Cancellation, Elliptic Surfaces and the Topology of certain Four-Manifolds

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This is the third in a series of three papers about cancellation problems (referred to as [I], [II] and [III]). In this part we use the techniques and results developed in the preceeding papers to give some further applications to the topology of four-manifolds. The main results concern smooth structures on elliptic surfaces with finite fundamental group, and the topological classification of four-manifolds with special fundamental groups.

One motivation for studying the classification of four-manifolds up to homeomorphism is to get information about smooth structures. S. K. Donaldson has proved [5] that simply-connected compact algebraic surfaces are often indecomposable as a smooth connected sum. It is however possible for such a surface to be *homeomorphic* to a connected sum of smooth 4-manifolds (by M. Freedman's results [7] in the simply-connected case), and thus the same underlying topological manifold can have distinct smooth structures (see [10, §1] for a survey of such results).

More generally, we conjectured in [10] that an algebraic surface with any finite fundamental group has at least two smooth structures, which remain distinct after blow-up (topologically just connected sum with $\overline{CP}^2$). We showed that for each non-trivial finite group G, there exists a constant $c(G)$ such that the conjecture holds for all algebraic surfaces X with $\pi_1(X) = G$, Euler characteristic $e(X) \geq c(G)$ and $c_1^2(X) \geq 0$.

We can now verify the conjecture for many non-simply connected elliptic surfaces. In the statement, p_g denotes the geometric genus.

Theorem A: *Let X be an elliptic surface with finite fundamental group. If $p_g > 0$ then X has at least two smooth structures which remain distinct under blow-ups. If $p_g = 0$ then $X \sharp\, \overline{CP}^2$ has at least two smooth structures which remain distinct under further blow-ups.*

The full conjecture for elliptic surfaces with cyclic fundamental group and $p_g = 0$ follows from [9, Cor.5]. Note that in this case, there are homeomorphic surfaces which are not diffeomorphic. In contrast, two elliptic surfaces with non-cyclic fundamental group which are homeomorphic are also diffeomorphic [15]. To prove Theorem A, we construct a smooth manifold M which is homeomorphic to X and whose universal covering decomposes as a connected sum of manifolds with indefinite intersection forms. By the result of Donaldson [5] mentioned above, M and X are not diffeomorphic.

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In §2 we discuss metabolic forms over group rings $\mathbf{Z}\pi$. This theory is used in §3 to prove that topological 4-manifolds with odd order fundamental group, and large Euler characteristic are classified up to homeomorphism by explicit invariants. The precise statement includes a lower bound for the Euler characteristic in terms of an integer $d(\pi)$ depending on the group.

For any finite group π, let $d(\pi)$ denote the minimal $\mathbf{Z}$-rank for the abelian group $\Omega^3\mathbf{Z} \otimes_{\mathbf{Z}\pi} \mathbf{Z}$. Here we minimize over all representatives of $\Omega^3\mathbf{Z}$, obtained from a free resolution of length three [**I**, (0.1)] of $\mathbf{Z}$ over the ring $\mathbf{Z}\pi$.

Theorem B: *Let M be a closed oriented manifold of dimension four, and let $\pi_1(M) = \pi$ be a finite group of odd order. When $w_2(\tilde{M}) = 0$ (resp. $w_2(\tilde{M}) \neq 0$), assume that $e(M) - |\sigma(M)| > 2d(\pi)$, (resp. $> 2d(\pi) + 2$). Then M is classified up to homeomorphism by the signature, Euler characteristic, type, Kirby-Siebenmann invariant, and fundamental class in $H_4(\pi, \mathbf{Z})/Out(\pi)$.*

The *type* is the type (even or odd) of the intersection form on M.

In §4, we obtain a classification theorem for manifolds with cyclic fundamental groups generalizing [**8**, Thm. B], [**9**, Thm. 3]:

Theorem C: *Let M be a closed, oriented 4-manifold with finite cyclic fundamental group. Then M is classified up to homeomorphism by the fundamental group, the intersection form on $H_2(M, \mathbf{Z})/Tors$, the w_2-type, and the Kirby-Siebenmann invariant. Moreover, any isometry of the intersection form can be realized by a homeomorphism.*

Corollary D: *An algebraic surface with non-trivial cyclic fundamental group has at least two distinct smooth structures which are stable under blowups.*

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§1: Applications to Elliptic Surfaces

In this section we will prove Theorem A. To begin, let us recall the construction of elliptic surfaces given in [**15**]. Since we are interested only in surfaces with finite fundamental group, we consider those which admit a fibration over the 2-sphere, with generic fibre a 2-torus. Let $p : V_g \to S^2$ denote the elliptic surface with geometric genus g (i.e. $e(V_g) = 12(g+1)$) and no multiple fibres. Let $B^2 \subset S^2$ be a small 2-disk which contains no singular locus of V_g, and let $V_g^0 = p^{-1}(S^2 - \operatorname{int} B^2)$. Then $V_g = (T^2 \times D^2) \cup V_g^0$, and any other elliptic surface with the same geometric genus is obtained from this by performing log transforms in the $T^2 \times D^2$ part, to create multiple fibres. In order to have finite fundamental group, there must be ≤ 3 multiple fibres. From this description, the fundamental group

$$\pi_1 X_{(m_1,m_2,m_3)} = \{q_1, q_2, q_3 \mid q_1^{m_1} = q_2^{m_2} = q_3^{m_3} = q_1 q_2 q_3 = 1\}$$

for 3 multiple fibres and multiplicities (m_1, m_2, m_3). The possible finite fundamental groups are the finite subgroups of $SO(3)$: cyclic $\mathbf{Z}/m$, dihedral D_m, A_4, S_4 and A_5, corresponding to the multiplicities (mp, mq) with g.c.d.$(p,q) = 1$, $(2,2,m)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$.

Proposition 1.1: *If X is a minimal elliptic surface with finite fundamental group, then $w_2(X)$ is non-trivial if $\sigma(X) \equiv 8 \pmod{16}$ or $\sigma(X) \equiv 0 \pmod{16}$ and one of the multiple fibres has even multiplicity. If $\pi_1(X)$ is non-cyclic, then $w_2(\tilde{X}) = 0$, where $\tilde{X}$ is the universal covering, and $w_2(X') \neq 0$ for every intermediate covering $X' \to X$ with even order fundamental group.*

Proof: This follows from [1, Chap. V, 12.3]. ∎

This information determines the normal 1-type of elliptic surfaces [11] with non-cyclic fundamental group (see [9] for the cyclic case). If $w_2(\tilde{X}) \neq 0$ the normal 1-type is $B = K(\pi,1) \times BSO \xrightarrow{p_2} BO$. If $w_2(\tilde{X}) = 0$ let $w \in H^2(K(\pi,1), \mathbf{Z}/2)$ such that $c*(w) = w_2(X)$, where $c : X \to K(\pi,1)$ classifies the universal covering. This class w determines the normal 1-type and is itself determined by its restriction to the 2-Sylow subgroup of π.

From Proposition 1.1 we know that the restriction of w is non-zero for each non-trivial subgroup of π. This implies that for $\pi = \mathbf{Z}/2 \times \mathbf{Z}/2$, $w = x^2 + xy + y^2$ and for $\pi = D_{2^r}$, $w = y^2 + z$. Here we write $H^*(\mathbf{Z}/2 \times \mathbf{Z}/2, \mathbf{Z}/2) = \mathbf{Z}/2[x,y]$ and $H^*(D_{2^r}, \mathbf{Z}/2) = \mathbf{Z}/2[x,y,z]/(x^2+xy)$ where z is the second Stiefel-Whitney class of the vector bundle given by the standard representation

$$D_{2^r} = \langle s, b \mid s^{2^r} = b^2 = (bs)^2 = 1 \rangle \to O(2).$$

In the case when π is a 2-group, there exists an oriented vector bundle E over $K(\pi,1)$ with $w_2(E) = w$, and the normal 1-type is the fibration $B = B(E) = K(\pi,1) \times BSpin \to BO$. The map to BO is the classifying map of the of the bundle $E \times \gamma$, where γ is the universal bundle over $BSpin$.

We need the following information about $\Omega_3(B)$.

