EFFECTIVE HIRONAKA RESOLUTION AND ITS COMPLEXITY

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Abstract. Building upon works of Hironaka, Bierstone-Milman, Villamayor and Włodarczyk we give apriori estimate for the complexity of the simplified Hironaka algorithm.

Dedicated to Professor Heisuke Hironaka on the occasion of his 80th birthday

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0. Introduction

In the present paper we discuss the complexity of the Hironaka theorem on resolution of singularities of a marked ideal. Recall that approach to the problem of embedded resolution was originated by Hironaka (see [31]) and later developed and simplified by Bierstone-Milman (see [7], [8], [9]) and Villamayor (see [46], [47]), and others. In particular, we also use some elements from the recent development by Kollár ([37]).

It seems easier to estimate the complexity of the resolution algorithm from the recursive descriptions in Włodarczyk [51] or Bierstone-Milman [12] than from the earlier iterative versions. The algorithms in [51] and [12] (or [10]) lead to identical blowing-up sequences; whether one proof is preferable to the other is partly a matter of taste. In this article, we estimate the complexity of the "weak-strong desingularization" algorithm (see Section 1) using the construction of [51], though [12] could also be used (see Remark in this section below). In a subsequent paper, we plan to use [12] to give a comparable complexity estimate for the algorithm of "strong desingularization" (where the centres of blowing up are smooth subvarieties of the successive strict transforms).

The basic question which arises is in what terms to estimate apriori the complexity? We recall (see e. g. [50], [26]) that the complexity is usually measured as a function on the bit-size of the input. In particular, in the paper we study varieties and ideals which are represented by families of polynomials with integer coefficients, and the vector of all these coefficients (for an initial variety and an ideal) is treated as an input. Hironaka's algorithm consists of many steps of elementary calculations, but they are organized in several (nested) recursions when the resolution of an object (a variety or a marked ideal, see below) is reduced to resolutions of suitable objects with less values of appropriate parameters (like dimension or multiplicity). It is instructive to represent Hironaka algorithm as a tree to each of its nodes a corresponds a marked ideal. The marked ideals which correspond to child nodes of a have either less multiplicity of an ideal or less dimension of a variety. An initial marked ideal corresponds to the root of the tree. The depth of the tree is bounded

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by $2 \cdot m$ where m denotes the dimension of the initial variety, while the number of the nested recursions does not exceed $m+3$. It appears that just the number of nested recursions brings the overwhelming contribution into the complexity of the Hironaka's algorithm.

That is why as a relevant language for expressing a complexity bound we have chosen the Grzegorczyk's classes $\mathcal{E}^l$, $l \geq 0$ [27], [50] which consists of (integer) functions whose construction requires l nested primitive recursions. Classes $\mathcal{E}^l$, $l \geq 0$ provide a hierarchy of the set of all primitive-recursive functions $\cup_{l<\infty}\mathcal{E}^l$. In particular, $\mathcal{E}^2$ contains all the (integer) polynomials and $\mathcal{E}^3$ contains all finite compositions of the exponential function. Thus, the principal complexity result of the paper (Theorem 6.4.2) states that the complexity of resolution of an ideal on m-dimensional variety is bounded by a function from class $\mathcal{E}^{m+3}$. We mention also that the complexity of (much simpler from the pure mathematical point of view) the Hilbert's Idealbasissatz for polynomial ideals in n variables belongs to class $\mathcal{E}^{n+1}$ (cf. [42], [44], where the latter was formulated in different languages), and moreover number $n+1$ is sharp. This shows that these two quite different algorithmic problems have a common feature in the recursion on the dimension which mainly determines their complexities.

Remark. The main differences between the proofs in [12] and [51] come from the notions of derivative ideal that are used ([12] uses only derivatives that preserve the ideal of the exceptional divisor) and from passage to a "homogenized ideal" in [51] (see §2.8). The latter has the advantage that any two maximal contact hypersurfaces for the homogenized ideal are related by an automorphism, while [12] provides a stronger version of functoriality that is needed for strong desingularization. Since [12] does not involve homogenization, certain complexity estimates can be improved (see Remark after Corollary 5.0.10), although the overall Grzegorczyk complexity class $\mathcal{E}^{m+3}$ is unchanged.

We mention that in [45] a polynomial complexity algorithm for resolution of a curve is exhibited.

Below in Section 1 we formulate the results on the canonical principalization of a sheaf of ideals and on the embedded desingularization. In Section 2 we give basic definitions like marked ideals, hypersurfaces of maximal contact, coefficient ideals and formulate their properties (the omitted proofs one can find in [51]). In Section 3 we describe the resolution algorithm. In Section 4 we provide bounds on the degrees and on the number of polynomials which describe a single blow-up. In Section 5 we give some auxiliary bounds: on the multiplicity of an ideal in terms of degrees of describing polynomials, on the degree of a hypersurface of the maximal contact, and on the number of generators and on their degrees of the coefficient ideal. Finally in Section 6 we estimate the complexity of the resolution algorithm in terms of the Grzegorczyk's classes (their definition is also provided in Section 6).

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1. Formulation of the Hironaka resolution theorems

All algebraic varieties in this paper are defined over a ground field of characteristic zero. The assumption of characteristic zero is only needed for the local existence of a hypersurface of maximal contact (Lemma 2.6.4).

We give a proof of the following Hironaka Theorems (see [31]):

(1) **Canonical Principalization**

Theorem 1.0.1. *Let $\mathcal{I}$ be a sheaf of ideals on a smooth algebraic variety X. There exists a principalization of $\mathcal{I}$, that is, a sequence*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \longleftarrow \ldots \longleftarrow X_i \longleftarrow \ldots \longleftarrow X_r = \widetilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ of smooth centers $C_{i-1} \subset X_{i-1}$ such that

(a) *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X$ has only simple normal crossings and C_i has simple normal crossings with E_i.*

(b) *The total transform $\sigma^{r*}(\mathcal{I})$ is the ideal of a simple normal crossing divisor $\widetilde{E}$ which is a natural combination of the irreducible components of the divisor E_r.*

The morphism $(\widetilde{X}, \widetilde{\mathcal{I}}) \to (X, \mathcal{I})$ defined by the above principalization commutes with smooth morphisms and embeddings of ambient varieties. It is equivariant with respect to any group action not necessarily preserving the ground field K.

(2) **Weak-Strong Hironaka Embedded Desingularization**

Theorem 1.0.2. *Let Y be a subvariety of a smooth variety X over a field of characteristic zero. There exists a sequence*

$$X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \longleftarrow \ldots \longleftarrow X_i \longleftarrow \ldots \longleftarrow X_r = \widetilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \longleftarrow X_i$ of smooth centers $C_{i-1} \subset X_{i-1}$ such that

(a) *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X$ has only simple normal crossings and C_i has simple normal crossings with E_i.*
(b) *Let $Y_i \subset X_i$ be the strict transform of Y. All centers C_i are disjoint from the set* $\mathrm{Reg}(Y) \subset Y_i$ *of points where Y (not Y_i) is smooth (and are not necessarily contained in Y_i).*
(c) *The strict transform $\widetilde{Y} := Y_r$ of Y is smooth and has only simple normal crossings with the exceptional divisor E_r.*
(d) *The morphism $(X, Y) \leftarrow (\widetilde{X}, \widetilde{Y})$ defined by the embedded desingularization commutes with smooth morphisms and embeddings of ambient varieties. It is equivariant with respect to any group action not necessarily preserving the ground K.*

(3) **Canonical Resolution of Singularities**

Theorem 1.0.3. *Let Y be an algebraic variety over a field of characteristic zero.*

There exists a canonical desingularization of Y that is a smooth variety $\widetilde{Y}$ together with a proper birational morphism $\mathrm{res}_Y : \widetilde{Y} \to Y$ *which is functorial with respect to smooth morphisms. For any smooth morphism $\phi : Y' \to Y$ there is a natural lifting $\widetilde{\phi} : \widetilde{Y'} \to \widetilde{Y}$ which is a smooth morphism.*

In particular $\mathrm{res}_Y : \widetilde{Y} \to Y$ *is an isomorphism over the nonsingular part of Y. Moreover* res_Y *is equivariant with respect to any group action not necessarily preserving the ground field.*

Remark. Note that the blow-up of codimension one components is an isomorphism. However it defines a nontrivial transformation of marked ideals. In the actual desingularization process this kind of blow-up may occur for some marked ideals induced on subvarieties of ambient varieties. Though they define isomorphisms of those subvarieties they determine blow-ups of ambient varieties which are not isomorphisms.

Remarks. (1) By the exceptional divisor of the blow-up $\sigma : X' \to X$ with a smooth center C we mean the inverse image $E := \sigma^{-1}(C)$ of the center C. By the exceptional divisor of the composite of blow-ups σ_i with smooth centers C_{i-1} we mean the union of the strict transforms of the exceptional divisors of σ_i. This definition coincides with the standard definition of the exceptional set of points of the birational morphism in the case when $\mathrm{codim}(C_i) \geq 2$ (as in Theorem 1.0.2). If $\mathrm{codim}(C_{i-1}) = 1$ the blow-up of C_{i-1} is an identical isomorphism and defines a formal operation of converting a subvariety $C_{i-1} \subset X_{i-1}$ into a component of the exceptional divisor E_i on X_i. This formalism is convenient for the proofs. In particular it indicates that C_{i-1} identified via σ_i with a component of E_i has simple normal crossings with other components of E_i.

(2) In the Theorem 1.0.2 we blow up centers of codimension ≥ 2 and both definitions coincide.

2. Marked ideals, coefficient ideals and hypersurfaces of maximal contact

We shall assume that the ground field is algebraically closed.

2.1. **Resolution of marked ideals.** For any sheaf of ideals $\mathcal{I}$ on a smooth variety X and any point $x \in X$ we denote by

$$\mathrm{ord}_x(\mathcal{I}) := \max\{i \mid \mathcal{I} \subset m_x^i\}$$

the *order* of $\mathcal{I}$ at x. (Here m_x denotes the maximal ideal of x.)

Definition 2.1.1. (Hironaka (see [31], [33]), Bierstone-Milman (see [8]),Villamayor (see [46])) A *marked ideal* (originally a *basic object* of Villamayor) is a collection $(X, \mathcal{I}, E, \mu)$, where X is a smooth variety, $\mathcal{I}$ is a sheaf of ideals on X, μ is a nonnegative integer and E is a totally ordered collection of divisors whose irreducible components are pairwise disjoint and all have multiplicity one. Moreover the irreducible components of divisors in E have simultaneously simple normal crossings.

Definition 2.1.2. (Hironaka ([31], [33]), Bierstone-Milman (see [8]),Villamayor (see [46])) By the *support* (originally *singular locus*) of $(X, \mathcal{I}, E, \mu)$ we mean

$$\mathrm{supp}(X, \mathcal{I}, E, \mu) := \{x \in X \mid \mathrm{ord}_x(\mathcal{I}) \geq \mu\}.$$

Remarks. (1) Sometimes for simplicity we shall represent marked ideals $(X, \mathcal{I}, E, \mu)$ as couples $(\mathcal{I}, \mu)$ or even ideals $\mathcal{I}$.
(2) For any sheaf of ideals $\mathcal{I}$ on X we have $\operatorname{supp}(\mathcal{I}, 1) = \operatorname{supp}(\mathcal{I})$.
(3) For any marked ideals $(\mathcal{I}, \mu)$ on X, $\operatorname{supp}(\mathcal{I}, \mu)$ is a closed subset of X (Lemma 2.5.2).

Definition 2.1.3. (Hironaka (see [31], [33]), Bierstone-Milman (see [8]),Villamayor (see [46])) By a *resolution* of $(X, \mathcal{I}, E, \mu)$ we mean a sequence of blow-ups $\sigma_i : X_i \to X_{i-1}$ of disjoint unions of smooth centers $C_{i-1} \subset X_{i-1}$,

$$X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \xleftarrow{\sigma_3} \ldots X_i \longleftarrow \ldots \xleftarrow{\sigma_r} X_r,$$

which defines a sequence of marked ideals $(X_i, \mathcal{I}_i, E_i, \mu)$ where

(1) $C_i \subset \operatorname{supp}(X_i, \mathcal{I}_i, E_i, \mu)$.
(2) C_i has simple normal crossings with E_i.
(3) $\mathcal{I}_i = \mathcal{I}(D_i)^{-\mu}\sigma_i^*(\mathcal{I}_{i-1})$, where $\mathcal{I}(D_i)$ is the ideal of the exceptional divisor D_i of σ_i.
(4) $E_i = \sigma_i^{\mathrm{c}}(E_{i-1}) \cup \{D_i\}$, where $\sigma_i^{\mathrm{c}}(E_{i-1})$ is the set of strict transforms of divisors in E_{i-1}.
(5) The order on $\sigma_i^{\mathrm{c}}(E_{i-1})$ is defined by the order on E_{i-1} while D_i is the maximal element of E_i.
(6) $\operatorname{supp}(X_r, \mathcal{I}_r, E_r, \mu) = \emptyset$.

Definition 2.1.4. The sequence of morphisms which are either isomorphisms or blow-ups satisfying conditions (1)-(5) is called a *multiple blow-up*. The number of morphisms in a multiple blow-up will be called its *length.*

Definition 2.1.5. An *extension* of a multiple blow-up (or a resolution) $(X_i)_{0\leq i\leq m}$ is a sequence $(X'_j)_{0\leq j\leq m'}$ of blow-ups and isomorphisms $X'_0 = X'_{j_0} = \ldots = X'_{j_1-1} \leftarrow X'_{j_1} = \ldots = X'_{j_2-1} \leftarrow \ldots X'_{j_m} = \ldots = X'_{m'}$, where $X'_{j_i} = X_i$.

Remarks. (1) The definition of extension arises naturally when we pass to open subsets of the considered ambient variety X.
(2) The notion of a *multiple blow-up* is analogous to the notions of or *admissible* blow-ups considered by Hironaka, Bierstone-Milman and Villamayor.

2.2. **Transforms of marked ideal and controlled transforms of functions.** In the setting of the above definition we shall call

$$(\mathcal{I}_i, \mu) := \sigma_i^{\mathrm{c}}(\mathcal{I}_{i-1}, \mu)$$

a *transform of the marked ideal* or *controlled transform* of $(\mathcal{I}, \mu)$. It makes sense for a single blow-up in a multiple blow-up as well as for a multiple blow-up. Let $\sigma^i := \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X$ be a composition of consecutive morphisms of a multiple blow-up. Then in the above setting

$$(\mathcal{I}_i, \mu) = \sigma^{i\mathrm{c}}(\mathcal{I}, \mu).$$

We shall also denote the controlled transform $\sigma^{i\mathrm{c}}(\mathcal{I}, \mu)$ by $(\mathcal{I}, \mu)_i$ or $[\mathcal{I}, \mu]_i$.

The controlled transform can also be defined for local sections $f \in \mathcal{I}(U)$. Let $\sigma : X \leftarrow X'$ be a blow-up of a smooth center $C \subset \operatorname{supp}(\mathcal{I}, \mu)$ defining transformation of marked ideals $\sigma^{\mathrm{c}}(\mathcal{I}, \mu) = (\mathcal{I}', \mu)$. Let $f \in \mathcal{I}(U)$ be a section of a sheaf of ideals. Let $U' \subseteq \sigma^{-1}(U)$ be an open subset for which the sheaf of ideals of the exceptional divisor is generated by a function y. The function

$$g = y^{-\mu}(f \circ \sigma) \in \mathcal{I}(U')$$

is a *controlled transform* of f on U' (defined up to an invertible function). As before we extend it to any multiple blow-up.

The following lemma shows that the notion of controlled transform is well defined.

Lemma 2.2.1. *Let $C \subset \operatorname{supp}(\mathcal{I}, \mu)$ be a smooth center of the blow-up $\sigma : X \leftarrow X'$ and let D denote the exceptional divisor. Let $\mathcal{I}_C$ denote the sheaf of ideals defined by C. Then*

(1) $\mathcal{I} \subset \mathcal{I}_C^{\mu}$.
(2) $\sigma^*(\mathcal{I}) \subset (\mathcal{I}_D)^{\mu}$.

Proof. (1) We can assume that the ambient variety X is affine. Let $u_1, \ldots, u_k$ be parameters generating $\mathcal{I}_C$ Suppose $f \in \mathcal{I} \setminus \mathcal{I}_C^{\mu}$. Then we can write $f = \sum_\alpha c_\alpha u^\alpha$, where either $|\alpha| \geq \mu$ or $|\alpha| < \mu$ and $c_\alpha \notin \mathcal{I}_C$. By the assumption there is α with $|\alpha| < \mu$ such that $c_\alpha \notin \mathcal{I}_C$. Take α with the smallest $|\alpha|$. There is a point $x \in C$

for which $c_\alpha(x) \neq 0$ and in the Taylor expansion of f at x there is a term $c_\alpha(x)u^\alpha$. Thus $\mathrm{ord}_x(\mathcal{I}) < \mu$. This contradicts to the assumption $C \subset \mathrm{supp}(\mathcal{I}, \mu)$.

