On the Cohen-Macaulay and Gorenstein properties of multigraded Rees algebras

M. Herrmann, E. Hyry, J. Ribbe

M. Herrmann
J. Ribbe
Mathematisches Institut
der Universität zu Köln
Weyertal 86-90
D-5000 Köln 41
Germany

Eero Hyry
Department of Mathematics
University of Helsinki
Hallituskatu 15
00100 Helsinki
Finland

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 92-66

ON THE COHEN-MACAULAY AND GORENSTEIN PROPERTIES OF MULTIGRADED REES ALGEBRAS

M. Herrmann, E. Hyry, J. Ribbe

0. Introduction

Let A be a ring and $I_1, \dots, I_r \subset A$ ideals. The multigraded Rees algebra (or multi-Rees algebra for short) corresponding to $\mathbf{I} = (I_1, \dots, I_r)$ is by definition $R_A(\mathbf{I}) = A[I_1 t_1, \dots, I_r t_r]$. The geometric object associated to $R_A(\mathbf{I})$ is the multiprojective scheme $\operatorname{Proj} R_A(\mathbf{I})$. Given a $\mathbf{N}^r$-graded ring $S = \bigoplus_{\mathbf{n} \in \mathbf{N}^r} S_{\mathbf{n}}$ finitely generated over $A = S_{\mathbf{0}}$ by elements of degree $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ the corresponding multiprojective scheme $\operatorname{Proj} S$ is defined as follows. Put $S^{++} = \bigoplus_{n_1, \dots, n_r > 0} S_{\mathbf{n}}$. Let $\operatorname{Proj} S$ denote the set of those homogenous primes in S which do not contain S^{++}. As in the ordinary graded case one can give $\operatorname{Proj} S$ a scheme structure and a given set of generators of S determines a closed embedding into some space $\mathbf{P}_A^{k_1} \times_A \dots \times_A \mathbf{P}_A^{k_r}$. Let S^Δ be the subring $\bigoplus_{n \in \mathbf{N}} S_{(n, \dots, n)}$ of S. One then easily sees that the inclusion $S^\Delta \longrightarrow S$ induces an isomorphism of schemes $\operatorname{Proj} S \longrightarrow \operatorname{Proj} S^\Delta$. If $S = R_A(\mathbf{I})$, we have $S^\Delta = R_A(I_1 \cdots I_r)$ The scheme $\operatorname{Proj} R_A(\mathbf{I})$ is thus isomorphic to $\operatorname{Proj} R_A(I_1 \cdots I_r)$, which is the blow-up of $\operatorname{Spec} A$ along the subscheme $V(I_1 \cdots I_r)$.

In this paper we mainly concentrate to the case where all ideals $I_1, \dots, I_r$ are powers of the same ideal I. Let $I_1 = I^{k_1}, \dots, I_r = I^{k_r}$. In this case we see that $\operatorname{Proj} R_A(\mathbf{I})$ is isomorphic to $\operatorname{Proj} R_A(I^{k_1 + \dots + k_r})$. The multi-Rees algebras are thus closely connected to the Rees algebras of powers of ideals. The main purpose of this paper is to study this connection with respect to the Cohen-Macaulay and Gorenstein properties and extend the results of [HRZ] and [R] concerning the Gorenstein properties of Rees algebras of powers of ideals to the case of multi-Rees algebras.

We first have to generalize the result of Trung and Ikeda about the Cohen-Macaulay property of Rees algebras ([TI]) to our situation (Theorem 2.2). If A is a local ring and $I \subset A$ an equimultiple ideal of $\operatorname{ht} I > 0$, it can be shown that the Cohen-Macaulayness of $R_A(\mathbf{I})$ implies that of $R_A(I^{k_1 + \dots + k_r})$. By an example we show that the converse does not hold in general (Example 2.5).

To calculate the local cohomology and canonical modules we use the concept of the Segre product. In studying the Gorenstein properties of multi-Rees algebras our main tool is a structure theorem for the canonical module (Theorem 3.1) similar to that given by Herzog, Simis and Vasconcelos for ordinary Rees algebras in [HSV]. Ikeda characterized the Gorenstein property of a Rees algebra in terms of the canonical modules of the base ring and the associated graded ring ([I]). As a corollary of the structure theorem we are able to extend this characterization to the multigraded case at least when the ideal in question is primary (Theorem 5.3). Our main result (Theorem 5.11) says that if A is a local Cohen-Macaulay ring and $I \subset A$ is an equimultiple ideal of ht > 1 such that $R_A(I)$ is Cohen-Macaulay, then $R_A(\mathbf{I})$ is Gorenstein if and only if $R_A(I^{k_1+\dots+k_r})$ is Gorenstein . The results concerning the Gorensteiness of a Rees algebra of a power of an ideal then say that this is equivalent to $k_1+\ldots+k_r = -a(gr_A(I))-1$. As an application we consider the case where $I = m$ is the maximal ideal of A. It is known that A is regular if and only if $R_A(m^{\dim A-1})$ is Gorenstein. By the result above we are now able to say that A is regular if and only if $R_A(\mathbf{m})$ is Gorenstein for some $k_1,\ldots,k_r$ such that $k_1+\ldots+k_r = \dim A - 1$ (Theorem 5.12). In the case $\dim A = 2$ this was shown us by Shimoda without using cohomological methods.

The main results of this paper were worked out by E. Hyry.

After finishing this manuscript we received a preprint of Goto and Nishida ([GN]). In the theorem (6.15) of this paper they prove a result similar to our Theorem 5.5 for any ideal I of $\operatorname{ht} I > 0$ in a local ring A such that $R_A(\mathbf{I})$ is Cohen-Macaulay, but under the assumptions that $r = 2$ and $k_1 = k_2 = 1$.

1. Preliminaries

The local cohomology theory of multi-graded rings and modules is analogous to that of graded rings and modules. We first fix some notation and recall certain basic facts (for details see [GW1], [GW2] and [HIO]).

We use the following multi-index notation. The norm of a multi-index $\mathbf{n} \in \mathbf{Z}^r$ is $|\mathbf{n}| = n_1+\ldots+n_r$. If $\mathbf{m},\mathbf{n} \in \mathbf{Z}^r$ are multi-indexes, their product $\mathbf{mn} = (m_1n_1,\ldots,m_rn_r)$ and dot-product $\mathbf{m}\cdot\mathbf{n} = m_1n_1+\ldots+m_rn_r$. If $m_i < n_i$ for every i, we set $\mathbf{m} < \mathbf{n}$. Let $\mathbf{1}_i = (0,\ldots,1,\ldots,0)$ $(i = 1,\ldots,r)$ be the canonical base vectors of $\mathbf{Z}^r$. Also denote $\mathbf{1} = (1,\ldots,1)$.

In the following we call $\mathbf{Z}^r$-graded rings and modules r-graded or simply multi-graded. Rings are always assumed to be Noetherian and $\mathbf{N}^r$-graded. Let $S = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} S_\mathbf{n}$ be a r-graded ring. We set $S^+ = \bigoplus_{\mathbf{n}\neq\mathbf{0}} S_\mathbf{n}$. For each $i = 1,\ldots,r$ put also $S_i^+ = \bigoplus_{n_i>0} S_\mathbf{n}$ and $S_i = S/S_i^+$. Let denote $M^r(S)$ be the category of r-graded S-modules.

Sometimes it is also useful to consider the ring S endowed with a different grading. Given a homomorphism $\varphi\colon \mathbf{Z}^r \to \mathbf{Z}^q$ satisfying $\varphi(\mathbf{N}^r) \subset \mathbf{N}^q$ put

$$S^\varphi = \bigoplus_{\mathbf{m}\in\mathbf{N}^q} \Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} S_\mathbf{n} \Big).$$

For any r-graded S-module M denote

$$M^\varphi = \bigoplus_{\mathbf{m}\in\mathbf{Z}^q} \Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} M_\mathbf{n} \Big).$$

We then get a functor $(\ \cdot\ )^\varphi\colon M^r(S) \to M^q(S^\varphi)$. Especially, by choosing φ to be a homomorphism $\mathbf{Z}^r \to \mathbf{Z}$, we can consider S as an ordinary graded ring. In the case φ is the map $\mathbf{n} \mapsto |\mathbf{n}|, \mathbf{Z}^r \to \mathbf{Z}$, we denote $S^{gr} = S^{\varphi}$. When $s \in S_{\mathbf{n}}$, we call $|\mathbf{n}|$ the total degree of s. If φ is the map $\mathbf{n} \mapsto n_i, \mathbf{Z}^r \to \mathbf{Z}$ $(i = 1, \dots, r)$, we denote $S_i = S^{\varphi}$. If $s \in S_{\mathbf{n}}$, we call n_i the i:th partial degree of s.

If $M = \bigoplus_{\mathbf{n} \in \mathbf{Z}^r} M_{\mathbf{n}}$ is an r-graded S-module and $\mathbf{k} \in \mathbf{Z}^r$, set $M(\mathbf{k}) = \bigoplus_{\mathbf{n} \in \mathbf{Z}^r} M_{\mathbf{n}+\mathbf{k}}$. If M, N are r-graded modules, we denote by $[\underline{\mathrm{Hom}}_S(M, N)]_{\mathbf{0}}$ the Abelian group of all degree $\mathbf{0}$ homomorphisms from M into N. We set

$$\underline{\mathrm{Hom}}_S(M, N) = \bigoplus_{\mathbf{n} \in \mathbf{Z}^r} [\underline{\mathrm{Hom}}_S(M, N(\mathbf{n}))]_{\mathbf{0}}.$$

The derived functors of $\underline{\mathrm{Hom}}_S(\cdot, \cdot)$ are $\underline{\mathrm{Ext}}^i_S(\cdot, \cdot)$ $(i \in \mathbf{N})$. If M is finitely generated, it is easy to see that for all r-graded S-modules N and every homomorphism $\varphi: \mathbf{Z}^r \to \mathbf{Z}^q$ satisfying $\varphi(\mathbf{N}^r) \subset \mathbf{N}^q$ $(\underline{\mathrm{Ext}}^i_S(M, N))^{\varphi} = \underline{\mathrm{Ext}}^i_S(M^{\varphi}, N^{\varphi})$.

From now on we assume that $S = \bigoplus_{\mathbf{n} \in \mathbf{N}^r} S_{\mathbf{n}}$, where $S_0 = A$ is a local ring. If m is the maximal ideal of A, the ring S now has a unique homogenous maximal ideal $\mathfrak{M} = m \oplus S^+$. It can be shown that S is Cohen-Macaulay or Gorenstein if and only $S_{\mathfrak{M}}$ is. The local cohomology groups $\underline{H}^i_{\mathfrak{M}}(M)$ are defined in the usual way and their standard properties generalize to the multi-graded case. It is useful to observe that they are compatible with a change of grading:

1.1. Lemma. *Let S be an r-graded ring defined over a local ring and let $\mathfrak{M}$ be the homogenous maximal ideal of S. Let M be an r-graded S-module. If $\varphi: \mathbf{Z}^r \to \mathbf{Z}^q$ is a homomorphism satisfying $\varphi(\mathbf{N}^r) \subset \mathbf{N}^q$, we have*

$$(\underline{H}^i_{\mathfrak{M}}(M))^{\varphi} = \underline{H}^i_{\mathfrak{M}^{\varphi}}(M^{\varphi}).$$

1.2. Remark. If φ is an isomorphism $\mathbf{Z}^r \to \mathbf{Z}^r$ such that $S^{\varphi} = S$ and $M^{\varphi} = M$, it especially follows that $(\underline{H}^i_{\mathfrak{M}}(M))^{\varphi} = \underline{H}^i_{\mathfrak{M}}(M)$.

We shall also often make use of the following fact

1.3. Lemma. *Let S be an r-graded ring defined over a local ring and let $\mathfrak{M}$ be the homogenous maximal ideal of S. Let M be an r-graded S-module such that for some $i \in \{1, \dots, r\}$ $M_{\mathbf{n}} = 0$ if $n_i \neq 0$. Then $[\underline{H}^i_{\mathfrak{M}}(M)]_{\mathbf{n}} = 0$ if $n_i \neq 0$.*

The injective envelope $\underline{E}_S(M)$ of an r-graded S-module M is defined as usual. The injective envelope $\underline{E}_S(k)$ of $k = S/\mathfrak{M}$ is $\underline{\mathrm{Hom}}_A(S, E_A(k))$, where $E_A(k)$ is the ordinary injective envelope of k in the category of A-modules. Here A is interpretated as an r-graded ring concentrated in degree $\mathbf{0}$ and both S and $E_A(k)$ are considered as r-graded A-modules. Every injective module in $M^r(S)$ can be expressed as a direct sum of the modules $\underline{E}_S(S/P)(\mathbf{n})$, where P is an r-homogenous prime of S and $\mathbf{n} \in \mathbf{Z}^r$. Recall also the r-graded version of the theorem of Matlis duality, which says that if M is a Noetherian or Artinian r-graded S-module, we have

$$\underline{\mathrm{Hom}}_S(\underline{\mathrm{Hom}}_S(M, \underline{E}_S), \underline{E}_S) \cong M \otimes_A \widehat{A}.$$

1.4. Definition. *Let S be an r-graded ring defined over a local ring A and let $\mathfrak{M}$ be the homogenous maximal ideal of S. An r-graded S-module ω_S is called a canonical module of S if*

$$\underline{\mathrm{Hom}}_S(\underline{H}^d_{\mathfrak{M}}(S), \underline{E}_S(k)) \cong \omega_S \otimes_A \widehat{A}.$$

If a canonical module exists, it is finitely generated and unique up to an isomorphism.

