $n + q$ on $\mathbb{R}^{n+p}$. If x_t is a geodesic in (M^n, ∇), we see that $f(x_t)$ lies in the affine $(n+q)$-space $\mathbb{R}^{n+q}$ through $x_0 \in M^n$ and tangent to Δ. It follows that $f(M^n) \subset \mathbb{R}^{n+q}$. □

The next result is known in the Riemannian case (for example, [10], Lemma 28, p.362; see [2] for its further generalization).

Proposition 4. *Let* $f: (M^n, \nabla) \to (\mathbb{R}^{n+p}, \tilde{\nabla})$ *be an affine immersion. Suppose there exists a nonzero normal vector field* ξ *and a bilinear symmetric function* h *on* M^n *such that* $\alpha(X,Y) = h(X,Y)\xi$ *for all tangent vectors* X *and* Y. *Assume furthermore that* $\operatorname{rank} h \geq 2$ *at every point. Then* $f(M^n)$ *is contained in an* $(n+1)$*-dimensional affine space* $\mathbb{R}^{n+1}$ *of* $\mathbb{R}^{n+p}$.

Proof. Let $\{X_1,\ldots,X_r,X_{r+1},\ldots,X_n\}$ be a basis in $T_x(M^n)$ such that $\{X_{r+1},\ldots,X_n\}$ is a basis of $\operatorname{Ker} h_x$ and $h(X_i,X_j) = \pm\delta_{ij}$ for $1 \leq i,j \leq r$, where by assumption $r \geq 2$. For any $X = X_i$, $1 \leq i \leq n$, there is $Y \neq X$ among $X_1,\ldots,X_r$ so that $h(X,Y) = 0$ and $h(Y,Y) \neq 0$. Now from Codazzi's equation (5) we get

$$(\nabla_X h)(Y,Z)\xi + h(Y,Z)\,\nabla^\perp{}_X\xi = (\nabla_Y h)(X,Z)\xi + h(X,Z)\,\nabla^\perp{}_Y\xi.$$

Set $Z = Y$ and consider this equation modulo span $\{\xi\}$. We obtain

$$h(Y,Y)\,\nabla^\perp{}_X\xi = 0 \mod \operatorname{span}\{\xi\} \quad \text{and hence} \quad \nabla^\perp{}_X\xi \in \operatorname{span}\{\xi\}.$$

This being true for every X_i, $1 \leq i \leq n$, and thus for every $X \in T_x(M^n)$, it follows that $N_1 = \operatorname{span}\{\xi\}$ is parallel relative to the normal connection. We may now apply Proposition 3 to N_1. □

Suppose an affine immersion $f: (M^n, \nabla) \to (\mathbb{R}^{n+p}, \tilde{\nabla})$ has the second osculating dimension $n+1$. Then around each point we may choose a normal vector field ξ such that $\alpha(X,Y) = h(X,Y)\xi$. The rank of h is independent of the choice of such ξ, and we define it as the *rank* of α.

Corollary. *Suppose that the second osculating dimension of an affine immersion* $f: (M^n, \nabla) \to (\mathbb{R}^{n+p}, \tilde{\nabla})$ *is* $n+1$ *and that the rank of* α *is* ≥ 2 *at every point. Then* $f(M^n)$ *is contained in an* $(n+1)$*-dimensional affine subspace* $\mathbb{R}^{n+1}$ *of* $\mathbb{R}^{n+p}$.

3. Cubic form

For an affine immersion $f: (M, \nabla) \to (\tilde{M}, \tilde{\nabla})$, where $\tilde{\nabla}$ is projectively flat, we define the cubic form to be

(11) $\nabla\alpha : T(M) \times T(M) \times T(M) \to N$

that is,

(11a) $(\nabla\alpha)(X,Y,Z) = (\nabla_X \alpha)(Y,Z).$

By (5), $\nabla\alpha$ is symmetric in X, Y, and Z.

We explain briefly our motivation and goal. For an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\tilde{M}$ of constant curvature, the condition that $\nabla\alpha = 0$ has a significant geometric meaning [4]. For the geometry of affine immersions, we might first consider the weaker condition that $\nabla\alpha$ is divisible by α. (Actually, this is a projective notion as we we shall further study in a subsequent paper.) In the present paper we deal with the case where the osculating dimension for $f: (M^n, \nabla) \to (\mathbb{R}^{n+p}, \tilde{\nabla})$ is $n+1$. In this case, it turns out that the condition $\alpha | \nabla\alpha$ depends only on the image $f(M^n)$ and not on the connection ∇ (induced from $\tilde{\nabla}$ by choosing a normal vector field ξ along $f(M^n)$). Furthermore this condition characterizes a quadric when the rank of α is ≥ 2. Now the detail follows.

We say that $\nabla\alpha$ <u>is divisible by</u> α (denoted by $\alpha \mid \nabla\alpha$) if there is a 1-form ρ on M such that

(12) $\alpha(X,Y,Z) = \rho(X)\,\alpha(Y,Z) + \rho(Y)\,\alpha(Z,X) + \rho(Z)\,\alpha(X,Y)$

for all tangent vectors X, Y and Z; or equivalently

(12a) $\alpha(X,X,X) = 3\,\rho(X)\,\alpha(X,X)$

for all tangent vectors X.

