Correlation functions and quantum Knizhnik-Zamolodchikov equation on level -4

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Abstract

This paper is continuation of our previous papers [12] and [13]. We discuss in more detail a new form of solution to the quantum Knizhnik-Zamolodchikov equation [qKZ] on level -4 obtianed in the paper [13] for the Heisenberg XXX spin chain. The main advantage of this form is it's explicit reducibility to one-dimensional integrals. We argue that the deep mathematical reason for this is some special cohomologies of deformed Jacobi varieties. We apply this new form of solution to the correlation functions using the Jimbo-Miwa conjecture [3]. The formula (47) for the correlation functions obtained in this way is in a good agreement with the ansatz for the emptiness formation probability from the paper [12]. Our previous conjecture on a structure of correlation functions of the XXX model in the homogeneous limit through the Riemann zeta functions at odd arguments is a corollary of the formula (47).

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1 Introduction.

In the paper [13] we have suggested a new form of solution to the quantum Knizhnik-Zamolodchikov equation [qKZ] on level -4 [2, 3]. The main idea was to use a duality between the solutions to the qKZ on level -4 and level 0. The latter solutions first appeared in a context of the form-factors of integrable models of quantum field theory [1]. They have much more simple structure in comparison with the qKZ solutions on level -4 because they all can be reduced to single integrals. The main point is to get solutions to the qKZ on level -4 from the known qKZ solutions on level 0 inverting some special matrix $\mathcal{F}$ of dimension $\binom{2n}{n} - \binom{2n}{n-1}$ built up from different level 0 solutions [1, 7]. The matrix may be represented as a product of two matrices $\mathcal{P}$ and $\mathcal{H}$ which carry respectively transcendental and rational dependence on the rapidities $\beta_1, \ldots, \beta_{2n}$. It is possible to invert the transcendental part using some facts about deformed hyper-elliptic integrals, in particular, the deformed Riemann bilinear relation [1, 4, 5]. But the problem to invert the rational part appeared to be rather complicated. In order to do this we need a theorem about a special form of solutions to the qKZ on level -4, namely, an ansatz given by the formula (20) from [13]. We have proved that the functions $\widetilde{h}$ that appear in the above formula are polynomials of all arguments. We know how these polynomials look like for few first cases. Unfortunately, an explicit form of these polynomials in general case is still to be found. In this paper we are trying to make some steps in this direction, namely, we have found a representation for the polynomials $\widetilde{h}$ in terms of contour integrals of a special form. The formula obtained has a cohomological meaning. In some sense it corresponds to a deformation of the cohomologies described by the theorem from the sixth section of [13] conjectured in [15] and proved by Nakayashiki in [16].

The next important issue is to apply the solution to the qKZ on level -4 to the correlation functions of the Heisenberg XXX spin chain. This is possible due to the nice conjecture suggested by Jimbo and Miwa in 1996 [3]. This conjecture says that there is a direct connection of the solution to the qKZ on level -4 with any correlation function for the Heisenberg XXZ model in the massless regime and, in particular, for the XXX model. The integral representation that follows from this conjecture was later confirmed by the algebraic Bethe ansatz approach [8], [9]. Our main statement is that using the Jimbo-Miwa conjecture together with our formulae for the solution to the qKZ on level -4 we come to the result that any correlation function of the XXX model can be expressed through the function G defined by the formulae (3.6) and (3.13) of the paper [12] with coefficients being rational functions like in the ansatz for the emptiness formation probability given by the formula (3.20) of the above mentioned paper. Our previous conjecture [10, 11, 14] that any correlation function in the homogeneous limit can be expressed in terms of the Riemann ζ-function at odd arguments with rational coefficients follows from the above statement which is valid for more general case when the spectral parameters are different.

The paper is organized as follows. In Section 2 we discuss in more detail the new form of solutions to the qKZ on level -4 which is similar to the Smirnov's solution to the qKZ on level 0, namely, it also has a property of reducibility to one-dimensional integrals. In Section 3 we apply this soution to the correlation functions of the XXX model using the Jimbo-Miwa conjecture. Some particular cases are discussed in detail in the Appendix.

2 Solutions to qKZ on level -4

Let us remind the reader the main statement of our previous work [13], namely, the formula from the theorem of the section 5 for the solutions to the qKZ on level -4 counted by integers $\{k_1,\cdots,k_{n-1}\}$, with $|k_j|\leq n-1,\forall j$

$$g^{\{k_1,\cdots,k_{n-1}\}}(\beta_1,\cdots,\beta_{2n})=\frac{e^{\frac{1}{2}\sum\beta_j}}{\sum e^{\beta_j}}\prod_{i<j}\frac{1}{\zeta(\beta_i-\beta_j)}\int\limits_{-\infty}^{\infty}d\alpha_1\cdots\int\limits_{-\infty}^{\infty}d\alpha_{n-1}\prod_{i,j}\varphi(\alpha_i-\beta_j)$$
$$\times\ \mathrm{det}|e^{k_i\alpha_j}|_{1\leq i,j\leq n-1}\ \widetilde{h}(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n}) \tag{1}$$

where

$$\varphi(\alpha)=\Gamma\left(\frac{1}{4}+\frac{\alpha}{2\pi i}\right)\Gamma\left(\frac{1}{4}-\frac{\alpha}{2\pi i}\right) \tag{2}$$

$$\zeta(\beta)=\exp\left(-\int\limits_0^{\infty}\frac{\sin^2\frac{1}{2}(\beta+\pi i)k\ e^{-\frac{\pi k}{2}}}{k\sinh(\pi k)\cosh\left(\frac{\pi k}{2}\right)}\right) \tag{3}$$

and $\widetilde{h}(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n})$ is a polynomial of all its arguments taking values in $(\mathbb{C}^2)^{\otimes 2n}$ which is skew-symmetric with respect to $\alpha_1,\cdots\alpha_{n-1}$.

As was explained in [13] one can present $\widetilde{h}$ in the following form:

$$\widetilde{h}(\alpha_1,\cdots,\alpha_{n-1}|\beta_1,\cdots,\beta_{2n})=\sum_{\{1,\cdots,2n\}=\{i_1,\cdots,i_n\}\cup\{j_1,\cdots,j_n\}}v(\alpha_1,\cdots\alpha_{n-1}|\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n})$$
$$\times\prod_{p,q=1}^{n}\frac{\beta_{i_p}-\beta_{j_q}+\pi i}{\beta_{i_p}-\beta_{j_q}}\ w^{\dagger}_{\epsilon_1,\cdots,\epsilon_{2n}}(\beta_1,\cdots\beta_{2n})$$

where $\epsilon_{i_p}=-$ and $\epsilon_{j_q}=+$. Of course, the function $w^{\dagger}$ also depends on $2n$ external indices. Each of them takes two possible values ±1.

In order to find the function v we should solve two sets of equations. The first one follows from requirement that $\widetilde{h}$ must belong to singlet subspace:

$$\sum_{p=1}^{n+1}v(\alpha_1,\cdots,\alpha_{n-1}|\beta_{i_1},\cdots,\beta_{i_{n-1}},\beta_{j_p}|\beta_{j_1},\cdots,\widehat{\beta_{j_p}},\cdots,\beta_{j_{n+1}})\prod_{q\neq p}\frac{\beta_{j_p}-\beta_{j_q}-\pi i}{\beta_{j_p}-\beta_{j_q}}=0 \tag{4}$$

The second equation is equivalent to the fact that $\widetilde{h}$ is obtained by inverting the matrix $\mathcal{H}$:

$$\sum_{\{1,\cdots,2n\}=\{i_1,\cdots,i_n\}\cup\{j_1,\cdots,j_n\}}v(\alpha_1,\cdots,\alpha_{n-1}|\beta_{j_1},\cdots,\beta_{j_n}|\beta_{i_1},\cdots,\beta_{i_n})$$
$$\times\ u(\alpha_1',\cdots,\alpha_{n-1}'|\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n})\prod_{p,q=1}^{n}\frac{1}{\beta_{i_p}-\beta_{j_q}}=c(\alpha_1,\cdots,\alpha_{n-1}|\alpha_1',\cdots,\alpha_{n-1}') \tag{5}$$

where

$$u(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n}) = \det(A_i(\alpha_j|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n}))|_{i,j=1,\cdots,n-1} \quad (6)$$

and the polynomials $A_i(\alpha)$ depend on β_j as parameters and may be defined through the generating function:

$$\sum_{i=1}^{n-1}\gamma^{n-i-1}A_i(\alpha|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n}) = \frac{\prod_{j=1}^{2n}(\alpha-\beta_j+\frac{\pi i}{2})}{\alpha-\gamma+\pi i} - \frac{\prod_{j=1}^{2n}(\alpha-\beta_j-\frac{\pi i}{2})}{\alpha-\gamma-\pi i} +$$
$$+\frac{\pi i\prod_{j=1}^{n}(\alpha-\beta_j-\frac{\pi i}{2})(\gamma-\beta_{n+j}+\frac{\pi i}{2})}{(\alpha-\gamma)(\alpha-\gamma-\pi i)} + \frac{\pi i\prod_{j=1}^{n}(\gamma-\beta_j-\frac{\pi i}{2})(\alpha-\beta_{n+j}+\frac{\pi i}{2})}{(\alpha-\gamma)(\alpha-\gamma+\pi i)} \quad (7)$$

The function $c(\alpha_1,\cdots\alpha_{n-1}|\alpha_1',\cdots\alpha_{n-1}')$ is the "intersection form". Essential part of this "intersection form" is det $\left|c(\alpha_i,\alpha_j')\right|$ with

$$c(\alpha_1,\alpha_2) = \frac{\prod_{j=1}^{2n}(\alpha_1-\beta_j+\frac{\pi i}{2})}{\alpha_1-\alpha_2+\pi i} - \frac{\prod_{j=1}^{2n}(\alpha_1-\beta_j-\frac{\pi i}{2})}{\alpha_1-\alpha_2-\pi i} - \frac{\prod_{j=1}^{2n}(\alpha_2-\beta_j+\frac{\pi i}{2})}{\alpha_2-\alpha_1+\pi i} + \frac{\prod_{j=1}^{2n}(\alpha_2-\beta_j-\frac{\pi i}{2})}{\alpha_2-\alpha_1-\pi i} \quad (8)$$

The rigorous definition of the "intersection form" was given in the end of section 5 of [13]. Now we do not need it.

Let us consider $v(\alpha_1,\cdots\alpha_{n-1}|\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n})$ for different partitions as $\binom{2n}{n}$ independent unknowns.

Actually, we have $\binom{2n}{n-1}$ equations from (4). In order to analyse the number of equations which follow from the second requirement (5) we need the formula (17) from [13]

$$A_k(\alpha|\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n}) = s_k(\alpha) + \sum_{l=1}^{n-1}c_{kl}(\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n})s_{-l}(\alpha) \quad (9)$$

where

$$s_k(\alpha) = A_k(\alpha|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n}), \quad s_{-k}(\alpha) = \alpha^{n-k-1}, \quad k=1,\dots n-1 \quad (10)$$

and the matrix c_{kl} is symmetric. The formula (9) follows from the fact that

$$c(\alpha_1,\alpha_2) = \sum_{k=1}^{n-1}(s_k(\alpha_1)s_{-k}(\alpha_2) - s_k(\alpha_2)s_{-k}(\alpha_1)) \quad (11)$$

and due to (8) it is symmetric function of all $2n$ variables $\beta_1,\dots,\beta_{2n}$. Using the equation (9) one can express the function $u(\alpha_1',\dots,\alpha_{n-1}'|\beta_{i_1},\dots,\beta_{i_n}|\beta_{j_1},\dots,\beta_{j_n})$ for an arbitrary partition as a linear combination of the determinants det$|s_{j_k}(\alpha_l)|_{1\le k,l\le n-1}$ with
$j_k = -(n-1),\dots,-1,1,\dots,(n-1)$. One can show that due to the relation (11) only $\binom{2n}{n}-\binom{2n}{n-1}$ of such determinants are linearly independent.

Thus two requirements (4) and (5) together provide us with sufficient number of linear equations, namely, $\binom{2n}{n}$ which, in priciple, may be solved and their solution is unique.

From the equations (5) we can deduce recurrent relations for $v(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n})$ which look as follows:

$$
\begin{aligned}
&v(\alpha_1,\cdots\alpha_{n-2},\beta|\beta_1,\cdots,\beta_{n-2},\beta-\frac{\pi i}{2},\beta+\frac{\pi i}{2}|\beta_{n-1},\cdots,\beta_{2n-2}) \;=\; \text{exact form}\\
&v(\alpha_1,\cdots\alpha_{n-2},\beta|\beta_1,\cdots,\beta_n,|\beta_{n+1},\cdots,\beta_{2n-2},\beta-\frac{\pi i}{2},\beta+\frac{\pi i}{2}) \;=\text{exact form}\\
&v(\alpha_1,\cdots\alpha_{n-2},\beta|\beta_1,\cdots,\beta_{n-1},\beta-\frac{\pi i}{2}|\beta_n,\cdots,\beta_{2n-2},\beta+\frac{\pi i}{2})=\text{exact form}\\
&v(\alpha_1,\cdots\alpha_{n-2},\beta|\beta_1,\cdots,\beta_{n-1},\beta+\frac{\pi i}{2}|\beta_n,\cdots,\beta_{2n-2},\beta-\frac{\pi i}{2})=\\
&=\prod_{j=1}^{n-2}(\alpha_j-\beta)\prod_{k=1}^{n-1}\left(\beta-\beta_k-\frac{\pi i}{2}\right)\left(\beta-\beta_{n+k-1}+\frac{\pi i}{2}\right)\\
&\times v(\alpha_1,\cdots\alpha_{n-2}|\beta_1,\cdots,\beta_{n-1}|\beta_n,\cdots,\beta_{2n-2})+\text{exact form}
\end{aligned}
\tag{12}
$$

where "exact form" stands for an expression of the kind:

$$
\begin{aligned}
&\prod_{j=1}^{n-2}(\alpha_j-\beta)\sum_{j=1}^{n-2}(-1)^jE(\alpha_j)m(\alpha_1,\cdots,\widehat{\alpha_j},\cdots,\alpha_{n-2}),\\
&E(\alpha)=\prod_{j=1}^{2n-2}\left(\alpha-\beta_k+\frac{\pi i}{2}\right)-\prod_{j=1}^{2n-2}\left(\alpha-\beta_k-\frac{\pi i}{2}\right)
\end{aligned}
$$

for some skew-symmetric polynomial of $n-2$ variables m. The relations (12) look undetermined because of presence of unknown "exact forms" in the RHS. However, knowing *a priori* degrees of polynomials one can show that this system defines v completely being in fact even overdetermined. Still we have no key for solving this system. At this point the idea arises of trying to find the polynomials v using original Jimbo-Miwa formula. Let us explain this point.