We shall also need the fact that the canonical module behaves well under a change of grading:

1.5. Lemma. *Let S be an r-graded ring defined over a local ring. Suppose $\varphi\colon \mathbf{Z}^r \to \mathbf{Z}^q$ is a homomorphism satisfying $\varphi(\mathbf{N}^r) \subset \mathbf{N}^q$ and $\varphi^{-1}(0) \cap \mathbf{N}^r = 0$. If S has a canonical module ω_S, so does S^φ and the canonical module of S^φ is*

$$\omega_{S^\varphi} = (\omega_S)^\varphi.$$

Proof. Denote $A = S_0$. Because $\varphi^{-1}(0) \cap \mathbf{N}^r = 0$, we have $[S^\varphi]_0 = A$. Set $d = \dim S$ and let $\mathfrak{M}$ be the homogenous maximal ideal of S. By Lemma 1.1

$$\begin{aligned}
\omega_{S^\varphi} &= \underline{\mathrm{Hom}}_{S^\varphi}\left(\underline{H}^d_{\mathfrak{M}^\varphi}(S^\varphi), \underline{E}_{S^\varphi}(k)\right)\\
&= \underline{\mathrm{Hom}}_A\left(\underline{H}^d_{\mathfrak{M}^\varphi}(S^\varphi), E_A(k)\right)\\
&= \underline{\mathrm{Hom}}_A\left((\underline{H}^d_{\mathfrak{M}}(S))^\varphi, E_A(k)\right)\\
&= \bigoplus_{\mathbf{m}\in\mathbf{Z}^q} \mathrm{Hom}_A\left([(\underline{H}^d_{\mathfrak{M}}(S))^\varphi]_{-\mathbf{m}}, E_A(k)\right)\\
&= \bigoplus_{\mathbf{m}\in\mathbf{Z}^q} \mathrm{Hom}_A\Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} [\underline{H}^d_{\mathfrak{M}}(S)]_{-\mathbf{n}}, E_A(k)\Big)\\
&= \bigoplus_{\mathbf{m}\in\mathbf{Z}^q} \Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} \mathrm{Hom}_A\left([\underline{H}^d_{\mathfrak{M}}(S)]_{-\mathbf{n}}, E_A(k)\right)\Big)\\
&= \bigoplus_{\mathbf{m}\in\mathbf{Z}^q} \Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} [\underline{\mathrm{Hom}}_A(\underline{H}^d_{\mathfrak{M}}(S), E_A(k))]_{\mathbf{n}}\Big)\\
&= \left(\underline{\mathrm{Hom}}_A\left(\underline{H}^d_{\mathfrak{M}}(S), E_A(k)\right)\right)^\varphi\\
&= \left(\underline{\mathrm{Hom}}_S\left(\underline{H}^d_{\mathfrak{M}}(S), \underline{E}_S(k)\right)\right)^\varphi\\
&= (\omega_S)^\varphi.
\end{aligned}$$

Recall next the theorem of local duality ([GW2], Theorem 2.2.2):

1.6. Theorem. *Let S be an r-graded ring of dimension d defined over a complete local ring and let $\mathfrak{M}$ be the homogenous maximal ideal of S. Then S is Cohen-Macaulay if and only if for any finitely generated r-graded S-module M*

$$\underline{\mathrm{Hom}}_S(\underline{H}^i_{\mathfrak{M}}(M), \underline{E}_S(k)) = \underline{\mathrm{Ext}}^{d-i}_S(M, \omega_S).$$

The local duality theorem implies the following corollaries:

1.7. Corollary. *If S is an r-graded Cohen-Macaulay ring having a canonical module ω_S, ω_S is a Cohen-Macaulay S-module with* $\operatorname{depth} \omega_S = \dim S$.

1.8. Corollary. *Let S and T be r-graded rings defined over local rings such that there exists a finite ring homomorphism $S \longrightarrow T$. If S is Cohen-Macaulay and has a canonical module ω_S, also T has a canonical module and*

$$\omega_T = \underline{\operatorname{Ext}}_S^e(T, \omega_S),$$

where $e = \dim S - \dim T$.

1.9. Corollary. *Let S be an r-graded ring defined over a local ring. Then S is Gorenstein if and only if it is Cohen-Macaulay and $\omega_S \cong S(\mathbf{n})$ for some $\mathbf{n} \in \mathbf{Z}^r$.*

Recall that for a non-negatively graded d-dimensional ring R which is defined over a local ring and has the homogenous maximal ideal $\mathfrak{N}$, the a-invariant $a(R)$ is defined as

$$a(R) = \max\{m \in \mathbf{N} | [\underline{H}_{\mathfrak{N}}^d(R)]_m \neq 0\}.$$

Then also

$$a(R) = -\min\{m \in \mathbf{N} | [\omega_R]_m \neq 0\}.$$

1.10. Definition. *Let S be an r-graded ring defined over a local ring. The element $(a(S_1), \dots, a(S_r)) \in \mathbf{Z}^r$, where S_j $(j = 1, \dots, r)$ denotes the ring S graded by the j:th partial degree, is called the a-invariant of S and is denoted by* $\mathbf{a}(S)$.

If S is of dimension d, it is meant in this definition that

$$a(S_j) = \max\{m \in \mathbf{N} | [\underline{H}_{\mathfrak{M}_j}^d(S_j)]_m \neq 0\} \quad (j = 1, \dots, r).$$

Note that by Lemma 1.1

$$[\underline{H}_{\mathfrak{M}_j}^d(S_j)]_m = \bigoplus_{n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_r} [\underline{H}_{\mathfrak{M}}^d(S)]_{(n_1, \dots, n_{j-1}, m, n_{j+1}, \dots, n_r)}.$$

Thus we can say that $\mathbf{a}(S) = (a_1, \dots, a_r)$, where

$$a_j = \max\{n_j | \mathbf{n} \in \mathbf{Z}^r \text{ and } [\underline{H}_{\mathfrak{M}}^d(S)]_{\mathbf{n}} \neq 0\} = -\min\{n_j | \mathbf{n} \in \mathbf{Z}^r \text{ and } [\omega_S]_{\mathbf{n}} \neq 0\}.$$

If S is Gorenstein, it follows that $\omega_S \cong S(\mathbf{a}(S))$.

1.11. Definition. *If S is an r-graded ring, the diagonal subring S^Δ of S is defined as*

$$S^\Delta = \bigoplus_{n \in \mathbf{N}} S_{(n, \dots, n)}.$$

If M is an r-graded S-module, the diagonal submodule M^Δ of M is the r-graded S^Δ-module

$$M^\Delta = \bigoplus_{n\in\mathbf{Z}} M_{(n,\dots,n)}.$$

The correspondence $M \mapsto M^\Delta$ defines a functor $M^r(S) \to M^1(S^\Delta)$.

1.12. Definition. *If S is an r-graded ring and $\mathbf{k} \in (\mathbf{N}^*)^r$, the Veronesian subring $S^{(\mathbf{k})}$ of S is defined as*

$$S^{(\mathbf{k})} = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} S_{\mathbf{kn}}.$$

If M is an r-graded S-module, the Veronesian submodule $M^{(\mathbf{k})}$ of M is the r-graded $S^{(\mathbf{k})}$-module

$$M^{(\mathbf{k})} = \bigoplus_{\mathbf{n}\in\mathbf{Z}^r} M_{\mathbf{kn}}.$$

Also now we obtain a functor $M^r(S) \to M^r(S^{(\mathbf{k})})$.

Let S be an r-graded ring defined over a local ring A with the homogenous maximal ideal $\mathfrak{M}$. If M is an r-graded S-module, we say that M has the property (A) if either of the following conditions hold:

(A1) Every $x \in M$ is annihilated by some power of $\mathfrak{M}$ (which is of course equivalent to requiring that $\operatorname{Supp} M = \{\mathfrak{M}\}$).

(A2) For some $\mathbf{n} \in \mathbf{N}^r$ there exists an element $s \in \mathfrak{M}_{\mathbf{n}}$ such that the induced multiplication map $M(-\mathbf{n}) \to M$ is an isomorphism.

If M is an r-graded S-module satisfying the property (A), it easily follows that $H^i_{\mathfrak{M}}(M) = 0$ for $i > 0$.

1.13. Lemma. *Let S be an r-graded ring generated by elements of total degree one over a local ring S_0 and let I be an injective S-module. Let $\mathbf{k} \in (\mathbf{N}^*)^r$. Then $I^{(\mathbf{k})}$ has the property (A).*

Proof. Let $\mathfrak{M}$ be the homogenous maximal ideal of S. Let m be the maximal ideal of S_0. It is enough to consider the case $I = \underline{E}_S(S/P)$, where P is an r-homogenous prime of S. If $P = \mathfrak{M}$, I and thus also $I^{(\mathbf{k})}$ clearly satisfy the condition (A1). Suppose then that $P \neq \mathfrak{M}$. We shall show that $I^{(\mathbf{k})}$ satisfies the condition (A2). If $P_0 \neq m$, multiplication by an element $a \in m$, $a \notin P$ gives an isomorphisn $I \to I$, which induces an isomorphism $I^{(\mathbf{k})} \to I^{(\mathbf{k})}$. Thus assume $P_0 = m$. Since $P \neq \mathfrak{M}$, there is $i \in \{1,\dots,r\}$ such that $S_{\mathbf{1}_i} \not\subset P$. Choose $a \in S_{\mathbf{1}_i}$ $a \notin P$. Multiplication by a^{k_i} then gives an isomorphism $I(-k_i\mathbf{1}_i) \to I$ and hence also an isomorphism $I^{(\mathbf{k})}(-\mathbf{1}_i) \to I^{(\mathbf{k})}$.

1.14. Lemma. *Let S be an r-graded ring generated by elements of total degree one over a local ring S_0 and let $\mathfrak{M}$ be the homogenous maximal ideal of S. Let $\mathbf{k} \in (\mathbf{N}^*)^r$. Then $(\underline{H}^i_{\mathfrak{M}}(M))^{(\mathbf{k})} = \underline{H}^i_{\mathfrak{M}^{(\mathbf{k})}}(M^{(\mathbf{k})})$.*

Proof. Let $0 \longrightarrow M \longrightarrow \mathbf{I}$ be an injective resolution of M. By Lemma 1.13 $0 \longrightarrow M^{(\mathbf{k})} \longrightarrow \mathbf{I}^{(\mathbf{k})}$ is a resolution of $M^{(\mathbf{k})}$ which satisfies the condition (A). Then $\underline{H}^i_{\mathfrak{M}^{(\mathbf{k})}}((I^j)^{(\mathbf{k})}) = 0$ when $i > 0$ for every $j \in \mathbf{N}$, so that we can use this resolution to compute the local cohomology of $M^{(\mathbf{k})}$. Because $\underline{H}^0_{\mathfrak{M}^{(\mathbf{k})}}((I^j)^{(\mathbf{k})}) = (\underline{H}^0_{\mathfrak{M}}(I^j))^{(\mathbf{k})}$, the claim follows.

1.15. Corollary. *Let S be an r-graded ring generated by elements of total degree one over a local ring S_0. Let $\mathbf{k} \in (\mathbf{N}^*)^r$. If $\dim S = \dim S^{(\mathbf{k})}$ and S has a canonical module ω_S, so does $S^{(\mathbf{k})}$ and the canonical module of $S^{(\mathbf{k})}$ is*

$$\omega_{S^{(\mathbf{k})}} = (\omega_S)^{(\mathbf{k})}.$$

Proof. Set $d = \dim S$. Denote $A = S_0$ and let $\mathfrak{M}$ be the homogenous maximal ideal of S. According to Lemma 1.14 we have

$$\begin{aligned}
\omega_{S^{(\mathbf{k})}} &= \underline{\mathrm{Hom}}_{S^{(\mathbf{k})}}\big(\underline{H}^d_{\mathfrak{M}^{(\mathbf{k})}}(S^{(\mathbf{k})}), \underline{E}_{S^{(\mathbf{k})}}(k)\big) \\
&= \underline{\mathrm{Hom}}_A\big(\underline{H}^d_{\mathfrak{M}^{(\mathbf{k})}}(S^{(\mathbf{k})}), E_A(k)\big) \\
&= \underline{\mathrm{Hom}}_A\big((\underline{H}^d_{\mathfrak{M}}(S))^{(\mathbf{k})}, E_A(k)\big) \\
&= \bigoplus_{\mathbf{m}\in\mathbf{Z}^r} \mathrm{Hom}_A\big([(\underline{H}^d_{\mathfrak{M}}(S))^{(\mathbf{k})}]_{-\mathbf{m}}, E_A(k)\big) \\
&= \bigoplus_{\mathbf{m}\in\mathbf{Z}^r} \mathrm{Hom}_A\big([\underline{H}^d_{\mathfrak{M}}(S)]_{-\mathbf{km}}, E_A(k)\big) \\
&= \bigoplus_{\mathbf{m}\in\mathbf{Z}^r} [\underline{\mathrm{Hom}}_A\big(\underline{H}^d_{\mathfrak{M}}(S), E_A(k)\big)]_{\mathbf{km}} \\
&= \big(\underline{\mathrm{Hom}}_A\big(\underline{H}^d_{\mathfrak{M}}(S), E_A(k)\big)\big)^{(\mathbf{k})} \\
&= \big(\underline{\mathrm{Hom}}_S\big(\underline{H}^d_{\mathfrak{M}}(S), \underline{E}_S(k)\big)\big)^{(\mathbf{k})} \\
&= (\omega_S)^{(\mathbf{k})}.
\end{aligned}$$

1.16. Definition. *Let A be a ring and let $I_1, \ldots, I_r \subset A$ be ideals. Set $\mathbf{I} = (I_1, \ldots, I_r)$. If $\mathbf{n} \in \mathbf{N}^r$, denote the product $I_1^{n_1} \cdots I_r^{n_r}$ by $\mathbf{I}^{\mathbf{n}}$. The multi-Rees ring $R_A(\mathbf{I})$ is the r-graded ring*

$$R_A(\mathbf{I}) = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} \mathbf{I}^{\mathbf{n}}.$$

Furthermore, for every $i = 1, \ldots, r$ the i:th associated r-graded ring is defined as

$$gr_A(\mathbf{I}; I_i) = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} \mathbf{I}^{\mathbf{n}}/\mathbf{I}^{\mathbf{n}+\mathbf{1}_i}.$$

Clearly $gr_A(\mathbf{I}; I_i) = R_A(\mathbf{I})/I_i R_A(\mathbf{I})$. We shall often identify $R_A(\mathbf{I})$ with the subring $A[I_1t_1, \ldots, I_rt_r]$ of $A[t_1, \ldots, t_r]$.

1.17. Proposition. *Let A be a ring and let $I_1, \ldots, I_r \subset A$ be ideals such that* $\operatorname{ht} I_i > 0$ *for every* $i = 1, \ldots, r$. *Set* $\mathbf{I} = (I_1, \ldots, I_r)$. *Then* $\dim R_A(\mathbf{I}) = d+r$. *Moreover, if A is local, we have* $\dim gr_A(\mathbf{I}; I_i) = d + r - 1$ $(i = 1, \ldots, r)$.

