Type III_1 factors generated by regular representations of infinite dimensional nilpotent group $B_0^{\mathbb{N}}$

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Abstract

We study the von Neumann algebra, generated by the unitary representations of infinite-dimensional groups nilpotent group $B_0^{\mathbb{N}}$. The conditions of the irreducibility of the regular and quasiregular representations of infinite-dimensional groups (associated with some quasi-invariant measures) are given by the so-called Ismagilov conjecture (see [1,2,9–11]). In this case the corresponding von Neumann algebra is type I_∞ factor. When the regular representation is reducible we find the sufficient conditions on the measure for the von Neumann algebra to be factor (see [13,14]). In the present article we determine the type of corresponding factors. Namely we prove that the von Neumann algebra generated by the regular representations of infinite-dimensional nilpotent group $B_0^{\mathbb{N}}$ is type III_1 hyperfinite factor. The case of the nilpotent group $B_0^{\mathbb{Z}}$ of infinite in both directions matrices will be studied in [6].

Key words: von Neumann algebra, type III_1 factor, unitary representation, infinite-dimensional groups, nilpotent groups, regular representations, irreducibility, infinite tensor products, Gaussian measures, Ismagilov conjecture
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Contents

1 Regular representations

Let us consider the group $\tilde{G} = B^{\mathbb{N}}$ of all upper-triangular real matrices of infinite order with unities on the diagonal

$$\tilde{G} = B^{\mathbb{N}} = \{I + x \mid x = \sum_{1\leq k<n} x_{kn}E_{kn}\},$$

and its subgroup

$$G = B_0^{\mathbb{N}} = \{I + x \in B^{\mathbb{N}} \mid \ x \text{ is finite}\},$$

where E_{kn} is an infinite-dimensional matrix with 1 at the place $k, n \in \mathbb{N}$ and zeros elsewhere, $x = (x_{kn})_{k<n}$ is *finite* means that $x_{kn} = 0$ for all (k, n) except for a finite number of indices $k, n \in \mathbb{N}$.

Obviously, $B_0^{\mathbb{N}} = \varinjlim_n B(n,\mathbb{R})$ is the inductive limit of the group $B(n,\mathbb{R})$ of real upper-triangular matrices with units on the principal diagonal

$$B(n,\mathbb{R}) = \{I + \sum_{1\leq k<r\leq n} x_{kr}E_{kr} \mid x_{kr} \in \mathbb{R}\}$$

with respect to the imbedding $B(n,\mathbb{R}) \ni x \mapsto x + E_{n+1n+1} \in B(n + 1,\mathbb{R})$.

We define the Gaussian measure μ_b on the group $B^{\mathbb{N}}$ in the following way

$$d\mu_b(x) = \otimes_{1\leq k<n}(b_{kn}/\pi)^{1/2} \exp(-b_{kn}x_{kn}^2)dx_{kn} = \otimes_{k<n}d\mu_{b_{kn}}(x_{kn}), \quad (1)$$

where $b = (b_{kn})_{k<n}$ is some set of positive numbers.

Let us denote by R and L the right and the left action of the group $B^{\mathbb{N}}$ on itself: $R_s(t) = ts^{-1}$, $L_s(t) = st$, $s, t \in B^{\mathbb{N}}$ and by $\Phi : B^{\mathbb{N}} \mapsto B^{\mathbb{N}}$, $\Phi(I + x) := (I + x)^{-1}$ the inverse mapping. It is known [9,10] that

Lemma 1 $\mu_b^{R_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{N}}$ *for any set* $b = (b_{kn})_{k<n}$.

Lemma 2 $\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{N}}$ *if and only if* $S_{kn}^L(b) < \infty$, $\forall k < n$, *where*

$$S_{kn}^L(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{b_{nm}}.$$

Lemma 3 $\mu_b^{L_t} \perp \mu_b$ $\forall t \in B_0^{\mathbb{N}} \backslash \{e\} \Leftrightarrow S_{kn}^L(b) = \infty$ $\forall k < n$.

Lemma 4 [12] *If* $E(b) = \sum_{k<n} S_{kn}^L(b)(b_{kn})^{-1} < \infty$, *then* $\mu_b^{\Phi} \sim \mu_b$.

Lemma 5 [12] *The measure* μ_b *on* $B^{\mathbb{N}}$ *is* $B_0^{\mathbb{N}}$ *ergodic with respect to the right action.*

Let $\alpha : G \to \mathrm{Aut}(X)$ be a measurable action of a group G on the measurable space X. We recall that a measure μ on the space X is G-ergodic if $f(\alpha_t(x)) = f(x)$ $\forall t \in G$ implies $f(x) = const$ μ a.e. for all functions $f \in L^1(X, \mu)$.

Remark 6 [13] *If* $\mu_b^{\Phi} \sim \mu_b$ *then* $\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{N}}$.

PROOF. This follows from the fact that the inversion Φ replace the right and the left action: $R_t \circ \Phi = \Phi \circ L_t$ $\forall t \in B^{\mathbb{N}}$. Indeed, if we denote $\mu^f(\cdot) = \mu(f^{-1}(\cdot))$ we have $(\mu^f)^g = \mu^{f \circ g}$. Hence

$$\mu_b \sim \mu_b^{R_t} \sim (\mu_b^{R_t})^{\Phi} = \mu_b^{R_t \circ \Phi} = \mu_b^{\Phi \circ L_t} = (\mu_b^{\Phi})^{L_t} \sim \mu_b^{L_t}.$$

□

If $\mu_b^{R_t} \sim \mu_b$ and $\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{N}}$, one can define in a natural way (see [9,10]), an analogue of the right $T^{R,b}$ and left $T^{L,b}$ representation of the group $B_0^{\mathbb{N}}$ in Hilbert space $H_b = L_2(B^{\mathbb{N}}, d\mu_b)$

$$T^{R,b},\ T^{L,b} : B_0^{\mathbb{N}} \to U(H_b = L_2(B^{\mathbb{N}}, d\mu_b)),$$

$$(T_t^{R,b} f)(x) = (d\mu_b(xt)/d\mu_b(x))^{1/2} f(xt),$$

$$(T_s^{L,b} f)(x) = (d\mu_b(s^{-1}x)/d\mu_b(x))^{1/2} f(s^{-1}x).$$

2 Von Neuman algebras generated by the regular representations

Let $\mathfrak{A}^{R,b} = (T_t^{R,b} \mid t \in B_0^{\mathbb{N}})''$ (resp. $\mathfrak{A}^{L,b} = (T_s^{L,b} \mid s \in B_0^{\mathbb{N}})''$) be the von Neumann algebras generated by the right $T^{R,b}$ (resp. the left $T^{L,b}$) regular representation of the group $B_0^{\mathbb{N}}$.

Theorem 7 [12] *If $E(b) < \infty$ then $\mu_b^{\Phi} \sim \mu_b$. In this case the left regular representation is well defined and the commutation theorem holds:*

$$(\mathfrak{A}^{R,b})' = \mathfrak{A}^{L,b}. \tag{2}$$

Moreover, the operator J_{μ_b} given by

$$(J_{\mu_b} f)(x) = (d\mu_b(x^{-1})/d\mu_b(x))^{1/2}\overline{f(x^{-1})} \tag{3}$$

is an intertwining operator:

$$T_t^{L,b} = J_{\mu_b} T_t^{R,b} J_{\mu_b},\ t \in B_0^{\mathbb{N}} \quad \text{and} \quad J_{\mu_b}\mathfrak{A}^{R,b} J_{\mu_b} = \mathfrak{A}^{L,b}.$$

If $\mu_b^{L_t} \perp \mu_b\ \forall t \in B_0^{\mathbb{N}}\backslash\{e\}$ one can't define the left regular representation of the group $B_0^{\mathbb{N}}$. Moreover the following theorem holds

Theorem 8 *The right regular representation $T^{R,b} : B_0^{\mathbb{N}} \mapsto U(H_b)$ is irreducible if and only if $\mu_b^{L_s} \perp \mu_b\ \forall s \in B_0^{\mathbb{N}}\backslash\{0\}$.*

Corollary 9 *The von Neumann algebra $\mathfrak{A}^{R,b}$ is a type I_∞ factor if*

$\mu_b^{L_s} \perp \mu_b\ \forall s \in B_0^{\mathbb{N}}\backslash\{0\}$.

Let us assume now that $\mu_b^{L_t} \sim \mu_b\ \forall t \in B_0^{\mathbb{N}}\backslash\{e\}$. In this case the right regular representation and the left regular representation of the group $B_0^{\mathbb{N}}$ are well defined.

In [13] the condition were studied *when the von Neumann algebra $\mathfrak{A}^{R,b}$ is factor*, i.e.

$$\mathfrak{A}^{R,b} \cap (\mathfrak{A}^{R,b})' = \{\lambda \mathbf{I} | \lambda \in \mathbb{C}^1\}.$$

Since $T_t^{L,b} \in (\mathfrak{A}^{R,b})'\ \forall t \in B_0^{\mathbb{N}}$, we have $\mathfrak{A}^{L,b} \subset (\mathfrak{A}^{R,b})'$, hence

$$\mathfrak{A}^{R,b} \cap (\mathfrak{A}^{R,b})' \subset (\mathfrak{A}^{L,b})' \cap (\mathfrak{A}^{R,b})' = (\mathfrak{A}^{R,b} \cup \mathfrak{A}^{L,b})'. \tag{4}$$

The last relation shows that $\mathfrak{A}^{R,b}$ is factor if the representation

$$B_0^{\mathbb{N}} \times B_0^{\mathbb{N}} \ni (t,\, s) \to T_t^{R,b} T_s^{L,b} \in U(H_b)$$

is irreducible.