The polynomials v can be decomposed with respect to s_a:

$$
v(\alpha_1,\cdots,\alpha_{n-1})=\sum_{-(n-1)\le j_1<\cdots<j_{n-1}\le n-1}K_{j_1,\cdots,j_{n-1}}\det\left|s_{j_p}(\alpha_q)\right|_{p,q=1,\cdots,n-1}
\tag{13}
$$

where irrelevant arguments are omitted. Let us look for $K_{j_1,\cdots,j_{n-1}}$ in the following form:

$$
K_{j_1,\cdots,j_{n-1}}=\prod_{p=1}^{n-1}\text{sgn}(j_p)\mathcal{L}(s_{-j_1},\cdots,s_{-j_{n-1}})
\tag{14}
$$

where $\mathcal{L}$ is a skew-symmetric multi-linear functional. This functional has to satisfy certain requirements. Recall that it depends upon β_j as parameters, we shall take into account this dependence writing

$$
K_{j_1,\cdots,j_{n-1}}(\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n})
$$

Our main concern is the recurrence relations (12). Now, according to (12), we want to consider the case $\beta_i=\beta+\frac{\pi i}{2}$, $\beta_j=\beta-\frac{\pi i}{2}$ for different choices of i and j. Notice that the formula for $c(\alpha_1,\alpha_2)$ is symmetric with respect to β_j. So, whatever is the choice of β_i and β_j we have:

$$c^{(n)}(\alpha_1,\alpha_2)\big|_{\beta_i=\beta+\frac{\pi i}{2},\beta_j=\beta-\frac{\pi i}{2}}=(\alpha_1-\beta)(\alpha_2-\beta)c^{(n-1)}(\alpha_1,\alpha_2)+(\alpha_1-\beta)E(\alpha_1)-(\alpha_2-\beta)E(\alpha_2)$$

where we wrote down $c^{(n)}$ for the bilinear form depending on $2n$ variables β_k while $c^{(n-1)}$ is similar form depending on 2n-2 β's with β_i and β_j omitted. So, for any choice of β_i, β_j there is a change of basis

$$\widehat{s}_a(\alpha)=C_{ab}\ s_b(\alpha)$$

such that $C\in Sp(2n-2)$ and

$$\begin{aligned}
&\widehat{s}_a(\alpha)|_{\beta_i=\beta+\frac{\pi i}{2},\beta_j=\beta-\frac{\pi i}{2}}=(\alpha-\beta)s_a^{(n-1)}(\alpha),\quad a=-(n-2),\cdots,n-2\\
&\widehat{s}_{-(n-1)}(\alpha)\big|_{\beta_i=\beta+\frac{\pi i}{2},\beta_j=\beta-\frac{\pi i}{2}}=1,\\
&\widehat{s}_{n-1}(\alpha)|_{\beta_i=\beta+\frac{\pi i}{2},\beta_j=\beta-\frac{\pi i}{2}}=(\alpha-\beta)E(\alpha)
\end{aligned}$$

In the formulae (13, 14) we can change simultaneously all s_a to $\widehat{s}_a$. The functional $\mathcal{L}$ depends on β_j as on parameters. The recurrence relations (12) are equivalent to the following ones:

$$\begin{aligned}
&\mathcal{L}(\widehat{s}_{a_1},\cdots,\widehat{s}_{a_{n-2}},\widehat{s}_{n-1})(\beta_1,\cdots,\beta_{n-2},\beta-\frac{\pi i}{2},\beta+\frac{\pi i}{2}|\beta_{n-1},\cdots,\beta_{2n-2})\ =0\\
&\mathcal{L}(\widehat{s}_{a_1},\cdots,\widehat{s}_{a_{n-2}},\widehat{s}_{n-1})(\beta_1,\cdots,\beta_n,|\beta_{n+1},\cdots,\beta_{2n-2},\beta-\frac{\pi i}{2},\beta+\frac{\pi i}{2})\ =0\\
&\mathcal{L}(\widehat{s}_{a_1},\cdots,\widehat{s}_{a_{n-2}},\widehat{s}_{n-1})(\beta_1,\cdots,\beta_{n-1},\beta-\frac{\pi i}{2}|\beta_n,\cdots,\beta_{2n-2},\beta+\frac{\pi i}{2})=0\\
&\mathcal{L}(\widehat{s}_{a_1},\cdots,\widehat{s}_{a_{n-2}},\widehat{s}_{n-1})(\beta_1,\cdots,\beta_{n-1},\beta+\frac{\pi i}{2}|\beta_n,\cdots,\beta_{2n-2},\beta-\frac{\pi i}{2})=\\
&=\prod_{k=1}^{n-1}\left(\beta-\beta_k-\frac{\pi i}{2}\right)\left(\beta-\beta_{n+k-1}+\frac{\pi i}{2}\right)\mathcal{L}(s_{a_1}^{(n-1)},\cdots,s_{a_{n-2}}^{(n-1)})(\beta_1,\cdots,\beta_{n-1}|\beta_n,\cdots,\beta_{2n-2})
\end{aligned}\tag{15}$$

where $a_p>-(n-1)\ \forall p$, the polynomials $\widehat{s}_a$ are constructed according to above procedure.

In fact the original Jimbo-Miwa formula satisfies some kind of similar recurrence relations. We cannot go into much details at this point, but careful development of this idea leads to the following formula for v. Let us define the function $\Phi(\sigma)$

$$\Phi(\sigma)=\prod_{j=1}^{2n}\phi(\sigma-\beta_j)\tag{16}$$

and the function $\phi(\sigma)$ satisfies properties which look similar to those of the function $\varphi(\sigma)$ given by (2), namely, it also satisfies the property

$$\phi(\sigma-\beta+2\pi i)=\phi(\sigma-\beta)\frac{\sigma-\beta+\frac{\pi i}{2}}{\sigma-\beta+\frac{3\pi i}{2}}\tag{17}$$

and the following expansion

$$\phi(\sigma-\beta)=\sigma^{-1/2}(1+c_1(\beta)\sigma^{-1}+c_2(\beta)\sigma^{-2}+\cdots) \tag{18}$$

is valid for $|\sigma| \gg 1$ and β fixed. Combining these two conditions (17) and (18) one can define reccursively the coefficients $c_j(\beta)$ to any given order j. One can also show that the function ϕ satisfies one more condition which will be very important for us, namely,

$$\phi(\sigma)\phi(\sigma+\pi i)=\frac{1}{\sigma+\frac{\pi i}{2}}$$

For the function Φ one has

$$\begin{aligned}\Phi(\sigma+2\pi i)&=\Phi(\alpha)\frac{P(\sigma+\frac{\pi i}{2})}{P(\sigma+\frac{3\pi i}{2})}\\ \Phi(\sigma)\Phi(\sigma+\pi i)&=\frac{1}{P(\sigma+\frac{\pi i}{2})}\end{aligned} \tag{19}$$

where

$$P(\sigma)=\prod_{j=1}^{2n}(\sigma-\beta_j) \tag{20}$$

Consider polynomials $p_1,\cdots,p_{n-1}$ which are taken as a subset of s_a. Using the properties of the function Φ one can check that for such a polynomials

$$\mathrm{res}_\infty(p_i(\sigma)\Phi(\sigma))=0 \tag{21}$$

Then for every p_i let us define:

$$X_i(\sigma)=\Delta^{-1}(p_i(\sigma)\Phi(\sigma)) \tag{22}$$

where by the defenition

$$(\Delta f)(\sigma)=f(\sigma+\pi i)-f(\sigma-\pi i) \tag{23}$$

The r.h.s. of eq. (22) is well defined as asymptotic series since $p_i\Phi$ does not have residue (21). Our first idea for definition of $\mathcal{L}(p_1,\cdots,p_{n-1})$ was as follows:

$$\mathcal{L}^0(p_1,\cdots,p_{n-1})=\int\limits_{\Gamma_1}\frac{d\sigma_1}{2\pi i}\cdots\int\limits_{\Gamma_{n-1}}\frac{d\sigma_{n-1}}{2\pi i}\,h_0(\sigma_1,\cdots,\sigma_{n-1})\prod_{j=1}^{n-1}X_j(\sigma_j)\Phi(\sigma_j)$$

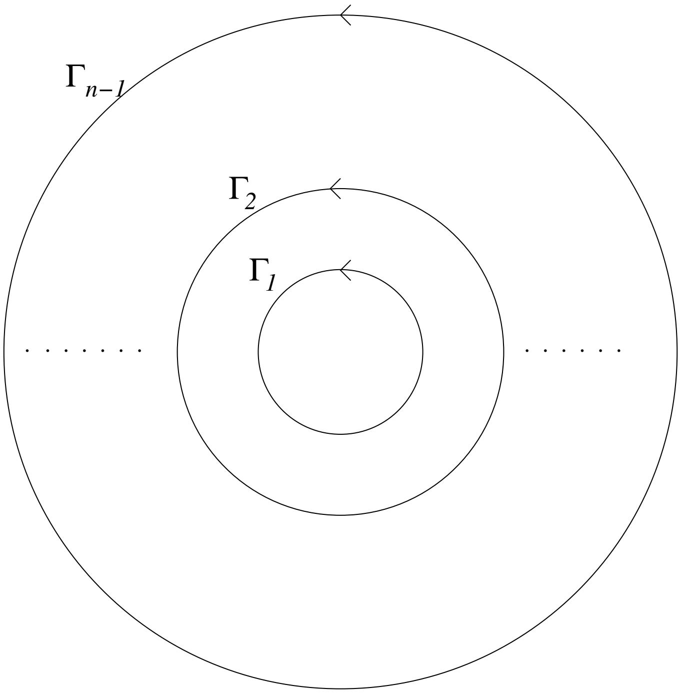
where

$$h_0(\sigma_1,\cdots,\sigma_{n-1})=\sum_{\pi\in S_{n-1}}(-1)^{\mathrm{sgn}\pi}D(\sigma_{\pi\{1\}},\cdots\sigma_{\pi\{n-1\}}|\beta_1,\cdots,\beta_{2n})_{-\cdots-+\cdots+} \tag{24}$$

and the function D is given by the formula (5) of the paper [13]

$$D(\alpha_1, \cdots \alpha_{n-1}|\beta_1, \cdots, \beta_{2n})_{-\cdots-+\cdots+} =$$
$$= \prod_{r>s} \frac{1}{\alpha_r - \alpha_s - \pi i} \prod_k \prod_{j>k} \left(\alpha_k - \beta_j + \frac{\pi i}{2}\right) \prod_{j<k} \left(\alpha_k - \beta_j - \frac{\pi i}{2}\right)$$
$$\times \sum_{l=1}^{n} \left(2 \sum \alpha_k + \beta_l + \pi i(2l-1)\right) \prod_{j \geq l} \frac{\alpha_j - \beta_j - \frac{\pi i}{2}}{\alpha_j - \beta_{j+1} + \frac{\pi i}{2}} \tag{25}$$

Due to the properties of D which ensure correct form of residues in Jimbo-Miwa [3] formulae one can expect that the formula (24) satisfies correct recurrence relations. It is almost the case, but some corrections are needed. The trouble is that h_0 has poles at $\sigma_i = \sigma_j \pm \pi i$. So, first of all we have to explain how the contours Γ_j are drawn. Let us use the following prescription:



Namely, we require that for any j $|\sigma_j| \ll |\sigma_{j+1}|$ where $\sigma_j \in \Gamma_j$ and $\sigma_{j+1} \in \Gamma_{j+1}$. This definition, however brakes the skew-symmetry with respect to polynomials p_i which is crucial for us. Necessary improvement is done as follows. Let us introduce some notations. Let U be a set $U = \{1, \ldots, n-1\}$ which is decomposed into the union of three subsets

$$U = S \bigcup \overline{S} \bigcup T$$

where S and $\overline{S}$ are the subsets of U of the length p with $0 \leq p \leq [\frac{n-1}{2}]$ and T is their complementary subset of the length $n-1-2p$

$$S = \{i_1 < i_2 < \ldots < i_p\}$$
$$\overline{S} = \{\bar{i_1},\ \bar{i_2},\ \ldots,\ \bar{i_p}\} \qquad i_1 < \bar{i_1},\ i_2 < \bar{i_2}, \ldots, i_p < \bar{i_p}$$
$$T = \{j_1 < j_2 < \ldots < j_{n-1-2p}\} \tag{26}$$

Let us note that there is the ordering in the subset S while for the subset $\overline{S}$ the ordering is not required.