Proof. Set $\mathbf{J} = (I_1, \ldots, I_{r-1})$ and $B = R_A(\mathbf{J})$. Since

$$A[I_1 t_1, \ldots, I_r t_r] = A[I_1 t_1, \ldots, I_{r-1} t_{r-1}][I_r t_r],$$

we clearly have $R_A(\mathbf{I}) = R_B(I_r B)$. Because the case $r = 1$ is well known (See [M], Theorem 15.7 and its proof), the claim follows easily by using induction on r.

In this paper we concentrate to the case where all the ideals $I_1, \ldots, I_r$ are powers of the same ideal $I \subset A$. We use the notation $\mathbf{I}_r$ for the r-tuple $(I, \ldots, I)$. In this case all the associated r-graded rings coincide and we denote $gr_A(\mathbf{I}) = gr_A(\mathbf{I}; I_i)$ for $i = 1, \ldots, r$.

2. On the Cohen-Macaulay property of multi-Rees algebras

In this section we shall show that the theorem of Trung and Ikeda concerning the Cohen-Macaulay property of Rees algebras ([TI]) can be generalized to the case of multi-Rees algebras. We need the following variant of the original version of this theorem. For the convenience of the reader we repeat the details of the proof.

2.1. Lemma. *Let A be a multi-graded ring of dimension d defined over a local ring and let $I \subset A$ be a homogenous ideal of* $\operatorname{ht} I > 0$. *If $\mathfrak{m}$ is the homogenous maximal ideal of A, denote* $\mathfrak{M} = \mathfrak{m} \oplus (R_A(I))^+$.

a) The following conditions are equivalent for every $q \in \mathbb{N}$:

(1) $[\underline{H}^i_{\mathfrak{M}}(R_A(I))]_n = 0$ *when* $i < d + 1$ *and* $n \notin \{-q, \ldots, -1\}$.

(2) $[\underline{H}^i_{\mathfrak{M}}(gr_A(I))]_n = 0$ *when* $i < d$ *and* $n \notin \{-q-1, \ldots, -1\}$, $a(gr_A(I)) < 0$.

b) We have $a(R_A(I)) = -1$.

Proof. Put $S = R_A(I)$ and $G = gr_A(I)$. We first show that A can be replaced by a local ring. By flatness $\underline{H}^i_{\mathfrak{M}}(S) \otimes_A A_{\mathfrak{m}} = \underline{H}^i_{\mathfrak{M} \otimes_A A_{\mathfrak{m}}}(S \otimes_A A_{\mathfrak{m}})$. Let A be r-graded. We may then consider S also as a $(r+1)$-graded ring with the homogenous maximal ideal $\mathfrak{M}$. According to Lemma 1.1 we have for every $n \in \mathbb{N}$

$$[\underline{H}^i_{\mathfrak{M}}(S)]_n = \bigoplus_{\mathbf{n} \in \mathbf{Z}^r} [\underline{H}^i_{\mathfrak{M}}(S)]_{(n_1, \ldots, n_r, n)},$$

so that each $[\underline{H}^i_{\mathfrak{M}}(S)]_n$ is an r-graded A-module. We observe that $[\underline{H}^i_{\mathfrak{M}}(S)]_n = 0$ if and only if $[\underline{H}^i_{\mathfrak{M} \otimes_A A_{\mathfrak{m}}}(S \otimes_A A_{\mathfrak{m}})]_n = 0$. Similarly we get $[\underline{H}^i_{\mathfrak{M}}(G)]_n = 0$ if and only if $[\underline{H}^i_{\mathfrak{M}\otimes_A A_\mathfrak{m}}(G \otimes_A A_\mathfrak{m})]_n = 0$. Since $S \otimes_A A_\mathfrak{m} = R_{A_\mathfrak{m}}(I_\mathfrak{m})$ and $G \otimes_A A_\mathfrak{m} = gr_{A_\mathfrak{m}}(I_\mathfrak{m})$, this means that A can be assumed local.

(1) $\Rightarrow$ (2) Consider the exact sequences

$$0 \longrightarrow S^+ \longrightarrow S \longrightarrow A \longrightarrow 0,$$

$$0 \longrightarrow S^+(1) \longrightarrow S \longrightarrow G \longrightarrow 0.$$

We get for all $i < d$ and $n \in \mathbb{Z}$ the exact sequences

$$[\underline{H}^i_{\mathfrak{M}}(S)]_n \longrightarrow [\underline{H}^i_{\mathfrak{M}}(A)]_n \longrightarrow [\underline{H}^{i+1}_{\mathfrak{M}}(S^+)]_n \longrightarrow [\underline{H}^{i+1}_{\mathfrak{M}}(S)]_n,$$

$$[\underline{H}^i_{\mathfrak{M}}(S)]_n \longrightarrow [\underline{H}^i_{\mathfrak{M}}(G)]_n \longrightarrow [\underline{H}^{i+1}_{\mathfrak{M}}(S^+)]_{n+1} \longrightarrow [\underline{H}^{i+1}_{\mathfrak{M}}(S)]_n.$$

Since $[H^i_{\mathfrak{M}}(S)]_n = 0$ for $n < -q$ or $n \geq 0$, these sequences imply for $n < -q-1$ or $n \geq 0$ the isomorphisms

$$[\underline{H}^i_{\mathfrak{M}}(G)]_n \cong [\underline{H}^{i+1}_{\mathfrak{M}}(S^+)]_{n+1} \cong [\underline{H}^i_{\mathfrak{M}}(A)]_{n+1}.$$

Because $[\underline{H}^i_{\mathfrak{M}}(A)]_{n+1} = 0$ if $n \neq -1$, it thus follows that $[\underline{H}^i_{\mathfrak{M}}(G)]_n = 0$ if $n < -q-1$ or ≥ 0.

We must then show that $[H^d_{\mathfrak{M}}(G)]_n = 0$ if $n \geq 0$. We use the exact sequences

$$0 \longrightarrow [\underline{H}^d_{\mathfrak{M}}(A)]_n \longrightarrow [\underline{H}^{d+1}_{\mathfrak{M}}(S^+)]_{n+1} \longrightarrow [\underline{H}^{d+1}_{\mathfrak{M}}(S)]_n \longrightarrow 0,$$

$$0 \longrightarrow [\underline{H}^d_{\mathfrak{M}}(G)]_n \longrightarrow [\underline{H}^{d+1}_{\mathfrak{M}}(S^+)]_{n+1} \longrightarrow [\underline{H}^{d+1}_{\mathfrak{M}}(S)]_n \longrightarrow 0.$$

From these we obtain the isomorphisms

$$[\underline{H}^{d+1}_{\mathfrak{M}}(S^+)]_n \cong [\underline{H}^{d+1}_{\mathfrak{M}}(S)]_n \quad (n \neq 0)$$

and the epimorphisms

$$[\underline{H}^{d+1}_{\mathfrak{M}}(S^+)]_{n+1} \longrightarrow [\underline{H}^{d+1}_{\mathfrak{M}}(S)]_n \longrightarrow 0.$$

Since $[\underline{H}^{d+1}_{\mathfrak{M}}(S)]_n = 0$ for $n \gg 0$, diagram chasing gives $[\underline{H}^{d+1}_{\mathfrak{M}}(S^+)]_n = 0$ for $n > 0$. The second exact sequence then implies that $[\underline{H}^d_{\mathfrak{M}}(G)]_n = 0$ for $n \geq 0$.

(2) $\Rightarrow$ (1) Consider then again the exact sequences

$$0 \longrightarrow S^+ \longrightarrow S \longrightarrow A \longrightarrow 0,$$

$$0 \longrightarrow S^+(1) \longrightarrow S \longrightarrow G \longrightarrow 0.$$

For $i < d+1$ and $n \in \mathbf{Z}$, we get the exact sequences

$$[\underline{H}_{\mathfrak{M}}^{i-1}(A)]_n \longrightarrow [\underline{H}_{\mathfrak{M}}^{i}(S^+)]_n \longrightarrow [\underline{H}_{\mathfrak{M}}^{i}(S)]_n \longrightarrow [\underline{H}_{\mathfrak{M}}^{i}(A)]_n,$$

$$[\underline{H}_{\mathfrak{M}}^{i-1}(G)]_n \longrightarrow [\underline{H}_{\mathfrak{M}}^{i}(S^+)]_{n+1} \longrightarrow [\underline{H}_{\mathfrak{M}}^{i}(S)]_n \longrightarrow [\underline{H}_{\mathfrak{M}}^{i}(G)].$$

It follows that there are isomorphisms

$$[\underline{H}_{\mathfrak{M}}^{i}(S^+)]_n \cong [\underline{H}_{\mathfrak{M}}^{i}(S)]_n \quad (n \neq 0),$$

epimorphisms

$$[\underline{H}_{\mathfrak{M}}^{i}(S^+)]_{n+1} \longrightarrow [\underline{H}_{\mathfrak{M}}^{i}(S)]_n \quad (n \geq 0)$$

and monomorphisms

$$[\underline{H}_{\mathfrak{M}}^{i}(S^+)]_{n+1} \longrightarrow [\underline{H}_{\mathfrak{M}}^{i}(S)]_n \quad (n < -q-1).$$

From these sequences it comes out by diagram chasing that $[\underline{H}_{\mathfrak{M}}^{i}(S)]_n = 0$ implies $[\underline{H}_{\mathfrak{M}}^{i}(S)]_{n-1} = 0$ for $n > 0$. It also follows that $[\underline{H}_{\mathfrak{M}}^{i}(S)]_n = 0$ implies $[\underline{H}_{\mathfrak{M}}^{i}(S)]_{n+1} = 0$ for $n < -q-1$. We have $[\underline{H}_{\mathfrak{M}}^{i}(S)]_n = 0$ if $n \gg 0$. By [TI], Lemma 2.2 and Theorem 3.3 one knows that $[\underline{H}_{\mathfrak{M}}^{i}(G)]_n = 0$ for $n \ll 0$ implies also $[\underline{H}_{\mathfrak{M}}^{i}(S)]_n = 0$ for $n \ll 0$. It is then easy to see that we have $[\underline{H}_{\mathfrak{M}}^{i}(S)]_n = 0$ for $n < -q$ or $n \geq 0$ as wanted.

The last claim follows similarly by considering the isomorphisms

$$[\underline{H}_{\mathfrak{M}}^{d+1}(S^+)]_n \cong [\underline{H}_{\mathfrak{M}}^{d+1}(S)]_n \quad (n \neq 0),$$

and the epimorphisms

$$[\underline{H}_{\mathfrak{M}}^{d+1}(S^+)]_{n+1} \longrightarrow [\underline{H}_{\mathfrak{M}}^{d+1}(S)]_n \quad (n \in \mathbf{Z}).$$

We must have $[\underline{H}_{\mathfrak{M}}^{d+1}(S)]_{-1} \neq 0$, since from $[\underline{H}_{\mathfrak{M}}^{d+1}(S)]_{-1} = 0$ it would follow that $[\underline{H}_{\mathfrak{M}}^{d+1}(S)]_n = 0$ for $n \leq -1$ and thus $[\underline{H}_{\mathfrak{M}}^{d+1}(S)]_n = 0$ for all $n \in \mathbf{Z}$ which is impossible.

2.2. Theorem. *Let A be a local ring of dimension d and $I \subset A$ an ideal of* $\operatorname{ht} I > 0$. *Let $\mathfrak{M}$ be the homogenous maximal ideal of $R_A(\mathbf{I}_r)$.*

a) The following conditions are equivalent.

(1) $R_A(\mathbf{I}_r)$ is Cohen-Macaulay.

(2) $[\underline{H}_{\mathfrak{M}}^{i}(gr_A(\mathbf{I}_r)]_{\mathbf{n}} = 0$ when $i < d+r-1$ and $\mathbf{n} \neq -\mathbf{1}$,
$\mathbf{a}(gr_A(\mathbf{I}_r)) < \mathbf{0}$.

(3) $[\underline{H}_{\mathfrak{M}}^{i}(R_A(I)]_n = 0$ when $i < d+1$ and $n \notin \{-r+1, \ldots, -1\}$.

(4) $[\underline{H}_{\mathfrak{M}}^{i}(gr_A(I)]_n = 0$ when $i < d$ and $n \notin \{-r, \ldots, -1\}$,
$a(gr_A(I)) < 0$.

b) We have $\mathbf{a}(R_A(\mathbf{I}_r)) = -\mathbf{1}$.

Proof.

(1) $\Leftrightarrow$ (2) Let $S = R_A(\mathbf{I}_r)$ and $G = gr_A(\mathbf{I}_r)$. Let $j \in \{1, \ldots, r\}$. Let S_j (resp. G_j) denote $R_A(\mathbf{I}_r)$ (resp. $gr_A(\mathbf{I}_r)$) graded by the j:th partial degree. If $B = R_A(I_1, \ldots, I_{j-1}, I_{j+1}, \ldots, I_r)$ and $J = I_j B$, we clearly have $S_j = R_B(J)$ and $G_j = gr_B(J)$. By Lemma 2.1 S_j is Cohen-Macaulay if and only if $[\underline{H}^i(G_j)]_n = 0$ for $i < d + r - 1$, $n \neq -1$ and $a(G_j) < 0$. According to Lemma 1.1

$$[\underline{H}^i(G_j)]_n = \bigoplus_{n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_r} [\underline{H}^i(G)]_{(n_1, \ldots, n_{j-1}, n, n_{j+1}, \ldots, n_r)}.$$

Since $j \in \{1, \ldots, r\}$ was arbitrary and $\mathbf{a}(G) = (a(G_1), \ldots, a(G_r))$, the claim follows. Because always $a(S_j) = -1$, we also obtain $\mathbf{a}(S) = (a(S_1), \ldots, a(S_r)) = -1$.

(3) $\Leftrightarrow$ (4) This is an immediate consequence of Lemma 2.1.

(1) $\Leftrightarrow$ (3) Let $q \in \{1, \ldots, r\}$. We shall prove by induction on q that (3) is equivalent to the condition:

(*) $[\underline{H}^i_{\mathfrak{M}}(R_A(\mathbf{I}_q))]_{\mathbf{n}} = 0$ when $i < d + q$ and some $n_j \notin \{-r + q, \ldots, -1\}$.