Lemma 1.2: *The map $\Omega_3(B) \to H_3(\pi, \mathbf{Z})$, induced by projection on the first factor, is injective.*

Proof: It is enough to prove this for π a 2-group, when we may take $B = B(E)$ as described above. Consider the Atiyah-Hirzebruch spectral sequence with E_2-term $H_*(M(E \times \gamma), \Omega_*^{Spin})$. After applying the Thom isomorphism, the first differential $H_3(\pi, \Omega_1^{Spin}) \to H_1(\pi, \Omega_2^{Spin})$ is dual to $H^1(\pi, \mathbf{Z}/2) \xrightarrow{w} H^3(\pi, \mathbf{Z}/2)$. Thus $E^3_{1,2} \cong \ker w$. Similarly, $E^3_{2,1}$ is isomorphic to the homology of the complex

$$H^0(\pi, \mathbf{Z}) \xrightarrow{w} H^2(\pi, \mathbf{Z}/2) \xrightarrow{Sq^1(Sq^2+w)} H^5(\pi, \mathbf{Z}/2).$$

One can easily check for $\pi = \mathbf{Z}/2^r$, $\pi = \mathbf{Z}/2 \times \mathbf{Z}/2$ or $\pi = D_{2^r}$, that $E^3_{1,2} = E^3_{2,1} = 0$. To see this we use the identification of $w \in H^2(\pi, \mathbf{Z}/2)$ given above and carry out the indicated calculation. ∎

The method of proof of Theorem A for a particular elliptic surface X is as follows. We construct a smooth 4-manifold M (i) which is stably homeomorphic to X, and (ii) such that the universal covering admits a smooth decomposition as a connected sum $\tilde{M} = N_1 \sharp N_2$, such that neither N_1 nor N_2 has a negative definite intersection form. By a result of S.K.Donaldson [**5**] X is not diffeomorphic to M. On the other hand, if one of our cancellation theorems applies, we can conclude that X is homeomorphic to M.

From now on we assume that $\pi_1(X)$ is **non-cyclic** unless stated otherwise. Suppose that X is a minimal elliptic surface with $p_g = g$. Then $e(X) = 12(g+1)$ and $\sigma(X) = -8(g+1)$. If $p_g(X) = g \geq 2$, a suitable model for X is

$$M = \mathsf{E}_{g-2} \sharp \mathsf{K} \sharp (S^2 \times S^2) \tag{1.3}$$

where K is the Kummer surface, and E_{g-2} denotes an elliptic surface with the same fundamental group $\pi_1(X)$ and $e(\mathsf{E}_{g-2}) = 12(g-1)$.

To prove that M and $X = \mathsf{E}_g$ are B-bordant we need the following lemma. Let η (resp. θ) denote the non-trivial (resp. trivial) spin structure on the circle S^1.

Lemma 1.4: *Let V_g be a 1-connected elliptic surface without multiple fibres and $p_g = g$. Then V_g^0 is a 1-connected spin manifold and the induced spin structure on $T^3 = \partial V_g^0$ is η^3 if g is even, and $\theta \times \eta^2$ if g is odd.*

Proof: The fact that V_g^0 is 1-connected follows from [**15**]. For g odd V_g is spin implying that V_g^0 is also spin. For g even, V_g^0 is contained in the universal covering of any elliptic surface with that geometric genus. For appropriate choice of log transforms (see Proposition 1.1) the universal covering can be spin.

Next consider the induced spin structure on $T^3 = \partial V_g^0$. If g even, $\sigma(V_g) \equiv 8 \pmod{16}$ and so the Rochlin invariant of ∂V_g^0 is non-trivial. This implies that the spin structure is η^3. If g is odd, the spin structure on T^3 must extend over $D^2 \times T^2$ and so has the form $\theta \times \alpha \times \beta$. Now introduce a multiple fibre with even multiplicity by performing a log transform on V_g. The resulting elliptic surface E is non-spin by (1.1). Since the glueing map for the multiple fibre is

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(see [**15**, p. 637]), if α were trivial the result of the log transform would be spin. However, the diffeomorphism type of E is independent of the parametrization chosen for the torus T^2 and so β is non-trivial also. ▪

We also need another description of E_g. Start with $S^3/\tilde{\pi}$, where $\tilde{\pi} \subset SU(2)$ is a finite subgroup, and hence acts freely on S^3. Every finite subgroup π of $SO(3)$ arises as the quotient of such a group $\tilde{\pi}$ by a central subgroup of order two.

Since $S^3/\tilde{\pi}$ admits a Seifert fibering over S^2, with ≤ 3 multiple fibres of multiplicity (m_1, m_2, m_3), we can decompose $Y = S^3/\tilde{\pi} \times S^1 = (T^2 \times D^2) \cup_f Y_0$, where Y_0 has a T^2 fibration over D^2 containing all the singular fibres of the product fibering $(S^3/\tilde{\pi} \to S^2) \times (S^1 \to *)$. The fundamental group $\tilde{\pi}$ has the presentation [**13**, Chap.6]

$$\tilde{\pi} = \{q_1, q_2, q_3, h \mid q_j^{m_j} h = 1, q_1 q_2 q_3 = h^b\},$$

where $b = -1$ if π is non-cyclic and $q_3 = 1, b = 0$ if π is cyclic. The $S^1 \times D^2$ neighbourhood of the fibre h is glued by a homeomorphism f of T^2 depending on b. By comparing this description with that of the elliptic surfaces above, we get

Lemma 1.5: *Let X be a minimal elliptic surface with finite fundamental group π and geometric genus $p_g = g$. Then X is diffeomorphic to $(Y - T^2 \times D^2) \cup V_g^0$, where $Y = S^3/\tilde{\pi} \times S^1$.*

Proof: This is clear from the discussion above, using the fact (from **[15]**) that one S^1 factor is preserved in the log transforms used to construct X. ▪

Proposition 1.6: *Let X be a minimal elliptic surface with non-cyclic finite fundamental group π and geometric genus $p_g = g \geq 2$. Then X is stably homeomorphic to $\mathsf{E}_{g-2} \sharp \mathsf{K} \sharp (S^2 \times S^2)$, where $\pi_1(\mathsf{E}_{g-2}) = \pi$.*

Proof: Both manifolds admit normal 1-smoothings into the same normal 1-type B. It is therefore enough to show that E_g and $\mathsf{E}_{g-2} \sharp \mathsf{K} \sharp (S^2 \times S^2)$ are B-bordant. From (1.5) we have the decompositions $\mathsf{E}_g = (Y - T^2 \times D^2) \cup V_g^0$ and

$$\mathsf{E}_{g-2} \sharp \mathsf{K} \sharp (S^2 \times S^2) = (Y - T^2 \times D^2) \cup V_{g-2}^0 \sharp \mathsf{K} \sharp (S^2 \times S^2).$$

However, the B-structure on the $T^3 = \partial V_g^0$ and the $T^3 = \partial V_{g-2}^0$ is the same, by Lemma 1.4. Hence the difference of the two manifolds is B-bordant to

$$-V_g^0 \cup V_{g-2}^0 \sharp \mathsf{K} \sharp (S^2 \times S^2),$$

which is a 1-connected spin manifold with signature zero. Since this manifold is B-bordant to zero, the result follows. ▪

When $p_g = 1$ we consider the following model for X. As above, E_0 denotes an elliptic surface with the same fundamental group $\pi = \pi_1 X$ and $e(\mathsf{E}_0) = 12$. Let D_2 denote an Enriques surface (with fundamental group $\mathbf{Z}/2$ and universal covering spin). Choose a non-trivial homomorphism $f : \mathbf{Z}/2 \to \pi$, and note that the normal 1-type of D_2 is $B(f^*w) = (K(\mathbf{Z}/2, 1) \times BSpin \to BO)$. Now consider embeddings of $S^1 \times D^3$ into E_0 and D_2 representing $f(1)$ and the non-trivial element in $\pi_1 \mathsf{D}_2$, compatible with the B and $B(f^*w)$ structures (with opposite orientation). Then our model for X is

$$M = (\mathsf{E}_0 - S^1 \times D^3) \cup (\mathsf{D}_2 - S^1 \times D^3). \tag{1.7}$$

By construction, the normal 1-type of M is again $B = B(w)$.

Proposition 1.8: *Let X be a minimal elliptic surface with non-cyclic finite fundamental group π and geometric genus $p_g = 1$. Then X is stably homeomorphic to M.*

Proof: Let $S^3/\tilde{\pi} = \Sigma$ and $L^3(\mathbf{Z}/4) = \Sigma'$. To shorten the notation we let $(\Sigma \times S^1)^0 = (\Sigma \times S^1 - D^2 \times T^2)$ and $(\Sigma' \times S^1)^0 = (\Sigma' \times S^1 - D^2 \times T^2)$. We are again using the

Seifert fibering structure of Σ and Σ' to remove a small neigbourhood of a regular fibre. We have B-bordisms:

$$M \sim \mathsf{E}_0 + \mathsf{D}_2 = (\Sigma \times S^1 - D^2 \times T^2) \cup V_0^0 + (\Sigma' \times S^1 - D^2 \times T^2) \cup V_0^0$$

by (1.5). We can write $\mathsf{K} = V_0^0 \cup V_0^0$ where the glueing diffeomorphism is orientation-reversing. This leads to B-bordisms:

$$\begin{aligned} M + (-\mathsf{K}) &\sim (\Sigma \times S^1)^0 \cup V_0^0 + (\Sigma' \times S^1)^0 \cup V_0^0 + (-V_0^0 \cup -V_0^0) \\ &\sim (\Sigma \times S^1)^0 \cup (\Sigma' \times S^1)^0 \\ &= \left[(\Sigma - D^2 \times S^1) \cup (\Sigma' - D^2 \times S^1)\right] \times S^1 \end{aligned}$$

where the final S^1 factor has the non-trivial spin structure by Lemma 1.4 and "$\cup$" denotes identifying along a common boundary.

