AREA OF SMALL DISKS

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Abstract. We consider Riemannian metrics on two dimensional disks where all geodesics are minimizing. We prove a sharp reverse isoperimetric inequality which yields near optimal bounds for the area of disks as well as near optimal upper bounds on the first nonzero Newmann eigenvalue of the Laplacian in terms only of the radius.

1. Introduction

In this paper we are considering "small" Riemannian disks. These are Riemannian manifolds M^2 diffeomorphic to the two dimensional disk that are "small" in the sense that all geodesics in M hit ∂M at both ends and minimize the distance between the endpoints. The use of the term small comes from the fact that this will hold for any such M that is a subdomain of a complete manifold N as long as it lies inside a ball of radius half the injectivity radius of N. In fact our main interest will be in small metric balls $B(x,R)$.

There is a long standing conjecture, in all dimensions n, that hemispheres have the smallest volume among small balls of a fixed dimension and radius. By a hemisphere we will mean a ball $B(x,R)$ of radius R and with constant curvature $(\frac{\pi}{2R})^2$. For example, when $R=\frac{\pi}{2}$ this is isometric to a hemisphere of the unit sphere.

Conjecture 1.1. *If $B(x,R)$ is a small metric ball then*

$$Vol(B(x,R)) \geq \frac{\alpha(n)}{2}(\frac{2R}{\pi})^n$$

where $\alpha(n)$ represents the volume of the unit n-sphere. Further equality holds if and only if $B(x,R)$ is isometric to a hemisphere of (intrinsic) radius R.

Although in all dimensions there are known (nonsharp) constants $C(n)$ such that $Vol(B(x,R)) \geq C(n)(\frac{2R}{\pi})^n$ (see [Be] for $n=2,3$ and [Cr] for all n) even the two dimensional case of the conjecture is open:

Conjecture 1.2. *If $B(x,R)$ is a 2-dimensional small metric ball of area A then*

$$A \geq \frac{8}{\pi}R^2.$$

Further equality holds if and only if $B(x,R)$ is isometric to a hemisphere of (intrinsic) radius R.

One of the goals of this paper is to give a good constant C_0 (though still $C_0 \neq \frac{8}{\pi}$) for the inequality $A \geq C_0R^2$. We shall get at this estimate by proving a sharp reverse isoperimetric inequality for small balls:

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Theorem 1.3. *If $B(x, R)$ is a 2-dimensional small metric ball of area A and boundary length L then*

$$\pi A \geq -\frac{1}{2}L^2 + 4RL.$$

Further equality holds if and only if $B(x, R)$ is isometric to a hemisphere of (intrinsic) radius R.

In fact we will prove this result (see Theorem 3.1) in the more general setting of a small Riemannian disk M^2 where the radius R is replaced by the "antipodal radius" R_a. For a given map $a : \partial M \to \partial M$, a continuous fixed-point-free map with $a^2 = Id$, (i.e. an antipodal map on the boundary) we define R_a as the minimum of $\frac{1}{2}d(x, a(x))$ over boundary points x. For small metric balls when a is the usual antipodal map R_a is just the radius R.

In the case of small balls we will use this to show:

Corollary 1.4. *If $B(x, R)$ is a 2-dimensional small metric ball of area A then*

$$A \geq \frac{8 - \pi}{2}R^2.$$

Although this is not sharp it is pretty good since the conjectured sharp constant, $\frac{8}{\pi}$, is approximately 2.5465 while $\frac{8-\pi}{2}$ is approximately 2.4292. We point out in section 3 that there is no corresponding general lower bound for the area of small disks in terms of R_a.

For a small ball $B(x, R)$ let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3, ...$ be the spectrum of the Laplace operator with Dirichlet boundary conditions and $0 = \mu_1 < \mu_2 \leq \mu_3 \leq ...$ be the spectrum for Newmann boundary conditions. The unit hemisphere (i.e. a ball of intrinsic radius $\frac{\pi}{2}$ in the unit sphere) has $\lambda_1 = \mu_2 = \mu_3 = 2$. Thus a hemisphere of intrinsic radius R has $\lambda_1 = \mu_2 = \mu_3 = 2(\frac{\pi}{2R})^2$ where the corresponding eigenfunctions are the coordinate functions (from the embedding in R^3).

