GRIDS AS UNKNOTTING OPERATIONS

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Abstract. We show that certain "grid" local moves are unknotting operations. For that we use and extend ideas involved in [A2], [A3]. We find basic estimates for the knot invariants these unknotting operations define working as in [A3].

1. Introduction

In [A2] we defined what we called "multi-# local moves" and we showed them to be unknotting operations. In [A3] (a revised version of [A2]) we strengthened the result and defined some more unknotting operations. Among others as an immediate consequence of the ideas involved in [A2] we proved in [A3] that certain grid moves ($2 \times n$ all n) are also unknotting operations. Let us for the convenience of the reader recall the definition of these grid moves.

Definition 1.1. *A grid local move of type* $(d_1, r_1) \times (d_2, r_2)$ *will be a local move as in the picture below (and its inverse).*

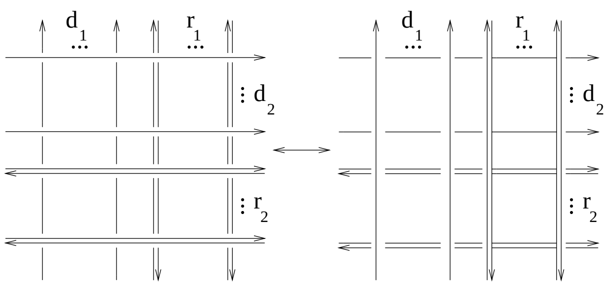

Figure 1

This can in the obvious way be written, thought and talked about alternatively in terms of congruence classes of knots as in [FM]. We will not use this language here but in our last section we will make some relevant comments. It is the fact that a multi-# local move of type $(2, s, d)$ (which we have proven to be an unknotting operation), is locally expressible as a grid move of type $(2, 0) \times (2d + 1, 2(s - d))$ (from now on we will write $d_1 \times d_2$ for all $m(d_1, 0) \times (d_2, 0)$) which allowed us in [A3] (Theorem 3.1) to show that that a $(2, 0) \times (d_2, r_2)$ is an unknotting operation. Here we will show that for $d = 3, 4$, a $d \times m$ grid is an unknotting operation for all $m \geq 1$, (modulo the fact that a $4 \times 2m$ grid has to preserve the Arf invariant). As in [A3] we will draw our proofs so, in order to simplify them, and (unfortunately) at

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the expense of making it a bit more painful to read them we need to develop some notation first. Specifically we will be dealing with bundles of strings so we need to abbreviate the way we draw them. The figure below contains an explanation of our notation. It basically makes a conventions on how to deal with crossings when a single drawn strand represents more than one strings. A string with a little vertical line decorated with a pair of integers (x, y) will stand for a bundle of x strings lying on a band which is twisted y half times. The bottom part of the figure deals with how to abbreviate twists between decorated strings.

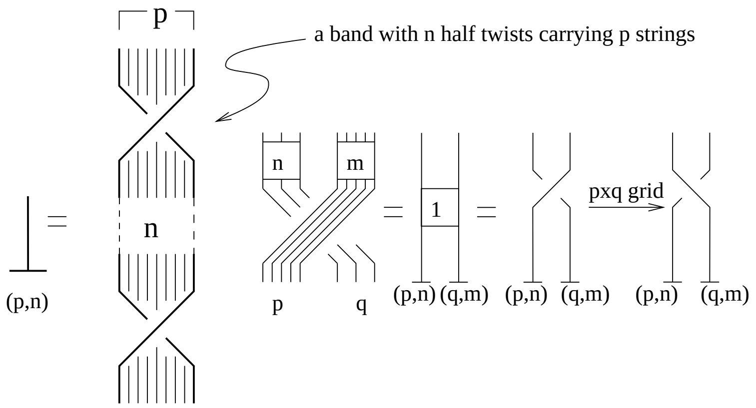

FIGURE 2

A grid move can also be expressed in terms of surgeries but we will deal with them here from the view point of local equivalence. The figure below is written in the notation we already introduced and consists of (besides isotopies) two grids as indicated. We will be referring to this as **the key sequence**.