Then let us define an object

$$I_{S,\overline{S},T} \equiv \int_{\Gamma_{i_1}} \frac{d\sigma_{i_1}}{2\pi i} \cdots \int_{\Gamma_{i_p}} \frac{d\sigma_{i_p}}{2\pi i} \int_{\Gamma_{j_1}} \frac{d\sigma_{j_1}}{2\pi i} \cdots \int_{\Gamma_{j_{n-1-2p}}} \frac{d\sigma_{j_{n-1-2p}}}{2\pi i}$$
$$\prod_{q=1}^{p} X_{i_q}(\sigma_{i_q}) p_{\bar{i}_q}(\sigma_{i_q}) \Phi(\sigma_{i_q}) \prod_{q=1}^{n-1-2p} X_{j_q}(\sigma_{j_q}) \Phi(\sigma_{j_q}) h_p(\sigma_{i_1}, \ldots, \sigma_{i_p} | \sigma_{j_1}, \ldots, \sigma_{j_{n-1-2p}}) \tag{27}$$

where the functions h_p are obtained from h_0 by a procedure which will be described later.

$$\mathcal{L}(p_1, \cdots, p_{n-1}) = \sum_{U = S \bigcup \overline{S} \bigcup T} \epsilon(S, \overline{S}, T) I_{S,\overline{S},T} \tag{28}$$

where $\epsilon(S, \overline{S}, T)$ is the sign of the permutation π:

$$\pi : \quad (1, \ldots, n-1) \to (i_1, \bar{i}_1, i_2, \bar{i}_2, \ldots, i_p, \bar{i}_p, j_1, \ldots, j_{n-1-2p})$$

Let us return to h_p. Introduce the following notations:
δ for the finite difference:

$$\delta(f)(\sigma) = f(\sigma + \pi i) - f(\sigma),$$

R_k for normalized residue

$$R_k = \frac{1}{P(\sigma_k + \frac{\pi i}{2})} \text{res}_{\sigma_{k+1} = \sigma_k + \pi i}$$

and D_l is the "exact form" taken in the variable σ_l

$$D_l f(\ldots|\sigma_l|\ldots) = \frac{f(\ldots|\sigma_l|\ldots)}{P(\sigma_l + \frac{3\pi i}{2})} - \frac{f(\ldots|\sigma_l - 2\pi i|\ldots)}{P(\sigma_l - \frac{3\pi i}{2})} \tag{29}$$

Then

$$h_p(\sigma_1, \sigma_3, \ldots, \sigma_{2p-1} | \sigma_{2p+1}, \sigma_{2p+2}, \ldots, \sigma_{n-1}) = \prod_{k=1}^{p} \delta^{-1}_{\sigma_{2k-1}} \Bigg[\prod_{k=1}^{p} R_{2k-1}\, h_0(\sigma_1, \sigma_2, \ldots, \sigma_{n-1})$$
$$- \sum_{r=1}^{\min(p,[\frac{n-2}{2}])} (-1)^r \sum_{1 \le m_1 < \ldots < m_r \le p} \prod_{\substack{1 \le k \le p \\ k \neq \{m_1, \ldots, m_r\}}} R_{2k-1} \sum_{\substack{1 \le l \le n-1 \\ l \neq \{2m_1 - 1, 2m_1, \ldots, 2m_r - 1, 2m_r\}}}$$
$$(-1)^l D_l f_r(\sigma_{2m_1-1}, \ldots, \sigma_{2m_r-1} | \sigma_l | \sigma_1, \ldots, \widehat{\sigma_{2m_1-1}}, \widehat{\sigma_{2m_1}}, \ldots, \widehat{\sigma_l}, \ldots, \widehat{\sigma_{2m_r-1}}, \widehat{\sigma_{2m_r}}, \ldots, \sigma_{n-1}) \Bigg] \tag{30}$$

The functions f_r are chosen in such a way that
1. Taking "primitive" 's δ^{-1} is possible.
2. $h_p(\sigma_1, \sigma_3, \ldots, \sigma_{2p-1} | \sigma_{2p+1}, \sigma_{2p+2}, \ldots, \sigma_{n-1})$ is symmetric with respect to $\sigma_1, \sigma_3, \ldots, \sigma_{2p-1}$ and skew-symmetric with respect to $\sigma_{2p+1}, \sigma_{2p+2}, \ldots, \sigma_{n-1}$ or equivalently $f_r(\sigma_1, \sigma_3, \ldots, \sigma_{2r-1} | \sigma_{2r} | \sigma_{2r+1}, \sigma_{2r+2}, \ldots, \sigma_{n-2r-2})$ is symmetric w.r.t. $\sigma_1, \sigma_3, \ldots, \sigma_{2r-1}$ and skew-symmetric w.r.t. $\sigma_{2r+1}, \sigma_{2r+2}, \ldots, \sigma_{n-2r-2}$

3. For the pair of variables $\sigma_{2r}, \sigma_{2r+1}$ the only non-zero residues are in the points $\sigma_{2r+1} = \sigma_{2r} + 3\pi i$, $\sigma_{2r+1} = \sigma_{2r} + \pi i$ and $\sigma_{2r+1} = \sigma_{2r} - \pi i$ and

$$\mathrm{res}_{\sigma_{2r+1}=\sigma_{2r}+3\pi i} f_r(\sigma_1, \sigma_3, \ldots, \sigma_{2r-1}|\sigma_{2r}|\sigma_{2r+1}, \sigma_{2r+2}, \ldots, \sigma_{n-2r-2}) = \mathrm{res}_{\sigma_{2r+1}=\sigma_{2r}} f_r(\sigma_1, \sigma_3, \ldots, \sigma_{2r-1}|\sigma_{2r} + \pi i|\sigma_{2r+1}, \sigma_{2r+2}, \ldots, \sigma_{n-2r-2}) \tag{31}$$

Some additional requirements on these functions f_r must be satisfied. For the moment we were not able to construct these functions completely for arbitrary n, formulae for $n = 2, 3, 4, 5, 6$ are given in Appendix.

Notice two remarkable properties of the formulae (27 , 28). First, in order to achieve skew-symmetry we need to add some integral with coupled p_i and $p_{\bar{i}}$. But we do not need to consider more complicated integrals in which three or more polynomials are bounded together. Second, due to the fact that $p_i \circ p_j = 0$ the ambiguity in definition of δ^{-1} in the formula for h_p is irrelevant. The definition of the pairing $\circ$ is defined by the formula (15) of the paper [13].

3 The correlation functions

Let us remind the reader the formula from [3] which relates a solution of the qKZ on level -4 to any correlation function of the XXX model:

$$\langle \prod_{j=1}^{n} E_{\epsilon_j, \epsilon'_j} \rangle = g_n(\beta_1, \ldots, \beta_n, \beta_n + \pi i, \ldots, \beta_1 + \pi i)_{-\epsilon'_1, \ldots, -\epsilon'_n, \epsilon_n, \ldots, \epsilon_1} \tag{32}$$

Let us also note that this solution should also satisfy some additional constraint, namely, that

$$g_n(\ldots, \beta_{j-1}, \beta_j, \beta_j - \pi i, \beta_{j+2}, \ldots)_{\ldots, \epsilon_{j-1}, \epsilon_j, \epsilon_{j+1}, \epsilon_{j+2}, \ldots} = \delta_{\epsilon_j, -\epsilon_{j+1}}\ g_{n-1}(\ldots, \beta_{j-1}, \beta_{j+2}, \ldots)_{\ldots, \epsilon_{j-1}, \epsilon_{j+2}, \ldots} \tag{33}$$

One can check that the singlet solution to qKZ equations on level -4 that satisfy the property (33) can be written in the following:

$$g(\beta_1, \cdots, \beta_{2n})_{\epsilon_1, \cdots, \epsilon_{2n}} = \frac{e^{\frac{1}{2}\sum \beta_j}}{\sum e^{\beta_j}} \prod_{i<j} \frac{1}{\zeta(\beta_i - \beta_j)} \times \sum_{-(n-1) \leq j_1 < j_2 < \cdots < j_{n-1} \leq n-1} P^{\epsilon_1, \cdots, \epsilon_{2n}}_{j_1, \cdots, j_{n-1}}(\beta_1, \cdots, \beta_{2n}) \det\left(I_{j_l, m}\right)_{l,m=1,\cdots,n-1} \tag{34}$$

where polynomials

$$P^{\epsilon_1, \cdots, \epsilon_{2n}}_{j_1, \cdots, j_{n-1}}(\beta_1, \cdots, \beta_{2n}) = \sum_{\{1, \cdots, 2n\} = \{k_1, \cdots, k_n\} \cup \{l_1, \cdots, l_n\}} K_{j_1, \cdots, j_{n-1}}(\beta_{k_1}, \cdots, \beta_{k_n}|\beta_{l_1}, \cdots, \beta_{l_n}) \times \prod_{p,q=1}^{n} \frac{\beta_{l_q} - \beta_{k_p} + \pi i}{\beta_{l_q} - \beta_{k_p}} w^{\dagger \epsilon_1, \cdots, \epsilon_{2n}}_{\epsilon'_1, \cdots, \epsilon'_{2n}}(\beta_1, \cdots, \beta_{2n}) \tag{35}$$

and as above $\epsilon'_{k_p}=-$ and $\epsilon'_{l_q}=+$. Here we have introduced the integrals:

$$I_{j,m}=\int\limits_{-\infty}^{\infty}d\alpha\prod_{p=1}^{2n}\varphi(\alpha-\beta_p)s_j(\alpha)e^{(n-1-2m)\alpha}$$

where the functions φ and ζ are given by the formulae (2) and (3) respectively. We want to split β'a into pairs:

$$\beta_k=\lambda_k-\frac{\pi i}{2}+i\delta_k,\quad \beta_{2n-k+1}=\lambda_k+\frac{\pi i}{2}-i\delta_k \tag{36}$$

and to take limit $\delta_k\to 0$. It is very convenient to keep δ_k different.

The function $\varphi(\alpha)$ satisfies the equation:

$$\varphi(\alpha+\frac{\pi i}{2})\varphi(\alpha-\frac{\pi i}{2})=\frac{4\pi^3}{\alpha\sinh(\alpha)}$$

In the limit $\delta_k\to 0$ the contour of integration is pinched by two poles which may give a singularity. Let us study the leading behaviour. For s_{-k} the situation is simple:

$$I_{-k,m}=\pi(4\pi)^n\sum_{p=1}^{n}\frac{1}{\delta_p}\prod_{q\neq p}\frac{1}{(\lambda_p-\lambda_q)\sinh(\lambda_p-\lambda_q)}e^{(n-1-2m)\lambda_p}\lambda_p^{n-1-k}+\mathcal{O}(1)$$

so, the leading part of the integrals is expressed in terms of elementary functions.

For s_k with $k>0$ the situation is more complicated. It is more convenient to use here the explicit formula for the polynomials

$$s_k(\alpha)\equiv A_k(\alpha|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n})$$

which follows form the formula (7) for the generating function

$$\begin{aligned}&A_k(\alpha|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n})=\\&\prod_{j=1}^{n}(\alpha-\beta_j-\frac{\pi i}{2})Q_k(\alpha-\frac{\pi i}{2}|\beta_{n+1}-\frac{\pi i}{2},\cdots,\beta_{2n}-\frac{\pi i}{2})+\\&+\prod_{j=1}^{n}(\alpha-\beta_{n+j}+\frac{\pi i}{2})Q_k(\alpha+\frac{\pi i}{2}|\beta_1+\frac{\pi i}{2},\cdots,\beta_n+\frac{\pi i}{2})\end{aligned} \tag{37}$$

with the polynomials

$$Q_k(\alpha|\lambda_1,\cdots,\lambda_n)=\sum_{l=1}^{k}\left((\alpha+\frac{\pi i}{2})^l-(\alpha-\frac{\pi i}{2})^l\right)(-1)^{k-l}\sigma_{k-l}(\lambda_1,\cdots,\lambda_n) \tag{38}$$

and the standard symmetric functions of n variables $\sigma_k(\lambda_1,\cdots,\lambda_n)$ defined as follows

$$\prod_{j=1}^{n}(\lambda+\lambda_j)=\sum_{l=0}^{n}\lambda^l\sigma_{k-l}(\lambda_1,\ldots,\lambda_n)$$

Actually, the multipliers $\prod(\alpha-\beta_j\pm\frac{\pi i}{2})$ in (37) cancel singularities of $\varphi(\alpha-\beta_j)$ closest to the real axis. So, the pinching does not take place and, hence, the result is finite. Denote:

$$I_{k,m}=I^0_{k,m}+\sum i\delta_p I^p_{k,m}+\mathcal{O}(\delta^2)$$

For $I^0_{k,m}$ we have:

$$\begin{aligned}I^0_{k,m}=&\int\limits_{\mathbb{R}+i0}d\alpha\prod_{j=1}^n\frac{4\pi^3}{\sinh(\alpha-\lambda_j)}e^{(n-1-2m)\alpha}Q_k(\alpha-\frac{\pi i}{2}|\lambda_1,\cdots,\lambda_n)+\\&+\int\limits_{\mathbb{R}-i0}d\alpha\prod_{j=1}^n\frac{4\pi^3}{\sinh(\alpha-\lambda_j)}e^{(n-1-2m)\alpha}Q_k(\alpha+\frac{\pi i}{2}|\lambda_1,\cdots,\lambda_n)=0\end{aligned}$$

So, the leading contribution is of the first order in δ's. Consider $I^p_{k,m}$:

$$\begin{aligned}&I^p_{k,m}=\\&\int\limits_{\mathbb{R}+i0}d\alpha\prod_{j=1}^n\frac{4\pi^3}{\sinh(\alpha-\lambda_j)}e^{(n-1-2m)\alpha}\left(\chi(\alpha-\lambda_p)-\frac{1}{\alpha-\lambda_p}-\frac{\partial}{\partial\lambda_p}\right)Q_k(\alpha-\frac{\pi i}{2}|\lambda_1,\cdots,\lambda_n)+\\&+\int\limits_{\mathbb{R}-i0}d\alpha\prod_{j=1}^n\frac{4\pi^3}{\sinh(\alpha-\lambda_j)}e^{(n-1-2m)\alpha}\left(\chi(\alpha-\lambda_p)+\frac{1}{\alpha-\lambda_p}+\frac{\partial}{\partial\lambda_p}\right)Q_k(\alpha+\frac{\pi i}{2}|\lambda_1,\cdots,\lambda_n)\end{aligned}$$

where

$$\chi(\alpha)=\frac{d}{d\alpha}\left(\log\frac{\varphi(\alpha-\frac{\pi i}{2})}{\varphi(\alpha+\frac{\pi i}{2})}\right)=2\pi i\sum_{k=1}^{\infty}(-1)^k\frac{k}{\alpha^2+(\pi k)^2}$$

Using the definition (38) of Q_k we present $I^p_{k,m}$ as follows:

$$I^p_{k,m}=\sum_{l=1}^k(-1)^{k-l}\sigma_{k-l}(\lambda_1,\cdots,\lambda_n)J^p_{l,m}$$

Then

$$\begin{aligned}&J^p_{l,m}=\\&\int\limits_{\mathbb{R}+i0}d\alpha\prod_{j=1}^n\frac{4\pi^3}{\sinh(\alpha-\lambda_j)}e^{(n-1-2m)\alpha}\left(\chi(\alpha-\lambda_p)\left((\alpha-\pi i)^l-\alpha^l\right)+\pi i\frac{(\alpha-\pi i)^l-\lambda_p^l}{(\alpha-\lambda_p)(\alpha-\lambda_p-\pi i)}\right)+\\&+\int\limits_{\mathbb{R}-i0}d\alpha\prod_{j=1}^n\frac{4\pi^3}{\sinh(\alpha-\lambda_j)}e^{(n-1-2m)\alpha}\left(\chi(\alpha-\lambda_p)\left(\alpha^l-(\alpha+\pi i)^l\right)-\pi i\frac{(\alpha+\pi i)^l-\lambda_p^l}{(\alpha-\lambda_p)(\alpha-\lambda_p+\pi i)}\right)\end{aligned}$$

Here we have used the following identity

$$\frac{\partial}{\partial\lambda_p}\sigma_k(\lambda_1,\cdots,\lambda_n)=\sum_{l=0}^{k-1}(-\lambda_p)^{k-l-1}\sigma_l(\lambda_1,\ldots,\lambda_n)$$