(2) $\sigma^*(\mathcal{I}) \subset \sigma^*(\mathcal{I}_C)^\mu = (\mathcal{I}_D)^\mu$. □

2.3. **Hironaka resolution principle.** Our proof is based upon the following principle which can be traced back to Hironaka and was used by Villamayor in his simplification of Hironaka's algorithm:

(1) **(Canonical) Resolution of marked ideals** $(X, \mathcal{I}, E, \mu)$

$\Downarrow$

(2) **(Canonical) Principalization of the sheaves** $\mathcal{I}$ **on** X

$\Downarrow$

(3) **(Canonical) Weak Embedded Desingularization of subvarieties** $Y \subset X$

$\Downarrow$

(4) **(Canonical) Desingularization**

(1)⇒(2) It follows immediately from the definition that a resolution of $(X, \mathcal{I}, \emptyset, 1)$ determines a principalization of $\mathcal{I}$. Denote by $\sigma : X \leftarrow \widetilde{X}$ the morphism defined by a resolution of $(X, \mathcal{I}, \emptyset, 1)$. The controlled transform $(\widetilde{\mathcal{I}}, 1) := \sigma^{\mathrm{c}}(\mathcal{I}, 1)$ has the empty support. Consequently, $V(\widetilde{\mathcal{I}}) = \emptyset$, and thus $\widetilde{\mathcal{I}}$ is equal to the structural sheaf $\mathcal{O}_{\widetilde{X}}$. This implies that the full transform $\sigma^*(\mathcal{I})$ is principal and generated by the sheaf of ideal of a divisor whose components are the exceptional divisors. The actual process of desingularization is often achieved before $(X, \mathcal{I}, E, 1)$ has been resolved (see [51]) .

(2)⇒(3) Let $Y \subset X$ be an irreducible subvariety. Assume there is a principalization of sheaves of ideals $\mathcal{I}_Y$ subject to conditions (a) and (b) from Theorem 1.0.1. Then in the course of the principalization of $\mathcal{I}_Y$ the strict transform Y_i of Y on some X_i is the center of a blow-up. At this stage Y_i is nonsingular and has simple normal crossing with the exceptional divisors.

(3)⇒(4) Every algebraic variety admits locally an embedding into an affine space. Thus we can show that the existence of canonical embedded desingularization independent of the embedding defines a canonical desingularization.

For more details see [51].

2.4. **Equivalence relation for marked ideals.** Let us introduce the following equivalence relation for marked ideals:

Definition 2.4.1. Let $(X, \mathcal{I}, E_\mathcal{I}, \mu_\mathcal{I})$ and $(X, \mathcal{J}, E_\mathcal{J}, \mu_\mathcal{J})$ be two marked ideals on the smooth variety X. Then $(X, \mathcal{I}, E_\mathcal{I}, \mu_\mathcal{I}) \simeq (X, \mathcal{J}, E_\mathcal{J}, \mu_\mathcal{J})$ if

1. $E_\mathcal{I} = E_\mathcal{J}$ and the orders on $E_\mathcal{I}$ and on $E_\mathcal{J}$ coincide.
2. $\mathrm{supp}(\mathcal{I}, \mu_\mathcal{I}) = \mathrm{supp}(\mathcal{J}, \mu_\mathcal{J})$.
3. The multiple blow-ups $(X_i)_{i=0,\dots,k}$ are the same for both marked ideals and $\mathrm{supp}(\mathcal{I}_i, \mu_\mathcal{I}) = \mathrm{supp}(\mathcal{J}_i, \mu_\mathcal{J})$.

Example 2.4.2. For any $k \in \mathbf{N}$, $(\mathcal{I}, \mu) \simeq (\mathcal{I}^k, k\mu)$.

Remark. The marked ideals considered in this paper satisfy a stronger equivalence condition: For any smooth morphisms $\phi : X' \to X$, $\phi^*(\mathcal{I}, \mu) \simeq \phi^*(\mathcal{J}, \mu)$. This condition will follow and is not added in the definition.

2.5. **Ideals of derivatives.** Ideals of derivatives were first introduced and studied in the resolution context by Giraud. Villamayor developed and applied this language to his *basic objects.*

Definition 2.5.1. (Giraud, Villamayor) Let $\mathcal{I}$ be a coherent sheaf of ideals on a smooth variety X. By the *first derivative* (originally *extension*) $\mathcal{D}(\mathcal{I})$ of $\mathcal{I}$ we mean the coherent sheaf of ideals generated by all functions $f \in \mathcal{I}$ with their first derivatives. Then the *i-th derivative* $\mathcal{D}^i(\mathcal{I})$ is defined to be $\mathcal{D}(\mathcal{D}^{i-1}(\mathcal{I}))$. If $(\mathcal{I}, \mu)$ is a marked ideal and $i \leq \mu$ then we define

$$\mathcal{D}^i(\mathcal{I}, \mu) := (\mathcal{D}^i(\mathcal{I}), \mu - i).$$

Recall that on a smooth variety X there is a locally free sheaf of differentials $\Omega_{X/K}$ over K generated locally by $du_1, \ldots, du_n$ for a set of local parameters $u_1, \ldots, u_n$. The dual sheaf of derivations $\mathrm{Der}_K(\mathcal{O}_X)$ is locally generated by the derivations $\frac{\partial}{\partial u_i}$. Immediately from the definition we observe that $\mathcal{D}(\mathcal{I})$ is a coherent sheaf defined locally by generators f_j of $\mathcal{I}$ and all their partial derivatives $\frac{\partial f_j}{\partial u_i}$. We see by induction that $\mathcal{D}^i(\mathcal{I})$ is a coherent sheaf defined locally by the generators f_j of $\mathcal{I}$ and their derivatives $\frac{\partial^{|\alpha|} f_j}{\partial u^\alpha}$ for all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $|\alpha| := \alpha_1 + \ldots + \alpha_n \leq i$.

Lemma 2.5.2. *(Giraud, Villamayor) For any $i \leq \mu - 1$,*

$$\mathrm{supp}(\mathcal{I}, \mu) = \mathrm{supp}(\mathcal{D}^i(\mathcal{I}), \mu - i).$$

In particular $\mathrm{supp}(\mathcal{I}, \mu) = \mathrm{supp}(\mathcal{D}^{\mu-1}(\mathcal{I}), 1) = V(\mathcal{D}^{\mu-1}(\mathcal{I}))$ *is a closed set.* □

We write $(\mathcal{I}, \mu) \subset (\mathcal{J}, \mu)$ if $\mathcal{I} \subset \mathcal{J}$.

Lemma 2.5.3. *(Giraud, Villamayor) Let $(\mathcal{I}, \mu)$ be a marked ideal and $C \subset \mathrm{supp}(\mathcal{I}, \mu)$ be a smooth center and $r \leq \mu$. Let $\sigma : X \leftarrow X'$ be a blow-up at C. Then*

$$\sigma^{\mathrm{c}}(\mathcal{D}^r(\mathcal{I}, \mu)) \subseteq \mathcal{D}^r(\sigma^{\mathrm{c}}(\mathcal{I}, \mu)).$$

Proof. See simple computations in [49], [51].

2.6. **Hypersurfaces of maximal contact.** The concept of the *hypersurfaces of maximal contact* is one of the key points of this proof. It was originated by Hironaka, Abhyankhar and Giraud and developed in the papers of Bierstone-Milman and Villamayor.

In our terminology we are looking for a smooth hypersurface containing the supports of marked ideals and whose strict transforms under multiple blow-ups contain the supports of the induced marked ideals. Existence of such hypersurfaces allows a reduction of the resolution problem to codimension 1.

First we introduce marked ideals which locally admit hypersurfaces of maximal contact.

Definition 2.6.1. (Villamayor (see [46])) We say that a marked ideal $(\mathcal{I}, \mu)$ is of *maximal order* (originally *simple basic object*) if $\max\{\mathrm{ord}_x(\mathcal{I}) \mid x \in X\} \leq \mu$ or equivalently $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_X$.

Lemma 2.6.2. *(Villamayor (see [46])) Let $(\mathcal{I}, \mu)$ be a marked ideal of maximal order and $C \subset \mathrm{supp}(\mathcal{I}, \mu)$ be a smooth center. Let $\sigma : X \leftarrow X'$ be a blow-up at $C \subset \mathrm{supp}(\mathcal{I}, \mu)$. Then $\sigma^{\mathrm{c}}(\mathcal{I}, \mu)$ is of maximal order.*

Proof. If $(\mathcal{I}, \mu)$ is a marked ideal of maximal order then $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_X$. Then by Lemma 2.5.3, $\mathcal{D}^\mu(\sigma^{\mathrm{c}}(\mathcal{I}, \mu)) \supset \sigma^{\mathrm{c}}(\mathcal{D}^\mu(\mathcal{I}), 0) = \mathcal{O}_X$. □

Lemma 2.6.3. *(Villamayor (see [46])) If $(\mathcal{I}, \mu)$ is a marked ideal of maximal order and $0 \leq i \leq \mu$ then $\mathcal{D}^i(\mathcal{I}, \mu)$ is of maximal order.*

Proof. $\mathcal{D}^{\mu-i}(\mathcal{D}^i(\mathcal{I}, \mu)) = \mathcal{D}^\mu(\mathcal{I}, \mu) = \mathcal{O}_X$. □

Lemma 2.6.4. *(Giraud (see [22])) Let $(\mathcal{I}, \mu)$ be the marked ideal of maximal order. Let $\sigma : X \leftarrow X'$ be a blow-up at a smooth center $C \subset \mathrm{supp}(\mathcal{I}, \mu)$. Let $u \in \mathcal{D}^{\mu-1}(\mathcal{I}, \mu)(U)$ be a function such that, for any $x \in V(u)$, $\mathrm{ord}_x(u) = 1$. Then*

1. *$V(u)$ is smooth.*
2. *$\mathrm{supp}(\mathcal{I}, \mu) \cap U \subset V(u)$*

Let $U' \subset \sigma^{-1}(U) \subset X'$ be an open set where the exceptional divisor is described by y. Let $u' := \sigma^{\mathrm{c}}(u) = y^{-1}\sigma^*(u)$ be the controlled transform of u. Then

1. *$u' \in \mathcal{D}^{\mu-1}(\sigma^{\mathrm{c}}(\mathcal{I}_{|U'}, \mu))$.*
2. *$V(u')$ is smooth.*
3. *$\mathrm{supp}(\mathcal{I}', \mu) \cap U' \subset V(u')$*
4. *$V(u')$ is the restriction of the strict transform of $V(u)$ to U'.*

Proof. (1) $u' = \sigma^{\mathrm{c}}(u) = u/y \in \sigma^{\mathrm{c}}(\mathcal{D}^{\mu-1}(\mathcal{I})) \subset \mathcal{D}^{\mu-1}(\sigma^{\mathrm{c}}(\mathcal{I}))$.

(2) Since u was one of the local parameters describing the center of blow-ups, $u' = u/y$ is a parameter, that is, a function of order one.

(3) follows from (2). □

Definition 2.6.5. We shall call a function

$$u \in T(\mathcal{I})(U) := \mathcal{D}^{\mu-1}(\mathcal{I}(U))$$

of multiplicity one a *tangent direction* of $(\mathcal{I}, \mu)$ on U.

As a corollary from the above we obtain the following lemma:

Lemma 2.6.6. *(Giraud) Let $u \in T(\mathcal{I})(U)$ be a tangent direction of $(\mathcal{I}, \mu)$ on U. Then for any multiple blow-up (U_i) of $(\mathcal{I}_{|U}, \mu)$ all the supports of the induced marked ideals* $\mathrm{supp}(\mathcal{I}_i, \mu)$ *are contained in the strict transforms $V(u)_i$ of $V(u)$.* □

Remarks. (1) Tangent directions are functions defining locally hypersurfaces of maximal contact.

(2) The main problem leading to complexity of the proofs is that of noncanonical choice of the tangent directions. We overcome this difficulty by introducing *homogenized ideals.*

2.7. **Arithmetical operations on marked ideals.** In this sections all marked ideals are defined for the smooth variety X and the same set of exceptional divisors E. Define the following operations of addition and multiplication of marked ideals:

(1) $(\mathcal{I}, \mu_{\mathcal{I}}) + (\mathcal{J}, \mu_{\mathcal{J}}) := (\mathcal{I}^{\mu_{\mathcal{I}}} + \mathcal{J}^{\mu_{\mathcal{I}}}, \mu_{\mathcal{I}}\mu_{\mathcal{J}})$, or more generally (the operation of addition is not associative)

$$(\mathcal{I}_1, \mu_1) + \ldots + (\mathcal{I}_m, \mu_m) := (\mathcal{I}_1^{\mu_2 \cdot \ldots \cdot \mu_m} + \mathcal{I}_2^{\mu_1\mu_3 \cdot \ldots \cdot \mu_m} + \ldots + \mathcal{I}_m^{\mu_1 \ldots \mu_{k-1}}, \mu_1\mu_2 \ldots \mu_m).$$

(2) $(\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}}) := (\mathcal{I} \cdot J, \mu_{\mathcal{I}} + \mu_{\mathcal{J}})$.

Lemma 2.7.1. (1) $\mathrm{supp}((\mathcal{I}_1, \mu_1) + \ldots + (\mathcal{I}_m, \mu_m)) = \mathrm{supp}(\mathcal{I}_1, \mu_1) \cap \ldots \cap \mathrm{supp}(\mathcal{I}_m, \mu_m)$. *Moreover multiple blow-ups (X_k) of $(\mathcal{I}_1, \mu_1) + \ldots + (\mathcal{I}_m, \mu_m)$ are exactly those which are simultaneous multiple blow-ups for all $(\mathcal{I}_j, \mu_j)$ and for any k we have the equality for the controlled transforms $(\mathcal{I}_j, \mu_{\mathcal{I}})_k$*

$$(\mathcal{I}_1, \mu_1)_k + \ldots + (\mathcal{I}_m, \mu_m)_k = [(\mathcal{I}_1, \mu_1) + \ldots + (\mathcal{I}_m, \mu_m)]_k$$

(2)

$$\mathrm{supp}(\mathcal{I}, \mu_{\mathcal{I}}) \cap \mathrm{supp}(\mathcal{J}, \mu_{\mathcal{J}}) \supseteq \mathrm{supp}((\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}})).$$

Moreover any simultaneous multiple blow-up X_i of both ideals $(\mathcal{I}, \mu_{\mathcal{I}})$ and $(\mathcal{J}, \mu_{\mathcal{J}})$ is a multiple blow-up for $(\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}})$, and for the controlled transforms $(\mathcal{I}_k, \mu_{\mathcal{I}})$ and $(\mathcal{J}_k, \mu_{\mathcal{J}})$ we have the equality

$$(\mathcal{I}_k, \mu_{\mathcal{I}}) \cdot (\mathcal{J}_k, \mu_{\mathcal{J}}) = [(\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}})]_k.$$

2.8. **Homogenized ideals and tangent directions.** Let $(\mathcal{I}, \mu)$ be a marked ideal of maximal order. Set $T(\mathcal{I}) := \mathcal{D}^{\mu-1}\mathcal{I}$. By the *homogenized ideal* we mean

$$\mathcal{H}(\mathcal{I}, \mu) := (\mathcal{H}(\mathcal{I}), \mu) = (\mathcal{I} + \mathcal{D}\mathcal{I} \cdot T(\mathcal{I}) + \ldots + \mathcal{D}^i\mathcal{I} \cdot T(\mathcal{I})^i + \ldots + \mathcal{D}^{\mu-1}\mathcal{I} \cdot T(\mathcal{I})^{\mu-1}, \mu)$$

Remark. A homogenized ideal features two important properties:

(1) It is equivalent to the given ideal.
(2) It "looks the same" from all possible tangent directions.

By the first property we can use the homogenized ideal to construct resolution via the Giraud Lemma 2.6.6. By the second property such a construction does not depend on the choice of tangent directions.

Lemma 2.8.1. *Let $(\mathcal{I}, \mu)$ be a marked ideal of maximal order. Then*

(1) $(\mathcal{I}, \mu) \simeq (\mathcal{H}(\mathcal{I}), \mu)$.
(2) *For any multiple blow-up (X_k) of $(\mathcal{I}, \mu)$,*
$(\mathcal{H}(\mathcal{I}), \mu)_k = (\mathcal{I}, \mu)_k + [\mathcal{D}(\mathcal{I}, \mu)]_k \cdot [(T(\mathcal{I}), 1)]_k + \ldots [\mathcal{D}^{\mu-1}(\mathcal{I}, \mu)]_k \cdot +[(T(\mathcal{I}), 1)]_k^{\mu-1}$.

Although the following Lemmas 2.8.2 and 2.8.3 are used in this paper only in the case $E = \emptyset$ we formulate them in slightly more general versions.

Lemma 2.8.2. *Let $(X, \mathcal{I}, E, \mu)$ be a marked ideal of maximal order. Assume there exist tangent directions $u, v \in T(\mathcal{I}, \mu)_x = \mathcal{D}^{\mu-1}(\mathcal{I}, \mu)_x$ at $x \in \mathrm{supp}(\mathcal{I}, \mu)$ which are transversal to E. Then there exists an automorphism $\widehat{\phi}_{uv}$ of $\widehat{X}_x := \mathrm{Spec}(\widehat{\mathcal{O}}_{x,X})$ such that*

(1) $\widehat{\phi}^*_{uv}(\mathcal{H}\widehat{\mathcal{I}})_x = (\mathcal{H}\widehat{\mathcal{I}})_x$.
(2) $\widehat{\phi}^*_{uv}(E) = E$.

(3) $\widehat{\phi}^*_{uv}(u) = v$.
(4) $\operatorname{supp}(\widehat{\mathcal{I}}, \mu) := V(T(\widehat{\mathcal{I}}, \mu))$ *is contained in the fixed point set of* ϕ.

