CR–TRANSFORMATIONS OF REAL MANIFOLDS IN $\mathbb{C}^n$

by

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§ 1. Introduction.

The notions of CR–manifold and CR–function are now fundamental in Several Complex Variables. The development of the theory of CR–manifolds and CR–functins naturally requires also to introduce and to study CR–transformations.

Let M_1, M_2 are CR–manifolds in $\mathbb{C}^n$. A mapping $f = (f_1,\dots,f_n) : M_1 \longrightarrow M_2$ is called CR–mapping if all components f_j are CR–functions on M_1, i.e. they satisfy tangential Cauchy–Riemann conditions. Here the functions f_j are not necessary differentiable because weak tangential Cauchy–Riemann conditions have sense for continuous functions and even for distributions. We restrict ourselfs by continuous CR–mappings to make sense the expression $f : M_1 \longrightarrow M_2$.

Definition 1 A mapping $f : M_1 \longrightarrow M_2$ is called CR–homeomorphism if

(i) f is homeomorphism,

(ii) f is CR–mapping,

(iii) f^{-1} is also CR–mapping.

Definition 2 A CR–manifold M in $\mathbb{C}^n$ is called locally k–CR–straightened near a point $p^0 \in M$ if there exist CR–manifold M_0 in $\mathbb{C}^{n-k}$ and CR–homeomorphism

$$f : (M_0 \times \mathbb{C}^k) \cap V \longrightarrow M \cap U$$

where $U, V \subset \mathbb{C}^n$ are some neighborhoods of the points $p^0, q^0 = f^{-1}(p^0)$ respectively.

In § 2 we discuss the problem of CR–straightening. Each k–CR–straightened manifold M is foliated by complex varieties of dimension k. We shall assume everywhere below that these varieties are non–singular, i.e. they are complex manifolds in $\mathbb{C}^n$. But the author doesn't know if some real manifold in $\mathbb{C}^n$ can be foliated by singular complex varieties.

Each point p of k–CR–straightened manifold M belongs to unique leaf S of the foliation. We denote by π_p the tangent plane $T_p(S) \subset T_p^c(M)$ to S at p. Therefore k–CR–straightening of M induces the distribution $\mathscr{L} : p \longrightarrow \pi_p$ of complex k–planes on M.

Let M be of class C^2, r is real codimension of M and $\rho_1, \dots, \rho_r \in C^2(U)$ are defining functions of M in some neighborhood $U \supset M$, i.e.

$$M = \{z \in U : \rho_1(z) = \dots = \rho_r(z) = 0\}$$

and $d\rho_1 \wedge \dots \wedge d\rho_r \neq 0$. Let

$$L_p(\rho_j, u, v) = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 \rho_j}{\partial z_\mu \partial \bar{z}_\nu}(p)\, u_\mu \bar{v}_\nu$$

$(u, v \in T_p^c(M))$ is the Levi form of function ρ_j at point $p \in M$ and let

$$N_p = \{v \in T_p^c(M) : L_p(\rho_j, u, v) = 0 \text{ for all } u \in T_p^c(M)\ ,\ j = 1, \dots, v\}$$

is the null space of Levi form of M at point p. Obviously k–CR–straightening is possible only along null directions of Levi form, i.e. $\pi_p \subset N_p$.

Another obvious restriction on $\mathscr{L}$ which is necessary for CR–straightening is integrability of $\mathscr{L}$. According to Frobenius theorem this means that $\mathscr{L}$ is involutive (if $\mathscr{L} \in C^1$) (see [7]). Integrability of $\mathscr{L}$ provides foliation of M by complex manifolds of dimension k, but it is not sufficient for holomorphic or CR–straightening.

M. Freeman [5] found other necessary conditions for holomorphic straightening of manifold of class C^{∞} which are sufficient in real analytic case. These conditions are expressed in terms of modules of special vector fields on M and sometimes it requires some affort to verify these conditions. For the boundary of so called "future tube" domain

$$\tau_+ = \{z = (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} : y_0^2 > y_1^2 + \dots + y_n^2 ,\ y_0 > 0\}$$

the calculations were fulfilled by A. Sergeev [11]. He proved that the boundary $\partial\tau_+$ is not 1–holomorphically straightened even locally though it is foliated by complex lines.

Another approach to the discussed problem is based on the consideration of $\mathscr{L}$ as a mapping from M to the Grassmannian $G(k,n)$ of complex k–subspaces in $\mathbb{C}^n$. C. Rea [10] seems was the first who noticed that for k–holomorphic straightening of real hypersurface $M \subset \mathbb{C}^n$ with constant rank of Levi form it is necessary that $\mathscr{L}: p \in M \longrightarrow \pi_p \in G(\kappa,n)$ is CR–mapping. He also proved in some cases that the last condition together with involutiveness of $\mathscr{L}$ is sufficient for holomorphic straightening of M.

In this paper we prove in rather general situation that the conditions

(i) $\mathscr{L}$ is integrable (involutive)

(ii) $\mathscr{L}: M \longrightarrow G(\kappa,n)$ is CR–mapping

are necessary and sufficient for the local k–CR–straightening of M (theorem 2.1). The necessity of the condition (ii) was proved by S. Tsyganov [9]. For the completeness of exposition we include the short proof of this fact.