Let us denote

$$f(\alpha) = \alpha^l - \lambda_p^l$$

then using the identities:

$$\begin{aligned}
\chi(\alpha + \pi i) + \chi(\alpha) &= \frac{1}{\alpha + \pi i} - \frac{1}{\alpha} \\
\chi(\alpha) + \chi(\alpha - \pi i) &= \frac{1}{\alpha} - \frac{1}{\alpha - \pi i}
\end{aligned} \tag{39}$$

one finds:

$$\begin{aligned}
J_{l,m}^p = &\int\limits_{\mathbb{R}+i0} d\alpha \prod_{j=1}^n \frac{4\pi^3}{\sinh(\alpha - \lambda_j)} e^{(n-1-2m)\alpha} \left(f(\alpha - \pi i)\chi(\alpha - \lambda_p - \pi i) + f(\alpha)\chi(\alpha - \lambda_p)\right) - \\
&- \int\limits_{\mathbb{R}-i0} d\alpha \prod_{j=1}^n \frac{4\pi^3}{\sinh(\alpha - \lambda_j)} e^{(n-1-2m)\alpha} \left(f(\alpha + \pi i)\chi(\alpha - \lambda_p + \pi i) + f(\alpha)\chi(\alpha - \lambda_p)\right) = \\
&= 2 \left(\int\limits_{\mathbb{R}+i0} - \int\limits_{\mathbb{R}-i0} \right) d\alpha \prod_{j=1}^n \frac{4\pi^3}{\sinh(\alpha - \lambda_j)} e^{(n-1-2m)\alpha} \chi(\alpha - \lambda_p) \left(\alpha^l - \lambda_p^l\right) = \\
&= 4\pi i (4\pi^3)^n \sum_{q \neq p} \prod_{j \neq q} \prod \frac{4\pi^3}{\sinh(\alpha - \lambda_j)} e^{(n-1-2m)\lambda_q} \chi(\lambda_p - \lambda_q)(\lambda_p^l - \lambda_q^l)
\end{aligned}$$

Returning to $I_{k,m}^p$:

$$\begin{aligned}
I_{k,m}^p = 4\pi i (4\pi^3)^n (-1)^k &\sum_{q \neq p} \prod_{j \neq q} \frac{1}{\sinh(\lambda_q - \lambda_j)} e^{(n-1-2m)\lambda_q} \chi(\lambda_p - \lambda_q) \\
&\times \left(\sigma_k(\lambda_1, \cdots, \widehat{\lambda_p}, \cdots, \lambda_n) - \sigma_k(\lambda_1, \cdots, \widehat{\lambda_q}, \cdots, \lambda_n)\right)
\end{aligned}$$

Let $\tilde{I}_{j,m}$ ($|j| = 1, \cdots, n-1$) correspond to a leading term of $I_{j,m}$ in δ's. It can be rewritten in the following form:

$$\tilde{I}_{j,m} = \sum_{p=1}^n A_{j,p} B_{p,m}$$

where A and B are, respectively, $2(n-1) \times n$ and $n \times (n-1)$ matrices with the following matrix elements:

$$\begin{aligned}
A_{-k,p} &= \pi (4\pi^3)^n \prod_{q \neq p} \frac{1}{\lambda_p - \lambda_q} \lambda_p^{n-1-k}, \\
A_{k,p} &= (-1)^{k-1} 4\pi (4\pi^3)^n \sum_{q \neq p} \delta_p \delta_q \chi(\lambda_p - \lambda_q)(\sigma_k(\lambda_1, \cdots, \widehat{\lambda_p}, \cdots, \lambda_n) - \sigma_k(\lambda_1, \cdots, \widehat{\lambda_q}, \cdots, \lambda_n)) \\
B_{p,m} &= \frac{1}{\delta_p} \prod_{j \neq p} \frac{1}{\sinh(\lambda_p - \lambda_j)} e^{(n-1-2m)\lambda_p}
\end{aligned}$$

Notice the following simple property:

$$\sum_{p=1}^{n} A_{j,p} = 0 \tag{40}$$

Take a partition $\{j_1, \cdots, j_{n-1}\}$ and introduce the following object

$$A_{j_1,\cdots,j_l} \equiv \det(A_{j_l,r})|_{r\neq p}$$

Then the identity (40) implies that this definition does not depend on the choice of p. There is one more identity which replaces Riemann bilinear relation:

$$\sum_{k=1}^{n-1} (A_{-k,p}A_{k,q} - A_{-k,q}A_{k,p}) = 0 \tag{41}$$

Calculating minors of the matrix B one finds:

$$\det(\tilde{I}_{j_l,m})|_{l,m=1,\cdots n-1} = 2^{\frac{(n-1)(n-2)}{2}} A_{j_1,\cdots,j_l} \prod_{p=1}^{n} \frac{1}{\delta_p} \frac{\sum(\delta_p e^{\lambda_p}) \exp(-\sum \lambda_p)}{\prod\limits_{i<j} \sinh(\lambda_i - \lambda_j)} \tag{42}$$

Let us return to the formula (34). It can be shown that

$$\zeta(\lambda)\zeta(-\lambda)\zeta(\lambda - \pi i)\zeta(-\lambda - \pi i) = C^2 \frac{\lambda}{\sinh(\lambda)}$$

where the constant

$$C = \frac{\zeta(\frac{\pi i}{2})\zeta(-\frac{\pi i}{2})}{\Gamma(\frac{1}{4})^2(\frac{\pi}{2})^2}$$

So, the formula (34) can be rewritten as follows:

$$\begin{aligned} g(\lambda_1 - \frac{\pi i}{2} + i\delta_1, \cdots, \lambda_1 + \frac{\pi i}{2} - i\delta_1)_{\epsilon_1,\cdots,\epsilon_{2n}} &= \left[\prod_{p=1}^{n} \frac{1}{\delta_p}\right] \frac{2^{n(n-3)/2}}{C^{n(n-1)}} \prod_{i<j} \frac{1}{\lambda_i - \lambda_j} \\ \times \sum_{-(n-1)\leq j_1 < j_2 < \cdots < j_{n-1} \leq n-1} P^{\epsilon_1,\cdots,\epsilon_{2n}}_{j_1,\cdots,j_{n-1}}(\lambda_1 - \frac{\pi i}{2} + i\delta_1, \cdots, \lambda_1 - \frac{\pi i}{2} + i\delta_1) A_{j_1,\cdots,j_l} &(1 + \mathcal{O}(\delta)) \end{aligned} \tag{43}$$

A priori we know that the result of specification $\beta_k = \lambda_k - \frac{\pi i}{2}$, $\beta_{2n-k-1} = \lambda_k + \frac{\pi i}{2}$ is finite. The only source of possible singularities in δ's is the multiplier $\prod \frac{1}{\delta_p}$ in RHS of (43). The sum is Tailor series in δ's. Hence, in the sum

1. All the coefficients in front of those terms which are not divisible by $\prod \delta_p$ must vanish.
2. In the final result only one term is important: that containing $\prod \delta_p$. It means, in particular that the terms divisible by δ_p^2 for some p do not contribute into result.

Recall that $A_{j_1,\cdots,j_{n-1}}$ is constructed out of $A_{j,p}$. Let us denote the number of positive j's in the multiindex by $l(\{j\})$. Obviously

$$A_{\{j\}} = \mathcal{O}(\delta^{2l(\{j\})}) \tag{44}$$

The second property shows that $A_{\{j\}}$ for which

$$l(\{j\}) > \left[\frac{n}{2}\right]$$

do not contribute to the final result.

Now we need the following

Assumption. If $l(\{j\}) \leq \left[\frac{n}{2}\right]$ then

$$P^{\{j\}}_{\{\epsilon\}}(\lambda_1 - \frac{\pi i}{2} + i\delta_1, \cdots, \lambda_1 - \frac{\pi i}{2} + i\delta_1) = \mathcal{O}(\delta^{n-2l(\{j\})}) \tag{45}$$

Unfortunately, for the moment we are not able to prove it. Let us mention at least some arguments why we believe in this Assumption? First, suppose the degree in δ is smaller then in (45). Then some huge cancellation must take place because all contributions from leading term of asymptotic described by $A_{\{j\}}$ are singular in the limit. The estimation (45) is valid for $n = 2, 3$, these cases we were able to calculate by Maple. Certainly, the best way to prove the Assumption is to calculate the polynomials $P^{\{\epsilon\}}_{\{j\}}$. These polynomials are related to the polynomials $K_{j_1,\cdots,j_{n-1}}$ through the formula (35). The latter polynomials are given by the formulae (14), (27) and (28). The main problem here is to take the limit $\delta_k \to 0$ for the integrals (27) when the spectral parameters β_k are taken in the form (36). We leave this question for further publication.

Let us suppose that the Assumption is true. Then still important cancellations must occur because only one kind of terms of the order δ^n must remain in the Tailor series: that with $\prod \delta_p$. The result will look like:

$$\begin{aligned} & g(\lambda_1 - \frac{\pi i}{2}, \cdots, \lambda_1 + \frac{\pi i}{2})_{\epsilon_1,\cdots,\epsilon_{2n}} = \\ & = \sum_{m=0}^{\left[\frac{n}{2}\right]} \sum_{k_1,\cdots,k_{2m}} Q^{k_1,\cdots,k_{2m}}_{\epsilon_1,\cdots,\epsilon_{2n}}(\lambda_1, \cdots, \lambda_n)\chi(\lambda_{k_1} - \lambda_{k_2}) \cdots \chi(\lambda_{k_{2m-1}} - \lambda_{k_{2m}}) \end{aligned} \tag{46}$$

where $Q^{k_1,\cdots,k_{2m}}_{\epsilon_1,\cdots,\epsilon_{2n}}(\lambda_1, \cdots \lambda_n)$ are rational functions with denominators containing only products of $\lambda_i - \lambda_j$. The requirement that terms in Tailor series with δ_p^2 must disappear implies that $k_i \neq k_j$ $\forall i, j$. We have checked for $n = 2$ and $n = 3$ that the functions $Q^{k_1,\cdots,k_{2m}}_{\epsilon_1,\cdots,\epsilon_{2n}}$ have additional multipliers, namely,

$$Q^{k_1,\cdots,k_{2m}}_{\epsilon_1,\cdots,\epsilon_{2n}}(\lambda_1, \cdots, \lambda_n) = \prod_{j=1}^{m}((\lambda_{k_{2j-1}} - \lambda_{k_{2j}})^2 + \pi^2)\tilde{A}^{k_1,\cdots,k_{2m}}_{\epsilon_1,\cdots,\epsilon_{2n}}(\lambda_1, \cdots, \lambda_n)$$

where $\tilde{A}^{k_1,\cdots,k_{2m}}_{\epsilon_1,\cdots,\epsilon_{2n}}$ are some rational functions. Doing a substitution like $\lambda_k \to \pi(z_k - \frac{i}{2})$ and using the formula (32) we arrive at the formula

$$\langle \prod_{j=1}^{n} E_{\epsilon_j,\epsilon'_j} \rangle = \sum_{m=0}^{\left[\frac{n}{2}\right]} \sum_{k_1 \neq \cdots \neq k_{2m}} A^{k_1,\cdots,k_{2m}}_{-\epsilon'_1,\cdots,-\epsilon'_n,\epsilon_n,\cdots,\epsilon_1}(z_1, \cdots, z_n) G(z_{k_1} - z_{k_2}) \cdots G(z_{k_{2m-1}} - z_{k_{2m}}) \tag{47}$$

where $A^{k_1,\cdots,k_{2m}}_{\epsilon_1,\cdots,\epsilon_{2n}}(z_1, \cdots, z_n)$ are rational functions related in an obvious way to the functions $\tilde{A}^{k_1,\cdots,k_{2m}}_{\epsilon_1,\cdots,\epsilon_{2n}}(\lambda_1, \cdots, \lambda_n)$ and the function G is defined by the formula (3.6) of the work [12]. We see wonderful agreement with the ansatz (3.20) from the above paper for the emptiness formation probability $P_n(z_1, \cdots, z_n)$ which was based on quite different arguments.

4 Conclusion

In this paper we have applied a new form of solution to the qKZ on level -4 for the correlation functions of the XXX model. They are given by the formula (47) in a full agreement with the result of paper [12]. The mathematical reasons for a reducibility of solutions of the qKZ on level -4 and the correlation functions to one-dimensional integrals is the special kind of the cohomologies of the deformed Jacobi varieties. Our previous conjecture that in the homogeneous limit $z_j \to 0$ both the emptiness formation probability and other correlation function may be expressed in terms of the Riemann zeta function at odd arguments and rational coefficients [10, 11] follows from the formula (47) and the expansion (3.13) of the paper [12].

Unfortuantely, there are still some technical problems that were not solved or solved only partially in this paper. First we are not completely satisfied with the formulae (14), (27) and (28) which are necessary for defining the polynomials $K_{j_1,\cdots,j_{n-1}}$. In particular, we have not succeeded in finding a general formula for the functions f_r from the formula (30). We have done it completely for $n \leq 6$ (see Appendix). For a generic case we have only formulated some general conditions for these functions. We may hope that, probably, some more explicit formulae should exist for those polynomials.

Also the challenging problem is to define the rational functions $A^{k_1,\cdots,k_{2m}}_{\epsilon_1,\cdots,\epsilon_{2n}}(z_1,\cdots,z_n)$ explicitly using the results of the Section 2.

We hope to do it in our further publication. We also hope to generalize all the above results to the case of the XXZ spin chain.

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6 Appendix

Here we show how the scheme generally described in the previous Section works for the cases $n = 2, 3, \ldots, 6$.

1. **The case** $n = 2$

This case is rather trivial because the function D defined in (25) does not have any poles and

$$h_0(\sigma) = D(\sigma|\beta_1,\beta_2,\beta_3,\beta_4)_{--++}$$

The formula (28) contains only one term for $p = 0$ which is the one-fold integral:

$$\mathcal{L}(p_1) = \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)h_0(\sigma_1) \tag{48}$$

and the function X_1 is defined by the formula (22) through a polynomial p_1 which in this case is one of the polynomials s_1 or s_{-1}.

2. **The case** $n = 3$

This is the first non-trivial case because one faces two-fold integrals for which the integration order is essential.