Let us denote by $\mathfrak{A}^{R,L,b}$ the the von Neumann algebras generated by the right $T^{R,b}$ and the left $T^{L,b}$ regular representations of the group $B_0^{\mathbb{N}}$:

$$\mathfrak{A}^{R,L,b} = (T_t^{R,b}, T_s^{L,b} \mid t, s \in B_0^{\mathbb{N}})'' = (\mathfrak{A}^{R,b} \cup \mathfrak{A}^{L,b})''.$$

Let us denote

$$S_{kn}^{R,L}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{S_{nm}^{L}(b)}, \quad k < n.$$

Theorem 10 [13] *The representation*

$$B_0^{\mathbb{N}} \times B_0^{\mathbb{N}} \ni (t,\, s) \to T_t^{R,b} T_s^{L,b} \in U(H_b)$$

is irreducible if $S_{kn}^{R,L}(b) = \infty, \ \forall k < n.$

Corollary 11 *The von Neumann algebra* $\mathfrak{A}^{R,b}$ *is factor if* $S_{kn}^{R,L}(b) = \infty \ \forall k < n.$

3 Type III_1 factor

Let us denote as before $M = \mathfrak{A}^{L,b} = (T_s^{L,b} \mid s \in B_0^{\mathbb{N}})'', \ \mathfrak{A}^{R,b} = (T_t^{R,b} \mid t \in B_0^{\mathbb{N}})''.$

Theorem 12 *If* $S_{kn}^{R,L}(b) = \infty, \ \forall k < n$ *then the von Neumann algebra* $\mathfrak{A}^{L,b}$ *(and hence* $\mathfrak{A}^{R,b}$*) is* III_1 *factor.*

PROOF. The proof is based on Lemma 13 and 14, we shall prove them later.

Using (3) we conclude that the modular operator Δ is defined as follows

$$(\Delta f)(x) = (d\mu_b(x)/d\mu_b(x^{-1}))f(x). \tag{5}$$

Lemma 13 *We have*

$$Sp\Delta = [0, \infty).$$

We have $Sp\Delta\phi = Sp\Delta = [0, \infty)$, where $\phi(a) = (a\mathbf{1},\, \mathbf{1})_{H_b}, \ a \in M = \mathfrak{A}^{L,b}$. The centralizer M_ϕ of ϕ is defined by the equality

$$M_\phi = \{a \in M \mid \sigma_t^{\phi}(a) \ \forall t \in \mathbb{R}\}$$

where $\sigma_t^{\phi}(a) = \Delta^{it} a \Delta^{-it}$. For every projection $e \neq 0$, $e \in M_\phi$, a faithful semifinite normal weight ϕ_e on the reduced von Neumann algebra $eMe = \{a \in M; \ ea = ae = a\}$ is defined by the equality

$$\phi_e(a) = \phi(a) \ \ \forall a \in eMe, a \geq 0.$$

One has the formula

$$S(M) = \bigcap_{e\neq 0} Sp\Delta_{\phi_e}, \tag{6}$$

where e varies over the nonzero projection of M_ϕ (see[4] p.472).

Lemma 14 *The von Neumann algebra M_ϕ is trivial.*

In this case

$$S(M) = Sp\Delta = [0,\, \infty),$$

so the von Neumann algebra $\mathfrak{A}^{L,b}$ (and hence algebra $\mathfrak{A}^{R,b}$) is type III_1 factor. □

Proof of Lemma 14. We show that

$$M_\phi = (\Delta^{it},\, T_s^{R,b} \mid t \in \mathbb{R},\, s \in B_0^{\mathbb{N}})'. \tag{7}$$

So M_ϕ is trivial means that the set of operators

$$(\Delta^{it},\, T_s^{R,b} \mid t \in \mathbb{R},\, s \in B_0^{\mathbb{N}}) \tag{8}$$

is **irreducible**. To prove (7) we get

$$M_\phi = (a \in \mathfrak{A}^{L,b} \mid \Delta^{it} a = a\Delta^{it},\ \forall t \in \mathbb{R}) = (\Delta^{it} \mid t \in \mathbb{R})' \cap \mathfrak{A}^{L,b}$$

$$= (\Delta^{it} \mid t \in \mathbb{R})' \cap (\mathfrak{A}^{R,b})' = (\Delta^{it} \mid t \in \mathbb{R})' \cap (T_s^{R,b} \mid s \in B_0^{\mathbb{N}})' =$$

$$(\Delta^{it},\, T_s^{R,b} \mid t \in \mathbb{R},\, s \in B_0^{\mathbb{N}})'$$

Definition. Recall (c.f. e.g. [5]) that a non necessarily bounded self-adjoint operator A in a Hilbert space H is said to be *affiliated* with a von Neumann algebra M of operators in this Hilbert space H, if $\exp(itA) \in M$ for all $t \in \mathbb{R}$. One then writes $A\ \eta\ M$.

To prove the irreducibility of $(\Delta^{it},\, T_s^{R,b} \mid t \in \mathbb{R},\, s \in B_0^{\mathbb{N}})$ it is sufficient to prove (see [10] p.258) that operators $f(x) \mapsto x_{kn} f(x)$ of multiplication in the space H_b by the independent variables x_{kn} are affiliated to the von Neumann algebra

$$(M_\phi)' = (\Delta^{it},\, T_s^{R,b} \mid t \in \mathbb{R},\, s \in B_0^{\mathbb{N}})''.$$

In this case the operator A commuting with Δ^{it} and $T_s^{R,b}$ is operator of multiplication by some function $a(x)$. If we use commutation relation $[A, T_s^{R,b}] = 0,\ s \in B_0^{\mathbb{N}}$ we obtain $a(x) = a(xs) \bmod \mu$. Using the ergodocity of the measure μ_b with respect of the right action of the group $B_0^{\mathbb{N}}$ we conclude that $a(x) = \mathrm{const} \bmod \mu$ i.e. A is scalar operator.

If we denote

$$A_{kn}^R = (d/dt) T_{I+tE_{kn}}^{R,b} \mid_{t=o}$$

we have (see for example [9–11])

$$A^R_{kn}=\sum_{r=1}^{k-1}x_{kr}D_{rn}+D_{kn},\quad 1\leq k<n. \tag{9}$$

The direct calculation shows that

$$[A^R_{13},[A^R_{23},\,\ln\Delta]]=2b_{13}x_{12}, \tag{10}$$

$$[A^R_{12},[A^R_{23},\,\ln\Delta]]=2b_{13}x_{13}. \tag{11}$$

Idea: to obtain in a similar way all variables x_{kn}.

Let us denote by X^{-1} the inverse matrix to the upper triangular matrix $X=I+x=I+\sum_{k<n}x_{kn}E_{kn}\in B^{\mathbb{N}}$

$$X^{-1}=(I+x)^{-1}=I+\sum_{k<n}x^{-1}_{kn}E_{kn}\in B^{\mathbb{N}}.$$

We have by definition $X^{-1}X=XX^{-1}=I$ hence

$$\Big(XX^{-1}\Big)_{kn}=\sum_{r=k}^{n}x_{kr}x^{-1}_{rn}=\delta_{kn}=\sum_{r=k}^{n}x^{-1}_{kr}x_{rn}=\Big(X^{-1}X\Big)_{kn},\quad k\leq n, \tag{12}$$

hence

$$x^{-1}_{kn}+\sum_{r=k+1}^{n-1}x_{kr}x^{-1}_{rn}+x_{kn}=0=x_{kn}+\sum_{r=k+1}^{n-1}x_{kr}x^{-1}_{rn}+x^{-1}_{kn},\quad k<n,$$

and

$$x^{-1}_{kn}=-x_{kn}-\sum_{r=k+1}^{n-1}x_{kr}x^{-1}_{rn}=-x_{kn}-\sum_{r=k+1}^{n-1}x^{-1}_{kr}x_{rn}. \tag{13}$$

We can write also

$$x^{-1}_{kn}=-\sum_{r=k+1}^{n}x_{kr}x^{-1}_{rn}=-\sum_{r=k}^{n-1}x^{-1}_{kr}x_{rn}. \tag{14}$$

There is also the explicit formula for x^{-1}_{kn} (see [8] formula (4.4)) $x^{-1}_{kk+1}=-x_{kk+1}$

$$x^{-1}_{kn}=-x_{kn}+\sum_{r=1}^{n-k-1}(-1)^{r+1}\sum_{k\leq i_1<i_2<\ldots<i_r\leq n}x_{ki_1}x_{i_1i_2}\ldots x_{i_rn},\quad k<n-1. \tag{15}$$

Remark 15 *Using (15) we see that x^{-1}_{kn} depends only on x_{rs} with $k\leq r<s\leq n$.*

