DEFORMATION BOUNDARY RIGIDITY FOR SGM-MANIFOLDS AND INTEGRAL GEOMETRY PROBLEMS FOR NONTRAPPING MANIFOLDS

NURLAN S. DAIRBEKOV

Abstract. We consider the deformation boundary rigidity problem for SGM-manifolds and corresponding integral geometry problems for nontrapping manifolds. We prove that an SGM-manifold with a certain upper bound of curvature is deformation boundary rigid. For nontrapping manifolds with a similar curvature bound, we prove that a smooth symmetric tensor field of rank m, $m \geq 1$, on such a manifold, integrating to zero over geodesics with endpoints on the boundary, is potential. For functions and 1-forms, this result is shown to be true for arbitrary nontrapping manifolds having no conjugate points.

1. Introduction

Let (M, g) be a (smooth) compact Riemannian manifold with boundary. The *boundary rigidity problem* asks whether the Riemannian metric g on M is determined by the boundary distance function d_g uniquely up to isometries that are the identity on the boundary. Here $d_g : \partial M \times \partial M \to \mathbb{R}$, with $d_g(x, y)$ the distance in M between x and y. If so, (M, g) is called *boundary rigid*.

Not all compact manifolds with boundary are boundary rigid. Corresponding examples can be found in [1]. There are two natural classes of Riemannian manifolds to consider, the "simple" manifolds and the SGM-manifolds.

(M, g) is said to be *simple* if the boundary ∂M is strictly convex and there is a unique geodesic between any pair of boundary points.

The *SGM condition* (formulated in terms of d_g only) is harder to define. A precise definition is given in [1], but loosely speaking it means that (M, g) is nontrapping and all nongrazing geodesic segments strongly minimize.

(M, g) is called *nontrapping* if every geodesic ray hits the boundary ∂M and there is no complete geodesic in M.

By a *nongrazing geodesic* we mean a segment of a geodesic which lies in the interior of M except possibly the endpoints. A segment is said to strongly minimize if its length equals the distance between the endpoints and it is a unique such path.

Most papers deal with the case of "simple" manifolds, and there are a few papers devoted to the SGM-manifolds.

In this paper we consider the linearized problem for SGM-manifolds. Assume (M, g) to be an SGM-manifold. Let g_τ, $-\varepsilon < \tau < \varepsilon$, be a deformation of g preserving the boundary distances, i.e., g_τ depends smoothly on τ, $g_0 = g$, and $d_{g_\tau} = d_g$ for all τ. Since the validity of the condition SGM depends only on the boundary distance function, all (M, g_τ) will be SGM. Furthermore, they are equivalent as geodesic lenses (see [1]). Therefore, a straightforward calculation shows (cf. [4]) that for any nongrazing g_0-geodesic $\gamma : [a, b] \to M$ with endpoints on the boundary, $\gamma(a) \in \partial M$, $\gamma(b) \in \partial M$, we have

$$\int_a^b f_{ij}(\gamma(t)\dot\gamma^i(t)\dot\gamma^j(t)\,dt = 0, \tag{1.1}$$

where $f = (f_{ij}) = \frac{dg_\tau}{d\tau}\big|_{\tau=0}$. So f *integrates to zero over geodesics with endpoints on the boundary.*

Let $Z(S_2M)$ be the space of all smooth symmetric 2-fields on M integrating to zero over geodesics with endpoints on the boundary. This space is nonzero and contains the subspace of *potential fields* $f = \sigma\nabla p$, where $\sigma\nabla p$ is the symmetrization of the covariant derivative of a smooth covector field p vanishing on the boundary, $p|_{\partial M} = 0$.

(M, g) is called *deformation boundary rigid* [6] if each field $f \in Z(S_2M)$ appears in this way from some smooth covector field p on M which vanishes on ∂M. (See [4] for a detailed discussion on the relationship between the boundary rigidity problem and the deformation boundary rigidity problem.)

Introduce one more notation. Let UM denote the unit tangent bundle of M. Given $x \in UM$ and a 2-plane $\sigma \subset T_xM$, denote by $K(x, \sigma)$ the sectional curvature of σ at x. For $v \in U_xM$, put

$$K(v) = \sup_{\sigma,\, v\in\sigma} K(x, \sigma), \quad K^+(v) = \max\{K(v), 0\}.$$

Define the following characteristic:

$$k^+(M, g) = \sup_\gamma \int_a^b tK^+(\dot\gamma(t))\,dt,$$

where γ, $\gamma : [a, b] \to M$, ranges in the set of all unit speed geodesics in M.

We prove the following theorem.

Theorem 1.1. *An n-dimensional SGM-manifold (M, g), $n \geq 2$, is deformation boundary rigid whenever $k^+(M, g) < \frac{n+3}{2(n+2)}$.*

This theorem follows as a special case ($m = 2$) from a corollary to our general theorem in which we deal with tensor fields of an arbitrary rank m on nontrapping manifolds.

As before, let (M, g) be a compact Riemannian manifold with boundary. For $m = 0, 1, 2, \dots$, let $C^\infty(S_mM)$ denote the space of symmetric covariant tensor fields on M of rank m (for $m = 0$, we abbreviate this to $C^\infty(M)$). Given a field $f \in C^\infty(S_mM)$ and m tangent vectors $v_j = (x, \xi_j) \in TM$, $x \in M$, $\xi_j \in T_xM$, $j = 1, \dots, m$, we write

$$\langle f; v_1, \dots, v_m\rangle = f_{i_1\dots i_m}(x)\xi_1^{i_1} \dots \xi_m^{i_m},$$

and also introduce the notation

$$\langle f, v^m\rangle = \langle f(x), \xi^m\rangle = f_{i_1\dots i_m}(x)\xi^{i_1} \dots \xi^{i_m}$$

for $v = (x, \xi) \in TM$.

We say that f *integrates to zero over geodesics with endpoints on the boundary*, and write $f \in Z(S_mM)$, if the equality

$$\int_a^b \langle f, \dot\gamma^m(t)\rangle\,dt = 0 \tag{1.2}$$

holds for all nongrazing geodesics $\gamma : [a, b] \to M$ with $\gamma(a) \in \partial M$ and $\gamma(b) \in \partial M$.