When the codimension p is 1, choose a normal vector field ξ and write $\alpha(Y,Z) = h(Y,Z)\,\xi$. We have

$$(\nabla_X \alpha)(Y,Z) = (\nabla_X h)(Y,Z)\, \xi + h(Y,Z)\, (\nabla^{\perp}{}_X \xi)$$

$$= [(\nabla_X h)(Y,Z) + \tau(X)\, h(Y,Z)]\, \xi = C(X,Y,Z)\xi,$$

where τ is the transversal (normal) connection form and C is the cubic form as already defined in [7]. Thus $\alpha \mid \nabla\alpha$ if and only if

(13) $C(X,Y,Z) = \rho(X)h(Y,Z) + \rho(Y)h(Z,X) + \rho(Z)h(X,Y)$

for all tangent vectors X,Y and Z. We may write (13) as $h \mid C$. In the special case where ξ is equiaffine so that f is an affine immersion in the sense of relative geometry (i.e. $\tau = 0$), (13) may be expressed by writing $h \mid \nabla h$.

We prove

Lemma 1. Let $f: (M^n, \nabla) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ be an affine immersion with a normal vector field ξ. If we change ξ to

(14) $\hat{\xi} = (\xi + U)/\lambda$

where U is a vector field on M^n and $\lambda : M^n \to \mathbb{R} - \{0\}$, then writing

$$\tilde{\nabla}_X f_*(Y) = f_*(\hat{\nabla}_X Y) + \hat{h}(X,Y)\hat{\xi}$$

we have an affine immersion $f: (M^n, \hat{\nabla}) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ and

(15) $\hat{\nabla}_X Y = \nabla_X Y - h(X,Y)U$

(16) $\hat{h} = \lambda h$

(17) $\hat{\tau} = \tau + \eta - d(\log \lambda)$

(18) $\hat{C}(X,Y,Z)/\lambda = C(X,Y,Z) + \eta(X)h(Y,Z) + \eta(Y)h(Z,X) + \eta(Z)h(X,Y)$,

where η is the 1-form such that $\eta(X) = h(X,U)$ for all X.

Proof. The verification is straightforward if we note

$$(\hat{\nabla}_X \hat{h})(Y,Z) = X\, \hat{h}(Y,Z) - \hat{h}(\hat{\nabla}_X Y, Z) - \hat{h}(Y, \hat{\nabla}_X, Z)$$

$$\tilde{\nabla}_X \hat{\xi} = - f_*(\hat{S}X) + \hat{\tau}(X)\hat{\xi}$$

and

$$\hat{C}(X,Y,Z) = (\hat{\nabla}_X \hat{h}) + \hat{\tau}(X)\hat{h}(Y,Z). \qquad \square$$

Now observe that if $f: M^n \to \mathbb{R}^{n+1}$ is an immersion which admits a transversal vector field ξ, then we may induce an affine connection ∇ in such a way that $f: (M^n, \nabla) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ is an affine immersion. As a consequence of Lemma 1 we have

Proposition 5. If an immersion $f: M^n \to \mathbb{R}^{n+1}$ has the property that $h \mid C$ for some choice of normal vector field ξ, then it has the same property for any choice of normal vector field. Also the rank of h does not depend on the choice of ξ.

In particular, the property that h is nondegenerate does not depend on the choice of ξ; we say that f is nondegenerate if h is.

In the case where the second fundamental form h of an affine immersion $f: (M^n, \nabla) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ is indefinite, we can give the following geometric interpretation of the condition $h \mid C$.

Proposition 6. If h is indefinite, the following statements are equivalent:

1) $h \mid C$;

2) a geodesic in (M^n, ∇) whose initial tangent vector is null is a null curve (relative to h);

3) all geodesics in (M^n, ∇) with null initial tangent vectors are geodesics in $\mathbb{R}^{n+1}$.

Proof.

1) → 2) : Assume $C(X,X,X) = 3\,\rho(X)h(X,X)$ for all $X \in TM$, where ρ is a certain 1-form. Then

$$(\nabla_X h)(X,X) = (3\rho - \tau)(X)\, h(X,X).$$

Suppose x_t is a geodesic in (M^n, ∇) such that $h(\vec{x}_0, \vec{x}_0) = 0$. The above equation implies $(d/dt)h(\vec{x}_t, \vec{x}_t) = (3\rho - \tau)(\vec{x}_t)\, h(\vec{x}_t, \vec{x}_t)$. Thus the function $\varphi(t) = h(\vec{x}_t, \vec{x}_t)$ satisfies the differential equation

$d\varphi/dt = \psi(t)\varphi(t)$, where $\psi(t) = (3\rho - \tau)(\vec{x}_t)$.

We know that a solution $\varphi(t)$ of this equation with $\varphi(0) = 0$ must be identically 0. Thus x_t is a null curve.

2) → 3): This is obvious from $\tilde{\nabla}_t \vec{x}_t = \nabla_t \vec{x}_t + h(\vec{x}_t, \vec{x}_t)$.