On the other hand, $\mathsf{E}_1 + (-\mathsf{K})$ is B-bordant to $\Sigma \times S^1$, with the non-trivial spin structure on the S^1. Thus we are finished if Σ and $(\Sigma - D^2 \times S^1) \cup (\Sigma' - D^2 \times S^1)$ are B-bordant in $\Omega_3(B)$. By Lemma 1.4 this follows if both have the same fundamental class in $H_3(\pi, \mathbf{Z})$. But, by construction, in both cases the fundamental class factors through $H_3(\tilde{\pi}, \mathbf{Z}) = \mathbf{Z}/|\pi|$ and is non-trivial there. Since the map induced by projection $H_3(\mathbf{Z}/4, \mathbf{Z}) \to H_3(\mathbf{Z}/2, \mathbf{Z})$ is zero, we are done. ∎

If $p_g = 0$ and $\pi_1(X)$ is non-cyclic, then we only know how to construct a suitable decomposable model for $X \,\sharp\, \overline{CP}^2$. This will be done in the Proof of Theorem A, after some further preparation. For rest of the discussion up to the Proof of Theorem A, we will assume that $\pi = \pi_1(X)$ itself does not act freely on S^3. In other words, π is not cyclic, or dihedral of order $2k$, with k odd. As above, we let $\tilde{\pi} \subset SU(2)$ be the two-fold covering group of π.

To begin, let Y_+ be the result of doing two surgeries on $Y = S^3/\tilde{\pi} \times S^1$, one to kill the class represented by $* \times S^1$, and the other to kill the central element $<z>$ of order 2 in $\tilde{\pi}$. We fix a spin structure on Y as the product of any spin structure on $S^3/\tilde{\pi}$ with the null-bordant spin structure on S^1. The surgeries are done preserving this spin structure. The result is a smooth spin 4-manifold Y_+ with $e(Y_+) = 4$ and $\pi_1(Y_+) = \pi$, where π is the quotient of $\tilde{\pi}$ by the central $\mathbf{Z}/2$, and hence is a finite subgroup of $SO(3)$.

Note that Y_+ has another description. It is the double of a suitable thickening of a finite 2-complex K with fundamental group π. Namely, Y_+ is the double of

$$(S^3/\tilde{\pi} - D^3) \times I \cup_{S^1 \times D^2} D^2 \times D^2 \simeq K.$$

Since the Euler characteristic of Y_+ is four, and we assume that π is not periodic, $\pi_2(K) = \mathfrak{N}$ is a minimal representative for $\Omega^3\mathbf{Z}$. In other words,

$$0 \to \mathfrak{N} \to C_2 \to C_1 \to C_0 \to \mathbf{Z} \to 0$$

with $C_i = C_i(\tilde{K})$ finitely generated free $\mathbf{Z}\pi$ modules, and $\mathfrak{N}$ has the minimal $\mathbf{Z}$-rank for $\pi_2(K)$ of a two-complex with the given π_1.

Moreover, we have an isomorphism $H_2(Y_+) \cong \mathfrak{N} \oplus \bar{\mathfrak{N}}$, and $\bar{\mathfrak{N}}$ denotes the dual left module $\mathfrak{N}^*$ made into a right A-module in the usual way. The intersection form $S_{Y_+} = Met(\mathfrak{N})$, where $Met(\mathfrak{N})$ denotes a metabolic (weakly) quadratic form on $\mathfrak{N} \oplus \overline{\mathfrak{N}}$ with $0 \oplus \overline{\mathfrak{N}}$ totally isotropic. The exact sequences for $\mathfrak{N}$, $\bar{\mathfrak{N}}$ were considered in [I,§2]. Let $\mathfrak{I} = \mathfrak{I}(\pi)$ denote the augmentation ideal of A, and observe that the ideal $\langle \mathfrak{I}, 2\rangle$ used in [I,§2] sits in an exact sequence

$$0 \to \langle \mathfrak{I}, 2\rangle \to A \to \mathbf{Z}/2 \to 0 \tag{1.9.}$$

In particular, by [I, Lemma 2.4] there is a short exact sequence

$$0 \to \bar{\mathfrak{I}} \to \mathfrak{N} \to \langle \mathfrak{I}, 2\rangle \to 0 \tag{1.10}$$

and its dual extension

$$0 \to \overline{\langle \mathfrak{I}, 2\rangle} \to \overline{\mathfrak{N}} \to \mathfrak{I} \to 0 \tag{1.11}$$

is classified by an element $\theta_{\mathfrak{N}} \in \mathrm{Ext}^1_A(\mathfrak{I}, \overline{\langle \mathfrak{I}, 2\rangle}) \cong H^2(\pi, \mathbf{Z}/2)$.

Now let W denote the cobordism from Y to Y_+ given by the trace of the surgeries described above. We have the following diagram of exact sequences arising from the the triple (W, Y_+, Y),

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \bar{\mathfrak{I}} \oplus A & \to & H_3(W, \partial W) & \to & \mathbf{Z}/2 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \bar{\mathfrak{I}} \oplus A & \to & H_2(Y_+) & \to & H_2(W) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathfrak{I} \oplus \langle \mathfrak{I}, 2\rangle & = & \mathfrak{I} \oplus \langle \mathfrak{I}, 2\rangle & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{1.12}$$

where the homology is taken with $A = \mathbf{Z}[\pi]$ coefficients.

Next we need to recall some of the notation and results of [I, §2]. Our goal is to find a quadratic submodule of $H_2(Y_+)$ which also embeds in a non-singular quadratic module of the form $H(p_1A) \perp Met(L)$. Here (as in [II, §1]), $H(p_1A)$ is the hyperbolic form on the module $p_1A \oplus q_1\bar{A}$. We use p_1A (resp. $q_1\bar{A}$ to denote the free rank one module (dual module) with basis elements p_1 (resp. q_1). In [I, (2.7] we proved the existence of a submodule $\mathfrak{K} = \mathfrak{K}(\pi) \subset \mathfrak{I}(\pi)$ such that the extension (1.11) splits when pulled back over $\mathfrak{K}$. From this we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & q_1\bar{A} & \to & q_1\bar{A} \oplus \mathfrak{K} & \to & \mathfrak{K} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{\langle \mathfrak{I}, 2\rangle} & \to & \overline{\mathfrak{N}} & \to & \mathfrak{I} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}/2 & \to & T & \to & \mathfrak{I}/\mathfrak{K} & \to & 0 \end{array}$$

The cokernel T has exponent two (see [I, (2.7)]), and we define its dual module to be $\hat{T} = \mathrm{Hom}_{\mathbf{Z}}(T, \mathbf{Q}/\mathbf{Z})$. Let $\mathfrak{W}$ denote the module obtained from $\mathfrak{N}$ by the pushout of (1.11) using the projection $\overline{\langle \mathfrak{J}, 2\rangle} \to \mathbf{Z}/2$.

We obtain exact sequences

$$(1.13) \qquad 0 \to \mathbf{Z}/2 \to \mathfrak{W} \to \mathfrak{J} \to 0$$

and

$$(1.14) \qquad 0 \to q_1\bar{A} \to \bar{\mathfrak{N}} \to \mathfrak{W} \to 0,$$

and an identification $T \cong \mathfrak{W}/\mathfrak{K}$.

From the diagram above, we also constructed a short exact sequence

$$0 \to \mathfrak{N} \to p_1 A \oplus \bar{\mathfrak{K}} \to \hat{T} \to 0$$

by dualizing. This was used in [I, (2.9)] to show that $\mathfrak{N}$ had an alternate description, as an extension

$$(1.15) \qquad 0 \to p_1\mathfrak{A} \to \mathfrak{N} \to L \to 0,$$

where $p_1\mathfrak{A} \subset p_1 A$ is a (two-sided) ideal. The embedding $\mathfrak{N} \subset p_1 A \oplus \bar{\mathfrak{K}}$, induces an injection of the quotient modules $L \subset \bar{\mathfrak{K}}$. The submodule $\bar{\mathfrak{J}} \subset \mathfrak{N}$ is mapped injectively into the second factor $0 \oplus \bar{\mathfrak{K}}$. It remains to see how these sequences relate to the maps in our surgery diagram.