Given $m \geq 1$ and $f \in C^\infty(S_m M)$, we say that f is *potential*, and write $f \in P(S_m M)$, if $f = dp$ for some field $p \in C^\infty(S_{m-1}M)$ vanishing on the boundary, $p|_{\partial M} = 0$. Here $d = \sigma\nabla$ is the *inner derivative*, with ∇ denoting the covariant derivative with respect to g and σ the symmetrization (see details in [4]).

If $f = dp$ then the integrand in (1.2) is a total derivative with respect to t, $\langle f, \dot\gamma^m(t)\rangle = \frac{d}{dt}\langle p, \dot\gamma^{m-1}(t)\rangle$. Therefore, for potential fields we always have (1.2), i.e,

$$P(S_m M) \subset Z(S_m M). \tag{1.3}$$

The problem we address here is to indicate conditions on (M, g) and m in order to have equality in (1.3).

This problem was earlier considered in the case when M is a simple manifold (see [4, 2] and references therein) or, more generally, when M lies in the interior of a simple manifold [5].

To formulate our result, we define the number

$$s_{n,m} = \frac{m(m+n)}{n+2m-1} \tag{1.4}$$

for $m \geq 1$. Note that $s_{n,1} = 1$. Following [2], we say that (M, g) *has no s-conjugate points* if the equation

$$D_t^2 X + sR_{\dot\gamma}X = 0 \tag{1.5}$$

has no pair of conjugate points on any geodesic γ in M. Here D_t stands for the covariant derivative along γ and $R_{\dot\gamma}$ is the curvature operator, $R_{\dot\gamma}X = R(\dot\gamma, X)\dot\gamma$.

As remarked by Sharafutdinov [4, p. 120], it is not very hard to justify the following:

Remark 1.2. If $k^+(M, g) < 1/s$, then (M, g) has no s-conjugate points.

Our main result in this paper is the following theorem.

Theorem 1.3. *Let n, m be integers such that $n \geq 2$, $m \geq 1$. Suppose that (M, g) is a nontrapping n-dimensional manifold having no $s_{n,m}$-conjugate points. Then*

$$P(S_m M) = Z(S_m M).$$

In the case of simple manifolds, this result was established in [2].

For $s = 1$, equation (1.5) is nothing but the Jacobi equation. So (M, g) has no 1-conjugate points if and only if it has no conjugate points in the ordinary sense. Therefore, for $m = 1$ Theorem 1.3 yields the following corollary.

Corollary 1.4. *A smooth 1-form ω on a nontrapping manifold M without conjugate points, integrating to zero over geodesics with endpoints on the boundary, is exact, $\omega = d\phi$, with ϕ a smooth function on M vanishing on ∂M, $\phi|_{\partial M} = 0$.*

In view of Remark 1.2, we also have the following corollary to Theorem 1.3.

Corollary 1.5. *Let n, m be integers such that $n \geq 2$, $m \geq 1$. If (M, g) is a nontrapping n-dimensional manifold such that $k^+(M, g) < \frac{n+2m-1}{m(m+n)}$, then $P(S_m M) = Z(S_m M)$.*

Theorem 1.1 follows now from Corollary 1.5 with $m = 2$.

For $m = 0$, i.e., for functions, we establish the following:

Theorem 1.6. *A smooth function on a nontrapping manifold without conjugate points, integrating to zero over geodesics with endpoints on the boundary, is itself zero.*

The paper is further organized as follows. In Section 2 we perform a reduction of the proofs of Theorems 1.3 and 1.6 to the kinetic equation. This reduction is a crucial step in our argument and it became possible due to Sharafutdinov's observations of [5] which we formulate below in Lemmas 2.1 and 2.2. Afterwards, the matter reduces to the analysis of the kinetic equation by the scheme of [3, 4, 2]. The main tools in this analysis are Pestov's identity for semibasic tensor fields and Sharafutdinov's decomposition of tensor fields into potential and solenoidal parts. Section 3 is devoted to Pestov's identity and Section 4, to Sharafutdinov's decomposition of tensor fields. In Section 5 we complete the proof of Theorem 1.3, as well as prove Theorem 1.6. The important idea, used in this section, of introducing the "tangent" vertical derivative $\overset{v}{\nabla}{}^{||}$ and utilizing the extremal properties of Jacobi fields on manifolds having no $s_{n,m}$-conjugate points is borrowed from [2].

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2. Reduction to the kinetic equation

Lemma 2.1. *Let (M, g) be a compact Riemannian manifold with boundary such that every geodesic ray in M hits the boundary. If $f \in Z(S_mM)$, then*

$$\langle f, v^m \rangle = 0 \quad \textit{for all} \quad v \in T(\partial M). \tag{2.1}$$

The proof of this lemma is essentially given in [5, Lemma 2.3], although the assumptions there are stronger. Since the proof of Lemma 2.1 repeats that of Lemma 2.3 in [5] almost word for word, we omit it.

For a function $f \in Z(S_0M)$, the lemma yields $f|_{\partial M} = 0$. The next lemma shows that, for a tensor field $f \in Z(S_mM)$ of higher rank, a certain correction can be added to f to make it vanish on ∂M.

Lemma 2.2 ([5, Lemma 2.2])**.** *Let (M, g) be a compact Riemannian manifold with boundary and let $f \in C^\infty(S_mM)$ be a smooth symmetric tensor field. There exists a field $p \in C^\infty(S_{m-1}M)$, vanishing on the boundary, $p|_{\partial M} = 0$, and a neighborhood $W \subset M$ of ∂M, such that the field $\tilde{f} = f - dp$ has the following property: For every geodesic $\gamma : [a, b] \to W$ starting from the boundary at the right angle to ∂M, $\gamma(a) \in \partial M$, $\dot{\gamma}(a) \perp \partial M$, the equality*

$$\langle \tilde{f}; \dot{\gamma}(t), v_1, \dots, v_{m-1} \rangle = 0$$

holds for all $t \in [a, b]$ and all $v_i \in T_{\gamma(t)}M$ $(1 \le i \le m - 1)$.

In particular,

$$\langle \tilde{f}; \nu, v_1, \dots, v_{m-1} \rangle = 0 \tag{2.2}$$

for every $x \in \partial M$, all $v_1, \dots, v_{m-1} \in T_xM$, and a normal ν to ∂M at x.