Using (14) we have

$$x_{kn}+x^{-1}_{kn}=-\sum_{r=k+1}^{n-1}x_{kr}x^{-1}_{rn},\quad x_{kn}-x^{-1}_{kn}=2x_{kn}-\sum_{r=k+1}^{n-1}x_{kr}x^{-1}_{rn}. \tag{16}$$

Let us denote

$$w_{kn}:=w_{kn}(x):=(x_{kn}+x_{kn}^{-1})(x_{kn}-x_{kn}^{-1}). \tag{17}$$

Using (1) we get

$$\Delta(x)=\frac{d\mu_b(x)}{d\mu_b(x^{-1})}=\exp\left[-\sum_{k<n}b_{kn}\left(x_{kn}^2-(x_{kn}^{-1})^2\right)\right]=\exp\left[-\sum_{k<n}b_{kn}w_{kn}(x)\right]. \tag{18}$$

$$-\ln\Delta(x)=\sum_{k<n}b_{kn}\left[x_{kn}^2-(x_{kn}^{-1})^2\right]=\sum_{k<n}b_{kn}(x_{kn}+x_{kn}^{-1})(x_{kn}-x_{kn}^{-1})$$

$$\sum_{k<n}b_{kn}(x_{kn}+x_{kn}^{-1})[2x_{kn}-(x_{kn}+x_{kn}^{-1})]=\sum_{k<n}b_{kn}w_{kn}(x).$$

To study the action of the operators $A_{kn}^R=\sum_{r=1}^{k-1}x_{rk}D_{rn}+D_{kn}$ on the function $\ln\Delta(x)$ we need to know the action of D_{pq} on x_{kn}^{-1}.

Lemma 16 *We have*

$$[D_{pq},x_{kn}^{-1}]=\begin{cases}-x_{kp}^{-1}x_{qn}^{-1}, & \textit{if } k\leq p<q\leq n,\\ 0, & \textit{otherwise}.\end{cases} \tag{19}$$

PROOF. We prove (19) by induction in $p: k\leq p<q\leq n$. For $p=k$ using (16) we have

$$[D_{kq},x_{kn}^{-1}]=-[D_{kq},x_{kn}+\sum_{r=k+1}^{n-1}x_{kr}x_{rn}^{-1}]=-[D_{kq},x_{kq}x_{qn}^{-1}]=-x_{qn}^{-1}=-x_{kk}^{-1}x_{qn}^{-1},$$

so (19) holds for $p=k$.

Let us suppose that (19) holds for all (p,q) with $k\leq p<s\leq n,\ k\leq p<q\leq n$. We prove that than (19) holds also for $(s,q)\ :\ s<q\leq n$. Indeed we have

$$[D_{sq},x_{kn}^{-1}]=-[D_{sq},x_{kn}^{-1}+\sum_{r=k+1}^{n-1}x_{kr}x_{rn}^{-1}]=-\sum_{r=k+1}^{s}x_{kr}[D_{sq},x_{rn}^{-1}]$$

$$=\sum_{r=k+1}^{s}x_{kr}x_{rs}^{-1}x_{qn}^{-1}\overset{(13)}{=}x_{ks}^{-1}x_{qn}^{-1}.$$

□

Using (19) we get

$$[D_{pq},x_{kn}+x_{kn}^{-1}]=\begin{cases}-x_{kp}^{-1}x_{qn}^{-1}, & \text{if } k\leq p<q\leq n,\ (p,q)\neq(k,n)\\ 0, & \text{otherwise}.\end{cases} \tag{20}$$

Using (20) we have

$$[D_{pq},(x_{kn}+x_{kn}^{-1})(x_{kn}-x_{kn}^{-1})] = \begin{cases} 2\,x_{kp}^{-1}x_{qn}^{-1}x_{kn}^{-1}, & \text{if } k \le p < q \le n,\ (p,q) \neq (k,n) \\ 2(x_{kn}+x_{kn}^{-1}), & \text{if } \quad (p,q)=(k,n) \\ 0, & \text{otherwise}. \end{cases} \tag{21}$$

Indeed, if $k \le p < q \le n$, $(p,q) \neq (k,n)$ we have

$$[D_{pq},(x_{kn}+x_{kn}^{-1})(x_{kn}-x_{kn}^{-1})] = [D_{pq},(x_{kn}+x_{kn}^{-1})(2x_{kn}-(x_{kn}+x_{kn}^{-1}))]$$

$$= [D_{pq},(x_{kn}+x_{kn}^{-1})](2x_{kn}-(x_{kn}+x_{kn}^{-1})) - (x_{kn}+x_{kn}^{-1})[D_{pq},(x_{kn}+x_{kn}^{-1})] =$$

$$-2x_{kn}^{-1}[D_{pq},(x_{kn}+x_{kn}^{-1})] \overset{(20)}{=} 2x_{kp}^{-1}x_{qn}^{-1}x_{kn}^{-1}.$$

Lemma 17 *We have*

$$[A_{mm+1}^R,\, w_{kn}] = \begin{cases} 0, & \text{if } \ k < n \le m \\ 2x_{km}x_{km+1} & \text{if } \ n = m+1,\ 1 \le k \le m-1 \\ 0, & \text{if } \ 1 \le k \le m-1,\ m+1 < n \\ 2x_{mn}^{-1}x_{m+1n}^{-1}, & \text{if } \ k = m,\ n \ge m+2 \\ 0, & \text{if } \ \ m+1 \le k < n. \end{cases} \tag{22}$$

hence

$$-[A_{mm+1}^R,\, \ln\Delta] = 2\sum_{r=1}^{m-1} b_{rm+1}x_{rm}x_{rm+1} + 2\sum_{n=m+2}^{\infty} b_{mn}x_{mn}^{-1}x_{m+1n}^{-1}. \tag{23}$$

PROOF. Since

$$A_{mm+1}^R = \sum_{r=1}^{m-1} x_{rm}D_{rm+1} + D_{mm+1}$$

and $w_{kn},\ k < n \le m$ do not depend on $x_{rm+1},\ 1 \le r \le m+1$ we conclude that $[A_{mm+1}^R,\, w_{kn}] = 0$ for $k < n \le m$ and $m+1 \le k < n$.

Let $n = m+1$, since $[D_{rm+1},\, w_{km+1}] = 0$ for $1 \le r < k$ we get

$$[A_{mm+1}^R,\, w_{km+1}] = \sum_{r=k}^{m-1} x_{rm}[D_{rm+1},\, w_{km+1}] + [D_{mm+1},\, w_{km+1}] =$$

$$2\left(x_{km}(x_{km+1}+x_{km+1}^{-1}) + \sum_{r=k+1}^{m-1} x_{rm}x_{kr}^{-1}x_{km+1}^{-1} + x_{km}^{-1}x_{km+1}^{-1} \right) =$$

$$2\left(x_{km}x_{km+1}+\left(x_{km}+\sum_{r=k+1}^{m-1}x_{kr}^{-1}x_{rm}+x_{km}^{-1}\right)x_{km+1}^{-1}\right)\overset{(13)}{=}2x_{km}x_{km+1}.$$

Similarly, for $1\leq k\leq m-1,\ m+1<n$ we get

$$[A^R_{mm+1},\,w_{kn}]=\sum_{r=k}^{m-1}x_{rm}[D_{rm+1},\,w_{kn}]+[D_{mm+1},\,w_{kn}]=$$

$$2\left(x_{km}x_{m+1n}^{-1}+\sum_{r=k+1}^{m-1}x_{rm}x_{kr}^{-1}x_{m+1n}^{-1}+x_{km}^{-1}x_{m+1n}^{-1}\right)$$

$$2\left(x_{km}+\sum_{r=k+1}^{m-1}x_{rm}x_{kr}^{-1}+x_{km}^{-1}\right)x_{m+1n}^{-1}\overset{(13)}{=}0.$$

Finally if $k=m$ and $n\geq m+2$ we have as before

$$[A^R_{mm+1},\,w_{mn}]=[D_{mm+1},\,w_{mn}]\overset{(21)}{=}2x_{mn}^{-1}x_{m+1n}^{-1}.$$

□

We consider the action of A^R_{mm+1} on $\ln\Delta$.

Let $m=2$. Since

$$[A^R_{23},\,w_{13}]=2b_{13}x_{12}x_{13},\ \ [A^R_{23},\,w_{1n}]=0,\ n\geq 4,\ \ [A^R_{23},\,w_{kn}]=0,\ 3\leq k<n,$$

we have

$$-[A^R_{23},\,\ln\Delta]=2b_{13}x_{12}x_{13}+2\sum_{n=4}^{\infty}b_{2n}x_{2n}^{-1}x_{3n}^{-1},$$

hence

$$-[A^R_{12},[A^R_{23},\,\ln\Delta]]=2b_{13}x_{13},$$
$$-[A^R_{13},[A^R_{23},\,\ln\Delta]]=2b_{13}x_{12}.$$

The last two equations gives us $x_{12},x_{13}\,\eta\,\mathfrak{A}$.

