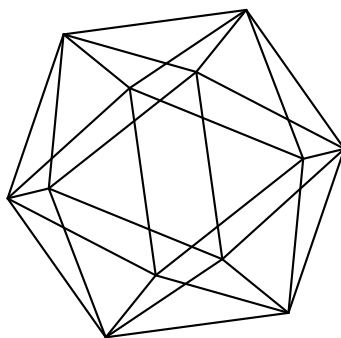


Max-Planck-Institut für Mathematik Bonn

Rational version of Archimedes symplectomorphism
and birational Darboux coordinates on coadjoint orbit of
 $GL(N, \mathbb{C})$

by

Mikhail V. Babich



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Mikhail V. Babich

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

PDMI
Fontanka 27
191023 St. Petersburg
Russia

**RATIONAL VERSION OF ARCHIMEDES
SYMPLECTOMORPHISM AND
BIRATIONAL DARBOUX COORDINATES ON
COADJOINT ORBIT OF $GL(N, \mathbb{C})$**

MIKHAIL V. BABICH

ABSTRACT. A set of all linear transformations with a fixed Jordan structure J is a symplectic manifold isomorphic to the coadjoint orbit $\mathcal{O}(J)$ of $GL(N, \mathbb{C})$.

Any linear transformation may be projected along its eigenspace to (at least one) coordinate subspace of the complement dimension. The Jordan structure \tilde{J} of the image is defined by the Jordan structure J of the pre-image, consequently the projection $\mathcal{O}(J) \rightarrow \mathcal{O}(\tilde{J})$ is the mapping of the symplectic manifolds.

It is proved that the fiber \mathcal{E} of the projection is a linear symplectic space and the map $\mathcal{O}(J) \xrightarrow{\sim} \mathcal{E} \times \mathcal{O}(\tilde{J})$ is a birational symplectomorphism.

The iteration of the procedure gives the isomorphism between $\mathcal{O}(J)$ and the linear symplectic space, which is the direct product of all the fibers of the projections. The Darboux coordinates on $\mathcal{O}(J)$ are pull-backs of the canonical coordinates on the linear spaces in question.

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Key words and phrases. Standard Jordan form, symplectic fibration, Lie-Poisson-Kirillov-Kostant form, rational symplectic coordinates.

0. INTRODUCTION

It was Archimedes who found that the “proper” coordinates for the element of the area of the sphere $\mathbf{O}(R)$ have a specific dual nature.

One coordinate is *the length*. It gives the position of the orthogonal projection of the parameterizing point on the diameter \mathcal{P} connecting poles of the sphere.

The conjugated coordinate is *the angle*. This angle parametrizes the elements of 1-parametric subgroup $\mathcal{Q} \subset \mathbf{O}(3)$, preserving the fibration $\mathbf{O}(R) \rightarrow \mathcal{P}$ of the sphere on the circles \mathcal{C}_p by the planes orthogonal to the diameter $\mathcal{P} \ni p$.

It is the cylindrical coordinates and the famous Archimedes area-preserving correspondence between the sphere $\mathbf{O}(R)$ and its circumscribing cylinder $\mathcal{P} \times \mathcal{C}_R$. These sphere and cylinder were placed on the tomb of Archimedes at his request [1].

Let us demonstrate how the Archimedes method introduces (the standard) birational Darboux coordinates on the coadjoint orbit \mathcal{O} of $\mathrm{GL}(2, \mathbb{C})$.

We identify Lie algebra with its dual using a form $\langle A, B \rangle = \mathrm{tr} AB$, and treat elements $A \in \mathcal{O}$ as matrices with the given Jordan structure. Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be eigenvalues of $A \in \mathcal{O}$. We put $\lambda_1 = 0$ and consider matrices with one zero eigenvalue, it is always possible to add a matrix proportional to the unit matrix to the answer.

Let us consider any orbit $\mathcal{O}(J_R) \subset \mathfrak{gl}(2, \mathbb{C})$, fixed by non-zero J_R with the eigenvalues 0 and R :

$$J_R = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}, \text{ if } R \neq 0, \quad J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The orbit $\mathcal{O}(J_R)$ coincides with the (non-singular part of the) affine quadric

$$A \in \mathcal{O}(J_R) \Leftrightarrow \det A = 0, \operatorname{tr} A = R, A \neq 0.$$

It is visual to consider the one-sheet hyperboloid $pY = X(R - X)$ in the 3-space of X, Y, p . The hyperboloid is fibrated on the parabolas \mathcal{C}_p :

$$\mathcal{C}_p : pY = X(R - X), \quad p = \mathbf{const}, \quad A = \begin{pmatrix} X & p \\ Y & R - X \end{pmatrix} \in \mathcal{O}(J_R)$$

Consider the one-parametric subgroup $\mathcal{Q} \in \operatorname{GL}(N, \mathbb{C})$ preserving sections $p = \mathbf{const}$. It is a subgroup of untriangular matrices $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \in \mathcal{Q}$. Its elements shift the natural parameter on each parabola like the rotations of the Archimedes sphere shift the natural parameter (angle) of its sections (circles).

Variables p, q parameterize a Zariski-open subset of $\mathcal{O}(J_R)$:

$$(1) \quad A(p, q) = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} 0 & p \\ 0 & R \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = \begin{pmatrix} -pq & p \\ -q(pq + R) & pq + R \end{pmatrix}$$

The calculation of the standard symplectic Lie-Poisson-Kirillov-Kostant form shows that p, q are Darboux coordinates.

In the present article we prove that the same trick in the combination with the simple iteration procedure gives the birational Darboux coordinates on the (co)adjoined orbit $\mathcal{O}(J)$ corresponding to the matrix J of any size and any Jordan structure.

Parameterizations of coadjoint orbits have attracted attention of many authors. I want to mention the papers [2, 3, 4, 5, 6, 7] that brought primary influence to bear on the author. All these works were initiated by needs

of the theory of integrable systems, where coadjoint orbits are used for the construction of the phase spaces within the framework of the Hamiltonian formalism.

The papers [4, 7] use the Gelfand-Zeitlin method. The explicit formulae are presented in [4], where the authors introduce the nice parameterization of the generic orbit, where one family of the coordinates is formed by the eigenvalues of the diagonal blocks of the matrix which is parameterized. The canonically conjugated coordinates can be easily calculated using vector-columns of the matrix and the corresponding eigenvectors of the blocks. These coordinates are not rational but algebraic because it is necessary to find the eigenvalues of the matrices of all sizes smaller than the initial one.

Another method was introduced in [2]. Really it is not method of the parameterization of $GL(N, \mathbb{C})$ orbit, but some general scheme that can be applied to different problems. The corresponding coordinates are called *the Spectral Darboux Coordinates*, see [5] where they are considered in detail. The method was applied to the parameterization of the $SL(N, \mathbb{C})$ -orbit in [3]. It is not rational but algebraic too, it is necessary to solve algebraic equations of high order.

Consider the so called *Isomonodromic Coordinates* introduced in [5]. They are parameterize not an orbit but some manifold related to orbits again. These coordinates are constructed for the orbits of the generic type and they are not rational but algebraic again. Nevertheless the method of [5] uses the cyclic process of the reduction of the matrix equation of the first order to the scalar equation of the high order. This process is somewhat similar to the method developed in the present paper, but the connection between these two parameterizations is not understood yet, it should be a subject for a future investigation.

It is the rationality that is the fundamental property of the presented coordinates which is very important, at least for the investigation of isomonodromic deformations of systems of linear differential equations.

In this context the orbit is a Zariski-open domain of the phase-space of the corresponding algebraic Hamiltonian system. The birationality of the transition functions is deeply connected with the famous Painlevé-property of the isomonodromic deformations.

We consider not a generic but *the general* case of the structure of the orbit here. There are the orbits swept by degenerated matrices that are the cases of high importance because of the small dimensions of the corresponding orbits. Such orbits can be treated as the phase spaces of “more classical”, low-dimensional systems immersed into “more roomy” high-dimensional spaces of matrices of higher sizes, that makes possible to find new approaches to old problems, see [8].

Let us turn to the subject of the paper. The crucial idea is the factorization of the matrix from the orbit on *the proper* triangular factors of a different nature. The idea belongs to S. E. Derkachov and A. N. Manashov, they use it for the needs of quantum field theory [9]. Recently the method was applied for the parameterization of the orbits swept by the diagonalizable matrices [10]. In the present article we are giving the evaluation of the method of [9, 10] to the general Jordan case.

The main idea of the method can be demonstrated on the toy-example of 2×2 case. Formula (2) shows that, if we transform the first vector of the basis of the space to the eigenvector of the matrix A by the transformation

$\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$, the Jordan form of the resulting matrix $\begin{pmatrix} 0 & p \\ 0 & R \end{pmatrix}$ almost does not depend on p .

Here “almost does not depend” means just “does not depend” if $R \neq 0$ and “does not depend for $p \neq 0$ ” if $R = 0$. In the general case there is a similar non-degenerate condition.

If we consider q, p, R as a blocks of proper dimensions, we arrive to the general case. The parameterization of A is reduced to the parameterization of R with given Jordan structure. Symbolically the final formula can be written as

$$\omega_A = \text{tr } dp \wedge dq + \omega_R,$$

where ω_A is a symplectic form on the “given” orbit that contains A , ω_R is a symplectic form on the orbit that contains R , it has the strictly smaller dimension.