Theorem 2.1 illustrates a phenomenon that more general results sometimes have more simple proofs.

In § 3 we discuss another problem which is connected with the definition 1 of CR–homeomorphism. The analogy with biholomorphic mappings makes natural the following

Conjecture 1 Let M_1, M_2 be CR–manifolds in $\mathbb{C}^n$ and $f : M_1 \longrightarrow M_2$ is CR–mapping and homeomorphism. Then the inverse mapping f^{-1} is also CR.

Unfortunately this conjecture is not true without soe additional assumptions. The simplest counterexample is the following. Let $L = \{z = (z_1,z_2) \in \mathbb{C}^2 : z_2 = 0\}$ and $\mathbb{R}^2 \subset \mathbb{C}^2$ is a real subspace in $\mathbb{C}^2$. The mapping $f : \mathbb{R}^2 \longrightarrow L$ defined by $(x_1,x_2) \longrightarrow (x_1+ix_2,0)$ is homeomorphism and even real analytic diffeomorphism. It is also CR because $\mathbb{R}^2$ is totally real submanifold of $\mathbb{C}^2$ and each continuous function on such manifold is CR–function. But the inverse mapping f^{-1} is not CR because L is complex submanifold of $\mathbb{C}^2$ and CR–functions on L are holomorphic.

Therefore the natural additional assumption is that $\mathrm{CRdim}\, M_1 = \mathrm{CRdim}\, M_2$. In this case conjecture is obviously true if f is diffeomorphism. But in continuous situation we can't use the formulas for the derivatives of the inverse mapping f^{-1} and some new difficulties also arise. One of them was noticed by S. Bell [2].

Let $M = \{(z_1,z_2) \in \mathbb{C}^2 : y_2 = 0\}$ and $f : M \longrightarrow M$ is defined by $(z_1,z_2) \longrightarrow (z_1,z_2^3)$. Then f is CR–mapping and homeomorphism. The inverse mapping f^{-1} is also CR because a continuous function on Levi flat hypersurface is CR if and only if it is holomorphic along the leaves of Levi–foliation. The new phenomenon which we observe in this example is that the holomorphic extension of f to one–side neighborhood of M is not biholomorphic mapping onto one–side neighborhood of M.

In § 3 we prove the conjecture 1 for rather general class of real hypersurfaces in $\mathbb{C}^n$.

This class includes all Levi–flat hypersurfaces and real hypersurfaces which contain only isolated complex hypersurfaces. Hence the conjecture is true for all real analytic hypersurfaces and for hypersurfaces of finite type. We cannot prove the conjecture 1 only for those real hypersurfaces M which have strange pathological structure of complex hypersurface inside.

The author believes that the conjecture 1 is true for arbitrary real hypersurfaces of class C^2 in $\mathbb{C}^n$. It is a consequence of another natural conjecture about CR–functions which is formulated in § 3.

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§ 2 CR–straightening.

Theorem 2.1 a) Let CR–manifold $M \subset \mathbb{C}^n$ of class C^1 is locally k–CR–straightened near a point $p^0 \in M$. Then the induced distribution $\mathscr{L} : M \cap U \longrightarrow G(k,n)$ is CR–mapping.

b) Conversely, if M is CR–manifold of class $C^m (m \geq 1)$ in $\mathbb{C}^n$ and $\mathscr{L} \in C^m$ is involutive CR–distribution of complex k–planes $\pi_p \subset T_p^c(M)$, $p \in M$. Then M is locally k–CR–straightened near arbitrary point $p^0 \in M$. The straightening f is C^m–diffeomorphism and $\mathscr{L}$ is induced by f. Moreover if M and $\mathscr{L}$ are real analytic then f is holomorphic near p^0.

Proof of part a) Let $f : (\mathbb{C}^k \times M_0) \cap V \longrightarrow M \cap U$ is a local k–CR–straightening. We shall denote the points in $\mathbb{C}^k \times M_0$ by (z,ω), $z = (z_1,\dots,z_\kappa) \in \mathbb{C}^k$, $\omega \in M_0$ and the points in M by $w = (w', w'')$, $w' = (w_1,\dots,w_k)$, $w'' = (w_{k+1},\dots,w_n)$. Corresponding coordinates in $T_w\mathbb{C}^n$ will be $t = (t', t'')$.

Without loss of generality we may assume that $p^0 = 0$, $f(0) = 0$ and

$$\pi_0 = \{t \in \mathbb{C}^n : t'' = 0\} .$$

By Weierstrass theorem the partial derivatives $\frac{\partial f_i}{\partial z_j}(z,\omega)$ are continuous on $(\mathbb{C}^k \times M_0) \cap V$. Therefore π_w continuously depends on $w \in M$ and there exists a neighborhood $U \ni 0$ such that for $w \in M \cup U$

$$\pi_w = \{t = (t', t'') : t'' = A(w)t'\} ,$$

where $A(w) = (a_{ij}(w))$ $(i = k+1,\dots,n,\ j = 1,\dots,k)$ is $(n-k)\times k$ matrix. The elements $a_{ij}(w)$ may be considered as local coordinates of π_w in $G(k,n)$ and we have to show that $a_{ij}(w)$ are CR–functions on M. We have the explicit formula

$$A(w) = \frac{\partial f''}{\partial z}(z,\omega) \cdot \left[\frac{\partial f'}{\partial z}(z,\omega)\right]^{-1} , \tag{1}$$

where $f' = (f_1,\dots,f_k)$, $f'' = (f_{k+1},\dots,f_n)$ and $(z,\omega) = f^{-1}(w)$.