Let $m=3$. Since

$$[A^R_{34},\,w_{13}]=0,\ [A^R_{34},\,w_{14}]=2x_{13}x_{14},\ [A^R_{34},\,w_{24}]=2x_{23}x_{24},$$

$$[A^R_{34},\,w_{1n}]=[A^R_{34},\,w_{1n}]=0,\ [A^R_{34},\,w_{3n}]=b_{3n}x_{3n}^{-1}x_{4n}^{-1},n\geq 5,$$
$$[A^R_{34},\,w_{kn}]=0,\ 4\leq k<n,$$

we have

$$-[A^R_{34},\,\ln\Delta]=2b_{14}x_{13}x_{14}+2b_{24}x_{23}x_{24}+2\sum_{n=5}^{\infty}b_{3n}x_{3n}^{-1}x_{4n}^{-1},$$

hence

$$-[A^R_{23},[A^R_{34},\,\ln\Delta]]=2b_{14}x_{12}x_{14}+2b_{24}x_{24}$$

$$-[A^R_{12}[A^R_{23},[A^R_{34},\,\ln\Delta]]]=2b_{14}x_{14},$$

$$-[A^R_{24},[A^R_{34},\,\ln\Delta]]=2[x_{12}D_{14}+D_{24},b_{14}x_{13}x_{14}+b_{24}x_{23}x_{24}]=2b_{14}x_{12}x_{13}+2b_{24}x_{23},$$

Since $x_{12},x_{13}\,\eta\,\mathfrak{A}$ from the latter equation we conclude that $x_{23}\,\eta\,\mathfrak{A}$. The previous equation gives us $x_{14}\,\eta\,\mathfrak{A}$ and the equation before gives $x_{24}\,\eta\,\mathfrak{A}$. Finally we conclude that $x_{14},x_{24},x_{23}\,\eta\,\mathfrak{A}$.

Let us suppose that we have obtained the variables $x_{rm}, 1\leq r\leq m-2$ and $x_{m-2,m-1}$. We prove that we can obtain the following variables $x_{rm+1}, 1\leq r\leq m-1$ and x_{m-1m}.

Indeed we calculate the action of the following sequence of operators on the result: $A^R_{m-1,m}$, $A^R_{m-2,m-1}$ etc. till A^R_{12}. We obtain

$$-[A^R_{m-1,m},[A^R_{mm+1},\,\ln\Delta]]=2\left(\sum_{r=1}^{m-2}b_{r,m+1}x_{r-1,m}x_{r,m+1}+b_{m-1,m+1}x_{m-1,m+1}\right),$$

$$-[A^R_{m-2,m-1},[A^R_{m-1,m},[A^R_{mm+1},\,\ln\Delta]]]$$
$$=2\left(\sum_{r=1}^{m-3}b_{r,m+1}x_{r-2,m}x_{r,m+1}+b_{m-2,m+1}x_{m-2,m+1}\right),$$
$$-[A^R_{m-s,m-s+1},[A^R_{m-s+1,m-s+2},...[A^R_{m-1,m}[A^R_{mm+1},\,\ln\Delta]]...]]$$
$$=2\left(\sum_{r=1}^{m-s-1}b_{r,m+1}x_{r,m-s}x_{r,m+1}+b_{m-s,m+1}x_{m-s,m+1}\right),\ 1\leq s\leq m,$$
$$-[A^R_{34},...[A^R_{mm+1},\,\ln\Delta]...]=2(b_{1,m+1}x_{13}x_{1,m+1}+b_{2,m+1}x_{23}x_{2,m+1}+b_{3,m+1}x_{3,m+1}),$$
$$-[A^R_{23},[A^R_{34},...[A^R_{mm+1},\,\ln\Delta]...]]=2(b_{1,m+1}x_{12}x_{1,m+1}+b_{2,m+1}x_{2,m+1}),$$
$$-[A^R_{12},[A^R_{23},[A^R_{34},...[A^R_{mm+1},\,\ln\Delta]...]]]=2b_{1,m+1}x_{1,m+1}.$$

From the latter equation we conclude that $x_{1,m+1}\ \eta\ \mathfrak{A}$. The last but one equation gives us $x_{2,m+1}\ \eta\ \mathfrak{A}$ (since $x_{12},x_{1,m+1}\ \eta\ \mathfrak{A}$) etc. i.e : $x_{rm+1}\ \eta\ \mathfrak{A},\ 1\leq r\leq m-1$.

$$-[A^R_{m-1m+1},[A^R_{mm+1},\,\ln\Delta]]=[\sum_{r=1}^{m-2}x_{rm-1}D_{rm+1}+D_{m-1m+1},2\sum_{r=1}^{m-1}b_{rm+1}x_{rm}x_{rm+1}]=$$

$$2\sum_{r=1}^{m-2}b_{rm+1}x_{rm-1}x_{rm}+b_{m-1,m+1}x_{m-1,m},$$

since $x_{rm-1},x_{rm}\ \eta\ \mathfrak{A}$ for $1\leq r\leq m-2$ hence $x_{m-1,m}\ \eta\ \mathfrak{A}$. □

To be sure that all this argument works we should prove that all involved operators are affiliated to the von Neumann algebra M'_ϕ defined by (7). For example if A^R_{23} and Δ (and hence $\ln\Delta$) are affiliated to the von Neumann algebra M'_ϕ, why the operator $[A^R_{23}, \ln\Delta]$ is also affiliated. In general, why the operators $[A^R_{12}, [A^R_{23}, [A^R_{34}, ...[A^R_{mm+1}, \ln\Delta]...]]]$ are affiliated?

Remark 18 *In general we do not know whether the commutator $[A, B]$ of two operators A and B affiliated to the von Neumann algebra is also affiliated.*

This is the reason, why we use another approach to prove that the algebra M_ϕ is trivial.

4 The von Neumann algebra M_ϕ is trivial

Since $M_\phi = (\Delta^{it}, T^{R,b}_s \mid t \in \mathbb{R},\ s \in B^{\mathbb{N}}_0)'$ (see (7)) it is sufficient to prove that the set of operators

$$(\Delta^{is}, T^{R,b}_t \mid s \in \mathbb{R},\ t \in B^{\mathbb{N}}_0) \subset M'_\phi$$

is irreducible.

Idea of the proof. *We show that the von Neumann subalgebra in the algebra M'_ϕ, generated by the following operators*

$$(\{T^R_{t_n}, \{T^R_{t_{n-1}}, ...\{T^R_{t_1}, \Delta^{is}\}...\}\} \mid s \in \mathbb{R},\ t_1, ..., t_n \in B^{\mathbb{N}}_0), \tag{24}$$

where $\{a, b\} := aba^{-1}b^{-1}$ is **the maximal abelian subalgebra**. *More precisely we prove that this subalgebra contains all functions* $\exp(isx_{kn}),\ k < n,\ s \in \mathbb{R}$.

To prove the irreducibility of the algebra M'_ϕ (see proof of the Lemma 14) we observe that if an bounded operator commute with all $\exp(isx_{kn}),\ k < n,\ s \in \mathbb{R}$ then this operator itself is an operator of multiplication by some essentially bounded function $A = a(x)$. Commutation relation $[T^{R,b}_t, A] = 0$ for all $t \in B^{\mathbb{N}}_0$ gives us $a(xt) = a(x) \bmod \mu_b$ for all t. Since the measure μ_b is $B^{\mathbb{N}}_0-$**right ergodic** we conclude that A is trivial i.e. $A = a(x) = CI$.

We note that expressions in (24) are the "right" analog of the left hand side of the expressions (10) and (11)

$$[A^R_{13}, [A^R_{23}, \ln\Delta]] = 2b_{13}x_{12},$$

$$[A^R_{12}, [A^R_{23}, \ln\Delta]] = 2b_{13}x_{13},$$

involving generators A^R_{kn}. In general, if we have two subgroups of unitary operators $U(t)$ and $V(s)$ with the generators A and B, to obtain the commutator $[iA, iB]$ it is sufficient to differentiate the following expression $U(t)V(s)U(-t)$:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}U(t)V(s)U(-t) \mid_{t=s=0} = [iA, iB].$$

Indeed we have

$$\frac{\partial}{\partial s}U(t)V(s)U(-t)=U(t)iBV(s)U(-t),\quad \frac{\partial}{\partial t}U(t)iBV(s)U(-t)\mid_{t=s=0}=$$

$$(iAU(t)iBV(s)U(-t)-U(t)iBV(s)iAU(-t))\mid_{t=s=0}=[iA,iB].$$

We show that more convenient analog of the commutator $[iA,iB]$ is *commutator* (in the group sence) of two one-parameter groups

$$\{U(t),V(s)\}:=U(t)V(s)U(t)^{-1}V(s)^{-1}=U(t)V(s)U(-t)V(-s).$$

Lemma 19 *For the operator g of multiplication on the function $g: f(x)\mapsto g(x)f(x)$ in the space $H_b=L_2(B^{\mathbb{N}},d\mu_b)$ we have*

$$T_t^R g(x)T_{t^{-1}}^R=g(xt),\ t\in B_0^{\mathbb{N}}.$$

PROOF. We have

$$f(x)\overset{g(x)T_{t^{-1}}^R}{\mapsto}g(x)\left(\frac{d\mu(xt^{-1})}{d\mu(x)}\right)^{1/2}f(xt^{-1})\overset{T_t^R}{\mapsto}$$

$$\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2}g(xt)\left(\frac{d\mu(x)}{d\mu(xt)}\right)^{1/2}f(x)=g(xt)f(x).$$

□

Using the lemma we have

$$T_t^R\Delta^{is}(x)T_{t^{-1}}^R=\Delta^{is}(xt).$$

Using (18) we have

$$\Delta^{is}(x)=\exp\left(-is\sum_{k+1<n}b_{kn}(x_{kn}+x_{kn}^{-1})[2x_{kn}-(x_{kn}+x_{kn}^{-1})]\right)=$$

$$\exp\left(-is\sum_{k+1<n}b_{kn}w_{kn}(x)\right),\tag{25}$$

where $w_{kn}(x)=(x_{kn}+x_{kn}^{-1})[2x_{kn}-(x_{kn}+x_{kn}^{-1})]$ (see (17)).