3) → 1): Let $X \in T_x(M)$ be null, i.e. $h(X,X) = 0$. If x_t is a geodesic in (M^n, ∇) with initial tangent vector X, then by assumption 3) we have

$$0 = \tilde{\nabla}_t \vec{x}_t = \nabla_t \vec{x}_t + h(\vec{x}_t, \vec{x}_t)\xi = h(\vec{x}_t, \vec{x}_t)\xi$$

so that $h(\vec{x}_t, \vec{x}_t) = 0$. Hence

$$(\nabla_t h)(\vec{x}_t, \vec{x}_t) = (d/dt)\, h(\vec{x}_t, \vec{x}_t) - 2\, h(\nabla_t \vec{x}_t, \vec{x}_t) = 0.$$

At $t = 0$ we have

$$(\nabla_X h)(X,X) = 0$$

and hence $C(X,X,X) = (\nabla_X h)(X,X) + \tau(X)h(X,X) = 0$. What we have shown is that $h(X,X) = 0$ for $X \in TM$ implies $C(X,X,X) = 0$. It follows that $h \mid C$. □

We now state a number of generalizations of the classical result. The proofs will be given in subsequent sections.

<u>Theorem</u> 7. <u>Let</u> $f: (M^n, \nabla) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ <u>be an affine immersion with a normal vector field</u> ξ <u>for which</u> $\tau = 0$. <u>If</u> rank $h \geq 2$ <u>and</u> $\nabla h = 0$ <u>at every point, then</u> $f(M^n)$ <u>lies in a quadric</u>.

<u>Remark</u> 1. More precisely, $f(M^n)$ lies in a cylinder $Q^r \times \mathbb{R}^{n-r}$, where Q^r is a nondegenerate quadric in an affine subspace $\mathbb{R}^{r+1}$ and $\mathbb{R}^{n-r}$ is an affine subspace transversal to $\mathbb{R}^{r+1}$.

<u>Remark</u> 2. This theorem extends the classical Pick-Berwald theorem (see [1] as well as the result in relative geometry (see [8]), which are for nondegenerate hypersurfaces. See also [9].

The formulations of the following Theorems 8 and 10 are based on the observations in Proposition 5.

Theorem 8. Let $f: M^n \to \mathbb{R}^{n+1}$ be a nondegenerate immersion. Then $f(M^n)$ lies in a quadric if and only if $h \mid C$.

We examine the following question: given (M^n, ∇), under what conditions can we find an affine immersion $f: (M^n, \nabla) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ such that $f(M^n)$ lies in a nondegenerate quadric in $\mathbb{R}^{n+1}$?

We proceed as follows. If there is an affine immersion $f: (M^n, \nabla) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ such that $f(M^n)$ lies in a nondegenerate quadric Q^n in $\mathbb{R}^{n+1}$, then we can choose a normal vector field ξ^0 and obtain the second fundamental form h^0 and the induced affine connection ∇^0 on M^n from

$$\tilde{\nabla}_X Y = \nabla^0_X Y + h^0(X,Y)\,\xi^0$$

such that h^0 is a pseudo-Riemannian metric and ∇^0 is the Levi-Civita connection of h^0. We may write, as in Lemma 1, $\xi = (\xi^0 + U)/\lambda$, where U is a certain vector field on M^n and λ a nonzero function. We find

(19) $\quad \nabla_X Y = \nabla^0_X Y - h^0(X,Y)U.$

In the case where Q^n is not locally convex, h^0 is indefinite. A geometric interpretation of (19) is the following. A null geodesic of ∇^0 is a geodesic of ∇. Conversely, an affine connection ∇ with this property relative to (h^0, ∇^0) must be of the form (19) for a certain vector field U.

In order to prove this, let K be the difference tensor: $K(X,Y) = \nabla_X Y - \nabla^0_X Y$. Take any $X \in T_x(M)$ with $h^0(X,X) = 0$. If x_t is a geodesic for ∇^0 with initial tangent vector X, then it is a null geodesic and, by assumption, it is a geodesic for ∇. Thus $\nabla_t \vec{x}_t = 0$, which implies $K(\vec{x}_t, \vec{x}_t) = 0$, in particular, $K(X,X) = 0$. We have shown that $K(X,X) = 0$ whenever $h^0(X,X) = 0$. By taking a basis $\{X_1, \ldots, X_n\}$ in $T_x(M^n)$, write $K(X,Y) = \Sigma^n_{i=1} K^i(X,X)X_i$. Since

$h^0(X,X) = 0$ implies $K^i(X,X) = 0$, we have $K^i(X,Y) = a^i h^0(X,Y)$, $1 \le i \le n$. Then $K(X,Y) = (\Sigma^n_{i=1} a^i X_i)\, h^0(X,Y)$. Thus we have (19) with $Z = -\Sigma^n_{i=1} a^i X_i$.

We can now state

Proposition 9. A differentiable manifold with an affine connection (M^n, ∇^n) admits an affine immersion into a (not locally convex) nondegenerate quadric Q^n in $\mathbb{R}^{n+1}$ if and only if M^n admits a pseudo-Riemannian (not positive-definite) metric of constant sectional curvature whose null geodesics are geodesics of ∇.

Theorem 10. Let $f: M^n \to \mathbb{R}^{n+1}$ be an immersion with rank $h \ge 2$ everywhere. Then $f(M^n)$ lies in a quadric if and only if $h \mid C$.