Due to Baouendi–Trèves theorem [1] a continuous function on CR–manifold is CR–function if and only if it can be locally uniformly approximated by holomorphic polynomials. Hence each component f_i of mapping f can be locally approximated by polynomials $\{P_i^\nu\}$, $\nu = 1,2,\dots$. By Weierstrass theorem the derivatives $\frac{\partial f_i}{\partial z_j}(z,\omega)$ are approximated by $\left\{\frac{\partial p_i^\nu}{\partial z_j}(z,\omega)\right\}$, $\nu = 1,2,\dots$ and therefore the elements of the matrixes

$$\frac{\partial f'}{\partial z}(z,\omega)\ ,\ \frac{\partial f''}{\partial z}(z,\omega)\ ,\ \frac{\partial f''}{\partial z}(z,\omega)\cdot\left[\frac{\partial f'}{\partial z}(z,\omega)\right]^{-1}$$

are CR–functions on $(\mathbb{C}^k \times M_0) \cap V$. By (1) $A(w)$ is the composition of $f^{-1}(w)$ and the last matrix. Its elements are CR–functions because they can be approximated by polynomials of the components of $f^{-1}(w)$. These prove the part a).

Proof of part b) Let $\mathcal{L} : M \longrightarrow G(k,n)$ is involutive and CR. We want to show that M is k–CR–straightened near arbitrary point $p^0 \in M$. As before we assume $p^0 = 0$ and $\pi_0 = \{t \in \mathbb{C}^n : t'' = 0\}$. But now it is more convenient to denote the coordinates of points on M by $(z,w) = (z_1,\dots,z_k, w_1,\dots,w_{n-k})$. Near the origin the distribution $\mathcal{L}$ is defined by

$$\pi_{z,w} = \{t = (t',t'') : t'' = A(z,w)t'\}\ ,$$

where the elements $a_{ij}(z,w)$ of $(n-k)\times k$ matrix $A(z,w)$ are CR–functions on M of class C^m.

The involutiveness of $\mathcal{L}$ implies that M is foliated by complex manifolds of dimension k. The leaves are the solutions of system

$$\begin{cases} \dfrac{\partial w}{\partial z} = A(z,w) \\ \dfrac{\partial w}{\partial \bar{z}} = 0 \end{cases} \tag{2}$$

We denote by M_0 the intersection $M \cap \{z = 0\}$ and we shall consider M_0 as a manifold in $\mathbb{C}_w^{n-k}$. Easy to see that near the origin $\mathrm{CRdim}\, M_0 = \mathrm{CRdim}\, M - k$.

The system (2) is overdetermined and we can't hope to solve it for arbitrary initial values $w(0) = \omega$. But we can do it if $\omega \in M_0$ because $\mathcal{L}$ is involutive on M. Such solution can be obtained by the following procedure.

Let us consider at first only those equations in (2) which contain the derivatives with respect to z_1 or $\bar{z}_1$ and assume $z_2 = z_3 = \dots = z_k = 0$. Integrating these equations with initial values $w(0) = \omega \in M_0$ we obtain the solution (more exactly the family of solutions) $w^1(z_1,\omega)$ $(z_1 \in U_1 \subset \mathbb{C})$ and CR–manifold

$$M_1 = \{(z_1,w) \in \mathbb{C}^{n-k+1} : w = w^1(z_1,\omega)\ ,\ \omega \in M_0\ ,\ z_1 \in U_1\}\ .$$

We can do it because of integrability of $\mathscr{L}$. In fact $M_1 = M \cap \{z_2 = z_3 = \dots = z_k = 0\}$ and the graphs of $w = w^1(z_1,\omega)$ for different $\omega \in M_0$ are the intersections of the leaves of foliation of M with the plane $\{z_2 = z_3 = \dots = z_k = 0\}$.

Now we can take only those equations in (2) which contain the derivatives with respect to $z_2, \bar{z}_2$ and assume $z_3 = z_4 = \dots = z_k = 0$. Integrating we obatin the solution

$$w = w^2(z_1,z_2,\omega) \quad (z_2 \in U_2 \subset \mathbb{C})$$

which satisfies the initial conditions

$$w^2(z_1,0,\omega) = w^1(z_1,\omega)$$

and CR–manifold

$$M_2 = \{z_1,z_2,w) \in \mathbb{C}^{n-k+2} : w = w^2(z_1,z_2,\omega),\ \omega \in M_0,\ z_1 \in U_1,\ z_2 \in U_2\}\ .$$

Repeating this procedure we obtain finally the solution $w = w^k(z,\omega)$ of the system (2) which is defined for $z \in U_1 \times\dots\times U_k = U$ and satisfies the initial condition $w^k(0,\omega) = \omega \in M_0$. To finish the proof of part b) we need to prove the following

<u>Proposition 2.1</u> The solution $w = w^k(z,\omega)$ is a CR–(vector)–function of class C^m on the manifold $U \times M_0$. The mapping $(z,\omega) \longrightarrow (z,w^k(z,\omega))$ is a CR–diffeomorphism from a neighborhood of the origin in $U \times M_0$ onto a neighborhood of the origin in M.