We will reduce the proof of Theorems 1.3 and 1.6 to a certain assertion about the kinetic equation. To this end, we fix a nontrapping manifold (M, g) and a field $f \in Z(S_mM)$, $m \ge 0$.

In view of (2.1) and (2.2), we may henceforth assume that the field f itself vanishes on the boundary:

$$f|_{\partial M} = 0. \tag{2.3}$$

Further, losing no generality we assume that (M, g) is a smooth submanifold of an n-dimensional compact smooth Riemannian manifold N without boundary (for example, by taking N to be the double of M and extending the Riemannian metric from M to N smoothly). We preserve the notation g for the Riemannian metric on N.

Now, we extend f from M to all of N by the zero field, denoting the resulting field by f again. The boundary condition (2.3) guarantees that the so-obtained field f is continuous on the whole manifold N and belongs to the Sobolev space $H^1(S_mN)$ of tensor fields with square integrable first-order derivatives.

Given $v \in UN$, let γ_v be the complete geodesic in N issuing from v, $\dot\gamma_v(0) = v$. Since (M, g) is nontrapping, there is no complete geodesic which would be contained entirely in M. Therefore, for any v there is a number l_v such that $\gamma_v(l_v) \notin M$. We define a function $\varphi : UN \to \mathbb{R}$ by

$$\varphi(v) = \int_{l_v}^{0} \langle f, \dot\gamma_v^m(t)\rangle \, dt. \tag{2.4}$$

Observe that the value of $\varphi(v)$ is independent of the choice of l_v subject to the above requirement. Indeed, the difference between the two integrals (2.4) with different values of l_v is the sum of integrals over nongrazing geodesic segments in M with endpoints on the boundary ∂M of M and integrals over geodesic segments disjoint from the interior of M. The integrals over the segments of the first type vanish by condition (1.2), and those over the segments of the second type vanish since f vanishes on ∂M and in the exterior of M.

Call a point $v \in UM$ *regular* if the geodesic γ_v intersects ∂M transversally from either side and the open segment of γ_v between the basepoint of v and the point of intersection lies entirely in the interior of M. Denote by $RM \subset UM$ the set of all regular points. It is clear that RM is open in UM and has full measure in UM.

Lemma 2.3. *The function $\varphi : UN \to \mathbb{R}$ has the following properties:*

(1) $\varphi|_{U(N\setminus M)} = 0$.
(2) $\varphi \in H^1(UN) \cap C(UN) \cap C^\infty(RM)$.
(3) *φ is C^1 smooth along the lifts of geodesics to UN and satisfies the kinetic equation on UN:*

$$H\varphi(v) = \langle f, v^m\rangle, \tag{2.5}$$

where H is the geodesic spray.

Proof. Item (1) is immediate from (2.4), since for $v \in U(N \setminus M)$ we can choose l_v so that the geodesic γ_v will never enter M for the values of the parameter t between l_v and 0, but then the integrand in (2.4) vanishes since f vanishes in the exterior of M.

Prove item (2). Take any point $v \in UN$ and the corresponding geodesic γ_v. Since $\gamma_v(l_v) \notin M$, we can take a small hypersurface Φ in N transversally intersecting γ_v at the point $\gamma_v(l_v)$ and disjoint from M. Then there is a neighborhood of v in UN such that for every v' in this neighborhood the geodesic $\gamma_{v'}$ will hit Φ at the time $l'_{v'}$ smoothly depending on v' and such that $l'_v = l_v$. For these v' we can therefore take $l'_{v'}$ as $l_{v'}$ while defining φ by (2.4). So locally the lower limit of integration in (2.4) can be chosen to be a smooth function. Since f is continuous and lies in the Sobolev space $H^1(S_mN)$, this observation allows us to prove routinely that φ is continuous and belongs to $H^1(UN)$.

If $v \in RM$, then there is a neighborhood of v in UM such that for all v' in this neighborhood the geodesics $\gamma_{v'}$ intersect ∂M transversally from the same side as γ_v so that the interior of the segment between the basepoint of v' and the point of intersection lies in the interior of M for each of these geodesics, moreover the parameter values $\tau_{v'}$ of the intersection points are smooth functions of v'. Obviously, we can calculate the values of φ at these points v', taking the lower limits in (2.4) to be $\tau_{v'}$. Since f is smooth in M, we conclude that φ is smooth in the chosen neighborhood of v and therefore smooth on RM.

To prove item (3), consider the geodesic flow $G^t : UN \to UN$, $G^t(v) = \dot\gamma_v(t)$ for $v \in UN$. The geodesic spray H is the vector field on UN generating the flow G^t. In coordinates, for $(x,\xi) \in UN$, $x \in N$, $\xi \in U_xN$,

$$H = \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}\xi^j\xi^k \frac{\partial}{\partial \xi^i}, \tag{2.6}$$

where Γ^i_{jk} are the Christoffel symbols.

For $\psi \in C^\infty(UN)$ and $v \in UN$, we have

$$H\psi(v) = \frac{d}{dt}\psi(G^t(v))\Big|_{t=0} = \frac{d}{dt}\psi(\dot\gamma_v(t))\big|_{t=0}. \tag{2.7}$$

The vector field H on TN is tangent to the submanifold UN of TN. Therefore, it can (and will) be considered also as a vector field on UN. In this case, it is the generating vector field of the geodesic flow on UN, $G^t : UN \to UN$.

By way of approximation, it is easy to see that the rule (2.7) works equally well for the functions $\psi \in H^1(UN) \cap C(UN)$. We apply it to our function φ.

Take $v \in UN$. For $s \in \mathbb{R}$, put $v_s = G^s(v) = \dot\gamma_v(s)$. Then $\gamma_{v_s}(t) = \gamma_v(t+s)$ and we can take $l_{v_s} = l_v - s$. Then by (2.4)

$$\varphi(G^s(v)) = \int_{l_{v_s}}^0 \langle f, \dot\gamma^m_{v_s}(t)\rangle\, dt = \int_{l_v}^s \langle f, \dot\gamma^m_v(t)\rangle\, dt.$$

Taking the derivative at $s = 0$ gives (2.5) and finishes the proof of the lemma. □

The further argument in the proofs of Theorems 1.3 and 1.6 consists in proving that the validity of the kinetic equation (2.5), with φ having the properties listed in items (1) and (2) of Lemma 2.3, forces the membership $f \in P(S_mM)$ if $m \geq 1$ and forces f to vanish if $m = 0$.