Proof. (0) **Construction of the automorphism $\widehat{\phi}_{uv}$.**

Find parameters $u_2, \ldots, u_n$ transversal to u and v such that $u = u_1, u_2, \ldots, u_n$ and $v, u_2, \ldots, u_n$ form two sets of parameters at x and divisors in E are described by some parameters u_i where $i \geq 2$. Set

$$\widehat{\phi}_{uv}(u_1) = v, \quad \widehat{\phi}_{uv}(u_i) = u_i \quad \text{for} \quad i > 1.$$

(1) Let $h := v - u \in T(\mathcal{I})$. For any $f \in \widehat{\mathcal{I}}$,

$$\widehat{\phi}^*_{uv}(f) = f(u_1 + h, u_2, \ldots, u_n) = f(u_1, \ldots, u_n) + \frac{\partial f}{\partial u_1} \cdot h + \frac{1}{2!} \frac{\partial^2 f}{\partial u_1^2} \cdot h^2 + \ldots + \frac{1}{i!} \frac{\partial^i f}{\partial u_1^i} \cdot h^i + \ldots$$

The latter element belongs to

$$\widehat{\mathcal{I}} + \mathcal{D}\widehat{\mathcal{I}} \cdot \widehat{T(\mathcal{I})} + \ldots + \mathcal{D}^i \widehat{\mathcal{I}} \cdot \widehat{T(\mathcal{I})}^i + \ldots + \mathcal{D}^{\mu-1} \widehat{\mathcal{I}} \cdot \widehat{T(\mathcal{I})}^{\mu-1} = \mathcal{H}\widehat{\mathcal{I}}.$$

Hence $\widehat{\phi}^*_{uv}(\widehat{\mathcal{I}}) \subset \mathcal{H}\widehat{\mathcal{I}}$. (2)(3) Follow from the construction.

(4) The fixed point set of $\widehat{\phi}^*_{uv}$ is defined by $u_i = \widehat{\phi}^*_{uv}(u_i)$, $i = 1, \ldots, n$, that is, $h = 0$. But $h \in \mathcal{D}^{\mu-1}(\mathcal{I})$ is 0 on $\operatorname{supp}(\mathcal{I}, \mu)$. $\square$

Lemma 2.8.3. *(***Glueing Lemma***) Let $(X, \mathcal{I}, E, \mu)$ be a marked ideal of maximal order for which there exist tangent directions $u, v \in T(\mathcal{I}, \mu)$ at $x \in \operatorname{supp}(\mathcal{I}, \mu)$ which are transversal to E. Then there exist étale neighborhoods $\phi_u, \phi_v : \overline{X} \to X$ of $x = \phi_u(\overline{x}) = \phi_v(\overline{x}) \in X$, where $\overline{x} \in \overline{X}$, such that*

(1) $\phi^*_u(\mathcal{H}(\mathcal{I}) = \phi^*_v(\mathcal{H}(\mathcal{I}))$.
(2) $\phi^*_u(E) = \phi^*_v(E)$.
(3) $\phi^*_u(u) = \phi^*_v(v)$.

Set $(\overline{X}, \overline{\mathcal{I}}, \overline{E}, \mu) := \phi^*_u(X, \mathcal{H}(\mathcal{I}), E, \mu) = \phi^*_v(X, \mathcal{H}(\mathcal{I}, E, \mu))$.

(4) *For any* $\overline{y} \in \operatorname{supp}(\overline{X}, \overline{\mathcal{I}}, \overline{E}, \mu)$, $\phi_u(\overline{y}) = \phi_v(\overline{y})$.
(5) *For any multiple blow-up* (X_i) *of* $(X, \mathcal{I}, \emptyset, \mu)$ *the induced multiple blow-ups* $\phi^*_u(X_i)$ *and* $\phi^*_v(X_i)$ *of* $(\overline{X}, \overline{\mathcal{I}}, \overline{E}, \mu)$ *are the same (defined by the same centers).*
Set $(\overline{X}_i) := \phi^*_u(X_i) = \phi^*_v(X_i)$.
(6) *For any* $\overline{y}_i \in \operatorname{supp}(\overline{X}_i, \overline{\mathcal{I}}_i, \overline{E}_i, \mu)$ *and the induced morphisms* $\phi_{ui}, \phi_{vi} : \overline{X}_i \to X_i$, $\phi_{ui}(\overline{y}_i) = \phi_{vi}(\overline{y}_i)$.

Proof. (0) **Construction of étale neighborhoods $\phi_u, \phi_v : U \to X$.**

Let $U \subset X$ be an open subset for which there exist $u_2, \ldots, u_n$ which are transversal to u and v on U such that $u = u_1, u_2, \ldots, u_n$ and $v, u_2, \ldots, u_n$ form two sets of parameters on U and divisors in E are described by some u_i, where $i \geq 2$. Let $\mathbf{A}^n$ be the affine space with coordinates $x_1, \ldots, x_n$. Construct first étale morphisms $\phi_1, \phi_2 : U \to \mathbf{A}^n$ with

$$\phi_1^*(x_i) = u_i \quad \text{for all } i \qquad \text{and} \quad \phi_2^*(x_1) = v, \quad \phi_2^*(x_i) = u_i \quad \text{for} \quad i > 1.$$

Then

$$\overline{X} := U \times_{\mathbf{A}^n} U$$

is a fiber product for the morphisms ϕ_1 and ϕ_2. The morphisms ϕ_u, ϕ_v are defined to be the natural projections $\phi_u, \phi_v : \overline{X} \to U$ such that $\phi_1 \phi_u = \phi_2 \phi_v$. Set

$$w_1 := \phi^*_u(u) = (\phi_1 \phi_u)^*(x_1) = (\phi_2 \phi_v)^*(x_1) = \phi^*_v(v),$$

$$w_i = \phi^*_u(u_i) = \phi^*_v(u_i) \quad \text{for } i \geq 2.$$

(1), (2), (3) follow from the construction

(4) Let $h := v - u$. By the above the morphisms ϕ_u and ϕ_v coincide on $\phi_u^{-1}(V(h)) = \phi_v^{-1}(V(h))$.

By (4) the blow-ups of the centers $C \subset \operatorname{supp}(\mathcal{H}(\mathcal{I}))$ lifts to the blow-ups at the same center $\phi_u^{-1}(C) = \phi_u^{-1}(C)$. Thus (5), (6) follow. (see [51] for details).

2.9. **Coefficient ideals and Giraud Lemma.** The idea of coefficient ideals was originated by Hironaka and then developed in papers of Villamayor and Bierstone-Milman. The following definition modifies and generalizes the definition of Villamayor.

Definition 2.9.1. Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order. By the *coefficient ideal* we mean

$$\mathcal{C}(\mathcal{I},\mu)=\sum_{i=1}^{\mu}(\mathcal{D}^i\mathcal{I},\mu-i).$$

Remark. The coefficient ideals $\mathcal{C}(\mathcal{I})$ feature two important properties.

(1) $\mathcal{C}(\mathcal{I})$ is equivalent to $\mathcal{I}$.
(2) The intersection of the support of $(\mathcal{I},\mu)$ with any smooth subvariety S is the support of the restriction of $\mathcal{C}(\mathcal{I})$ to S:
$$\mathrm{supp}(\mathcal{I})\cap S=\mathrm{supp}(\mathcal{C}(\mathcal{I})_{|S}).$$
Moreover this condition is persistent under relevant multiple blow-ups.

These properties allow one to control and modify the part of support of $(\mathcal{I},\mu)$ contained in S by applying multiple blow-ups of $\mathcal{C}(\mathcal{I})_{|S}$.

Lemma 2.9.2. $\mathcal{C}(\mathcal{I},\mu)\simeq(\mathcal{I},\mu)$.

Proof. By Lemma 2.7.1 multiple blow-ups of $\mathcal{C}(\mathcal{I},\mu)$ are simultaneous multiple blow-ups of $\mathcal{D}^i(\mathcal{I},\mu)$ for $0\le i\le\mu-1$. By Lemma 2.5.3 multiple blow-ups of $(\mathcal{I},\mu)$ define the multiple blow-up of all $\mathcal{D}^i(\mathcal{I},\mu)$. Thus multiple blow-ups of $(\mathcal{I},\mu)$ and $\mathcal{C}(\mathcal{I},\mu)$ are the same and $\mathrm{supp}(\mathcal{C}(\mathcal{I},\mu))_k=\bigcap\mathrm{supp}(\mathcal{D}^i\mathcal{I},\mu-i)_k=\mathrm{supp}(\mathcal{I}_k,\mu)$. $\square$

Lemma 2.9.3. *Let $(X,\mathcal{I},E,\mu)$ be a marked ideal of maximal order. Assume that S has only simple normal crossings with E. Then*
$$\mathrm{supp}(\mathcal{I},\mu)\cap S=\mathrm{supp}(\mathcal{C}(\mathcal{I},\mu)_{|S}).$$
Moreover let (X_i) be a multiple blow-up with centers C_i contained in the strict transforms $S_i\subset X_i$ of S. Then

(1) *The restrictions $\sigma_{i|S_i}:S_i\to S_{i-1}$ of the morphisms $\sigma_i:X_i\to X_{i-1}$ define a multiple blow-up (S_i) of $\mathcal{C}(\mathcal{I},\mu)_{|S}$.*
(2) $\mathrm{supp}(\mathcal{I}_i,\mu)\cap S_i=\mathrm{supp}[\mathcal{C}(\mathcal{I},\mu)_{|S}]_i$.
(3) *Every multiple blow-up (S_i) of $\mathcal{C}(\mathcal{I},\mu)_{|S}$ defines a multiple blow-up (X_i) of $(\mathcal{I},\mu)$ with centers C_i contained in the strict transforms $S_i\subset X_i$ of $S\subset X$.*

Proof. By Lemma 2.9.2, $\mathrm{supp}(\mathcal{I},\mu)\cap S=\mathrm{supp}(\mathcal{C}(\mathcal{I},\mu))\cap S\subseteq\mathrm{supp}(\mathcal{C}(\mathcal{I},\mu)_{|S})$.

Let $x_1,\ldots,x_k,y_1,\ldots,y_{n-k}$ be local parameters at x such that $\{x_1=0,\ldots,x_k=0\}$ describes S. Then any function $f\in\mathcal{I}$ can be written as
$$f=\sum c_{\alpha f}(y)x^\alpha,$$
where $c_{\alpha f}(y)$ are formal power series in y_i.

Now $x\in\mathrm{supp}(\mathcal{I},\mu)\cap S$ iff $\mathrm{ord}_x(c_\alpha)\ge\mu-|\alpha|$ for all $f\in\mathcal{I}$ and $|\alpha|\le\mu$. Note that
$$c_{\alpha f|S}=\left(\frac{1}{\alpha!}\frac{\partial^{|\alpha|}(f)}{\partial x^\alpha}\right)_{|S}\in\mathcal{D}^{|\alpha|}(\mathcal{I})_{|S}$$
and consequently $\mathrm{supp}(\mathcal{I},\mu)\cap S=\bigcap_{f\in\mathcal{I},|\alpha|\le\mu}\mathrm{supp}(c_{\alpha f|S},\mu-|\alpha|)\supset\mathrm{supp}(\mathcal{C}(\mathcal{I},\mu)_{|S})$.

The above relation is preserved by multiple blow-ups of $(\mathcal{I},\mu)$. For the details see [51].

Lemma 2.9.4. *Let $\phi:X'\to X$ be an étale morphism of smooth varieties and let $(X,\mathcal{I},\emptyset,\mu)$ be a marked ideal. Then*

(1) $\phi^*(\mathcal{D}(\mathcal{I}))=\mathcal{D}(\phi^*(\mathcal{I}))$.
(2) $\phi^*(\mathcal{H}(\mathcal{I}))=\mathcal{H}(\phi^*(\mathcal{I}))$.
(3) $\phi^*(\mathcal{C}(\mathcal{I}))=\mathcal{C}(\phi^*(\mathcal{I}))$.

Proof. Note that for any point $x\in X$ the completion $\widehat{\phi^*_x}$ is an isomorphism. Thus $\widehat{\phi^*_x}(\widehat{\mathcal{D}(\mathcal{I})})=\widehat{\mathcal{D}((\mathcal{I})})$ and therefore $\phi^*(\mathcal{D}(\mathcal{I}))=\mathcal{D}(\phi^*(\mathcal{I}))$. (2) and (3) follow from (1). $\square$

3. Resolution algorithm

The presentation of the following Hironaka resolution algorithm builds upon Bierstone-Milman's, Villamayor's and Wlodarczyk's algorithms which are simplifications of the original Hironaka proof. We also use Kollár's trick allowing to completely eliminate the use of invariants.

Remarks. (1) Note that the blow-up of codimension one components is an isomorphism. However it defines a nontrivial transformation of marked ideals. The inverse image of the center is still called the exceptional divisor.

(2) In the actual desingularization process this kind of blow-up may occur for some marked ideals induced on subvarieties of ambient varieties. Though they define isomorphisms of those subvarieties they determine blow-ups of ambient varieties which are not isomorphisms.

(3) The blow-ups of the center C which coincides with the whole variety X is an empty set. The main feature which characterizes is given by the restriction property:

If X is a smooth variety containing a smooth subvariety $Y \subset X$, which contains the center $C \subset Y$ then the blow-up $\sigma_{C,Y} : \tilde{Y} \to Y$ at C coincides with the strict transform of Y under the blow-up $\sigma_{C,X} : \tilde{X} \to X$, i.e

$$\tilde{Y} \simeq \overline{\sigma_{C,X}^{-1}(Y \setminus C)}$$

Inductive assumption For any marked ideal $(X, \mathcal{I}, E, \mu)$ such that $\mathcal{I}$ there is an associated resolution $(X_i)_{0 \leq i \leq m_X}$, called <u>canonical</u>, satisfying the following conditions:

(1) For any surjective locally étale morphism $\phi : X' \to X$ the induced sequence $(X_i') = \phi^*(X_i)$ is the canonical resolution of $(X', \mathcal{I}', E', \mu) := \phi^*(X, \mathcal{I}, E, \mu)$.

(2) For any locally analytic isomophism $\phi : M' \to M$ the induced sequence $(X_i') = \phi^*(X_i)$ is an extension of the canonical resolution of $(X', \mathcal{I}', E', \mu) := \phi^*(X, \mathcal{I}, E, \mu)$.

Proof. If $\mathcal{I} = 0$ and $\mu > 0$ then $\mathrm{supp}(X, \mathcal{I}, \mu) = X$, and the blow-up of X is the empty set and thus it defines a unique resolution. Assume that $\mathcal{I} \neq 0$.

We shall use the induction on the dimension of X. If X is 0-dimensional, $\mathcal{I} \neq 0$ and $\mu > 0$ then $\mathrm{supp}(X, \mathcal{I}, \mu) = \emptyset$ and all resolutions are trivial.

Step 1 Resolving a marked ideal $(X, \mathcal{J}, E, \mu)$ of maximal order.

Before performing the resolution algorithm for the marked ideal $(\mathcal{J}, \mu)$ of maximal order in Step 1 we shall replace it with the equivalent homogenized ideal $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$. Resolving the ideal $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$ defines a resolution of $(\mathcal{J}, \mu)$ at this step. To simplify notation we shall denote $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$ by $(\overline{\mathcal{J}}, \overline{\mu})$.

Step 1a Reduction to the nonboundary case. Moving $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$ and H^s_α apart . For any multiple blow-up (X_i) of $(X, \overline{\mathcal{J}}, E, \overline{\mu})$ we shall identify (for simplicity) strict transforms of E on X_i with E.

For any $x \in X_i$, let $s(x)$ denote the number of divisors in E through x and set

$$s_i = \max\{s(x) \mid x \in \mathrm{supp}(\overline{\mathcal{J}}_i)\}.$$

Let $s = s_0$. By assumption the intersections of any $s > s_0$ components of the exceptional divisors are disjoint from $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$. Each intersection of divisors in E is locally defined by intersection of some irreducible components of these divisors. Find all intersections $H^s_\alpha, \alpha \in A$, of s irreducible components of divisors E such that $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu}) \cap H^s_\alpha \neq \emptyset$. By the maximality of s, the supports $\mathrm{supp}(\overline{\mathcal{J}}_{|H^s_\alpha}) \subset H^s_\alpha$ are disjoint from $H^s_{\alpha'}$, where $\alpha' \neq \alpha$.

Set

$$H^s := \bigcup_\alpha H^s_\alpha, \quad U^s := X \setminus H^{s+1}, \quad \underline{H^s} := H^s \setminus H^{s+1}.$$

Then $\underline{H^s} \subset U_s$ is a smooth closed subset U_s. Moreover $\underline{H^s} \cap \mathrm{supp}(\mathcal{I}) = H^s \cap \mathrm{supp}(\mathcal{I})$ is closed.

Construct the canonical resolution of $\overline{\mathcal{J}}_{|\underline{H^s}}$. By Lemma 2.9.3, it defines a multiple blow-up of $(\overline{\mathcal{J}}, \overline{\mu})$ such that

$$\mathrm{supp}(\overline{\mathcal{J}}_{j_1}, \overline{\mu}) \cap H^s_{j_1} = \emptyset.$$

In particular the number of the strict tranforms of E passing through a single point of the support drops $s_{j_1} < s$. Now we put $s = s_{j_1}$ and repeat the procedure. We continue the above process till $s_{j_k} = s_r = 0$. Then $(X_j)_{0 \leq j \leq r}$ is a multiple blow-up of $(X, \overline{\mathcal{J}}, E, \overline{\mu})$ such that $\mathrm{supp}(\overline{\mathcal{J}}_r, \overline{\mu})$ does not intersect any divisor in E.

Therefore $(X_j)_{0\leq j\leq r}$ and further longer multiple blow-ups $(X_j)_{0\leq j\leq m}$ for any $m\geq r$ can be considered as multiple blow-ups of $(X,\overline{\mathcal{J}},\emptyset,\overline{\mu})$ since starting from X_r the strict transforms of E play no further role in the resolution process since they do not intersect $\mathrm{supp}(\overline{\mathcal{J}}_j,\overline{\mu})$ for $j\geq r$. We reduce the situation to the "nonboundary case".

Step 1b. **Nonboundary case**

Let $(X_j)_{0\leq j\leq r}$ be the multiple blow-up of $(X,\overline{\mathcal{J}},\emptyset,\overline{\mu})$ defined in Step 1a.