To prove this proposition we successively verify that

$$w^\nu(z_1,\dots,z_\nu,\omega)$$

are CR–functions of class C^m on $U_1 \times \dots \times U_\nu \times M_0$ for $\nu = 1,\dots,k$. The mappings

$$(z_1,\dots,z_\nu,\omega) \longrightarrow (z_1,\dots,z_\nu,w^\nu(z_1,\dots,z_\nu,\omega))$$

are local diffeomorphisms from $U_1 \times \dots \times U_\nu \times M_0$ to M_ν because of Frobenius theorem.

<u>Lemma 2.1</u> Let $A(z,\zeta,w)$ is a (vector)–function of class C^m in a neighborhood of the origin in the space $\mathbb{C}^{1+s+r}_{z,\zeta,w} = \mathbb{C}_z \times \mathbb{C}^s_\zeta \times \mathbb{C}^r_w$ with values in $\mathbb{C}^r$.

Let $\tilde{M}_0$ be a CR–manifold in a neighborhood of the origin in $\mathbb{C}^{s+r}$ and $\varphi(\zeta,\omega)$ is a CR–(vector)–function of class C^m on $\tilde{M}_0$ with values in $\mathbb{C}^r$. Assume that for arbitrary $(\zeta,\omega) \in \tilde{M}_0$ there exists the solution $w(z,\zeta,\omega)$ $(z \in U)$ of the system

$$\begin{cases} \dfrac{\partial w}{\partial z} = A(z,\zeta,w) \\ \dfrac{\partial w}{\partial \bar{z}} = 0 \end{cases} \tag{3}$$

which satisfies the initial condition

$$w(0,\zeta,\omega) = \varphi(\zeta,\omega) \ . \tag{4}$$

We also assume that

$$\tilde{M} = \{(z,\zeta,w) : w = w(z,\zeta,\omega),\ (\zeta,\omega) \in \tilde{M}_0,\ z \in U\}$$

is CR–manifold of class C^m and the restriction of $A(z,\zeta,w)$ to $\tilde{M}$ is a CR–function. Then $w(z,\zeta,\omega)$ is a CR–function on $U \times \tilde{M}_0$ of class C^m.

<u>Proof of lemma 2.1</u> We can extend the functions $A(z,\zeta,w)$, $\varphi(\zeta,\omega)$ to the neighborhood of $\tilde{M}$, $\tilde{M}_0$ respectively as functions of class C^m such that

$$\bar{\partial}A\,|_{\tilde{M}} = 0\ ,\ \bar{\partial}\varphi\,|_{\tilde{M}_0} = 0 \tag{5}$$

The solution of (3–4) satisfies the integral equation

$$w(z,\zeta,\omega) = \varphi(\zeta,\omega) + \int_0^z A(\tau,\zeta,w(\tau,\zeta,\omega))d\tau \tag{6}$$

$z \in U$, $(\zeta,\omega) \in \tilde{M}_0$. The integral doesn't depend on the path between 0 and z because $w(z,\zeta,\omega)$ is holomorphic in z. We shall consider (6) for (ζ,ω) from a neighborhood of $\tilde{M}_0$ in $\mathbb{C}^{s+r}$, where the integral is taken over the segment $[0,z]$. For $(\zeta,\omega) \notin \tilde{M}_0$ (6) is not equivalent to (3–4) but the solution of (6) gives us the extension of $w(z,\zeta,\omega)$ from $U \times \tilde{M}_0$ to the neighborhood of the origin in $\mathbb{C}^{1+s+r}_{z,\zeta,\omega}$. Differentiating $w(z,\zeta,\omega)$ with respect to $\overline{\zeta}_i(i = 1,...,s,)$, $\overline{\omega}_j(j = 1,...,r)$ and using (5), (6) we obtain for $(z,\zeta,\omega) \in U \times \tilde{M}_0$

$$\frac{\partial w}{\partial \overline{\zeta}_i}(z,\zeta,\omega) = \int_0^z \sum_{\mu=1}^r \frac{\partial A}{\partial w_\mu}(\tau,\zeta,w(\tau,\zeta,\omega)) \frac{\partial w}{\partial \overline{\zeta}_i}(\tau,\zeta,\omega)d\tau \,, \tag{7}$$

$$\frac{\partial w}{\partial \overline{\omega}_i}(z,\zeta,\omega) = \int_0^z \sum_{\mu=1}^r \frac{\partial A}{\partial w_\mu}(\tau,\zeta,w(\tau,\zeta,\omega)) \frac{\partial w}{\partial \overline{\omega}_i}(\tau,\zeta,\omega)d\tau \,, \tag{8}$$

$i = 1,...,s$, $j = 1,...,r$.