Theorem 16 of [Cr] shows that for small surfaces of diameter D, $\lambda_1 \geq 2(\frac{\pi}{D})^2$, with equality holding only for hemispheres. This (see section 4), combined with a result of Hersch [He] gives the sharp

Corollary 1.5. *If M^2 is a small disk of area A and diameter D then*

$$\mu_2 \leq \frac{8\pi^2}{3\pi A - 2D^2}.$$

Further equality holds if and only if M^2 is isometric to a hemisphere of (intrinsic) diameter D.

For metric balls $D = 2R$. Hence a lower bound for A would give an upper bound for μ_2. In particular, Conjecture 1.2 would (along with Theorem 16 of [Cr]) imply.

Conjecture 1.6. *On a 2-dimensional small metric ball $B(x, R)$*

$$\mu_2 \leq 2(\frac{\pi}{2R})^2 \leq \lambda_1.$$

Further equality holds in either (and hence both) inequality if and only if $B(x, R)$ is isometric to a hemisphere of (intrinsic) radius R.

On the other hand, Corollary 1.4 gives a good, but not sharp, upper bound

Corollary 1.7. *On a 2-dimensional small metric ball $B(x,R)$*

$$\mu_2 < 2.1485(\frac{\pi}{2R})^2.$$

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2. The Space of Geodesics

We will assume throughout this section that M is a small disk of area A and boundary length L. The unit tangent bundle, UM, of M has a natural measure, du, that is locally a product measure. In particular $Vol(UM) = 2\pi A$. Let τ be a curve in M with arclength s and unit normal ν. We can let $\{(\theta, s)\}$ parameterize the set of geodesics γ that intersect τ, by $\gamma(0) = \tau(s)$ and the unit tangent $\gamma'(0)$ makes angle θ with ν. For all t such that $\gamma(t)$ is defined we let $\{(\theta, s, t)\}$ correspond to the unit vector $u = \gamma'(t)$. Then Santaló's formula [Sa1], [Sa2, Chap. 19] tells us that $du = |cos(\theta)|d\theta ds dt$.

It is easy to describe the (standard) measure space, Γ, of complete unit speed oriented geodesics by using the boundary ∂M as our τ above. The measure is $d\gamma = |cos(\theta)|d\theta ds$. Note that the parametrization (θ, s, t) of UM is one-to-one and onto if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, s is an arc length parametrization of ∂M and $0 \le t \le L(\gamma)$ (where γ is the geodesic determined by (θ, s)). In particular,

$$2\pi A = Vol(UM) = \int_\Gamma L(\gamma) d\gamma. \tag{2.1}$$

Now consider a curve τ. We claim that $\int_\Gamma i(\tau,\gamma)d(\gamma) = 4L(\tau)$ where $i(\tau,\gamma)$ represents the number of times the geodesic γ intersects τ. (This is known as Crofton's formula.) It follows directly from Santaló's formula when we note that the parametrization (θ, s) of the unit vectors along τ counts each $\gamma \in \Gamma$ exactly as often as $i(\tau,\gamma)$ and hence (since $\int_0^{2\pi} |cos(\theta)|d\theta = 4$)

$$4L(\tau) = \int\int |cos(\theta)|d\theta ds = \int_\Gamma i(\tau,\gamma)d\gamma. \tag{2.2}$$