For any $x\in \mathrm{supp}(\overline{\mathcal{J}},\overline{\mu})\subset X$ find a tangent direction $u_\alpha\in \mathcal{D}^{\overline{\mu}-1}(\overline{\mathcal{J}})$ on some neighborhood U_α of x. Then $V(u_\alpha)\subset U_\alpha$ is a hypersurface of maximal contact. By the quasicompactness of X we can assume that the covering defined by U_α is finite. Let $U_{i\alpha}\subset X_i$ be the inverse image of $U_{i\alpha}$ and let $H_{i\alpha}:=V(u_\alpha)_i\subset U_{i\alpha}$ denote the strict transform of $H_\alpha:=V(u_\alpha)$.

Set (see also [37])

$$\widetilde{X}:=\coprod U_\alpha \qquad \widetilde{H}:=\coprod H_\alpha\subseteq \widetilde{X}.$$

The closed embeddings $H_\alpha\subseteq U_\alpha$ define the closed embedding $\widetilde{H}\subset\widetilde{M}$ of a hypersurface of maximal contact $\widetilde{H}$.

Consider the surjective étale morphism

$$\phi_U:\widetilde{X}:=\coprod U_\alpha\to X$$

Denote by $\widetilde{J}$ the pull back of the ideal sheaf $\overline{\mathcal{J}}$ via ϕ_U. The multiple blow-up $(X_i)_{0\leq i\leq r}$ of $\overline{J}$ defines a multiple blow-up $(\widetilde{X}_{0\leq i\leq r})$ of $\widetilde{J}$ and a multiple blow-up $(\widetilde{H}_i)_{0\leq i\leq r}$ of $\widetilde{J}_{|H}$.

Construct the canonical resolution of $(\widetilde{H}_i)_{r\leq i\leq m}$ of the marked ideal $\widetilde{\mathcal{J}}_{r|\widetilde{H}_r}$ on $\widetilde{H}_r$. It defines, by Lemma 2.9.2, a resolution $(\widetilde{X}_{r\leq i\leq m})$ of $\widetilde{\mathcal{J}}_r$ and thus also a resolution $(\widetilde{X}_i)_{0\leq i\leq m}$ of $(\widetilde{X},\widetilde{\mathcal{J}},\emptyset,\overline{\mu})$. Moreover both resolutions are related by the property

$$\mathrm{supp}(\widetilde{\mathcal{J}}_i)=\mathrm{supp}(\widetilde{\mathcal{J}}_{i|\widetilde{H}_i}).$$

Consider a (possible) lifting of ϕ_U:

$$\phi_{iU}:\widetilde{X}_i:=\coprod U_{i\alpha}\to X_i,$$

which is a surjective locally étale morphism. The lifting is constructed for $0\leq i\leq r$.

For $r\leq i\leq m$ the resolution $\widetilde{X}_i$ is induced by the canonical resolution $(\widetilde{H}_i)_{r\leq i\leq m}$ of $\overline{\mathcal{J}}_{r|\widetilde{H}_r}$

We show that the resolution $(\widetilde{X_i})_{r\leq i\leq m}$ descends to the resolution $(X_i)_{r\leq i\leq m}$.

Let $\widetilde{C}_{j_0}=\coprod C_{j_0\alpha}$ be the center of the blow-up $\widetilde{\sigma}_{j_0}:\widetilde{X}_{j_0+1}\to\widetilde{X}_{j_0}$. The closed subset $C_{j_0\alpha}\subset U_{j_0\alpha}$ defines the center of an extension of the canonical resolution $(H_{j\alpha})_{r\leq j\leq m}$.

If $C_{j_0\alpha}\cap U_{j_0\beta}\neq\emptyset$ then by the canonicity and condition (2) of the inductive assumption, the subset $C_{j_0\alpha\beta}:=C_{j_0\alpha}\cap U_{j_0\beta}$ defines the center of an extension of of the canonical resolution $H_{j\alpha\beta}:=((H_{j\alpha}\cap U_{j\beta}))_{r\leq j\leq m}$. On the other hand $C_{j_0\beta\alpha}:=C_{j_0\beta}\cap U_{j\alpha}$ defines the center of an extension of the canonical resolution $((H_{j\beta\alpha}:=H_{j\beta}\cap U_{j\alpha}))_{r\leq j\leq m}$.

By Glueing Lemma 2.8.3 for the tangent directions u_α and u_β we find there exist étale neighborhoods $\phi_{u_\alpha},\phi_{u_\beta}:\overline{U}_{\alpha\beta}\to U_{\alpha\beta}:=U_\alpha\cap U_\beta$ of $x=\phi_u(\overline{x})=\phi_v(\overline{x})\in X$, where $\overline{x}\in\overline{X}$, such that

(1) $\phi^*_{u_\alpha}(\mathcal{J})=\phi^*_{u_\beta}(\mathcal{J})$.
(2) $\phi^*_{u_\alpha}(E)=\phi^*_{u_\beta}(E)$.
(3) $\phi^{-1}_{u_\alpha}(H_{j\alpha\beta})=\phi^{-1}_{u_\beta}(H_{j\beta\alpha})$.
(4) $\phi_{u_\alpha}(\bar{x})=\phi_{u_\beta}(\bar{x})$ for $\overline{x}\in\mathrm{supp}(\phi^*_{u_\alpha}(\mathcal{J}))$.

Moreover all the properties lift to the relevant étale morphisms $\phi_{u_{\alpha i}},\phi_{u_{\beta i}}:\overline{U}_{\alpha\beta i}\to U_{\alpha\beta i}$. Consequently, by canonicity $\phi^{-1}_{u_\alpha j_0}(C_{j_0\alpha\beta})$ and $\phi^{-1}_{u_\beta j_0}(C_{j_0\beta\alpha})$ define both the next center of the extension of the canonical resolution $\phi^{-1}_{u_\alpha}(H_{j_0\alpha\beta})=\phi^{-1}_{u_\beta}(H_{j_0\beta\alpha})$ of $\phi^*_{u_\alpha j_0}(\mathcal{J}_{|H_{\alpha\beta}})=\phi^*_{u_\beta}(\mathcal{J}_{|H_{\beta\alpha}})$.

Thus

$$\phi^{-1}_{u_\alpha}(C_{j_0\alpha\beta})=\phi^{-1}_{u_\beta}(C_{j_0\beta\alpha}),$$

and finally, by property (4),

$$C_{j_0\alpha\beta} = C_{j_0\beta\alpha}.$$

Consequently $\widetilde{C}_{j_0}$ descends to the smooth closed center $C_{j_0} = \bigcup C_{j_0\alpha} \subset X_{j_0}$ and the resolution $(\widetilde{X_i})_{r\leq i\leq m}$ descends to the resolution $(X_i)_{r\leq i\leq m}$.

Step 2. Resolving of marked ideals $(X,\mathcal{I},E,\mu)$.

For any marked ideal $(X,\mathcal{I},E,\mu)$ write

$$I = \mathcal{M}(\mathcal{I})\mathcal{N}(\mathcal{I}),$$

where $\mathcal{M}(\mathcal{I})$ is the *monomial part* of $\mathcal{I}$, that is, the product of the principal ideals defining the irreducible components of the divisors in E, and $\mathcal{N}(\mathcal{I})$ is a *nonmonomial part* which is not divisible by any ideal of a divisor in E. Let

$$\mathrm{ord}_{\mathcal{N}(\mathcal{I})} := \max\{\mathrm{ord}_x(\mathcal{N}(\mathcal{I})) \mid x \in \mathrm{supp}(\mathcal{I},\mu)\}.$$

Definition 3.0.5. (Hironaka, Bierstone-Milman,Villamayor, Encinas-Hauser) By the *companion ideal* of $(\mathcal{I},\mu)$ where $I = \mathcal{N}(\mathcal{I})\mathcal{M}(\mathcal{I})$ we mean the marked ideal of maximal order

$$O(\mathcal{I},\mu) = \begin{cases} (\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) \quad + \quad (\mathcal{M}(\mathcal{I}),\mu-\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \mathrm{ord}_{\mathcal{N}(\mathcal{I})} < \mu, \\ (\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \mathrm{ord}_{\mathcal{N}(\mathcal{I})} \geq \mu. \end{cases}$$

In particular $O(\mathcal{I},\mu) = (\mathcal{I},\mu)$ for ideals $(\mathcal{I},\mu)$ of maximal order.

Step 2a. Reduction to the monomial case by using companion ideals

By Step 1 we can resolve the marked ideal of maximal order $(\mathcal{J},\mu_{\mathcal{J}}) := O(\mathcal{I},\mu)$. By Lemma 2.7.1, for any multiple blow-up of $O(\mathcal{I},\mu)$,

$$\begin{aligned}\mathrm{supp}(O(\mathcal{I},\mu))_i &= \mathrm{supp}[\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}]_i \,\cap\, \mathrm{supp}[M(\mathcal{I}),\mu-\mathrm{ord}_{\mathcal{N}(H\mathcal{I})}]_i = \\ &\mathrm{supp}[\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}]_i \,\cap\, \mathrm{supp}(\mathcal{I}_i,\mu).\end{aligned}$$

Consequently, such a resolution leads to the ideal $(\mathcal{I}_{r_1},\mu)$ such that $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_1})} < \mathrm{ord}_{\mathcal{N}(\mathcal{I})}$. Then we repeat the procedure for $(\mathcal{I}_{r_1},\mu)$. We find marked ideals $(\mathcal{I}_{r_0},\mu) = (\mathcal{I},\mu), (\mathcal{I}_{r_1},\mu), \ldots, (\mathcal{I}_{r_m},\mu)$ such that $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_0)} > \mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_1})} > \ldots > \mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_m})}$. The procedure terminates after a finite number of steps when we arrive at the ideal $(\mathcal{I}_{r_m},\mu)$ with $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_m})} = 0$ or with $\mathrm{supp}(\mathcal{I}_{r_m},\mu) = \emptyset$. In the second case we get the resolution. In the first case $\mathcal{I}_{r_m} = \mathcal{M}(\mathcal{I}_{r_m})$ is monomial.

Step 2b. Monomial case $\mathcal{I} = \mathcal{M}(\mathcal{I})$.

Let $x_1,\ldots,x_k$ define equations of the components $D^x_1,\ldots,D^x_k \in E$ through $x \in \mathrm{supp}(X,\mathcal{I},E,\mu)$ and $\mathcal{I}$ be generated by the monomial $x^{a_1,\ldots,a_k}$ at x. In particular

$$\mathrm{ord}_x(\mathcal{I})(x) := a_1 + \ldots + a_k.$$

Let $\rho(x) = \{D_{i_1},\ldots,D_{i_l}\} \in \mathrm{Sub}(E)$ be the maximal subset satisfying the properties

(1) $a_{i_1} + \ldots + a_{i_l} \geq \mu$.
(2) For any $j = 1,\ldots,l$, $a_{i_1} + \ldots + \check{a}_{i_j} + \ldots + a_{i_l} < \mu$.

Let $R(x)$ denote the subsets in $\mathrm{Sub}(E)$ satisfying the properties (1) and (2). The maximal components of the $\mathrm{supp}(\mathcal{I},\mu)$ through x are described by the intersections $\bigcap_{D\in A} D$ where $A \in R(x)$. The maximal locus of ρ determines at most o one maximal component of $\mathrm{supp}(\mathcal{I},\mu)$ through each x.

After the blow-up at the maximal locus $C = \{x_{i_1} = \ldots = x_{i_l} = 0\}$ of ρ, the ideal $\mathcal{I} = (x^{a_1,\ldots,a_k})$ is equal to $\mathcal{I}' = ({x'}^{a_1,\ldots,a_{i_j-1},a,a_{i_j+1},\ldots,a_k})$ in the neighborhood corresponding to x_{i_j}, where $a = a_{i_1} + \ldots + a_{i_l} - \mu < a_{i_j}$. In particular the invariant $\mathrm{ord}_x(\mathcal{I})$ drops for all points of some maximal components of $\mathrm{supp}(\mathcal{I},\mu)$. Thus the maximal value of $\mathrm{ord}_x(\mathcal{I})$ on the maximal components of $\mathrm{supp}(\mathcal{I},\mu)$ which were blown up is bigger than the maximal value of $\mathrm{ord}_x(\mathcal{I})$ on the new maximal components of $\mathrm{supp}(\mathcal{I},\mu)$. The algorithm terminates after a finite number of steps.

□

3.1. **Summary of the resolution algorithm.** The resolution algorithm can be represented by the following scheme.

Step 2. Resolve $(\mathcal{I},\mu)$.

Step 2a. Reduce $(\mathcal{I},\mu)$ to the monomial marked ideal $\mathcal{I}=\mathcal{M}(\mathcal{I})$.

$$\Downarrow$$

If $\mathcal{I}\neq\mathcal{M}(\mathcal{I})$, decrease the maximal order of the nonmonomial part $\mathcal{N}(\mathcal{I})$ by resolving the companion ideal $O(\mathcal{I},\mu)$.

Step 1. Resolve the companion ideal $(\mathcal{J},\mu_{\mathcal{J}}):=O(\mathcal{I},\mu)$:

Replace $\mathcal{J}$ with $\overline{\mathcal{J}}:=\mathcal{C}(\mathcal{H}(\mathcal{J}))\simeq\mathcal{J}$. (*)

Step 1a. Move apart all strict transforms of E and $\mathrm{supp}(\overline{\mathcal{J}},\mu)$.

$$\Downarrow$$

Move apart all intersections H^s_α of s divisors in E

(where s is the maximal number of divisors in E through points in $\mathrm{supp}(\mathcal{I},\mu)$).

$$\Updownarrow$$

For any α, resolve $\overline{\mathcal{J}}_{|(\bigcup_\alpha H^s_\alpha)}$.

Step 1b If the strict transforms of E do not intersect $\mathrm{supp}(\overline{\mathcal{J}},\mu)$, resolve $(\overline{\mathcal{J}},\mu)$.

$$\Updownarrow$$

Simultaneously resolve all $\overline{\mathcal{J}}_{|V(u)}$, where $V(u)$ is a hypersurface of maximal contact.

(Use the property of homogenization ([51]), and Kollár's trick ([37]).

Step 2b. Resolve the monomial marked ideal $\mathcal{I}=\mathcal{M}(\mathcal{I})$.

Remarks. (1) (*) The ideal $\mathcal{J}$ is replaced with $\mathcal{H}(\mathcal{J})$ to ensure that the algprithm constructed in Step 1b is independent of the choice of the tangent direction u.
We replace $\mathcal{H}(\mathcal{J})$ with $\mathcal{C}(\mathcal{H}(\mathcal{J}))$ to ensure the equalities $\mathrm{supp}(\mathcal{J}_{|S})=\mathrm{supp}(\mathcal{J})\cap S$, where $S=H^s_\alpha$ in Step 1a and $S=V(u)$ in Step 1b.

(2) If $\mu=1$ the companion ideal is equal to $O(\mathcal{I},1)=(\mathcal{N}(\mathcal{I}),\mu_{\mathcal{N}(\mathcal{I})})$ so the general strategy of the resolution of $(\mathcal{I},\mu)$ is to decrease the order of the nonmonomial part and then to resolve the monomial part.

(3) In particular if we desingularize Y we put $\mu=1$ and $\mathcal{I}=\mathcal{I}_Y$ to be equal to the sheaf of the subvariety Y and we resolve the marked ideal $(X,\mathcal{I},\emptyset,\mu)$. The nonmonomial part $\mathcal{N}(\mathcal{I}_i)$ is nothing but the weak transform $(\sigma^i)^{\mathrm{w}}(\mathcal{I})$ of $\mathcal{I}$.

In the next sections we provide a complexity bound for the algorithm.

4. Complexity bounds on a blow-up

Our purpose for the rest of the paper is to estimate the complexity of the desingularization algorithm described in the previous sections.

4.1. **Preliminary setup.**

4.1.1. *Affine marked ideals.* An input of the algorithm is an *affine marked ideal* that is a collection of tuples

$$\mathcal{T}:=(\{X_{\alpha,\beta},\mathcal{I}_{\alpha,\beta},E_{\alpha,\beta},U_{\alpha,\beta},(\mathbb{C}^{n_\alpha})_\alpha\mid\alpha\in A,\beta\in B_\alpha\},\mu),$$

where

(1) $(\mathbb{C}^{n_\alpha})_\alpha\simeq\mathbb{C}^{n_\alpha}$
(2) $\{U_{\alpha,\beta}\mid\beta\in B_\alpha\}$ is an open cover of $(\mathbb{C}^{n_\alpha})_\alpha$.
(3) $U_{\alpha,\beta}\subset(\mathbb{C}^{n_\alpha})_\alpha$ is an open subset whose complement is given by $f_{\alpha,\beta}=0$.

(4) $X_{\alpha,\beta} \subset (\mathbb{C}^{n_\alpha})_\alpha$ is a closed subset such that $X_{\alpha,\beta} \cap U_{\alpha,\beta}$ is a nonsingular m-dimensional variety (possibly reducible). Moreover there exists a set of parameters (coordinates) on $U_{\alpha,\beta}$,

$$u_{\alpha,\beta,1}, \ldots, u_{\alpha,\beta,n_\alpha} \in \mathbb{C}[x_{\alpha,1}, \ldots, x_{\alpha,n_\alpha}],$$

such that $u_{\alpha,\beta,i}$ is a coordinate $x_{\alpha,j}$ describing exceptional divisor or it is transversal to the exceptional divisors (over $U_{\alpha,\beta}$) , and, moreover $\mathcal{I}_{X_{\alpha,\beta}} = (u_{\alpha,\beta,i_1}, \ldots, u_{\alpha,\beta,i_k}) \subset \mathbb{C}[x_{\alpha,1}, ..., x_{\alpha,n_\alpha}]$, for certain subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n_\alpha\}$.