Let us represent the function D from (25) in the following form:

$$D(\sigma_1, \sigma_2|\beta_1, \ldots, \beta_6)_{---+++} = \frac{g(\sigma_1, \sigma_2)}{\sigma_2 - \sigma_1 - \pi i} \tag{49}$$

where g is a polynomial for which the dependence on the rapidities $\beta_1, \ldots, \beta_6$ is implied. Then according to the formula (24)

$$h_0(\sigma_1, \sigma_2) = \frac{g(\sigma_1, \sigma_2)}{\sigma_2 - \sigma_1 - \pi i} - \frac{g(\sigma_2, \sigma_1)}{\sigma_1 - \sigma_2 - \pi i} \tag{50}$$

The formula (28) contains two terms for $p = 0$ and $p = 1$ respectively:

$$\mathcal{L}(p_1, p_2) = I_0 + I_1 \tag{51}$$

with polynomials p_1 and p_2 which belong to the set of polynomials s_{-2}, s_{-1}, s_1, s_2. The term I_0 corresponds to the case when partitions

$$S = \overline{S} = \emptyset \quad T = (1, 2)$$

$$I_0 \equiv I_{\emptyset,\emptyset,(1,2)} = \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2)h_0(\sigma_1, \sigma_2) \tag{52}$$

If one tries to transpose indices 1 and 2 in this formula:

$$\begin{aligned} I_{0|_{1\leftrightarrow 2}} &= \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} X_2(\sigma_1)\Phi(\sigma_1)X_1(\sigma_2)\Phi(\sigma_2)h_0(\sigma_1, \sigma_2) \\ &= -\int_{\Gamma_2} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_1} \frac{d\sigma_2}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2)h_0(\sigma_1, \sigma_2) \end{aligned} \tag{53}$$

If it were not necessary to transpose the contours Γ_1 and Γ_2 the last expression would be $-I_0$. Due to the transposition of the contours we get one additional term:

$$I_{0|_{1\leftrightarrow 2}} = -I_0 + J_0 \tag{54}$$

where

$$J_0 = \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} B_{12}(\sigma_1) R_1(h_0(\sigma_1, \sigma_2)) \tag{55}$$

and

$$B_{12}(\sigma_1) = X_1(\sigma_1)X_2(\sigma_1 + \pi i) + X_2(\sigma_1)X_1(\sigma_1 + \pi i) \tag{56}$$

$$R_1 = \frac{1}{P(\sigma_1 + \frac{\pi i}{2})}\mathrm{Res}_{\sigma_2=\sigma_1+\pi i}$$

Here we have used the property

$$\Phi(\sigma)\Phi(\sigma+\pi i) = \frac{1}{P(\sigma + \frac{\pi i}{2})} \tag{57}$$

In order to compensate this additional term J_0 we need the second term I_1 in the formula (51). In this case there is only one partition corresponding to $p = 1$ which satisfies the conditions (26)

$$S = (1) \quad \overline{S} = (2) \quad T = \emptyset$$

$$I_1 \equiv I_{(1),(2),\emptyset} = \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} X_1(\sigma_1)p_2(\sigma_1)\Phi(\sigma_1)h_1(\sigma_1) \tag{58}$$

where

$$h_1(\sigma_1) = \delta_{\sigma_1}^{-1}[R_1 h_0(\sigma_1,\sigma_2)] \tag{59}$$

Using the formula (50) one gets

$$[R_1 h_0(\sigma_1,\sigma_2)] = \frac{g(\sigma_1,\sigma_1+\pi i)}{P(\sigma_1 + \frac{\pi i}{2})}$$

One can check that due to the formula (25) the polynomial $g(\sigma_1, \sigma_1 + \pi i)$ has a divisor $P(\sigma_1 + \frac{\pi i}{2})$ i.e. $[R_1 h_0(\sigma_1, \sigma_2)]$ is a polynomial. Therefore the operator $\delta_{\sigma_1}^{-1}$ is well-defined in this case and may be calculated explicitly.

In order to check that the term I_1 defined in (58) is suitable for compensation of the additional term J_0 given by (55) let us substitute to the r.h.s. of (55)

$$R_1 h_0(\sigma_1,\sigma_2) = \delta_{\sigma_1} h_1(\sigma_1)$$

which is an obvious consequnce of the formula (59). Then we get

$$\begin{aligned} J_0 &= \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} B_{12}(\sigma_1)\delta_{\sigma_1} h_1(\sigma_1) \\ &= -\int_{\Gamma_1} \frac{d\sigma_1}{2\pi i}(X_1(\sigma_1)p_2(\sigma_1) + X_2(\sigma_1)p_1(\sigma_1))\Phi(\sigma_1)h_1(\sigma_1) \end{aligned} \tag{60}$$

Actually the last expression is just

$$-I_1 - I_{1|_{1\leftrightarrow 2}}$$

and we come to the result that the r.h.s. of the formula (51) is the skew-symmetric function.

3. **The case** $n = 4$

First let us represent the function D from (25) in analogous to (49) form:

$$\begin{gathered} D(\sigma_1,\sigma_2,\sigma_3|\beta_1,\ldots,\beta_8)_{----++++} = \\ \frac{g(\sigma_1,\sigma_2,\sigma_3)}{(\sigma_2-\sigma_1-\pi i)(\sigma_3-\sigma_1-\pi i)(\sigma_3-\sigma_2-\pi i)} \end{gathered} \tag{61}$$

where g is a polynomial of both variables $\sigma_1, \sigma_2, \sigma_3$ and of the rapidities $\beta_1, \ldots, \beta_8$. An explicit form of this polynomial is defined by the formula (25). For us it is important that this polynomial has the property that three polynomials
$g(\sigma_1, \sigma_1 + \pi i, \sigma_3), g(\sigma_1, \sigma_3, \sigma_1 + \pi i), g(\sigma_3, \sigma_1, \sigma_1 + \pi i)$ contain divisor
$P(\sigma_1 + \frac{\pi i}{2})$ while the polynomial $g(\sigma_1, \sigma_1 + \pi i, \sigma_1 + 2\pi i)$ is divisible by
$P(\sigma_1 + \frac{\pi i}{2})P(\sigma_1 + \frac{3\pi i}{2})$. Then according to the formula (24)

$$
\begin{aligned}
&h_0(\sigma_1, \sigma_2, \sigma_3) = \\
&\frac{g(\sigma_1, \sigma_2, \sigma_3)}{(\sigma_{21} - \pi i)(\sigma_{31} - \pi i)(\sigma_{32} - \pi i)} - \frac{g(\sigma_2, \sigma_1, \sigma_3)}{(\sigma_{12} - \pi i)(\sigma_{31} - \pi i)(\sigma_{32} - \pi i)} + \\
&\frac{g(\sigma_2, \sigma_3, \sigma_1)}{(\sigma_{32} - \pi i)(\sigma_{12} - \pi i)(\sigma_{13} - \pi i)} - \frac{g(\sigma_3, \sigma_2, \sigma_1)}{(\sigma_{23} - \pi i)(\sigma_{12} - \pi i)(\sigma_{13} - \pi i)} + \\
&\frac{g(\sigma_3, \sigma_1, \sigma_2)}{(\sigma_{13} - \pi i)(\sigma_{23} - \pi i)(\sigma_{21} - \pi i)} - \frac{g(\sigma_1, \sigma_3, \sigma_2)}{(\sigma_{31} - \pi i)(\sigma_{23} - \pi i)(\sigma_{21} - \pi i)}
\end{aligned}
\tag{62}
$$

As in the previous case the formula (28) contains two terms for $p = 0$ and $p = 1$ respectively:

$$
\mathcal{L}(p_1, p_2, p_3) = I_0 + I_1 \tag{63}
$$

with polynomials p_1, p_2, p_3 from the set $s_{-3}, s_{-2}, s_{-1}, s_1, s_2, s_3$.

The term I_0 corresponds to the case when partitions

$$
S = \overline{S} = \emptyset \quad T = (1, 2, 3)
$$

$$
\begin{aligned}
&I_0 \equiv I_{\emptyset, \emptyset, (1,2,3)} = \\
&\int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2)X_3(\sigma_3)\Phi(\sigma_3)h_0(\sigma_1, \sigma_2, \sigma_3)
\end{aligned}
\tag{64}
$$

For the next term corresponding to $p = 1$ there are three contributions because one can find three different partitions that satisfy the conditions (26)

$$
\begin{aligned}
S = (1) \quad \overline{S} = (2) \quad T = (3) \\
S = (1) \quad \overline{S} = (3) \quad T = (2) \\
S = (2) \quad \overline{S} = (3) \quad T = (1)
\end{aligned}
$$

and according to the formulae (27,28)

$$
I_1 = I_{(1),(2),(3)} - I_{(1),(3),(2)} + I_{(2),(3),(1)} \tag{65}
$$

where

$$
\begin{aligned}
I_{(1),(2),(3)} &= \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_1(\sigma_1)p_2(\sigma_1)\Phi(\sigma_1)X_3(\sigma_3)\Phi(\sigma_3)h_1(\sigma_1|\sigma_3) \\
I_{(1),(3),(2)} &= \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} X_1(\sigma_1)p_3(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2)h_1(\sigma_1|\sigma_2) \\
I_{(2),(3),(1)} &= \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} X_2(\sigma_2)p_3(\sigma_2)\Phi(\sigma_2)X_1(\sigma_1)\Phi(\sigma_1)h_1(\sigma_2|\sigma_1)
\end{aligned}
\tag{66}
$$

and due to the formula (30)

$$h_1(\sigma_1|\sigma_3)=\delta_{\sigma_1}^{-1}\Big(R_1h_0(\sigma_1,\sigma_2,\sigma_3)-D_3f_1(\sigma_1|\sigma_3)\Big) \tag{67}$$

with the "exact form" given by (29)

$$D_3f_1(\sigma_1|\sigma_3)=\frac{f_1(\sigma_1|\sigma_3)}{P(\sigma_3+\frac{3\pi i}{2})}-\frac{f_1(\sigma_1|\sigma_3-2\pi i)}{P(\sigma_3-\frac{3\pi i}{2})} \tag{68}$$

In order to define the function f_1 we need some facts about the structure of non-zero residues. Let us introduce the notation

$$\mathrm{Res}_{\sigma_2=\sigma_1+\pi i}h_0(\sigma_1,\sigma_2,\sigma_3)\equiv V_1(\sigma_1|\sigma_3)$$

then it is not difficult to check that

$$\mathrm{Res}_{\sigma_2=\sigma_1-\pi i}h_0(\sigma_1,\sigma_2,\sigma_3)=V_1(\sigma_1-\pi i|\sigma_3)$$

and

$$\mathrm{Res}_{\sigma_2=\sigma_1}h_0(\sigma_1,\sigma_2,\sigma_3)=0$$

Explicitly,

$$V_1(s_1|\sigma_3)=\frac{g(\sigma_1,\sigma_1+\pi i,\sigma_3)}{(\sigma_{31}-\pi i)(\sigma_{31}-2\pi i)}+\frac{g(\sigma_3,\sigma_1,\sigma_1+\pi i)}{(\sigma_{31}-\pi i)\sigma_{13}}-\frac{g(\sigma_1,\sigma_3,\sigma_1+\pi i)}{(\sigma_{31}-\pi i)\sigma_{13}} \tag{69}$$

We shall also need

$$\mathrm{Res}_{\sigma_3=\sigma_1+2\pi i}V_1(\sigma_1|\sigma_3)\equiv V_2(\sigma_1) \tag{70}$$

which due to (69) is

$$V_2(\sigma_1)=\frac{g(\sigma_1,\sigma_1+\pi i,\sigma_1+2\pi i)}{\pi i} \tag{71}$$

It is easy to check that

$$\mathrm{Res}_{\sigma_3=\sigma_1-\pi i}V_1(\sigma_1|\sigma_3)=-V_2(\sigma_1-\pi i) \tag{72}$$

and

$$\mathrm{Res}_{\sigma_3=\sigma_1+\pi i}V_1(\sigma_1|\sigma_3)=\mathrm{Res}_{\sigma_3=\sigma_1}V_1(\sigma_1|\sigma_3)=0$$

Our claim is that

$$f_1(\sigma_1|\sigma_3)=\frac{V_2(\sigma_3)}{\sigma_{13}} \tag{73}$$

Let us briefly explain this result. First we should check that the r.h.s. of (67) is well-defined. Indeed, using (70,72) one can establish that the expression

$$\frac{V_1(\sigma_1|\sigma_3)}{P(\sigma_1+\frac{\pi i}{2})}-\frac{V_2(\sigma_3)}{(\sigma_{13}-\pi i)P(\sigma_3+\frac{3\pi i}{2})}+\frac{V_2(\sigma_3-2\pi i)}{(\sigma_{13}+2\pi i)P(\sigma_3-\frac{3\pi i}{2})}$$

is a polynomial of σ_1 i.e. the operator $\delta_{\sigma_1}^{-1}$ is well-defined for it. Here we also used the above property of the polynomial g which provides that the ratio $\frac{V_1(\sigma_1|\sigma_3)}{P(\sigma_1+\frac{\pi i}{2})}$ has poles only for the variable σ_{13} because the numerator $V_1(\sigma_1|\sigma_3)$ is divisible by the polynomial $P(\sigma_1+\frac{\pi i}{2})$.