3. Pestov's identity

We first recall the notion of a semibasic tensor field and its vertical and horizontal derivatives [4, 2].

Let τ^r_sN denote the bundle of tensors of degree (r,s) over N. As usual, the presence of the Riemannian metric allows us to identify all bundles τ^r_sN of the same total degree $r+s$ by fixing a certain rule for raising and lowering indices. Therefore, we will not distinguish between co- and contravariant tensors, preferring to speak of co- and contravariant components of the same tensor.

Let $\pi : TN \to N$ be the natural projection. The bundle $\beta^r_sN = \pi^*(\tau^r_sN)$ over TN is called the *bundle of semibasic tensor fields of degree* (r,s) and its sections are called *semibasic tensor fields*. For such a field u, the coordinate representation

$$u = \big(u^{i_1\dots i_r}_{j_1\dots j_s}\big)(x,\xi)$$

holds in the domain of a local coordinate system on TN. Under a change of a local coordinate system, the components of a semibasic tensor field are transformed by the same formula as those of an ordinary tensor field on N. Like the ordinary tensors on N, we can identify all bundles $\beta^r_s N$ having the same total degree $r+s$. This clearly allows us to identify the sections of these bundles as well.

If $\mathcal{F}$ is a notation for a function space (C^k, L^p, H^k, etc.), then we denote by $\mathcal{F}(\beta^r_s N)$ the space of semibasic tensor fields on TN whose components belong to the corresponding space $\mathcal{F}$ in each local coordinate system. In particular, $C^\infty(\beta^0_0 N) = C^\infty(TN)$.

We now define certain differential operators for semibasic tensor fields.

The *vertical derivative*

$$\overset{v}{\nabla} : C^\infty(\beta^r_s N) \to C^\infty(\beta^r_{s+1} N)$$

is defined in coordinates by

$$\overset{v}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s}(x,\xi) = \frac{\partial}{\partial \xi^k} u^{i_1\dots i_r}_{j_1\dots j_s}.$$

The *horizontal derivative*

$$\overset{h}{\nabla} : C^\infty(\beta^r_s N) \to C^\infty(\beta^r_{s+1} N)$$

is defined in coordinates by

$$\begin{aligned}\overset{h}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s}(x,\xi) =& \frac{\partial}{\partial x^k} u^{i_1\dots i_r}_{j_1\dots j_s} - \Gamma^p_{kq}\xi^q \frac{\partial}{\partial \xi^p} u^{i_1\dots i_r}_{j_1\dots j_s} \\ &+ \sum_{m=1}^{r} \Gamma^{i_m}_{kp} u^{i_1\dots i_{m-1} p i_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s} \Gamma^p_{kj_m} u^{i_1\dots i_r}_{j_1\dots j_{m-1} p j_{m+1}\dots j_s}.\end{aligned}$$

For a function $u \in C^\infty(TN)$, we in particular have

$$\overset{h}{\nabla} u(x,\xi) = \frac{\partial}{\partial x^k} u - \Gamma^p_{kq}\xi^q \frac{\partial}{\partial \xi^p} u.$$

A look at (2.6) shows that

$$Hu(x,\xi) = \xi^i \overset{h}{\nabla}_i u. \tag{3.1}$$

In this form H, regarded as a first-order partial differential operator, extends to the semibasic tensor fields $u \in C^\infty(\beta^r_s N)$. It is easy to see that for $u \in C^\infty(\beta^r_s N)$ the rule (2.7) changes to

$$Hu(v) = D_t u(\dot\gamma_v(t))|_{t=0}, \tag{3.2}$$

where D_t is the covariant derivative of the tensor field $u(\dot\gamma(t))$ along the geodesic γ_v.

The *vertical divergence*, $\overset{v}{\operatorname{div}}$, and the *horizontal divergence* , $\overset{h}{\operatorname{div}}$, of a semibasic vector field u are defined by

$$\overset{v}{\operatorname{div}}\, u = \overset{v}{\nabla}_i u^i, \quad \overset{h}{\operatorname{div}}\, u = \overset{h}{\nabla}_i u^i.$$

As can be easily shown, $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$ are well-defined first-order differential operators. In particular, they extend naturally to the spaces $H^1(\beta^r_s N)$.

With the above notations, *Pestov's identity* reads [4]:

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\operatorname{div}}\, V + \overset{v}{\operatorname{div}}\, W - \langle R(\xi, \overset{v}{\nabla} u)\xi, \overset{v}{\nabla} u\rangle \tag{3.3}$$

for every function $u \in C^\infty(TN)$. Here H is the geodesic spay on TN, R is the curvature tensor, and the semibasic vector fields V and W are given by

$$V = \langle \overset{h}{\nabla} u, \overset{v}{\nabla} u\rangle \xi - \langle \overset{h}{\nabla} u, \xi\rangle \overset{v}{\nabla} u, \tag{3.4}$$

$$W = \langle \overset{h}{\nabla} u, \xi\rangle \overset{h}{\nabla} u. \tag{3.5}$$

Let D be an n-dimensional submanifold of N with boundary ∂D. Given $u \in C^\infty(UD)$ such that $u|_{\partial(UD)} = 0$, extend the function $u(x,\xi)$ to $TD \setminus 0$ so that the function becomes homogeneous of degree $m-1$ in the second argument. Integrate (3.3) over UD with respect to the natural (local product) measure $d\Sigma$ on UN. Using the Gauss-Ostrogradskiĭ formulas for horizontal and vertical divergences [4, Theorem 3.6.3], we arrive at the relation

$$\int_{UD} 2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle \, d\Sigma = \int_{UD} |\overset{h}{\nabla} u|^2 \, d\Sigma + (-1)^n \int_{\partial(UD)} \langle V, \nu\rangle \, d\Sigma^{2n-2}$$
$$+ (n+2m-2) \int_{UD} \langle W, \xi\rangle \, d\Sigma - \int_{UD} \langle R(\xi, \overset{v}{\nabla} u)\xi, \overset{v}{\nabla} u\rangle \, d\Sigma, \tag{3.6}$$

where $d\Sigma^{2n-2}$ is the natural local product measure on $\partial(UD) = \{(x,\xi) : x \in \partial D,\ \xi \in U_x N\}$ and ν is the outward unit normal to ∂D.