We would like **to obtain the functions** $\exp(isx_{kn})$ using the expressions (24). To simplify the situation we consider firstly the projections of all considered object: the measure $\mu_b^{(k)}$, the generators $A_{kn}^{R,(k)}$, operator $\Delta_{(k)}$ algebra $M^{(k)}:=(M'_\phi)^{(k)}$ etc. on the following subspace $X^{(k)},\ k\geq 2$ of the space $B^{\mathbb{N}}$:

$$X^{(2)}=\begin{pmatrix}1&x_{12}&x_{13}&\dots&x_{1n}&\dots\\0&1&x_{23}&\dots&x_{2n}&\dots\end{pmatrix},\quad X^{(3)}=\begin{pmatrix}1&x_{12}&x_{13}&x_{14}&\dots&x_{1n}&\dots\\0&1&x_{23}&x_{14}&\dots&x_{2n}&\dots\\0&0&1&x_{34}&\dots&x_{3n}&\dots\end{pmatrix},\ \text{etc.}$$

Note that

$$\begin{pmatrix} 1 & x_{12} & x_{13} & ... & x_{1n} & ... \\ 0 & 1 & x_{23} & ... & x_{2n} & ... \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x_{12} & -x_{13}+x_{12}x_{23} & ... & -x_{1n}+x_{12}x_{2n} & ... \\ 0 & 1 & -x_{23} & ... & -x_{2n} & ... \end{pmatrix}. \tag{26}$$

We have for the corresponding projections on $X^{(2)}$:

$$A^R_{1n} = D_{1n}, \quad A^R_{2n} = x_{12}D_{1n} + D_{2n}, \quad A^{R,(2)}_{kn} = x_{1k}D_{1n} + x_{2k}D_{2n},\ 2 < k < n,$$

$$w_{1n}(x) = (x_{1n} + x^{-1}_{1n})(x_{1n} - x^{-1}_{1n}) = x_{12}x_{2n}(2x_{1n} - x_{12}x_{2n}),\ w_{2n}(x) = 0,$$

hence

$$\Delta^{is}_{(2)}(x) := \exp\left(-is\sum_{k=3}^{\infty} b_{1n}w_{1n}(x)\right) = \exp\left(-is\sum_{k=3}^{\infty} b_{1n}x_{12}x_{2n}(2x_{1n} - x_{12}x_{2n})\right).$$

Let us denote by

$$E_{kn}(t) := I + tE_{kn}, \quad T_{kn}(t) = T^R_{E_{kn}(t)},\ k < n,\ t \in \mathbb{R} \tag{27}$$

the corresponding one-parameter subgroups. We have

$$\begin{pmatrix} x_{12} & x_{1m} \\ 1 & x_{2m} \end{pmatrix} \overset{E_{2m}(t)}{\mapsto} \begin{pmatrix} x_{12} & x_{1m}+tx_{12} \\ 1 & x_{2m}+t \end{pmatrix},\ w_{1n}(xE_{2m}(t)) = \begin{cases} w_{1n}(x) & \text{if } n\neq m \\ w_{1m}(xE_{2m}(t)) & \text{if } n=m \end{cases}$$

so using Lemma 19 we get

$$\{T_{2m}(t), \Delta^{is}_{(2)}(x)\} = T_{2m}(t)\Delta^{is}_{(2)}(x)T_{2m}(-t)\Delta^{-is}_{(2)}(x) = \Delta^{is}_{(2)}(xE_{2m}(t))\Delta^{-is}_{(2)}(x) =$$

$$\exp\left(-is\left[\sum_{k=3,k\neq m}^{\infty} b_{1n}w_{1n}(x) + b_{1m}w_{1m}(xE_{2m}(t))\right]\right)\exp\left(is\sum_{k=3}^{\infty} b_{1n}w_{1n}(x)\right) =$$

$$\exp\left(-isb_{1m}[w_{1m}(xE_{2m}(t)) - w_{1m}(x)]\right) = \exp\left(isb_{1m}(2tx_{12}x_{1m} + t^2x^2_{12})\right),$$

since

$$w_{1m}(xE_{2m}(t)) - w_{1m}(x) = x_{12}(x_{2m} + t)[2(x_{1m} + tx_{12}) - x_{12}(x_{2m} + t)] -$$

$$x_{12}x_{2m}(2x_{1m} - x_{12}x_{2m}) = x_{12}[tx_{12}x_{2m} + t(2x_{1m} - x_{12}x_{2m}) + t^2x_{12}] = 2tx_{12}x_{1m} + t^2x^2_{12}.$$

Let us denote

$$\phi_{t,s}(x) := \{T_{2m}(t), \Delta^{is}_{(2)}(x)\} = \exp\left(isb_{1m}(2tx_{12}x_{1m} + t^2x^2_{12})\right). \tag{28}$$

Using Lemma 19 we get

$$\{T_{1m}(t_1), \{T_{2m}(t), \Delta^{is}_{(2)}(x)\}\} = \{T_{1m}(t_1), \phi_{t,s}(x)\} =$$

$$T_{1m}(t_1)\phi_{t,s}(x)T_{1m}(-t_1)(\phi_{t,s}(x))^{-1} = \phi_{t,s}(xE_{1m}(t_1))(\phi_{t,s}(x))^{-1} =$$

$$\exp\left[isb_{1m}(2tx_{12}(x_{1m} + t_1) + t^2x^2_{12}) - isb_{1m}(2tx_{12}x_{1m} + t^2x^2_{12})\right] =$$

$$\exp\left(isb_{1m}x_{12}2tt_1\right).$$

Finally we get for $X^{(2)}$

$$\exp(isx_{12}) \in M^{(2)} := (M'_\phi)^{(2)}.$$

Using (28) we conclude that

$$\exp(isx_{12}x_{1m}) \in M^{(2)}.$$

Applying again $T_{12}(t)$ and $T_{1m}(t)$ we get

$$\{T_{12}(t), \exp(isx_{12}x_{1m})\} = T_{12}(t)\exp(isx_{12}x_{1m})T_{12}(-t)\exp(-isx_{12}x_{1m}) =$$

$$\exp(is(x_{12}+t)x_{1m} - isx_{12}x_{1m}) = \exp(istx_{12}),$$

$$\{T_{1m}(t), \exp(isx_{12}x_{1m})\} = T_{1m}(t)\exp(isx_{12}x_{1m})T_{1m}(-t), \exp(-isx_{12}x_{1m}) =$$

$$\exp(isx_{12}(x_{1m}+t) - isx_{12}x_{1m}) = \exp(istx_{1m}).$$

At last we conclude that for $X^{(2)}$ we have $\exp(isx_{12}),\ \exp(isx_{1m}) \in M^{(2)}$ in particular

$$\exp(isx_{12}),\ \exp(isx_{13}) \in M^{(2)}. \tag{29}$$

For $X^{(3)}$ and the corresponding projections we have

$$\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & \dots & x_{1n} & \dots \\ 0 & 1 & x_{23} & x_{14} & \dots & x_{2n} & \dots \\ 0 & 0 & 1 & x_{34} & \dots & x_{3n} & \dots \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} 1 & -x_{12} & -x_{13}+x_{12}x_{23} & -x_{14}+x_{12}x_{24}+x_{13}x_{34}+x_{12}x_{23}x_{34} & \dots & -x_{1n}+x_{12}x_{2n}+x_{13}x_{3n}+x_{12}x_{23}x_{3n} & \dots \\ 0 & 1 & -x_{23} & -x_{24}+x_{23}x_{34} & \dots & -x_{2n}+x_{23}x_{3n} & \dots \\ 0 & 0 & 1 & -x_{34} & \dots & x_{3n} & \dots \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -x_{12} & -x_{13}-x_{12}^{-1}x_{23} & -x_{14}-x_{12}^{-1}x_{24}-x_{13}^{-1}x_{34} & \dots & -x_{1n}-x_{12}^{-1}x_{2n}-x_{13}^{-1}x_{3n} & \dots \\ 0 & 1 & -x_{23} & -x_{24}-x_{23}^{-1}x_{34} & \dots & -x_{2n}-x_{23}^{-1}x_{3n} & \dots \\ 0 & 0 & 1 & -x_{34} & \dots & -x_{3n} & \dots \end{pmatrix}, \tag{30}$$

$$A^R_{1n} = D_{1n}, \quad A^R_{2n} = x_{12}D_{1n} + D_{2n}, \quad A^R_{3n} = x_{13}D_{1n} + x_{23}D_{2n} + D_{3n},\ 3 < n.$$