In the case 2×2 the step “parameterization of R ” is not visible, we should “parameterize” the 0-dimensional orbit of the 1×1 matrix R .

All the procedure uses the operation of solving linear equations systems only, consequently it is rational.

The whole atlas for the orbit consists of the maps in question numerated by the permutations of the basic vectors. The transition functions are rational, they are given by the formulae for the parameterization of the already parameterized matrix, but conjugated by the matrices of the permutations of N basic vectors.

After all we make a good point. As was noted at the very beginning the canonically conjugated families the constructed Archimedes-type coordinates have special dual structure. The family that is an analog of the angle is generated by the projections to the Grassmanians. It has a natural

global structure as a set of affine coordinates on the projective manifold.

The analog of the projection on the diameter of the sphere in the original Archimedes scheme form the second family of the constructed coordinates. It has no evident global structure. At the same time it has a remarkable *local* structure. The elements of the matrix on the orbit depend on these coordinates *linearly*.

1. DEFINITIONS AND NOTATIONS

It is well known that any coadjoint orbit of a semisimple Lie group is equipped with a standard symplectic (Lie-Poisson-Kirillov-Kostant) form.

In the partial case of $GL(N, \mathbb{C})$ there are simplifications, at least in a terminology. We can use the widely known language of elementary linear algebra in spite of the much less known language of Hamiltonian systems on Lie-algebras [13, 14].

We use the Zariski topology, *open* set means *Zariski-open* set, *closed* set means *algebraically closed* set.

Let us treat elements $A \in gl(N, \mathbb{C})$ as linear transformations $\mathcal{A} \in \text{End } V$ of some complex linear N -dimensional space $V \simeq \mathbb{C}^N$, equipped with a basis $\mathbf{e}_v = e^1, \dots, e^N$:

$$A \in gl(N, \mathbb{C}) \leftrightarrow \mathcal{A} : \mathbf{e}_v x \rightarrow \mathbf{e}_v Ax, \quad \mathbf{e}_v x, \mathbf{e}_v Ax \in V, \quad x, Ax \in \mathbb{C}^N.$$

We identify the Lie algebra $gl(N)$ and its dual $gl^*(N)$ using the non-degenerate pairing (scalar product) $gl(N, \mathbb{C}) \times gl(N, \mathbb{C}) \rightarrow \mathbb{C} : \langle A, B \rangle \rightarrow \text{tr } AB$. The Lie group $GL(N, \mathbb{C}) \ni g$ acts on $A \in gl(N, \mathbb{C})$ by usual similarity transformations $A \rightarrow g^{-1}Ag$ induced by changes of basis $\mathbf{e}_v \rightarrow \mathbf{e}_v g$, consequently an orbit of the coadjoint action can be identified with a manifold of all matrices similar to each other in this case. Let us choose one element of the orbit, say J that is the Jordan normal form of the matrices from the orbit,

and denote

$$\mathcal{O}(J) = \bigcup_{g \in \text{GL}(N, \mathbb{C})} gJg^{-1}.$$

It is the subject of our investigations.

The canonical symplectic Lie-Poisson-Kirillov-Kostant form $\omega_{\mathcal{O}(J)}$ on the orbit can be introduced by the equality

$$(2) \quad \omega_{\mathcal{O}(J)}(\xi_1, \xi_2) = \text{tr} J[g^{-1}\dot{g}_1, g^{-1}\dot{g}_2],$$

where the vectors ξ_1, ξ_2 are tangent to the trajectories $A_i(t) = g_i(t)Jg_i^{-1}(t)$, $i = 1, 2$ that intersect each other at $t = 0$:

$$g_1(0) = g_2(0) = g, \dot{g}_i = \left. \frac{d}{dt} \right|_{t=0} g_i(t), \dot{A}_i = \left. \frac{d}{dt} \right|_{t=0} A_i(t).$$

We will use the following version (see [12]) of the previous formula

$$(3) \quad \omega_{\mathcal{O}(J)}(\xi_1, \xi_2) = \text{tr}(\dot{g}_1 g^{-1}) \dot{A}_2$$

The following observation (see [9, 10]) forms a basement of the construction: the canonical symplectic structure on an orbit and the hierarchic structure (12) which I present below are coordinated.

Let $K \subset V$ be a subspace. Denote by V/K the factor-space. It is a linear space of the dimension $\dim V - \dim K = \dim V/K$. The linear structure is inherited from any $\dim V/K$ -dimensional subspace of V which is transverse to K . We will denote by $\text{Pr}^{\parallel K}$ the projection

$$(4) \quad V \xrightarrow{\text{Pr}^{\parallel K}} V/K.$$

Space V has a structure of a trivial fiber bundle. Its fibers are subspaces parallel to K .

Let \mathcal{A} be a linear transformation of V and let its eigenspace corresponding to the eigenvalue λ_0 be $K := \ker(\mathcal{A} - \lambda_0 id)$.

Let

$$0 < \dim K < \dim V.$$

The submanifold of all such $\mathcal{A} \in \text{End } V$ will be denoted by $\text{End } V|_{\lambda_0, K}$:

$$\mathcal{A} \in \text{End } V|_{\lambda_0, K} \iff \ker(\mathcal{A} - \lambda_0 id) = K.$$

Let $\lambda_0 = 0$, $K = \ker \mathcal{A}$. The transformation \mathcal{A} has the same value on all $X \in V$ from one equivalence class V/K that means that there is a linear transformation $\tilde{\mathcal{A}} \in \text{Hom}(V/K, V)$ such that

$$\mathcal{A} = (\text{Pr}^{\parallel K})^* \tilde{\mathcal{A}}.$$

Let us denote by $((\text{Pr}^{\parallel K})^*)^{-1}$ the corresponding map $\text{End } V|_{0, K} \rightarrow \text{Hom}(V/K, V)$:

$$(5) \quad \tilde{\mathcal{A}} = ((\text{Pr}^{\parallel K})^*)^{-1} \mathcal{A} \iff \mathcal{A} = (\text{Pr}^{\parallel K})^* \tilde{\mathcal{A}}$$

The space V has the structure of the fiber bundle $V \xrightarrow{\text{Pr}^{\parallel K}} V/K$ consequently $((\text{Pr}^{\parallel K})^*)^{-1}$ can be projected back to V/K by $\text{Pr}^{\parallel K}$ that gives some $\tilde{\mathcal{A}} \in \text{End } V/K$:

$$\text{Pr}^{\parallel K} \circ ((\text{Pr}^{\parallel K})^*)^{-1} : \text{End } V|_{0, K} \rightarrow \text{End } V/K.$$

Notation 1. Let π denote $\text{Pr}^{\parallel K} \circ ((\text{Pr}^{\parallel K})^*)^{-1}$.

To reconstruct the initial \mathcal{A} from $\tilde{\mathcal{A}} = \pi \mathcal{A}$ we need to know the position of the \mathcal{A} -image on the assigned fiber of $V \xrightarrow{\text{Pr}^{\parallel K}} V/K$.

Any subspace $M : M \oplus K = V$ sets the isomorphism $V/K \simeq M$ and defines the structure of the direct product on $\text{End } V|_{0, K}$:

$$(6) \quad \text{End } V|_{0, K} \xrightarrow{\sim} \pi(\text{End } V|_{0, K}) \times \text{Hom}(V/K, K).$$

2. FILTRATION OF ORBIT

The area of our exploration will be a modification of the Jordan structure by the action of the projection π .

By the Jordan structure \mathcal{J} of a transformation \mathcal{A} we mean the set of the eigenvalues of \mathcal{A} and the information

about the Jordan chains corresponding to each eigenvalue, namely the number of the chains and their lengths. By J we denote a matrix (the normal Jordan form of \mathcal{A}) of the transformation \mathcal{A} in some basis collecting from the vectors of the Jordan chains with the structure \mathcal{J} . We will specify the order of the vectors later.

An important property of the projection $\pi = \text{Pr}^{\parallel K} \circ ((\text{Pr}^{\parallel K})^*)^{-1}$ on the first Cartesian factor of the target of (6) we serve as the theorem.

Theorem 1. *Let \mathcal{A} be a linear transformation with non-trivial kernel $K : 0 < \dim K < \dim V$.*

The Jordan structure $\tilde{\mathcal{J}}$ of $\tilde{\mathcal{A}} := \pi\mathcal{A}$ is defined by the Jordan structure \mathcal{J} of \mathcal{A} , namely

- *the Jordan chains corresponding to the non-zero eigenvalues for \mathcal{J} and for $\tilde{\mathcal{J}}$ coincide.*
- *the Jordan chains corresponding to the zero eigenvalue of $\tilde{\mathcal{J}}$ are in one-to-one correspondence with those chains of \mathcal{J} that have non-unit length. The chains of $\tilde{\mathcal{J}}$ are one unit shorter than corresponding chains of \mathcal{J} .*
- *the chains of the unit length (without generalized eigenvectors) form the kernel of the map (projection) of the set of Jordan chains of \mathcal{J} to the set of Jordan chains of $\tilde{\mathcal{J}}$.*

Proof

First of all let us note that by the definition of $((\text{Pr}^{\parallel K})^*)^{-1}$

$$b = \mathcal{A}a \Rightarrow b = (((\text{Pr}^{\parallel K})^*)^{-1}\mathcal{A})(\text{Pr}^{\parallel K} a)$$

consequently

$$(7) \quad \tilde{b} := \text{Pr}^{\parallel K} b = (\pi\mathcal{A})(\text{Pr}^{\parallel K} a) = \tilde{\mathcal{A}}\tilde{a}$$

It implies that the cyclic law of the construction of Jordan chains takes place.