If $q = 1$, this is the same as (4). Thus assume $q > 1$. Set $S = R_A(\mathbf{I}_q)$ and $T = R_A(\mathbf{I}_{q-1})$. The homogenous components $S_{\mathbf{n}}$ of S are symmetric with respect to $n_1, \ldots, n_q$. By Remark 1.2 the same holds for the homogenous components of $\underline{H}^i_{\mathfrak{M}}(S)$. It follows that we can assume $j \in \{1, \ldots, q-1\}$ in condition (*). Let $\widetilde{S}$ denote S endowed with the grading

$$\widetilde{S} = \bigoplus_{m \in \mathbf{N}^{q-1}} \Big(\bigoplus_{k \geq 0} S_{(m_1, \ldots, m_{q-1}, k)} \Big).$$

According to Lemma 1.1 we have

$$[\underline{H}^i_{\mathfrak{M}}(\widetilde{S})]_{\mathbf{m}} = \bigoplus_k [\underline{H}^i_{\mathfrak{M}}(S)]_{(m_1, \ldots, m_{q-1}, k)}$$

for all $\mathbf{m} \in \mathbf{N}^{q-1}$. This implies that we may now replace S by $\widetilde{S}$ in condition (*). Denote $B = R_A(I)$ and $J = R^+$. Because

$$\widetilde{S} = \bigoplus_{m \in \mathbf{N}^{q-1}} \Big(\bigoplus_{k \geq 0} I^{|\mathbf{m}|+k} \Big) = \bigoplus_{m \in \mathbf{N}^{q-1}} \Big(\bigoplus_{k \geq |\mathbf{m}|} I^k \Big),$$

we observe that $\widetilde{S} = R_B(\mathbf{J}_{q-1})$. Moreover, since

$$J\widetilde{S} = \bigoplus_{m \in \mathbf{N}^{q-1}} \Big(\bigoplus_{k \geq |\mathbf{m}|+1} I^k \Big)$$

we also have $T = gr_B(\mathbf{J}_r)$. Now $\mathbf{a}(T) < 0$. We then conclude by using Lemma 2.1 similarly as in proving (1) $\Leftrightarrow$ (2).

2.3. Corollary. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{ht} I > 0$. *If $R_A(I)$ is Cohen-Macaulay, then $R_A(\mathbf{I}_r)$ is Cohen-Macaulay for all $r \in \mathbb{N}^*$.*

2.4. Corollary. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{ht} I > 0$. *If $R_A(\mathbf{I}_r)$ is Cohen-Macaulay for some $r \in \mathbb{N}^*$, then $R_A(I^q)$ is Cohen-Macaulay for all $q \geq r$.*

Proof. Let $\mathfrak{M}$ be the homogenous maximal ideal of $R_A(I)$. The corollary is an immediate consequence of Theorem 2.2 and the fact that $R_A(I^q) = (R_A(I))^{(q)}$ and $\underline{H}^i_{\mathfrak{M}^{(q)}}((R_A(I))^{(q)}) = (\underline{H}^i_{\mathfrak{M}}(R_A(I)))^{(q)}$ for all $q \in \mathbb{N}^*$

The following example from [HRS] shows that the converse of Corollary 2.4 does not hold in general.

2.5. Example. Let k be a field. Consider the ring $A = k[[x_1, \ldots, x_{11}]]/(x_1^2)$, where $k[[x_1, \ldots, x_{11}]]$ is the formal power series ring over k. Let I denote the ideal generated by all monomials of degree 4 in $x_2, \ldots, x_{11}$ different from $x_2^2 x_3^2$. Let m be the maximal ideal of A. Because A is a hypersurface ring of multiplicity 2 and dimension 10, we know that $R_A(m)$ is Cohen-Macaulay (See [HIO], Corollary (26.5)). One now easily sees that there exists an short exact sequence

$$0 \longrightarrow R_A(I) \longrightarrow R_A(m^4) \longrightarrow k x_2^2 x_3^2(-1) \longrightarrow 0.$$

Let $\mathfrak{M}$ be the homogenous maximal ideal of $R_A(I)$. Since now also $R_A(m^4)$ is Cohen-Macaulay, the corresponding cohomology sequence implies $\underline{H}^i_{\mathfrak{M}}(R_A(I)) = 0$ for $i \neq 1, 11$, but $\underline{H}^1_{\mathfrak{M}}(R_A(I)) \cong k(-1)$. Let $r > 1$. Since $R_A(I^r) = (R_A(I))^{(r)}$ and $\underline{H}^i_{\mathfrak{M}^{(r)}}((R_A(I))^{(r)}) = (\underline{H}^i_{\mathfrak{M}}(R_A(I)))^{(r)}$, we obtain $\underline{H}^i_{\mathfrak{M}}(R_A(I^r)) = 0$ for $i < 11$ so that $R_A(I^r)$ must be Cohen-Macaulay. On the other hand, since $[\underline{H}^i_{\mathfrak{M}}(R_A(I))]_1 \neq 0$, it follows from Theorem 2.2 that $R_A(\mathbf{I}_r)$ can not be Cohen-Macaulay for any $r \in \mathbb{N}^*$.

3. A structure theorem for the canonical module

We shall next show that the theorem of Herzog, Simis and Vasconcelos concerning the structure of the canonical module of an ordinary Rees algebra ([HSV]) generalizes to the case of multi-Rees algebras. Let A be a local ring and $I \subset A$ an ideal. As usual interpret the Rees algebra $R_A(I)$ as a subring of the polynomial ring $A[t]$. The above theorem deals with the situation when the canonical module of $R_A(I)$ is up to shift by -1 isomorphic to the $R_A(I)$-submodule of $A[t]$ generated by $1, \ldots, t^m$, where $m \geq 0$. This submodule is denoted by $(1,t)^m$ and it is then said that the canonical module of $R_A(I)$ is of the expected form. Consider now the multi-Rees algebra $R_A(\mathbf{I}_r)$ and interpret it as a subring of $A[t_1, \ldots, t_r]$. Analogously denote by $(1, t_1, \ldots, t_r)^m$ the $R_A(\mathbf{I}_r)$-submodule of $A[t_1, \ldots, t_r]$ generated by the monomials $\mathbf{t}^{\mathbf{n}} = t_1^{n_1} \cdots t_r^{n_r}$, where $|\mathbf{n}| \leq m$. One easily sees that

$$(1, t_1, \ldots, t_r)^m(-1) = \bigoplus_{\mathbf{n} \geq 1} I^{|\mathbf{n}| - m - r} \mathbf{t}^{\mathbf{n}}.$$

The proof of the following theorem follows the proof presented by S. Zarzuela in the ordinary graded case ([Z]).

3.1. Theorem. *Let A be a local ring and $I \subset A$ an ideal of* ht $I > 0$ *such that $R_A(I)$ is Cohen-Macaulay. Let $r \in \mathbf{N}^*$. Suppose that $a((gr_A(\mathbf{I}_q))^{gr}) < -q$ for all $q \leq r$. Set $S = R_A(\mathbf{I}_r)$, $G = gr_A(\mathbf{I}_r)$ and $a = -a(G^{gr})$. If S has a canonical module, then the following conditions are equivalent:*

$$(a) \qquad \omega_S = \bigoplus_{\mathbf{n} \geq 1} I^{|\mathbf{n}|-a+1} \omega_A \mathbf{t}^{\mathbf{n}}$$

$$(b) \qquad \omega_G = \bigoplus_{\mathbf{n} \geq 1} I^{|\mathbf{n}|-a} \omega_A / I^{|\mathbf{n}|-a+1} \omega_A .$$

Proof. We may assume that A is complete. Corollary 2.3 implies that $R_A(\mathbf{I}_q)$ is Cohen-Macaulay for every $q \in \mathbf{N}^*$. For all $i \in \{1, \dots, r\}$ denote

$$\omega_S^{(i)} = \bigoplus_{n_i \geq 2} [\omega_S]_{\mathbf{n}} .$$

We shall first show that there exists a homomorphism $\tau_i : \omega_S^{(i)} \longrightarrow \omega_S$ of degree $-\mathbf{1}_i$ with the following properties

(i) For all $\alpha \in \omega_S$ and $s \in IS$ $\tau_i(st_i\alpha) = s\alpha$.

(ii) For all $\alpha \in \omega_S$ and $a \in I^{|\mathbf{n}|}$, where $n_i > 0$, $\tau_i(a\mathbf{t}^{\mathbf{n}}\alpha) = (a\mathbf{t}^{\mathbf{n}-\mathbf{1}_i})\alpha$.

(iii) For $|\mathbf{n}| < a - 1$ τ_i induces an isomorphism $[\omega_S]_{\mathbf{n}+\mathbf{1}_i} \longrightarrow [\omega_S]_{\mathbf{n}}$.

(iv) For all $i, j \in \{1, \dots, r\}$ and $\beta \in \omega_S^{(i)} \cap \omega_S^{(j)}$ $(\tau_i\tau_j)\beta = (\tau_j\tau_i)\beta$.

Consider the short exact sequences

$$0 \longrightarrow S_i^+ \longrightarrow S \longrightarrow S_i \longrightarrow 0,$$

$$0 \longrightarrow IS \longrightarrow S \longrightarrow G \longrightarrow 0.$$

By dualizing with ω_S we get the exact sequences

$$0 \longrightarrow \underline{\mathrm{Hom}}_S(S_i, \omega_S) \longrightarrow \underline{\mathrm{Hom}}_S(S, \omega_S) \longrightarrow \underline{\mathrm{Hom}}_S(S_i^+, \omega_S) \longrightarrow$$

$$\longrightarrow \underline{\mathrm{Ext}}^1(S_i, \omega_S) \longrightarrow \underline{\mathrm{Ext}}^1(S, \omega_S) = 0,$$

$$0 \longrightarrow \underline{\mathrm{Hom}}_S(G, \omega_S) \longrightarrow \underline{\mathrm{Hom}}_S(S, \omega_S) \longrightarrow \underline{\mathrm{Hom}}_S(IS, \omega_S) \longrightarrow$$

$$\longrightarrow \underline{\mathrm{Ext}}^1(G, \omega_S) \longrightarrow \underline{\mathrm{Ext}}^1(S, \omega_S) = 0.$$

Since $\dim S_i = \dim G = d + r - 1$, the local duality gives

$$\underline{\mathrm{Hom}}_S(S_i, \omega_S) = \underline{\mathrm{Hom}}_S(\underline{H}^{d+r}_{\mathfrak{M}}(S_i), \underline{E}_S(k)) = 0$$

and

$$\underline{\mathrm{Hom}}_S(G, \omega_S) = \underline{\mathrm{Hom}}_S(\underline{H}^{d+r}_{\mathfrak{M}}(G), \underline{E}_S(k)) = 0.$$

Moreover, we know that

$$\omega_{S_i} = \underline{\mathrm{Ext}}^1_S(S_i, \omega_S) \quad \text{and} \quad \omega_G = \underline{\mathrm{Ext}}^1_S(G, \omega_S).$$

Identify $\underline{\mathrm{Hom}}_S(S, \omega_S)$ with ω_S and let $\Lambda_i : \underline{\mathrm{Hom}}_S(S_i^+, \omega_S) \longrightarrow \underline{\mathrm{Hom}}_S(IS, \omega_S)$ be the degree $\mathbf{1}_i$ homomorphism induced by the homomorphism $s \mapsto st_i, IS \longrightarrow S_i^+$. We obtain the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_S & \xrightarrow{\pi_i} & \underline{\mathrm{Hom}}_S(S_i^+, \omega_S) & \longrightarrow & \omega_{S_i} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \Lambda_i} & & & & \\ 0 & \longrightarrow & \omega_S & \underset{\sigma}{\longrightarrow} & \underline{\mathrm{Hom}}_S(IS, \omega_S) & \longrightarrow & \omega_G & \longrightarrow & 0 \end{array} \tag{$*$}$$

Theorem 2.2 implies that $[\omega_S]_{\mathbf{n}} = 0$ if $n_i \leq 0$. Furthermore, $[\omega_{S_i}]_{\mathbf{n}} = 0$ for $n_i \neq 0$. It then follows that π_i induces an isomorphism

$$\omega_S \longrightarrow \bigoplus_{n_i \geq 1} [\underline{\mathrm{Hom}}_S(S_i^+, \omega_S)]_{\mathbf{n}}.$$

Since Λ gives a degree $\mathbf{1}_i$ isomorphism

$$\bigoplus_{n_i \geq 1} [\underline{\mathrm{Hom}}_S(S_i^+, \omega_S)]_{\mathbf{n}} \longrightarrow \bigoplus_{n_i \geq 2} [\underline{\mathrm{Hom}}_S(IS, \omega_S)]_{\mathbf{n}},$$

we obtain an isomorphism

$$\Omega_i : \omega_S \longrightarrow \bigoplus_{n_i \geq 2} [\underline{\mathrm{Hom}}_S(IS, \omega_S)]_{\mathbf{n}}$$

of degree $\mathbf{1}_i$. We now define $\tau_i : \omega_S^{(i)} \longrightarrow \omega_S$ by setting $\tau_i = \Omega_i^{-1}\sigma$.

The property (i) is now easy to check from the definition. The property (ii) follows from (i). Since $[\omega_G]_{\mathbf{n}+\mathbf{1}_i} = 0$ for $|\mathbf{n}| < a - 1$, it follows that σ induces an isomorphism $[\omega_S]_{\mathbf{n}+\mathbf{1}_i} \longrightarrow [\underline{\mathrm{Hom}}_S(IS, \omega_S)]_{\mathbf{n}+\mathbf{1}_i}$. This implies (iii). To prove (iv) we first note that by (i) we have for all $s \in I^2S$ and $\beta \in \omega_S^{(i)}$

$$(t_i t_j s)((\tau_i \tau_j)\beta) = (t_j s)(\tau_j \beta) = s\beta$$

and

$$(t_jt_is)((\tau_j\tau_i)\beta) = (t_is)(\tau_i\beta) = s\beta,$$

so that

$$(t_it_js)\big((\tau_i\tau_j)\beta - (\tau_j\tau_i)\beta\big) = 0.$$

Denote $\alpha = (\tau_i\tau_j)\beta - (\tau_j\tau_i)\beta$. Now $st_it_j\alpha = 0$ for all $s \in I^2S$ implies that $S_i^+ \cap S_j^+ \subset \operatorname{Ann}\alpha$. Suppose that we would have $\alpha \neq 0$. There would then exist an associated prime P of ω_S such that $\operatorname{Ann}\alpha \subset P$. This would imply $S_i^+ \cap S_j^+ \subset P$, so that $\dim S/P \leq \dim S/S_i^+ \cap S_j^+ < \dim S$. But this is impossible, since ω_S is Cohen-Macaulay. So we must have $\alpha = 0$ and the property (iv) is thus proved.