We should notice that (7), (8) are well–known vector Volterra equations with respect to $\frac{\partial w}{\partial \overline{\zeta}}$, $\frac{\partial w}{\partial \overline{\omega}}$. They have only zero solutions

$$\frac{\partial w}{\partial \overline{\zeta}_i}(z,\zeta,\omega) \equiv 0 \,, \; \frac{\partial w}{\partial \overline{\omega}_j}(z,\zeta,\omega) \equiv 0 \,,$$

under zero initial values

$$\frac{\partial w}{\partial \overline{\zeta}_i}(0,\zeta,\omega) = 0 \,, \; \frac{\partial w}{\partial \overline{\omega}_j}(0,\zeta,\omega) = 0$$

$(i = 1,...,s$, $j = 1,...,r)$. These completes the proofs of Lemma 2.1, Proposition 2.1 and Theorem 2.1.

<u>Example</u> Let us consider the "future tube"

$$\tau_+ = \{z = (z_0,z_1,...,z_n) \in \mathbb{C}^{n+1} : y_0^2 > y_1^2 + ... + y_n^2, \; y_0 > 0\}$$

where $z_\nu = x_\nu + iy_\nu$. The boundary of τ_+ consists of singular part $\mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$ and regular part $\partial\tau_+ \backslash \mathbb{R}^{n+1}$. In regular points Levi form of $\partial\tau_+$ is nonnegative and has one–dimensional null space

$$N_z = \{t = (t_0, t_1, \dots, t_n) \in \mathbb{C}^{n+1} : t_\nu = (y_\nu / y_0) t_0 \, , \ \nu = 1, \dots, n\}$$

The distribution $z \longrightarrow N_z$ on $\partial\tau_+$ is integrable because $\partial\tau_+$ is foliated by complex lines

$$\ell_{a,b} = \{z = a + b\zeta, \ \zeta \in \mathbb{C}, \ \mathrm{Im}\,\zeta > 0\} \ ,$$

where $a \in \mathbb{R}^{n+1}$, $b = (1, b_1, \dots, b_n) \in \mathbb{R}^{n+1}$, $b_1^2 + \dots + b_n^2 = 1$. So k–CR–straightening of $\partial\tau_+$ could be possible only for $k = 1$ and along N_z. But local coordiantes of N_z in $G(1,n+1)$ are y_ν / y_0, $\nu = 1, \dots, n$ and it is very easy to check that they are not CR–functions on $\partial\tau_+$. Therefore $\partial\tau_+$ is not CR–straightened. This strengthens A. Sergeev's result [11].

§ 3. CR–homeomorphisms.

In this section we discuss the conjecture 1 from the introduction for real hypersurfaces in $\mathbb{C}^n$. There are few partial cases for which the conjecture is known to be true. These cases are the following

1) M_1, M_2 are strictly pseudoconvex hypersurfaces (S. Pinchuk, S. Tsyganov [8]). Actually in [8] it was proved that if M_1, M_2 are strictly pseudoconvex hypersurfaces of class $C^m (m > 2)$ and $f : M_1 \longrightarrow M_2$ is nonconstant CR–mapping then f is local

CR–diffeomorphism of class C^{m-1-0} and even $C^{m-1/2-0}$ (see [6]).

2) M_1, M_2 are pseudoconvex hypersurfaces of class C^{∞} and of finite type. S. Bell [2] proved that these imply f to be C^{∞}–diffeomorphism.

3) M_1, M_2 are Levi flat.

In the last case the statement of conjecture is almost obvious. Actually M_1, M_2 are foliated by complex hypersurfaces (see [14]) and f maps biholomorphically the leaves in M_1 onto the leaves in M_2. Therefore f^{-1} is CR–mapping because it is holomorphic along the leaves of foliation of M_2.

The following facts will be useful in the study of CR–homeomorphisms of real hypersurfaces.

Proposition 3.1 Let M be a real hypersurface in $\mathbb{C}^n$ of class C^2 and $S \subset M$ is (n–1)–dimensional complex variety. Then S is complex manifold.

The proof of this proposition for real analytic hypersurfaces in $\mathbb{C}^2$ is contained in the proof of theorem 2 of paper by K. Diederich and J.E. Fornaess [4]. Actually their proof is valid for general situation. Nevertheless for the completeness of exposition we give the proof of proposition 3.1.

Take an arbitrary point $p \in S$ and choose the coordinates in $\mathbb{C}^n$ such that $p = 0$ and $T_0^c(M) = \{z_n = 0\}$. We have C^1–distribution

$$\pi : z \in M \longrightarrow \pi_z = T_z^c(M)$$

of complex hyperplanes on M and we can extend it as a C^1–distribution to a neighborhood of the origin in $\mathbb{C}^n$. In this neighborhood S is defined by some pseudopolynomial

$$z_n^k + a_1(z')z_n^{k-1} + \dots + a_k(z') = 0 \tag{9}$$

with holomorphic coefficients $a_j(z')$, where $z' = (z_1,\dots,z_{n-1})$, $k \geq 1$. Generally for each admissible z' there are k solutions of (9) and we need to show that $k = 1$. Let $\ell' \ni 0'$ is an arbitrary real line in the space $\mathbb{C}^{n-1}$ of variables $z' = (z_1,\dots,z_{n-1})$ and

$$\ell = \{z = (z',z_n) \in \mathbb{C}^n : z' \in \ell'\} \ .$$

In a neighborhood of the origin in ℓ we have C^1–distribution of real lines $\tilde{\pi} : z \longrightarrow \ell \cap \pi_z$ and each component of $S \cap \ell$ is obviously integral curve for $\tilde{\pi}$. The differentiability of $\tilde{\pi}$ implies that there is only one integral curve through the origin. This means that S is a graph of a singlevalued analytic function $z_n = h(z')$ near the origin and hence $p = 0$ is a regular point of S. ∎

<u>Proposition 3.2</u> Let M be a real hypersurface of class C^1 in $\mathbb{C}^n$, $S \subset M$ is a real submanifold of dimension $\leq 2n-2$ and $h \in C(M) \cap CR(M\setminus S)$. Then $h \in CR(M)$.