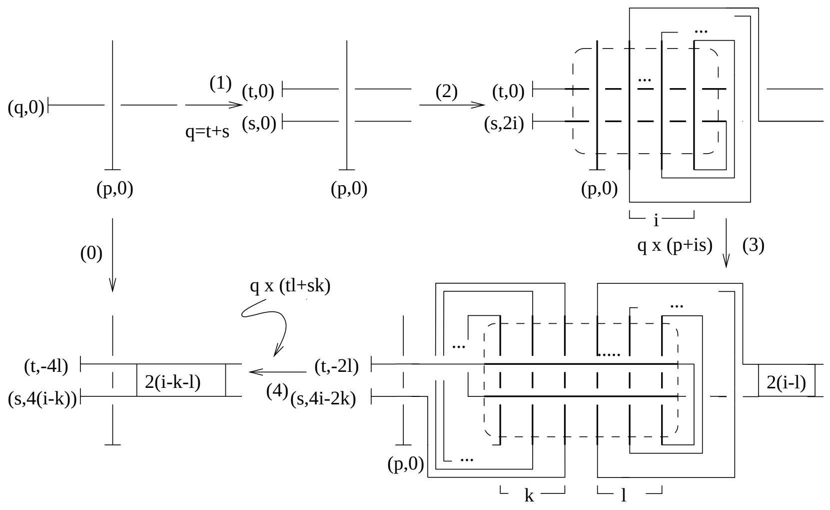

FIGURE 3

What we then wish to prove is:

Theorem 1.1. *A grid local move of type $a \times b$ is an unknotting operation for $a = 3, 4$ and any $b \geq 1$ modulo the fact that a move of the form $4 \times 2b$ has to preserve the Arf invariant.*

The paper is organized as follows. In section 2 we prove Theorem 1.1. The proof consists by applying our key sequence to special situations. In Section 3 we discuss the metrics on the space of knots and knot invariants resulting from these unknotting operations and write down some basic estimates. In the last section we explain some ideas involved. This is a preview of work in preparation.

2. Grid Local Moves

In this section we want to prove our Theorem 1.1. The proof in the case $a = 3$ is immediate. For $b = 1$ it is trivial and for $b = 2$ it was proven in [A3]. The proof for $b \geq 3$ is an immediate consequence of:

Proposition 2.1. *A grid local move of type $3 \times b$ is locally equivalent to the ordinary crossing change (a 1×1 grid).*

Proof We only need to show that a $3 \times b$ grid can be expressed in terms of a 1×1 grid. To see this plug the key sequence, for $p = 2$, $s = 1$, $t = 2$, $l = 2$ and $k = i - 2$, into the dotted box of the figure below. Arrow (2) is then the key sequence (two $3 \times (i + 2)$ grids for $i \geq 2$). The other arrows are isotopies. That and the mirror image of the same figures show that a $3 \times b$ grid $b \geq 4$ induces a $\pm$-full twist on two strings.

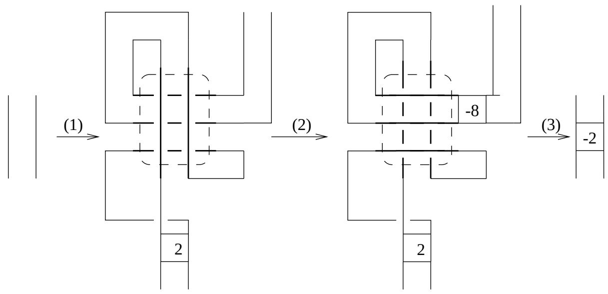

FIGURE 4

To express a 1×1 grid in terms of 3×2 and 3×3 grids just notice that a $a \times bc$ grid is nothing but a sequence of c grids of type $a \times b$. This finishes the proof of the proposition.

To prove Theorem 1.1 in the case $a = 4$ we will have to work somewhat harder. We will first use our key sequence to get various lemmas which will then be assembled into a proof. The upshot of the following series of lemmas will be to show that a grid move of type $4 \times b$ reduces every knot to a connected sum of trefoils. From that point on we will use the ideas and tricks developed here to update and modify certain lemmas in [A3]. The latter ones will then allow us to finish the proof.

Lemma 2.1. *Two $4 \times m$ grids (any $m \geq 6$) have the effect in (A) in the figure below:*

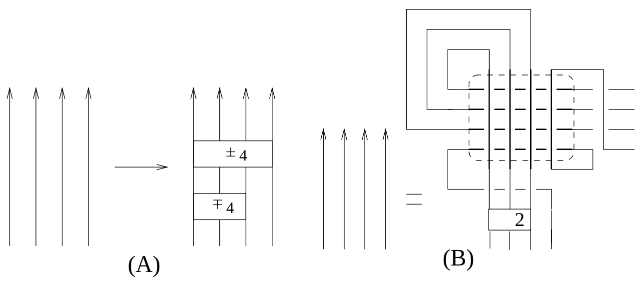

FIGURE 5

Proof Apply the key sequence for $p = q = 4$, $t = 3$, $s = 1$, $l = 2$ and $k = i - 2$ in the figure above in (B).