Observe that property (iii) implies $[\omega_S]_{\mathbf{n}} \cong \omega_A$ if $\mathbf{n} \geq \mathbf{1}$ and $|\mathbf{n}| < a$. Indeed, we immediately obtain $[\omega_S]_{\mathbf{n}} \cong [\omega_S]_{\mathbf{1}}$. Since $a > r$, we have $[\omega_G]_{\mathbf{1}} = 0$. It follows from the diagram $(*)$ that $[\omega_S]_{\mathbf{1}} \cong [\omega_{S_r}]_{\mathbf{1}'}$, where $\mathbf{1}' = (1,\dots,1) \in \mathbb{N}^{r-1}$. By induction we then easily get $[\omega_S]_{\mathbf{1}} \cong \omega_A$, which proves the above claim.

We are now ready to prove that (b) implies (a). We thus assume that

$$\omega_G = \bigoplus_{\mathbf{n}\geq\mathbf{1}} I^{|\mathbf{n}|-a}\omega_A/I^{|\mathbf{n}|-a+1}\omega_A.$$

We shall first prove by induction on $|\mathbf{n}|$ that for $\mathbf{n} \geq \mathbf{1}$ and $|\mathbf{n}| \geq a-1$

$$[\omega_S]_{\mathbf{n}+\mathbf{1}_i} = It_i[\omega_S]_{\mathbf{n}}.$$

By the inductive assumption we have $[\omega_S]_{\mathbf{n}} = (I^{|\mathbf{n}|-a+1}\mathbf{t}^{\mathbf{l}})[\omega_S]_{\mathbf{m}}$ for some $|\mathbf{l}| = |\mathbf{n}|-a+1$ and $|\mathbf{m}| = a-1$. By applying the map $\tau^{l_1}\cdots\tau^{l_r}$ we get an isomorphism $[\omega_S]_{\mathbf{n}} \cong I^{|\mathbf{n}|-a+1}[\omega_S]_{\mathbf{m}}$. Since $[\omega_S]_{\mathbf{m}} \cong \omega_A$, we finally get $[\omega_S]_{\mathbf{n}} \cong I^{|\mathbf{n}|-a+1}\omega_A$. Consider the diagram:

$$\begin{array}{lllll}
[\omega_S]_{\mathbf{n}} & & & & \\
 & \searrow^{\Omega_i} & & & \\
 & & [\underline{\operatorname{Hom}}_S(IS,\omega_S)]_{\mathbf{n}+\mathbf{1}} \stackrel{\varrho}{\longrightarrow} & [\omega_G]_{\mathbf{n}+\mathbf{1}_i} & \longrightarrow 0. \\
 & \nearrow_{\sigma} & & & \\
[\omega_S]_{\mathbf{n}+\mathbf{1}_i} & & & &
\end{array}$$

Now

$$\tau_i([\omega_S]_{\mathbf{n}+\mathbf{1}_i}) = \Omega_i^{-1}(\sigma([\omega_S]_{\mathbf{n}+\mathbf{1}_i})) = \Omega_i^{-1}(\operatorname{Ker}\varrho) = \operatorname{Ker}\varrho\Omega_i.$$

Since

$$[\omega_G]_{\mathbf{n}+\mathbf{1}_i} = I^{|\mathbf{n}|-a+1}\omega_A/I^{|\mathbf{n}|-a+2}\omega_A,$$

we get $I[\omega_S]_{\mathbf{n}} \subset \operatorname{Ker}\varrho\Omega_i$. As

$$[\omega_S]_{\mathbf{n}}/I[\omega_S]_{\mathbf{n}} \cong [\omega_G]_{\mathbf{n}+\mathbf{1}_i} \cong [\omega_S]_{\mathbf{n}}/\operatorname{Ker}\varrho\Omega_i,$$

there exists an exact sequence

$$0 \longrightarrow \operatorname{Ker} \varrho\Omega_i / I[\omega_S]_{\mathbf{n}} \longrightarrow [\omega_G]_{\mathbf{n}+\mathbf{1}_i} \longrightarrow [\omega_G]_{\mathbf{n}+\mathbf{1}_i} \longrightarrow 0.$$

Since any epimorphism from a Noetherian module onto the module itself is an isomorphism, it follows that $\operatorname{Ker} \varrho\Omega_i = I[\omega_S]_{\mathbf{n}}$. Hence

$$\tau_i([\omega_S]_{\mathbf{n}+\mathbf{1}}) = I[\omega_S]_{\mathbf{n}} = \tau_i(It_i[\omega_S]_{\mathbf{n}})$$

Because τ_i is injective, we finally get $[\omega_S]_{\mathbf{n}+\mathbf{1}} = It_i[\omega_S]_{\mathbf{n}}$.

We shall next construct an isomorphism of r-graded S-modules

$$\varepsilon: \omega_S \longrightarrow \bigoplus_{\mathbf{n}\geq\mathbf{1}} I^{|\mathbf{n}|-a+1}\omega_A \mathbf{t}^{\mathbf{n}}.$$

We first show by induction on $|\mathbf{n}|$ that there are A-module isomorphisms

$$\varepsilon_{\mathbf{n}}: [\omega_S]_{\mathbf{n}} \longrightarrow I^{|\mathbf{n}|-a+1}\omega_A \mathbf{t}^{\mathbf{n}} \quad (\mathbf{n} \geq \mathbf{1})$$

satisfying for $\mathbf{n} > \mathbf{1}_i$ and $\alpha \in [\omega_S]_{\mathbf{n}}$ $\varepsilon_{\mathbf{n}}(\alpha) = t_i \varepsilon_{\mathbf{n}-\mathbf{1}_i}(\tau_i(\alpha))$.

If $|\mathbf{n}| = r$ ($\mathbf{n} = \mathbf{1}$), set $\varepsilon_{\mathbf{n}} = 1_{\omega_A}$. Let $|\mathbf{n}| > r$ and suppose that A-module isomorphisms $\varepsilon_{\mathbf{m}}$ have been defined for $|\mathbf{m}| < |\mathbf{n}|$. To define $\varepsilon_{\mathbf{n}}$ by the above formula, we must check that $\varepsilon_{\mathbf{n}}$ is really a map to $I^{|\mathbf{n}|-a+1}\omega_A \mathbf{t}^{\mathbf{n}}$ and the definition is independent of i. If $|\mathbf{n}| \leq a-1$, the first statement is immediately clear since $\tau_i([\omega_{\mathbf{n}}]) = [\omega_S]_{\mathbf{n}-\mathbf{1}_i}$. Let $|\mathbf{n}| > a-1$. Since $[\omega_S]_{\mathbf{n}} = It_i[\omega_S]_{\mathbf{n}-\mathbf{1}_i}$, we have

$$\tau_i([\omega_S]_{\mathbf{n}}) = \tau_i(It_i[\omega_S]_{\mathbf{n}-\mathbf{1}_i}) = I[\omega_S]_{\mathbf{n}-\mathbf{1}_i}$$

and so

$$\varepsilon_{\mathbf{n}}([\omega_S]_{\mathbf{n}}) = t_i\varepsilon_{\mathbf{n}-\mathbf{1}_i}(\tau_i[\omega_S]_{\mathbf{n}}) = It_i\varepsilon_{\mathbf{n}-\mathbf{1}_i}([\omega_S]_{\mathbf{n}-\mathbf{1}_i}) = I^{|\mathbf{n}|-a+1}\omega_A \mathbf{t}^{\mathbf{n}}.$$

To show that the definition does not depend on i suppose that $\mathbf{n} > \mathbf{1}_i$ and $\mathbf{n} > \mathbf{1}_j$. Then for all $\alpha \in [\omega_S]_{\mathbf{n}}$

$$\begin{aligned} t_i\varepsilon_{\mathbf{n}-\mathbf{1}_i}(\tau_i(\alpha)) &= t_i t_j \varepsilon_{\mathbf{n}-\mathbf{1}_i-\mathbf{1}_j}(\tau_j\tau_i(\alpha)) \\ &= t_j t_i \varepsilon_{\mathbf{n}-\mathbf{1}_j-\mathbf{1}_i}(\tau_i\tau_j(\alpha)) \\ &= t_j \varepsilon_{\mathbf{n}-\mathbf{1}_j}(\tau_j(\alpha)). \end{aligned}$$

By the above arguments the bijectivity of $\varepsilon_{\mathbf{n}}$ is clear.

The homomorphisms $\varepsilon_{\mathbf{n}}$ now define an A-linear map

$$\varepsilon: \omega_S \longrightarrow \bigoplus_{\mathbf{n}\geq\mathbf{1}} I^{|\mathbf{n}|-a+1}\omega_A \mathbf{t}^{\mathbf{n}}.$$

To show that ε is S-linear consider $\alpha \in [\omega_S]_{\mathbf{n}}$ and $a \in I^{|\mathbf{m}|}$. Using induction on $|\mathbf{m}|$ and assuming $\mathbf{m} \geq \mathbf{1}_i$ we get

$$\begin{aligned}\varepsilon_{\mathbf{m}+\mathbf{n}}((a\mathbf{t}^{\mathbf{m}})\alpha) &= t_i \varepsilon_{\mathbf{m}+\mathbf{n}-\mathbf{1}_i}(\tau_i((a\mathbf{t}^{\mathbf{m}})\alpha)) \\ &= t_i \varepsilon_{\mathbf{m}+\mathbf{n}-\mathbf{1}_i}((a\mathbf{t}^{\mathbf{m}-\mathbf{1}_i})\alpha) \\ &= (a\mathbf{t}^{\mathbf{m}})\varepsilon_{\mathbf{n}}(\alpha).\end{aligned}$$

We shall now prove that (b) implies (a). From the basic diagram $(*)$ we get the isomorphisms

$$\begin{aligned}\omega_G^{(i)} &\cong \bigoplus_{n_i \geq 2} [\underline{\mathrm{Hom}}_S(IS, \omega_S)]_{\mathbf{n}} / [\sigma(\omega_S^{(i)})]_{\mathbf{n}} \\ &\cong \bigoplus_{n_i \geq 2} [\underline{\mathrm{Hom}}_S(S_i^+, \omega_S)]_{\mathbf{n}-\mathbf{1}_i} / [(\Lambda_i^{-1}\sigma)(\omega_S^{(i)})]_{\mathbf{n}-\mathbf{1}_i} \\ &\cong \bigoplus_{n_i \geq 2} [\omega_S]_{\mathbf{n}-\mathbf{1}_i} / [\tau_i(\omega_S^{(i)})]_{\mathbf{n}-\mathbf{1}_i} \\ &\cong (\omega_S / \tau_i(\omega_S^{(i)}))(-\mathbf{1}_i).\end{aligned}$$

If now

$$\omega_S = \bigoplus_{\mathbf{n} \geq \mathbf{1}} I^{|\mathbf{n}|-a+1} \omega_A \mathbf{t}^{\mathbf{n}},$$

we have

$$\tau_i(\omega_S^{(i)}) = \bigoplus_{\mathbf{n} \geq \mathbf{1}} I^{|\mathbf{n}|-a+2} \omega_A \mathbf{t}^{\mathbf{n}},$$

so that

$$\omega_G^{(i)} \cong \bigoplus_{\mathbf{n} \geq \mathbf{1}, n_i \geq 2} I^{|\mathbf{n}|-a} \omega_A / I^{|\mathbf{n}|-a+1} \omega_A.$$

Since $[\omega_G]_{\mathbf{1}} = 0$, we have

$$\omega_G = \sum_{i=1}^{r} \omega_G^{(i)}.$$

It is now easy to verify that the above isomorphisms are compatible, so that

$$\omega_G \cong \bigoplus_{\mathbf{n} \geq \mathbf{1}} I^{|\mathbf{n}|-a} \omega_A / I^{|\mathbf{n}|-a+1} \omega_A$$

as desired.

3.2. Remark. Let A be a local ring and $I \subset A$ an ideal of ht $I > 0$ such that $R_A(I)$ is Cohen-Macaulay. Assume, moreover, that $\mathrm{Hom}(I, A) = A$. As above set $S = R_A(\mathbf{I}_r)$ and $G = gr_A(\mathbf{I}_r)$. If S has a canonical module, then $[\omega_S]_{\mathbf{1}} \cong A$ implies $a((gr_A(\mathbf{I}_q))^{gr}) < -q$ for all $q \leq r$. Indeed, from $\operatorname{Hom}(I, A) = A$ it follows that every element of $[\operatorname{Hom}_S(IS, \omega_S)]_{\mathbf{1}}$ arises from multiplication of some element of ω_S. The morphism σ in the basic diagram $(*)$ of the preceding proof is then an isomorphism in degree $\mathbf{1}$, so that $[\omega_G]_{\mathbf{1}} = 0$. Then $a(G^{gr}) < -r$, since $[\omega_G]_{\mathbf{n}} = 0$ for $|\mathbf{n}| < r$ by Theorem 2.2. In the the proof of Theorem 3.1 we saw that if σ is an isomorphism we also have an isomorphism $[\omega_S]_{\mathbf{1}} \cong [\omega_{S_t}]_{\mathbf{1}'}$, where $\mathbf{1}' = (1, \ldots, 1) \in \mathbf{N}^{r-1}$. But this means that we may continue the above reasoning to get the claim.

4. Calculation of local cohomology and canonical modules

4.1. Definition. *Let R be a graded ring. We call the r-graded ring*

$$\bigoplus_{\mathbf{n} \in \mathbf{N}^r} R_{|\mathbf{n}|}$$

the r-graded ring corresponding to R and denote it by R^{r-gr}.