In particular (since each geodesic hits ∂M twice) we see that

$$Vol(\Gamma) = 2L. \tag{2.3}$$

Let $a : \partial M \to \partial M$ be a fixed antipodal map. For a given geodesic γ from $p \in \partial M$ to $q \in \partial M$ we want to choose $S(\gamma) \subset \partial M$ (the shadow of γ) an interval of ∂M between p and q. (There are two choices since ∂M is homeomorphic to S^1.) When $q \ne a(p)$ one of the intervals will contain both $a(p)$ and $a(q)$ (since a is an antipodal map). We take $S(\gamma)$ to be the closed interval without $a(p)$ and $a(q)$. Note that $a(S(\gamma))$ is disjoint from $S(\gamma)$. When $q = a(p)$, $S(\gamma)$ will not be defined, however we will interpret $\int_{S(\gamma)} f(s)ds$ to be $\frac{1}{2}\int_{\partial M} f(s)ds$.

For a small ball $B(x,R)$ we define $S(\gamma)$ for any γ that does not run through the center x by setting $S(\gamma)$ to be the set of $p \in \partial M$ such that the geodesic from x to p intersects γ. This agrees with the previous definition when the endpoints are not antipodal (with the usual antipodal map). In this case $S(\gamma)$ is defined for all but a set of measure 0 of Γ (which is not necessarily the case for small disks because the set of geodesics between antipodal points could have positive measure). The reader should note that for small balls $B(x,R)$ and geodesics whose endpoints are antipodal but that don't pass through the center x, $\int_{S(\gamma)} f(s)ds$ means something different if we think of $B(x,R)$ as a small ball or as a small disk with an antipodal map.

A useful new tool for the study of areas of small balls is the following formula which is just an application of Fubini's theorem.

Lemma 2.1. *Let $B(x,R)$ be a small ball with boundary of length L. Then for any continuous function $f : \partial M \to R$ we have*

$$\int_\Gamma \int_{S(\gamma)} f(s)dsd\gamma = 4R \int_{\partial M} f(s)ds.$$

In particular, for $f(s) = 1$ we see

$$\int_\Gamma L(S(\gamma))d\gamma = 4RL.$$

Proof: We consider the space $\Gamma \times \partial M$ with the product measure $d\gamma ds$ and let $F(\gamma, s) = 1$ if $s \in S(\gamma)$ and 0 otherwise. For any fixed boundary point s, $\int_\Gamma F(\gamma, s)d\gamma$ represents the measure of the space of geodesics γ such that $s \in S(\gamma)$. But this is just the measure of the space of geodesics that intersect the geodesic from the center x to s, which by 2.2 is just $4R$. Thus

$$\int_\Gamma \int_{S(\gamma)} f(s)dsd\gamma = \int_\Gamma \int_{\partial M} F(\gamma, s)f(s)dsd\gamma = \int_{\partial M} f(s)(\int_\Gamma F(\gamma, s)d\gamma)ds =$$

$$= 4R \int_{\partial M} f(s)ds.$$

□

For γ a geodesic in a small ball $B(x,R)$ that does not pass through x then we can measure the angle $\theta(\gamma)$ that γ spans as viewed from x. To be precise let θ be the parameterization of ∂M given by standard normal polar coordinates based at x. Then $\theta(\gamma)$ is the change ($\leq \pi$) of θ over $S(\gamma)$. Choosing $f(s)$ such that $d\theta = f(s)ds$ and applying Lemma 2.1 one gets a curious consequence for the average value of $\theta(\gamma)$.

Corollary 2.2.

$$\frac{1}{vol(\Gamma)} \int_\Gamma \theta(\gamma)d\gamma = \pi \frac{4R}{L}.$$

There is a lemma corresponding to Lemma 2.1 for small Riemannian disks with an antipodal map a, which is only an inequality rather than a formula. This is the version we will use to prove Theorem 3.1.