(5) $\mathcal{I}_{\alpha,\beta} = \langle g_{\alpha,\beta,1}, \ldots, g_{\alpha,\beta,\overline{j}} \rangle \subset \mathbb{C}[x_{\alpha,1}, \ldots, x_{\alpha,n_\alpha}]$ is an ideal,

(6) $E_{\alpha,\beta}$ is a collection of smooth divisors in $(\mathbb{C}^{n_\alpha})_\alpha$ described by some $x_{\alpha,j} = 0$ where $j = s, s+1, \ldots, n$ for some $0 \leq s \leq m - n_\alpha$.

(7) There exist birational maps $i_{\alpha_1\beta_1,\alpha_2,\beta_2} : X_{\alpha_1,\beta_1} \dashrightarrow X_{\alpha_2,\beta_2}$ given by

$$X_{\alpha_1,\beta_1} \ni x \mapsto i_{\alpha_1\beta_1,\alpha_2,\beta_2}(x) = (\frac{v_{\alpha_1\beta_1,\alpha_2,\beta_2,1}}{w_{\alpha_1\beta_1,\alpha_2,\beta_2,1}}, \ldots, \frac{v_{\alpha_1\beta_1,\alpha_2,\beta_2,n_{\alpha_2}}}{w_{\alpha_1\beta_1,\alpha_2,\beta_2,n_{\alpha_2}}})(x) \in X_{\alpha_2,\beta_2}$$

for regular functions $v_{\alpha_1\beta_1,\alpha_2,\beta_2,1}, \ldots, v_{\alpha_1\beta_1,\alpha_2,\beta_2,n_{\alpha_2}}, w_{\alpha_1\beta_1,\alpha_2,\beta_2,1}, \ldots, w_{\alpha_1\beta_1,\alpha_2,\beta_2,n_{\alpha_2}} \in \mathbb{C}[x_1, \ldots, x_{n_{\alpha_1}}]$

(8) The birational maps $i_{\alpha\beta,\alpha',\beta'}$ determine uniquely up to an isomorphism a variety $X_\mathcal{T}$ in the following sense: There exist open embeddings $j_{\alpha,\beta} : X_{\alpha,\beta} \cap U_{\alpha,\beta} \hookrightarrow X_\mathcal{T}$ defining an open cover of $X_\mathcal{T}$, and satisfying

$$j^{-1}_{\alpha_2,\beta_2} j_{\alpha_1,\beta_1} = i_{\alpha_1\beta_1,\alpha_2,\beta_2}$$

(9) $\mu \geq 0$ is an integer.

(10) $\mathrm{supp}(\mathcal{I}_{\alpha,\beta}, \mu) \cap U_{\alpha,\beta} \cap U_{\alpha,\beta'} = \mathrm{supp}(\mathcal{I}_{\alpha,\beta'}, \mu) \cap U_{\alpha,\beta} \cap U_{\alpha,\beta'}$

Remarks. (1) The objects $X_{\alpha,\beta}, E_{\alpha,\beta}, \mathcal{I}_{\alpha,\beta}$, as well as corresponding functions $g_{\alpha,\beta,i}, u_{\alpha,\beta,j}, x_{\alpha,i}$ are relevant for the algorithm after they are restricted to $U_{\alpha,\beta}$. Their behavior in the complement $(\mathbb{C}^{n_\alpha})_\alpha \setminus U_{\alpha,\beta}$ has no relevance.

(2) The operation of restricting to the maximal contact leads to considering open subsets $U_{\alpha,\beta} \subset (\mathbb{C}^{n_\alpha})_\alpha$.

(3) While studying the complexity of the algorithm we shall assume that the coefficients of the input polynomials belong just to $\mathbb{Z}$. All general considerations are given for coefficients from $\mathbb{C}$, and remain valid for any algebraically closed field of zero characteristic.

(4) The open embeddings $j_{\alpha,\beta}$ can be constructed from $i_{\alpha_1\beta_1,\alpha_2,\beta_2}$ after performing the algorithm but we don't dwell on it.

Definition 4.1.1. By the support of $\mathcal{T}$ we mean the collection of the sets

$$\mathrm{supp}(\mathcal{T}) := \{\mathrm{supp}(X_{\alpha,\beta} \cap U_{\alpha,\beta}, \mathcal{I}_{\alpha,\beta}, E_{\alpha,\beta} \cap U_{\alpha,\beta} \cap X_{\alpha,\beta}, \mu)\}_{\alpha \in A, \beta \in B}$$

Definition 4.1.2. Given an affine marked ideal $\mathcal{T} := (\{X_{\alpha,\beta}, \mathcal{I}_{\alpha,\beta}, E_{\alpha,\beta}, U_{\alpha,\beta}, (\mathbb{C}^{n_\alpha})_\alpha \mid \alpha \in A, \beta \in B_\alpha\}, \mu)$, we say that an affine marked ideal $\mathcal{T}' := (\{X_{\alpha',\beta'}, \mathcal{I}_{\alpha,\beta}, E_{\alpha,\beta}, U_{\alpha,\beta}, (\mathbb{C}^{n_\alpha})_\alpha \mid \alpha \in A', \beta \in B'_\alpha\}, \mu)$, *is defined over* $\mathcal{T}$, provided

(1) There exist maps of set of indices $p : A' \to A$, and $p_{\alpha'} : B'_{\alpha'} \to B_{p(\alpha')}$

(2) The canonical projection on the first $\alpha = p(\alpha')$ components

$$\pi'_\alpha : (\mathbb{C}^{n_{\alpha'}})_{\alpha'} \to (\mathbb{C}^{n_\alpha})_\alpha$$

determine birational morphisms $\pi_{\alpha',\beta'} := \pi'_{|X_{\alpha',\beta'}} : X_{\alpha',\beta'} \to X_{\alpha,\beta}$ commuting with $i_{\alpha_1\beta_1,\alpha_2,\beta_2}$ and $i_{\alpha'_1\beta'_1,\alpha'_2,\beta'_2}$.

(3) There exist natural birational morphisms $X_{\mathcal{T}'} \to X_\mathcal{T}$ commuting with $j_{\alpha,\beta}$, and $j_{\alpha',\beta'}$.

We introduce the following functions to characterize the affine marked ideal

$$\mathcal{T} := (\{X_{\alpha,\beta}, \mathcal{I}_{\alpha,\beta}, E_{\alpha,\beta}, U_{\alpha,\beta}\}_{\alpha,\beta}, (\mathbb{C}^{n_\alpha})_\alpha, \mu) :$$

(1) $m(\mathcal{T}) = \dim(X_\mathcal{T})$,

(2) $\mu(\mathcal{T}) = \mu$,

(3) $d(\mathcal{T})$ is the maximal degree of all polynomials in

$$\Psi(\mathcal{T}) := \{u_{\alpha,\beta,i}, g_{\alpha,\beta,i}, f_{\alpha,\beta}, v_{\alpha,\beta,\alpha_2,\beta_2,i}, w_{\alpha,\beta,\alpha_2,\beta_2,j}.\}$$

(4) $n(\mathcal{T}) = \max n_\alpha$,

(5) $l(\mathcal{T})$ is the maximal number of all polynomials in $\Psi(\mathcal{T})$.

(6) $q(\mathcal{T})$ is the number of neighborhoods $U_{\alpha,\beta}$ in $\mathcal{T}$ (i. e the number of the indices s.t $\alpha \in A, \beta \in B$).
(7) $b(\mathcal{T})$ the maximum of bit size of any (integer) coefficient of each of the polynomials in $\Psi(\mathcal{T})$

Remark. The function $b(\mathcal{T})$ is used only for the estimation of the total complexity of the algorithm. In particular it has no relevance for the estimates of the number of blow-ups, the maximal embedding dimension $n(\mathcal{T})$, or the number of neighborhoods.

Algorithmically the input is represented by the coefficients of polynomials describing an affine marked ideal $\mathcal{T}_0$. We assume

$$m(\mathcal{T}_0) = m, \quad d(\mathcal{T}_0) \leq d_0, \quad n(\mathcal{T}_0) \leq n_0, \quad l(\mathcal{T}_0) \leq l_0, \quad b(\mathcal{T}_0) \leq b_0.$$

Then in particular, the total bit-size of the input does not exceed $b_0 \cdot l_0 \cdot d_0^{O(n_0)}$, cf. [26].

4.1.2. *Resolution of singularities.* For simplicity consider an irreducible affine variety $Y \subset \mathbb{C}^n$ described by some equations. The algorithm resolves Y by the following procedure:

Step A Find the generators of $\mathcal{I}_Y = \langle g_1, \ldots, g_{\overline{j}} \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ and construct affine marked ideal

$$\mathcal{T} := (X = \mathbb{C}^n, \mathcal{I}_Y, E = \emptyset, U = \mathbb{C}^n, \mathbb{C}^n, \mu = 1)$$

Step B Start the resolving procedure for the affine marked ideal $(\{X = \mathbb{C}^n, \mathcal{I}_Y, E = \emptyset, U = \mathbb{C}^n\}, \mathbb{C}^n, \mu = 1)$. (see below)

Step C Pick a nonsingular point $p \in Y \subset \mathbb{C}^n$. Stop the resolution procedure when the constructed center of the following blow-up in the algorithm passes through the inverse image of p. As an output of the resolution algorithm we get an affine marked ideal

$$\mathcal{T}' := (\{X_{\alpha',\beta'}, \mathcal{I}_{\alpha',\beta'}, E_{\alpha',\beta'}, U_{\alpha,\beta}, \mathbb{C}^n_{\alpha'}\}_{\alpha',\beta'}, 1)$$

over $\mathcal{T}$. In particular we have a collection of projections

$$\pi_{\alpha'} : (\mathbb{C}^{n_{\alpha'}})_{\alpha'} \to \mathbb{C}^n$$

(projection on the first n coordinates.). (Note that the restriction of π_α defines a birational morphism $\pi_{\alpha',\beta'} : X_{\alpha',\beta'} \to X$ which is an isomorphism in a neighborhood of $p \in X$.)

The center of the following blow-up is described on some open subcover $\{U_{\alpha',\beta''}\}_{\alpha' \in A', \beta'' \in B''_{\alpha'}}$ of $\{U_{\alpha',\beta'}\}_{\alpha' \in A', \beta' \in B'_{\alpha'}}$ by $C_{\alpha',\beta''} \cap U_{\alpha',\beta''}$ for the closed subsets $C_{\alpha',\beta''} \subset (\mathbb{C}^{n_{\alpha'}})_{\alpha'}$. Consider the unique irreducible component $\tilde{C}_{\alpha',\beta''}$ of $C_{\alpha',\beta''}$ containing the inverse image of the point p. Then

$$\overline{\pi}_{\alpha',\beta''} := \pi_{|\tilde{C}_{\alpha',\beta''} \cap U_{\alpha',\beta''}} : \tilde{Y}_{\alpha',\beta''} := \tilde{C}_{\alpha',\beta''} \cap U_{\alpha',\beta''} \to Y$$

is a local resolution of Y. The resolution space $\tilde{Y}$ is described by an open cover $\{\tilde{Y}_{\alpha',\beta''}, \overline{\pi}_{\alpha',\beta''}\}_{\alpha',\beta''}$. The sets $\tilde{Y}_{\alpha',\beta''}$ are represented as closed subsets of open subset $U_{\alpha',\beta''} \subset \mathbb{C}^{n_{\alpha'}}$.

4.1.3. *Principalization.* Given a smooth affine variety $X \subset \mathbb{C}^n$, described by equations $u_{X,i} = 0$, and an ideal $\mathcal{I} = (g_1, \ldots, g_k)$ on X and on $\mathbb{C}^n$.

Step A First the algorithm finds affine neighbourhoods $U_{\alpha,\beta}$ in $\mathbb{C}^n$ each given by an inequality $f_{\alpha,\beta} \neq 0$ in which X is represented by a family of local parameters

$$u_1 = \cdots = u_{n-m} = 0.$$

Moreover it finds the coordinates x_i on $\mathbb{C}^n$, such that (up to index permutation)

$$u_1, \cdots, u_{n-m}, x_{n-m+1}, \ldots, x_n$$

is a complete set of parameters.

The local parameters u_i are chosen among the input polynomials. To this end the algorithm can for each choice of $\{i_1, \ldots, i_{n-m}\} \subset \{1, \ldots, \overline{i}\}$ pick a non-vanishing identically minor $f_{\alpha,\beta}$ of the Jacobian matrix of $\{u_i\}_i$.

We construct an affine marked ideal given by input tuple by

$$\mathcal{T} := (\{X_\beta := X, \mathcal{I}_\beta := \mathcal{I}, E_\beta = \emptyset, U_\beta, \mathbb{C}^n\}_\beta, \mu = 1)$$

Step B The algorithm resolves $\mathcal{T} = (\{X_\beta := X, \mathcal{I}_\beta := \mathcal{I}, E_\beta = \emptyset, U_\beta, \mathbb{C}^n\}_\beta, \mu = 1)$. As an output we get

$$\mathcal{T}' := (\{X_{\alpha',\beta'}, \mathcal{I}_{\alpha',\beta'}, E_{\alpha',\beta'}, U_{\alpha',\beta'}, (\mathbb{C}^{n_{\alpha'}})_{\alpha'}\}_{\alpha',\beta'}, 1),$$

over $\mathcal{T}$.

Step C The variety $X' := X_{\mathcal{T}'}$ is described by an open cover $\{X_{\alpha',\beta'} \cap U_{\alpha',\beta'}\}_{\alpha',\beta'}$ for closed subsets $X_{\alpha',\beta'} \subset \mathbb{C}^{n_{\alpha'}}$, and open subsets $U_{\alpha',\beta'} \subset \mathbb{C}^{n_{\alpha'}}$ (see (8) from 4.1.1). Moreover, we have a collection of birational morphisms $\pi_{\alpha',\beta'} : X_{\alpha',\beta'} \cap U_{\alpha',\beta'} \to X \subset \mathbb{C}^n$. The principal ideal on $X_{\alpha',\beta'}$ is generated by

$$(g_1 \circ \pi_{\alpha',\beta'} \ldots, g_k \circ \pi_{\alpha',\beta'}) = (x_1^{a_1} \ldots x_{n_{\alpha'}}^{a_{n_{\alpha'}}})$$

4.2. **Description of blow-up.** Consider an affine marked ideal $\mathcal{T} := (\{X_{\alpha,\beta}, \mathcal{I}_{\alpha,\beta}, E_{\alpha,\beta}, U_{\alpha,\beta}, (\mathbb{C}^{n_\alpha})_\alpha \mid \alpha \in A, \beta \in B_\alpha\}, \mu)$ corresponding to a marked ideal $(X, \mathcal{I}, E, \mu)$. Let $C \subset X$ be a smooth center described as follows:

We assume that there is an open subcover cover $\{U_{\alpha,\beta'}\}_{\alpha\in A,\beta'\in B'_\alpha} \subset (\mathbb{C}^{n_\alpha})_\alpha \simeq \mathbb{C}^n$ of $U_{\alpha,\beta}$, together with map of indices $\rho : B'_\alpha \to B_\alpha$, and the collection of the closed subvarieties $C_{\alpha,\beta'} \subset (\mathbb{C}^{n_\alpha})_\alpha$ (of dimension $k_{\alpha\beta'} \le m$), such that

(1) $\bigcup_{\rho(\beta')=\beta} U_{\alpha,\beta'} = U_{\alpha,\beta}$
(2) $C_{\alpha,\beta'} \cap U_{\alpha,\beta'} \subset \mathrm{supp}(\mathcal{I}_{\alpha,\beta',\mu)} \cap U_{\alpha,\beta'}$
(3) $C_{\alpha,\beta'}$ is described on each $U_{\alpha,\beta'}$ by the set of local parameters

$$u_{\alpha,\beta',1}, \ldots, u_{\alpha,\beta',n_\alpha-m}, u_{\alpha\beta',n_\alpha-m+1}, \ldots, u_{\alpha,\beta',n_\alpha-k_{\alpha\beta'}} \in \mathbb{C}[x_1, \ldots, x_{n_\alpha}], \quad \text{i. e.}$$

$$u_{\alpha,\beta',1} = \ldots = u_{\alpha,\beta',n_\alpha-m} = u_{\alpha\beta',n_\alpha-m+1} = \ldots = u_{\alpha,\beta',n_\alpha-k_{\alpha\beta'}} = 0$$

where $X_{\alpha,\beta}$ is described on $U_{\alpha,\beta} \supset U_{\alpha,\beta'}$ by $u_{\alpha,\beta',1} = \ldots = u_{\alpha,\beta',n_\alpha-m} = 0$,
(4) $u_{\alpha,\beta',1}, \ldots, u_{\alpha,\beta',n_\alpha-k_{\alpha\beta'}}$ are transversal to the exceptional divisors (over $U_{\alpha,\beta'}$), or coincide with coordinate functions describing the exceptional divisors.

Denote by

$$\mathcal{T}' := (\{X_{\alpha',\beta'}, \mathcal{I}_{\alpha',\beta'}, E_{\alpha',\beta'}, U_{\alpha',\beta'}, (\mathbb{C}^{n_{\alpha'}})_{\alpha'} \mid \alpha' \in A', \beta' \in B'_{\alpha'}\}, \mu)$$

the resulting affine marked ideal obtained from $\mathcal{T}$ by the blow-up with the center C. Below we describe more precisely the ingredients of $\mathcal{T}'$.

The open cover after blow-up.

The blow-up creates a new collection of ambient affine spaces $(\mathbb{C}^{n_{\alpha'}})_{\alpha'}$. Namely, we can associate with functions $u_{\alpha,\beta',i}$ on $(\mathbb{C}^{n_\alpha})_\alpha$, where $i = 1, \ldots, n_\alpha - k_{\alpha\beta'}$, the $n_\alpha - k_{\alpha\beta'}$ affine charts

$$(\mathbb{C}^{n_{\alpha'}})_{\alpha'}, \quad \text{where} \quad \alpha' := (\alpha, i), \quad i = 1, \ldots, n_\alpha - k_{\alpha\beta'}, \quad n_{\alpha'} := 2n_\alpha - k_{\alpha\beta'}$$

We also create new collection of open subsets $U_{\alpha',\beta'} \subset (\mathbb{C}^{n_{\alpha'}})_{\alpha'}$ by taking the inverse images of $U_{\alpha,\beta'} \subset (\mathbb{C}^{n_\alpha})_\alpha$ under the morphisms $(\mathbb{C}^{n_{\alpha'}})_{\alpha'} \to (\mathbb{C}^{n_\alpha})_\alpha$.