Let A be a ring, $I \subset A$ an ideal, $R_A(I)$ the Rees algebra and $gr_A(I)$ the associated graded ring. We now observe that the r-graded rings corresponding to $R_A(I)$ and $gr_A(I)$ are the multi-Rees algebra

$$R_A(\mathbf{I}_r) = \bigoplus_{\mathbf{n} \in \mathbf{N}^r} I^{|\mathbf{n}|} = (R_A(I))^{r-gr}$$

and the associated r-graded ring

$$gr_A(\mathbf{I}_r) = \bigoplus_{\mathbf{n} \in \mathbf{N}^r} I^{|\mathbf{n}|}/I^{|\mathbf{n}|+1} = (gr_A(I))^{r-gr}.$$

Given a graded ring R with a canonical module it would therefore be nice to express the canonical module of R^{r-gr} in terms of the canonical module of R. To that purpose recall the notion of Segre product of two graded rings. We generalize this concept slightly and define the Segre product of a graded ring and an r-graded ring.

4.2. Definition. *If R is a graded ring and T is an r-graded ring defined over a ring A, their Segre product is the r-graded ring*

$$R \sharp T = \bigoplus_{\mathbf{n} \in \mathbf{N}^r} R_{|\mathbf{n}|} \otimes_A T_{\mathbf{n}}.$$

If M is a graded R-module and N a graded T-module, the Segre product $M \sharp N$ is the r-graded $R \sharp T$-module

$$M \sharp N = \bigoplus_{\mathbf{n} \in \mathbf{Z}^r} M_{|\mathbf{n}|} \otimes_A N_{\mathbf{n}}.$$

4.3. Remark. Note that $(R\sharp T)^{gr} = R\sharp T^{gr}$. Here $R\sharp T^{gr}$ is the usual Segre product of the graded rings R and T^{gr}. For the properties of Segre product see [GW1].

Let R be a graded ring defined over a ring A. If $A[t_1, \dots, t_r]$ is the polynomial ring over A, we clearly have

$$R^{r-gr} = R\sharp A[t_1, \dots, t_r].$$

We are thus interested about Segre products of type $R\sharp A[t_1, \dots, t_r]$. Goto and Watanabe have calculated the local cohomology of the Segre product of two graded rings defined over a field ([GW1], Theorem (4.1.5)). We shall show that their arguments can be generalized to the above situation in the case A is an Artinian ring. First we need the following elementary lemma and some further notation.

4.4. Lemma. *Let A be a local ring and let*

$$\mathbf{F}: \quad F^0 \longrightarrow \dots \longrightarrow F^i \longrightarrow F^{i+1} \longrightarrow \dots \longrightarrow F^n \longrightarrow 0$$

be a finite free complex of A-modules. If the complex $H(\mathbf{F})$ is also free, so are the complexes $Z(\mathbf{F})$ and $B(\mathbf{F})$.

Proof. Set $Z^i = Z^i(\mathrm{F}), B^i = B^i(\mathrm{F})$ and $H^i = H^i(\mathrm{F})$. We shall show by descending induction on i that Z^i and B^i are free. The case $i > n$ being clear, we assume that Z^i and B^i are free. Because a module over a local ring is free if and only if it is projective, we get from the exact sequence

$$0 \longrightarrow Z^{i-1} \longrightarrow F^{i-1} \longrightarrow B^i \longrightarrow 0,$$

that Z^{i-1} is free. The exact sequence

$$0 \longrightarrow B^{i-1} \longrightarrow Z^{i-1} \longrightarrow H^{i-1} \longrightarrow 0$$

then implies that also B^{i-1} is free.

Let S be an r-graded ring defined over a local ring A. Let $\mathbf{E}$ be a complex of r-graded S-modules. Recall the property (A) and the properties (A1) and (A2) mentioned in Chapter 1. Suppose that each E^i is a direct sum of r-graded S-modules satisfying the property (A) (for example, any injective resolution). The following notation is then used. Let $'E^i$ denote the direct sum of those summands which satisfy the property (A1). The $'E^i$:s now form a subcomplex $'\mathbf{E} = ('E^i)$. Let $''\mathbf{E}$ be the quotient complex $\mathbf{E}/'\mathbf{E}$. Each $''E^i$ is then isomorphic to the direct sum of those summands of E^i which satisfy the property (A2).

4.5. Lemma. *Let R be a graded ring and T an r-graded ring defined over a local ring A. Let M be a graded R-module and N an r-graded T-module. Consider the r-graded $R\sharp T$-module $M\sharp N$.*

a) If M or N satisfy the property (A1), so does $M\sharp N$.

b) If M and N satisfy the property (A2), so does $M\sharp N$.

Proof. The first claim follows immediately from the definitions. Let us prove the second claim. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the homogenous maximal ideals of R and T respectively. By assumption there are for some $k \in \mathbf{N}$ and $\mathbf{n} \in \mathbf{N}^r$ elements $x \in \mathfrak{M}_k$ and $y \in \mathfrak{N}_{\mathbf{n}}$ such that the induced multiplication maps $M(-k) \to M$ and $N(-\mathbf{n}) \to N$ are isomorphisms. If $k = 0$ or $\mathbf{n} = 0$, multiplication by x or y respectively induces an isomorphism $M\sharp N \to M\sharp N$. Otherwise multiplication by $x^{|\mathbf{n}|}\sharp y^k$ gives an isomorphism $(M\sharp N)(-k\mathbf{n}) \to M\sharp N$.

4.6. Theorem. *Let R be a graded ring and T an r-graded ring defined over a local ring A. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the homogenous maximal ideals of R and T respectively. Let $\mathfrak{B}$ be the homogenous maximal ideal of the Segre product $R\sharp T$. Let M be a graded R-module and N an r-graded T-module. Assume that N is free as an A-module and $\underline{H}^i_{\mathfrak{N}}(N) = 0$ for $i = 0, 1$. Also assume that there exists a finite resolution $0 \to N \to \mathbf{F}$ such that $\mathbf{F}$ is free as a complex of A-modules, each F^i is a direct sum of r-graded S-modules satisfying the property (A) and that the complexes $H(\mathbf{F})$ and $H(''\mathbf{F})$ are free as complexes of A-modules. We then have for all $i \in \mathbf{N}$*

$$\underline{H}^i_{\mathfrak{B}}(M\sharp N) = \big(M''\sharp \underline{H}^i_{\mathfrak{N}}(N)\big) \oplus \big(\underline{H}^i_{\mathfrak{M}}(M)\sharp N\big) \oplus \Big(\bigoplus_{p,q>1,p+q=i+1} \underline{H}^p_{\mathfrak{M}}(M)\sharp \underline{H}^q_{\mathfrak{N}}(N) \Big),$$

where M'' is a certain r-graded S-module such that there is an exact sequence

$$0 \longrightarrow \underline{H}^0_{\mathfrak{M}}(M) \longrightarrow M \longrightarrow M'' \longrightarrow \underline{H}^1_{\mathfrak{M}}(M) \longrightarrow 0.$$

Proof. We first remark that if $\mathbf{C}_1$ and $\mathbf{C}_2$ are complexes of r-graded S-modules, one can define their Segre-product $\mathbf{C}_1\sharp\mathbf{C}_2$ by setting

$$(\mathbf{C}_1\sharp\mathbf{C}_2)^i = \bigoplus_{p+q=i} C_1^p\sharp C_2^q$$

and defining the differential $d: \mathbf{C}_1\sharp\mathbf{C}_2 \to \mathbf{C}_1\sharp\mathbf{C}_2$ by the formula

$$d = \bigoplus_{p+q=i} (d_1^p\sharp 1 + (-1)^p 1\sharp d_2^q)$$

where $d_1: \mathbf{C}_1 \to \mathbf{C}_1$ and $d_2: \mathbf{C}_2 \to \mathbf{C}_2$ are the differentials of $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively.

Let $\mathbf{I} = (I^i)$ be an injective resolution of M. We then have $\underline{H}^i_{\mathfrak{M}}(M) = H^i(\underline{H}^0_{\mathfrak{M}}(\mathbf{I})) = H^i('\mathbf{I})$ and $\underline{H}^i_{\mathfrak{N}}(N) = H^i(\underline{H}^0_{\mathfrak{M}}(\mathbf{F})) = H^i('\mathbf{F})$ for all $i \geq 0$. Let $\mathbf{E} = \mathbf{I}\sharp\mathbf{F}$. The assumptions together with Lemma 4.4 now imply that we can apply the Künneth formula ([B], §4, N°7, Corollaire 4) to $H(\mathbf{I}\sharp\mathbf{F})$. It follows that the complex $\mathbf{E} = (E^i)$ is a resolution of $M\sharp N$. By Lemma 4.5 one sees that for every $i \geq 0$

$$'E^i = \bigoplus_{p+q=i} \left(('I^p\sharp' F^q) \oplus (''I^p\sharp' F^q) \oplus ('I^p\sharp'' F^q)\right)$$

and

$$''E^i = \bigoplus_{p+q=i} {''I^p}\sharp'' F^q.$$

Moreover, it is easy to check that we in fact have $''\mathbf{E} ='' \mathbf{I}\sharp''\mathbf{F}$. We now get

$$H^i_{\mathfrak{B}}(M\sharp N) = H^i(\underline{H}^0_{\mathfrak{M}}(\mathbf{E})) = H^i('\mathbf{E}).$$

Consider the exact sequence

$$0 \longrightarrow {'\mathbf{E}} \longrightarrow \mathbf{E} \longrightarrow {''\mathbf{E}} \longrightarrow 0.$$

Because $H^i(\mathbf{E}) = 0$ for $i \geq 1$, it follows that we have for $i \geq 2$ the isomorphisms $H^i('\mathbf{E}) = H^{i-1}(''\mathbf{E})$ and that there exists an exact sequence

$$0 \longrightarrow H^0('\mathbf{E}) \longrightarrow H^0(\mathbf{E}) \longrightarrow H^0(''\mathbf{E}) \longrightarrow H^1('\mathbf{E}) \longrightarrow 0.$$

By the Künneth formula we get

$$H^i(''\mathbf{E}) = \bigoplus_{p+q=i} H^p(''\mathbf{I})\sharp H^q(''\mathbf{F})$$

for all $i \geq 0$ so that

$$H^i_{\mathfrak{B}}(M\sharp N) = \bigoplus_{p+q=i-1} H^p(''\mathbf{I})\sharp H^q(''\mathbf{F})$$

for $i \geq 2$ and we have an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{B}}(M\sharp N) \longrightarrow M\sharp N \longrightarrow H^0(''\mathbf{I})\sharp H^0(''\mathbf{F}) \longrightarrow H^1_{\mathfrak{B}}(M\sharp N) \longrightarrow 0.$$

The exact sequences

$$0 \longrightarrow {'\mathbf{I}} \longrightarrow \mathbf{I} \longrightarrow {''\mathbf{I}} \longrightarrow 0,$$

$$0 \longrightarrow {'\mathbf{F}} \longrightarrow \mathbf{F} \longrightarrow {''\mathbf{F}} \longrightarrow 0$$

now give the isomorphisms

$$H^p(''\mathbf{I}) = H^{p+1}('\mathbf{I}), H^q(''\mathbf{F}) = H^{q+1}('\mathbf{F}) \quad (p, q \geq 1)$$

and the exact sequences

$$0 \longrightarrow H^0('\mathbf{I}) \longrightarrow H^0(\mathbf{I}) \longrightarrow H^0(''\mathbf{I}) \longrightarrow H^1('\mathbf{I}) \longrightarrow 0,$$

$$0 \longrightarrow H^0('\mathbf{F}) \longrightarrow H^0(\mathbf{F}) \longrightarrow H^0(''\mathbf{F}) \longrightarrow H^1('\mathbf{F}) \longrightarrow 0.$$

If $M'' = H^0(''\mathbf{I})$, we thus obtain an exact sequence

$$0 \longrightarrow \underline{H}^0_{\mathfrak{M}}(M) \longrightarrow M \longrightarrow M'' \longrightarrow \underline{H}^1_{\mathfrak{M}}(M) \longrightarrow 0.$$

Since $\underline{H}^i_{\mathfrak{N}}(N) = 0$ for $i = 0, 1$, we get $H^0(''\mathbf{F}) = H^0(\mathbf{F}) = N$. This implies the claim if $i \geq 2$. To prove the claim in the case $i < 2$ we compare the exact sequences

$$0 \longrightarrow H^0_{\mathfrak{B}}(M \sharp N) \longrightarrow M \sharp N \longrightarrow M'' \sharp N \longrightarrow H^1_{\mathfrak{B}}(M \sharp N) \longrightarrow 0$$

and

$$0 \longrightarrow \underline{H}^0_{\mathfrak{M}}(M) \sharp N \longrightarrow M \sharp N \longrightarrow M'' \sharp N \longrightarrow \underline{H}^1_{\mathfrak{M}}(M) \sharp N \longrightarrow 0$$

to get

$$H^0_{\mathfrak{B}}(M \sharp N) = \underline{H}^0_{\mathfrak{M}}(M) \sharp N, H^1_{\mathfrak{B}}(M \sharp N) = \underline{H}^1_{\mathfrak{M}}(M) \sharp N$$

as wanted.

4.7. Lemma. *Let A be a local Artinian ring. Consider the corresponding polynomial ring $T = A[t_1, \ldots, t_r]$. There exists a finite r-graded resolution*

$$0 \longrightarrow T \longrightarrow F^0 \longrightarrow \ldots \longrightarrow F^i \longrightarrow F^{i+1} \longrightarrow \ldots \longrightarrow F^r \longrightarrow 0$$

such that $\mathbf{F}$ and $H(\mathbf{F})$ are free as complexes of A-modules. Moreover, F^r satisfies the property (A1) and for each $i < r$ F_i is a direct sum of r-graded T- modules satisfying the property (A2).

Proof. Take, for example, the Cech-complex

$$\mathbf{C}: \quad T \longrightarrow \bigoplus_i T_{t_i} \longrightarrow \ldots \longrightarrow \bigoplus_i T_{t_1 \cdots \widehat{t_i} \cdots t_r} \longrightarrow T_{t_1 \cdots t_r} \longrightarrow 0.$$

Let $\mathfrak{N}$ be the maximal homogenous ideal of T. Now $H^i(\mathbf{C}) = \underline{H}^i_{\mathfrak{N}}(T)$ for $i \geq 0$. Furthermore, it is well known that for $i < r$ $\underline{H}^i_{\mathfrak{N}}(T) = 0$ and that

$$\underline{H}^r_{\mathfrak{N}}(T) = \bigoplus_{\mathbf{n} < 0} A t_1^{n_1} \cdots t_r^{n_r}.$$

We then observe that

$$0 \longrightarrow T \longrightarrow C_1 \longrightarrow \ldots \longrightarrow C_r \longrightarrow H^r(\mathrm{C}) \longrightarrow 0$$

is a resolution of T with the desired properties.