Lemma 1.16: *There are A-module isomorphisms $H_3(W, \partial W) \cong \bar{\mathfrak{J}} \oplus \overline{\langle \mathfrak{J}, 2\rangle}$, $H_2(Y_+) \cong \mathfrak{N} \oplus \bar{\mathfrak{N}}$ and $H_2(W) \cong \mathfrak{W} \oplus \langle \mathfrak{J}, 2\rangle$, such that for the sequences in* (1.12)*:*
(i) the upper horizontal sequence is the dual of (1.9) *direct sum with $id : \bar{\mathfrak{J}} \to \bar{\mathfrak{J}}$*
(ii) the middle horizontal sequence is the direct sum of (1.14) *and* (1.10)
(iii) the middle vertical sequence is the direct sum of (1.10) *and* (1.11)
(iv) the right-hand vertical sequence is the direct sum of (1.13) *and the map $id : \langle \mathfrak{J}, 2\rangle \to \langle \mathfrak{J}, 2\rangle$.*

Proof: For part (i), we note that the upper sequence is part of the exact sequence of the triple:

$$0 \to H_3(Y) \to H_3(W, Y_+) \to H_3(W, \partial W) \to H_2(Y) \to 0.$$

The maps in this sequence are dual to the attaching maps of our handles in the surgery given by $A \oplus A \cong H_2(W, Y) \to H_1(Y) \cong \mathbf{Z} \oplus \mathbf{Z}/2$.

For part (ii), we use the embedding $K \mapsto Y_+$ to split $H_2(Y_+)$ in a natural way. Parts (iii) and (iv) then follow. ∎

Let $\gamma : H_2(W) \to T$ be defined as the composite

$$\gamma : H_2(W) \cong \mathfrak{W} \oplus \langle \mathfrak{J}, 2\rangle \to \mathfrak{W}/\mathfrak{K} = T,$$

where the last map is the obvious projection.

Under the isomorphisms of (1.16), the composite γ_+ of γ with the geometrically induced map $H_2(Y_+) \to H_2(W)$ is just the composite of the second factor projection $H_2(Y_+) = \mathfrak{N} \oplus \bar{\mathfrak{N}} \to \bar{\mathfrak{N}}$ followed by the quotient map $\bar{\mathfrak{N}} \to \mathfrak{W} \to T$ given above. Recall that he intersection form S_{Y_+} is the metabolic (weakly) quadratic form $Met(\mathfrak{N})$. We now use the description (1.15) for $\mathfrak{N}$ to (i) fix the pushout embedding $\mathfrak{N} \subset p_1 A \oplus L$, and (ii) identify the rational space $S_{Y_+} \otimes \mathbf{Q}$ with $Met(p_1 A \otimes \mathbf{Q}) \perp Met(L \otimes \mathbf{Q})$. This rational space also contains the form $Met(p_1 A) \perp Met(L)$, with basis $\{p_1, q_1\}$ for the first factor $Met(p_1 A)$.

Lemma 1.17: *With the identifications of* Lemma 1.16, $\ker \gamma_+ = \mathfrak{N} \oplus (q_1 \bar{A} \oplus \mathfrak{K})$. *The second factor is* $\ker \gamma_+ \cap \bar{\mathfrak{N}}$ *and* $\ker \gamma_+$ *is a quadratic submodule of* $Met(p_1 A) \perp Met(L)$, *where the first factor* $\mathfrak{N} \subset p_1 A \oplus L$ *has the fixed embedding, and the second factor embeds via the* $\mathfrak{K} \subset \bar{L}$.

Proof: Direct from the definitions and (1.16). ∎

The Proof of Theorem A: The procedure described above has already been carried out to prove Theorem A for cyclic groups [9, Cor. 5]. For non-cyclic fundamental groups we first consider the cases when $p_g > 0$. In these cases, we have shown in (1.6) or (1.8) that X is stably homeomorphic to one of the models M from (1.3) or (1.7).

When $p_g \geq 2$ M contains one $S^2 \times S^2$ factor, and our cancellation theorem [**II**, Thm. B] applies.

When $p_g = 1$ we need to use the fact that the Enriques surface $\mathbf{D_2}$ decompose topologically. First note that for $\pi = \mathbf{Z}/2$, there exists a rational homology 4-sphere Σ with $\pi_1(\Sigma) = \pi$ and $w_2(\Sigma) \neq 0$. From surgery theory, we can also construct such rational homology 4-spheres Σ' with non-trivial KS invariant. Now $\mathbf{D_2}$ is stably homeomorphic to $\Sigma' \sharp M(E_8) \sharp (S^2 \times S^2)$. Since the latter topological model splits off an $S^2 \times S^2$ factor, we apply [**II**, Thm. B] twice to finish the proof in this case.

We now consider an elliptic surface X with $\pi = \pi_1(X)$ non-cyclic and $p_g = 0$. If π is periodic dihedral, we compare $X \sharp \overline{CP}^2$ with $\Sigma \sharp (S^2 \times S^2) \sharp 9\overline{CP}^2$, where Σ is a suitable rational homology 4-sphere. These two smooth manifolds are stably homeomorphic, and we are done again by [**II**, Thm. B].

It remains to consider the case when $\pi_1(X)$ is non-periodic and $p_g = 0$. For any manifold V' cobordant to V_g, a cobordism Z between $X \sharp \overline{CP}^2$ and $Y_+ \sharp V' \sharp \overline{CP}^2$ can be constructed by attaching W to $V_g \times I$ along $T^2 \times D^2 \subset V_g \times 0$, and then glueing on any 1-connected cobordism U between $V_g \times 1$ and V'. To the result, we attach $\overline{CP}^2 \times I$ by "connected sum along I". In the present situation, $V_g = 9\overline{CP}^2 \sharp CP^2$ and we take $V' = 8\overline{CP}^2$.

After the connected sum with $\overline{CP}^2$ the normal 2-type is $B = K(\pi, 1) \times BSO$ and Z is a bordism between the two normal 1-smoothings. The next step is to extend the map $\gamma : H_2(W) \to T$ over Z so that its restriction to $H_2(X)$ is also surjective. Since T has at most three generators over A (see [**I**, (2.7)]) this is straightforward. Note that since the $T^2 \times D^2$ used to attach W and V_g has a simply connected complement in V_g, the module $H_2(V_g^0)$ is a free A-module in the induced coefficient system. The extended map γ is induced by a geometric map $W \to K(\pi,1) \times K(T,2)$, since the first k-invariant of W vanishes under the induced homomorphism $\gamma_* : H^3(\pi, H_2(W)) \to H^3(\pi, T)$ by Lemma 1.16. From this we conclude that Z is a bordism of two 1-smoothings into $B' = B \times K(T,2)$, and hence $X \sharp \overline{CP}^2$ and $Y_+ \sharp V' \sharp \overline{CP}^2$ are stably homeomorphic over B'.

By induction, we can assume that there is a homeomorphism

$$h : X \sharp \overline{CP}^2 \sharp (S^2 \times S^2) \xrightarrow{\approx} Y_+ \sharp V' \sharp \overline{CP}^2 \sharp (S^2 \times S^2)$$

We will now apply the results of **[II]** to geometrically cancel the last $(S^2 \times S^2)$-factor.

By Lemma 1.17, the submodule $\ker \gamma_+ \subset (H_2(Y_+), S_{Y_+})$ is a quadratic submodule of $Met(p_1 A) \perp Met(L)$. We fix an isometry $\tau : Met(p_1 A) \cong H(p_1 A)$ and use it to identify these quadratic modules. Let $H(p_0 A)$ denote the intersection form of the last $(S^2 \times S^2)$-factor, and define

$$N = H(p_0 A) \perp \ker \gamma_+ \perp (K\pi_2(V' \sharp \overline{CP}^2),$$

where $K\pi_2(V' \sharp \overline{CP}^2)$ is the kernel of w_2 in $\pi_2(V' \sharp \overline{CP}^2)$. Now let

$$M = H(p_0 A) \perp H(p_1 A) \perp Met(L) \perp K\pi_2(V' \sharp \overline{CP}^2)$$

and embed $N \subset M$ as a quadratic submodule using Lemma 1.17 and τ.

We will now check that N and $\mathfrak{D} = \mathrm{Ann}(M/N)$ satisfy the assumptions of **[II**, Theorem 1.19]. The ideal $\mathfrak{D} = \mathfrak{A}$ from (1.15), by construction. To find a subgroup $G_0 \subseteq U(H(P))$ which is $(N, H(P), \epsilon)$-transitive, we apply **[I**, (2.9)] and **[II**, (1.17)]. The form $Met(L)$ has (A, B)-hyperbolic rank ≥ 1 by construction.