The birational maps

The natural projection $\pi_\alpha : (\mathbb{C}^{n_{\alpha'}})_{\alpha'} \to (\mathbb{C}^{n_\alpha})_\alpha$ on the first n_α components defines the birational morphism $\pi_{\alpha',\beta'} = \pi_{\alpha|X_{\alpha',\beta'}} : X_{\alpha',\beta'} \to X_{\alpha,\beta'}$, for any α, β', such that $X_{\alpha,\beta'} \neq \emptyset$. This defines birational morphisms

$$i_{\alpha'_1\beta'_1,\alpha'_2,\beta'_2} : X_{\alpha'_1,\beta'_1} \stackrel{\pi_{\alpha'_1,\beta'_1}}{\to} X_{\alpha_1,\beta'_1} \stackrel{i_{\alpha_1\beta'_1,\alpha'_2,\beta_2}}{\dashrightarrow} X_{\alpha_2,\beta'_2} \stackrel{\pi^{-1}_{\alpha'_2,\beta'_2}}{\leftarrow} X_{\alpha'_2,\beta'_2}.$$

Consider a blow-up $X_{\mathcal{T}'}$ of $C \subset X_{\mathcal{T}}$. There exist open embeddings $j_{\alpha',\beta'} : X_{\alpha',\beta'} \cap U_{\alpha',\beta'} \hookrightarrow X_{\mathcal{T}'}$ induced by $j_{\alpha,\beta} : X_{\alpha,\beta'} \cap U_{\alpha,\beta'} \hookrightarrow X_{\mathcal{T}}$, defining an open cover of $X_{\mathcal{T}'}$, and satisfying

$$j^{-1}_{\alpha'_2,\beta'_2} \circ j_{\alpha'_1,\beta'_1} = i_{\alpha'_1\beta'_1,\alpha'_2,\beta'_2}$$

Equations of blow-up.

Without loss of generality the blow-up in each of $n_\alpha - k_{\alpha\beta'}$ affine charts $(\mathbb{C}^{n_{\alpha'}})_{\alpha'}$, where $\alpha' := (\alpha, i)$, $\quad i = 1, \ldots, n_\alpha - k_{\alpha\beta'}$, can be described as follows: (For simplicity drop α, β indices below.)

Assume that the function u_{i_0}, $i_0 \le n - k$, defines the chart of the blow-up. The blow-up of $\mathbb{C}^n$ is a closed subset $bl(\mathbb{C}^n)$ of $\mathbb{C}^{2n-k}$ described by the following equations

$$u_j - u_{i_0} x_{j+n} = 0 \quad \text{for} \quad 0 < j \le n-k, j \neq i_0$$
$$u_{i_0} - x_{i_0+n} = 0$$

The exceptional divisors.

The exceptional divisor for this blow-up is given by $u_{i_0} = 0$ on $bl(\mathbb{C}^n)$. Since $u_{i_0} = x_{i_0+n}$ we may represent it by the coordinate x_{i_0+n} on $\mathbb{C}^{2n-k}$.

The previous exceptional divisors keep their form $x_j = 0$ if they do not describe C, or they convert to $x_{j+n}(= u_j/u_{i_0})$ if they were described by the function $u_j \equiv x_j$.

The strict transform of X.

Recall that X is described by $u_1 = \ldots = u_{n-m} = 0$ on $U \subset \mathbb{C}^n$. The blow-up of $X = X_{\alpha,\beta}$ is a closed subset $X' \subset \mathbb{C}^{2n-k}$ which is described by a new set of equations:

(1) $u_j - u_{i_0} x_{j+n} = 0 \quad \text{for} \quad 0 < j \leq n-k, \quad j \neq i_0$
(2) $u_{i_0} - x_{i_0+n} = 0$
(3) $x_{j+n} = 0 \quad \text{for} \quad 0 < j \leq n-m, \quad j \neq i_0.$
In some situations we consider additionally the induced equation
(4) $1 = 0 \quad \text{if} \quad 0 < i_0 \leq n-m$

(Note that the equations of the first two types describe the blow-up $bl(\mathbb{C}^n)$ of $\mathbb{C}^n$. The third and the fourth types of the equations $x_{j+n} = u_j/u_{i_0} = 0$, $j \neq i_0$ (or $1 = u_{i_0}/u_{i_0} = 0$) describe the strict transform of X inside $bl(\mathbb{C}^n)$. In the latter case if $0 < j = i_0 \leq n-m$ the strict transform is an empty set in the relevant chart. Still we shall keep the uniform description of the objects and their transformations, and do not eliminate any equations in the description of the empty set.)

4.2.1. *The generators of $\mathcal{I}_{\alpha,\beta}$ after blow-up.* We will not compute the controlled transforms of the generators of $\mathcal{I}$ $(= \mathcal{I}_{\alpha,\beta})$ (over U) directly. Instead we modify them first. The generators g_i of $\mathcal{I}$ satisfy, by Lemma 2.2.1, the condition

$$g_i \cdot f^{r_i} \in \mathcal{I}_C^{\mu} + \mathcal{I}_X,$$

where $V(f) = \mathbb{C}^n \setminus U \subset \mathbb{C}^n$ (we dropped indices α, β here).

For any generator g_i write

$$g_i \cdot f^{r_i} = \sum_{a_{n-m+1}+\ldots+a_{n-k}=\mu} h_{(a_{n-m+1},\ldots,a_{n-k}),i} u_{n-m+1}^{a_{n-m+1}} \cdot \ldots \cdot u_{n-k}^{a_{n-k}} + \sum_{j:=1\ldots,m-n} h_{ij} u_j$$

Set $\bar{a} := (a_{n-m+1}, \ldots, a_{n-k}), \bar{u}^{\bar{a}} := u_{n-m+1}^{a_{n-m+1}} \cdot \ldots \cdot u_{n-k}^{a_{n-k}}$. Then we can rewrite above as

$$g_i \cdot f^{r_i} = \sum_{|\bar{a}|=\mu} h_{\bar{a}} \bar{u}^{\bar{a}} + \sum_{j:=1\ldots,m-n} h_{ij} u_j$$

To bound r_i and $deg(h_{\bar{a},i})$, $deg(h_{ij})$ we first consider a similar equality

$$g_i \cdot f^{R_i} = \sum_{|\bar{a}|=\mu} H_{\bar{a}} \bar{u}^{\bar{a}} + \sum_{j:=1\ldots,m-n} H_{ij} u_j$$

for certain R_i, $H_{\bar{a},i}$, H_{ij}. We introduce a new variable z and we get an equality

$$g_i = z^{R_i} \cdot (\sum_{|\bar{a}|=\mu} H_{\bar{a}} \bar{u}^{\bar{a}} + \sum_{j:=1\ldots,m-n} H_{ij} u_j) + g_i \cdot (\sum_{0 \leq j \leq r-1} (z \cdot f)^j) \cdot (1 - z \cdot f),$$

in other words g_i belongs to the ideal generated by $\{\bar{u}^{\bar{a}}\}$, $\{u_j\}$, $1 - z \cdot f$. Therefore one can represent

$$g_i = \sum_{|\bar{a}|=\mu} \widetilde{H}_{\bar{a},i} \cdot \bar{u}^{\bar{a}} + \sum_{j:=1\ldots,m-n} \widetilde{H}_{ij} \cdot u_j + \widetilde{H} \cdot (1 - z \cdot f)$$

for suitable polynomials $\widetilde{H}_{\bar{a},i}$, $\widetilde{H}_{ij}$, $\widetilde{H}$ with degrees less than $(d \cdot \mu)^{2^{O(n)}}$ due to [43], [24], [40]. Hence substituting in the latter equality $z = 1/f$ and cleaning the denominator we obtain the bound $(d \cdot \mu)^{2^{O(n)}}$ on r_i, $deg(h_{\bar{a},i})$, $deg(h_{ij})$.

The generators after blow-up and their degree.

Using the above we can describe the controlled transform of $\mathcal{I}$ in terms of controlled transforms of its modified generators. Define the modified generators of $\mathcal{I}$ to be

$$\bar{g}_i = \sum_{|\bar{a}|=\mu} h_{\bar{a}} \bar{u}^{\bar{a}}$$

Then their controlled transforms are given by

$$\sigma^c(\bar{g}_i) = u_{n-k}^{-\mu} \sigma^*(h_{\bar{a},i} u^{\bar{a}})$$

Denote by $G(d,n,\mu)$ the bound on the degree of the resulting marked ideal $\mathcal{T}'$ after a blow up applied to a marked ideal $\mathcal{T}$, provided that $d(\mathcal{T}) \leq d$, $n(\mathcal{T}) \leq n$. Thus, by the above:

Lemma 4.2.1. $G(n,d,\mu) < (d\cdot\mu)^{2^{O(n)}}$.

4.3. **Elementary operations and elementary auxillary functions.** To estimate the complexity of the desingularization algorithm we introduce few auxiliary functions related to the ingredients of $\mathcal{T}$. It is convenient to associate with $\mathcal{T}$ with data (m,d,n,l,q,μ) the vector

$$\gamma := (r,m,d,n,l,q,\mu) \in \mathbf{Z}^7_{\geq 0},$$

where r is the subscript of the element of the resolution $(\mathcal{T}_r)_{r=0,1,\ldots}$,

4.3.1. *The effect of a single blow-up.* Summarizing the above we get the following

Lemma 4.3.1. *Consider the object* $\mathcal{T} := (\{X_{\alpha,\beta}, \mathcal{I}_{\alpha,\beta}, E_{\alpha,\beta}, U_{\alpha,\beta}\}_{\alpha,\beta}, n, \mu)$ *with data* (m,μ,d,n,l,q). *Let* $\mathcal{T}' := (\{X'_{\alpha',\beta'}, \mathcal{I}'_{\alpha',\beta'}, E'_{\alpha',\beta'}, U'_{\alpha',\beta'}\}_{\alpha',\beta'}, n_{\alpha'}, \mu)$ *denote the object with data* (m,μ,d',n',l',q') *obtained from* $\mathcal{T}$ *by a single blow-up at the center* C *represented by the collection of closed sets* $\{C_{\alpha\beta''} \subset (\mathbb{C}^{n_\alpha})_\alpha\}$ *describing the center in open subsets* $U_{\alpha\beta''} \subset U_{\alpha\beta}$. *Assume that the maximal degree of of the polynomials describing center is less than* d. *Assume that* q *gives also a bound for the number of open neighborhoods* $U_{\alpha\beta''}$. *Then*

(1) $d' \leq G(n,d,\mu) < (d\cdot\mu)^{2^{O(n)}}$.
(2) $n' \leq 2n$
(3) $l' \leq l+n$
(4) $q' \leq n\cdot q$

The effect of the single blow-up can be measured by the function.

$$Bl(r,m,d,n,l,q,\mu) := (r+1,m,G(n,d,\mu),2n,l+n,n\cdot q,\mu)$$

The multiple effect of the t blow-ups can be measured by the recursive function.

By

$$\overline{Bl}(r,m,d,n,l,q,\mu,t) = Bl\circ\overline{Bl}(r,m,d,n,l,q,\mu,t-1)$$

or shortly

$$\overline{Bl}(\gamma,t) = \overline{Bl}(Bl(\gamma,t-1))$$

Note that

$$\overline{Bl}(r,m,d,n,l,q,\mu,t) = (r+t,m,\overline{G}(n,d,\mu,t),2^t n,l+2^{t-1}n,(2^{t+1}-1)\cdot n^t q,\mu)$$

for the relevant function $\overline{G}(n,d,\mu,t)$.

5. Bounds on multiplicities and degrees of coefficient ideals

5.0.2. *The maximal multiplicity of* $\mathcal{I}$ *on the subvariety* $X\subset\mathbb{C}^n$. Let d be the maximal degree of $(X,\mathcal{I})$ on $\mathbb{C}^n$. Denote by $M(d,n)$ a bound on the multiplicities of ideals $\mathcal{I}$ on X. To verify a bound on $M(d,n)$ we may assume (after a linear transformation of the coordinates) that the order of the polynomial $u_{i_j}-x_j$ is at least 2 for all $1\leq j\leq n-m$. For any polynomial $g\in\mathcal{I}_{\alpha,\beta}$ one can find polynomials $h_j\in\mathbb{C}[x_1,\ldots,x_n]$, $0\leq j\leq n-m$ and $h\in\mathbb{C}[x_{n-m+1},\ldots,x_n]$ such that

$$h_0\cdot g + \sum_{1\leq j\leq n-m} h_j\cdot u_{i_j} \ = \ h(x_{n-m+1},\ldots,x_n). \tag{5}$$

Moreover, rewriting the latter equality over the field $\mathbb{C}(x_{n-m+1},\ldots,x_n)$ in the form

$$\tilde{h}_0\cdot g + \sum_{1\leq j\leq n-m} \tilde{h}_j\cdot u_{i_j} \ = \ 1, \tag{6}$$

where $\tilde{h}_j = \frac{h_i}{h} \in \mathbb{C}(x_{n-m+1},\ldots,x_n)[x_1,\ldots,x_{n-m}]$, with the common denominator in $\mathbb{C}[x_{n-m+1},\ldots,x_n]$ for $0\leq j\leq n-m$.

We apply to (6) the Effective Nullstellensatz (see e.g. [15], [23], [36], [34]). This gives us the bound $d^{O(n)}$ on the degrees of $\tilde{h_i} = h_i/h$ with respect to variables $x_1, \ldots, x_{n-m}$ (for certain solutions) . To find h_i/h one can solve the latter equality considering it as a linear system over the field $\mathbb{C}(x_{n-m+1}, \ldots, x_n)$.

The algorithm can find

$$\tilde{h_j} = \sum a_{I,j} x^I$$

with indeterminates $a_{I,j} \in \mathbb{C}(x_{n-m+1}, \ldots, x_n)$, and monomials $x^I = x_1^{i_1} \cdot \ldots \cdot x_{n-m}^{i_{n-m}}$ with degrees $i_1 + \ldots + i_{n-m} \leq d^{O(n)}$ substituting $\tilde{h_j}$ in (6) and solving linear system over the field $\mathbb{C}(x_{n-m+1}, \ldots, x_n)$. Clearing the common denominator in (6) gives (5) with

$$deg(h), deg(h_j) \leq d^{O(n^2)}, \quad 1 \leq j \leq n-m$$

Now we have to estimate the maximal multiplicity $\mathrm{ord}_x(g_{|X})$ for $g \in \mathbb{C}[x_1, \ldots, x_n]$, such that $deg(g) \leq d$ and $x \in X$. We use (5).

We get immediately by above

Lemma 5.0.2. *The maximal multiplicity* $\mathrm{ord}_x(g_{|X})$ *on* X *for any function* $g \in \mathbb{C}[x_1, \ldots, x_n]$*, such that* $deg(g) \leq d$ *and* $x \in X$ *is bounded by the function* $M(d,n) = d^{O(n^2)}$ *constructed as above.*

$$\mathrm{ord}_x(g_{|X}) \leq M(d,n)$$

Proof.

$$\mathrm{ord}_x(g_{|X}) \leq \mathrm{ord}_x(h_0 \cdot g)_{|X} = \mathrm{ord}_x h(x_{n-m+1}, \ldots, x_n)_{|X} = \mathrm{ord}_x h(x_{n-m+1}, \ldots, x_n) \leq deg(h) \leq M(d,n) = d^{O(n^2)}.$$

□

5.0.3. **Derivations on the subvariety $X \subset \mathbb{C}^n$.** In order to follow the construction of the algorithm we use the language of derivations Der_X on X. Since our X is embedded into $\mathbb{C}^n$ it is natural to represent all objects on X as the restriction of the relevant objects on $\mathbb{C}^n$ to $X \subset \mathbb{C}^n$. Unfortunately the sheaf of derivations on $\mathbb{C}^n$ does not restrict well to X:

$$\mathrm{Der}_{\mathbb{C}^n|X} \neq Der_X.$$

Instead we consider

$$\mathrm{Der}_{\mathbb{C}^n,X} := \{D \in \mathrm{Der}_{\mathbb{C}^n} \mid D(\mathcal{I}_X) \subset (\mathcal{I}_X)\}$$

Lemma 5.0.3. *Let* $u_1 = 0 \ldots, u_{n-m} = 0$ *be describe* $X \subset \mathbf{C^n}$*. Then the ring* $\mathrm{Der}_{\mathbb{C}^n,X}$ *is generated by*

$$\{u_i \cdot d_{u_j}\}_{1 \leq i \leq n-m,\, 1 \leq j \leq n-m} \cup \{d_{u_j}\}_{n-m < j \leq n}.$$

In particular the restriction $\mathrm{Der}_{\mathbb{C}^n,X|X} = Der_X$ *is generated by*

$$\{d_{u_j}\}_{n-m < j \leq n}.$$

Proof. Follows form the definition. □

Note that since we have

$$(d_{u_j})_j = ((\partial u_j/\partial x_i)_{i,j})^{-1} \cdot (d_{x_i})_i = \frac{1}{det((\partial u_j/\partial x_i)_{i,j})} adj((\partial u_j/\partial x_i)_{i,j}) \cdot (d_{x_i})_i$$

we get immediately

Lemma 5.0.4. *Let* $\mathcal{U} := \{x \in \mathbb{C}^n \mid det((\partial u_j/\partial x_i)_{i,j}) \neq 0\}$.