We have

$$\Delta^{is}_{(3)}(x) = \exp\left(-is\left[\sum_{n=3}^{\infty} b_{1n}w_{1n}(x) + \sum_{n=4}^{\infty} b_{2n}w_{2n}(x)\right]\right) =$$

$$\exp\left(-is\left[\sum_{n=3}^{\infty} b_{1n}(x_{1n}+x_{1n}^{-1})[2x_{1n} - (x_{1n}+x_{1n}^{-1})]\right]\right) \times$$

$$\exp\left(-is\left[\sum_{n=4}^{\infty} b_{2n}(x_{2n}+x_{2n}^{-1})[2x_{2n} - (x_{2n}+x_{2n}^{-1})]\right]\right).$$

By the same procedure as in the case of the space $X^{(2)}$ we can obtain that

$$\exp(isx_{12}),\ \exp(isx_{13}) \in M^{(3)}. \tag{31}$$

We show that

$$\{T_{34}(t), \Delta^{is}_{(3)}(x)\} = \exp\left(is\left[b_{14}(2tx_{13}x_{14} + t^2x_{13}^2) + b_{24}(2tx_{23}x_{24} + t^2x_{23}^2)\right]\right). \tag{32}$$

(compare with (28)). Indeed we have

$$\{T_{34}(t),\Delta^{is}_{(3)}(x)\}=T_{34}(t)\Delta^{is}_{(3)}(x)T_{34}(-t)\Delta^{-is}_{(3)}(x)=$$

$$\Delta^{is}_{(3)}(xE_{34}(t))\Delta^{-is}_{(3)}(x)=$$

$$\exp\left(-is\left(b_{14}[w_{14}(xE_{34}(t))-w_{14}(x)]+b_{24}[w_{24}(xE_{34}(t))-w_{24}(x)]\right)\right),$$

which implies (32), since

$$w_{14}(x)=(x_{14}+x^{-1}_{14})[2x_{14}-(x_{14}+x^{-1}_{14})]=-(x^{-1}_{12}x_{24}+x^{-1}_{13}x_{34})[2x_{14}+x^{-1}_{12}x_{24}+x^{-1}_{13}x_{34}],$$

and

$$w_{14}(xE_{34}(t))-w_{14}(x)=$$
$$-[x^{-1}_{12}(x_{24}+tx_{23})+x^{-1}_{13}(x_{34}+t)][2(x_{14}+tx_{13})+x^{-1}_{12}(x_{24}+tx_{23})+x^{-1}_{13}(x_{34}+t)]$$
$$+(x^{-1}_{12}x_{24}+x^{-1}_{13}x_{34})[2x_{14}+x^{-1}_{12}x_{24}+x^{-1}_{13}x_{34}]=$$
$$-t\left[(x^{-1}_{12}x_{24}+x^{-1}_{13}x_{34})(2x_{13}+x^{-1}_{12}x_{23}+x^{-1}_{13})+(x^{-1}_{12}x_{23}+x^{-1}_{13})(2x_{14}+x^{-1}_{12}x_{24}+x^{-1}_{13}x_{34})\right]$$
$$-t^2(x^{-1}_{12}x_{23}+x^{-1}_{13})(2x_{13}+x^{-1}_{12}x_{23}+x^{-1}_{13})=-t[-(x_{14}+x^{-1}_{14})x_{13}-x_{13}(x_{14}+x^{-1}_{14})]+t^2x_{13}x_{13}=$$
$$2tx_{13}x_{14}+t^2x^2_{13}.$$

Using (31) and (32) we get

$$\phi^{(3)}_{t,s}(x):=\exp\left(is\left[b_{14}2tx_{13}x_{14}+b_{24}(2tx_{23}x_{24}+t^2x^2_{23})\right]\right)\in M^{(3)},$$

hence

$$\{T_{13}(t_1),\phi^{(3)}_{t,s}(x)\}=T_{13}(t_1)\phi^{(3)}_{t,s}(x)T_{13}(-t_1)(\phi^{(3)}_{t,s}(x))^{-1}=\exp\left(istt_1b_{14}2tx_{14}\right),$$

so $\exp(isx_{14})\in M^{(3)}$ and $\exp[isb_{24}(2tx_{23}x_{24}+t^2x^2_{23})]\in M^{(3)}$. Similarly we get

$$\{T_{24}(t_1),\exp[isb_{24}(2tx_{23}x_{24}+t^2x^2_{23})]\}=\exp(isb_{24}tt_1x_{23}),$$

so $\exp(isx_{23}),\exp(isx_{23}x_{24})\in M^{(3)}$. At last we get

$$\{T_{24}(t_1),\exp(isx_{23}x_{24})\}=\exp(ist_1x_{24}).$$

Finally we can obtain $\exp(isx_{kn})$ in the following order on the **first step**:

$$\exp(isx_{12}),\ \exp(isx_{13});$$

on the **second step**:

$$\exp(isx_{14}),\ \exp(isx_{23}),\ \exp(isx_{24})\in M^{(3)},$$

or symbolically in the following **order**:

$$\begin{pmatrix}1&x_{12}&x_{13}&x_{14}\\0&1&x_{23}&x_{24}\\0&0&1&\end{pmatrix},\quad\begin{pmatrix}0&1_1&2_1&1_2\\0&0&2_2&3_2\\0&0&0&\end{pmatrix}.$$

In general we get **the order**

$$\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} \\ 0 & 1 & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} \\ 0 & 0 & 1 & x_{34} & x_{35} & x_{36} & x_{37} \\ 0 & 0 & 0 & 1 & x_{45} & x_{46} & x_{47} \\ 0 & 0 & 0 & 0 & 1 & x_{56} & x_{57} \\ 0 & 0 & 0 & 0 & 0 & 1 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1_1 & 2_1 & 1_2 & 1_3 & 1_4 & 1_5 \\ 0 & 0 & 2_2 & 3_2 & 2_3 & 2_4 & 2_5 \\ 0 & 0 & 0 & 3_3 & 4_3 & 3_4 & 3_5 \\ 0 & 0 & 0 & 0 & 4_4 & 5_4 & 4_5 \\ 0 & 0 & 0 & 0 & 0 & 5_5 & \end{pmatrix}. \tag{33}$$

This **order is right in the general case** (without any projections on $X^{(k)}$). To obtain $\exp(isx_{12})$ and $\exp(isx_{13})$ on the **first step** we get by Lemma 19

$$\{T_{23}(t), \Delta^{is}(x)\} = T_{23}(t)\Delta^{is}(x)T_{23}(-t)\Delta^{-is}(x) = \Delta^{is}(xE_{23}(t))\Delta^{-is}(x) =$$

$$\exp\left\{-is\left(\sum_{n=3}^{\infty} b_{1n}[w_{1n}(xE_{23}(t)) - w_{1n}(x)] + \sum_{n=4}^{\infty} b_{2n}[w_{2n}(xE_{23}(t)) - w_{2n}(x)]\right)\right\}. \tag{34}$$

Now we shall calculate $w_{1n}(xE_{23}(t)) - w_{1n}(x)$ and $w_{2n}(xE_{23}(t)) - w_{2n}(x)$. We have by (16)

$$x_{1n} + x^{-1}_{1n} = -\sum_{r=2}^{n-1} x_{1r}x^{-1}_{rn}, \quad x_{2n} + x^{-1}_{2n} = -\sum_{r=3}^{n-1} x_{2r}x^{-1}_{rn}$$

so we conclude that for $n > 3$ holds

$$(x_{1n}+x^{-1}_{1n})^{E_{23}(t)} = -\left(\sum_{r=2}^{n-1} x_{1r}x^{-1}_{rn}\right)^{E_{23}(t)} = -\left(x_{12}x^{-1}_{2n} + x_{13}x^{-1}_{3n} + \sum_{r=4}^{n-1} x_{1r}x^{-1}_{rn}\right)^{E_{23}(t)} =$$

$$-\left(x_{12}(-x_{2n} - [x_{23}+t]x^{-1}_{3n} - \sum_{r=4}^{n-1} x_{2r}x^{-1}_{rn}) + [x_{13}+tx_{12}]x^{-1}_{3n} + \sum_{r=4}^{n-1} x_{1r}x^{-1}_{rn}\right) =$$

$$-\left(\sum_{r=2}^{n-1} x_{1r}x^{-1}_{rn} - tx_{12}x^{-1}_{3n} + tx_{12}x^{-1}_{3n}\right) = x_{1n} + x^{-1}_{1n}.$$

For $n = 3$ we get $x_{13} + x^{-1}_{13} = -x_{12}x^{-1}_{23} = x_{12}x_{23}$ hence

$$(x_{13} + x^{-1}_{13})^{E_{23}(t)} = (x_{12}x_{23})^{E_{23}(t)} =$$

$$x_{12}[x_{23} + t] = x_{12}x_{23} + tx_{12} = x_{13} + x^{-1}_{13} - tx^{-1}_{12}.$$