Then if (73) is correct then the expression in brackets of the r.h.s. of (67) is

$$\frac{V_1(\sigma_1|\sigma_3)}{P(\sigma_1+\frac{\pi i}{2})}-\frac{V_2(\sigma_3)}{\sigma_{13}P(\sigma_3+\frac{3\pi i}{2})}+\frac{V_2(\sigma_3-2\pi i)}{(\sigma_{13}+2\pi i)P(\sigma_3-\frac{3\pi i}{2})}$$

which up to the above polynomial of σ_1 is as follows:

$$(\frac{1}{\sigma_{13}-\pi i}-\frac{1}{\sigma_{13}})\frac{V_2(\sigma_3)}{P(\sigma_3+\frac{3\pi i}{2})}=-\delta_{\sigma_1}\left(\frac{V_2(\sigma_3)}{(\sigma_{13}-\pi i)P(\sigma_3+\frac{3\pi i}{2})}\right)$$

Now let us check the skew-symmetry of the formula (63). Actually it is enough to do for two transpositions $1\leftrightarrow 2$ and $2\leftrightarrow 3$. Let us check it, for instance, for the transposition $2\leftrightarrow 3$. Proceeding in analogous way as we did for the previous case for derivation of the formula (54) we can get for the first term I_0 defined by (64)

$$I_0+I_{0|_{2\leftrightarrow 3}}=J_0 \tag{74}$$

where now

$$J_0=\int_{\Gamma_1}\frac{d\sigma_1}{2\pi i}\int_{\Gamma_2}\frac{d\sigma_2}{2\pi i}X_1(\sigma_1)\Phi(\sigma_1)B_{23}(\sigma_2)R_2h_0(\sigma_1,\sigma_2,\sigma_3) \tag{75}$$

and

$$B_{23}(\sigma_2)=X_2(\sigma_2)X_3(\sigma_2+\pi i)+X_3(\sigma_2)X_2(\sigma_2+\pi i) \tag{76}$$

$$R_2=\frac{1}{P(\sigma_2+\frac{\pi i}{2})}\mathrm{Res}_{\sigma_3=\sigma_2+\pi i}$$

We can rewrite (67) in the following form

$$R_2h_0(\sigma_1,\sigma_2,\sigma_3)=\delta_{\sigma_2}h_1(\sigma_2|\sigma_1)+D_1f_1(\sigma_2|\sigma_1) \tag{77}$$

Substituting this expression into the formula (75) we come to

$$J_0=J_0'+\mathcal{F}_1 \tag{78}$$

where

$$\begin{aligned}J_0'&=\int_{\Gamma_1}\frac{d\sigma_1}{2\pi i}\int_{\Gamma_2}\frac{d\sigma_2}{2\pi i}X_1(\sigma_1)\Phi(\sigma_1)B_{23}(\sigma_2)\delta_{\sigma_2}h_1(\sigma_2|\sigma_1)\\&=-\int_{\Gamma_1}\frac{d\sigma_1}{2\pi i}\int_{\Gamma_2}\frac{d\sigma_2}{2\pi i}X_1(\sigma_1)\Phi(\sigma_1)(X_2(\sigma_2)p_3(\sigma_2)+X_3(\sigma_2)p_2(\sigma_2))\Phi(\sigma_2)h_1(\sigma_2|\sigma_1)\\&=-I_{(2),(3),(1)}-I_{(2),(3),(1)}{}_{|_{2\leftrightarrow 3}}\end{aligned} \tag{79}$$

$$\mathcal{F}_1 = \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)B_{23}(\sigma_2)D_1 f_1(\sigma_2|\sigma_1)$$
$$= -\int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} B_{23}(\sigma_2) \frac{p_1(\sigma_1)f_1(\sigma_2|\sigma_1)}{P(\sigma_1+\frac{\pi i}{2})P(\sigma_1+\frac{3\pi i}{2})} = 0 \tag{80}$$

Here we have got zero because the ratio

$$\frac{p_1(\sigma_1)f_1(\sigma_2|\sigma_1)}{P(\sigma_1+\frac{\pi i}{2})P(\sigma_1+\frac{3\pi i}{2})}$$

can have poles only for the variable σ_{12} and since σ_2 belongs to the contour Γ_2 which contains the contour Γ_1 those poles can be only outside of the contour Γ_1. Thus we come to the following result

$$J_0 = -I_{(2),(3),(1)} - I_{(2),(3),(1)}{}_{|2\leftrightarrow 3} \tag{81}$$

It is left to note that the combination of two residual terms in the r.h.s. of (65), namely, $I_{(1),(2),(3)} - I_{(1),(3),(2)}$ is already skew-symmetric w.r.t. the transposition $2 \leftrightarrow 3$.

Thus we have checked the skew-symmetry for the transposition $2 \leftrightarrow 3$. For another transposition $1 \leftrightarrow 2$ it is a bit more difficult but also possible to do. The strategy is more or less repeats that for the transposition $2 \leftrightarrow 3$. We shall not do it here.

4. **The case** $n = 5$

It is straightforward to write down the function D from (25) in the form like (49) for $n = 3$ and (61) for $n = 4$ defining a polynomial $g(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ which also depends on the rapidities $\beta_1, \ldots, \beta_{10}$. Again the main property of this polynomial is that all polynomails of the form $g(\ldots, \sigma_1, \ldots, \sigma_1 + \pi i, \ldots)$ contain as a divisor
$P(\sigma_1+\frac{\pi i}{2})$, the polynomials $g(\ldots, \sigma_1, \ldots, \sigma_1+\pi i, \ldots, \sigma_1+2\pi i)$ are divisible by $P(\sigma_1+\frac{\pi i}{2})P(\sigma_1+\frac{3\pi i}{2})$ etc.

Now the formula (28) will contain three terms for $p = 0, p = 1$ and $p = 2$:

$$\mathcal{L}(p_1, p_2, p_3, p_4) = I_0 + I_1 + I_2 \tag{82}$$

with polynomials p_1, p_2, p_3, p_4 from the set $s_{-4}, s_{-3}, s_{-2}, s_{-1}, s_1, s_2, s_3, s_4$.

The main term I_0 corresponds to the partitions

$$S = \overline{S} = \emptyset \quad T = (1, 2, 3, 4)$$

$$I_0 \equiv I_{\emptyset,\emptyset,(1,2,3,4)} =$$
$$\int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} \int_{\Gamma_4} \frac{d\sigma_4}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2)$$
$$\cdot X_3(\sigma_3)\Phi(\sigma_3)X_4(\sigma_4)\Phi(\sigma_4)h_0(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{83}$$

where as above the function $h_0(\sigma_1, \ldots, \sigma_4)$ is defined by the formula (24).

For the next term with $p = 1$ there are six contributions corresponding to six different partitions that satisfy the conditions (26)

$$S = (1) \quad \overline{S} = (2) \quad T = (34)$$

$$
\begin{aligned}
&S=(1) \quad \overline{S}=(3) \quad T=(24)\\
&S=(1) \quad \overline{S}=(4) \quad T=(23)\\
&S=(2) \quad \overline{S}=(3) \quad T=(14)\\
&S=(2) \quad \overline{S}=(4) \quad T=(13)\\
&S=(3) \quad \overline{S}=(4) \quad T=(12)
\end{aligned}
$$

and according to the formulae (27,28)

$$
I_1 = I_{(1),(2),(34)} - I_{(1),(3),(24)} + I_{(1),(4),(23)} + I_{(2),(3),(14)} - I_{(2),(4),(13)} + I_{(3),(4),(12)} \tag{84}
$$

where

$$
\begin{aligned}
I_{(i_1),(i_2),(i_3\, i_4)} = &\int_{\Gamma_{i_1}} \frac{d\sigma_{i_1}}{2\pi i} \int_{\Gamma_{i_3}} \frac{d\sigma_{i_3}}{2\pi i} \int_{\Gamma_{i_4}} \frac{d\sigma_{i_4}}{2\pi i} X_{i_1}(\sigma_{i_1}) p_{i_2}(\sigma_{i_1}) \Phi(\sigma_{i_1})\\
&\cdot X_{i_3}(\sigma_{i_3}) \Phi(\sigma_{i_3}) X_{i_4}(\sigma_{i_4}) \Phi(\sigma_{i_4}) h_1(\sigma_{i_1}|\sigma_{i_3}, \sigma_{i_4})
\end{aligned} \tag{85}
$$

and due to the formula (30)

$$
h_1(\sigma_1|\sigma_3,\sigma_4) = \delta_{\sigma_1}^{-1}\Big(R_1 h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4) - D_3 f_1(\sigma_1|\sigma_3|\sigma_4) + D_4 f_1(\sigma_1|\sigma_4|\sigma_3)\Big) \tag{86}
$$

with the "exact forms" D_3 and D_4 defined by (29). Let us note that the function $h_1(\sigma_1|\sigma_3,\sigma_4)$ is manifestly anti-symmetric w.r.t. the transposition $\sigma_3 \leftrightarrow \sigma_4$.

For the last term with $p=2$ there are three contributions which correspond to the three possible partitions that satisfy the conditions (26)

$$
\begin{aligned}
&S=(12) \quad \overline{S}=(34) \quad T=\emptyset\\
&S=(12) \quad \overline{S}=(43) \quad T=\emptyset\\
&S=(13) \quad \overline{S}=(24) \quad T=\emptyset
\end{aligned}
$$

and

$$
I_2 = -I_{(12),(34),\emptyset} + I_{(12),(43),\emptyset} + I_{(13),(24),\emptyset} \tag{87}
$$

where

$$
\begin{aligned}
I_{(i_1\, i_2),(i_3\, i_4),\emptyset} = &\int_{\Gamma_{i_1}} \frac{d\sigma_{i_1}}{2\pi i} \int_{\Gamma_{i_2}} \frac{d\sigma_{i_2}}{2\pi i} X_{i_1}(\sigma_{i_1}) p_{i_3}(\sigma_{i_1}) \Phi(\sigma_{i_1})\\
&\cdot X_{i_2}(\sigma_{i_2}) p_{i_4}(\sigma_{i_2}) \Phi(\sigma_{i_2}) h_2(\sigma_{i_1},\sigma_{i_2})
\end{aligned} \tag{88}
$$

and due to the formula (30)

$$
\begin{aligned}
h_2(\sigma_1,\sigma_3) = &\,\delta_{\sigma_1}^{-1}\delta_{\sigma_3}^{-1}\Big(R_1 R_3 h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4)\\
&- R_1[D_1 f_1(\sigma_3|\sigma_1|\sigma_2) - D_2 f_1(\sigma_3|\sigma_2|\sigma_1)]\\
&- R_3[D_3 f_1(\sigma_1|\sigma_3|\sigma_4) - D_4 f_1(\sigma_1|\sigma_4|\sigma_3)]\Big)
\end{aligned} \tag{89}
$$

Actually this formula for $h_2(\sigma_1,\sigma_3)$ is symmetric w.r.t. the transposition $\sigma_1 \leftrightarrow \sigma_3$.

In order to define the function f_1 we need notation which are similar to those for the case $n=4$. Let us write only non-zero residues

$$\mathrm{Res}_{\sigma_2=\sigma_1+\pi i} h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \equiv V_1(\sigma_1|\sigma_3,\sigma_4)$$

$$\mathrm{Res}_{\sigma_2=\sigma_1-\pi i} h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4) = V_1(\sigma_1-\pi i|\sigma_3,\sigma_4)$$

$$\begin{aligned} &\mathrm{Res}_{\sigma_3=\sigma_1+2\pi i} V_1(\sigma_1|\sigma_3,\sigma_4) \equiv V_2(\sigma_1|\sigma_4) \\ &\mathrm{Res}_{\sigma_3=\sigma_1-\pi i} V_1(\sigma_1|\sigma_3,\sigma_4) = -V_2(\sigma_1-\pi i|\sigma_4) \end{aligned} \tag{90}$$

$$\begin{aligned} &\mathrm{Res}_{\sigma_4=\sigma_1+3\pi i} V_2(\sigma_1|\sigma_4) \equiv V_3(\sigma_1) \\ &\mathrm{Res}_{\sigma_4=\sigma_1+\pi i} V_2(\sigma_1|\sigma_4) \equiv V_{2,1}(\sigma_1) \\ &\mathrm{Res}_{\sigma_4=\sigma_1-\pi i} V_2(\sigma_1|\sigma_4) = V_3(\sigma_1-\pi i) \end{aligned} \tag{91}$$

The result for the function f_1 looks as follows:

$$f_1(\sigma_1|\sigma_3|\sigma_4) = \frac{V_2(\sigma_3|\sigma_4)}{\sigma_{13}} + (\frac{1}{\sigma_{13}} - \frac{1}{\sigma_{13}-\pi i})\frac{V_3(\sigma_3)}{\sigma_{34}+3\pi i} \tag{92}$$

An important property of the function f_1 is that non-zero residues w.r.t. to the pair of the variables σ_3,σ_4 are only in three points $\sigma_4=\sigma_3+3\pi i, \sigma_4=\sigma_3+\pi i, \sigma_4=\sigma_3-\pi i$. Let us denote

$$\begin{aligned} &\mathrm{Res}_{\sigma_4=\sigma_3+3\pi i} f_1(\sigma_1|\sigma_3|\sigma_4) \equiv \chi_3(\sigma_1|\sigma_3) \\ &\mathrm{Res}_{\sigma_4=\sigma_3+\pi i} f_1(\sigma_1|\sigma_3|\sigma_4) \equiv \chi_1(\sigma_1|\sigma_3) \end{aligned} \tag{93}$$

One can also check that

$$\mathrm{Res}_{\sigma_4=\sigma_3-\pi i} f_1(\sigma_1|\sigma_3|\sigma_4) = \chi_3(\sigma_1|\sigma_3-\pi i) \tag{94}$$

First we should be convinced that the expressions (86) and (89) for the functions h_1 and h_2 respectively are well-defined. For the case of the function h_1 one can proceed in the same line as it was done above for $n=4$. In order to check that (89) is well-defined also we need the following relation which can be verified straightforwardly using the above properties (93,94) of the function f_1

$$\begin{aligned} &R_1\Big(D_1 f_1(\sigma_3|\sigma_1|\sigma_2) - D_2 f_1(\sigma_3|\sigma_2|\sigma_1)\Big) = \delta_{\sigma_1}\Big(\frac{\chi_1(\sigma_3|\sigma_1-\pi i)}{P(\sigma_1-\frac{\pi i}{2})P(\sigma_1+\frac{\pi i}{2})} \\ &+ \frac{\chi_3(\sigma_3|\sigma_1-\pi i)}{P(\sigma_1-\frac{\pi i}{2})P(\sigma_1+\frac{3\pi i}{2})} + \frac{\chi_3(\sigma_3|\sigma_1-2\pi i)}{P(\sigma_1-\frac{3\pi i}{2})P(\sigma_1+\frac{\pi i}{2})}\Big) \end{aligned} \tag{95}$$

Then the bracket in the r.h.s. of (89) can be rewritten as follows

$$\begin{aligned} &R_3\Big(R_1 h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4) - D_3 f_1(\sigma_1|\sigma_3|\sigma_4) + D_4 f_1(\sigma_1|\sigma_4|\sigma_3)\Big) \\ &- R_1\Big(D_1 f_1(\sigma_3|\sigma_1|\sigma_2) - D_2 f_1(\sigma_3|\sigma_2|\sigma_1)\Big) \end{aligned} \tag{96}$$

For the first term the operator $\delta_{\sigma_1}^{-1}$ is well-defined just because it is $\delta_{\sigma_1} R_3 h_1(\sigma_1|\sigma_3,\sigma_4)$ according to the formula (86). For the second term in the formula (96) $\delta_{\sigma_1}^{-1}$ does exist due to the identity (95). Thus we have proved that the operator $\delta_{\sigma_1}^{-1}$ is well-defined for the whole expression (96). On the other hand it is manifestly symmetric w.r.t. the transposition $\sigma_1 \leftrightarrow \sigma_3$. It means that the operator $\delta_{\sigma_3}^{-1}$ is well-defined as well.