The sheaf $\mathrm{Der}_{\mathbb{C}^n,X}$ *is generated over* $U \subset \mathbb{C}^n$ *by*

$$\{u_i \cdot d'_{u_j}\}_{1 \leq i \leq n-m,\, 1 \leq j \leq n-m} \cup \{d'_{u_j}\}_{n-m < j \leq n}, \tag{7}$$

where

$$(d'_{u_j})_j = adj((\partial u_j/\partial x_i)_{i,j}) \cdot (d_{x_i})_i$$

The derivations (7) generate a subsheaf $\overline{\mathrm{Der}_{\mathbb{C}^n,X}}$ of $\mathrm{Der}_{\mathbb{C}^n,X}$ over $\mathbb{C}^n$. Both sheaves coincide over U. Thus we shall replace $\mathrm{Der}_{\mathbb{C}^n,X}$ with $\overline{\mathrm{Der}_{\mathbb{C}^n,X}}$ for our computations over U.

Lemma 5.0.5. *Let $\mathcal{I}$ be any ideal on $\mathbf{C}^n$. Assume the maximal degree of some generating set of $\mathcal{I}$ is $\leq d_1$, and the maximal degree of u_i is less than d_2 then the maximal degree of generators of $\overline{\mathrm{Der}_{\mathbb{C}^n,X}}(\mathcal{I})$ is bounded by $d_1 + nd_2$.*

5.0.4. *Construction of the coefficient homogenized companion ideal.* Recall then in the step 2 for the marked ideal $(\mathcal{I}, \mu)$, we find the maximal multiplicity $\bar{\mu} \leq M(d, n)$ and construct companion ideal $\mathcal{O}(\mathcal{I})$, for which immediately we take homogenized coefficient ideal $\mathcal{J} := \mathcal{C}(\mathcal{H}(\mathcal{O}(\mathcal{I}))))$. In our situation of the set $X \subset \mathbb{C}^n$ defined by set of parameters $\{u_i\}_{1\leq i\leq n-m}$ on the open set $U \subset \mathbb{C}^n$ we shall use $\overline{\mathrm{Der}_{\mathbb{C}^n,X}}$ instead of Der_X for the above devinition. Denote the relevant operations of homogenization and coefficient ideal by $\bar{\mathcal{H}}()$, and $\bar{\mathcal{C}}()$. Immediately from the definition we get the formula for a bound $A(d, n, \mu)$ on the degrees of generators of the marked ideal $\mathcal{C}(\mathcal{H}(\mathcal{I}_q))$. Note, first,that we have the the following bounds on the multiplicities:

Lemma 5.0.6. $\mu(\mathcal{N}(\mathcal{I})) = \bar{\mu}$, $\quad \mu(\mathcal{O}(\mathcal{I})) \leq \mu \cdot \bar{\mu}$ $\quad$ *and* $\mu(\mathcal{J}) \leq (\mu \cdot \bar{\mu})! \leq (\mu \cdot M(d, n))!$

As a Corollary we obtain

Lemma 5.0.7. *The maximal degree of generators of*

$\mathcal{J} := \bar{\mathcal{C}}(\bar{\mathcal{H}}(\mathcal{O}(\mathcal{I}))))$ *is bounded by* $A(d, n, \mu) := (\mu \cdot \bar{\mu})!nd \leq (\mu \cdot M(d, n))!nd \leq (d^{O(n^2)})!$.

Proof. Follows from Lemma 5.0.5 □

5.0.5. **Restriction to hypersurface of maximal contacts, and to exceptional divisors.** In the step 1 we restrict $\mathcal{I}$ to intersections of the exceptional divisors and maximal contact.

We need to estimate a bound $B(d, n, \mu)$ for the degree of the maximal contact $u \in \overline{\mathrm{Der}_{\mathbb{C}^n,X}}^{\bar{\mu}-1}(\mathcal{N}(\mathcal{I}))$. It follows immediately form Lemma 5.0.5 that

Lemma 5.0.8. *The maximal degree of any maximal contact $u \in \overline{\mathrm{Der}_{\mathbb{C}^n,X}}^{\bar{\mu}-1}(\mathcal{N}(\mathcal{I}))$ is bounded by*

$$B(d, n, \mu) := \bar{\mu}nd \leq M(d, n) \cdot nd \leq d^{O(n^2)}.$$

5.0.6. *The bound for the number of generators of J.* First, state the basic properties for the number of generators of the ideal in the lemma:

Lemma 5.0.9.
(1) *The number of generators of $\overline{\mathrm{Der}_{\mathbb{C}^n,X}}(\mathcal{I})$ is given by $(n + 1)l(\mathcal{I})$.*
(2) *The number of generators of $\overline{\mathrm{Der}_{\mathbb{C}^n,X}}^i(\mathcal{I})$ is given $(n + 1)^i l(\mathcal{I})$*
(3) *The number of generators of $\mathcal{I}^i$ is bounded by $l(\mathcal{I})^i$*
(4) *The number of generators of $\mathcal{I}' = \mathcal{O}(\mathcal{I})$ can be bounded by $L_{\mathcal{O}}(l(\mathcal{I}), \mu) := l(\mathcal{I})^\mu + 1$.*
(5) *The number of generators of $\mathcal{I}' = \mathcal{H}(\mathcal{I}, \mu)$ can be bounded by $L_{\mathcal{H}}(l(\mathcal{I}), \mu) := \mu(n + 1)^{\mu^2} l(\mathcal{I})^\mu$.*
(6) *The number of generators of $\mathcal{I}' = \mathcal{C}(\mathcal{I}, \mu)$ can be bounded by $L_{\mathcal{C}}(l(\mathcal{I}), \mu)) := \mu(n + 1)^{\mu!} l(\mathcal{I})^{\mu!}$.*

Proof. Immediately from the definition. □

By the Lemma we get

$$l(\mathcal{H}(\mathcal{O}(\mathcal{I}))) \leq L_{\mathcal{H}}(L_{\mathcal{O}}(l(\mathcal{I}), \mu), \mu \cdot M(d, n))$$
$$l(\mathcal{J}) = l(\mathcal{C}(\mathcal{H}(\mathcal{O}(\mathcal{I})))) \leq L_{\mathcal{C}}(L_{\mathcal{H}}(L_{\mathcal{O}}(l(\mathcal{I}), \mu), \mu \cdot M(d, n))), \mu \cdot M(d, n))$$

Thus we get

Corollary 5.0.10. $l(\mathcal{J}) \leq F(d, n, \mu)$, *where*

$$F(d, n, \mu) := L_{\mathcal{C}}(L_{\mathcal{H}}(L_{\mathcal{O}}(l(\mathcal{I}), \mu), \mu \cdot M(d, n))), \mu \cdot M(d, n))$$

Remark. The algorithm of [12] does not involve the homogenization step and therefore gives better estimates for the introduced elementary functions. In particular,

-The degree of generators of $\mathcal{J}$ is bounded by $B(d, n, \mu)$ (which improves the bound $A(d, n, \mu)$, cf. Lemma 5.0.7),

- The number of generators $l(\mathcal{J})$ can be bounded by $L_{\mathcal{C}}(L_{\mathcal{O}}(l(\mathcal{I}), \mu), \mu \cdot M(d, n))$ (which improves the bound $F(d, n, \mu)$, cf. Corollary 5.0.10),

However, the above improvements do not affect overall Grzegorczyk complexity class $\mathcal{E}^{m+3}$. (See Theorem 6.4.2.)

Summarizing,

Lemma 5.0.11. *The effect of passing from $\mathcal{I}$ to $\mathcal{J} = (\mathcal{C}(\mathcal{H}(\mathcal{O}(\mathcal{I}))))$ as in the Step 2a/Step1 can be described by the function*

$$\Delta_{2a}(r, m, d, n, l, q, \mu) := (r, m, A(d, n, \mu), n, F(d, n, \mu, l), q, (\mu \cdot M(d, n))!)$$

$\square$

5.0.7. *The bound for the number of maximal contacts and the relevant neighborhoods.* We shall construct maximal contacts along with the open neighborhood for which it is defined. Each maximal contact we consider $u \in \overline{\mathrm{Der}_{\mathbb{C}^n,X}}^{\bar{\mu}-1}(\mathcal{N}(\mathcal{I}))$ is of the form $u = D^{\bar{\mu}-1}(g_i)$, where $D^{\bar{\mu}-1} = D_1^{a_1} \dots D_n^{a_n}$ is a certain composition of $\bar{\mu} - 1$ differential operators (7). (i.e. D_i are of the from as in (7), and $a_1 + \dots + a_n = \bar{\mu} - 1$.

Consider all differential operators $\{D_r^{\bar{\mu}}\}_{r \in R}$ which are certain compositions of $\bar{\mu}$ differential operators (7) and take all the corresponding functions $f_{r,i} := D_r^{\bar{\mu}}(g_i)$. On the open set $U_{r,i} = U \setminus V(f_{r,i})$ consider the maximal contact $u_{r,i} = D^{\bar{\mu}-1}(g_i)$, where $D^{\bar{\mu}-1} = D_1^{a_1} \dots D_n^{a_n}$ is a certain composition of $\bar{\mu} - 1$ differential operators (7) obtained from $D_r^{\bar{\mu}} = D_1^{b_1} \dots D_n^{b_n}$ by replacing one of the positive b_i with $b_i - 1$ (i.e $a_i := b_i - 1$ for some $b_i > 0$ and $a_j = b_j$ for $j \neq i$.)

Lemma 5.0.12. *The number of the maximal contacts $u_{r,i} \in \overline{\mathrm{Der}_{\mathbb{C}^n,X}}^{\bar{\mu}-1}(\mathcal{N}(\mathcal{I}))$ and at the same time the number of neighborhoods $U_{i,r} \subset U$ can be bounded by*

$$C(d, n, \mu) \cdot l(\mathcal{I}),$$

where $l(\mathcal{I})$ is the number of generators of $\mathcal{I}$, and

$$C(d, n, \mu) := \binom{M(d, n) + n}{n}$$

Proof. The number of the maximal contacts is bounded by $\binom{\bar{\mu}+n}{n} \cdot l(\mathcal{I}) \leq \quad C(d, n, \mu) := \binom{M(d,n)+n}{n}$ $\square$

Summarizing,

Lemma 5.0.13. *The effect of passing from $\mathcal{I}$ to $\mathcal{J} = (\mathcal{C}(\mathcal{H}(\mathcal{O}(\mathcal{I}))))$ and then to $\mathcal{J}_{|H_s}$ in Step 1a or to $\mathcal{J}_{|V(u)}$ as in the Step 1b or can be described by the function*

$$\Delta_1(r, m, d, n, l, q, \mu) := (r, m - 1, A(d, n, \mu), n, F(d, n, \mu, l), q \cdot C(d, n, \mu), (\mu \cdot M(d, n))!)$$

$\square$

Note that the restriction to the maximal contact does not affect the degree since the function $B(d, n, \mu)$ measuring the degree of maximal contact is smaller than $A(d, n, \mu)$

Remark. A particular form of the obtained bounds does not influence much on the Theorem 6.4.2: we need only that the functions belong to class $\mathcal{E}^3$. (See the beginning of the next section.)

6. Complexity bound of the resolution algorithm in terms of Grzegorczyk's classes

6.1. **Language of Grzegorczyk's classes.** The complexity estimate of the desingularization algorithm which we provide in this Section is given in terms of the Grzegorczyk's classes $\mathcal{E}^l$, $l \geq 0$ [27], [50] of primitive-recursive functions. For the sake of self-containdness we provide the definition of $\mathcal{E}^l$ by induction on l (informally speaking, $\mathcal{E}^l$ consists of integer functions $\mathbb{Z}^s \to \mathbb{Z}^t$ whose construction requires l nested primitive recursions).

For the base of definition $\mathcal{E}^0$ contains constant functions $x_k \mapsto c$, functions $x_k \mapsto x_k + c$ and projections $(x_1, \dots, x_n) \mapsto x_k$ for any variables $x_1, \dots, x_n$.

Class $\mathcal{E}^1$ contains linear functions $x_k \mapsto c \cdot x_k$ and $(x_{k_1}, x_{k_2}) \mapsto x_{k_1} + x_{k_2}$.

For the inductive step of the definition assume that functions $G(x_1, \dots, x_n), H(x_1, \dots, x_n, y, z) \in \mathcal{E}^l$. Then function $F(x_1, \dots, x_n, y)$ defined by the following primitive recursion

$$F(x_1, \dots, x_n, 0) = G(x_1, \dots, x_n) \tag{8}$$

$$F(x_1, \dots, x_n, y + 1) = H(x_1, \dots, x_n, y, F(x_1, \dots, x_n, y)) \tag{9}$$

belongs to $\mathcal{E}^{l+1}$.

To complete the definition of $\mathcal{E}^l$, $l \geq 0$ take the closure with respect to the composition and the following *limited primitive recursion*:

let $G(x_1,\dots,x_n), H(x_1,\dots,x_n,y,z), Q(x_1,\dots,x_n,y) \in \mathcal{E}^l$, then function $F(x_1,\dots,x_n,y)$ defined by (8),(9) also belongs to $\mathcal{E}^l$, *provided that* $F(x_1,\dots,x_n,y) \leq Q(x_1,\dots,x_n,y)$.

Clearly, $\mathcal{E}^{l+1} \supset \mathcal{E}^l$.

Observe that $\mathcal{E}^2$ contains all the polynomials with integer coefficients.

$\mathcal{E}^3$ contains all the towers of exponential functions.

$\mathcal{E}^4$ contains all tetration functions [27], [50].

The union $\cup_{l<\infty}\mathcal{E}^l$ coincides with the set of all primitive-recursive functions.

6.2. **Resolution algorithm as a graph.** It is instructive to represent the resolution algorithm in a form of a tree T as in the following Figure.

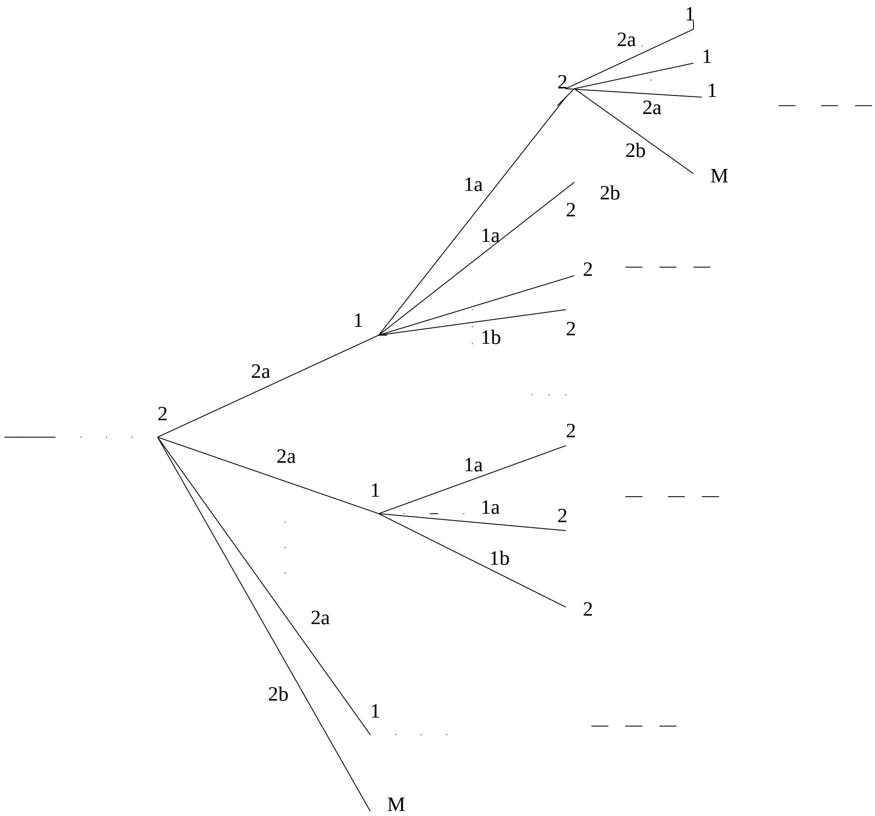

FIGURE 1.

Each node a of T corresponds to an intermediate object $T_a = (\{X_{\alpha,\beta}, \mathcal{I}_{\alpha,\beta}, E_{\alpha,\beta}, U_{\alpha,\beta}\}_{\alpha,\beta}, n, \mu)$. Each node a is labeled either by 1 or 2 depending on whether it corresponds to step 1 or 2 in the description of the algorithm (see the previous sections). An edge from a node labeled by 1 leads to its child node labeled by 2 and the edge is labeled in its turn either by 2a or by 2b depending on to which step it corresponds. Similarly, an edge from a node labeled by 2 leads to its child node labeled by 1 and is labeled in its turn either by 1a or by 1b. In the Figure a child node is always located to the right from a node.

The algorithm yields T by recursion starting with its root. Assume that a and T_a is already yielded. The next task of the algorithm is to resolve object T_a. To this end the algorithm first constructs the child nodes $a_1,\dots,a_{\overline{t}}$ of a according to the algorithm. The order of yielding $a_1,\dots,a_{\overline{t}}$ goes from up to down in the Figure. The algorithm resolves objects $T_{a_1},\dots,T_{a_{t-1}}$ by recursion on $0 \leq t \leq \overline{t}$ and in the process modifies $T_a := T_a(0)$ obtaining current object $T_a(t-1)$. Then the algorithm yields a_t, T_{a_t}, resolves T_{a_t} and collects all the blow ups produced while resolving T_{a_t} and applies them (with the same centers) to the current object $T_a(t-1)$, the resulting object we denote by $T_a(t)$. This allows the algorithm to yield a child node a_{t+1} and $T_{a_{t+1}}$ following the description from the previous sections.