For $(x,\xi) \in \partial(UD)$, put

$$\overset{h}{\nabla}{}^{||} u = \overset{h}{\nabla} u - \langle \overset{h}{\nabla} u, \nu\rangle \nu.$$

Then $\overset{h}{\nabla}{}^{||}$ is in fact a differential operator on $\partial(UD)$ and $\overset{h}{\nabla}{}^{||} u$ is completely determined by the restriction $u|_{\partial(UD)}$ (see [2]).

For $(x,\xi) \in \partial(UD)$, by a straightforward calculation we deduce from (3.4) that

$$\langle V, \nu\rangle = \langle \overset{h}{\nabla}{}^{||} u, \overset{v}{\nabla} u\rangle \langle \xi, \nu\rangle - \langle \overset{h}{\nabla}{}^{||} u, \xi\rangle \langle \overset{v}{\nabla} u, \nu\rangle.$$

In view of the boundary condition $u|_{\partial(UD)} = 0$, we therefore have $\langle V, \nu\rangle = 0$.

Using also the equality $\langle W, \xi\rangle = (Hu)^2$, from (3.6) we obtain

$$\int_{UD} 2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle \, d\Sigma = \int_{UD} |\overset{h}{\nabla} u|^2 \, d\Sigma + (n+2m-2) \int_{UD} (Hu)^2 \, d\Sigma$$
$$- \int_{UD} \langle R(\xi, \overset{v}{\nabla} u)\xi, \overset{v}{\nabla} u\rangle \, d\Sigma. \tag{3.7}$$

Lemma 3.1. *Let D be an n-dimensional submanifold of N with boundary ∂D. Let a function $u : UD \to \mathbb{R}$ be such that $u \in H^1(UD)$, $Hu \in H^1(UD)$, u is smooth in some neighborhood of $\partial(UD)$ in UD, and $u|_{\partial(UD)} = 0$. Then the integral identity* (3.7) *is valid for u.*

Proof. We will construct a sequence of smooth functions u_k coinciding with u in a neighborhood of $\partial(UD)$ and such that $u_k \to u$ in $H^1(UD)$ and $Hu_k \to Hu$ in $H^1(UD)$ as $k \to \infty$. Applying (3.7) to each function u_k and passing to the limit as $k \to \infty$, we will come to the desired conclusion.

The construction of such an approximation is quite standard and we merely sketch it. Call a chart (O, ψ) of the manifold UN *straightening* the vector field H if $\psi_* H$ coincides with a coordinate vector field on the range $\psi(O) \subset \mathbb{R}^{2n-1}$. Cover UD by a finite number of sufficiently small straightening charts O_i, so that the charts having nonempty intersection with $\partial(UD)$ lie entirely in that neighborhood of $\partial(UD)$ where u is smooth. Choose a partition of unity $\{\mu_i\}$ subordinate to this covering. Then $u = \sum_i \mu_i u$, where each $\mu_i u$ has support in O_i Choose those indices i for which O_i is disjoint from $\partial(UD)$. For such an index i, approximate the function $(\mu_i u) \circ \psi_i^{-1}$ using the Sobolev mollification whose radius is small enough and lift the result back to O_i by means of ψ_i. For the other indices, take the functions $\mu_i u$ themselves. Then the sum of the so-obtained functions gives the sought approximation to u. □

4. Decomposition of tensor fields into potential and solenoidal parts

For $m \geq 1$, let δ be the *divergence* [4],

$$\delta : C^\infty(S_m N) \to C^\infty(S_{m-1}N), \quad (\delta T)_{i_1 \dots i_{m-1}} = g^{jk}\nabla_k T_{j i_1 \dots i_{m-1}}.$$

The operator $-\delta$ is the formal dual to the inner derivative d with respect to the natural L^2-product on the space of tensor fields. In an obvious way, it extends to the spaces $H^1(S_m N)$.

A a field $T \in H^1(S_m N)$, $m \geq 1$, is called *solenoidal* [4] if it satisfies the equation

$$\delta T = 0. \tag{4.1}$$

In the following lemma we recall Sharafutdinov's decomposition of tensor fields into potential and solenoidal parts.

Lemma 4.1 (see [4, Theorem 3.3.2])**.** *Let (D, g) be an arbitrary smooth compact Riemannian manifold with boundary ∂D. Each field $f \in H^k(S_m D)$, $m \geq 1$, can be uniquely decomposed as*

$$f = dp + \tilde f \tag{4.2}$$

with some $p \in H^{k+1}(S_{m-1}D)$ and $\tilde f \in H^k(S_m D)$ such that $p|_{\partial D} = 0$ and $\delta \tilde f = 0$. The summands depend continuously on f in the following sense: the estimates

$$\|p\|_{H^{k+1}(S_{m-1}D)} \leq C\|f\|_{H^k(S_m D)}, \quad \|\tilde f\|_{H^k(S_m D)} \leq C\|f\|_{H^k(S_m D)}$$

hold with some constant C independent of f. In particular, if $f \in C^\infty(S_m D)$ then $p \in C^\infty(S_{m-1}D)$ and $\tilde f \in C^\infty(S_m D)$.

We will apply this decomposition over a nested family of subdomains in N whose intersection is our manifold M.

Let $W \subset N$ be a collar neighborhood of ∂M in N. This means that there is a diffeomorphism $\Phi : \partial M \times (-1, 1) \to W$ such that the restriction $\Phi|_{\partial M \times \{0\}}$ is the identity. We also assume that $\Phi(\partial M \times (-1, 0)) \subset M$ and $\Phi(\partial M \times (0, 1)) \subset N \setminus M$.

Put $M_\varepsilon = M \cup \Phi(\partial M \times [0, \varepsilon])$, $0 \leq \varepsilon < 1$, obtaining a nested family of subdomains in N, with $M_0 = M$.

Now, let f be the symmetric field on N that we used in Section 2 while defining the function u of Lemma 2.3. Applying Lemma 4.1 to $D = M_\varepsilon$ and the field $f|_{M_\varepsilon}$, we obtain for every $\varepsilon \in [0, 1)$ the corresponding fields $p_\varepsilon \in H^2(S_{m-1}M_\varepsilon)$ and $\tilde f_\varepsilon \in H^1(S_{m-1}M_\varepsilon)$ and the decomposition

$$f = dp_\varepsilon + \tilde f_\varepsilon. \tag{4.3}$$

Lemma 4.2. *With the above notation, the following holds:*

(1) $p := p_0 \in C^\infty(S_{m-1}M)$, $p|_{\partial M} = 0$.