Let us check the skew-symmetry of (82) for the transposition $3 \leftrightarrow 4$. An analog of the formulae (54) for $n=3$ and (74) for $n=4$ for the first term I_0 defined by (83) is

$$I_0 + I_{0|_{3\leftrightarrow 4}} = J_0 \tag{97}$$

with a new

$$J_0 = \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2)B_{34}(\sigma_3) \\ \cdot R_3 h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \tag{98}$$

and analogously to (56) and (76)

$$B_{34}(\sigma_2) = X_3(\sigma_3)X_4(\sigma_3+\pi i) + X_4(\sigma_3)X_3(\sigma_3+\pi i) \tag{99}$$

$$R_3 = \frac{1}{P(\sigma_3+\frac{\pi i}{2})}\text{Res}_{\sigma_4=\sigma_3+\pi i}$$

As we did above we rewrite (86) in the following form

$$R_3 h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4) = \delta_{\sigma_3} h_1(\sigma_3|\sigma_1,\sigma_2) + D_1 f_1(\sigma_3|\sigma_1|\sigma_2) - D_2 f_1(\sigma_3|\sigma_2|\sigma_1) \tag{100}$$

Substituting this expression into the formula (98) we get

$$J_0 = J_0' + \mathcal{F}_1 + \mathcal{F}_2 \tag{101}$$

where

$$\begin{aligned} J_0' &= \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2)B_{34}(\sigma_3)\delta_{\sigma_3} h_1(\sigma_3|\sigma_1,\sigma_2) \\ &= -\int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2) \\ &\cdot (X_3(\sigma_3)p_4(\sigma_3) + X_4(\sigma_3)p_3(\sigma_3))\Phi(\sigma_3)h_1(\sigma_3|\sigma_1,\sigma_2) \\ &= -I_{(3),(4),(12)} - I_{(3),(4),(12)_{|3\leftrightarrow 4}} \end{aligned} \tag{102}$$

$$\begin{aligned} \mathcal{F}_1 &= \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_1(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2)B_{34}(\sigma_3)D_1 f_1(\sigma_3|\sigma_1|\sigma_2) \\ &= -\int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_2} \frac{d\sigma_2}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_2(\sigma_2)\Phi(\sigma_2)B_{34}(\sigma_3)\frac{p_1(\sigma_1)f_1(\sigma_3|\sigma_1|\sigma_2)}{P(\sigma_1+\frac{\pi i}{2})P(\sigma_1+\frac{3\pi i}{2})} = 0 \end{aligned} \tag{103}$$

The reason why we get zero here is similar to that of the previous case, namely, because the ratio

$$\frac{p_1(\sigma_1)f_1(\sigma_3|\sigma_1|\sigma_2)}{P(\sigma_1+\frac{\pi i}{2})P(\sigma_1+\frac{3\pi i}{2})}$$

can have poles only for the variables σ_{12} and σ_{13}. In both cases these poles are outside of the contour Γ_1.

The situation with the third term in (101) $\mathcal{F}_2$ is different

$$\begin{aligned}\mathcal{F}_2 &= -\int_{\Gamma_1}\frac{d\sigma_1}{2\pi i}\int_{\Gamma_2}\frac{d\sigma_2}{2\pi i}\int_{\Gamma_3}\frac{d\sigma_3}{2\pi i}X_1(\sigma_1)\Phi(\sigma_1)X_2(\sigma_2)\Phi(\sigma_2)B_{34}(\sigma_3)D_2f_1(\sigma_3|\sigma_2|\sigma_1)\\ &= \int_{\Gamma_1}\frac{d\sigma_1}{2\pi i}\int_{\Gamma_2}\frac{d\sigma_2}{2\pi i}\int_{\Gamma_3}\frac{d\sigma_3}{2\pi i}X_1(\sigma_1)\Phi(\sigma_1)B_{34}(\sigma_3)\frac{p_2(\sigma_2)f_1(\sigma_3|\sigma_2|\sigma_1)}{P(\sigma_2+\frac{\pi i}{2})P(\sigma_2+\frac{3\pi i}{2})}\end{aligned}\tag{104}$$

This is not zero because now there is a contribution from the poles for the variable σ_{12}. Using the properties of the function f_1 (93,94) we can get

$$\begin{aligned}&\int_{\Gamma_2}\frac{d\sigma_2}{2\pi i}\frac{p_2(\sigma_2)f_1(\sigma_3|\sigma_2|\sigma_1)}{P(\sigma_2+\frac{\pi i}{2})P(\sigma_2+\frac{3\pi i}{2})}\\ &= D_1\left(p_2(\sigma_1)\frac{\chi_3(\sigma_3|\sigma_1-\pi i)}{P(\sigma_1-\frac{\pi i}{2})} - p_2(\sigma_1+2\pi i)\frac{\chi_3(\sigma_3|\sigma_1)}{P(\sigma_1+\frac{5\pi i}{2})}\right)\\ &- p_2(\sigma_1)\left(\frac{\chi_1(\sigma_3|\sigma_1-\pi i)}{P(\sigma_1-\frac{\pi i}{2})P(\sigma_1+\frac{\pi i}{2})}+\frac{\chi_3(\sigma_3|\sigma_1-\pi i)}{P(\sigma_1-\frac{\pi i}{2})P(\sigma_1+\frac{3\pi i}{2})}\right.\\ &\left.+\frac{\chi_3(\sigma_3|\sigma_1-2\pi i)}{P(\sigma_1-\frac{3\pi i}{2})P(\sigma_1+\frac{\pi i}{2})}\right)\\ &= D_1\left(p_2(\sigma_1)\frac{\chi_3(\sigma_3|\sigma_1-\pi i)}{P(\sigma_1-\frac{\pi i}{2})} - p_2(\sigma_1+2\pi i)\frac{\chi_3(\sigma_3|\sigma_1)}{P(\sigma_1+\frac{5\pi i}{2})}\right)\\ &- p_2(\sigma_1)\delta_{\sigma_1}^{-1}R_1\Big(D_1f_1(\sigma_3|\sigma_1|\sigma_2)-D_2f_1(\sigma_3|\sigma_2|\sigma_1)\Big)\end{aligned}\tag{105}$$

The last equality follows from the relation (95). Substituting this result for the integration over σ_2 into (104) and using the fact that the term $D_1(\ldots)$ in (105) does not contribute one arrives at

$$\begin{aligned}\mathcal{F}_2 &= \int_{\Gamma_1}\frac{d\sigma_1}{2\pi i}\int_{\Gamma_3}\frac{d\sigma_3}{2\pi i}X_1(\sigma_1)p_2(\sigma_1)\Phi(\sigma_1)B_{34}(\sigma_3)\\ &\cdot\delta_{\sigma_1}^{-1}R_1\Big(-D_1f_1(\sigma_3|\sigma_1|\sigma_2)+D_2f_1(\sigma_3|\sigma_2|\sigma_1)\Big)\end{aligned}\tag{106}$$

Thus we come to the following result

$$J_0 = -I_{(3),(4),(12)} - I_{(3),(4),(12)}{}_{|3\leftrightarrow 4} + \mathcal{F}_2\tag{107}$$

with $\mathcal{F}_2$ given by (106).

Now let us note that the part of (84)

$$-I_{(1),(3),(24)}+I_{(1),(4),(23)}+I_{(2),(3),(14)}-I_{(2),(4),(13)}$$

is already skew-symmetric w.r.t. the transposition $3\leftrightarrow 4$. It is left to treat the very first term in (84)

$$I_{(1),(2),(34)}$$

Proceeding in a similar way as we treated above the term I_0 by the derivation of the formula (98) and using the formula (86) for the function h_1 we can obtain

$$I_{(1),(2),(34)} + I_{(1),(2),(34)}{}_{|3\leftrightarrow 4} + \mathcal{F}_2 = J_1 \tag{108}$$

where

$$J_1 = \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_1(\sigma_1) p_2(\sigma_1) \Phi(\sigma_1) B_{34}(\sigma_3) \delta_{\sigma_1}^{-1} \Big(R_1 R_3 h_0(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$$
$$+ R_1[-D_1 f_1(\sigma_3|\sigma_1|\sigma_2) + D_2 f_1(\sigma_3|\sigma_2|\sigma_1)]$$
$$+ R_3[-D_3 f_1(\sigma_1|\sigma_3|\sigma_4) + D_4 f_1(\sigma_1|\sigma_4|\sigma_3)] \Big) \tag{109}$$

Actually, expression in the curled bracket coincides with that from the formula (89). Therefore

$$J_1 = \int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_1(\sigma_1) p_2(\sigma_1) \Phi(\sigma_1) B_{34}(\sigma_3) \delta_{\sigma_3} h_2(\sigma_1, \sigma_3)$$
$$= -\int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_1(\sigma_1) p_2(\sigma_1) \Phi(\sigma_1)$$
$$\cdot (X_3(\sigma_3) p_4(\sigma_3) + X_4(\sigma_3) p_3(\sigma_3)) \Phi(\sigma_3) h_2(\sigma_1, \sigma_3)$$
$$= -I_{(13),(24),\emptyset} - I_{(13),(24),\emptyset}{}_{|3\leftrightarrow 4} \tag{110}$$

What is left is to note that the part of (87) which contains the residual two terms

$$-I_{(12),(34),\emptyset} + I_{(12),(43),\emptyset}$$

is already skew-symmetric w.r.t. the transposition $3 \leftrightarrow 4$.

Thus we have checked the skew-symmetry of the whole expression (82) for the transposition $3 \leftrightarrow 4$. For other transpositions $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$ this can also be done.

5. **The case** $n = 6$

As in the previous case the formula (28) contains three terms for $p = 0, p = 1$ and $p = 2$:

$$\mathcal{L}(p_1, p_2, p_3, p_4, p_5) = I_0 + I_1 + I_2 \tag{111}$$

where p_1, p_2, p_3, p_4, p_5 belong to the set of polynomials
$s_{-5}, \ldots, s_{-2}, s_{-1}, s_1, s_2, \ldots, s_5$.

The main term I_0 corresponds to the partitions

$$S = \overline{S} = \emptyset \quad T = (1, 2, 3, 4, 5)$$

$$I_0 \equiv I_{\emptyset,\emptyset,(1,2,3,4,5)} =$$
$$\int_{\Gamma_1} \frac{d\sigma_1}{2\pi i} \cdots \int_{\Gamma_5} \frac{d\sigma_5}{2\pi i} X_1(\sigma_1) \Phi(\sigma_1) X_2(\sigma_2) \Phi(\sigma_2) X_3(\sigma_3) \Phi(\sigma_3)$$
$$\cdot X_4(\sigma_4) \Phi(\sigma_4) X_5(\sigma_5) \Phi(\sigma_5) h_0(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \tag{112}$$

where as above the function $h_0(\sigma_1, \ldots, \sigma_5)$ is defined by the formula (24).

For the next term with $p = 1$ there are ten contributions corresponding to ten partitions

$$S = (1) \quad \overline{S} = (2) \quad T = (345)$$

$$\begin{gathered} S = (1) \quad \overline{S} = (3) \quad T = (245) \\ S = (1) \quad \overline{S} = (4) \quad T = (235) \\ S = (1) \quad \overline{S} = (5) \quad T = (234) \\ S = (2) \quad \overline{S} = (3) \quad T = (145) \\ S = (2) \quad \overline{S} = (4) \quad T = (135) \\ S = (2) \quad \overline{S} = (5) \quad T = (134) \\ S = (3) \quad \overline{S} = (4) \quad T = (125) \\ S = (3) \quad \overline{S} = (5) \quad T = (124) \\ S = (4) \quad \overline{S} = (5) \quad T = (123) \end{gathered}$$

and according to the formulae (27,28)

$$\begin{aligned} I_1 = {} & I_{(1),(2),(345)} - I_{(1),(3),(245)} + I_{(1),(4),(235)} - I_{(1),(5),(234)} + I_{(2),(3),(145)} \\ & - I_{(2),(4),(135)} + I_{(2),(5),(134)} + I_{(3),(4),(125)} - I_{(3),(5),(124)} + I_{(4),(5),(123)} \end{aligned} \tag{113}$$

where

$$\begin{aligned} I_{(i_1),(i_2),(i_3\, i_4\, i_5)} = {} & \int_{\Gamma_{i_1}} \frac{d\sigma_{i_1}}{2\pi i} \int_{\Gamma_{i_3}} \frac{d\sigma_{i_3}}{2\pi i} \int_{\Gamma_{i_4}} \frac{d\sigma_{i_4}}{2\pi i} \int_{\Gamma_{i_5}} \frac{d\sigma_{i_5}}{2\pi i} X_{i_1}(\sigma_{i_1}) p_{i_2}(\sigma_{i_1}) \Phi(\sigma_{i_1}) \\ \cdot\, & X_{i_3}(\sigma_{i_3}) \Phi(\sigma_{i_3}) X_{i_4}(\sigma_{i_4}) \Phi(\sigma_{i_4}) X_{i_5}(\sigma_{i_5}) \Phi(\sigma_{i_5}) h_1(\sigma_{i_1}|\sigma_{i_3}, \sigma_{i_4}, \sigma_{i_5}) \end{aligned} \tag{114}$$

and due to the formula (30)

$$\begin{aligned} h_1(\sigma_1|\sigma_3, \sigma_4, \sigma_5) = {} & \delta_{\sigma_1}^{-1} \Bigg(R_1 h_0(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \\ & - D_3 f_1(\sigma_1|\sigma_3|\sigma_4, \sigma_5) + D_4 f_1(\sigma_1|\sigma_4|\sigma_3, \sigma_5) - D_5 f_1(\sigma_1|\sigma_5|\sigma_3, \sigma_4) \Bigg) \end{aligned} \tag{115}$$

Obviously the function $h_1(\sigma_1|\sigma_3, \sigma_4, \sigma_5)$ is fully anti-symmetric for the variables $\sigma_3, \sigma_4, \sigma_5$.