Lemma 2.2. *Two $4 \times m$ grids (any $m \geq 6$) have the effect in (A) in the figure below:*

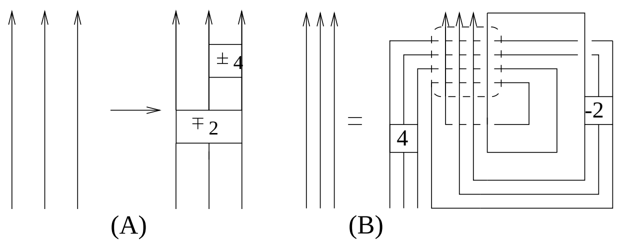

FIGURE 6

Proof Apply the key sequence for $p = q = 4$, $t = 3$, $s = 1$, $l = 2$ and $k = i - 2$ in the figure above in (B).

Lemma 2.3. *Two $4 \times 2m$ grids (any $m \geq 1$) have the effect in (A) in the figure below:*

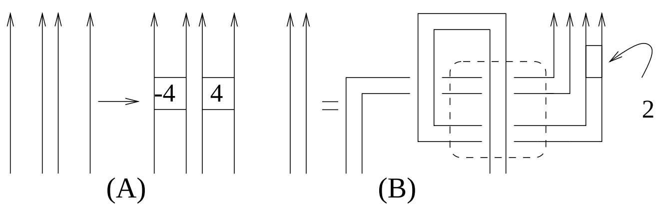

FIGURE 7

Proof Apply the key sequence for $p = 2, q = 4$, $t = 2$, $s = 2$, $l = 1$ and $k = i$ in the figure above in (B).

Now we are ready to start putting together the proof of theorem 1.1 for $a = 4$. Look at the figure below. Arrow (1) is simply isotopy. Arrow (2) is a $4 \times (2i+2)$ (any $i \geq 1$) grid followed by isotopy. Arrow (3) is i applications of lemma 2.1 (i.e a number of $4 \times m$ grids (any $m \geq 6$)). Arrow (4) is $2i$ applications of lemma 2.2 (i.e a number of $4 \times m$ grids (any $m \geq 6$)). Arrow (5) is isotopy and finally arrow (6) is once lemma 2.3 (i.e a number of $4 \times 2n$ grids (any $n \geq 1$)).

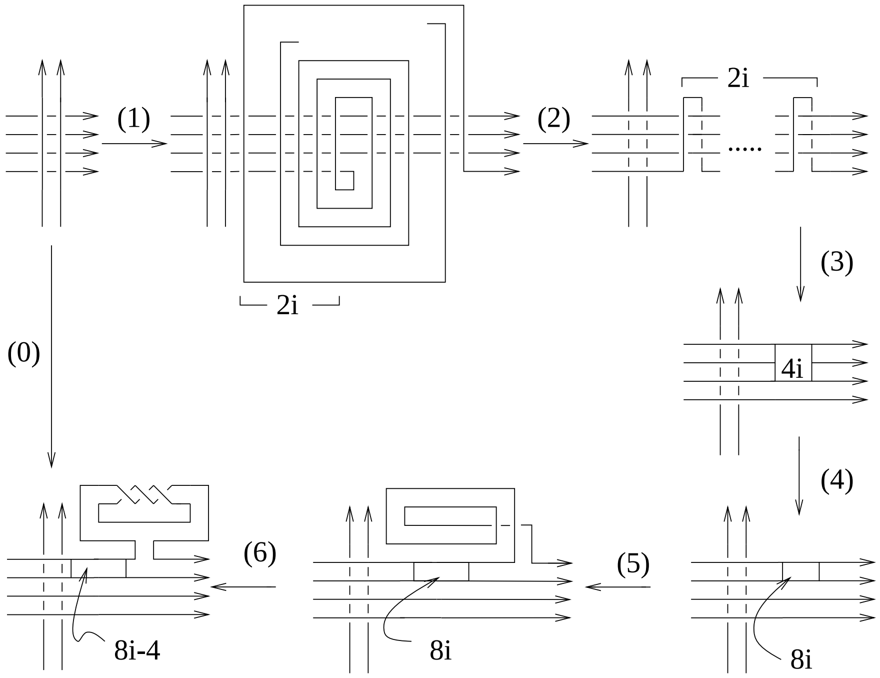

FIGURE 8

Since any knot can be reduced to the unknot or the trefoil via 2×4 grids this figure shows that by applying $4 \times 2b$ grids (any $b \geq 3$) any knot can be reduced to a connected sum of trefoil knots (keep applying arrow (6) $2i$ times). The number of connected summands is then odd or even according as the Arf invariant of the knot is 1 or 0. Hence any grid of the form $4 \times b$, any $b \geq 3$, achieves the same since a $a \times bc$ grid is locally nothing but c successive $a \times b$ grids. Notice that in the figure above we achieve a - to + 2×4 via any $4 \times b$, $b \geq 3$ grid modulo introducing an even number of $K(3,2)$ summands. The mirror image of the figure shows that a + to - 2×4 grid can be achieved via $4 \times b$ grids modulo introducing an even number of $K(-3,2)$ connect summands. Now we need to further show that any $4 \times 2b$ grid can reduce the number of summands to one or zero (according to Arf invariant) and that any grid of the form $4 \times b$, b odd, can unknot the trefoil as well. For this we need to modify some of our lemmas in [A3]. The idea is basically that the central lemma there, namely lemma 2.6, can be restated for $4 \times b$ grids. We will refer to that proof and show how to modify it.