Lemma 2.3. *Let M^2 be a small disk with boundary of length L and a an antipodal map. Then for any continuous nonnegative function $f : \partial M \to R$ such that $a^*(fds) = fds$ we have*

$$\int_\Gamma \int_{S(\gamma)} f(s)dsd\gamma \geq 4R_a \int_{\partial M} f(s)ds$$

In particular choosing f such that $f(s)ds = \frac{1}{2}(ds + a^*ds)$ we see:

$$\frac{1}{2}\int_\Gamma L(S(\gamma)) + L(a(S(\gamma)))d\gamma \geq 4R_a L.$$

Proof: We first note that by the symmetry assumption on f for any γ

$$\int_{S(\gamma)} f(s)ds = \frac{1}{2}\int_{S(\gamma)} f(s)ds + \int_{a(S(\gamma))} f(s)ds.$$

We now let $F(\gamma, s)$ be 1 if either $s \in S(\gamma)$ or $a(s) \in S(\gamma)$, and 0 otherwise. This is the same as saying that the endpoints of γ are separated along ∂M by s and $a(s)$(or when s or $a(s)$ is an endpoint of γ). We note that if γ is a geodesic between antipodal points then $F(\gamma, s) = 1$ for all $s \in \partial M$.

For a given point $s \in \partial M$ choose a length minimizing path τ from s to $a(s)$ then $\int_\Gamma F(\gamma, s)d\gamma$ is precisely the measure of the space of geodesics that intersect τ (each will intersect τ once by minimality), thus by 2.2 is precisely $4L(\tau)$. Thus by the definition of R_a,

$$\int_\Gamma F(\gamma, s)d\gamma = 4L(\tau) \geq 8R_a.$$

The proof now follows as a variation of the previous proof

$$\int_\Gamma\int_{S(\gamma)} f(s)dsd\gamma = \int_\Gamma \frac{1}{2}(\int_{S(\gamma)} f(s)ds + \int_{a(S(\gamma))} f(s)ds)d\gamma =$$

$$\int_\Gamma\int_{\partial M} \frac{1}{2}F(\gamma, s)f(s)dsd\gamma = \frac{1}{2}\int_{\partial M} f(s)(\int_\Gamma F(\gamma, s)d\gamma)ds \geq$$

$$\geq 4R_a\int_{\partial M} f(s)ds.$$

□

Remark 2.4. Equality will hold in Lemma 2.3 for all nonnegative f as long as $d(s, a(s)) = 2R_a$ for all $s \in \partial M$.

3. Isoperimetric Inequalities

The purpose of this section is to prove Theorem 3.1 and corollary 1.4.

First let us collect some known isoperimetric inequalities. For small disks M^2 of area A, boundary length L, and diameter D, we see from [Cr] Theorem 11

$$L^2 \geq 2\pi A, \tag{3.1}$$

while [Cr] Corollary 2(ii) gives

$$LD \geq \pi A. \tag{3.2}$$

Equality in either 3.1 or 3.2 holds if and only if M is isometric to a hemisphere. The equality case in 3.2 is not proved in [Cr] however the proof shows that equality holds if and only if all geodesics have length equal to D. This in turn implies it is a hemisphere by the result in [Ba].

If $a : \partial M \to \partial M$ is an antipodal map and $p \in \partial M$ then (from the definition of R_a) each of the two intervals of ∂M between p and $a(p)$ must have length $\geq 2R_a$ thus

$$L \geq 4R_a. \tag{3.3}$$

Since one of our goals is to get a lower bound on the area of small balls in terms of the radius, a first question might be if you can bound A below in terms of R_a for small disks. It turns out that one cannot do this. Consider the example MB_ϵ which is the ϵ neighborhood in the flat plane $\mathbb{R}^2$ of a tripod (three unit length line segments from the origin making angles $\frac{2\pi}{3}$ with each other). We let $a_\epsilon : \partial MB \to \partial MB$ take a boundary point to the boundary point half way around the boundary. It is not hard to see $R_{a_\epsilon} \geq \frac{1}{2}$. On the other hand, as ϵ goes to 0 the area goes to 0 while the length of the boundary goes to 6. (Note that the non convexity of MB_ϵ is not the point here since one could use instead equilateral triangles in a more and more negatively curved simply connected space form.)