4.8. Corollary. *Let R be a graded ring of dimension d defined over a local Artinian ring. Let $\mathfrak{M}$ be the homogenous maximal ideal R. Then*

$$\underline{H}^{d+r-1}_{\mathfrak{M}^{r-gr}}(R^{r-gr}) = \bigoplus_{\mathbf{n}<0} [\underline{H}^d_{\mathfrak{M}}(R)]_{|\mathbf{n}|}.$$

Proof. If $T = A[t_1, \ldots, t_r]$ is the polynomial ring over $A = R_0$, we have $R^{r-gr} = R\sharp T$. Since $\underline{H}^i_{\mathfrak{N}}(T) = 0$ for $i < r$ and

$$\underline{H}^r_{\mathfrak{N}}(T) = \bigoplus_{\mathbf{n}<0} At_1^{n_1}\cdots t_r^{n_r},$$

the claim is an immediate consequence of Theorem 4.6 and Lemma 4.7.

4.9. Corollary. *Let R be a graded ring defined over an Artinian ring. Suppose that* $\dim R^{r-gr} = \dim R + r - 1$. *If R has a canonical module, then so does R^{r-gr} and we have*

$$\omega_{R^{r-gr}} = \bigoplus_{\mathbf{n}>0} [\omega_R]_{|\mathbf{n}|}.$$

Proof. Set $d = \dim R$, so that $\dim R^{r-gr} = d + r - 1$. Denote $A = S_0$ and let $\mathfrak{M}$ be the homogenous maximal ideal R. According to Corollary 4.8 we have $[\underline{H}^{d+r-1}_{\mathfrak{M}^{r-gr}}(R^{r-gr})]_{\mathbf{n}} = [\underline{H}^d_{\mathfrak{M}}(R)]_{|\mathbf{n}|}$ if $\mathbf{n} < 0$ and 0 otherwise. Then

$$\begin{aligned}
\omega_{R^{r-gr}} &= \underline{\mathrm{Hom}}_{R^{r-gr}}\big(\underline{H}^{d+r-1}_{\mathfrak{M}^{r-gr}}(R^{r-gr}), \underline{E}_{R^{r-gr}}(k)\big)\\
&= \underline{\mathrm{Hom}}_A\big(\underline{H}^{d+r-1}_{\mathfrak{M}^{r-gr}}(R^{r-gr}), E_A(k)\big)\\
&= \bigoplus_{\mathbf{n}\in\mathbf{Z}^r} \mathrm{Hom}_A\big([\underline{H}^{d+r-1}_{\mathfrak{M}^{r-gr}}(R^{r-gr})]_{-\mathbf{n}}, E_A(k)\big)\\
&= \bigoplus_{\mathbf{n}>0} \mathrm{Hom}_A\big([\underline{H}^d_{\mathfrak{M}}(R)]_{-|\mathbf{n}|}, E_A(k)\big)\\
&= \bigoplus_{\mathbf{n}>0} [\underline{\mathrm{Hom}}_A\big(\underline{H}^d_{\mathfrak{M}}(R), E_A(k)\big)]_{|\mathbf{n}|}\\
&= \bigoplus_{\mathbf{n}>0} [\underline{\mathrm{Hom}}_R\big(\underline{H}^d_{\mathfrak{M}}(R), \underline{E}_R(k)\big)]_{|\mathbf{n}|}\\
&= \bigoplus_{\mathbf{n}>0} [\omega_R]_{|\mathbf{n}|}.
\end{aligned}$$

5. The Gorenstein property of multi-Rees algebras

We begin with the following lemma ([I], proof of Theorem (3.1)):

5.1. Lemma. *Let A be a local ring with an infinite residue field and $I \subset A$ an ideal of $\operatorname{ht} I > 0$ such that $R_A(I)$ is Cohen-Macaulay. There exists then an element $a \in I - I^2$ such that the corresponding degree one element $a^* \in gr_A(I)$ is a non zero-divisor.*

5.2. Lemma. *Let A be a local ring with an infinite residue field and $I \subset A$ an ideal of $\operatorname{grade} I > 1$ such that $R_A(I)$ is Cohen-Macaulay. Let $\mathbf{k} \in (\mathbf{N}^*)^r$. If $S = R_A(\mathbf{I}_r^{\mathbf{k}})$ has the canonical module*

$$\omega_S = \bigoplus_{\mathbf{n} \geq \mathbf{1}} I^{\mathbf{k} \cdot \mathbf{n} - q} \mathbf{t}^{\mathbf{n}},$$

where $q \in \mathbf{N}$, then the canonical module of $S' = R_A(I^{k_1}, \ldots, I^{k_{r-1}})$ is

$$\omega_{S'} = \bigoplus_{n_1, \ldots, n_{r-1} \geq 1} I^{k_1 n_1 + \ldots + k_{r-1} n_{r-1} - q} t_1^{n_1} \cdots t_{r-1}^{n_{r-1}}.$$

Proof. Since S is Cohen-Macaulay by Corollary 2.3, $\omega'_S = \underline{\operatorname{Ext}}_S^1(S/S_r^+, \omega_S)$. According to Lemma 5.1 there exists an element $a \in I - I^2$ such that the corresponding degree one element $a^* \in gr_A(I)$ is a non zero-divisor. Set $s = a^{k_r} t_r$. Multiplication by s gives an exact sequence

$$0 \longrightarrow \omega_S(-\mathbf{1}_r) \xrightarrow{\cdot s} \omega_S \longrightarrow \omega_S / s\omega_S \longrightarrow 0.$$

From this we get the long exact sequence

$$\begin{aligned} 0 \longrightarrow \underline{\operatorname{Hom}}_S(S/S_r^+, \omega_S(-\mathbf{1}_r)) &\longrightarrow \underline{\operatorname{Hom}}_S(S/S_r^+, \omega_S) \\ &\longrightarrow \underline{\operatorname{Hom}}_S(S/S_r^+, \omega_S/s\omega_S) \longrightarrow \underline{\operatorname{Ext}}_S^1(S/S_r^+, \omega_S(-\mathbf{1}_r)) \\ &\longrightarrow \underline{\operatorname{Ext}}_S^1(S/S_r^+, \omega_S) \longrightarrow \ldots \end{aligned}$$

Because $\underline{\operatorname{Hom}}_S(S/S_r^+, \omega_S(-\mathbf{1}_r)) \longrightarrow \underline{\operatorname{Hom}}_S(S/S_r^+, \omega_S)$ is a zero map, we obtain an isomorphism

$$\underline{\operatorname{Hom}}_S(S/S_r^+, \omega_S/s\omega_S) \cong \underline{\operatorname{Ext}}_S^1(S/S_r^+, \omega_S(-\mathbf{1}_r))$$

or

$$\underline{\operatorname{Ext}}_S^1(S/S_r^+, \omega_S) \cong \underline{\operatorname{Hom}}_S(S/S_r^+, \omega_S/s\omega_S)(\mathbf{1}_r).$$

Now

$$\underline{\operatorname{Hom}}_S(S/S_r^+, \omega_S/s\omega_S) \cong (s\omega_S : S_r^+)_{\omega_S} / s\omega_S.$$

Since grade $I > 1$, there exists $b \in I$ such that (a, b) is a regular sequence. If $x \in [(s\omega_S : S_r^+)_{\omega_S}]_{\mathbf{n}}$, we have $b^{k_r} t_r x \in s\omega_S$. Suppose that $x = u\mathbf{t}^{\mathbf{n}}$, where $u \in I^{\mathbf{k}\cdot\mathbf{n}-q}$. Then for some $v \in I^{\mathbf{k}\cdot\mathbf{n}-q}$ we have $b^{k_r} u = a^{k_r} v$, which implies that $u \in (a^{k_r}) \cap I^{\mathbf{k}\cdot\mathbf{n}-q}$. Since $a^* \in gr_A(I)$ is a non zero-divisor, we have

$$(a^{k_r}) \cap I^{\mathbf{k}\cdot\mathbf{n}-q} = I^{\mathbf{k}\cdot\mathbf{n}-q-k_r}.$$

Thus

$$[(s\omega_S : S_r^+)_{\omega_S}]_{\mathbf{n}} = a^{k_r} I^{\mathbf{k}\cdot\mathbf{n}-q-k_r} \mathbf{t}^{\mathbf{n}} \quad (\mathbf{n} \geq \mathbf{1})$$

and so

$$\omega_{S'} = ((s\omega_S : S_r^+)_{\omega_S}/s\omega_S)(\mathbf{1}_r) \cong \bigoplus_{\mathbf{n}\geq\mathbf{1}, n_r=0} I^{\mathbf{k}\cdot\mathbf{n}-q}\mathbf{t}^{\mathbf{n}},$$

which proves the lemma.

Recall the characterization given by Ikeda for the Gorenstein property of a Rees algebra: If I is an ideal in a local ring A such that grade $I > 1$ and $R_A(I)$ is Cohen-Macaulay, then $R_A(I)$ is Gorenstein if and only if $\omega_A \cong A$ and $\omega_{gr_A(I)} \cong gr_A(I)(-2)$. The following theorem generalizes this to the case of multi-Rees algebras.

5.3. Theorem. *Let A be a local ring and $I \subset A$ a primary ideal. Suppose that* grade $I > 1$ *and $R_A(I)$ is Cohen-Macaulay. Then the following conditions are equivalent*

(1) $R_A(\mathbf{I}_r)$ is Gorenstein.

(2) $\omega_A \cong A$ and $\omega_{gr_A(I)} \cong gr_A(I)(-(r+1))$.

Proof. Set $S_q = R_A(\mathbf{I}_q)$ and $G_q = gr_A(\mathbf{I}_q)$ for each $q \leq r$.

(1)$\Rightarrow$(2) We may assume that A has an infinite residue field. Since S_r is Gorenstein and $\mathbf{a}(S_r) = -\mathbf{1}$ by Theorem 2.2, the canonical module of S_r is

$$\omega_{S_r} = S_r(-\mathbf{1}) = \bigoplus_{\mathbf{n}\geq\mathbf{1}} I^{|\mathbf{n}|-r}.$$

It follows from Lemma 5.2 that

$$\omega_{S_q} = \bigoplus_{\mathbf{n}\geq\mathbf{1}, \mathbf{n}\in\mathbf{N}^q} I^{|\mathbf{n}|-r}$$

for every $q \leq r$. Especially we obtain

$$\omega_{S_1} = \bigoplus_{n\geq 1} I^{n-r}$$

and $\omega_A \cong A$. According to Theorem 3.1 and Remark 3.2 we then have

$$\omega_{G_1} = \bigoplus_{n\geq 1} I^{n-r-1}/I^{n-r} = G_1(-(r+1)).$$

(2)$\Rightarrow$(1) Since $G_q = G_1^{q-gr}$ for every $q \leq r$, we have by Corollary 4.9

$$\omega_{G_q} = (\omega_{G_1})^{q-gr} = \bigoplus_{\mathbf{n}\geq 1, \mathbf{n}\in \mathbf{N}^q} I^{|\mathbf{n}|-r-1}/I^{|\mathbf{n}|-r}.$$

By Lemma 1.5 one sees that $a(G_q^{gr}) = -(r+1)$. Theorem 3.1 then implies that

$$\omega_{S_r} = \bigoplus_{\mathbf{n}\geq 1} I^{|\mathbf{n}|-r} = S_r(-1).$$

Because S_r is Cohen-Macaulay by Corollary 2.3, it follows that S_r must be Gorenstein.

5.4. Remark. Suppose that $S = R_A(\mathbf{I}_r)$ is Gorenstein. Then $G = gr_A(\mathbf{I}_r)$ is not Gorenstein if $r > 1$. In fact, one sees from the preceeding proof that ω_G is generated by r elements of total degree $r+1$ and one easily sees that r is the minimal number of generators of ω_G. Hence G has CM-type r, which implies the above claim.

We now want to find out for which $\mathbf{k} \in (\mathbf{N}^*)^r$ the multi-Rees algebra $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein. We shall first show that there can only be one value of $|\mathbf{k}|$ such that $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein. This is based on the following lemma from [R]:

5.5. Lemma. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{ht} I > 1$. *If $I^r \cong I^s$ for some $r, s \in \mathbf{N}^*$, we have $r = s$.*

Proof. If m is the maximal ideal of A, the isomorphism $I^r \cong I^s$ induces an isomorphism $I^{rj}/mI^{rj} \cong I^{sj}/mI^{sj}$. Set $l = l(I)$. Since $\operatorname{ht} I > 1$, $l > 1$. There exists a polynomial $P \in \mathbf{Q}[t]$ of degree $l-1$ such that $P(j) = \text{lenght}(I^j/mI^j)$ for $j \gg 0$. Now $P(rj) = P(sj)$ for $j \gg 0$. If

$$P(t) = \sum_{i=0}^{l-1} a_i t^i,$$

we must have $a_{l-1}r^{l-1} = a_{l-1}s^{l-1}$, so that $r = s$.

5.6. Proposition. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{ht} I > 1$. *Let $k, l \in (\mathbf{N}^*)^r$. If $R_A(\mathbf{I}_r^{\mathbf{k}})$ and $R_A(\mathbf{I}_r^{\mathbf{l}})$ are both Gorenstein, then $|\mathbf{k}| = |\mathbf{l}|$.*

Proof. Since $R_A(\mathbf{I}_r^{\mathbf{k}})$ and $R_A(\mathbf{I}_r^{\mathbf{l}})$ are Gorenstein, we know by Theorem 2.2 that their canonical modules are $R_A(\mathbf{I}_r^{\mathbf{k}})(-1)$ and $R_A(\mathbf{I}_r^{\mathbf{l}})(-1)$ respectively. According to Corollary 1.15 the canonical modules of the Veronesians $\left(R_A(\mathbf{I}_r^{\mathbf{k}})\right)^{(\mathbf{l})}$ and $\left(R_A(\mathbf{I}_r^{\mathbf{l}})\right)^{(\mathbf{k})}$ are then $\left(R_A(\mathbf{I}_r^{\mathbf{k}})(-1)\right)^{(\mathbf{l})}$ and $\left(R_A(\mathbf{I}_r^{\mathbf{k}})(-1)\right)^{(\mathbf{l})}$. Now

$$\left(R_A(\mathbf{I}_r^{\mathbf{k}})(-1)\right)^{(\mathbf{l})} = \bigoplus_{\mathbf{n}\geq 1} I^{(\mathbf{kl})\cdot\mathbf{n}-|\mathbf{k}|}$$

and

$$\left(R_A(\mathbf{I}_r^{\mathbf{l}})(-1)\right)^{(\mathbf{k})} = \bigoplus_{\mathbf{n}\geq 1} I^{(\mathbf{kl})\cdot\mathbf{n}-|\mathbf{l}|}.$$

Thus $I^{(\mathbf{kl})\cdot\mathbf{n}-|\mathbf{k}|} \cong I^{(\mathbf{kl})\cdot\mathbf{n}-|\mathbf{l}|}$ for all $\mathbf{n} \in (\mathbb{N}^*)^r$, so that by the previous lemma we must have $|\mathbf{k}| = |\mathbf{l}|$.