Lemma 2.4. *We can pass from (a) to any of (b) or (c) in the figure below via a number of* $4 \times b$ *grid local moves for any* $b \geq 3$.

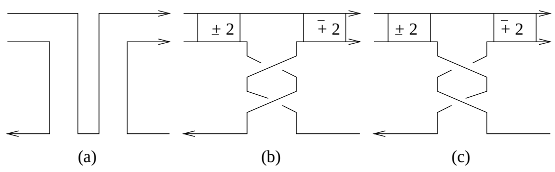

FIGURE 9

Proof Lemma 2.6 of [A3] is (stated for and) proven by applying grids of the form $2 \times n$, any $n \geq 2$. Moreover two such are required of opposite direction (- to + or + to - grid-crossing change). But our figure above shows that modulo connect summing with an even number of trefoil knots a 2×4 grid can be achieved by $4 \times b$ grids, any $b \geq 3$. Since we apply two 2×4 grids we have an even number of $K(3,2)$ and an even number of $K(-3,2)$ summands. Lemma 2.3 then eliminates summands of the form $K(3,2)\#K(-3,2)$. QED.

Now lemma 2.8 of [A3] holds for $4 \times b$ grids.

Lemma 2.5. *We can pass from* $K(3,2)$ *to* $K(-3,2)$ *via a sequence of* $4 \times b$ *grids, any* $b \geq 3$.

Putting together Lemmas 2.3, 2.4, 2.5 we now have proven:

Lemma 2.6. *Any knot* K *can be reduced to either the unknot (if* $Arf(K) = 0$*) or the trefoil (if* $Arf(K) = 1$*) via* $4 \times b$ *grids, any* $b \geq 3$.

For grids of the form $4 \times b$, b even this cannot be improved. It remains to show that for grids of the form $4 \times b$, b odd one can unknot the trefoil. Notice that so far we have proven that any knot can be reduced to a connected sum of a number of same sign trefoils and that this number can be changed by any even number. It suffices therefore to show that a grid of the form $4 \times b$, b odd can change the parity of the number of summands. This will be achieved with the help of:

Lemma 2.7. *A sequence of* $4 \times b$ *grids,* b *odd, achieves* $\pm(8k+2)$ *half twists, some* k *on two parallel like oriented strings.*

Proof The proof is contained in the figure below. We take the reader through the arrows. We will show that arrow (0) can be achieved via grid moves of the desired kind. This will finish the proof. Arrow (1) is just isotopy. Arrow (2) is a $4 \times (2i+3)$ grid followed by isotopy. Arrow (3) is i applications of lemma 2.1 (i.e. grids of type $4 \times m$, $m \geq 6$). Arrow (4) is just isotopy and finally arrow (5) is $2i+3$ applications of lemma 2.2 (i.e. $4 \times m$ grids, $m \geq 6$). So we can achieve this result by any $4 \times b$ grid with $b \geq 5$ and b odd.