We may now conclude that algebraic cancellation is possible, and geometric cancellation follows if we can realize the necessary self-automorphisms of N by homeomorphisms of $Y_+ \sharp V' \sharp \overline{CP}^2 \sharp (S^2 \times S^2)$. These are listed in **[II**, (1.18), (1.11)]. For the elements of G_0, we use the fact established in **[I**, Lemma 2.11] that the linear automorphisms of **[I**, (2.9)] are all realized by simple homotopy equivalences of the two-complex K used to construct Y_+. The argument was to check that the k-invariant of K is p-reserved by such linear automorphisms, and then use $SK_1(\mathbf{Z}\pi) = 0$ (valid for finite subgroups of $SO(3)$ by [**12**, 14.1, 14.5]) to show that the induced homotopy equivalence is simple. Since Y_+ is the boundary of a 5-dimensional thickening of $K \subset \mathbf{R}^5$, the s-cobordism theorem [7] implies that simple homotopy equivalences of K induce self-homomeomorphisms of Y_+. The effect on $\pi_2(Y_+)$ is to apply the hyperbolic functor to the original linear automorphism of $\pi_2(K)$.

To realize the elementary automorphisms in

$$EU(H(P), Q; Met(L) \perp K\pi_2(V' \sharp \overline{CP}^2)),$$

we can apply **[II**, 2.3] with $V_0 = H_2(Y_+)$. It follows that $X \sharp \overline{CP}^2$ and $Y_+ \sharp V' \sharp \overline{CP}^2$ are homeomorphic. ∎

§2: Metabolic Forms

In this section we return to our original algebraic setting. Let R be a Dedekind domain and F its field of quotients. and recall that a *lattice* over an R-order A is an A-module which is projective as an R-module. Let A be an order in a separable algebras over F [**4**, 71.1, 75.1]. In [**I**] we introduced the following definition: a finitely generated A-module L has *(A, B)-free rank ≥ 1 at a prime* $\mathfrak{p} \in R$, if there exists an integer r such that $(B^r \oplus L)_{\mathfrak{p}}$ has free rank ≥ 1 over $A_{\mathfrak{p}}$. Here $A_{\mathfrak{p}}$ denotes the localized order $A \otimes R_{(\mathfrak{p})}$.

Similarly, we will say that a quadratic module V has *(A, B)-hyperbolic rank ≥ 1 at a prime* $\mathfrak{p} \in R$ if there exists an integer r such that $(H(B^r) \oplus V)_{\mathfrak{p}}$ has free hyperbolic rank ≥ 1 over $A_{\mathfrak{p}}$. Our general reference for quadratic and hermitian forms is [**2**, pp. 80, 87].

One way to obtain quadratic modules V with (A, B)-hyperbolic rank ≥ 1 at all but finitely many primes is to assume that V has a submodule $Met(L)$ where L has (A, B)-free rank ≥ 1. A generalization of this would be to assume that V contains a "metabolic form" on a non-split extension of L and $\bar{L}$. In this section we define a notion of metabolic forms general enough for our applications to topological 4-manifolds in §3. The notation and conventions of [II,§1]will be used.

If N is an A-lattice and $g : \bar{N} \times \bar{N} \to A$ is an R-bilinear form, let

$$[g] = \{g_\tau | g_\tau(\phi, \phi') = g(\phi, \phi') + < \phi,\ \tau(\phi') >, \tau \in \mathrm{Hom}_R(\bar{N},\ N)\}.$$

Any $\theta \in \mathrm{Ext}^1_A(\bar{N},\ N)$ defines an extension

$$0 \to N \xrightarrow{i} E \xrightarrow{j} \bar{N} \to 0 \tag{2.1}$$

of A-lattices which splits over R. We say that $[g]$ is *θ-sesquilinear* if there is a cocycle $\gamma \in \mathrm{Hom}_R(\bar{N} \otimes_R \mathrm{A},\ N)$ representing θ, and A-maps $\alpha, \beta \in \mathrm{Hom}_A(N, N)$ such that for all $a \in A$:

$$\begin{aligned} g(\phi a, \phi') &= \bar{a} g(\phi, \phi') - \lambda\overline{\langle \alpha^*(\phi'), \gamma(\phi, a)\rangle} \\ g(\phi, \phi' a) &= g(\phi, \phi')a - \langle \beta^*(\phi), \gamma(\phi', a)\rangle. \end{aligned} \tag{2.2}$$

Note that any cocycle γ satisfies the relation:

$$\gamma(\phi, a_1 a_2) = \gamma(\phi, a_1)a_2 + \gamma(\phi a_1, a_2)$$

and serves as a way to specify the A-module structure E on the R-module $N \oplus \bar{N}$ given by θ. For $(x, \phi) \in N \oplus \bar{N}$ define

$$(x, \phi) \cdot a = (xa + \gamma(\phi, a), \phi a). \tag{2.3}$$

If we vary the choice of representative $g_\tau \in [g]$, then the new γ is $\gamma_\tau = \gamma + \delta\tau$, where

$$(\delta\tau)(\phi, a) = \tau(\phi)a - \tau(\phi a),$$

for some $\tau \in \mathrm{Hom}_R(\bar{N},\, N)$, and all $a \in A$. Then $g_\tau(\phi, \phi') = g(\phi, \phi') + < \phi,\, \tau(\phi') >$ satisfies (2.2). Given an extension $(N,\, \theta)$ and a θ-sesquilinear form $[g]$ with $\alpha + \beta = 1$, we define the *metabolic* (λ, Λ)-quadratic form $Met(N,\, \theta,\, [g]) = (E,\, [q])$ as follows: pick a compatible γ, g satisfying (2.2) and set

$$q((x,\phi),(x',\phi')) = \langle \beta^*(\phi), x'\rangle + \lambda\overline{\langle \alpha^*(\phi'), x\rangle} + g(\phi,\phi'). \tag{2.4}$$

It is easy to check that q is sesquilinear in the usual sense if $[g]$ is θ-sesquilinear. Since $\alpha + \beta = 1$ the associated hermitian form $q + \lambda q^*$ is non-singular. We remark that the special case $\alpha = 0, \beta = 1$ gives the usual definition of a quadratic metabolic form on the split extension.

An arbitrary extension need not admit any such form and we wish to determine the obstructions. Suppose that N is reflexive and let τ denote the involution on $\mathrm{Ext}^1_A(\bar{N}, N)$ given by dualizing exact sequences $(N, \theta) \mapsto (N, \theta)^*$. An extension $(N,\, \theta)$ is λ-*self-dual* (*i.e.* $(N,\theta)^* = \lambda(N,\theta)$) if N is reflexive and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \xrightarrow{i} & E & \xrightarrow{j} & \bar{N} & \to & 0 \\ & & \| & & \downarrow h & & \downarrow \cdot\lambda & & \\ 0 & \to & N & \xrightarrow{j^*} & \bar{E} & \xrightarrow{i^*} & \bar{N} & \to & 0 \end{array} \tag{2.5}$$

If $h^* = \lambda h$ then h is the adjoint of a metabolic hermitian form on E. We will define a homomorphism

$$\rho : \{(N,\theta)^* = \lambda(N,\theta)\} \subseteq \mathrm{Ext}^1_A(\bar{N},\, N) \to H^1(\mathbf{Z}/2;\, \mathrm{Hom}_A(\bar{N},\, N))$$

where $\mathrm{Hom}_A(\bar{N},\, N)$ has the involution $\alpha \mapsto \bar{\lambda}\alpha^*$. We will show that $\rho(N,\theta)$ is the obstruction for finding a λ-self-dual map h. Choose an R-section $s : \bar{N} \to E$ inducing a cocycle γ and identify $E = N \oplus \bar{N}$ as above. Then the lower sequence is split over R by s^* leading to an identification of $\bar{E} = N \oplus \bar{N}$. In these coordinates, for any A-map h making the diagram (2.5) commute,

$$h(x,\phi) = (x + s^*hs(\phi), \bar{\lambda}\phi)$$

and similarly

$$h^*(x,\phi) = (\lambda x + s^*h^*s(\phi), \phi).$$

Now $(h^*)^{-1} \circ \lambda h(x,\phi) = (x + \rho(h)(\phi),\ \phi)$ where $\rho(h) = s^*hs - \bar{\lambda}s^*h^*s$. Since $(h^*)^{-1} \circ \lambda h$ is an A-map, we can check using (2.3) that $\rho(h)$ is also an A-map. Similarly, by computing $h^* \circ (\bar{\lambda}h^{-1})$ and comparing with the formula for the dual, we see that $\rho(h)^* = -\lambda\rho(h)$. Moreover the cohomology class

$$[\rho(h)] \in H^1(\mathbf{Z}/2; \mathrm{Hom}_A(\bar{N}, N))$$

is independent of the choice of h and the choice of the section s. Define $\rho(N,\, \theta) = [\rho(h)]$ for any h making the diagram (2.5) commute.

Proposition 2.6: *If N is a reflexive A-module and $(N,\,\theta)$ is a λ-self-dual extension, then $(N,\,\theta)$ admits a metabolic λ-hermitian form if and only if $\rho(N,\,\theta) = 0 \in H^1(\mathbf{Z}/2;\ \mathrm{Hom}_A(\bar{N},\,N))$.*

Remark 2.7: A non-singular metabolic λ-hermitian form is unique up to isometry if it admits a quadratic refinement. This is easy if the maps α are the same for the two refinements, since the difference between the two quadratic maps is an A-module homomorphism which can be used to define the isometry.