(2) $\tilde{f} := \tilde{f}_0 \in C^\infty(S_m M)$, $\delta \tilde{f} = 0$.
(3) $p_\varepsilon \in H^2(S_{m-1}M_\varepsilon) \cap C^\infty(S_{m-1}(M_\varepsilon \setminus \partial M))$, $p_\varepsilon|_{\partial M_\varepsilon} = 0$.
(4) $\tilde{f}_\varepsilon \in H^1(S_m M_\varepsilon) \cap C^\infty(S_m(M_\varepsilon \setminus \partial M))$, $\delta \tilde{f}_\varepsilon = 0$.
(5) $\tilde{p}_\varepsilon|_M \to p$ *in* $H^2(S_m M)$ *as* $\varepsilon \to 0$, *and for* ε *small enough the estimate*

$$\|\tilde{p}_\varepsilon\|_{H^2(S_m M_\varepsilon)} \le C\|f\|_{H^1(S_m M)}$$

holds uniformly in f *and* ε.

(6) $\tilde{f}_\varepsilon|_M \to \tilde{f}$ *in* $H^1(S_m M)$ *as* $\varepsilon \to 0$, *and for* ε *small enough the estimate*

$$\|\tilde{f}_\varepsilon\|_{H^1(S_m M_\varepsilon)} \le C\|f\|_{H^1(S_m M)}$$

holds uniformly in f *and* ε.

Proof. Items (1) and (2) are straightforward from Lemma 4.1.

To prove the other items, let us recall the construction of p_ε and $\tilde{f}_\varepsilon$ provided by Lemma 4.1. The field p_ε is a solution of the boundary value problem

$$\delta d p_\varepsilon = v, \quad p_\varepsilon|_{\partial M_\varepsilon} = 0, \tag{4.4}$$

with $v = \delta f$. Afterwards, $\tilde{f}_\varepsilon$ is defined by $\tilde{f}_\varepsilon = f - dp_\varepsilon$.

Denote by $H_0^2(S_{m-1}M_\varepsilon)$ the subspace of tensor fields $h \in H^2(S_{m-1}M_\varepsilon)$ of zero trace on ∂M_ε, $h|_{\partial M} = 0$. It is shown in [4, Section 3.3] that the boundary value problem (4.4) is elliptic and the operator

$$\mathcal{D}_\varepsilon = \delta d : H_0^2(S_{m-1}M_\varepsilon) \to L^2(S_m M_\varepsilon)$$

is invertible.

Now, items (3) and (4) are obvious, since the field f is smooth on $M_\varepsilon \setminus \partial M$.

Item (6) is a direct consequence of item (5) and the formula $\tilde{f}_\varepsilon = f - dp_\varepsilon$.

We now prove item (5). Let $\Psi_\varepsilon : M \to M_\varepsilon$, $0 \le \varepsilon < 1$, be a smooth family of diffeomorphisms such that Ψ_0 is the identity of M. Such a family is easy to construct by using the above diffeomorphism Φ. Define the following operators:

$$\widetilde{\mathcal{D}}_\varepsilon : H_0^2(S_{m-1}M) \to L^2(S_m M),$$

$$\widetilde{\mathcal{D}}_\varepsilon p = \Phi_\varepsilon^{-1} \circ \mathcal{D}_\varepsilon(p \circ \Phi_\varepsilon).$$

It is easy to see that these operators are well-defined and are in fact second-order partial differential operators on M. Furthermore, as $\varepsilon \to 0$ they converge in the operator norm to the operator $\mathcal{D}_0$. Since $\mathcal{D}_0$ is invertible, the inverse operators $\widetilde{\mathcal{D}}_\varepsilon^{-1}$ converge in the operator norm to $\mathcal{D}_0^{-1}$. Now, the claims of item (5) follow trivially. □

5. End of the proof of Theorem 1.3

As before, we let φ be the function of Lemma 2.3 and consider the same M_ε, $\tilde{f}_\varepsilon$, and p_ε as in the previous section.

Put

$$u_\varepsilon(v) = \varphi(v) - \langle p_\varepsilon, v^{m-1} \rangle \tag{5.1}$$

and

$$F_\varepsilon(v) = \langle \tilde{f}_\varepsilon, v^m \rangle. \tag{5.2}$$

A direct computation in coordinates, using (2.6), yields

$$H\langle p_\varepsilon(x), \xi^{m-1} \rangle = \langle dp_\varepsilon(x), \xi^m \rangle. \tag{5.3}$$

For $v \in UM_\varepsilon$, (5.3) and (2.5) give

$$Hu_\varepsilon(v) = F_\varepsilon(v). \tag{5.4}$$

From the properties of φ listed in Lemma 2.3 and the properties of p_ε and $\tilde{f}_\varepsilon$ listed in Lemma 4.2 we deduce the next lemma.

Lemma 5.1. *The functions u_ε and F_ε have the following properties:*

(1) $u := u_0 \in H^1(UM) \cap C(UM) \cap C^\infty(RM)$, $u|_{\partial UM} = 0$.
(2) $F := F_0 \in C^\infty(UM)$, $\overset{h}{\operatorname{div}}\,(\overset{v}{\nabla} F) = 0$.
(3) $u_\varepsilon \in H^1(UM_\varepsilon) \cap C^\infty(M_\varepsilon \setminus M)$, $u_\varepsilon|_{\partial(UM_\varepsilon)} = 0$.
(4) $u_\varepsilon|_{UM} \to u$ *in* $H^1(UM)$ *as* $\varepsilon \to 0$, *and for* ε *small enough the estimate* $\|u_\varepsilon\|_{H^1(UM_\varepsilon)} \le C$ *holds uniformly in* ε.
(5) $F_\varepsilon|_{UM} \to F$ *in* $H^1(UM)$ *as* $\varepsilon \to 0$, *and for* ε *small enough the estimate* $\|F_\varepsilon\|_{H^1(UM_\varepsilon)} \le C$ *holds uniformly in* ε.