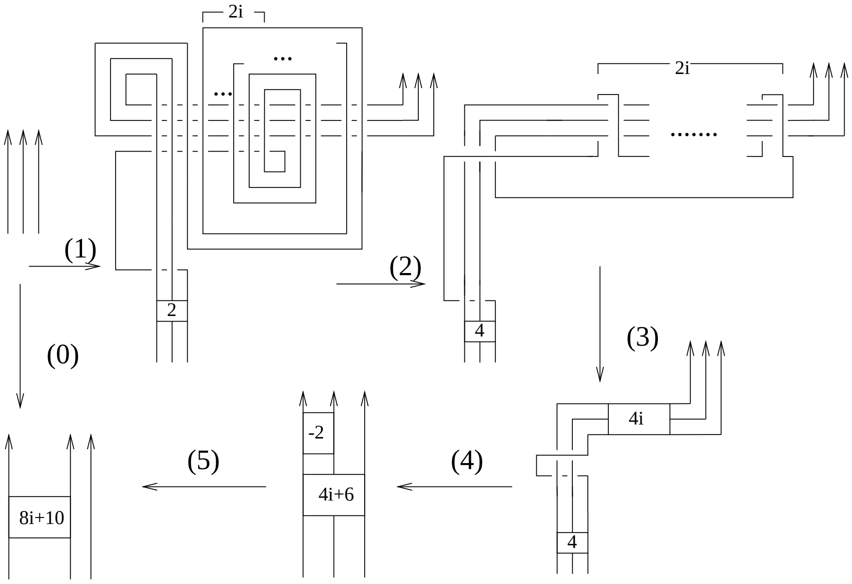

FIGURE 10

In the figure below we show how lemma 2.7 can be used to change the parity of the number of $K(3,2)$ summands. We describe the arrows. Arrow (1) is lemma 2.7 (i.e. any $4 \times b$ grid with $b \geq 5$ odd. Arrow (2) is lemma 2.3 (i.e. any $4 \times 2b$ grid). Arrow (3) finally is repeated applications of arrow (2). The final result is a connected sum of an odd number of trefoils.

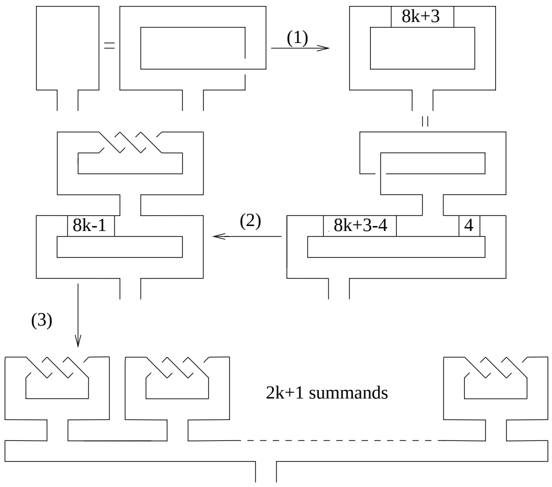

FIGURE 11

This finishes the proof.

3. Metrics, unknotting numbers and estimates

As in [A3] a grid unknotting operation defines a metric on the space of all knots and relevant "unknotting sets". We briefly recall their definition and we proceed to give some basic estimates. Before we do this let us discuss notation. In [A3] we talked about multi-# moves and grid local moves and used the symbol # decorated with a certain vector carrying the parameters. We introduced $\natural$ for a grid move. It is now clear that we should continue to use one symbol decorated with the parameters. We will use # for all of them and we will decorated as follows: $\#_{(d_1,r_1)\times(d_2\times r_2)}$ or $\#_{d_1\times d_2}$ when $r_i=0$. We can now define the relevant unknotting sets:

Definition 3.1. *By $\mathcal{U}^{\#}_{d_1\times d_2}(p,n)$ we will mean the set of all knots which can be unknotted by p + to - $d_1\times d_2$-grid crossing changes and n - to + $d_1\times d_2$-grid crossing changes. The relevant unknotting number $u^{\#}_{d_1\times d_2}(K)$ is then nothing but:*

$$\min\{p+n\colon K\in\mathcal{U}^{\#}_{d_1\times d_2}(p,n)\}$$

As in [A3] one could more generally define slicing numbers and sets as well as (when d_1 or d_2=3 or 4) metrics $d^{\#}_{d_1\times d_2}$ on the space $\mathcal{K}$ of all knots. Let us agree also on notation for some other standard moves. The relevant entities for one full twist one d strings will be decorated with d. So the classical unknotting number for example will be u^2 or simply u. We now would like to begin writing some estimates about these unknotting numbers. We prepare the ground for $u^{\#}_{3\times d}$. Our first lemma is the analogue of lemma 2.3 of [A3] and its proof is similar to the one for the latter.

Lemma 3.1. *Suppose that $K\in\mathcal{U}^{\#}_{(d_1,r_1)\times(d_2,r_2)}(p,n)$. Then there is a sphere embedded in $M^4_{p+n}=\#_{p+n}S^2\times S^2$ which is smooth everywhere except at one point where it is the cone on K and represents on homology $\oplus_p(d_1,d_2)\oplus_n(d_1-d_2)$ in the natural basis of $H_2(M^4_{p+n})$.*

Now we are ready to write down our basic estimate for $u^{\#}_{3\times d}$.