Consider any Jordan basis of V for \mathcal{A} , where the first $\dim K$ vectors $e^1, \dots, e^{\dim K}$ form a basis of $K = \ker A$.

Consider the projection of the remaining subset of the basic vectors $e^{\dim K+1}, \dots, e^{\dim V}$. It is a linear-independent set of the vectors, otherwise some linear combination $\sum_{k>\dim K} \alpha_k e^k$ would be a vector from the kernel $K = \cup_{\alpha_k} \sum_{k \leq \dim K} \alpha_k e^k$. It contradicts with the linear independence of e^k .

The number of vectors in the set $e^{\dim K+1}, \dots, e^{\dim V}$ is equal to $\dim V - \dim K$ that is the dimension of V/K , consequently the projection of the set

$$e^{\dim K+1}, \dots, e^{\dim V}$$

forms a basis of V/K . □

Note 1. *The corresponding transformation of the Jordan structures may be thought as the projection of the Jordan structures induced by the projection of a linear transformation along its kernel, see the definition on the page 14.*

Corollary 1. *All non-zero projections by $\text{Pr}^{\parallel K}$ of vectors forming any Jordan basis of V for \mathcal{A} form a Jordan basis of V/K for $\pi\mathcal{A}$.*

Corollary 2. *The projection of the set of the Jordan bases for \mathcal{A} to the set of the Jordan bases for $\pi\mathcal{A}$ is surjective, namely for any Jordan basis $\tilde{\mathbf{e}}_J$ of V/K for $\pi\mathcal{A}$ there exists such a Jordan basis \mathbf{e}_J of V for \mathcal{A} that the non-zero projections of its vectors form $\tilde{\mathbf{e}}_J$.*

Proof

By the definition of $\pi\mathcal{A}$ the statement that $\tilde{\mathbf{e}}_J$ form a Jordan basis of V/K for $\pi\mathcal{A}$ is equivalent to the existence of the pre-images, i.e. it is equivalent to the existence of the set of vectors of V connected by the Jordan cyclic law the projections of which are vectors of the set $\tilde{\mathbf{e}}_J$.

It follows from the formula (7) that the cyclic law takes place. We can start from the pre-images of starting vectors

of the Jordan chains of $\tilde{\mathbf{e}}_J$ and iterate the transformation \mathcal{A} in V . The projection gives the iterations of $\pi\mathcal{A}$. We get the set of vectors in V that can be complemented to the basis by the eigenvectors of \mathcal{A} without pre-images (i.e. without the generalized eigenvectors), in other words the set can be complemented by the vectors of Jordan chains of the unit lengths.

Only one thing has to be proved. It is the linear independence of the last non-zero iterations of \mathcal{A} . These iterations are already trivial for $\pi\mathcal{A}$, because their inverse images belong to the $\ker \mathcal{A}$.

The statement follows from the uniqueness of the Jordan form. The desired dimension of the envelope of the last iterations is an invariant-defined value, it is the dimension of the $\text{im } \mathcal{A} \cap \ker \mathcal{A}$. In other words it is the difference between the dimension of $\ker \mathcal{A}$ and the number of Jordan chains of the unit lengths.

This number does not depend on the bases which we use for calculation, consequently it coincides with the number which we have for the basis constructed as the projection of any Jordan basis for \mathcal{A} using the previous corollary. \square

Denote by $\text{End}_{\mathcal{J}} V$ a submanifold of the transformations with a fixed Jordan structure \mathcal{J} . It has a structure of the fiber-bundle

$$(8) \quad \text{End}_{\mathcal{J}} V \xrightarrow{\gamma} G(n, V), \quad n := \dim \ker J$$

over the Grassmanian.

The fiber $\gamma^{-1}(K)$ over any $K \in G(n, V)$ is formed by all \mathcal{A} from $\text{End}_{\mathcal{J}} V \cap \text{End } V|_{0,K}$.

It follows from (6) that a fiber is a subset of $\text{Hom}(V/K, K)$. The Jordan structure of \mathcal{A} obviously will not be changed if we add *any* vectors from K to the images of all vectors of a Jordan basis of V for \mathcal{A} if two restrictions are satisfied:

- the images of the vectors from K keep zero values,

- the last vectors of chains form a basis of K .

For the chains corresponding to the non-zero eigenvalues and for the chains corresponding to the zero eigenvalue but with the unit lengths these restrictions are trivial.

For the chains of the lengths longer than one we have just one non-degeneracy restriction, namely *the images of the generalized eigenvectors of the first order (next to the last vectors of the chains corresponding to the zero eigenvalue) must complete the set of vectors from the chains of the unit length to the basis of K* . The following theorem has been proved:

Theorem 2. *A fiber $(\pi)^{-1}\tilde{\mathcal{A}}$ of the bundle*

$$(9) \quad \text{End}_{\mathcal{J}} V|_{0,K} \xrightarrow{\pi} \text{End}_{\tilde{\mathcal{J}}} V/K$$

is isomorphic to the open subset of $\text{Hom}(V/K, K)$:

$$(10) \quad \mathcal{A} \in (\pi)^{-1}\tilde{\mathcal{A}} \Leftrightarrow \text{rank } \mathcal{A}|_{\tilde{K}} = \dim \tilde{K},$$

where $\tilde{K} := (\text{Pr}^{\parallel K})^{-1} \ker \tilde{\mathcal{A}}$ is an inverse image of the kernel of $\tilde{\mathcal{A}}$ under the projection $\text{Pr}^{\parallel K} : V \rightarrow V/K$:

$$x \in \tilde{K} \Leftrightarrow \text{Pr}^{\parallel K} x \in \ker \tilde{\mathcal{A}}$$

□

Note 2. *In the case $\ker \mathcal{A} \cap \text{im } \mathcal{A} = 0$, i.e. if \mathcal{A} has no generalized eigenvectors for the zero eigenvalue*

$$(11) \quad \text{End}_{\mathcal{J}} V|_{0,K} \simeq \text{End}_{\tilde{\mathcal{J}}} V/K \times \text{Hom}(V/K, K).$$

□

We see that the manifold $\text{End}|_{\mathcal{J}}$ has a structure of a fiber-bundle over the Grassmanian $G(n, V)$, where the fiber is in its turn the fibration described by the previous theorem i.e. by the equalities (9), (10). In the simplest case of absence of generalized eigenvectors it is given by the equality (11).

It is evident that for any eigenvalue λ_1 we can make the same construction with $\mathcal{A} - \lambda_1 id_v$, where by id_v we denoted the identical transformation in V . We get the similar representation, but from all the eigenvalues of all the chains the value λ_1 will be subtracted.

Let us add $\lambda_1 id_{v/k}$ back to the transformations of V/K in order that restores the initial set of eigenvalues. We introduce a special notation for such transformations of Jordan structures. The transformation $\mathcal{J} \rightarrow \tilde{\mathcal{J}}$ from the Theorem 1 is the partial case, when $\lambda_1 = 0$.

Definition 1. *The operation of the projection of the Jordan structure \mathcal{J} along the eigenspace, corresponding to the eigenvalue λ_1 is a transformation of the Jordan structure \mathcal{J} to the following Jordan structure denoted by $\mathcal{J} \setminus \{\lambda_1\}$:*

- *all the Jordan chains corresponding to all $\lambda_i \neq \lambda_1$ are the same for \mathcal{J} and for $\mathcal{J} \setminus \{\lambda_1\}$*
- *if all the Jordan chains of \mathcal{J} corresponding to λ_1 have the lengths equal to one, $\mathcal{J} \setminus \{\lambda_1\}$ has no chains corresponding to the eigenvalue λ_1 , it consists of all the Jordan chains of \mathcal{J} corresponding to $\lambda_i \neq \lambda_1$.*
- *if \mathcal{J} contains the Jordan chains corresponding to λ_1 of the lengths longer than one, $\mathcal{J} \setminus \{\lambda_1\}$ has chains corresponding to λ_1 . In this case the Jordan chains of $\mathcal{J} \setminus \{\lambda_1\}$ corresponding to the eigenvalue λ_1 are in one-to-one correspondence with the Jordan chains of \mathcal{J} corresponding to λ_1 with the lengths longer than one. They are one unit shorter.*

Let us denote

$$\mathcal{J} \setminus \{\lambda_1 \lambda_2 \dots \lambda_k\} := (\dots ((\mathcal{J} \setminus \{\lambda_1\}) \setminus \{\lambda_2\}) \dots) \setminus \{\lambda_k\}$$

Note that in the case of the presence of generalized eigenvectors, the set $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ may contains the corresponding eigenvalue several times. It means we may *project*

along the eigenspace, corresponding to one eigenvalue several times, but no more times than the length of the longest chain corresponding to this eigenvalue is.