Nevertheless, there is a sharp lower bound on A in terms of L and R_a for small disks. The lower bound will say nothing (since we already know $A \geq 0$) once $L \geq 8R_a$.

Theorem 3.1. *For any small disk and any antipodal map $a : \partial M \to \partial M$ we have:*

$$\pi A \geq -\frac{1}{2}L^2 + 4LR_a.$$

Rigidity*: Equality holds iff the metric is isometric to a hemisphere.*
Stability*: If L is close to $4R_a$ then A is close to $\frac{8}{\pi}R_a^2$. Specifically:*

$$\frac{L^2}{2\pi} \geq A \geq -\frac{1}{2\pi}L^2 + \frac{4}{\pi}R_a L.$$

Proof: Let γ be a geodesic in M from $p \in \partial M$ to $q \in \partial M$ where $p \neq a(q)$. $S(\gamma)$ is an interval along ∂M from p to q while $a(S(\gamma))$ is an interval (disjoint from $S(\gamma)$) from $a(p)$ to $a(q)$. Let $\overline{pa(q)}$ and $\overline{qa(p)}$ be the other two segments of the boundary. $\gamma \cup \overline{qa(p)}$ is a curve from p to $a(p)$ and hence

$$L(\gamma) + L(\overline{qa(p)}) \geq d(p, a(p)) \geq 2R_a. \tag{3.4}$$

Note for future reference that if γ is not tangent to the boundary at either end then equality holds in the inequality if and only if γ goes from p to $a(p)$ and has length $2R_a$ (i.e. $q = a(p)$). Similarly $L(\gamma) + L(\overline{pa(q)}) \geq d(q, a(q)) \geq 2R_a$.

In the case that $q = a(p)$ then the intervals $\overline{pa(q)}$ and $\overline{qa(p)}$ are empty intervals and although $S(\gamma)$ is not well defined it still makes sense to let $L(S(\gamma)) + L(a(S(\gamma))) = L$, and it is still the case that $L(\gamma) + L(\overline{qa(p)}) \geq d(p, a(p)) \geq 2R_a$ and $L(\gamma) + L(\overline{pa(q)}) \geq d(q, a(q)) \geq 2R_a$.

We thus have for any γ

$$\begin{aligned} L = L(S(\gamma)) + L(a(S(\gamma))) + L(\overline{pa(q)}) + L(\overline{pa(q)}) + 2L(\gamma) - 2L(\gamma) \geq \\ \geq L(S(\gamma)) + L(a(S(\gamma))) - 2L(\gamma) + 4R_a. \end{aligned}$$

Integrating over Γ, using lemma 2.3, equation 2.3, and equation 2.1 we see:

$$2L^2 = \int_\Gamma L \geq 8LR_a - 4\pi A + 8LR_a.$$

Hence the result

$$\pi A \geq -\frac{1}{2}L^2 + 4LR_a. \tag{3.5}$$

Assume now that equality holds in inequality 3.5. Since the set of geodesics that are tangent to the boundary at either end is a set of measure 0 the equality condition in equation 3.4 (which must hold for almost every γ) says that almost every geodesic γ would have length $2R_a$. Hence, by equation 2.1, $2\pi A = \int_\Gamma L(\gamma)d\gamma = 4LR_a$. Thus, by the equality assumption, we have $\pi A = -\frac{1}{2}L^2 + 2\pi A$ and hence $L^2 = 2\pi A$. But we know from the equality case of the isoperimetric inequality 3.1 that this implies that M is isometric to a hemisphere.