Recall that an ideal $I \subset A$ is called equimultiple, if the analytic spread $l(I) = \operatorname{ht} I$. The proof of the following lemma can be found from [HIO], p 407.

5.7. Lemma. *Let A be a local ring with an infinite residue field and $I \subset A$ an equimultiple ideal of $d > \operatorname{ht} I > 0$ such that $R_A(I)$ is Cohen-Macaulay. There exists then an element $b \in A$ such that the corresponding degree zero element $b \in gr_A(I)$ is a non zero-divisor.*

5.8. Theorem. *Let A be a local ring and $I \subset A$ an equimultiple ideal such that $\operatorname{ht} I > 1$. Assume that $gr_A(I)$ is Gorenstein. Let $k \in (\mathbb{N}^*)^r$. Then $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein if and only $|\mathbf{k}| = -a(gr_A(I)) - 1$.*

Proof. By Proposition 5.6 there can be only one value of $|\mathbf{k}|$ such that $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein. It is thus enough to show that $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein if $|\mathbf{k}| = -a(gr_A(I)) - 1$. We can assume that the residue field of A is infinite. Set $S = R_A(\mathbf{I}_r)$. Then $S^{(\mathbf{k})} = R_A(\mathbf{I}_r^{\mathbf{k}})$. Also set $G_q = gr_A(\mathbf{I}_q)$ for each $q \leq r$. Denote $a = -a(G_1)$. We use induction on $\dim A/I$.

If $\dim A/I = 0$, I is a primary ideal. Since $G_q = G_1^{q-gr}$ for every $q \leq r$ and G_1 is Gorenstein, Corollary 4.9 implies that G_q has the canonical module

$$\omega_{G_q} = \bigoplus_{\mathbf{n}\geq 1, \mathbf{n}\in\mathbb{N}^q} I^{|\mathbf{n}|-a}/I^{|\mathbf{n}|-a+1}.$$

By Lemma 1.5 $a(G_q^{gr}) = a$. The Gorensteiness of G_1 implies that of A. It then follows from Theorem 3.1 that S has the canonical module

$$\omega_S = \bigoplus_{\mathbf{n}\geq 1} I^{|\mathbf{n}|-a+1},$$

so that by Corollary 1.15 the canonical module of $S^{(\mathbf{k})}$ is

$$\omega_{S^{(\mathbf{k})}} = \bigoplus_{\mathbf{n}\geq 1} I^{\mathbf{k}\cdot\mathbf{n}-a+1}.$$

Because $|\mathbf{k}| = a-1$, we thus see that $\omega_{S^{(\mathbf{k})}} = S^{(\mathbf{k})}(-1)$. According to Theorem 2.2 S is Cohen-Macaulay. Lemma 1.13 then says that also $S^{(\mathbf{k})}$ is Cohen-Macaulay. So the claim follows.

Suppose then that $\dim A/I > 0$. According to Lemma 5.7 there exists an element $b \in A$ such that the corresponding degree zero element $b \in G_1$ is a non zero-divisor. Then $I^n \cap (b) = bI^n$ for all $n \in \mathbf{N}$. Set $\overline{A} = A/(b)$ and $\overline{I} = I\overline{A}$. If $\overline{S} = R_{\overline{A}}(\overline{\mathbf{I}})$ and $\overline{G}_1 = gr_{\overline{A}}(\overline{I})$, it follows that $\overline{S}^{(\mathbf{k})} = S^{(\mathbf{k})}/bS^{(\mathbf{k})}$ and $\overline{G}_1 = G_1/bG_1$. Because b is a regular element of degree zero in G_1, $\overline{G}_1$ must be Gorenstein with $a(\overline{G}_1) = a(G_1)$. By the induction hypothesis we get that $\overline{S}^{(\mathbf{k})}$ is Gorenstein. It then follows that $S^{(\mathbf{k})}$ is Gorenstein.

We shall next study the relationship between the Gorensteiness of $R_A(\mathbf{I}_r^{\mathbf{k}})$ and $R_A(I^{|\mathbf{k}|})$. Analogously to Lemma 5.2 one can prove the following:

5.9. Lemma. *Let A be a local ring with an infinite residue field and $I \subset A$ an ideal of* grade $I > 1$ *such that $R_A(I)$ is Cohen-Macaulay. Let $\mathbf{k} \in (\mathbf{N}^*)^r$. If $S = R_A(\mathbf{I}_r^{\mathbf{k}})$ has the canonical module*

$$\omega_S = \bigoplus_{\mathbf{n}\geq 1} I^{\mathbf{k}\cdot\mathbf{n}-q}\mathbf{t}^{\mathbf{n}},$$

where $q \in \mathbf{N}$, then the canonical module of $Q = S/I^{|\mathbf{k}|}S$ is

$$\omega_Q = \bigoplus_{\mathbf{n}\geq 1} I^{\mathbf{k}\cdot\mathbf{n}-q-|\mathbf{k}|}/I^{\mathbf{k}\cdot\mathbf{n}-q}.$$

5.10. Theorem. *Let A be a local ring and $I \subset A$ an equimultiple ideal of* grade $I > 1$ *such that $R_A(I)$ is Cohen-Macaulay. Let $\mathbf{k} \in (\mathbf{N}^*)^r$. If $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein, then so is $R_A(I^{|\mathbf{k}|})$.*

Proof. We use induction on $\dim A$. Suppose first that $\dim A/I = 0$, so that I is an primary ideal. We may assume that A is complete with an infinite residue field. Set $S = R_A(\mathbf{I}_r^{\mathbf{k}})$. Consider the ring

$$Q = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} I^{\mathbf{k}\cdot\mathbf{n}}/I^{\mathbf{k}\cdot(\mathbf{n}+1)} \quad (= S/I^{|\mathbf{k}|}S).$$

Denote

$$P = \bigoplus_{n\in\mathbf{N}} I^n/I^{n+|\mathbf{k}|}.$$

Then $Q = (P^{r-gr})^{(\mathbf{k})}$. It follows from Corollaries 1.11 and 4.9 that

$$\omega_Q = (\bigoplus_{\mathbf{n}>0} [\omega_P]_{\mathbf{n}})^{(\mathbf{k})} = \bigoplus_{\mathbf{n}>0} [\omega_P]_{\mathbf{k}\cdot\mathbf{n}}.$$

Since S is Gorenstein, we know by Theorem 2.2 that the canonical module of S is

$$\omega_S = S(-\mathbf{1}) = \bigoplus_{\mathbf{n}\geq\mathbf{1}} I^{\mathbf{k}\cdot(\mathbf{n}-\mathbf{1})}.$$

According to Lemma 5.10 we thus have

$$\omega_Q = \bigoplus_{\mathbf{n}\geq\mathbf{1}} I^{\mathbf{k}\cdot(\mathbf{n}-2\cdot\mathbf{1})}/I^{\mathbf{k}\cdot(\mathbf{n}-\mathbf{1})}.$$

An application of the diagonal functor Δ gives

$$(\omega_Q)^\Delta = \bigoplus_{n>0} [\omega_P]_{|\mathbf{k}|n} = \bigoplus_{n\geq 1} I^{|\mathbf{k}|(n-2)}/I^{|\mathbf{k}|(n-1)}.$$

Now $P^{(|\mathbf{k}|)} = gr_A(I^{|\mathbf{k}|})$. Since $R_A(I^{|\mathbf{k}|})$ is Cohen-Macaulay by Corollary 2.3, Theorem 2.2 implies that $a(gr_A(I^{|\mathbf{k}|}) < 0$. It follows that

$$\omega_{gr_A(I^{|\mathbf{k}|})} = (\omega_P)^{(|\mathbf{k}|)} = (gr_A(I^{|\mathbf{k}|}))(-2).$$

An application of Lemma 5.2 shows that $\omega_A \cong A$, so that we can use Theorem 5.3 to get the claim.

Assume then that $\dim A/I > 0$. By Lemma 5.7 we find an element $b \in A$ such that the corresponding degree zero element $b \in gr_A(I)$ is a non zero-divisor. Then $I^n \cap (b) = bI^n$ for all $n \in \mathbb{N}$. If $\overline{A} = A/(b)$, $\overline{I} = I\overline{A}$, we have $R_{\overline{A}}(\overline{\mathbf{I}}_r^{\mathbf{k}}) = R_A(\mathbf{I}_r^{\mathbf{k}})/bR_A(\mathbf{I}_r^{\mathbf{k}})$ and $R_{\overline{A}}(\overline{I}^{|\mathbf{k}|}) = R_A(I^{|\mathbf{k}|})/bR_A(I^{|\mathbf{k}|})$, so that the claim follows from the induction hypothesis by the regularity of b.

5.11. Theorem. *Let A be a local Cohen-Macaulay-ring and $I \subset A$ an equimultiple ideal of $\operatorname{ht} I > 1$ such that $R_A(I)$ is Cohen-Macaulay. Let $q \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein for any $r \in \mathbb{N}^*$ and $\mathbf{k} \in (\mathbb{N}^*)^r$ such that $|\mathbf{k}| = q$.

(2) $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein for some $r \in \mathbb{N}^*$ and $\mathbf{k} \in (\mathbb{N}^*)^r$ such that $|\mathbf{k}| = q$.

(3) $R_A(I^q)$ is Gorenstein.

(4) $gr_A(I)$ is Gorenstein with $a(gr_A(I)) = -(q+1)$.

Proof.

(1) $\Rightarrow$ (2) Trivial

(2) $\Rightarrow$ (3) This was proved in Theorem 5.10.

(3) $\Rightarrow$ (4) Because $R_A(I^q)$ is Gorenstein, it follows from [HRS], Theorem (2.3) that $gr_A(I)$ is Gorenstein with $a(gr_A(I)) = -(q+1)$.

(4) $\Rightarrow$ (1) This is a consequence of Theorem 5.8.

As an application we consider the case where $I = m$ is the maximal ideal of A and $q = \dim A - 1$.

5.12. Theorem. *Let A be a Cohen-Macaulay local ring of dimension $d > 1$ with the maximal ideal m such that $R_A(m)$ is Cohen-Macaulay. Then the following conditions are equivalent:*

(1) A is regular

(2) $R_A(m^{d-1})$ is Gorenstein.

(3) $R_A(m^{|\mathbf{k}|})$ is Gorenstein for some $r \in \mathbb{N}^*$ and $\mathbf{k} \in (\mathbb{N}^*)^r$ such that $|\mathbf{k}| = d-1$.

(4) $R_A(m^{|\mathbf{k}|})$ is Gorenstein for all $r \in \mathbb{N}^*$ and $\mathbf{k} \in (\mathbb{N}^*)^r$ such that $|\mathbf{k}| = d-1$.

Proof. The claim follows from Theorem 5.11, since the equivalence of (1) is (2) is known by [R], Folgerung (8.3.2).

Acknowledgements

This paper was written when the second author was visiting Germany. He is greatly indebted to Professor Manfred Herrmann and Dr. Jürgen Ribbe for the hospitality he has received. For the financial support he wants to thank the Alfred Kordelin Foundation (Helsinki).

References

[B] BOURBAKI N.: Algèbre, Ch. 10: Algèbre homologique - Masson, Paris, 1980.

[GN] GOTO S., and NISHIDA K.: Filtrations and the Gorenstein property of the associated Rees algebras. - Preprint.

[GW1] GOTO S., and WATANABE K.: On graded rings, I. - J. Math. Soc. Japan, 30, 1978, 179-213.

[GW2] GOTO S., and WATANABE K.: On graded rings, II. - Tokyo J. Math. 1, 1978, 237-260.

[I] IKEDA, S. : On the Gorensteiness of Rees algebras over local rings. - Nagoya Math J. 102, 1986, 135-154.

[HIO] HERRMANN, M. and IKEDA, S. and ORBANZ, U.: Equimultiplicity and blowing up. - Springer-Verlag, Berlin-Heidelberg, 1988.

[HK] HERZOG, J., and E. KUNZ: Der kanonische modul eines Cohen-Macaulay-rings. - Lecture Notes in Mathematics 238, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[HRS] HERRMANN, M. and RIBBE, J. and SCHENZEL, P.: On the Gorenstein property of form rings. - Preprint.

[HRZ] HERRMANN, M. and RIBBE, J. and ZARZUELA, S.: On Rees and form rings of almost complete intersections. - Preprint.

[HZV] HERZOG, J. and SIMIS, A. and VASCONCELOS W.V.: On the canonical module of the Rees algebra and the associated graded ring of an ideal. - Journal of Algebra, 105, 1987, 285-302.

[M] MATSUMURA, N.: Commutative ring theory. - Cambridge University Press, Cambridge, 1986.

[R] RIBBE, J. : Zur Gorenstein-Eigenschaft von Aufblasungsringen unter besonderer Berücksichtigung der Aufblasungsringe von Idealpotenzen. - Dissertation, Universität zu Köln 1991.

[TI] TRUNG N.V. and IKEDA S.: When is the Rees algebra Cohen-Macaulay?. - Communications in Alg, 17, 1989, 2893-2922

[Z] ZARZUELA, S. : On the structure of the canonical module of the Rees algebra and the associated graded ring of an ideal. - Preprint.

M. Herrmann
J. Ribbe
Mathematisches Institut
der Universität zu Köln
Weyertal 86-90
D-5000 Köln 41
Germany

Eero Hyry
Departement of Mathematics
University of Helsinki
Hallituskatu 15
00100 Helsinki
Finland