Consider a set $\lambda'_1, \lambda'_2, \dots, \lambda'_M$ of numbers collected from the set of eigenvalues of \mathcal{J} , where each eigenvalue λ_k is written such a number of times that is the length of the longest Jordan chain corresponding to it. Consider any $\mathcal{A} \in \text{End}_{\mathcal{J}} V$. It defines a point K_1 of the Grassmanian:

$$K_1 = \ker(\mathcal{A} - \lambda'_1 id_v) \in G(n_1, V), \quad n_1 := \dim K_1,$$

and the linear transformation of V/K_1 :

$$\mathcal{A}_1 := \lambda'_1 id_{v/k_1} + \text{Pr}^{\parallel K_1} \circ ((\text{Pr}^{\parallel K_1})^*)^{-1} (\mathcal{A} - \lambda'_1 id_v) \in \text{End}_{\mathcal{J} \setminus \{\lambda'_1\}} V/K_1,$$

where the sub-index near id indicates the space where it is defined.

Let us consider the number λ'_2 and $V/K_1 =: V_1$ where \mathcal{A}_1 acts. Due to the Theorem 1, λ'_2 is the eigenvalue of \mathcal{A}_1 , so we can make the same procedure. We get $V_2 := (V/K_1)/K_2$ and \mathcal{A}_2 :

$$K_2 = \ker(\mathcal{A}_1 - \lambda'_2 id_{v_1}) \in G(n_2, V_1), \quad n_2 := \dim K_2,$$

$$\mathcal{A}_2 := \lambda'_2 id_{v_1/k_2} + \text{Pr}^{\parallel K_2} \circ ((\text{Pr}^{\parallel K_2})^*)^{-1} (\mathcal{A}_1 - \lambda'_2 id_{v_1}) \in \text{End}_{\mathcal{J} \setminus \{\lambda'_1, \lambda'_2\}} V_2,$$

and so on, up to the last $\mathcal{J} \setminus \{\lambda'_1, \dots, \lambda'_{M-1}\}$ for which the corresponding transformation is proportional to identical

$$\mathcal{A}_{M-1} = \lambda_M id.$$

Denote a transformation of A_{k-1} to A_k by $\pi_{\{\lambda'_k\}}$ and consider a hierarchy

$$\begin{array}{ccc} \text{End}_{\mathcal{J}} V & \xrightarrow{\gamma_1} & G(n_1, V) \ni K_1 \\ \gamma_1^{-1}(K_1) & \xrightarrow{\pi_{\{\lambda'_1\}}} & \text{End}_{\mathcal{J} \setminus \{\lambda'_1\}} V_1 \xrightarrow{\gamma_2} G(n_2, V_1) \ni K_2 \\ & \dots & \dots \\ & & \dots \end{array}$$

(12)

$$\gamma_k^{-1}(K_k) \xrightarrow{\pi_{\{\lambda'_k\}}} \text{End}_{\mathcal{J} \setminus \{\lambda'_1 \dots \lambda'_k\}} V_k \xrightarrow{\gamma_{k+1}} G(n_{k+1}, V_k) \ni K_{k+1}$$

...

$$\begin{aligned} \gamma_{M-2}^{-1}(K_{M-2}) &\xrightarrow{\pi_{\{\lambda'_{M-2}\}}} \text{End}_{\mathcal{J} \setminus \{\lambda'_1 \dots \lambda'_{M-2}\}} V_{M-2} \\ &\xrightarrow{\gamma_{M-1}} G(n_{M-1}, V_{M-2}) \ni K_{M-1} \end{aligned}$$

$$\gamma_{M-1}^{-1}(K_{M-1}) \xrightarrow{\pi_{\{\lambda'_{M-1}\}}} \text{End}_{\mathcal{J} \setminus \{\lambda'_1 \dots \lambda'_{M-1}\}} V_{M-1},$$

where

$$\gamma_k : \text{End}_{\mathcal{J} \setminus \{\lambda'_1 \dots \lambda'_{k-1}\}} V_{k-1} \longrightarrow G(n_k, V_{k-1})$$

maps any transformation to its eigenspace corresponding to λ'_k . Subspace K_k is any n_k -dimensional subspace of V_{k-1} , $V_k := V_{k-1}/K_k$. Transformation $\pi_{\{\lambda'_k\}}$ of A_{k-1} to A_k is defined by

$$\begin{aligned} \pi_{\{\lambda'_k\}}(\mathcal{A}_{k-1}) &:= \lambda'_k \text{id} + \text{Pr}^{\parallel K_k} \circ ((\text{Pr}^{\parallel K_k})^*)^{-1} (\mathcal{A}_{k-1} - \lambda'_k \text{id}) \\ &=: \mathcal{A}_k \in \text{End}_{\mathcal{J} \setminus \{\lambda'_1 \dots \lambda'_k\}} V_k, \end{aligned}$$

the transformation $((\text{Pr}^{\parallel K_k})^*)^{-1}$ is defined by (5).

3. MATRIX REPRESENTATION

If a basis in V is fixed linear transformations of V get a matrix representation that identify $\text{End}_{\mathcal{J}} V$ and the manifold $\mathcal{O}(J)$ of all matrices similar to a given J .

Consider hierarchy (12). The basis in V does not induce neither matrix representations nor identifications

$$\text{End}_{\mathcal{J} \setminus \{\lambda'_1 \dots \lambda'_k\}} V_k \leftrightarrow \mathcal{O}(\mathcal{J} \setminus \{\lambda'_1 \dots \lambda'_k\})$$

on the levels of (12) automatically, because a projection of a basis is not a basis.

Consider any ordering of the vectors of the given basis \mathbf{e}_v . Denote by E_k the envelope of the last $m_k := \dim \text{im } \mathcal{A}_k$ vectors of \mathbf{e}_v , it is some coordinate subspace. The sequence of projections along eigenspaces maps this set of m_k vectors to V_k . For each \mathcal{A} from some open subset of $\mathcal{O}(J)$ this $\dim V_k$ vectors form a basis of V_k .

For each ordering of vectors of \mathbf{e}_v this process sets natural isomorphisms between the abstract linear spaces V_k and the coordinate subspaces E_k . The isomorphisms are defined for some open subset of the orbit $\mathcal{O}(J)$.

On the other hand a projection of a full set of vectors is a full set, consequently for any \mathcal{A} from the orbit we can put vectors of \mathbf{e}_v in such an order that the bases of all V_k will be formed by the images of the last several vectors of \mathbf{e}_v .

Proposition 1. *The covering of the whole orbit $\mathcal{O}(J)$ by the open domains numerated by the permutations of vectors of \mathbf{e}_v has been constructed.*

Let us fix some ordering and identify V_k with the corresponding subspaces E_k of V .

Note 3. *We will not distinguish V_k and E_k from now.*

Filtration (12) defines the sequence of the transformations \mathcal{A}_k of the coordinate subspaces $E_k \simeq V_k$. In the given bases of E_k the hierarchy has the transparent matrix representation:

$$A = \begin{pmatrix} \text{I} & 0 \\ Q_1 & \text{I} \end{pmatrix} \begin{pmatrix} \lambda'_1 \text{I} & P_1 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} \text{I} & 0 \\ Q_1 & \text{I} \end{pmatrix}^{-1}$$

$$A_1 = \begin{pmatrix} \text{I} & 0 \\ Q_2 & \text{I} \end{pmatrix} \begin{pmatrix} \lambda'_2 \text{I} & P_2 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} \text{I} & 0 \\ Q_2 & \text{I} \end{pmatrix}^{-1}$$

$$\dots \quad \dots \quad \dots$$

$$(13) \quad A_{k-1} = \begin{pmatrix} \mathbf{I} & 0 \\ Q_k & \mathbf{I} \end{pmatrix} \begin{pmatrix} \lambda'_k \mathbf{I} & P_k \\ 0 & A_k \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ Q_k & \mathbf{I} \end{pmatrix}^{-1}$$

...

$$A_{M-2} = \begin{pmatrix} \mathbf{I} & 0 \\ Q_{M-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \lambda'_{M-1} \mathbf{I} & P_{M-1} \\ 0 & \lambda_M \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ Q_{M-1} & \mathbf{I} \end{pmatrix}^{-1}$$

Consider one flight of the hierarchy. To simplify the notations and in order to reduce the number of indexes let us put $k = 0$, i.e. we consider the first flight. For the same reasons we put $\lambda_1 = 0$, the eigenspace K_1 becomes $\ker A =: K$, $\dim K =: n$, $\dim \operatorname{im} A =: m$.

The coordinate subspace enveloping last m basic vectors we denoted by $E_1 =: E$, let us denote the envelope of the first n vectors by F : $V = F \oplus E$.

The open subset where this expansion of V takes place consists of all the transformations

$$A \in \operatorname{End}_J V : \ker A \cap E = 0.$$

Let us denote such subset of the orbit by $((\mathcal{O}(J)))_E$. The corresponding subset of the Grassmanian we denote by $((G(n, V)))_E$:

$$K \in ((G(n, V)))_E \Leftrightarrow K \cap E = 0.$$

Let $(e^1 \dots e^n)$ be the set of the first vectors of \mathbf{e}_v . It forms the basis of F .