For the leaves of T there are two possibilities: either a leaf is labeled by 2 or a certain node a labeled by 2 could have a single edge (the lowest in the Figure among the edges originating at a) labeled by 2b which leads to a child node of a being a leaf corresponding to the monomial case (labeled by M). Note also that if a is labeled by 1 then few top edges originating at a are labeled by 1a and the remaining bottom ones are labeled by 1b (in the order from up to down in the Figure).

Observe that the dimension of varieties $X_{\alpha,\beta}$ corresponding to a drops while passing to any of its child nodes when a is labeled by 1, and the dimension does not increase when a is labeled by 2. Therefore, the depth of T does not exceed $2m$.

6.3. **Main recursive functions.** Now we proceed to the bounds of some recursive functions related to the ingredients of $\mathcal{T}$. Set

$$\gamma := (r, m, d, n, l, q, \mu) \in \mathbf{Z}^7_{\geq 0}$$

Let $\mathcal{T}_*$ be the canonical resolution of $\mathcal{T}$. For simpilicity of notation introduce following function defined on $\mathbf{Z}^7_{\geq 0}$:

Let

$$\Gamma^{(m)}(\gamma) := (r + R^{(m)}(\gamma), m, D^{(m)}(\gamma), N^{(m)}(\gamma), L^{(m)}(\gamma), Q^{(m)}(\gamma), \mu).$$

(Here the subscript r can be interpreted as the subscript in the resolution of $\mathcal{T}$),

where

(1) $R^{(m)}(\gamma)$ the number of blow-ups needed to resolve the initial marked ideal with data bounded by (m, d, n, l, q, μ).
(2) $D^{(m)}(\gamma)$ be a function bounding the maximum of the degrees of all the polynomials which represent T_* and all objects constructed on the way, in particular the centers.
(3) $N^{(m)}(\gamma)$ as the bound for the dimensions of the ambient affine spaces constructed on the way, .
(4) $L^{(m)}(\gamma)$ the bound for number of polynomials appearing in description of a single neighborhood $U_{\alpha,\beta}$ upon resolving T.
(5) $Q^{(m)}(\gamma)$ the number of neighborhoods in all the auxiliary objects, in particular the centers. appearing upon resolving T.

Remark. The functions $R^{(m)}(\gamma)$, $D^{(m)}(\gamma)$, $N^{(m)}(\gamma)$ do not depend upon l, q.

6.3.1. *Algorithm revisited.* Let $(\mathcal{I}, \mu)$ be a marked ideal on m-dimensional smooth variety X. Consider the corresponding object

$$\mathcal{T}^{(m)} = (\{X_{\alpha,\beta}, \mathcal{I}_{\alpha,\beta}, E_{\alpha,\beta}, U_{\alpha,\beta}, \mathbb{C}^n_\alpha \mid \alpha \in A, \beta \in B\}, \mu)$$

with the initial data

$$\gamma := (0, m, d, n, l, q, \mu).$$

Our next goal is two-fold. We will give recursive formula for $\Gamma^{(m)}, R^{(m)}, D^{(m)}, N^{(m)}, L^{(m)}, Q^{(m)}$ and prove by induction on m that functions $\Gamma^{(m)}$, and others belong to Grzegorczyk's class $\mathcal{E}^{m+3}$. In the base of induction for $m = 0$ functions

$$R^{(0)} = 1,\ D^{(0)} = O(dn),\ N^{(0)} \leq 2n,\ L^{(0)} \leq l \cdot (dn)^{O(n)},\ Q^{(0)} \leq nq$$

belong to class $\mathcal{E}^3$ as well as $\Gamma^{(0)}$.

Now we proceed to the inductive step. Assume that $\Gamma^{(m-1)}, R^{(m-1)}, D^{(m-1)}, N^{(m-1)}, L^{(m-1)}, Q^{(m-1)}$ belong to Grzegorczyk's class $\mathcal{E}^{m+2}$, and $m \geq 1$.

If $\mathcal{I} = 0$ then the resolution is done by the single blow-up at the center $C = X$ and the object $\mathcal{T}^{(m)}$ is transformed into an object $\mathcal{T}^{(m)}_1$ with $X_1 = \emptyset$ and data bounded by $Bl(\gamma) \in \mathcal{E}^3$ (see Lemma 4.3.1).

If $\mathcal{I} \neq 0$ the resolution algorithm can be represented by the following scheme.

Step 2. Resolve $(\mathcal{I}, \mu)$ on m-dimensional smooth variety X. Consider the corresponding object $\mathcal{T}^{(m)}$ with initial data $\gamma := (0, m, d, n, l, q, \mu)$. Let $\bar{\mu}$ denote the maximal order of $\mathcal{N}(\mathcal{I})$ on X. We have the following estimate for $\bar{\mu}$:

$$\bar{\mu} \leq M(d, n) \in \mathcal{E}^3$$

(cf Lemma 5.0.2).

Step 2a. In this Step we are going to decrease the maximal order of the nonmonomial part $\mathcal{N}(\mathcal{I})$ by resolving the companion ideal $O(\mathcal{I},\mu)$. In fact we perform additional modification of $\mathcal{O}(\mathcal{I})$ and construct the ideal $\mathcal{J} := \mathcal{C}(\mathcal{H}(\mathcal{O}(\mathcal{I})))$. This corresponds to the new object $\mathcal{T}_1^{(m)}$ with the initial data

$$\gamma^{(2a)} := \Delta_{2a}(\gamma) \in \mathcal{E}^3,$$

(see Lemma 5.0.11.)

The object $\mathcal{T}_1^{(m)}$ will be then resolved and its resolution will cause the maximal order to decrease. Resolution $\mathcal{T}_1^{(m)}$ is done by performing Step 1.

Step 1. In this step we resolve $\mathcal{J}$, i.e $\mathcal{T}_1^{(m)}$. The Step splits in two Steps (1a) and (1b).

Step 1a. Move apart all unions of the intersections $H^s_\alpha \subset \mathbb{C}^n_\alpha$ of s divisors in E, where s is the maximal number of divisors in E through points in $\mathrm{supp}(\mathcal{I},\mu))$. For any α, resolve all $\overline{\mathcal{J}}_{|H^s_\alpha}$. We construct new object

$$\mathcal{T}_2^{(s)} := (\{H^s_{\alpha,\beta}, \mathcal{I}_{\alpha,\beta}, E_{\alpha,\beta}, U_{\alpha,\beta}, \mathbb{C}^n_\alpha \mid \alpha \in A, \beta \in B\}, \mu),$$

with the initial data bounded by

$$\gamma^{(1a)} := \Delta_1(\gamma) \in \mathcal{E}^3$$

with $s \le m-1$. (See Lemma 5.0.13.)

By the inductive assumption, the resolution of $\mathcal{T}_2^{(s)}$, i.e the sequence $\mathcal{T}_{2*}^{(s)}$ of the induced intermediate objects determined by the blow-ups, requires at most $R^{(m-1)}(\gamma^{(1)})$ blow ups. The maximal degree of the polynomials of the centers and the objects $\mathcal{T}_{2*}^{(s)}$ describing the resolution does not exceed $D^{(m-1)}(\gamma^{(1)})$. The dimension n of the objects does not exceed $N^{(m-1)}(\gamma^{(1)})$.

Note that the resolution of $\mathcal{T}_{2*}^{(s)}$ determines a multiple blow-up $\mathcal{T}_{1*}^{(m)}$ of $\mathcal{T}_1^{(m)}$ consisting of $R^{(m-1)}(\gamma^{(1)})$ blow-ups. We have a direct correspondence between objects $\mathcal{T}_{1*}^{(m)}$, and $\mathcal{T}_{2*}^{(s)}$. The bound

$$\Gamma^{(m-1)}(\Delta_1(\gamma)) \in \mathcal{E}^{m+2},$$

for the data for the resolution $\mathcal{T}_{2*}^{(s)}$, given by the induction, remains valid for data for $\mathcal{T}_{1*}^{(m)}$ as we use the same centers, the same ambient affine spaces and others for these multiple blow-ups. Only the strict transforms of the current X are different which does not affect the bounds for data. We have additional equations for describing current X in $\mathcal{T}_{2*}^{(s)}$ comparing to those in $\mathcal{T}_{1*}^{(m)}$.

Step 1a is performed at most $s \le m$ times. Introduce the auxillary unknown $t = 0, 1, \dots, m$, and the function $\Gamma_{1a}^{(m)}(\gamma, t)$ which measures the possible effect after performing Step 1a t times.

$$\Gamma_{1a}^{(m)}(\gamma, 0) := \Delta_1(\gamma) \in \mathcal{E}^3$$
$$\Gamma_{1a}^{(m)}(\gamma, t+1) := \Gamma^{(m-1)}(\Gamma_{1a}^{(m)}(\gamma, t))$$

Since the Step 1a is performed at most m times its final effect after completing Step 1a and passing to Step 1b is the measured by the function

$$\Gamma_{1b}^{(m)}(\gamma) := \Gamma_{1a}^{(m)}(\gamma, m)$$

Note that for any fixed value of $t = t_0$ (in particular, for $t = m$) functions $\Gamma_{1a}^{(m)}(\gamma, t_0)$, belongs to class $\mathcal{E}^{m+2}$ due to the inductive hypothesis and by virtue of Lemma 4.3.1, Corollary 5.0.10. Therefore $\Gamma_{1b}^{(m)}$ belongs to the class $\mathcal{E}^{m+2}$. (We use here the property that Grzegorczyk classes are closed under the composition.) After performing Step 1a we moved apart all strict transforms of E and $\mathrm{supp}(\overline{\mathcal{J}},\mu)$.

Step 1b If the strict transforms of E do not intersect $\mathrm{supp}(\overline{\mathcal{J}},\mu)$, we resolve $(\overline{\mathcal{J}},\mu)$ i.e. the object $\mathcal{T}_1^{(m)}$. This is achieved by resolving $\overline{\mathcal{J}}_{|V(u)}$ (by induction), where $V(u)$ is a hypersurface of maximal contact. After completing Step 1a the bound $\gamma^{(1a)}$ is transformed to

$$\gamma^{(1b)} := \Gamma_{1b}^{(m)}(\gamma) = (r^{(1b)}, m, d^{(1b)}, n^{(1b)}, l^{(1b)}, q^{(1b)}, (\mu \cdot (M(d,n)))!)$$

(cf. Lemma 5.0.6.)

Passing $(\overline{\mathcal{J}},\mu)$ to $\overline{\mathcal{J}}_{|V(u)}$ we adjoin the equations of maximal contact as well as create new neighborhoods. This operation has been reflected in the construction of $\Delta_1(\gamma)$. By the construction of $\Gamma_{1b}^{(m)}(\gamma)$ and $\Delta_1(\gamma)$ the degree of the maximal contact does not exceed $d^{(1b)}$, while the number of neighborhoods does not exceed $q^{(1b)}$. In other words $\Gamma_{1b}^{(m)}(\gamma)$ bounds the initial data for $\overline{\mathcal{J}}_{|V(u)}$.

The resolution process $\overline{\mathcal{J}}_{|V(u)}$ leads eventually to the resolution of the object $\mathcal{T}_1^{(m)}$ corresponding to $\overline{\mathcal{J}}$ with the data bounded by

$$\Gamma_1^{(m)}(\gamma) := \Gamma^{(m-1)}(\Gamma_{1b}^{(m)}(\gamma)) = \Gamma_{1a}^{(m)}(\gamma, m+1) = (r^{(1)}, m, d^{(1)}, n^{(1)}, l^{(1)}, q^{(1)}, (\mu \cdot (M(d,n)))!),$$

for the relevant $r^{(1)}, d^{(1)}, n^{(1)}, l^{(1)}, q^{(1)}$.

Hence the function $\Gamma_1^{(m)}$ belongs to class $\mathcal{E}^{m+2}$ by inductive hypothesis and by virtue of Lemma 4.3.1 and Corollary 5.0.10. This completes Step 1.

The object $\mathcal{T}^{(m)}$ corresponding to $\mathcal{I}$ with initial data γ is transformed to the new object with the data bounded by

$$\Gamma_{2a}^{(m)}(\gamma) := (r^{(1)}, m, d^{(1)}, n^{(1)}, l^{(1)}, q^{(1)}, \mu)$$

with smaller $\bar{\mu}$ – the maximal order of $\mathcal{N}(\mathcal{I})$. (Note that $\bar{\mu} < M(d,n)$).

This completes Step 2a. The Step 2a is then repeated at most $M(d,n)$ times until the maximal order drops to zero as we arrived at the monomial case. The final effect of Step 2a is measured then by the recursive function

$$\Gamma_{2a}^{(m)}(\gamma, 0) = \gamma$$

$$\Gamma_{2a}^{(m)}(\gamma, t+1) = \Gamma_{2a}^{(m)}(\Gamma_{2a}^{(m)}(\gamma, t)).$$

Therefore the function $\Gamma_{2a}^{(m)}$ belongs to class $\mathcal{E}^{m+3}$ by definition of the Grzegorczyk's classes (see (8),(9)), and by virtue of Lemmas 4.3.1, 5.0.2.

Putting $t = M(d,n)$ gives the final effect after completing all necessary Steps 2a and subsequent passing to the Step 2b

$$\Gamma_{2b}^{(m)}(\gamma) := \Gamma_{2a}^{(m)}(\gamma, M(d,n)),$$

and thereby function $\Gamma_{2b}^{(m)}$ belongs to class $\mathcal{E}^{m+3}$ as well.

The procedure eventually reduces $(\mathcal{I}, \mu)$ to the monomial marked ideal $\mathcal{I} = \mathcal{M}(\mathcal{I})$.

Step 2b. Resolve the monomial marked ideal $\mathcal{I} = \mathcal{M}(\mathcal{I})$. The marked ideal corresponds to the object $\mathcal{T}^{(m)}$ with data

$$(r^{(2b)}, m, d^{(2b)}, n^{(2b)}, l^{(2b)}, q^{(2b)}, \mu) := \Gamma_{2b}^{(m)}(\gamma).$$

The resolving of $\mathcal{I} = (x^\alpha)$ consists of blow-ups each of which decreases the multiplicity $|x^\alpha| \leq d^{(2b)}$. The resolution of $\mathcal{I}$ requires at most $d^{(2b)}$ blow-ups. Thus the final data solution can be bounded by the function

$$\Gamma^{(m)}(\gamma) := \overline{Bl}(\Gamma_{2b}^{(m)}(\gamma), d^{(2b)}) \in \mathcal{E}^{m+3}$$

We summarize the achieved bounds in the following Corollary (recall that the notations one can find in subsection 4.1.1).

Corollary 6.3.1. *While resolving a marked ideal $(X, \mathcal{I}, E, \mu)$ on $X \subset \mathbb{C}^n$ by Hironaka algorithm the degree d, and the number l of the occurring polynomials, the embedding dimension n, the number r of the blow ups and the number q of the affine neighborhoods satisfy , for a fixed $m = \dim(X)$, the recursive equalities above and are majotated by a function*

$$(r, m, d, n, l, q, \mu) := \Gamma^{(m)}(0, m, d_0, n_0, l_0, q_0, \mu) \in \mathcal{E}^{m+3},$$

for the initial values $d = d_0$, $n = n_0$, $l = l_0$, $q = q_0$,

6.4. **Complexity of the algorithm.** The principal complexity result of the paper states that

Theorem 6.4.1. *While resolving a marked ideal $(X, \mathcal{I}, E, \mu)$ on $X \subset \mathbb{C}^n$ by Hironaka algorithm its complexity can be bounded , for a fixed $m = \dim(X)$, by*

$$b^{O(1)} \cdot \mathcal{F}^{(m)}(d_0, n_0, l_0, q_0, \mu)$$

for a certain function $\mathcal{F}^{(m)}(d_0, n_0, l_0, q_0, \mu) \in \mathcal{E}^{m+3}$.

Proof. Indeed, each step of the algorithm consists of solving a certain subroutine (basically, solving a linear system) over the coefficients of current polynomials. Therefore, Corollary 6.3.1 provides a bound on the number of arithmetic operations with the coefficients of current polynomials (being a function from class $\mathcal{E}^{m+3}$). On the other hand, all the coefficients of the polynomials for the next step are obtained as results of these arithmetic operations, so bit sizes of the coefficients grow at most as $b \cdot \mathcal{F}_1^{(m)}$ for a suitable function $\mathcal{F}_1^{(m)} \in \mathcal{E}^{m+3}$. A cost of a single arithmetic operation is obviously polynomial.

As a corollary we obtain the following theorem.

Theorem 6.4.2. *While*

(1) *resolving singularities of $X \subset \mathbb{C}^{n_0}$, or*
(2) *principalizing an ideal sheaf $\mathcal{I}$ on a nonsingular $X \subset \mathbb{C}^{n_0}$*

by Hironaka algorithm the degree d, and the number l of the occurring polynomials, the embedding dimension n, the number r of the blow ups and the number q of the affine neighborhoods satisfy , for a fixed $m = \dim(X)$, the recursive equalities above and are majotated by a function

$$(r, m, d, n, l, q, 1) := \Gamma^{(m)}(0, m, d_0, n_0, l_0, q_0, 1) \in \mathcal{E}^{m+3},$$

for the initial values $d = d_0$, $n = n_0$, $l = l_0$, $q = q_0$,

The complexity of the algorithm is bounded by

$$b^{O(1)} \cdot \mathcal{F}^{(m)}(d_0, n_0, l_0, q_0, \mu),$$

for a certain function $\mathcal{F}^{(m)}(d_0, n_0, l_0, q_0, \mu) \in \mathcal{E}^{m+3}$.

Remark. Above in the proof we gave a more explicit form of $\mathcal{F}^{(m)}$ providing an additional information on its dependance on r, d, n, l, q, μ. But the main consequence of the Theorem is that $m = \dim X$ brings the most significant contribution into the complexity bound.

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