Theorem 3.1. *Let $K\in\mathcal{K}$ be any knot. If $d\geq 4$ then $u^{\#}_{3\times d}(K)\leq 2u^2(K)$. Moreover if $d\equiv 0 \bmod 3$ then $(8\frac{d}{3}+2)u^{\#}_{3\times d}(K)\geq|\sigma_{\frac{1}{3}}(K)|$.*

Proof To prove the first inequality notice that the proof of our Proposition 2.1 implies that $\mathcal{U}^2(p,n)\subset\mathcal{U}^{\#}_{3\times(2+i)}(p+n,p+n)$ for any $i\geq 2$. To prove the second inequality use our lemma 3.1 and P. Gilmer's thesis as in [A3] to get that if $K\in\mathcal{U}^{\#}_{3\times d}(p,n)$ then $2(p+n)\geq|8\frac{d}{3}(p-n)+\sigma_{\frac{1}{3}}(K)$. This finishes the proof of this lemma.

Now we want to estimate $u^{\#}_{4\times d}$. As always the estimates will fall out of lemma 3.1 and P. Gilmer's theorem. In this way we can get estimates involving σ_x for $x=\frac{1}{2},\frac{1}{4}$ as follows:

Theorem 3.2. *Let $K\in\mathcal{K}$ be any knot. Then:*

$$d\equiv 0 \bmod 4 \implies (6d+2)u^{\#}_{4\times d}(K)\geq|\sigma_{\frac{1}{4}}(K)|,$$

$$d\equiv 0 \bmod 2 \implies (4d+2)u^{\#}_{4\times d}(K)\geq|\sigma_{\frac{1}{2}}(K)|,\quad\text{and}$$

$$d\equiv 1 \bmod 2 \implies (4d+6)u^{\#}_{4\times d}(K)\geq|\sigma_{\frac{1}{2}}(K)|$$

Proof They are all proved combining lemma 3.1 and P. Gilmer's theorem. We only need to make a remark in the third case when d is odd. In that case if $K \in \mathcal{U}^{\#}_{4\times d}(p,n)$ then there is an embedded sphere in $\#_p(S^2 \times S^2 \#_4 CP^2)\#_n(S^2 \times S^2 \#_4 \overline{CP}^2)$ which is smooth everywhere except at a point where it is a cone on K and on homology it represents $\oplus_p(4,(d-1)) \oplus_4 (2)) \oplus_p (-4,(d-1)) \oplus_4 (2))$ in the obvious standard basis. Then applying Gilmer's inequality we get respectively in these three cases:

$$2(p+n) \geq |6d(p-n) + \sigma_{\frac{1}{4}}(K)|$$
$$2(p+n) \geq |4d(p-n)\sigma_{\frac{1}{2}}(K)|,$$
$$6(p+n) \geq |4d(p-n)\sigma_{\frac{1}{2}}(K)|.$$

This finishes the proof.

We should remind the reader that in case d is even a $4 \times d$ grid unknots only Arf invariant zero knots. Estimates on unknotting number one knots in these unknotting operations can be found using minimal representing genus information which comes from gauge theory. We wish to finally explain part of the motivation for looking for these unknotting operations and bring up some issues we have not managed to resolve.

4. The background philosophy and some ideas

In [A3] I generalized H. Murakami's ([M]) #-unknotting operation to what I called multi-# operations. At the time I realized that multi-# local moves are unknotting operations just by working with non-orientable spanning surfaces as Murakami does and for a long time I tried to think of them in terms of canonical planar projections in the spirit of S. Suzuki's [S]. In this way I came to also realize that a ± 1-full twist on four parallel like-oriented strings is also an unknotting operation. Although this method cannot be replaced for the purpose of showing them to be unknotting operations it is worth noting that in time I realized that there was something more at work. In what follows I want to explain what I mean.