Let us denote the projection on the subspace L_1 along the subspace L_2 by $\operatorname{Pr}_{L_1}^{\parallel L_2}$. It is defined for any transversal subspaces: $L_1 \oplus L_2 = V$.

The projection along E sets the isomorphism between $\ker A$ and F . It is defined for the transformations A from

the domain $((\mathcal{O}(J)))_E$. Let us introduce the standard coordinates on the corresponding subset of the Grassmanian (see [15]):

$$K \leftrightarrow \Pr_E^{\parallel F} \circ \Pr_K^{\parallel E}(e^1 \dots e^n).$$

It is the isomorphism between the open subset of the Grassmanian in question and the set of all $m \times n$ matrices.

For our aims it is more natural to forget about the specification of the bases on E and F and use the decomposition $V = F \oplus E$ only. It gives the following isomorphism $((G(n, V)))_E \xrightarrow{\sim} \text{Hom}(F, E)$:

$$(14) \quad K \leftrightarrow \Pr_E^{\parallel F} \circ \Pr_K^{\parallel E} |_F$$

Consider the fibration that is given by Theorem 2. Its fiber is the open subset of $\text{Hom}(E, K)$. We know that the projection parallel to E sets the bijection between F and $K = \ker A$, on $((\mathcal{O}(J)))_E$, consequently we may replace $\text{Hom}(E, K)$ on $\text{Hom}(E, F)$. We should just compose the projection $\Pr_F^{\parallel E}$ with each element of $\text{Hom}(E, K)$.

We formulate the modified version of the Theorem 2 as the proposition.

Proposition 2. *The open set $((\mathcal{O}(J)))_E$ has a structure of a fiber-bundle*

$$((\mathcal{O}(J)))_E \rightarrow \text{Hom}(F, E) \times \mathcal{O}(J \setminus \{0\}),$$

where a fiber is the open subset of $\text{Hom}(E, F)$.

Theorem 2 states that some algebraically closed set of $\text{Hom}(E, F)$ does not belong to the fiber. It corresponds to the transformations which Jordan chains are shorter than necessary and their last units do not form a basis of the eigenspace.

This effect comes into particular prominence in 2×2 case. Let us parameterize the Zariski-open part $((\mathcal{O}(J)))_E$ of the

orbit¹ of $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ by the functions p, q :

$$\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -pq & p \\ -pq^2 & pq \end{pmatrix}.$$

We must exclude $p = 0$ because $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does not belong to the orbit.

Nevertheless in several important applications it is not naturally to exclude the divisor $p = 0, q \in \mathbb{C}$ from the chart. For example for the problems of isomonodromic deformations the corresponding Hamiltonian flows have no any special behavior on the line $p = 0, q = \mathbb{C}$, so the natural way to make a theory consistent is *to add this divisor* i.e. to expand the orbit $\mathcal{O}(J)$. It means that we introduce a new symplectic manifold $\mathcal{O}'(J)$ with almost the same fiber-bundle structure as $\mathcal{O}(J)$ has, but the fibers are just $\text{Hom}(E, F)$.

In other words to get $\mathcal{O}'(J)$ we glue some algebraically closed set to $\mathcal{O}(J)$ in such a way that $\mathcal{O}'(J)$ will be the symplectic manifold too and there will be the symplectic map $\mathcal{O}(J) \hookrightarrow \mathcal{O}'(J)$.

Note 4. *We assume that our orbits are already enlarged. We mark this enlarging by the prime. It means that we are investigating the symplectic manifolds $\mathcal{O}'(J)$ that are equipped with the symplectic mappings $\mathcal{O}'(J) \leftarrow \mathcal{O}(J)$.*

The complement to the image of $\mathcal{O}(J)$ in $\mathcal{O}'(J)$ is the algebraically closed set isomorphic to the set of the transformations which Jordan chains (their last units) do not form a basis of the corresponding eigenspace.

¹In this case the complement of $((\mathcal{O}(J))_E)$ to $((\mathcal{O}(J)))$ is formed by the lower triangular matrices $\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$. The subspace E is the coordinate subspace spanned the second basic vector e^2

From now we are in the conditions of the Note 2. We formulate the version of the Theorem 2 for the enlarged orbits \mathcal{O}' as the next proposition.

Coordinate subspace E_k is identified with V_k . In accordance to the identification let us change the notation of $((\Pr^{\|K})^*)^{-1}$ to $((\Pr_{E_k}^{\|K_k})^*)^{-1}$. We keep the previous notation $\pi_{\{\lambda'_k\}}$ for the transformation $A_{k-1} \rightarrow A_k$.

Proposition 3. *The open set $((\mathcal{O}'(J \setminus \{\lambda'_1 \dots \lambda'_{k-1}\}))_{E_k})$ is isomorphic to the direct product*

$$((\mathcal{O}'(J \setminus \{\lambda'_1 \dots \lambda'_{k-1}\}))_{E_k}) \xrightarrow{\sim} \text{Hom}(E_k, F_k) \times \text{Hom}(F_k, E_k) \times \mathcal{O}(J \setminus \{\lambda'_1 \dots \lambda'_k\})$$

A point $A_{k-1} \in ((\mathcal{O}'(J \setminus \{\lambda'_1 \dots \lambda'_{k-1}\}))_{E_k})$ has the following projections on the Cartesian factors

$$A_{k-1} \rightarrow (\mathcal{P}_k, \mathcal{Q}_k, A_k) :$$

$$\mathcal{P}_k = \Pr_{F_k}^{\|E_k} \circ ((\Pr_{E_k}^{\|K_k})^*)^{-1} (A_{k-1} - \lambda'_k \mathbf{I}),$$

$$\mathcal{Q}_k = \Pr_{E_k}^{\|F_k} \circ \Pr_{K_k}^{\|E_k} |_{F_k},$$

$$A_k = \lambda'_k \mathbf{I} + \Pr_{E_k}^{\|K_k} \circ ((\Pr_{E_k}^{\|K_k})^*)^{-1} (A_{k-1} - \lambda'_k \mathbf{I}) =: \pi_{\{\lambda'_k\}} A_{k-1}.$$

Note 5. *The formulae for the projections have sense for the points of \mathcal{O} only. The complement of the $\mathcal{O}(J \setminus \{\lambda'_1 \dots \lambda'_{k-1}\})$ with respect to the $\mathcal{O}'(J \setminus \{\lambda'_1 \dots \lambda'_{k-1}\})$ does not immersed into $\mathfrak{gl}(N_{k-1}) \simeq \text{End } V_{k-1}$.*

Note 6. *The proposition is equivalent to (13) where the restrictions on the matrices P_k are removed.*

Auxiliary symplectic space $\mathcal{E}(F \oplus E)$. Let $V = F \oplus E$ be a splitting of $n + m$ -dimensional space V into a direct sum of two subspaces F , and E , $\dim F = n, \dim E = m$.

Consider $\text{End } V$ as a linear $(n + m)^2$ -dimensional space. Let us define a skew-symmetrical scalar product $\omega_{\mathcal{E}} : \text{End } V \times \text{End } V \rightarrow \mathbb{C}$:

$$(15) \quad \omega_{\mathcal{E}}(\mathcal{B}_1, \mathcal{B}_2) = \text{tr } \Pr_E^{\|F} \mathcal{B}_2 \circ \Pr_F^{\|E} \mathcal{B}_1 |_E - \text{tr } \Pr_F^{\|E} \mathcal{B}_2 \circ \Pr_E^{\|F} \mathcal{B}_1 |_F.$$

It is obviously degenerated. Let us introduce a basis (\mathbf{fe}) , where \mathbf{f} is a basis of F and \mathbf{e} is a basis of E . Let \mathcal{B}_i have the matrix $\begin{pmatrix} b_{11}^i & b_{12}^i \\ b_{21}^i & b_{22}^i \end{pmatrix}$ in this basis. By b_{jk}^i we denoted the corresponding blocks.

In this coordinates

$$(16) \quad \omega_{\mathcal{E}}(\mathcal{B}_1, \mathcal{B}_2) = \text{tr } b_{21}^2 b_{12}^1 - \text{tr } b_{12}^2 b_{21}^1.$$

Let us introduce the $2nm$ -dimensional subspace $\mathcal{E}(F \oplus E) \subset \text{End } V$:

$$\mathcal{B} \in \mathcal{E}(F \oplus E) \Leftrightarrow \text{Pr}_E^{\parallel F} \mathcal{B}|_E = 0 = \text{Pr}_F^{\parallel E} \mathcal{B}|_F.$$

It consists of the matrices which are off-diagonal in the basis (\mathbf{fe}) , i.e. $b_{11} = 0 = b_{22}$. We keep the previous notation $\omega_{\mathcal{E}}$ for the restriction of $\omega_{\mathcal{E}}$ on $\mathcal{E}(F \oplus E)$.