The stability statement is simply the two inequalities 3.1 and 3.5 together. □

Remark 3.2. For small metric balls $B(x, R)$ there is, using inequality 3.2, a slightly better version of stability:

$$\frac{2LR}{\pi} \geq A \geq -\frac{1}{2\pi}L^2 + \frac{4}{\pi}RL.$$

Proof of Corollary 1.4: We will show that for small metric balls $\frac{A}{R^2} \geq \frac{8-\pi}{2}$. When $B(p, R)$ is small then $B(p, r)$ is also small for all $0 < r \leq R$. Hence for $Area(B(x, r)) = A(r)$ and $L(r)$ the length of the boundary of $B(x, r)$ Theorem 3.1 gives

$$\pi\frac{A(r)}{r^2} \geq -\frac{1}{2}(\frac{L(r)}{r})^2 + 4\frac{L(r)}{r}.$$

Since $\frac{d}{dr}A(r) = L(r)$,

$$\frac{d}{dr}(\frac{A(r)}{r^2}) = \frac{1}{r}(\frac{L(r)}{r} - 2\frac{A(r)}{r^2}).$$

Since $\frac{A(r)}{r^2}$ approaches π near $r = 0$ the minimum value of $\frac{A(r)}{r^2}$ for $r \in (0, R]$, if less than π, either occurs for $r_0 = R$ and $\frac{A(R)}{R^2} \geq \frac{1}{2}\frac{L(R)}{R}$ or at some r_0 where $\frac{A(r_0)}{r_0^2} = \frac{1}{2}\frac{L(r_0)}{r_0}$. In either case, since $\frac{2A(r_0)}{r_0^2} \geq \frac{L(r_0)}{r_0} \geq 4$, and $-\frac{1}{2}x^2+4x$ is decreasing for $x > 4$, we see that

$$\pi\frac{A(r_0)}{r_0^2} \geq -\frac{1}{2}(\frac{2A(r_0)}{r_0^2})^2 + 4(\frac{2A(r_0)}{r_0^2}).$$

Hence

$$\frac{A(R)}{R^2} \geq \frac{A(r_0)}{r_0^2} \geq \frac{8-\pi}{2}.$$

This proves the Corollary. □

4. Eigenvalues of the Laplacian

In this section we consider bounds on the eigenvalues for a small disk M^2 whose area is A, boundary length is L and diameter is D. As in the introduction we let λ_1 be the first Dirichlet eigenvalue and μ_2 and μ_3 the first two nonzero Newmann eigenvalues M.

Then from [Cr] Theorem 16 (which is an n-dimensional theorem) we have for small disks

$$\lambda_1 \geq 2\frac{\pi^2}{D^2}$$

with equality only for round hemispheres.

While from [He] equation (2) (which is a two dimensional result but holds more generally than for small disks) we have

$$\frac{1}{\lambda_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} \geq A\frac{3}{4\pi}.$$

Where equality holds for the round hemisphere.

Combining these (with the sharp $\mu_2 \leq \mu_3$) yields

$$\frac{D^2}{2\pi^2} + \frac{2}{\mu_2} \geq A\frac{3}{4\pi}.$$

and hence Corollary 1.5:

$$\mu_2 \leq \frac{8\pi^2}{3\pi A - 2D^2}.$$

Where equality will hold only for hemispheres from the previous equality results.

So a lower bounds on A and upper bounds on D give upper bounds on μ_2. Thus we can get such bounds from Theorem 3.1. However, the results have a nicer form in the case of small balls $B(x,R)$ where in particular $D = 2R$. Thus an application of Theorem 3.1 gives us

Corollary 4.1. *If $B(x,R)$ is a small ball of area A and boundary length L then*

$$\mu_2 \leq \frac{8\pi^2}{3\pi A - 8R^2},$$

and

$$\mu_2 \leq \frac{8\pi^2}{-\frac{3}{2}L^2 + 12RL - 8R^2}.$$

Further equality holds in either inequality if and only if $B(x,R)$ is isometric to a hemisphere of (intrinsic) radius R.

An application of Corollary 1.4 gives a non-sharp result (and corollary 1.7)

Corollary 4.2. *On a 2-dimensional small metric ball $B(x,R)$*

$$\mu_2 \leq \frac{4 \cdot 16}{-3\pi^2 + 24\pi - 16}(\frac{\pi}{2R})^2 < 2.1485(\frac{\pi}{2R})^2.$$

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