Finally we conclude that

$$(x_{1n} + x^{-1}_{1n})^{E_{23}(t)} = \begin{cases} x_{1n} + x^{-1}_{1n}, & \text{if } 3 < n, \\ x_{13} + x^{-1}_{13} + tx_{12}, & \text{if } n = 3 \end{cases} \tag{35}$$

and

$$(x_{1n} \pm x^{-1}_{1n})^{E_{23}(t)} = \begin{cases} x_{1n} \pm x^{-1}_{1n}, & \text{if } 3 < n, \\ x_{13} \pm x^{-1}_{13} + tx_{12}, & \text{if } n = 3 \end{cases} \tag{36}$$

since

$$(x_{13}-x_{13}^{-1})^{E_{23}(t)} = (2x_{13}-(x_{13}+x_{13}^{-1}))^{E_{23}(t)} = 2[x_{13}+tx_{12}]-(x_{13}+x_{13}^{-1}+tx_{12})$$

$$= x_{13}-x_{13}^{-1}+tx_{12}.$$

We have $w_{1n}(xE_{23}(t)) - w_{1n}(x) = 0$ for $n > 3$. For $n = 3$ holds

$$w_{13}(xE_{23}(t))-w_{13}(x) = (x_{13}+x_{13}^{-1}+tx_{12})(x_{13}-x_{13}^{-1}+tx_{12})-(x_{13}+x_{13}^{-1})(x_{13}-x_{13}^{-1})$$

$$= tx_{12}(x_{13}+x_{13}^{-1}+x_{13}-x_{13}^{-1})+t^2x_{12}^2 = 2tx_{12}x_{13}+t^2x_{12}^2.$$

Finally

$$w_{1n}(xE_{23}(t)) - w_{1n}(x) = \begin{cases} 0, & \text{if} \quad 3 < n \\ 2tx_{12}x_{13}+t^2x_{12}^2, & \text{if} \quad n = 3. \end{cases} \tag{37}$$

For $(x_{2n}+x_{2n}^{-1})^{E_{23}(t)}$ we have

$$(x_{2n}+x_{2n}^{-1})^{E_{23}(t)} = -\left(\sum_{r=3}^{n-1} x_{2r}x_{rn}^{-1}\right)^{E_{23}(t)} = -\left(x_{23}x_{3n}^{-1}+\sum_{r=4}^{n-1} x_{2r}x_{rn}^{-1}\right)^{E_{23}(t)} =$$

$$-\left([x_{23}+t]x_{3n}^{-1}+\sum_{r=4}^{n-1} x_{2r}x_{rn}^{-1}\right) = x_{2n}+x_{2n}^{-1}-tx_{3n}^{-1}.$$

Since

$$(x_{2n}-x_{2n}^{-1})^{E_{23}(t)} = [2x_{2n}-(x_{2n}+x_{2n}^{-1})]^{E_{23}(t)} = [2x_{2n}-(x_{2n}+x_{2n}^{-1}-tx_{3n}^{-1})]$$

$$= x_{2n}-x_{2n}^{-1}+tx_{3n}^{-1}$$

we conclude that

$$(x_{2n}\pm x_{2n}^{-1})^{E_{23}(t)} = x_{2n}\pm x_{2n}^{-1}\mp tx_{3n}^{-1}. \tag{38}$$

Finally we have

$$w_{2n}(xE_{23}(t))-w_{2n}(x) = (x_{2n}+x_{2n}^{-1}-tx_{3n}^{-1})(x_{2n}-x_{2n}^{-1}+tx_{3n}^{-1})-(x_{2n}+x_{2n}^{-1})(x_{2n}-x_{2n}^{-1}) =$$

$$tx_{3n}^{-1}(x_{2n}+x_{2n}^{-1}+x_{2n}-x_{2n}^{-1})-t^2(x_{3n}^{-1})^2 = 2tx_{2n}^{-1}x_{3n}^{-1}-t^2(x_{3n}^{-1})^2,$$

$$w_{2n}(xE_{23}(t))-w_{2n}(x) = 2tx_{2n}^{-1}x_{3n}^{-1}-t^2(x_{3n}^{-1})^2. \tag{39}$$

Using (37) and (39) we get

$$w_{kn}(xE_{23}(t))-w_{kn}(x) = \begin{cases} 2tx_{12}x_{13}+t^2x_{12}^2, & \text{if} \quad n=3,\ k=1 \\ 2tx_{2n}^{-1}x_{3n}^{-1}-t^2(x_{3n}^{-1})^2, & \text{if} \quad k=2,\ n\geq 4 \\ 0, & \text{otherwise}. \end{cases} \tag{40}$$

At last using (34) and (40) we have

$$\{T_{23}(t),\Delta^{is}(x)\}=\exp\left(-is\left[b_{13}(2tx_{12}x_{13}+t^2x_{12}^2)+\sum_{n=4}^{\infty}b_{2n}(2tx_{2n}^{-1}x_{3n}^{-1}-t^2(x_{3n}^{-1})^2)\right]\right).$$

Further we get

$$\{T_{13}(t_2)\{T_{23}(t_1),\Delta^{is}(x)\}\}=\exp\left(-isb_{13}2t_1t_2x_{12}\right). \tag{41}$$

Indeed

$$\{T_{13}(t_2)\{T_{23}(t_1),\Delta^{is}(x)\}\}=$$
$$\exp\left(-isb_{13}\left[(2t_1x_{12}[x_{13}+t_2]-t_1^2x_{12}^2)-(2t_1x_{12}x_{13}-t_1^2x_{12}^2)\right]\right)$$
$$=\exp\left(-isb_{13}2t_1t_2x_{12}\right),$$

compare with (10): $-[A^R_{13},[A^R_{23},\ln\Delta]]=2b_{13}x_{12}$! We have $\exp(itx_{12})\in M'_\phi$ and hence $\exp(itx_{12}^2)\in M'_\phi$. Using expression for $\{T_{23}(t_1),\Delta^{is}(x)\}$ we conclude that

$$M'_\phi\ni\{T_{23}(t_1),\Delta^{is}(x)\}\exp(isb_{13}t^2x_{12}^2)=$$
$$\exp\left(-is\left[b_{13}(2tx_{12}x_{13})+\sum_{n=4}^{\infty}b_{2n}(2tx_{2n}^{-1}x_{3n}^{-1}-t^2(x_{3n}^{-1})^2)\right]\right),$$

so

$$M'_\phi\ni\{T_{12}(t_2),\{T_{23}(t_1),\Delta^{is}(x)\}\exp(isb_{13}t^2x_{12}^2)\}=\exp\left(-isb_{13}2t_1t_2x_{13}\right).$$

Compare with the expression $-[A^R_{12},[A^R_{23},\ln\Delta]]=2b_{13}x_{13}$. Finally we conclude that

$$\exp(itx_{12}),\quad \exp(itx_{13})\in M'_\phi \tag{42}$$

In general (without any projections) the following lemma holds

Lemma 20 *We have*

$$w_{kn}(xE_{mm+1}(t))-w_{kn}(x)=\begin{cases}2tx_{rm}x_{rm+1}+t^2x_{rm+1}^2, & \text{if } n=m+1,\ 1\le k\le m-1\\ 2tx_{mn}^{-1}x_{m+1n}^{-1}-t^2(x_{m+1n}^{-1})^2, & \text{if } k=m,\ n\ge m+2\\ 0, & \text{otherwise},\end{cases} \tag{43}$$

hence

$$\{T_{mm+1}(t),\Delta^{is}(x)\}=$$
$$\exp\left(-is\left[\sum_{r=1}^{m-1}b_{rm+1}(2tx_{rm}x_{rm+1}+t^2x_{rm+1}^2)+\sum_{n=m+2}^{\infty}b_{mn}(2tx_{mn}^{-1}x_{m+1n}^{-1}-t^2(x_{m+1n}^{-1})^2)\right]\right). \tag{44}$$

PROOF. The proof is similar to the proof of the Lemma 17. □

To obtain another functions $\exp(itx_{kn})$ in the general case we should make all the steps as it was indicated before. For example to obtain $\exp(isx_{14})$, $\exp(isx_{23})$, $\exp(isx_{24})$ we should do **the second step** i.e. consider the operators

$$\{T_{34}(t), \Delta^{is}(x)\}$$

and all necessary combinations.

To obtain $\exp(isx_{15})$, $\exp(isx_{25})$, $\exp(isx_{34})$, $\exp(isx_{34})$ we should consider the following operators

$$\{T_{45}(t), \Delta^{is}(x)\},$$

and so on. Finally we shall obtain all functions $\exp(isx_{kn})$, $k < n$.