Proposition 4. *Space $(\mathcal{E}(F \oplus E), \omega_{\mathcal{E}})$ is $2nm$ -dimensional symplectic space. Canonical basis is formed by the set of couples P_{ij}, Q_{ji} , $1 \leq i \leq n, n+1 \leq j \leq n+m$, where $P_{ij}, Q_{ji} \in \mathcal{E}(F \oplus E)$ are the transformations with the following matrices*

$$(P_{ij})_{st} = \delta_{si} \delta_{tj}, \quad (Q_{ji})_{st} = \delta_{sj} \delta_{ti}$$

in the basis (\mathbf{fe}) where the first n vectors form basis F and the last m vectors form basis E .

Proof

The proof follows from the formula (16) that shows that P_{ij}, Q_{ji} is really the Darboux basis for $(\mathcal{E}(F \oplus E), \omega_{\mathcal{E}})$. \square

Proposition 5. *There is a natural isomorphism between the manifold $\text{Hom}(E, F) \times \text{Hom}(F, E)$ and the space $\mathcal{E}(F \oplus E)$.*

To construct the point of $\text{End } V \supset \mathcal{E}(F \oplus E)$ it is sufficient to assign its action on each of summands of $F \oplus E$. Let a couple \mathcal{P}, \mathcal{Q} be a point of $\text{Hom}(E, F) \times \text{Hom}(F, E)$.

We define the transformation $\mathcal{B} \in \text{End } V$ corresponding to the couple as the map which transforms the vectors from $E, F \subset V$ (the natural embedding) as it is assigned by \mathcal{P} and \mathcal{Q} .

Consider the opposite direction. Any $\mathcal{B} \in \text{End } V$ can be decomposed on $\mathcal{P} := \text{Pr}_F^{\parallel E} \mathcal{B}|_E$ and $\mathcal{Q} := \text{Pr}_E^{\parallel F} \mathcal{B}|_F$. For the transformations from $\mathcal{E}(F \oplus E) \subset \text{End } V$ these \mathcal{P} and \mathcal{Q} define \mathcal{B} uniquely. It is evidently an isomorphism. \square

Main theorem. We can see that out of some algebraically closed subset the symplectic manifold $\mathcal{O}(J)$ is isomorphic to the Cartesian product of the *linear* symplectic space $\mathcal{E}(F \oplus E)$ and the symplectic manifold $\mathcal{O}(J \setminus \{\lambda'\})$ of the smaller than $\mathcal{O}(J)$ dimension. Here λ' is some eigenvalue of J and

$$F \simeq \ker(J - \lambda' I), \quad E \simeq \text{im}(J - \lambda' I).$$

Let us denote the projection on the Cartesian factor $\mathcal{E}(F \oplus E)$ by $\pi_{\mathcal{E}(F \oplus E)}$ and the projection on $\mathcal{O}(J \setminus \{\lambda'\})$ by $\pi_{\{\lambda'\}}$.

We constructed the isomorphism between two symplectic spaces equipped with their own forms $\omega_{\mathcal{O}(J)}$ and $\omega_{\mathcal{E}(F \oplus E)} + \omega_{\mathcal{O}(J \setminus \{\lambda'\})}$.

Let us introduce the main theorem now.

Theorem 3. *The isomorphism*

$$(17) \quad ((\mathcal{O}'(J))_E) \xrightarrow{\sim} \mathcal{E}(F \oplus E) \times \mathcal{O}(J \setminus \{\lambda'\})$$

is birational and symplectic:

$$(18) \quad \omega_{\mathcal{O}(J)} = \pi_{\mathcal{E}(F \oplus E)}^* \omega_{\mathcal{E}(F \oplus E)} + \pi_{\{\lambda'\}}^* \omega_{\mathcal{O}(J \setminus \{\lambda'\})}.$$

The proof will be based on the following lemma.

Let the given basis \mathbf{e}_v be divided in two parts $\mathbf{e} = (\mathbf{f}\tilde{\mathbf{e}})$ in accordance with the dimensions of the kernel and the image of $A - \lambda' I$. Let E be the envelope of $\tilde{\mathbf{e}}$, $A \in ((\mathcal{O}(J))_E)$,

$\tilde{A} := \pi_{\{\lambda'\}} A \in \mathcal{O}(J \setminus \{\lambda'\})$, $\dim \ker(A - \lambda' I) = n$, $\dim \operatorname{im}(A - \lambda' I) = m$.

Lemma 1. *For any $g \in \operatorname{GL}(m, \mathbb{C})$ that transforms the fixed basis $\tilde{\mathbf{e}}$ of E to any Jordan basis $\tilde{\mathbf{e}}_J$ of E for \tilde{A} , there exist*

- *the set of vectors κ that form the basis of $\ker(A - \lambda' I)$,*
- *the matrix $\hat{P} \in \mathbb{C}^{n \times m}$*

such that \mathbf{e}_J :

$$(19) \quad \mathbf{e}_J = (\kappa \tilde{\mathbf{e}}) \begin{pmatrix} I & \hat{P} \\ 0 & g \end{pmatrix} = (\kappa \tilde{\mathbf{e}}_J) \begin{pmatrix} I & \hat{P} \\ 0 & I \end{pmatrix}$$

form a Jordan basis of V for A .

Proof of the lemma

For the simplification of the notations let us put $\lambda' = 0$. Consider $\mathcal{O}(J)$, where J is the Jordan normal form of the matrices from the orbit. Let us order the vectors of the Jordan basis for J in such a way that the first set κ of the vectors of the basis of V forms the basis of the root-space of J :

$$J = \begin{pmatrix} 0 & J_P \\ 0 & \tilde{J} \end{pmatrix}.$$

Consider such a part of the orbit where κ is completing some fixed linear independent set $\tilde{\mathbf{e}}$ to the basis of V .

In the basis $(\kappa, \tilde{\mathbf{e}})$ any A from the orbit has the form

$$\begin{pmatrix} 0 & P \\ 0 & \tilde{A} \end{pmatrix}.$$

The statement of the lemma is equivalent to the following:

if $\begin{pmatrix} 0 & P \\ 0 & \tilde{A} \end{pmatrix}$ is similar to the $\begin{pmatrix} 0 & J_P \\ 0 & \tilde{J} \end{pmatrix}$, and if the zero columns form the basis of the root-spaces of the matrices,

then for the given g : $g^{-1}\tilde{A}g = \tilde{J}$ there exist such $\hat{g} \in \text{GL}(n, \mathbb{C})$, and such \hat{P} that

$$(20) \quad \begin{pmatrix} 0 & P \\ 0 & \tilde{A} \end{pmatrix} = \begin{pmatrix} \hat{g} & \hat{P} \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & J_P \\ 0 & \tilde{J} \end{pmatrix} \begin{pmatrix} \hat{g} & \hat{P} \\ 0 & g \end{pmatrix}^{-1}.$$

It is equivalent to the solvability of the equation on \hat{g} and \hat{P}

$$P = (\hat{g}J_P + \hat{P}\tilde{J})g^{-1},$$

where P, g, J_P, \tilde{J} are given.

The equation is solvable for any P , because the number of the linear independent rows in $(n+m) \times m$ matrix $\begin{pmatrix} J_P \\ \tilde{J} \end{pmatrix}$ coincides with the number of the linear independent columns that is m . It is just the dimension of the the space \mathbb{C}^m of the rows of P .

To prove that the matrix \hat{g} can be chosen non-degenerated let us rewrite the equation:

$$(21) \quad Pg = \hat{g}J_P + \hat{P}\tilde{J}.$$

The matrices J_P and \tilde{J} are the blocks of the Jordan matrix $J = \begin{pmatrix} 0 & J_P \\ 0 & \tilde{J} \end{pmatrix}$, each column of $\begin{pmatrix} J_P \\ \tilde{J} \end{pmatrix}$ contains exactly one unit. It implies that the root space of \tilde{J} and the root space of J_P form a basis of columns \mathbb{C}^m . Consider the zero columns of the Jordan matrix \tilde{J} . The set of the corresponding columns of Pg has full dimension otherwise there will be a linear relation between the columns of $\begin{pmatrix} P \\ \tilde{A} \end{pmatrix}$:

$$P - \hat{P}\tilde{J}g^{-1} = P - \hat{P}g^{-1}\tilde{A} = \hat{g}J_Pg^{-1}.$$

Consider (21). From the linear independence of the columns of Pg in question it follows that on the places of the zero columns of \tilde{J} there are linear independent columns of Pg

that implies the linear independence of the corresponding columns of \hat{g} . Matrix $\hat{g}J_P$ does not depend on the other columns of \hat{g} because the corresponding columns of J_P vanishes, consequently the set of the linear independent columns of \hat{g} can be completed in an arbitrary way, we choose $\det \hat{g} \neq 0$.

The lemma has been proved

Note 7. *Lemma itself follows from the Corollary 2 directly, but for the future considerations we need the information about the introduced matrices.*

Let us proof the theorem.