<u>Proof</u> Without loss of generality we shall assume $\dim_R S = 2n-2$. The statement is local and it is enough to show that h is CR–function in a neighborhood of arbitrary point $p \in S$. The surface S locally devides M into two parts M^+ and M^-. We have to show that

$$\int_M h\, \bar{\partial}\varphi = 0 \ .$$

for any smooth $(n,n-2)$ form φ with compact support. Let $q \in M\setminus S$ is an arbitrary point. Let us assume $q \in M^+$ and $U \subset M^+$ is a small neighborhood of q with piece–wise smooth boundary. Then we have

$$\int_U h\, \bar\partial\varphi = \int_{\partial U} h\varphi \ . \tag{10}$$

Indeed, h can be uniformly approximated on $\bar U$ by polynomials h_ν and using the Stokes formula we obtain

$$\int_U h\, \bar\partial\varphi = \lim \int_U h_\nu \bar\partial\varphi = \lim \int_U \bar\partial(h_\nu\varphi) =$$

$$= \lim \int_U d(h_\nu\varphi) = \lim \int_{\partial U} h_\nu\varphi = \int_{\partial U} h\varphi \ .$$

The equality (10) obviously extends for arbitrary open subsets $U \subset M^+$ with piece–wise smooth boundaries. Therefore taking suitable exaustion of M^+ by open sets U_μ we obtain

$$\int_{M^+} h\bar\partial\varphi = \int_S h\,\varphi$$

and analogously

$$\int_{M^-} h\bar\partial\varphi = \int_S h\varphi \ .$$

But in the last integral S has the opposite orientation. So we conclude

$$\int_M h\bar{\partial}\varphi = 0 \ . \qquad \blacksquare$$

<u>Proposition 3.3</u> Let M_1, M_2 are real hypersurfaces of class C^2 in $\mathbb{C}^n$ and $f : M_1 \longrightarrow M_2$ is homeomorphism and CR–mapping. Suppose that f holomorphically extends to some neighborhood U of a point $p \in M_1$. Then f^{-1} is CR–mapping near the point $f(p) \in M_2$.

<u>Proof</u> Let F denotes the holomorphic extension of f, $J_F(z) = \det\left[\frac{\partial F_i}{\partial z_j}(z)\right]$ is the complex Jacobian of F and

$$E = \{z \in U : J_F(z) = 0\} \ .$$

The inverse mapping f^{-1} is CR on $M_2 \backslash f(E)$ because it can be locally extended from $M_2 \backslash f(E)$ as a holomorphic mapping. We only must prove that f^{-1} is CR near $M_2 \cap f(E)$. Since E is analytic set in U the image $f(E)$ can be represented near the point $f(p)$ as a countable union of complex manifolds (see § 3.8 of [3]). Moreover taking U small enough and repeating the arguments of [3] we obtain $f(E)$ as a finite union of complex manifolds. Therefore $f(E)$ can be stratified near $f(p)$. Take a strata N of maximal dimension and a point $q \in N \cap M_2$. There exists a holomorphic function h near q such that $dh \neq 0$ and $N \subset \{h = 0\}$. The sets $\Gamma_1 = \{\mathrm{Re}\, h = 0\}$, $\Gamma_2 = \{\mathrm{Im}\, h = 0\}$ are real manifolds near q and at least one of them is transversal to M_2 at q. Let it be Γ_1 and let $S_1 = \Gamma_1 \cap M_2$. Then $N \cap M_2 \subset S_1$ and f^{-1} is CR on $M_2 \backslash S_1$ near q. By proposition 3.2 f^{-1} is CR on $N \cap M_2$. Repeating this procedure we conclude that f^{-1} is CR in a neighborhood of the point $f(p)$. ■

Let M be a real hypersurface of class C^2 in $\mathbb{C}^n$ $(n > 1)$. In the problem under consideration the complex hypersurfaces in M are of particular importance. Following A. Tumanov [16] we call the point $p \in M$ to be minimal if M doesn't contain germs of complex hypersurfaces through p.

The principle result of this section is the following

Theorem 3.1 Let M_1, M_2 are real hypersurfaces of class C^2 in $\mathbb{C}^n$, $f : M_1 \longrightarrow M_2$ is minimal point. Then f^{-1} is CR in a neighborhood of the point $q = f(p)$.