5 Example of the measure

We show that the set $b = (b_{kn})_{k<n}$ for which

$$S^L_{kn}(b) < \infty, \quad E(b) < \infty, \quad \text{and } S^{R,L}_{kn}(b) = \infty, \ \ 1 \le k < n,$$

where

$$S^L_{kn}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{b_{nm}}, \ E(b) = \sum_{k<n} \frac{S^L_{kn}(b)}{b_{kn}}, \ S^{R,L}_{kn}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{S^L_{nm}(b)}.$$

is not empty. Indeed let us take $b_{kn} = (a_k)^n$. We have

$$S^L_{kn}(b) = \sum_{m=n+1}^{\infty} \left(\frac{a_k}{a_n}\right)^m = \left(\frac{a_k}{a_n}\right)^{n+1} \sum_{m=0}^{\infty} \left(\frac{a_k}{a_n}\right)^m = \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1 - \frac{a_k}{a_n}} < \infty$$

iff $a_k < a_{k+1}, \ k \in \mathbb{N}$, for example $a_k = s^k$ with $s > 1$. Further we get

$$E(b) = \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \frac{S^L_{kn}(b)}{b_{kn}} = \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1 - \frac{a_k}{a_n}} \frac{1}{a_k^n} =$$

$$\sum_{k=1}^{\infty} a_k \sum_{n=k+1}^{\infty} \left(\frac{1}{a_n}\right)^{n+1} \frac{1}{1 - \frac{a_k}{a_n}} < \sum_{k=1}^{\infty} \frac{a_k}{1 - \frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left(\frac{1}{a_n}\right)^{n+1}$$

$$< \sum_{k=1}^{\infty} \frac{a_k}{1 - \frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left(\frac{1}{a_{k+1}}\right)^{n+1} = \sum_{k=1}^{\infty} \frac{a_k}{1 - \frac{a_k}{a_{k+1}}} \left(\frac{1}{a_{k+1}}\right)^{k+2} \frac{1}{1 - \frac{1}{a_{k+1}}} =$$

$$\sum_{k=1}^{\infty} \frac{\frac{a_k}{a_{k+1}}}{1 - \frac{a_k}{a_{k+1}}} \left(\frac{1}{a_{k+1}}\right)^k \frac{1}{a_{k+1} - 1} < \sum_{k=1}^{\infty} \frac{\frac{a_k}{a_{k+1}}}{1 - \frac{a_k}{a_{k+1}}} \left(\frac{1}{a_2}\right)^k \frac{1}{a_2 - 1}.$$

If for example $a_k = s^k$ with $s > 1$ we have

$$E(b) < \frac{\frac{1}{s}}{1 - \frac{1}{s}} \sum_{k=1}^{\infty} \frac{1}{s^{k(k+1)}} \frac{1}{s^{k+1} - 1} < \infty.$$

At last

$$S_{kn}^{R,L}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{S_{nm}^{L}(b)} = \sum_{m=n+1}^{\infty} \frac{a_k^m \left(1 - \frac{a_n}{a_m}\right)}{\left(\frac{a_n}{a_m}\right)^{m+1}}$$

$$= \sum_{m=n+1}^{\infty} \left(\frac{a_k a_m}{a_n}\right)^m \left(\frac{a_m}{a_n}\right)\left(1 - \frac{a_n}{a_m}\right) = \sum_{m=n+1}^{\infty} \left(\frac{a_k a_m}{a_n}\right)^m \left(\frac{a_m}{a_n} - 1\right) = \infty,$$

if $\lim_m a_m = \infty$. For $a_k = s^k$ with $s > 1$ we have

$$S_{kn}^{R,L}(b) = \sum_{m=n+1}^{\infty} s^{(m+k-n)m}(s^{m-n} - 1) = \infty.$$

6 Modular operator

We recall how to find the modular operator and the operator of canonical conjugation for the von Neumann algebra $\mathfrak{A}_G^{\rho}$, generated by the right regular representation ρ of a locally compact Lie group G. Let h be a right invariant Haar measure on G and

$$\rho, \lambda : G \mapsto U(L^2(G, h))$$

be the right and the left regular representations of the group G defined by

$$(\rho_t f)(x) = f(xt), \ \ (\lambda_t f)(x) = (dh(t^{-1}x)/dh(x))^{-1/2} f(t^{-1}x).$$

To define the right Hilbert algebra on G we can proceed as follows. Let $M(G)$ be algebra of all probability measures on G with convolution

$$(\mu * \nu)(s) =$$

We define the homomorphism

$$M(G) \ni \mu \mapsto \rho^{\mu} = \int_G \rho_t \, \mathrm{d}\mu(t) \in B(L^2(G, h)).$$

We have $\rho^{\mu}\rho^{\nu} = \rho^{\mu*\nu}$, indeed

$$\rho^{\mu}\rho^{\nu} = \int_G \rho_t \, \mathrm{d}\mu(t) \int_G \rho_s \, \mathrm{d}\nu(s) = \int_G \int_G \rho_{ts} \, \mathrm{d}\mu(t)\nu(s) = \int_G \rho_t \, \mathrm{d}(\mu * \nu)(t) = \rho^{\mu*\nu}.$$

Let us consider a subalgebra $M_h(G) := (\nu \in M(G) \mid \nu \sim h)$ of the algebra $M_h(G)$ In the case when $\mu \in M_h(G)$ we can associate with the measure μ its Rodon-Nikodim derivative $d\nu(t)/dh(t) = f(t)$. When $f \in C_0^{\infty}(G)$ or $f \in L^1(G)$ we can write

$$\rho^f = \int_G f(t)\rho_t dh(t),$$

hence we can replace the algebra $M_h(G)$ by its subalgebra identified with algebra of functions $C_0^\infty(G)$ or $L^1(G,h)$ with convolutions. If we replace the Haar measure h with some measure $\mu\in M_h(G)$ we obtain the isomorphic image $T^{R,\mu}$ of the right regular representation ρ in the space $L^2(G,\mu)$: $T_t^{R,\mu}=U\rho_tU^{-1}$ where $U: L^2(G,h)\mapsto L^2(G,\mu)$ defined by $(Uf)(x)=\left(\frac{dh(x)}{d\mu(x)}\right)^{1/2}f(x)$. we have

$$(T_t^{R,\mu}f)(x)=\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2}f(xt),$$

and

$$T^f=\int_G f(t)T_t^{R,\mu}d\mu(t).$$

We have (see [4], p.462) (we shall write T_t instead of $T_t^{R,\mu}$)

$$S(T^f):=(T^f)^*=\int_G\overline{f(t)}T_{t^{-1}}d\mu(t)=\int_G\overline{f(t)}T_{t^{-1}}\frac{d\mu(t)}{d\mu(t^{-1})}d\mu(t^{-1})$$

$$\int_G\frac{d\mu(t^{-1})}{d\mu(t)}\overline{f(t^{-1})}T_td\mu(t).$$

Hence

$$(Sf)(t)=\frac{d\mu(t^{-1})}{d\mu(t)}\overline{f(t^{-1})}.$$

To calculate S^* we use the fact that S is antilinear so $(Sf,g)=(S^*g,f)$. We have

$$(Sf,g)=\int_G\frac{d\mu(t^{-1})}{d\mu(t)}\overline{f(t^{-1})g(t)}d\mu(t)=\int_G\overline{f(t^{-1})g(t)}d\mu(t^{-1})=$$

$$\int_G\overline{g(t^{-1})f(t)}d\mu(t)=(S^*g,f),$$

hence $(S^*g)(t)=\overline{g(t^{-1})}$. Finally the modular operator Δ defined by $\Delta=S^*S$ has the following form $(\Delta f)(t)=\frac{d\mu(t)}{d\mu(t^{-1})}f(t)$. Indeed we have

$$f(t)\overset{S}{\mapsto}\frac{d\mu(t^{-1})}{d\mu(t)}\overline{f(t^{-1})}\overset{S^*}{\mapsto}\frac{d\mu(t)}{d\mu(t^{-1})}f(t).$$

Finally , since $J=S\Delta^{-1/2}$ (see [4] p.462) we get

$$f(t)\overset{\Delta^{-1/2}}{\mapsto}\left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2}f(t)\overset{J}{\mapsto}\frac{d\mu(t^{-1})}{d\mu(t)}\left(\frac{d\mu(t)}{d\mu(t^{-1})}\right)^{1/2}\overline{f(t^{-1})}$$

$$=\left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2}\overline{f(t^{-1})}.$$

Hence

$$(Jf)(t)=\left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2}\overline{f(t^{-1})},\ \text{ and }\ (\Delta f)(t)=\frac{d\mu(t)}{d\mu(t^{-1})}f(t).$$

To prove that $JT_t^{R,\mu}J = T_t^{L,\mu}$ we get

$$f(t) \overset{J}{\mapsto} \left(\frac{d\mu(x^{-1})}{d\mu(x)}\right)^{1/2} \overline{f(x^{-1})} \overset{T_t^{R,\mu}}{\mapsto} \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} \left(\frac{d\mu((xt)^{-1})}{d\mu(xt)}\right)^{1/2} \overline{f((xt)^{-1})} =$$

$$\left(\frac{d\mu(t^{-1}x^{-1})}{d\mu(x)}\right)^{1/2} \overline{f(t^{-1}x^{-1})} \overset{J}{\mapsto} \left(\frac{d\mu(x^{-1})}{d\mu(x)}\right)^{1/2} \left(\frac{d\mu(t^{-1}x)}{d\mu(x^{-1})}\right)^{1/2} f(t^{-1}x) =$$

$$\left(\frac{d\mu(t^{-1}x)}{d\mu(x)}\right)^{1/2} f(t^{-1}x) = (T_t^{L,\mu}f)(x).$$

Remark 21 *The representation T^{R,μ_b} is the inductive limit of the representations T^{R,μ_b^m} of the group $B(m,\mathbb{R})$ where the measure μ_b^m is the projection of the measure μ_b onto subgroup $B(m,\mathbb{R})$. Obviously μ_b^m is equivalent with the Haar measure h_m on $B(m,\mathbb{R})$.*

7 The uniqueness of the constructed factor

Let G be a solvable separable locally compact group or a connected locally compact group. Then any representation π of G in a Hilbert space generates an approximately finite-dimensional von Neumann algebra (see [3]).

Theorem 15 from V.9 p. 504 [4] (Haagerup) There exists up to isomorphism only one amenable factor of type III_1, the factor R_∞ of Araki and Woods (see [7]).

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