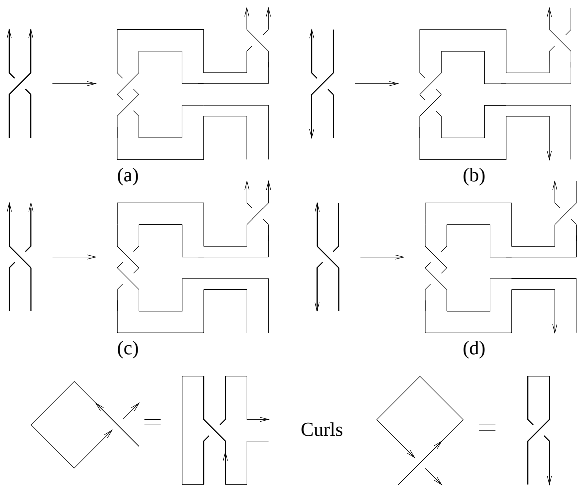

FIGURE 12

The classical unknotting operation (± 1 full twist on two strings) results, as it has over time been observed and used by various workers in low-dimensional topology, in making a knot the (unique) "singularity type" of an otherwise smoothly embedded sphere in a connected sum of CP^2's and $\overline{CP}^2$'s representing a homology class which is divisible by 2 (it is actually this very fact we are using in order to estimate their unknotting numbers). Moreover one can at will add CP^2's and $\overline{CP}^2$'s in this set-up by an inconsequential curl on the knot as well as choose whether the homology class has zero or two as a coordinate for every given connect summand. In the figure above (a) indicates how changing a positive crossing allows the knot to pick up $2 \in H_2(CP^2)$, (b) indicates how changing a negative crossing results in the knot picking up $0 \in H_2(CP^2)$, and similarly (c): change negative crossing and pick up $2 \in H_2(\overline{CP}^2)$ and (d): change positive crossing and pick up $0 \in H_2(\overline{CP}^2)$. The rest of the figure indicates the inconsequential curls.

This means that our knot can be the singularity type of any homology class just as long as we keep the square, divisibility and type of the class fixed. This can be seen by using C. T. C. Wall's results on the transitivity of the orthogonal group of the intersection form of the four manifold and their realizability by self diffeomorphisms.

We elaborate further. Suppose that $K \in \mathcal{U}_2(p,n)$ (i.e. it can be unknotted by changing p (resp. n) positive (resp. negative) crossings). Let us also assume that $u_2(K) = p + n$. Then $K \in \mathcal{U}_2(p + c_+, n + c_-)$ for all integers $c_+, c_- \geq 0$. (add c_+ positive and c_- negative curls). Now we can see a sphere in $M^4 = \#_{p_2} CP^2 \#_{n_0} CP^2 \#_{p_0} \overline{CP}^2 \#_{n_2} \overline{CP}^2$ smooth everywhere except at one point where it is the cone on K which on homology represents $\lambda = \oplus_{p_2}(2) \oplus_{n_0}(0) \oplus_{p_0}(0) \oplus_{n_2}(2) \in H_2(M^4)$ in the standard basis, where $p_2 + p_0 = p + c_+$ and $n_2 + n_0 = n + c_-$. This element has divisibility equal to 2 and square: $\lambda^2 = 4(p_2^2 - n_2^2)$. The underlying primitive class will be characteristic or ordinary according as $n_0 = p_0 = 0$ or not. Now we can make our choice of $\lambda' \in H_2(M^4)$ so that λ' and λ have the same divisibility, type of underlying primitive class and square, as well as make a choice of basis to express this element in. These two choices will yield "canonical" representatives of the concordance class of K and the form of these representatives will reveal certain local moves as *almost slicing operations* in the sense of [A3].

We need first to make some Morse theoretic considerations. Suppose we have a framed link $\cup_i (L_i, m_i)$ representing a 4-manifold with boundary S^3. Then our manifold is made of a 0-handle a 4-handle and some 2-handles (as many as the components of the framed link). Suppose also that we have a 2-sphere in the 4-manifold which is smooth everywhere except at one point where it is the cone on a knot K and represents a certain class $\alpha = \sum_i x_i [L_i]$ on homology. Then we can assume that K is in the boundary of the 0-handle and the cone point in the center of the latter. On the boundary of the 4-handle we see a number of oriented small circles dual to the components of the link so that algebraically they link L_i, x_i times. We can add pairs of circles of opposite orientation at will for each one of the components by changing the sphere as in the figure below, which shows the tubing of a cylinder linking a 2-sphere in the four manifold corresponding to L_i to the singular sphere.

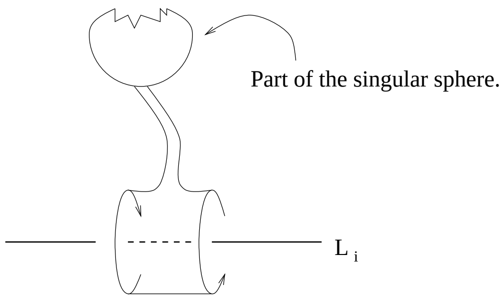

FIGURE 13

In the obvious way now this situation gives rise to a punctured disk in $S^3 \times I$ from our knot $KS^3 \times 0$ to a certain link L_α in $S^3 \times 1$ made out of the trivial link of small circles dual to our framed link as follows: desolve all the components of the framed link away from our trivial link by using Kirby calculus moves and follow how our trivial link of small circles changes.

Isotoping the embedding of this punctured disk so that the restriction of the projection $S^3 \times I \to I$ on it is a Morse function we see that the embedding has isolated critical points whose flow lines of index one are of two kinds: They are either "blind" (i.e flow into a local minimum or local maximum) or they flow all the way down to the other end to one of the components of L_α. We can assume that after a certain point of I there are no more blind critical values as in the figure below.