Proof We shall assume below that ρ_1, ρ_2 are defining functions of M_1, M_2 respectively, i.e. there exist two open sets Ω_1, Ω_2 in $\mathbb{C}^n$ and two real functions $\rho_1 \in C^2(\Omega_1)$, $\rho_2 \in C^2(\Omega_2)$ such that

$$M_j = \{z \in \Omega_j : \rho_j(z) = 0\}$$

and $d\rho_j \neq 0$ in Ω_j $(j = 1,2)$. Let

$$\Omega_j^{\pm} = \{z \in \Omega_j : \pm\, \rho_j(z) > 0\}\ ,\ \ j = 1,2\ .$$

By a result of Trepeau [15] f extends holomorphically to oneside neighborhood of p. The problem is local and we may assume that f extends to a mapping $F \in \mathcal{O}(\Omega_1^-) \cap C(\overline{\Omega_1^-})$.

Lemma 3.1 $F(\Omega_1^-) \not\subset M_2$.

Proof Take a sequence $z^\nu \longrightarrow p$, $z^\nu \in \Omega_1^-$ and consider the sets

$$E_\nu = \{z \in \Omega_1^- : F(z) = F(z^\nu)\} \ .$$

If $F(\Omega_1^-) \subset M_2$ then rank $F < n$ everywhere and E_ν are analytic sets in Ω_1^- of dimension ≥ 1. Each E_ν has not more than one limit point on M_1 because f is homeomorphic on M_1. By Shiffman's theorem [13] $\overline{E}_\nu$ are analytic sets in Ω_1. We have $d(p,E_\nu) \longrightarrow 0$ and since $f : M_1 \longrightarrow M_2$ is homeomorphism

$$\underline{\lim}_{\nu \longrightarrow \infty} d(\tilde{p},E_\nu) > 0$$

for any other point $\tilde{p} \neq p$ in M_1 (here we denote by $d(p,E_\nu)$ the distance between p and $\overline{E}_\nu$). By continuity principle [12] F holomorphically extends through point p. We obtain the contradiction because the restriction of F to M_1 can't be one–to–one. ■

Let $E = \{z \in \Omega_1^- : F(z) = q\}$.

Lemma 3.2 If there exists irreducible component E' of E of dimension ≥ 1 such that $p \in \overline{E'}$ then f^{-1} is CR near the point $q = f(p)$.

Proof As in the proof of the previous lemma we easily conclude that F holomoprhically extends to a neighborhood of p. Now by proposition 3.3 f^{-1} is CR near q. ■

So we may assume further that E is discrete and the distance

$$d(q,F(\partial\Omega_1^- \setminus M_1)) > 0 \ .$$

We may choose such neighborhood V of the point q in $\mathbb{C}^n$ that

$$d(V,F(\partial\Omega_1^-\setminus M_1)) > 0 \ . \tag{11}$$

Hypersurface M_2 divides V into two parts V^+, V^-. Due to lemma 3.1 we may assume $F(\Omega_1^-) \cap V^- \neq \phi$.

<u>Lemma 3.3</u> F properly maps $F^{-1}(V^-)$ onto V^-.

<u>Proof</u> Let $K \subset V^-$ be a compact. If $F^{-1}(K)$ is not compact in $F^{-1}(V^-)$ then there exists a point z^0 on the boundary of $F^{-1}(V^-)$ such that $F(z^0) \in K$. We obviously have $z^0 \notin M_1$. Therefore there exists a neighborhood $U \ni z^0$ such that $F(U) \subset V^-$ and hence z^0 is not a boundary point for $F^{-1}(V^-)$. ∎

Let m be the mulitplicity of the restriction of F to $F^{-1}(V^-)$. For any $w \in V^-$ the set $F^{-1}(w)$ consists of not more than m elements. We want to show that $E = F^{-1}(q)$ is finite. If it is not so, then there exists a sequence $z^\nu \longrightarrow p$, $z^\nu \in \Omega_1^-$ such that $F(z^\nu) = q$ for each $\nu = 1,2,\ldots$. Since E is discrete then F is open near each point z^ν and there exist such mutually disjoint neighborhoods U_ν of z^ν that all $F(U_\nu)$ are neighborhoods of point q. The set

$$\tilde{V} = \bigcap_{\nu=1}^{m+1} F(U_\nu)$$

is also the neighborhood of q and any point $w \in \tilde{V}^-$ has at least $m+1$ preimages. These prove that E is finite. Taking Ω_1 small enough we may assume that $E = \phi$ and (11) preserves. These imply that for any sequence $w^\nu \longrightarrow q$, $w^\nu \in V^-$ all preimages $F^{-1}(w^\nu)$ tend to p as $\nu \longrightarrow \infty$.

Lemma 3.4 The point $q = f(p)$ is minimal for M_2.

Proof If q is not minimal then there exist a germ $S \subset M_2$ of complex hypersurface through point q. Shrinking $V \ni q$ we may assume that S is closed in V and can be uniformly approximated by complex hypersurfaces $S_\nu \subset V^-$, which are also closed in V^-. Since $F : F^{-1}(V^-) \longrightarrow V^-$ is proper and holomorphic, we may consider $G = F^{-1}$ as algebroid mapping, i.e. the components g_κ of (multivalued mapping) G satisfy the equations

$$g_k^m(w) + a_{1,k}(w)g_k^{m-1}(w) + \dots + a_{m,k}(w) \equiv 0 \tag{12}$$

for $k = 1,\dots,n$, $w \in V^-$. Therefore $F^{-1}(S_\nu)$ are analytic sets in $F^{-1}(V^-)$ (and closed in $F^{-1}(V^-)$) for all ν. Moreover there exists a small ball $U \ni p$ such that all sets $T_\nu = F^{-1}(S_\nu) \cap U$ are closed in U. We also have $d(p,T_\nu) \longrightarrow 0$ as $\nu \longrightarrow \infty$. There exist such holomorphic functions $h_\nu \in \mathcal{O}(U)$ that

$$T_\nu = \{z \in U : h_\nu(z) = 0\}$$

for all ν. The functions $1/h_\nu$ are holomorphic on $M_1 \cap U$, but one can't find such neighborhood $U_1 \ni p$, that all $1/h_\nu$ are holomorphic in U_1^-. This contradicts to Trepeau theorem and hence $q \in M_2$ is minimal point. ∎

There are two possibilities.