Remark 3. If $h \mid C$ and if the affine connection ∇ induced by f relative to some choice of a transversal vector field is complete, then $f(M^n)$ is a cylinder as in Remark 1 above. Even for the standard $S^2 \subset \mathbb{R}^3$, ∇ is incomplete for most choices of ξ.

Theorem 11. Let $f: (M^n, \nabla) \to (\mathbb{R}^{n+p}, \tilde{\nabla})$ be an affine immersion, $n \ge 2$. Then $f(M^n)$ is contained in a quadric Q^n of an affine subspace $\mathbb{R}^{n+1}$ of $\mathbb{R}^{n+p}$ if and only if the osculating dimension is $n+1$, rank $\alpha \ge 2$, and $\alpha \mid \nabla\alpha$.

4. Proofs of Theorems 7 and 8

We start with a few lemmas.

Lemma 2. Let $f: (M^n, \nabla) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ be an affine immersion and assume that $\tau = 0$, $\nabla h = 0$ and rank $h \ge 2$ everywhere. Then

1) Ker h is a parallel distribution on (M^n, ∇);

2) $x \in M^n \to f_*(\mathrm{Ker}\, h_x)$ is a distribution along f which is parallel in $\mathbb{R}^{n+1}$;

3) There is a constant ρ such that $SX = \rho X$ mod Ker h for every $X \in TM$.

Proof. 1) Let Y_t and Z_t be parallel vector fields along a curve x_t in M^n.

Then $\nabla h = 0$ implies that

$$dh(Y,Z)/dt = h(\nabla_t Y, Z) + h(Y, \nabla_t Z) = 0.$$

Thus $h(Y_t, Z_t)$ is constant. If $Y_0 \in \ker h$ at x_0, then it follows that $Y_t \in \operatorname{Ker} h$ along the curve x_t. This shows that $\dim \operatorname{Ker} h$ is constant and the distribution $x \to \operatorname{Ker} h_x$ is parallel on M^n.

2) Let Y_t be a parallel vector field belonging to Ker h along a curve x_t. Then

$$\tilde{\nabla}_t f_*(Y_t) = f_*(\nabla_t Y_t) + h(\vec{x}_t, Y_t) = 0,$$

which shows that $f_*(Y_t)$ is parallel in $\mathbb{R}^{n+1}$. This proves that $x \mapsto f_*(\operatorname{Ker} h_x) \subset T_{f(x)}(\mathbb{R}^{n+1})$ is parallel in $\mathbb{R}^{n+1}$.

3) From $\nabla h = 0$ we get $h(R(X,Y)Y,Y) = 0$ for all $X, Y \in T_x(M^n)$. Using the equation of Gauss: $R(X,Y)Y = h(Y,Y)SX - h(X,Y)SY$, we get

(20) $$h(Y,Y)h(SX,Y) = h(X,Y)h(SY,Y).$$

In $T_x(M)$ choose a basis $\{X_1, \ldots, X_r, X_{r+1}, \ldots, X_n\}$ such that $\{X_{r+1}, \ldots, X_n\}$ is a basis of $\operatorname{Ker} h_x$ and $h(X_i, X_j) = \pm \delta_{ij}$ for $1 \leq i,j \leq r$. By assumption, $r \geq 2$.

For each X_i, $1 \leq i \leq r$, choose X_j, $1 \leq j \leq r$, $j \neq i$; we get $h(SX_i, X_j) = 0$ from (20). Thus $SX_i = \rho_i X_i \mod \operatorname{Ker} h_x$. We want to show that $\rho_1 = \cdots = \rho_r$. If $i \neq j$ among $1, \ldots, r$, then $Z = X_i + X_j$ or $X_i + 2X_j$ has the property that $h(Z,Z) \neq 0$ and may be chosen as part of an orthonormal basis (after normalization) of a supplementary subspace to Ker h. Thus by what we have seen above we get

$$S(X_i + X_j) = \rho (X_i + X_j) \quad \text{or} \quad S(X_i + 2X_j) = \rho (X_i + 2X_j),$$

with a certain constant ρ. Then we get

$$\rho_i X_i + \rho_j X_j = \rho X_i + \rho X_j \quad \text{or} \quad \rho_i X_i + 2\rho_j X_j = \rho X_i + 2\rho X_j .$$

It follows that $\rho_i = \rho_j = \rho$. We have thus shown that all ρ_i's are equal. Call this number ρ. We have shown $SX = \rho X \mod \operatorname{Ker} h$ for every $X = X_1, \ldots, X_r$.

Now let $1 \le j \le r$ and $r+1 \le i \le n$. (20) implies $h(SX_i, X_j) = 0$. This shows that $SX_i \in \operatorname{Ker} h$. So $S(\operatorname{Ker} h) \subset \operatorname{Ker} h$. We can write $SX = \rho X \mod \operatorname{Ker} h$ for every $X = X_{r+1}, \ldots, X_n$. Hence $SX = \rho X \mod \operatorname{Ker} h$ for all $X \in T_x(M)$.