We want to identify the obstruction to obtaining a quadratic refinement, given a metabolic λ-hermitian form h on the extension. Let

$$\mathrm{Hom}^0_A(E,\bar{E}) = \{g : E \to \bar{E} \mid i^*gi = 0, g \text{ an } A\text{-homomorphism}\}$$

Then define

$$\eta : \ker\,\rho \to \mathrm{coker}\{\hat{H}^0(\mathbf{Z}/2;\ \mathrm{Hom}_A(\bar{N},N)) \to \hat{H}^0(\mathbf{Z}/2; \mathrm{Hom}^0_A(E,\bar{E}))\}.$$

as the homomorphism $\eta(N,\theta) = [h]$. The map of Tate cohomology groups is induced by the homomorphism $\alpha \mapsto j^*\alpha j$, for any $\alpha \in \mathrm{Hom}_A(\bar{N},N)$.

Proposition 2.8: *Suppose that (N,θ) admits a metabolic λ-hermitian form. Then (N,θ) admits a metabolic (λ,Λ)-quadratic form with respect to the minimal form parameter if and only if $\eta(N,\theta) = 0$.*

Proof: If the obstruction is zero, we can write $h = q + \lambda q^*$, for some A-map q such that $i^*qi = 0$. Now q fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \xrightarrow{i} & E & \xrightarrow{j} & \bar{N} & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle\lambda\beta^*} & & \\ 0 & \to & N & \xrightarrow{j_*} & \bar{E} & \xrightarrow{i^*} & \bar{N} & \to & 0 \end{array}$$

for some A-maps α and β. It is easy to check that q is the adjoint of a θ- sesquilinear form as in (2.4). ∎

For our geometric applications it is useful to identify the obstruction to the existence of a (λ,Λ)-Quadratic refinement of a metabolic hermitian form. Let

$$\gamma : \ker\,\rho \to \mathrm{coker}\{\hat{H}^0(\mathbf{Z}/2;\ \mathrm{Hom}_A(\bar{N},N)) \to \hat{H}^0(\mathbf{Z}/2; \mathrm{Hom}_R(\bar{N},N))\}.$$

be the homomorphism defined by $\gamma(N,\theta) = [s^*hs]$, where h is a metabolic λ-hermitian form on the extension.

Proposition 2.9: *Suppose that (N, θ) admits a metabolic λ-hermitian form and that $H^1(\mathbf{Z}/2, A) = 0$, where the $\mathbf{Z}/2$ action on A is given by $a \mapsto \lambda\bar{a}$. Then (N, θ) admits a metabolic (λ, Λ)-Quadratic form with respect to the minimal form parameter if and only if $\gamma(N, \theta) = 0$.*

Proof: Under the assumption $H^1(\mathbf{Z}/2, A) = 0$, an element $a \in A$ such that $a = b + \lambda\bar{b}$ determines b uniquely modulo the minimal form parameter $\Lambda = \{c - \lambda\bar{c} \mid c \in A\}$. Now the condition that $\gamma(N, \theta) = 0$ is equivalent to the existence of a metabolic form such that for each $e \in E$ there exists an $b \in A$ with $h(e)(e) = b + \lambda\bar{b}$. We can define $q : E \to A/\Lambda$ by $q(e) = [b]$. ▪

Remark 2.10: Without the assumption that $H^1(\mathbf{Z}/2, A) = 0$, we get a quadratic refinement with respect to the maximal form-parameter if and only if $\gamma(N, \theta) = 0$. Suppose that $R = \mathbf{Z}$ and $A = \mathbf{Z}\pi$ where π is a finite group. If A has the involution induced by $g \mapsto g^{-1}$, for $g \in \pi$ and $\lambda = +1$, then $H^1(\mathbf{Z}/2, A) = 0$. Note that this is not always true for involutions on the group ring. For the standard involutions $g \mapsto w(g)g^{-1}$ arising from and orientation character $w : \pi \to \mathbf{Z}/2$, the maximal form-parameter is generated by $\{a - \lambda\bar{a} \mid a \in A\} \cup \{g \in \pi \mid g^2 = 1, w(g) = -\bar{\lambda}\}$.

Remark 2.11: Notice that from (2.4), a metabolic quadratic form has associated quadratic function $[q](x, \phi) = \langle \phi, x \rangle + g(\phi, \phi)$, $(\bmod \{a - \lambda\bar{a}\})$. This is exactly the usual formula for the split extension.

For the rest of this section we assume that $R = \mathbf{Z}$ and $A = \mathbf{Z}\pi$ where π is a finite group. Then each lattice L over A is reflexive. Let $N = \Omega^k\mathbf{Z}$, the kernel of a projective resolution F_* of $\mathbf{Z}$ of length k (see [I, (0.1)] for the case $k = 3$). We will show that every element of $\mathrm{Ext}^1_A(\bar{N}, N)$ is $(-1)^{k+1}$-self-dual.

Lemma 2.12: *Let $N = \Omega^k\mathbf{Z}$. The involution τ given by dualizing exact sequences induces multiplication by $(-1)^{k+1}$ on $\mathrm{Ext}^1_A(\bar{N}, N)$.*

Proof: Let $\bar{X}$ be a projective resolution of $\bar{N}$ and X the dual co-resolution of N. We have two isomorphisms $\alpha, \beta : \mathrm{Ext}^1(\bar{N}, N) \cong H^1(\mathrm{Hom}_A(\bar{X}, X))$ comparing an extension with $\bar{X}$ or X respectively. Note that over $A = \mathbf{Z}\pi$ we can use X instead of an injective co-resolution for computing $\mathrm{Ext}^i(\bar{N}, N)$. It is not difficult to see that $\alpha = -\beta$. Let t be the involution on $H^1(\mathrm{Hom}_A(\bar{X}, X))$ induced by dualization. By construction, $\alpha\tau = t\beta$ implying $\alpha\tau\alpha^{-1} = -t$.

Note that $\mathrm{Hom}_A(\bar{X}, X) \cong \mathrm{Hom}_{\mathbf{Z}}(\bar{X}, X) \otimes_A \mathbf{Z}$, and that $\mathrm{Hom}_{\mathbf{Z}}(\bar{X}, X)$ is a co-resolution of $\mathrm{Hom}_{\mathbf{Z}}(\bar{N}, N)$. Thus

$$H^i(\mathrm{Hom}_A(\bar{X}, X)) = H^i(\mathrm{Hom}_{\mathbf{Z}}(\bar{X}, X) \otimes_A \mathbf{Z}) = H^i(\pi; N \otimes_{\mathbf{Z}} N)$$

and under these identifications t corresponds to the involution induced by the flip map $s : x \otimes y \mapsto y \otimes x$ on $N \otimes N$. Hence $\alpha\tau\alpha^{-1} = -s$ and we finish by applying the following more general remark.

Sublemma 2.13: *The flip map $s : x \otimes y \mapsto y \otimes x$ on $N \otimes N$ induces multiplication by $(-1)^k$ on Tate cohomology $\hat{H}^i(\pi; N \otimes_{\mathbf{Z}} N)$ for each $i \geq 0$.*

Proof of the Sublemma: We follow an argument suggested by R. Swan (compare [3]). Extend the projective resolution F defining N to a projective resolution $\hat{F}$ of $\mathbf{Z}$. Let f be the chain map on $\hat{F} \otimes_{\mathbf{Z}} \hat{F}$ mapping $x \otimes y \mapsto (-1)^{deg(x)deg(y)} y \otimes x$. Since f induces the identity on $\mathbf{Z}$ it induces the identity on all the derived functors. We have the similar chain map on $F \otimes_{\mathbf{Z}} F$ which on $F_{2k} = N \otimes N$ is $(-1)^k s$. Now we consider $F \otimes_{\mathbf{Z}} F$ as part of a co-resolution of $N \otimes N$ ending in $\mathbf{Z}$. Similarly we consider $\hat{F} \otimes_{\mathbf{Z}} \hat{F}$ a part of a complete co-resolution of $\mathbf{Z}$. Then

$$H^i(\pi; N \otimes_{\mathbf{Z}} N) = H^i(\mathrm{Hom}_A(\mathbf{Z}, F \otimes_{\mathbf{Z}} F)) \cong H^i(\mathrm{Hom}_A(\mathbf{Z}, \hat{F} \otimes_{\mathbf{Z}} \hat{F}))$$

where the last isomorphism is induced by the obvious chain map $\hat{F} \to F$. Thus $s = (-1)^k f^* = (-1)^k$. ∎

Example 2.14: *Now we restrict to groups π of odd order. Since $\mathrm{Ext}^1_{\mathbf{Z}\pi}(\bar{N}, N)$ then has odd order $\rho(N, \theta)$ and $\eta(N, \theta)$ vanish for each λ-self-dual extension. In particular for $N = \Omega^k \mathbf{Z}$ and $\lambda = (-1)^{k+1}$, each extension (N, θ) admits a metabolic (λ, Λ)-quadratic form whose λ-symmetrization is unique up to isometry.*