1) $F(U^-) \cap \Omega_2^- \neq \emptyset$ and $F(U^-) \cap \Omega_2^+ \neq \emptyset$ for any sufficiently small neighborhood $U \ni p$.

2) $F(U^-) \subset \Omega_2^-$ for some neighborhood $U \ni p$.

To finish the proof of the theorem 3.1 we must study both of these possibilities. First consider the case 1). Since q is minimal point of M_2 , one of the sets V^+, V^- (for example, V^+) has the property that all holomorphic functions in V^+ extend holomorphically to a neighborhood pf p . According to lemma 3.3 $F(U^-) \cap \Omega_2^+ \neq \phi$ implies that F properly maps $F^{-1}(V^+)$ onto V^+ . The inverse mapping $G = F^{-1}$ is algebroid in V^+ and its components admit in V^+ representations analogous to (12). The coefficients of these representations extend holomorphically through q and therefore $G = F^{-1}$ is algebroid in a neighborhood of q . The mapping f^{-1} is obviously CR near those points $w \in M_2$, which are regular for F^{-1} . Using the same arguments as in the proof of proposition 3.3, we conclude that f^{-1} is CR near q .

Case 2) We can choose arbitrarily small neighborhoods $U \ni p$, $V \ni q$ such that F properly maps U^- onto V^- . Since we assume $\{z \in U^- : F(z) = q\} = \phi$, for any $w^0 \in M_2 \cap V$ and any sequence $w^\nu \longrightarrow w^0$, $w^\nu \in V^-$ all preimages $F^{-1}(w^\nu)$ tend to $f^{-1}(w^0)$ as $\nu \longrightarrow \infty$. Thus the discriminants of pseudopolynomials (12) have zero boundary values on M_2 . Therefore they are identically zero in V^- and $F^{-1} : V^- \longrightarrow U^-$ is singlevalued holomorphic mapping. The mapping f^{-1} is CR on $M_2 \cap V$ because it extends holomorphically to V^- . This completes the proof of the theorem 3.1.

Theorem 3.2 Let M_1, M_2 are real hypersurfaces of class C^2 in $\mathbb{C}^n$ and $f : M_1 \longrightarrow M_2$ is a homeomorphic CR–mapping. Then for arbitrary point $p_1 \in M_1$ the inverse mapping is CR near $p_2 = f(p_1)$ in each of the following cases

a) M_1 is Levi flat near p_1 or M_2 is Levi flat near p_2 ;

b) at least for one $j = 1,2$ hypersurface M_j doesn't contain germs of complex hypersurfaces through p_j ;

c) at least for one $j = 1,2$ hypersurface M_j contains only finite number of different complex hypersurfaces.

<u>Proof</u> a) If M_1 is Levi flat near p_1 , then f is holomorphic along each leaf of foliation and M_2 is Levi flat near p_2 due to proposition 3.1. Hence f^{-1} is CR near p_2 .

If M_2 is Levi flat, then it immediately follows from lemma 3.4 and proposition 3.1 that M_1 is also Levi flat and f^{-1} is CR.

b) The case p_1 is minimal for M_1 is covered by theorem 3.1. The property that $p_2 \in M_2$ is minimal immediately implies $p_1 \in M_1$ is minimal too.

The statement c) easily follows from the theorem 3.1, lemma 3.4 and proposition 3.2. ∎

Summarizing we can conclude that f^{-1} is CR near any minimal point $w \in M_2$. If $w \in M_2$ is not minimal then there exist a unique complex hypersurface $S \subset M_2$ through w and f^{-1} is holomorphic along S . Therefore the conjecture 1 for real hypersurfaces of class C^2 would be proved if the following conjecture is true.

<u>Conjecture 2</u> Let M be a real hypersurface of class C^2 in $\mathbb{C}^n$, $\{S_\alpha\}$, $\alpha \in A$, is a family of complex hypersurfaces in M and $S = \bigcup_\alpha S_\alpha$. Suppose that a function $f \in C(M)$ is CR near each point $p \in M \setminus S$ and that the restriction of f to any S_α is holomorphic. Then $f \in CR(M)$.

The conjecture 2 can be easily proved if the structure of hypersurfaces $S_\alpha \subset M$ is not very complicated, for example if A is finite or M is real analytic, etc. It can also be proved if $\{S_\alpha\}$ is a convergent sequence of complex hypersurfaces or if all S_α are closed in M . But in general case the structure of $\{S_\alpha\}$ can be very complicated and the problem requires more delicate consideration.

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