It now remains to show that ρ is a constant. Since $\tau = 0$, we have Codazzi's equation $(\nabla_X S)(Y) = (\nabla_Y S)(X)$ (see [7]). We extend a basis $\{X_1, \ldots, X_r, X_{r+1}, \ldots, X_n\}$ as before to vector fields in a neighborhood with the property that they still form a basis and $\{X_{r+1}, \ldots, X_n\}$ form a basis of Ker h at each point. Then

$$\begin{aligned}(\nabla_X S)(X_j) &= \nabla_X (SX_j) - S(\nabla_X X_j) = \nabla_X (\rho X_j + Z) - S(\nabla_X X_j) \\ &= (X_i \rho) X_j + \rho (\nabla_X X_j) + \nabla_X Z - S(\nabla_X X_j) \\ &\equiv (X_i \rho) X_j \mod \operatorname{Ker} h,\end{aligned}$$

where $Z \in \operatorname{Ker} h$ and $\nabla_X Z \in \operatorname{Ker} h$, since Ker h is parallel. Thus by Codazzi's equation, we have

(21) $\qquad (X_i \rho) X_j = (X_j \rho) X_i \mod \operatorname{Ker} h.$

This holds for all i and j. If $1 \le i \le r$, then, using $r \ge 2$, take $j \ne i$, $1 \le j \le r$. Then (21) implies that $X_i \rho = 0$. If $r+1 \le i \le n$, then take j, $1 \le j \le r$. Then

(21) implies $X_i \rho = 0$. We have thus shown that $X\rho = 0$ for every $X \in T_x(M)$.

Remark. In case rank $h = 1$ and $\{X_1, \ldots, X_n\}$ is a basis in $T_x(M)$, where $\{X_2, \ldots, X_n\}$ is a basis of Ker h, we cannot conclude $X_1\rho = 0$ (there is an example showing that ρ is not a constant).

Lemma 3. *Under the assumptions of Lemma 2 define for each $x \in M^n$ a bilinear symmetric function g in $T_{f(x)}(\mathbb{R}^{n+1})$ as follows:*

$$g(f_*X, f_*Y) = h(X,Y) \quad \text{for } X, Y \in T_x(M^n)$$

$$(22) \quad g(f_*X, \xi) = 0 \qquad \text{for } X \in T_x(M^n)$$

$$g(\xi, \xi) = \rho.$$

Then g is parallel relative to the connection $\tilde{\nabla}$ in $\mathbb{R}^{n+1}$.

Proof. We want to show that

$$X\, g(U,V) = g(\tilde{\nabla}_X U, V) + g(U, \tilde{\nabla}_X V)$$

for all vector fields U and V along f and for all $X \in T_x(M^n)$.

1) If $U = f_*(Y)$, $V = f_*(Z)$ for vector fields Y and Z on M^n, then the above identity follows from $\nabla_X h = 0$ and $g(\xi, U) = g(\xi, V) = 0$.

2) If $U = f_*(Y)$, and $V = \xi$, then

$$X\, g(U,\xi) = 0, \quad g(\tilde{\nabla}_X U, \xi) = g(f_*(\nabla_X Y) + h(X,Y)\xi, \xi) = h(X,Y)\rho$$

and

$$g(U, \tilde{\nabla}_X \xi) = g(U, -f_*(SX)) = g(U, -\rho f_*(X) + f_*(Z)) \quad (\text{where } Z \in \text{Ker } h)$$

$$= -\rho\, h(Y,X) + h(Y,Z) = -\rho\, h(Y,X).$$

So the above identity holds.

3) If $U = V = \xi$, then we have $X\, g(\xi,\xi) = X\rho = 0$ as well as $g(\tilde{\nabla}_X \xi, \xi) =$

$g(-f_*(SX)),\ \xi) = 0.$

Remark. At each $x \in M^n$,

$$\mathrm{Ker}\, g = f_*(\mathrm{Ker}\, h) \ \text{ if } \ \rho \neq 0 \ \text{ and } \ \mathrm{Ker}\, g = f_*(\mathrm{Ker}\, h) + \mathrm{span}\,(\xi) \ \text{ if } \rho = 0.$$

Lemma 4. We identify $f(x)$, $x \in M^n$, with the position vector and simply write it as x. Define a function φ on M^n by $\varphi(x) = g(x,x)/2$ and a 1-form λ on $T_{f(x)}(\mathbb{R}^{n+1})$ for $x \in M^n$ by

(23) $\quad \lambda(f_*X) = g\,(X,x) \quad$ for $\quad X \in T_x(M^n)$

$\qquad \lambda(\xi) = g(x,x) + 1.$

Then λ is parallel relative to $\tilde{\nabla}$ in $\mathbb{R}^{n+1}$.

Proof. We have

$$(\tilde{\nabla}_X\lambda)(\ f_*\, Y)$$

$$= \ X(\lambda(\ f_*\ Y)) - \lambda(\tilde{\nabla}_X\, f_*\, Y) = Xg(f_*(Y),x) - \lambda(f_*(\nabla_X Y) + h(X,Y)\xi)$$

$$= g(\tilde{\nabla}_X\, f_*\, Y,x) + g(f_*\, Y, f_*\, X) - g(\ f_*\nabla_X Y,x) - h(X,Y)(g(\xi,x) + 1) = 0$$

and

$$(\tilde{\nabla}_X\lambda)(\xi) = X(\lambda(\xi)) - \lambda(\tilde{\nabla}_X\xi) = X(g(\xi,x) + 1) - \lambda(\tilde{\nabla}_X\xi)$$

$$= g(\tilde{\nabla}_X\xi,x) + g(\xi,x) - \lambda(\tilde{\nabla}_X\xi) = 0.$$

Thus λ is parallel in $\mathbb{R}^{n+1}$.