In particular, each function u_ε with $\varepsilon > 0$ satisfies the conditions of Lemma 3.1 on M_ε. By this lemma and (5.4) we have

$$\int_{UM_\varepsilon} 2\langle \overset{h}{\nabla} u_\varepsilon, \overset{v}{\nabla} F_\varepsilon\rangle\, d\Sigma = \int_{UM_\varepsilon} |\overset{h}{\nabla} u_\varepsilon|^2\, d\Sigma + (n+2m-2)\int_{UM_\varepsilon} F_\varepsilon^2\, d\Sigma - \int_{UM_\varepsilon} \langle R(\xi, \overset{v}{\nabla} u_\varepsilon)\xi, \overset{v}{\nabla} u_\varepsilon\rangle\, d\Sigma.$$

Items (4) and (5) of Lemma 5.1 allow us to pass to the limit as $\varepsilon \to 0$ in this identity. We then arrive at

$$\int_{UM} 2\langle \overset{h}{\nabla} u, \overset{v}{\nabla} F\rangle\, d\Sigma = \int_{UM} |\overset{h}{\nabla} u|^2\, d\Sigma + (n+2m-2)\int_{UM} F^2\, d\Sigma - \int_{UM} \langle R(\xi, \overset{v}{\nabla} u)\xi, \overset{v}{\nabla} u\rangle\, d\Sigma. \tag{5.5}$$

Notice that

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla} F\rangle = \overset{h}{\operatorname{div}}\,(u\overset{v}{\nabla} F) - u\,\overset{h}{\operatorname{div}}\,(\overset{v}{\nabla} F) = \overset{h}{\operatorname{div}}\, V, \tag{5.6}$$

with

$$V := u\overset{v}{\nabla} F = mu\tilde{f}_{ji_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}. \tag{5.7}$$

Since $V \in H^1(\beta_1^0 M)$ and $V|_{\partial(UM)} = 0$, the Gauss-Ostrogradskiĭ formula for the horizontal divergence yields

$$\int_{UM} 2\langle \overset{h}{\nabla} u, \overset{v}{\nabla} F\rangle\, d\Sigma = \int_{UM} \overset{h}{\operatorname{div}}\, V\, d\Sigma = \int_{\partial(UM)} \langle V, \nu\rangle\, d\Sigma^{2n-2} = 0.$$

Hence, the left-hand side of (5.5) vanishes. We thus come to the identity

$$\int_{UM} \langle R(\xi, \overset{v}{\nabla} u)\xi, \overset{v}{\nabla} u\rangle = \int_{UM} |\overset{h}{\nabla} u|^2 + (n+2m-2)\int_{UM} F^2. \tag{5.8}$$

We will derive one more identity.

Following [2], for $(x,\xi) \in UM$ put

$$\overset{v}{\nabla}{}^{||} u = \overset{v}{\nabla} u - \langle \overset{v}{\nabla} u, \xi\rangle \xi. \tag{5.9}$$

It is easy to see that $\overset{v}{\nabla}{}^{||}$ is just gradient when restricted to U_xM; in particular, $\overset{v}{\nabla}{}^{||}u$ is independent of a particular choice of the extension of u from UM to TM.

From formula (3.1) for H one can easily deduce the commutation formula

$$H(\overset{v}{\nabla}u) = \overset{v}{\nabla}(Hu) - \overset{h}{\nabla}u. \tag{5.10}$$

(For smooth functions u this equality is valid pointwise; for $u \in H^1(UM)$ such that $Hu \in H^1(UM)$, the left-hand side should be understood as the distributional derivative of the field $\overset{v}{\nabla}u$; in our case, by item (1) of Lemma 5.1 the function u is smooth on the set RM of full measure in UM, and so we can understand the left-hand side of (5.10) pointwise as well.)

Since $Hu = F$, we infer that

$$H(\overset{v}{\nabla}u) = \overset{v}{\nabla}F - \overset{h}{\nabla}u. \tag{5.11}$$

Since u is positively homogeneous of degree $(m-1)$ in ξ, Euler's formula shows that $\langle\overset{v}{\nabla}u, \xi\rangle = (m-1)u$ and therefore from (5.9) and (5.11) we obtain

$$H(\overset{v}{\nabla}{}^{||}u) = \overset{v}{\nabla}F - \overset{h}{\nabla}u - (m-1)F\xi. \tag{5.12}$$

Now,

$$\begin{aligned}|H(\overset{v}{\nabla}{}^{||}u)|^2 &= |\overset{v}{\nabla}F|^2 + |\overset{h}{\nabla}u|^2 + (m-1)^2F^2 \\ &\quad - 2\langle\overset{v}{\nabla}F, \overset{h}{\nabla}u\rangle - 2(m-1)F\langle\overset{v}{\nabla}F, \xi\rangle + 2(m-1)F\langle\overset{h}{\nabla}u, \xi\rangle.\end{aligned} \tag{5.13}$$

By (5.6)

$$\langle\overset{v}{\nabla}F, \overset{h}{\nabla}u\rangle = \overset{h}{\operatorname{div}}\,V, \tag{5.14}$$

with V defined by (5.7). Next,

$$\langle\overset{v}{\nabla}F, \xi\rangle = mF \tag{5.15}$$

and

$$\langle\overset{h}{\nabla}u, \xi\rangle = Hu = F. \tag{5.16}$$

Inserting (5.14)–(5.16) in (5.13), we obtain

$$|H(\overset{v}{\nabla}{}^{||}u)|^2 = |\overset{v}{\nabla}F|^2 + |\overset{h}{\nabla}u|^2 - (m-1)^2F^2 - 2\overset{h}{\operatorname{div}}\,V. \tag{5.17}$$

Integrating and recalling that the integral of $\overset{h}{\operatorname{div}}\,V$ vanishes, we come to the second integral identity

$$\int_{UM} |H(\overset{v}{\nabla}{}^{||}u)|^2 = \int_{UM} |\overset{v}{\nabla}F|^2 + \int_{UM} |\overset{h}{\nabla}u|^2 - (m-1)^2 \int_{UM} F^2. \tag{5.18}$$