For the last term with $p = 2$ there are fifteen contributions corresponding to all possible partitions that satisfy the conditions (26)

$$S = (12) \quad \overline{S} = (34) \quad T = (5)$$

$$\begin{gathered} S = (12) \quad \overline{S} = (43) \quad T = (5) \\ S = (12) \quad \overline{S} = (35) \quad T = (4) \\ S = (12) \quad \overline{S} = (53) \quad T = (4) \\ S = (12) \quad \overline{S} = (45) \quad T = (3) \end{gathered}$$

$$S=(12)\quad \overline{S}=(54)\quad T=(3)$$
$$S=(13)\quad \overline{S}=(24)\quad T=(5)$$
$$S=(13)\quad \overline{S}=(25)\quad T=(4)$$
$$S=(13)\quad \overline{S}=(45)\quad T=(2)$$
$$S=(13)\quad \overline{S}=(54)\quad T=(2)$$
$$S=(14)\quad \overline{S}=(25)\quad T=(3)$$
$$S=(14)\quad \overline{S}=(35)\quad T=(2)$$
$$S=(23)\quad \overline{S}=(45)\quad T=(1)$$
$$S=(23)\quad \overline{S}=(54)\quad T=(1)$$
$$S=(24)\quad \overline{S}=(35)\quad T=(1)$$

and

$$\begin{aligned}
I_2 = &-I_{(12),(34),(5)} + I_{(12),(43),(5)} + I_{(12),(35),(4)} - I_{(12),(53),(4)} - I_{(12),(45),(3)} \\
&+ I_{(12),(54),(3)} + I_{(13),(24),(5)} - I_{(13),(25),(4)} + I_{(13),(45),(2)} - I_{(13),(54),(2)} \\
&+ I_{(14),(25),(3)} - I_{(14),(35),(2)} - I_{(23),(45),(1)} + I_{(23),(54),(1)} + I_{(24),(35),(1)}
\end{aligned} \tag{116}$$

where

$$\begin{aligned}
I_{(i_1\,i_2),(i_3\,i_4),(i_5)} = &\int_{\Gamma_{i_1}} \frac{d\sigma_{i_1}}{2\pi i} \int_{\Gamma_{i_2}} \frac{d\sigma_{i_2}}{2\pi i} \int_{\Gamma_{i_5}} \frac{d\sigma_{i_5}}{2\pi i} X_{i_1}(\sigma_{i_1}) p_{i_3}(\sigma_{i_1}) \Phi(\sigma_{i_1}) \\
&X_{i_2}(\sigma_{i_2}) p_{i_4}(\sigma_{i_2}) \Phi(\sigma_{i_2}) X_{i_5}(\sigma_{i_5}) \Phi(\sigma_{i_5}) h_2(\sigma_{i_1}, \sigma_{i_2}|\sigma_{i_5})
\end{aligned} \tag{117}$$

and due to the formula (30)

$$\begin{aligned}
h_2(\sigma_1,\sigma_3|\sigma_5) = \delta_{\sigma_1}^{-1}\delta_{\sigma_3}^{-1}\Big(&R_1 R_3 h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5) \\
&- R_1[D_1 f_1(\sigma_3|\sigma_1|\sigma_2,\sigma_5) - D_2 f_1(\sigma_3|\sigma_2|\sigma_1,\sigma_5) + D_5 f_1(\sigma_3|\sigma_5|\sigma_1,\sigma_2)] \\
&- R_3[D_3 f_1(\sigma_1|\sigma_3|\sigma_4,\sigma_5) - D_4 f_1(\sigma_1|\sigma_4|\sigma_3,\sigma_5) + D_5 f_1(\sigma_1|\sigma_5|\sigma_3,\sigma_4)] \\
&+ D_5 f_2(\sigma_1,\sigma_3|\sigma_5)\Big)
\end{aligned} \tag{118}$$

Like in the case $n=5$ the formula for $h_2(\sigma_1,\sigma_3|\sigma_5)$ is symmetric w.r.t. the transposition $\sigma_1 \leftrightarrow \sigma_3$.

In order to define the functions f_1 and f_2 we need some residues which, in principle, may be computed explicitly using the formula (24)

$$\mathrm{Res}_{\sigma_2=\sigma_1+\pi i} h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5) = V_1(\sigma_1|\sigma_3,\sigma_4,\sigma_5)$$
$$\mathrm{Res}_{\sigma_2=\sigma_1-\pi i} h_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5) = V_1(\sigma_1-\pi i|\sigma_3,\sigma_4,\sigma_5)$$

$$\mathrm{Res}_{\sigma_3=\sigma_1+2\pi i} V_1(\sigma_1|\sigma_3,\sigma_4,\sigma_5) = V_2(\sigma_1|\sigma_4,\sigma_5)$$
$$\mathrm{Res}_{\sigma_3=\sigma_1-\pi i} V_1(\sigma_1|\sigma_3,\sigma_4,\sigma_5) = -V_2(\sigma_1-\pi i|\sigma_4,\sigma_5)$$

$$\mathrm{Res}_{\sigma_4=\sigma_1+3\pi i}V_2(\sigma_1|\sigma_4,\sigma_5)=V_3(\sigma_1|\sigma_5)$$
$$\mathrm{Res}_{\sigma_4=\sigma_1+\pi i}V_2(\sigma_1|\sigma_4,\sigma_5)=V_{2,1}(\sigma_1|\sigma_5)$$
$$\mathrm{Res}_{\sigma_4=\sigma_1-\pi i}V_2(\sigma_1|\sigma_4,\sigma_5)=V_3(\sigma_1-\pi i|\sigma_5)$$
$$\mathrm{Res}_{\sigma_5=\sigma_4+\pi i}V_2(\sigma_1|\sigma_4,\sigma_5)=\tilde{V}_2(\sigma_1|\sigma_4)$$

$$\mathrm{Res}_{\sigma_5=\sigma_1+4\pi i}V_3(\sigma_1|\sigma_5)=V_4(\sigma_1)$$
$$\mathrm{Res}_{\sigma_5=\sigma_1+2\pi i}V_3(\sigma_1|\sigma_5)=V_{3,2}(\sigma_1)$$
$$\mathrm{Res}_{\sigma_5=\sigma_1+\pi i}V_3(\sigma_1|\sigma_5)=V_{3,1}(\sigma_1)$$
$$\mathrm{Res}_{\sigma_5=\sigma_1-\pi i}V_3(\sigma_1|\sigma_5)=-V_4(\sigma_1-\pi i)$$

$$\mathrm{Res}_{\sigma_4=\sigma_3+3\pi i}\tilde{V}_2(\sigma_3|\sigma_4)=V_4(\sigma_3)$$
$$\mathrm{Res}_{\sigma_4=\sigma_3+2\pi i}\tilde{V}_2(\sigma_3|\sigma_4)=V_{3,2}(\sigma_3)$$
$$\mathrm{Res}_{\sigma_4=\sigma_3+\pi i}\tilde{V}_2(\sigma_3|\sigma_4)=\tilde{V}_{2,1}(\sigma_3)$$
$$\mathrm{Res}_{\sigma_4=\sigma_3}\tilde{V}_2(\sigma_3|\sigma_4)=\tilde{V}_{2,0}(\sigma_3)$$
$$\mathrm{Res}_{\sigma_4=\sigma_3-\pi i}\tilde{V}_2(\sigma_3|\sigma_4)=V_{3,1}(\sigma_3-\pi i)$$
$$\mathrm{Res}_{\sigma_4=\sigma_3-2\pi i}\tilde{V}_2(\sigma_3|\sigma_4)=-V_4(\sigma_3-2\pi i)$$

The result for the function f_1 looks as follows:

$$\begin{aligned}&f_1(\sigma_1|\sigma_3|\sigma_4,\sigma_5)=\frac{V_2(\sigma_1|\sigma_4,\sigma_5)}{\sigma_{13}}\\&-(\frac{1}{\sigma_{13}}-\frac{1}{\sigma_{13}-\pi i})p(\sigma_3|\sigma_4,\sigma_5)-(\frac{1}{\sigma_{13}-\pi i}-\frac{1}{\sigma_{13}-2\pi i})q(\sigma_3|\sigma_4,\sigma_5)\end{aligned} \tag{119}$$

where

$$\begin{aligned}p(\sigma_3|\sigma_4,\sigma_5)=&\frac{V_3(\sigma_3|\sigma_4)}{\sigma_{35}+3\pi i}-\frac{V_3(\sigma_3|\sigma_5)}{\sigma_{34}+3\pi i}-\\&-\frac{V_4(\sigma_3)}{(\sigma_{35}+4\pi i)(\sigma_{54}-\pi i)}+\frac{V_4(\sigma_3)}{(\sigma_{34}+4\pi i)(\sigma_{45}-\pi i)}+\\&+\frac{V_{3,2}(\sigma_3)}{(\sigma_{35}+2\pi i)(\sigma_{45}-\pi i)}-\frac{V_{3,2}(\sigma_3)}{(\sigma_{34}+2\pi i)(\sigma_{54}-\pi i)}\end{aligned} \tag{120}$$

and

$$\begin{aligned}&q(\sigma_3|\sigma_4,\sigma_5)=\frac{V_4(\sigma_3)}{(\sigma_{35}+3\pi i)(\sigma_{45}-\pi i)}-\frac{V_4(\sigma_3)}{(\sigma_{34}+3\pi i)(\sigma_{54}-\pi i)}-\\&-\frac{V_4(\sigma_3)}{(\sigma_{35}+\pi i)(\sigma_{45}-\pi i)}\frac{P(\sigma_3+\frac{3\pi i}{2})}{P(\sigma_3+\frac{7\pi i}{2})}+\frac{V_4(\sigma_3)}{(\sigma_{34}+\pi i)(\sigma_{54}-\pi i)}\frac{P(\sigma_3+\frac{3\pi i}{2})}{P(\sigma_3+\frac{7\pi i}{2})}\end{aligned} \tag{121}$$

The answer for the function f_2 is

$$f_2(\sigma_1,\sigma_3|\sigma_5) = \frac{V_4(\sigma_5)}{(\sigma_{15}-\pi i)(\sigma_{35}-\pi i)P(\sigma_5+\frac{7\pi i}{2})} - \frac{V_4(\sigma_5)}{\sigma_{15}\sigma_{35}P(\sigma_5-\frac{3\pi i}{2})}$$
$$+\frac{V_{3,1}(\sigma_5-\pi i)}{\sigma_{15}\sigma_{35}P(\sigma_3-\frac{\pi i}{2})} + \frac{V_{3,2}(\sigma_5)}{(\sigma_{15}-\pi i)(\sigma_{35}-\pi i)P(\sigma_5+\frac{5\pi i}{2})}$$
$$+\frac{\tilde{V}_{2,0}(\sigma_5)}{\sigma_{15}\sigma_{35}P(\sigma_5+\frac{\pi i}{2})} + \frac{\tilde{V}_{2,1}(\sigma_5)}{\sigma_{15}\sigma_{35}P(\sigma_5+\frac{3\pi i}{2})} \tag{122}$$

Let us note that the function $f_1(\sigma_1|\sigma_3|\sigma_4,\sigma_5)$ is anti-symmetric w.r.t. the transposition of the last two variables $\sigma_4 \leftrightarrow \sigma_5$ while the function $f_2(\sigma_1,\sigma_3|\sigma_5)$ is symmetric w.r.t. the transposition of the first two variables $\sigma_1 \leftrightarrow \sigma_3$.

Let us also note that the function f_1 defined by (119) has a similar to the case $n = 5$ property, namely, that the only non-zero residues for the pair of the variables σ_3, σ_4 are as follows

$$\begin{aligned}
&\mathrm{Res}_{\sigma_4=\sigma_3+3\pi i} f_1(\sigma_1|\sigma_3|\sigma_4,\sigma_5) = \chi_3(\sigma_1|\sigma_3|\sigma_5)\\
&\mathrm{Res}_{\sigma_4=\sigma_3+\pi i} f_1(\sigma_1|\sigma_3|\sigma_4,\sigma_5) = \chi_1(\sigma_1|\sigma_3|\sigma_5)\\
&\mathrm{Res}_{\sigma_4=\sigma_3-\pi i} f_1(\sigma_1|\sigma_3|\sigma_4,\sigma_5) = \chi_3(\sigma_1|\sigma_3-\pi i|\sigma_5)
\end{aligned} \tag{123}$$

where the functions χ_1 and χ_3 may be calculated explicitly. Of course, such a property is true for the pair of variables σ_3, σ_5 also.

Similar to the case $n = 5$ arguments can be used in order to check that the expressions (115) and (118) are well-defined because due to the property (123) a similar to (95) relation is valid also

$$R_1\Bigg(D_1 f_1(\sigma_4|\sigma_1|\sigma_2,\sigma_3) - D_2 f_1(\sigma_4|\sigma_2|\sigma_1,\sigma_3)\Bigg) = \delta_{\sigma_1}\Bigg(\frac{\chi_1(\sigma_4|\sigma_1-\pi i|\sigma_3)}{P(\sigma_1-\frac{\pi i}{2})P(\sigma_1+\frac{\pi i}{2})}$$
$$+\frac{\chi_3(\sigma_4|\sigma_1-\pi i|\sigma_3)}{P(\sigma_1-\frac{\pi i}{2})P(\sigma_1+\frac{3\pi i}{2})} + \frac{\chi_3(\sigma_4|\sigma_1-2\pi i|\sigma_3)}{P(\sigma_1-\frac{3\pi i}{2})P(\sigma_1+\frac{\pi i}{2})}\Bigg) \tag{124}$$

In order to check the skew-symmetry of the answer (111) one can follow the same way like for the previous case $n = 5$. All necessary relations may be straightforwardly generalized for the case $n = 6$ also. For instance, the integral formula (105) may be generalized as follows

$$\int_{\Gamma_3} \frac{d\sigma_3}{2\pi i} X_3(\sigma_3)\Phi(\sigma_3) D_3 f_1(\sigma_4|\sigma_3|\sigma_1,\sigma_2)$$
$$= D_1\Bigg(p_3(\sigma_1)\frac{\chi_3(\sigma_4|\sigma_1-\pi i|\sigma_2)}{P(\sigma_1-\frac{\pi i}{2})} - p_3(\sigma_1+2\pi i)\frac{\chi_3(\sigma_4|\sigma_1|\sigma_2)}{P(\sigma_1+\frac{5\pi i}{2})}\Bigg)$$
$$- D_2\Bigg(p_3(\sigma_2)\frac{\chi_3(\sigma_4|\sigma_2-\pi i|\sigma_1)}{P(\sigma_2-\frac{\pi i}{2})} - p_3(\sigma_2+2\pi i)\frac{\chi_3(\sigma_4|\sigma_2|\sigma_1)}{P(\sigma_2+\frac{5\pi i}{2})}\Bigg)$$
$$- p_3(\sigma_1)\delta_{\sigma_1}^{-1}\Bigg[\frac{\mathrm{Res}_{\sigma 3=\sigma 1+\pi i}}{P(\sigma_1+\frac{\pi i}{2})}\Bigg(D_1 f_1(\sigma_4|\sigma_1|\sigma_3,\sigma_2) - D_3 f_1(\sigma_4|\sigma_3|\sigma_1,\sigma_2)\Bigg)\Bigg]$$
$$- p_3(\sigma_2)\delta_{\sigma_2}^{-1}\Bigg[\frac{\mathrm{Res}_{\sigma 3=\sigma 2+\pi i}}{P(\sigma_2+\frac{\pi i}{2})}\Bigg(D_2 f_1(\sigma_4|\sigma_2|\sigma_1,\sigma_3) - D_3 f_1(\sigma_4|\sigma_3|\sigma_1,\sigma_2)\Bigg)\Bigg] \tag{125}$$

Only one complication in comparison with $n = 5$ appears because of the second term with $D_2(\ldots)$ in the r.h.s. of (125) that makes a non-zero contribution after further integration. Because of this fact we had to include an additional terms with the ratio $P(\sigma_3 + \frac{3\pi i}{2})/P(\sigma_3 + \frac{7\pi i}{2})$ in (121) which appeared to be necessary for the correct definition of the function f_1.

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