We are now in position to prove Theorem 7.

Proof of Theorem 7. First we note that the parallel 1-form λ in Lemma 4 is nothing but a covector in the dual vector space $\mathbb{R}_{n+1}$. Thus there is an affine function ψ on $\mathbb{R}^{n+1}$ such that $d\psi = \lambda$. Moreover we may assume that $\psi(x_0) = \varphi(x_0)$ for some point x_0. Now obviously $d\varphi = d\psi$ on M^n. Hence $\varphi = \psi$ on M^n. This means that $f(M^n)$ lies in a quadric.

Remark. For any affine coordinate system in $\mathbb{R}^{n+1}$ we may write

$$\varphi(x) = \Sigma_{i,j=1}\ a_{ij}\, x^i x^j\ , \qquad \psi(x) = 2\ \Sigma_{i=1}\ a_i\, x^i + b.$$

Suppose rank $g = r+1$. Then we may retake an affine coordinate system so that $\varphi(x) = \Sigma_{i,j=1}\ a_{ij}\, x^i x^j$, where the matrix $[a_{ij}]$ is nondegenerate. We can further simplify the quadratic equation $\varphi(x) = \psi(x)$ for $f(M^n)$ into

$$\Sigma_{i=i}\, \varepsilon_i\, (x^i)^2 = \pm 1 \quad \text{or} \quad x^{r+2} = \Sigma_{i=1}\, \varepsilon_i\, (x^i)^2\ , \quad \text{where } \varepsilon_i = \pm 1$$

by a change of affine coordinate system.

Before we prove Theorem 9, we need two lemmas.

Lemma 5. *Let* $f: (M^n, \nabla) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ *be a nondegenerate affine immersion with a normal vector field* ξ *and second fundamental form* h. *Then we can change* ξ *to* $\hat{\xi} = \xi/\lambda$ *for some* $\lambda: M^n \to \mathbb{R}^+$ *so that the volume element* $\hat{\omega}$ *for the second fundamental form* $\hat{h}$ *for* $\hat{\xi}$ *coincides with the volume elment* ω *induced by* ξ *from the standard volume element* $\tilde{\omega}$ *in* $\mathbb{R}^{n+1}$.

Proof. Assume that the volume element ω_h for h is equal to $\mu\,\omega$, where $\mu: M^n \to \mathbb{R}^+$. Choose $\lambda = \mu^{-n/2}$. Then $\hat{h} = \lambda\, h$ implies that $\hat{\omega} = \lambda^{n/2}\omega_h = \mu^{-1}\omega_h = \omega$.

Lemma 6. *Let* $f: (M^n, \nabla) \to (\mathbb{R}^{n+1}, \tilde{\nabla})$ *be a nondegenerate affine immersion such that* $\omega = \omega_h$. *If the cubic form* C *vanishes, then* $\tau = 0$.

Proof. We recall from [7]

$$C(X,Y,Z) = (\nabla_X h)(Y,Z) + \tau(X)\, h(Y,Z) \quad \text{and} \quad \nabla_X \omega = \tau(X)\omega.$$

If $\hat{\nabla}$ denotes the Levi-Civita connection for h and if $K_X = \nabla_X - \hat{\nabla}_X$, then

$$(\nabla_X h)(Y,Z) = -\, h(K_X Y, Z) - h(Y, K_X Z),$$

because $\hat{\nabla}_X h = 0$. Using $C = 0$, we get

$$(24) \qquad \tau(X) h(Y,Z) = h(K_X Y, Z) + h(Y, K_X Z).$$

Take an orthonormal basis $\{X_1, \ldots, X_n\}$ for h, where $h(X_i, X_i) = \varepsilon_i = \pm 1$ and $h(X_i, X_j) = 0$ for $i \neq j$. Taking $Y = X_i$, $Z = \varepsilon_i X_i$ in (24) and summing over i, we get $n\,\tau(X) = 2$ trace K_X.

On the other hand, applying $\nabla_X = \hat{\nabla}_X + K_X$ on $\omega = \omega_h$ we obtain

$$\tau(X)\omega = \nabla_X\omega = K_X\omega_h = -(\text{trace } K_X)\omega_h = -(\text{trace } K_X)\omega,$$

that is, $\tau(X) = -$ trace K_X. Comparing this with the previous relation, we conclude that trace $K_X = 0$ and $\tau = 0$.

Now we can prove Theorem 8.

<u>Proof of Theorem</u> 8. Choose a normal vector field ξ and consider the given immersion f as an affine immersion $(M^n, \nabla) \rightarrow (\mathbb{R}^{n+1}, \tilde{\nabla})$. By assumption, $h \mid C$, that is, we have (13). By Lemma 4 we may change ξ to another normal vector field $\hat{\xi}$ and the corresponding cubic form as in (18) in Lemma 1. Since h is nondegenerate, we can choose U so that $\eta = -\rho$ and achieve $\hat{C} = 0$. Moreover, by choosing λ suitably as in Lemma 5, we can also make $\hat{\omega}$, volume element for $\hat{h}$, coincide with ω. Now we can apply Lemma 6 and conclude $\hat{\tau} = 0$. By Theorem 7 we conclude that $f(M^n)$ is a qudric.