Consider any point $A \in ((\mathcal{O}(J)))_E$, and the level-sets of the map (17): $(\cup A)|_{\mathcal{E}=\text{const}}$ and $(\cup A)|_{\mathcal{O}=\text{const}}$ passing this point. The map (17) is the isomorphism, consequently

$$T_A \mathcal{O}(J) = T_A(\cup A)|_{\mathcal{E}=\text{const}} \oplus T_A(\cup A)|_{\mathcal{O}=\text{const}}.$$

Let $\partial_{\mathcal{E}}$ and $\partial_{\mathcal{O}}$ be any vectors from the corresponding subspaces:

$$\partial_{\mathcal{E}} \in T_A(\cup A)|_{\mathcal{O}=\text{const}}, \quad \partial_{\mathcal{O}} \in T_A(\cup A)|_{\mathcal{E}=\text{const}}.$$

They are tangents to the lines

$$A_{\mathcal{O}}(t) = \begin{pmatrix} \mathbf{I} & 0 \\ Q(t) & \mathbf{I} \end{pmatrix} \begin{pmatrix} 0 & P(t) \\ 0 & \tilde{A} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ Q(t) & \mathbf{I} \end{pmatrix}^{-1} + \lambda' \mathbf{I}.$$

and

$$A_{\mathcal{E}}(t) = \begin{pmatrix} \mathbf{I} & 0 \\ Q & \mathbf{I} \end{pmatrix} \begin{pmatrix} 0 & P \\ 0 & \tilde{A}(t) \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ Q & \mathbf{I} \end{pmatrix}^{-1} + \lambda' \mathbf{I}$$

that belong to the corresponding level sets.

It follows from the lemma that on the level set $(\cup A)|_{\mathcal{O}=\text{const}}$ any curve $A_{\mathcal{O}}(t)$ can be parameterized in the following way:

$$\begin{pmatrix} \mathbf{I} & 0 \\ Q(t) & \mathbf{I} \end{pmatrix} \begin{pmatrix} \hat{g}(t) & \hat{P}(t) \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & J_P \\ 0 & \tilde{J} \end{pmatrix} (\dots)^{-1} (\dots)^{-1} + \lambda' \mathbf{I},$$

consequently

$$\begin{aligned}\frac{d}{dt}\Big|_A A_{\mathcal{O}}(t) &= \left[\begin{pmatrix} \mathbf{I} & 0 \\ Q & \mathbf{I} \end{pmatrix} \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ Q & \mathbf{I} \end{pmatrix}^{-1}, A \right] \\ \frac{d}{dt}\Big|_A A_{\mathcal{E}}(t) &= \begin{pmatrix} \mathbf{I} & 0 \\ Q & \mathbf{I} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ Q & \mathbf{I} \end{pmatrix}^{-1},\end{aligned}$$

where we denote the terms the values of which are unessential by stars.

The application of formula (3) gives the desired

$$(22) \quad \omega_{\mathcal{O}(J)}(\partial_{\mathcal{E}}, \partial_{\mathcal{O}}) = 0$$

Let $\partial_{\mathcal{Q}}^i \in \mathbb{T}_A(\cup A)|_{\mathcal{O}=\text{const}}$, $i = 1, 2$ be two vectors tangent to the level-set of function P i.e. they are tangents to the lines

$$A_{\mathcal{P}\mathcal{O}}^i(t) = \begin{pmatrix} \mathbf{I} & 0 \\ Q_i(t) & \mathbf{I} \end{pmatrix} \begin{pmatrix} 0 & P \\ 0 & \tilde{A} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ Q_i(t) & \mathbf{I} \end{pmatrix}^{-1} + \lambda' \mathbf{I}.$$

The calculation gives:

$$\omega_{\mathcal{O}(J)}(\partial_{\mathcal{Q}}^1, \partial_{\mathcal{Q}}^2) = \text{tr} \begin{pmatrix} 0 & 0 \\ \dot{Q}_1 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & 0 \\ \dot{Q}_2 & 0 \end{pmatrix}, A \right] = 0.$$

Let $\partial_{\mathcal{P}}^i \in \mathbb{T}_A(\cup A)|_{\mathcal{O}=\text{const}}$, $i = 1, 2$ be two vectors tangent to the level-set of function Q i.e. they are tangents to the lines

$$A_{\mathcal{Q}\mathcal{O}}^i(t) = \begin{pmatrix} \mathbf{I} & 0 \\ Q & \mathbf{I} \end{pmatrix} \begin{pmatrix} 0 & P_i(t) \\ 0 & \tilde{A} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ Q & \mathbf{I} \end{pmatrix}^{-1} + \lambda' \mathbf{I}.$$

We can set $Q = 0$ because trace does not depend on the conjugation of all the factors by one matrix. The calculation of the tangent vectors gives

$$\begin{aligned}\frac{d}{dt}\Big|_A \begin{pmatrix} 0 & P_i(t) \\ 0 & \tilde{A} \end{pmatrix} &= \begin{pmatrix} 0 & \dot{P}_i \\ 0 & 0 \end{pmatrix} \\ &= \frac{d}{dt}\Big|_A \begin{pmatrix} \hat{g}_i(t) & \hat{P}_i(t) \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & J_P \\ 0 & \tilde{J} \end{pmatrix} \begin{pmatrix} \hat{g}_i(t) & \hat{P}_i(t) \\ 0 & g \end{pmatrix}^{-1}\end{aligned}$$

$$= \left[\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & P \\ 0 & \tilde{A} \end{pmatrix} \right]$$

Consequently

$$\omega_{\mathcal{O}(J)}(\partial_{\mathcal{P}}^1, \partial_{\mathcal{P}}^2) = \text{tr} \begin{pmatrix} 0 & \dot{P}_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} = 0.$$

Simple calculation gives $\omega_{\mathcal{O}(J)}(\partial_{\mathcal{P}}, \partial_{\mathcal{Q}}) = \text{tr} PQ$, that means

$$(23) \quad \omega_{\mathcal{O}(J)}|_{\mathcal{O}=\text{const}} = \pi_{\mathcal{E}(F \oplus E)}^* \omega_{\mathcal{E}(F \oplus E)}.$$

Let us consider two tangents ∂^1, ∂^2 to the lines $A_i(t)$ on the level-set $(\cup A)|_{\mathcal{E}=\text{const}}$. For the previous reasons without the loss of generality we put $Q = 0$,

$$\frac{d}{dt} \Big|_A \begin{pmatrix} 0 & P \\ 0 & \tilde{A}_i(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \dot{\tilde{A}}_i \end{pmatrix}.$$

From the representation (20) we get

$$\frac{d}{dt} \Big|_A \begin{pmatrix} 0 & P \\ 0 & \tilde{A}_i(t) \end{pmatrix} = \left[\begin{pmatrix} * & * \\ 0 & g^{-1}\dot{g}_i \end{pmatrix}, \begin{pmatrix} 0 & P \\ 0 & \tilde{A} \end{pmatrix} \right].$$

The application of the formula (3) gives

$$\omega_{\mathcal{O}(J)}(\partial^1, \partial^2) = \text{tr} g^{-1}\dot{g}_1 \dot{\tilde{A}}_2,$$

that is the value of $\omega_{\mathcal{O}(\tilde{J})}$ on the projections of the vectors ∂^1, ∂^2 .

The equality (18) follows from (22), (23) and the last one.

Let us prove the birationality of the isomorphism (17). To find the images of the projections $\pi_{\mathcal{E}(F \oplus E)}$ and $\pi_{\{\lambda'\}}$ we have to find the eigenvectors corresponding to the given eigenvalue and project along the subspaces. The inverse operation is the multiplication of the matrices with the given blocks in formulae (13). All these operations are rational. \square

Let us present the final formulae for the map

$$\mathcal{E}(F_1 \oplus E_1) \times \mathcal{E}(F_2 \oplus E_2) \times \cdots \times \mathcal{E}(F_M \oplus E_M) \rightarrow \mathcal{O}'(J).$$

Let $\lambda'_1, \dots, \lambda'_M$ be a sequence of the eigenvalues of some Jordan matrix matrix J . Let each eigenvalue λ' be written in the sequence such a number of times $r_{\lambda'}$ as the length of the longest Jordan chain corresponding to this eigenvalue is:

$$\dim \ker(J - \lambda'I)^{r_{\lambda'}-1} < \dim \ker(J - \lambda'I)^{r_{\lambda'}} = \dim \ker(J - \lambda'I)^{r_{\lambda'}+1}.$$

Let us denote by n_k a number of the Jordan chains that are not shorter than the number of eigenvalues equal to this λ'_k in the subsequence $\lambda'_1, \dots, \lambda'_k$.

Proposition 6. *The full information about the Jordan structure of J is contained in the set of couples (λ'_k, n_k) , $k = 1, \dots, M$.*

Proof

If the eigenspace corresponding to λ' does not contain generalized eigenvectors we have $r_{\lambda'} = 1$. In this case there is only one $\lambda'_k = \lambda'$ in the sequence and the number n_k is the dimension of the eigenspace.