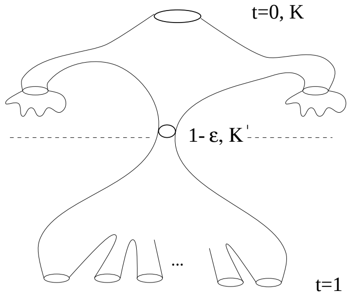

FIGURE 14

Up to $1-\epsilon$ we have arrive from K to $K^{'}$ within the same concordance class. Now we can assume that the critical points whose flow lines flow to components down to the other end appear all at once. This means in terms of isotopy of the knot $K^{'}$ that after the drastic introduction of bands which split $K^{'}$ to various components there is an isotopy of leading on the other side to L_α. If we then keep track of the dual bands we can get a canonical form of $K^{'}$ at the other end. In the figure below we have an example of $1 \in H_2(CP^2)$. We drew only one of the dual bands in the final form. It is clear that now in this example $K^{'}$ becomes ribbon after one negative full twist on all the strings (in terms of Fox's congruence $K^{'}$ is equivalent modulo $n = 1, q = 1$ to a ribbon knot via one move.

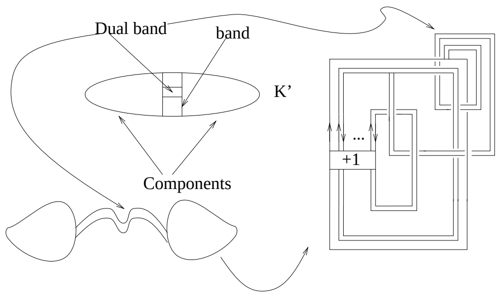

FIGURE 15

After introducing some notation we can state this as a theorem. Let $L = (L_i, m_i)$ be a framed link which represents a four manifold with boundary S^3. Let $\alpha = \sum_i x_i[L_i]$ be a homology class. Associated to this data we have L_α as above. And then the set $\mathcal{K}_\alpha$ of all knots obtained from the link L_α by banding the components in any way possible into a knot. Now we are ready to formulate our discussion above into:

Theorem 4.1. *Let M^4 be the four manifold corresponding to a framed link L as above. Suppose that there is a 2-sphere representing α on homology which is smooth everywhere except at a point where it is the cone on K. Then the concordance class of K contains a representative which belongs in $\mathcal{L}_\alpha$.*

As an example this theorem states that if M^4 is $S^2 \times S^2$ (i.e. L is the Hopf link of zero framed components) and $\alpha = (2, 2)$ then K has a representative which can be sliced via a $(2, r_1) \times (2, r_2)$ grid.

Now let us go back to the discussion we left off on page 10. Using what we have discussed in the mean time we can see that if $K \in \mathcal{U}_2(p + c_+, n + c_-)$ then we can look at our manifold M^4 as $\#_{\frac{p+c_++n+c_--1}{2}} S^2 \times S^2 \# CP^2$. as in the figure below where we have also written our homology class.

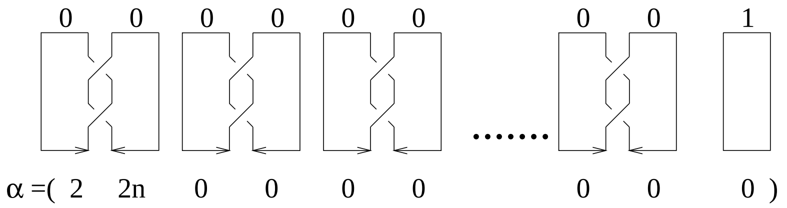

FIGURE 16

By our discussion above we have:

Theorem 4.2. *Every knot K is concordant to a knot K' which can be turned into a ribbon knot by grids of type $a \times b$ for any $a, b \geq 0$ and possibly one Fox move of type 1,2q.*

This motivates one to try to prove in terms of local moves on knot diagrams that grids are unknotting operations. In the same manner working out the number theoretic features of the situation one can get a statement similar with the one above for knot equivalence modulo $1, q$. Personally I know these to be unknotting operations for $q = 2, 3, 4$ but not higher. Also for the grids in this paper I managed to see that when $a = 3, 4$ then they are unknotting operations but modulo the fact of not having enough time to look into it I could not show it for the rest as all the techniques I managed to organize seem to fail somewhere. Using the same approach but now assuming Murakami's #-unknotting operation one can now see that all grids are slicing operations at least. Whether or not more of them are unknotting operations than we've managed to show is, as of the time of this write-up, unknown to the author. One can develop a more systematic theory of local moves in Fox's language of surgeries as elaborated in [M] for example and using the outline of our approach above show them to be slicing operations modulo perhaps one other move. The ideas in this section will be formalized and made more precise in subsequent work.

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