The converse is obvious from the following well known fact. If $f(M^n)$ is a nondegenerate quadric in $\mathbb{R}^{n+1}$, then with a suitable choice of affine coordinate system $f(M^n)$ is expressed either by

$$x^{n+1} = \Sigma_{i,j=1}\, a_{ij} x^i x^j, \quad \text{where } [a_{ij}] \text{ is a nonsingular matrix}$$

or by

$$\Sigma_{i=1}\, \varepsilon_i x_i^2 = 1, \quad \text{where} \quad \varepsilon_i = \pm 1.$$

In the first case, $\xi = (0, \ldots, 0, 1)$ is a normal vector field (called the affine normal in the classical theory, see [7], Proposition 6) for which $\tau = 0$,

$h(\partial/\partial x^i, \partial/\partial x^j) = a_{ij}$, and the induced affine connection ∇ on $M^n = \mathbb{R}^n$ (with affine coordinates $x^1, \ldots, x^n$) is flat. Thus $C = \nabla h = 0$. In the second case, by considering an appropriate flat pseudo-euclidean metric on $\mathbb{R}^{n+1}$, the affine normal ξ coincides with the metric normal. We have $\tau = 0$; h coincides with the usual second fundamental form in the metric sense and $\nabla h = 0$. Thus $C = 0$ again. □

5. Proofs of Theorems 10 and 11

We now give a proof of Theorem 10. Let Ω be the set of points x in M^n such that Ker h has constant dimension in a neighborhood of x. Then Ω is an open subset. It is dense for the following reason. Let x_0 be an arbitrary point in M^n and let U be any neighborhood of x_0. Let $x \in U$ be a point where dim Ker h attains the minimum on U. Then rank h_x is equal to the maximum of rank h on U and rank h_y = rank h_x and thus dim Ker h_y = dim Ker h_x for all points y in a neighborhood V of x. Thus $x \in \Omega$, showing that Ω is dense. For Theorem 10 it is sufficient to show that $f(M^n)$ is contained in a quadric around each point x of Ω.

Let $x_0 \in \Omega$. In a certain neighborhood of x_0, $x \to$ Ker h_x defines a distribution of dimension, say, $n - r$. We show that it is totally geodesic and integrable. Let X and Y be vector fields belonging to Ker h. For any tangent vector X we have by assumption (13)

$$X\, h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = \rho(X)h(Y,Z) + \rho(Y)h(Z,X) + \rho(Z)h(X,Y).$$

Since $X, Y \in$ Ker h, this equation is reduced to $h(\nabla_X Y, Z) = 0$. Since Z is arbitrary, it follows that $\nabla_X Y \in$ Ker h. Thus $[X,Y] = \nabla_X Y - \nabla_Y X \in$ Ker h.

Now let H an $(r+1)$-dimensional affine subspace in $\mathbb{R}^{n+1}$ through $f(x_0)$ and transversal to $f(L)$, where L is the leaf of the distribution Ker h through x_0. Then near x_0 the foliation $\tilde{F}$ of $\mathbb{R}^{n+1}$ by $(r+1)$-dimensional affine subspaces parallel to H gives rise to a foliation F of M^n by r-dimensional submanifolds.

Choose a convex neighborhood V of $f(x_0)$ such that the foliations F and Ker h are defined on the component U of $f^{-1}(V)$ that contains x_0. Set $N = f^{-1}(H) \cap U$. Then $f_N : N \to H$ is a nondegenerate hypersurface in H.

We choose a new normal vector field ξ for f_N that lies in H and translate it parallelly along each leaf in $\mathbb{R}^{n+1}$, thus getting a normal vector field ξ for $f: U \to \mathbb{R}^{n+1}$. For vector fields X and Y tangent to N the equation $\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi$ shows that $\nabla_X Y$ is tangent to N, because $\tilde{\nabla}_X Y$ and ξ lie on H. This means that N is totally geodesic in U (relative to the affine connection induced by f with the new normal vector field ξ). The same equation also shows that the second fundamental form h_N for f_N is simply the restriction of h for f and is nondegenerate. The affine immersion f_N also has the property that its cubic form C_N is divisible by h_N.

Now just as we have done to reduce the proof of Theorem 8 to Theorem 7, we take once more a new normal vector field to f_N such that $C = 0$, $\tau = 0$ and $\nabla h_N = 0$ and extend it to a normal vector field ξ for f by parallel translation in $\mathbb{R}^{n+1}$. Relative to this ξ, f still has the property that C is divisble by h, that is, $C(X,Y,Z) = \rho(X)h(Y,Z) + \rho(Y)h(Z,X) + \rho(Z)h(X,Y)$ for some 1-form ρ. We have $\rho(X) = 0$ for $X \in TN$.

The rest of the proof proceeds as follows. We shall show that

(i) N is umbilical in $\mathbb{R}^{n+1}$;

(ii) $(\nabla_X \rho)(Z) = 0$ for every $X \in TN$, $Z \in \operatorname{Ker} h$.