§3: Four-manifolds with odd order fundamental group

We now apply the results of §2 to prove Theorem B. The method of proof is to construct a model for M and then apply our cancellation theorem. First, let X denote a closed, oriented 4-manifold with $\pi_1(X) = \pi$ and $\sigma(X) = 0$, representing the fundamental class of a spin 4-manifold M in $H_4(\pi, \mathbf{Z})$. Note that since π has odd order, any class in $H_4(\pi, \mathbf{Z})$ can be realized in this way. We may assume (by forming the connected sum with enough copies of $S^2 \times S^2$ that there is a short exact sequence [8, 2.4(i)]:

$$0 \to \mathfrak{N} \to \pi_2(X) \to \bar{\mathfrak{N}} \to 0,$$

where $\mathfrak{N}$ is some representative of $\Omega^3 \mathbf{Z}$. Then $X = K \cup_\alpha D^4$, where K is a finite 3-complex. The attaching map $\alpha \in \pi_3(K)$ which sits in an exact sequence (see [8, §1])

$$0 \to \Gamma(\pi_2(X)) \to \pi_3(K) \to H_3(\tilde{K}, \mathbf{Z}) \to 0. \tag{3.1}$$

The module $H_3(\tilde{K}, \mathbf{Z}) \cong \mathfrak{I}(\pi)^*$ as an A-module and α maps to a generator (it is a cyclic module).

Lemma 3.2: *If π has odd order, then the Tate cohomology groups $\hat{H}^i(\pi, \Gamma(L)) = 0$ for L any representative of $\Omega^3 \mathbf{Z}$ or $S^3 \mathbf{Z}$.*

Proof: We apply Lemma 2.13 following the method of [3] to conclude that these Tate cohomology groups have exponent four for π any finite group. Since π here has odd order, they must vanish. ∎

We can now study $\hat{H}^i(\pi, \Gamma(\pi_2(X))$ using the exact sequences

$$0 \to \Gamma(\Omega^3\mathbf{Z}) \to \Gamma(\pi_2(X)) \to D \to 0,$$

$$0 \to \Omega^3\mathbf{Z} \otimes S^3\mathbf{Z} \to D \to \Gamma(S^3\mathbf{Z}) \to 0,$$

$$0 \to K \to \Gamma(\pi_2(X)) \to \Gamma(S^3\mathbf{Z}) \to 0.$$

These sequences can be combined into a commutative diagram. We use the short notation $\hat{H}^i(L) \equiv \hat{H}^i(\pi, L)$ for the Tate cohomology groups of π with coefficients in an A-module L. In particular, the group $H^0(\pi, L) = L^\pi/\Sigma L$, where Σ denotes the norm map (multiplication by the group ring element $\Sigma = \sum\{g \mid g \in \pi\}$).

$$\begin{tikzcd}
\hat{H}^0(\Gamma(\Omega^3\mathbf{Z})) \arrow[dr] \arrow[rr, bend left] & & \hat{H}^0(\Gamma(\pi_2)) \arrow[dr] \arrow[rr, bend left] & & \hat{H}^0(\Gamma(S^3\mathbf{Z})) \arrow[dr] & & 0 \\
& \hat{H}^0(K) \arrow[ur] \arrow[dr] & & \hat{H}^0(D) \arrow[ur] \arrow[dr] & & \hat{H}^1(K) \arrow[ur] & \\
0 \arrow[ur] \arrow[rr, bend right] & & \hat{H}^0(\Omega^3\mathbf{Z} \otimes S^3\mathbf{Z}) \arrow[ur] \arrow[rr, bend right] & & \hat{H}^1(\Gamma(\Omega^3\mathbf{Z})) \arrow[ur] & &
\end{tikzcd}$$

Since $\hat{H}^0(\pi, \Omega^3\mathbf{Z} \otimes S^3\mathbf{Z}) = \mathbf{Z}/|\pi|$, we get (for any group of odd order)

$$\hat{H}^0(\pi, \Gamma(\pi_2(X)) = \mathbf{Z}/|\pi|. \tag{3.3}$$

Let S_X denote the equivariant intersection form on $\pi_2(X)$. We can construct other complexes by varying the attaching map α. More precisely, we can attach the top cell by any element $\alpha + f$, where $f \in \Gamma(\pi_2(X))$. The equivariant intersection form on π_2 for the new complex X_f is $S_X + \Sigma(f)$. Since $f \in \Gamma(\pi_2(X))$, our new attaching map $\alpha + f$ has the same image in $H_3(\tilde{K}, \mathbf{Z})$ as α. Hence to obtain a new Poincaré complex it remains to arrange that the new intersection form is non-singular.

Lemma 3.4: *Suppose that $\pi_1(X)$ has odd order. There exists a closed topological 4-manifold Y with $\pi_1(Y) = \pi$, $\sigma(Y) = 0$ and the same w_2-type as X, such that when $w_2(\tilde{X}) = 0$ (resp. $w_2(\tilde{X}) \neq 0$), $e(Y) = 2d(\pi)$, (resp. $= 2d(\pi) + 2$)). Furthermore, Y represents the same class in $H_4(\pi, \mathbf{Z})$ as X.*

Proof: We will give the proof when X is spin; in the non-spin case we form the connected sum with $CP^2 \sharp \overline{CP}^2$ to finish. Our construction of Y will consist of attaching suitable cells of dimension ≥ 3 to X and hence a reference map $c : Y \to K(\pi, 1)$ is preserved. It follows that the image of the fundamental class $c_*[X] \in H_4(\pi, \mathbf{Z})$ is not changed by attaching cells using element of $\Gamma(\pi_2(X))$. This uses the fibration $\tilde{B} \to B \to K(\pi, 1)$, and the fact that $H_4(\tilde{B}, \mathbf{Z}) \cong \Gamma(\pi_2(X))$.

We need the following result [3]: when $\pi_1(X)$ has odd order, the sequence

$$0 \to \mathbf{Z} \to \Gamma(\pi_2(X)) \to \pi_3(X) \to 0$$

is split exact as a sequence of A-modules. The generator of $\mathbf{Z}$ maps to the element $\Sigma(\alpha)$. It follows that $\Sigma(\alpha)$ represents a generator of the group $\mathbf{Z}/|\pi$ under the isomorphism (3.3).

We begin by noting that the image of $[S_X]$ under the natural map $\hat{H}^0(\pi,\Gamma(\pi_2)) \to \hat{H}^0(\pi,\Gamma(S^3\mathbf{Z}))$ is just the restriction of the form to the submodule $\mathfrak{N} = \Omega^3\mathbf{Z}$ of $\pi_2(X)$. Since the group $\hat{H}^0(\pi,\Gamma(S^3\mathbf{Z})) = 0$, we can re-attach the top cell to get a complex X_1 with a metabolic intersection form (possibly singular). Let $\mathfrak{N} = L \oplus A^s$, where L has no projective direct summands. Then $\pi_2(X) = E \oplus A^s \oplus \bar{A}^s$, where E is a well-defined extension of $\bar{L}$ by L given by pulling-back the extension (3.1). Let S_L denote the form S_{X_1} restricted to the direct summand $E \subset \pi_2(X)$. Since the class of $[S_{X_1}] \in \hat{H}^0(\pi,\Gamma(\pi_2))$ is unchanged, it follows that S_L is non-degenerate, with determinant prime to $|\pi|$, after possibly varying by an element of Im Σ. This can be verified by considering the class in

$$\hat{H}^0(\pi,\Omega^3\mathbf{Z}\otimes S^3\mathbf{Z}) = \hat{H}^0(\pi,\mathrm{Hom}_A(L,L)) = \mathbf{Z}/|\pi|$$

represented by an off-diagonal block of S_L, using a splitting over $\mathbf{Z}$ to write the matrices. Now we complete at a prime dividing $|\pi|$ and we find that our class is represented by a unit in $\mathrm{Hom}_A(L,L)$ modulo Im Σ.

The next step is purely algebraic. Any non-degenerate metabolic form on the extension $E \oplus A^s \oplus \bar{A}^s$ with $N = L \oplus A^s$ totally isotropic is the restriction of a non-singular metabolic form on $E \oplus A^{2(r+s)}$ with $L \oplus A^{r+s}$ totally isotropic. To see this, note that since X was spin the form S_{X_1} admits a quadratic refinement (see Proposition 2.8). Now the form S_{X_1} is the pull-back of forms over $\hat{\mathbf{Z}}\pi$ and $\mathbf{Q}\pi$, glued together over $\hat{\mathbf{Q}}\pi$. This reduces our problem to forms over fields where it is trivial. The pull-back gives a stabilized form h on $E \oplus P \oplus \bar{P}$ for some projective module P. By forming the sum with $H(Q)$, where $P \oplus Q = A^{(r+s)}$, we are done.