Multiplying (5.8) by the number $s_{n,m}$ of (1.4), subtracting the result from (5.18), and using the inequality $|\overset{h}{\nabla}u| \geq |Hu| = |F|$, we arrive at the inequality

$$\begin{aligned}\int_{UM} |H(\overset{v}{\nabla}{}^{||}u)|^2 - s_{n,m} \int_{UM} \langle R(\xi, \overset{v}{\nabla}u)\xi, \overset{v}{\nabla}u\rangle \\ \leq \int_{UM} |\overset{v}{\nabla}F|^2 - m(n+2m-2) \int_{UM} F^2.\end{aligned} \tag{5.19}$$

Lemma 5.2 ([4, Lemma 4.5.3]). *The following inequality holds:*

$$\int_{UM} |\overset{v}{\nabla} F|^2 \, d\Sigma \le m(n+2m-2) \int_{UM} F^2 \, d\Sigma.$$

Thus, the right-hand side of (5.19) is nonpositive. Note also that by (5.9) and the properties of the curvature tensor, we have

$$\langle R(\xi, \overset{v}{\nabla} u)\xi, \overset{v}{\nabla} u\rangle = \langle R(\xi, \overset{v}{\nabla}{}^{||} u)\xi, \overset{v}{\nabla}{}^{||} u\rangle.$$

Therefore, (5.19) leads to the inequality

$$\int_{UM} \Big(|H(\overset{v}{\nabla}{}^{||} u)|^2 - s_{n,m}\langle R(\xi, \overset{v}{\nabla}{}^{||} u)\xi, \overset{v}{\nabla}{}^{||} u\rangle \Big) \, d\Sigma \le 0. \tag{5.20}$$

We will show that this inequality may hold only if $\overset{v}{\nabla}{}^{||} u = 0$.

Let $\partial_+(UM)$ be the set of inward unit tangent vectors of M above ∂M:

$$\begin{aligned} \partial_+(UM) &= \{v \in \partial(UM) : \langle v, \nu\rangle > 0\} \\ &= \{(x,\xi) : x \in \partial M,\ \xi \in U_x M,\ \langle \xi, \nu(x)\rangle > 0\}, \end{aligned}$$

where ν is the inward unit normal to ∂M.

For $v \in \partial_+(UM)$, let τ_v stand for the parameter value at which the geodesic γ_v meets ∂M for the first time (since (M,g) is nontrapping, τ_v is well defined).

Given an integrable function $\psi : UM \to \mathbb{R}$, the Santaló formula holds:

$$\int_{UM} \psi \, d\Sigma = \int_{\partial_+(UM)} \langle v, \nu\rangle \Big(\int_0^{\tau_v} \psi(\gamma_v(t)) \, dt \Big) \, d\Sigma^{2n-2}(v).$$

We apply it to the integral in (5.20).

Put

$$Y_v(t) = \overset{v}{\nabla}{}^{||} u(\gamma_v(t)).$$

Using Santaló's formula, we can rewrite (5.20) as follows:

$$\int_{\partial_+(UM)} \langle v, \nu\rangle \Big(\int_0^{\tau_v} \big(|D_t Y_v|^2 - s_{n,m}\langle R_{\dot\gamma} Y_v, Y_v\rangle \big) \, dt \Big) \, d\Sigma^{2n-2}(v) \le 0, \tag{5.21}$$

where $D_t Y_v$ is the covariant derivative along γ. Observe that by item (1) of Lemma 5.1 we have $u \in C^\infty(RM)$ and $u|_{\partial(UM)} = 0$. Therefore, for every $v \in \partial_+(UM)$ the field Y_v in (5.21) is a smooth field along γ_v vanishing at its endpoints, $Y_v(0) = Y_v(\tau_v) = 0$.

Lemma 5.3 (see [2, p. 46]). *A geodesic $\gamma : [a,b] \to M$ in an arbitrary Riemannian manifold (M,g) has no s-conjugate points if and only if*

$$E(X) := \int_a^b \big(|D_t X|^2 - s\langle R_\gamma X, X\rangle\big) \, dt > 0$$

for every nonzero smooth vector field X along γ vanishing at its endpoints, $X(a) = X(b) = 0$.

This lemma is just an analog of the corresponding extremal properties of Jacobi fields and is a consequence of the fact that (1.5) is the Euler–Lagrange equation for the functional $E(X)$.

Recalling that by assumption M has no $s_{n,m}$-conjugate points and applying Lemma 5.3, we see that $Y_v(t) = 0$ for all $v \in \partial_+(UM)$ and $t \in [0, \tau_v]$. Thus,

$$\overset{v}{\nabla}{}^{||} u = 0 \quad \text{a.e. on } UM.$$

Now, (5.8) yeilds $F = 0$, i.e., $\tilde{f} = 0$. This means that $f = dp$ with $p|_{\partial M} = 0$, i.e., $f \in P(S_m M)$.

The proof of Theorem 1.3 is over.

The proof of Theorem 1.6 proceeds along the same lines, but is slightly simpler, because in this case Sharafutdinov's decomposition of Lemma 4.1 is trivial.

Now, the kinetic equation takes the form $H\varphi(x, \xi) = f(x)$. On taking in the above arguments $m = 0$, $u_\varepsilon = \varphi$, and $F_\varepsilon = f$, identities (5.8) and (5.18) acquire the form

$$\int_{UM} \langle R(\xi, \overset{v}{\nabla} u)\xi, \overset{v}{\nabla} u\rangle = \int_{UM} |\overset{h}{\nabla} u|^2 + (n-2) \int_{UM} F^2,$$

$$\int_{UM} |H(\overset{v}{\nabla}{}^{||} u)|^2 = \int_{UM} |\overset{h}{\nabla} u|^2 - \int_{UM} F^2.$$

Subtracting the first of them from the second, we come to the identity

$$\int_{UM} |H(\overset{v}{\nabla}{}^{||} u)|^2 - \langle R(\xi, \overset{v}{\nabla} u)\xi, \overset{v}{\nabla} u\rangle = -(n-1) \int_{UM} F^2.$$

Now, the application of Lemma 5.3 gives $F = 0$, i.e., $f = 0$, finishing the proof of Theorem 1.6.

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Sobolev Institute of Mathematics, Koptyuga 4, 630090, Novosibirsk, Russia

Current address: Max Planck Institute of Mathematics, Vivatsgasse 7, D-53111, Bonn, Germany

E-mail address: nurlan@mpim-bonn.mpg.de, dair@math.nsc.ru