(iii) If $\rho \neq 0$, the images $f(L)$ of all leaves L meet in a certain affine $(n-r-1)$-dimensional subspace, say K, so that $f(M^n)$ lies on the cone with vertex K and base $f(N) \subset H$;

(iv) If $\rho = 0$, then all $f(L)$'s are parallel in $\mathbb{R}^{n+1}$ and $f(M^n)$ is a cylinder.

We now prove these statements.

(i) Since N satisfies $\tau = 0$ and $\nabla h_N = 0$, we know from Lemma 2 of Section 4 that $S = A_\xi$ is a constant multiple of I. We show that $A_X = \rho(X)\, I$ for every $X \in \operatorname{Ker} h$ (note that $\operatorname{Ker} h_x$ and ξ_x span the transversal space for N in $\mathbb{R}^{n+1}$). If $Y \in TN$, then extending X to a vector field in $\operatorname{Ker} h$, we see that the equation (13) reduces to $h(\rho(X)Y, Z) = -\, h(\nabla_Y X, Z)$. Since this holds for every $Z \in TN$ at every point of N, we see that $A_X = \rho(X)\, I$.

(ii) From $A_X = \rho(X) I$ on TN for every $X \in \operatorname{Ker} h$, and from Codazzi's equation for the submanifold N in $\mathbb{R}^{n+1}$ we get

$$(\nabla_X \rho)(Z)\, Y = (\nabla_Y \rho)(Z)\, X \quad \text{for } X, Y \in TN \text{ and } Z \in \operatorname{Ker} h.$$

Since $\dim N = \operatorname{rank} h \geq 2$, we may take X, Y to be linearly independent. Thus $(\nabla_X \rho)(Z) = 0$ for every $X \in TN$ and $Z \in \operatorname{Ker} h$.

(iii) We first show that $x \in N \rightarrow f_*(\operatorname{Ker} \rho_x \cap \operatorname{Ker} h_x)$ is parallel in $\mathbb{R}^{n+1}$ along N. Let $Z \in \operatorname{Ker} \rho_x \cap \operatorname{Ker} h_x$ be a vector field and let $X \in TN$. Then $(\nabla_X \rho)(Z) = 0$ implies that $X \rho(Z) - \rho(\nabla_X Z) = -\, \rho(\nabla_X Z) = 0$. Then $\tilde{\nabla}_X Z = \nabla_X Z \in \operatorname{Ker} \rho_x$. On the other hand, (13) implies

$$-\, h(Y, \nabla_X Z) = \rho(Z) h(X, Y) = 0 \quad \text{for every } Y \in TN$$

so that $\nabla_X Z \in \mathrm{Ker}\ h$. Thus $\tilde{\nabla}_X Z = \nabla_X Z \in \mathrm{Ker}\ h$. It follows that $\tilde{\nabla}_X Z \in \mathrm{Ker}\ \rho \cap \mathrm{Ker}\ h$. We have shown that $x \to f_*(\mathrm{Ker}\ \rho \cap \mathrm{Ker}\ h)$ is parallel in $\mathbb{R}^{n+1}$ so that these subspaces are all parallel, say, to a subspace W.

(iii) Assume $\rho \neq 0$ on N. Let X be a vector field $\neq 0$ on N belonging to Ker h at every point and consider

$$x \in N \to y = x + X/\rho(X).$$

For every $Y \in TN$, we have by a similar computation to that in Theorem 2 that $\rho(\tilde{\nabla}_X Y) = 0$. Also we show that

$$\tilde{\nabla}_Y y = -[(\nabla^{\perp}{}_Y X)/\rho(X)^2]X + (\nabla^{\perp}{}_Y X)/\rho(X)$$

is in Ker h. Here, of course, $\nabla^{\perp}{}_Y X$ is the Ker h-component of $\tilde{\nabla}_Y X$ for the submanifold N. But $\tilde{\nabla}_Y X = \nabla_Y X$ because $h(Y,X) = 0$. We know from Lemma 2 applicable to N that $\nabla_Y X \in \mathrm{Ker}\ h$. So $\tilde{\nabla}_Y X \in \mathrm{Ker}\ h$. Thus $\tilde{\nabla}_Y y \in \mathrm{Ker}\ \rho \cap \mathrm{Ker}\ h$.

Let x_0 be the point we started with and let $y_0 = x_0 + X/\rho(X)$ for any nonzero vector field X on N in Ker h. Then all points $y = x + X/\rho(X)$ lie in the affine subspace through y_0 and parallel to W. If X is replaced by any vector field Y in Ker h, this affine subspace does not change because $X/\rho(X) - Y/\rho(Y) \in \mathrm{Ker}\ \rho \cap \mathrm{Ker}\ h$.

(iv) Suppose $\rho = 0$ on N. Then $x \in N \to f_*(\mathrm{Ker}\ h_x)$ is parallel in $\mathbb{R}^{n+1}$, because if X is a vector field belonging to Ker h on N and $Y \in TN$, then $\tilde{\nabla}_Y X = \nabla_Y X$ Ker h as in Lemma 2 again. Thus there is an (n-r)-dimensional affine subspace to which all f(L)'s are parallel. Thus $f(M^n)$ is contained in the cylinder $f(N) \times W \subset \mathbb{R}^{n+1}$. We have completed the proof of Theorem 10.

Finally, Theorem 11 follows Proposition 4, its corollary and Theorem 10.

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