Next, observe that the difference $h - S_{X_1} = \Sigma(f)$ for some $f \in \Gamma(E \oplus A^{2r})$. It follows that we can stabilize X_1 by copies of $S^2 \times S^2$ and then re-attach the top cell to get a finite Poincaré 4-complex X_2 with metabolic intersection form $S_{X_2} = h$. Since $(\pi_2(X_2), h)$ contains the totally isotropic submodule $N \oplus A^r \cong L \oplus A^{r+s}$, we can write $h \cong h' \perp H(A^{r+s})$. The final step is to attach cells to X_2 to kill the hyperbolic summand $H(A^{r+s})$. The resulting Poincaré complex is called X'.

Our final step is to go from a Poincaré complex to a topological manifold. There exists a degree one normal map $Y_1 \to X'$ where Y_1 is a closed topological 4-manifold. The intersection form on Y_1 has signature zero and contains the intersection form of X' as an orthogonal direct summand.

Lemma 3.5: [16] *Let π be a finite group of odd order. Any element of $L_4^h(\mathbf{Z}\pi)$ with multisignature zero can be represented by a form $H(P)$, where P is a projective module over $\mathbf{Z}\pi$.*

Every projective module over $\mathbf{Z}\pi$ has the form $P = P_1 \oplus A^k$, where P_1 has rank one, by Swan's Theorem [**14**]. Using this result and our improvement of the Roiter Replacement Theorem [**I**, (1.19)], we see that $(\pi_2(Y_1), S_{Y_1})$ contains a hyperbolic summand $H(A^{k+1})$. Now we can surger away these hyperbolic planes in Y_1 to obtain the required 4-manifold Y. ∎

The Proof of Theorem B: The basic part of our model for M is provided by the manifold Y from Lemma 3.4. To obtain the rest we form the connected sum of Y with a suitable simply-connected 4-manifold, including at least one $S^2 \times S^2$. The proof is now finished by [**II**, Thm. B]. ∎

§4: Four-manifolds with cyclic fundamental group

The goal of this section is to prove Theorem C. We will fix the notation $\pi = C_n$ for the cyclic group of order n, and $\mathfrak{I}$ for the augmentation ideal in $A = \mathbf{Z}\pi$. By [**8**, Thm.B] we can assume that n is even. We showed in [**9**, p.57], or [**11**, §3] that the stable homeomorphism types are of the form $\Sigma \sharp\ Z$, where Σ is a rational homology sphere, and Z is a 1-connected closed 4-manifold. Recall that there are three w_2-types: (I) $w_2(\tilde{X}) \neq 0$, (II) $w_2(X) = 0$, and (III) $w_2(\tilde{X}) = 0$, but $w_2(X) \neq 0$.

Proposition 4.1: *For any n, there exist a rational homology spheres with fundamental group C_n, w_2-types* II *or* III, *and hyperbolic equivariant intersection form on the universal covering. In w_2-types* III, *there exist such rational homology spheres with either value of the Kirby-Siebenmann invariant.*

Proof: Since we may construct a rational homology sphere as the double of a suitable 4-dimensional thickening of a 2-complex with cyclic π_1, it is clear that the intersection form can always be chosen metabolic on $\mathfrak{I} \oplus \bar{\mathfrak{I}}$ (see [**8**, p.99]). Also, we have shown in [**8**, 4.5] that for $\pi = C_2$, both w_2-types can be realized with hyperbolic intersection forms $H(\mathfrak{I})$. For n odd it is also true since metabolic implies hyperbolic in this case by (2.14).

To handle the general case with n even, note that the obstruction to finding a quadratic refinement for the intersection form lies in $\hat{H}^0(\mathbf{Z}/2, \mathfrak{I} \otimes_A \mathfrak{I})$. The restriction map $C_2 \subseteq C_n$ induces an injection on $\hat{H}^0(\mathbf{Z}/2, \mathfrak{I} \otimes_A \mathfrak{I})$. But the covering of our rational homology sphere with fundamental group C_2 is just a rational homology sphere connected sum with $(k-1)$ copies of $S^2 \times S^2$. This has hyperbolic equivariant intersection form. It follows that our obstruction is zero, and from [**2**, p.85] that the rational homology sphere Y with $\pi_1(Y) = C_n$ has hyperbolic intersection form on $\tilde{Y}$. ∎

Proof of Theorem C : For a manifold X with cylic fundamental group π, we abbreviate $H_2(X)/Tors = H$. Then as in the proof of [**II**, (4.2)] we consider the following three fibrations $B(I)$, $B(II)$ and $B(III)$ over $BTop$, for w_2-type (I), (II) and (III) respectively. If X and Y are two manifolds satisfying the given conditions, by [**II**, (4.2)] there is a homeomophism

$$h : X \sharp r(S^2 \times S^2) \to Y \sharp r(S^2 \times S^2)$$

such that h_* restricted to $H_2(X)/Tors$ is a prescribed isometry $\theta : H_2(X)/Tors \to H_2(Y)/Tors$. This implies that the restriction to $H_2(r(S^2 \times S^2)) = H(\mathbf{Z}^r)$ is an isometry. By [**7**] or [**II**, (3.1)], any $\sigma \in U(H_2(X, \mathbf{Z}))$ can be realized by a self-homeomorphism f of $r(S^2 \times S^2)$, and we compose h with $Id_X \sharp f$ to get the restriction of h_* the identity on $H_2(r(S^2 \times S^2))$.

It is enough to prove the result for $X = \Sigma \sharp Z$, where Z is 1-connected, and Σ is a rational homology sphere with the same w_2-type as Y and hyperbolic intersection form on the universal cover.

First we will carry out algebraic cancellation. As usual, we can assume that $r = 1$. The intersection form on $X \sharp (S^2 \times S^2)$ is just $H(P_0) \perp H(\mathfrak{J}) \perp V$, where V is the intersection form of Z. Since $\theta(\epsilon_*(h_*))$ induces the identity on $H(\mathbf{Z}^r)$, we need only prove transitivity on hyperbolic elements in $N = H(P_0) \perp (p_1\mathfrak{J} \oplus \bar{P}_1) \perp V$. This is a quadratic submodule of $M = H(P_0 \oplus P_1) \perp V$ with $\mathrm{Ann}(M/N) = \mathfrak{O}$. We claim that the assumptions of [**II**, Lemma 3.2] are satisfied, with $A = \mathbf{Z}\pi$, $B = \mathbf{Z}[\mathbf{Z}/2]$ and $\mathfrak{O} = \mathfrak{J}$. Indeed, take the group $G_0 = \langle H(SL_2(A;\mathfrak{O}))\cdot EU(H(P);\mathfrak{O})\rangle$, using [**II**, Lemma 3.4] with $\mathfrak{O} = \epsilon_*(\mathfrak{J})$ to establish the condition [**II**, (1.15)(ii)]. The group $\Gamma = SL_2(A; \ker \epsilon)$ has the desired linear transitivity property by [**I**, Lemma 1.15]. Since $\ker \epsilon \subset \mathfrak{J}$, the group G resulting from [**II**, Lemma 3.2] is just the G_0 above. Now to finish the algebraic transitivity, we use [**II**, Theorem 1.11]. This last step uses automorphisms from the group $\langle EU(H(P), Q; V\mathfrak{O}), H(E(P;\mathfrak{O})) \cdot EU(H(P);\mathfrak{O})\rangle$.

For the automorphism g used in the algebraic cancellation, $g \oplus id_{3(S^2\times S^2)}$ can all be realized by self-homeomorphisms of $X \sharp 3(S^2 \times S^2)$. For $H(SL_2(A;\mathfrak{O}))$ we use the fact that Σ is the boundary of a thickening of a two-complex K in $\mathbf{R}^5$, and apply the same argument used in §1. For the elements of $EU(H(P), Q; V\mathfrak{O})$ or $EU(H(P);\mathfrak{O})$ we are done by [**II**, Corollary 2.3], applied with $V_0 = H(\mathfrak{J})$. ∎

The Proof of Corollary D: If X is an algebraic surface with cyclic fundamental group and $e(X) \neq 4$, the result was already proved in [**9**, Cor. 5]. If $e(X) = 4$, we apply Theorem C to conclude that X is homeomorphic to a smooth decomposable manifold of the form $\Sigma \sharp (S^2 \times S^2)$ or $\Sigma \sharp \overline{CP}^2 \sharp CP^2$, where Σ is a rational homology sphere with the correct fundamental group and appropriate w_2. ∎

Remark 4.2: Our methods give new proofs of [**6**, Thm. 1, Thm. 2]. For the $(4k+2)$-dimensional result, we use [**II**, (1.24)] and carry out geometric cancellation. For the $4k$-dimensional case, we use [**II**, (4.2)] and Theorem C.

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