Let the set of Jordan chains corresponding to λ' consists of m_1 chains of the length 1, m_2 chains of the length 2, \dots , $m_{r_{\lambda'}} \neq 0$ chains of the length $r_{\lambda'}$. In this case the set of the numbers n_k corresponding to these eigenvalue is the non-increasing sequence of $r_{\lambda'}$ integers $m_i + m_{i+1} + \cdots + m_{r_{\lambda'}}$, $i = 1, \dots, r_{\lambda'}$. The smallest n_k is the number of the longest Jordan chains. Their lengths $r_{\lambda'}$ are equal to the number of repetitions of λ' in the sequence. To reconstruct other m_i 's we should take the differences between neighbour n_k 's. □

Denote by Q the following lower-triangular block-matrix

$$Q := \begin{pmatrix} I_{n_1} & 0 & 0 & \dots & 0 & 0 & 0 \\ q_2^1 & I_{n_2} & 0 & \dots & 0 & 0 & 0 \\ q_3^1 & q_3^2 & I_{n_3} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ q_{M-2}^1 & q_{M-2}^2 & q_{M-2}^3 & \dots & I_{n_{M-2}} & 0 & 0 \\ q_{M-1}^1 & q_{M-1}^2 & q_{M-1}^3 & \dots & q_{M-1}^{M-2} & I_{n_{M-1}} & 0 \\ q_M^1 & q_M^2 & q_M^3 & \dots & q_M^{M-2} & q_M^{M-1} & I_{n_M} \end{pmatrix}.$$

Its diagonal is formed by the set of M square blocks $n_k \times n_k$. Each diagonal block is proportional to the unit matrix of the corresponding dimension. Block q_i^j is $n_i \times n_j$ matrix.

Let $[Q]_k$ be its diagonal lower $k \times k$ block

$$[Q]_k := \begin{pmatrix} I_{n_{M-k+1}} & 0 & 0 & \dots & 0 & 0 & 0 \\ q_{M-k+2}^{M-k+1} & I_{n_{M-k+2}} & 0 & \dots & 0 & 0 & 0 \\ q_{M-k+3}^{M-k+1} & q_{M-k+3}^{M-k+2} & I_{n_{M-k+3}} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ q_{M-2}^{M-k+1} & q_{M-2}^{M-k+2} & q_{M-2}^{M-k+3} & \dots & I_{n_{M-2}} & 0 & 0 \\ q_{M-1}^{M-k+1} & q_{M-1}^{M-k+2} & q_{M-1}^{M-k+3} & \dots & q_{M-1}^{M-2} & I_{n_{M-1}} & 0 \\ q_M^{M-k+1} & q_M^{M-k+2} & q_M^{M-k+3} & \dots & q_M^{M-2} & q_M^{M-1} & I_{n_M} \end{pmatrix},$$

so $Q = [Q]_M, [Q]_1 = I_{n_M}$.

Denote the non-trivial parts of the vector-column-blocks by \vec{q}_k :

$$\vec{q}^k := (q_{k+1}^k, q_{k+2}^k, \dots, q_{M-2}^k, q_{M-1}^k, q_M^k)^T.$$

They are rectangular matrices of the dimension $(n_{k+1} + n_{k+2} + \dots + n_{M-2} + n_{M-1} + n_M) \times n_k$. Consider the vector-row-blocks \vec{p}_k

$$\vec{p}_k := (p_k^{k+1}, p_k^{k+2}, \dots, p_k^{M-2}, p_k^{M-1}, p_k^M)$$

They are $n_k \times (n_{k+1} + n_{k+2} + \dots + n_{M-2} + n_{M-1} + n_M)$ matrices. The blocks q_i^j and p_j^i have the dimensions $n_i \times n_j$ and $n_j \times n_i$ correspondingly.

Consider upper-triangular matrix ρ

$$(24) \quad \rho := \begin{pmatrix} \lambda'_1 I_{n_1} & \rho_1^2 & \rho_1^3 & \cdots & \rho_1^{M-2} & \rho_1^{M-1} & \rho_1^M \\ 0 & \lambda'_2 I_{n_2} & \rho_2^3 & \cdots & \rho_2^{M-2} & \rho_2^{M-1} & \rho_2^M \\ 0 & 0 & \lambda'_3 I_{n_3} & \cdots & \rho_3^{M-2} & \rho_3^{M-1} & \rho_3^M \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda'_{n_{M-2}} I_{n_{M-2}} & \rho_{M-2}^{M-1} & \rho_{M-2}^M \\ 0 & 0 & 0 & \cdots & 0 & \lambda'_{n_{M-1}} I_{n_{M-1}} & \rho_{M-1}^M \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda'_{n_M} I_{n_M} \end{pmatrix}.$$

Denote non-trivial parts of the vector-raw-blocks by $\vec{\rho}_k$:

$$\vec{\rho}_k := (\rho_k^{k+1}, \rho_k^{k+2}, \dots, \rho_k^{M-2}, \rho_k^{M-1}, \rho_k^M)$$

Theorem 4. *Matrix A:*

$$(25) \quad A = Q\rho Q^{-1},$$

where the block-vector-raws of ρ are

$$\vec{\rho}_k := \vec{p}_k[Q]_{M-k}$$

provides the canonical parameterization of the orbit $\mathcal{O}(J) \ni$

A by the couples of matrix elements of blocks $p_j^i, q_i^j: (p_j^i)_{st}, (q_i^j)_{ts}, 1 \leq s \leq n_j, 1 \leq t \leq n_i, 1 \leq n_i, n_j \leq M$.

Proof

To find the Jordan structure of ρ we construct the hierarchy (13) for it.

Let us proof that on the open set of the matrix elements of ρ the hierarchy (13) gives the lower-diagonal blocks of ρ .

The first n_1 columns of $\rho - \lambda_1 I$ vanish. Consider the equality corresponding to (21) for this stair-flight:

$$Pg = \hat{g}J' + \hat{P}\tilde{J}_\rho.$$

Here \tilde{J}_ρ is the normal Jordan form of the lower diagonal block of ρ and J' complements \tilde{J}_ρ to the normal Jordan form of ρ .

We do not know the \tilde{J}_ρ and J' now, J' may have too many zero columns. Consider the columns corresponding to the zero columns of \tilde{J}_ρ . Denote matrices collected from these columns only by $\llbracket \dots \rrbracket$. It is the projection on the subset of columns:

$$\llbracket Pg \rrbracket = \llbracket \hat{g}J' \rrbracket.$$

Matrix g is non-degenerate, consequently on the open set of matrix elements of P matrix $\llbracket Pg \rrbracket$ has a full rank that is $m - \dim \ker \tilde{J}_\rho$. It implies $\text{rank } J' = m - \dim \ker \tilde{J}_\rho$ or

$$\text{rank} \begin{pmatrix} J' \\ \tilde{J}_\rho \end{pmatrix} = m.$$

Geometrically it means the following.

- No one of the Jordan chains of the lower block of ρ in question was finished on the flight of the hierarchy.
- We started new $n_k - \dim \ker \tilde{J}_\rho = n_k - n_{k+1}$ chains.

It proves that the Jordan structures of ρ and J coincide.

Let us construct the canonical coordinates for (25) using the method developed in the present paper. We proved that the kernel of $\rho - \lambda'_1 I$ is formed by the first n_1 columns on the open set of matrix elements of ρ .

It is easy to verify that

$$Q = \begin{pmatrix} I & 0 \\ \vec{q}^{-1} & [Q]_{M-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \vec{q}^{-1} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & [Q]_{M-1} \end{pmatrix},$$

consequently

$$Q^{-1} = \begin{pmatrix} I & 0 \\ 0 & ([Q]_{M-1})^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\vec{q}^{-1} & I \end{pmatrix}.$$

The substitution of these representations of Q and Q^{-1} to (25) gives the first flight of the hierarchy (13). The diagonal lower block has the same structure as (25). The iteration of the procedure gives the statement of the theorem. \square

The inverse map

$$\mathcal{O}(J) \rightarrow \mathcal{E}(F_1 \oplus E_1) \times \mathcal{E}(F_2 \oplus E_2) \times \dots \times \mathcal{E}(F_M \oplus E_M)$$

involves the construction of the hierarchy (13). It is a sequence of the couples of steps. We should find the eigenspace of the diagonal lower block and change the first part of the basic vectors to the normalized basis of the eigenspace.

4. EXAMPLES

Let us consider examples. The canonical parameterization of $A \in \mathcal{O}'$ is given by the product $Q\rho Q^{-1}$.

Example 1. Let $N = 4$, $\lambda_i = \lambda_j \Leftrightarrow i = j$.

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_4 & 1 & 0 & 0 \\ q_5 & q_2 & 1 & 0 \\ q_6 & q_3 & q_1 & 1 \end{pmatrix}$$

$$\begin{aligned} Q^{-1} &= \left(\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -q_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -q_2 & 1 & 0 \\ 0 & -q_3 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q_4 & 1 & 0 & 0 \\ -q_5 & 0 & 1 & 0 \\ -q_6 & 0 & 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q_4 & 1 & 0 & 0 \\ -q_5 + q_4q_2 & -q_2 & 1 & 0 \\ -q_6 + q_5q_1 - q_4(-q_3 + q_1q_2) & -q_3 + q_1q_2 & -q_1 & 1 \end{pmatrix} \end{aligned}$$

The corresponding matrix ρ is

$$\begin{pmatrix} \lambda_4 & p_4 + p_5q_2 + p_6q_3 & p_5 + p_6q_1 & p_6 \\ 0 & \lambda_3 & p_2 + p_3q_1 & p_3 \\ 0 & 0 & \lambda_2 & p_1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

Example 2. Let $N = 5$, $\lambda_i = \lambda_j \Leftrightarrow i = j$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q_7 & 1 & 0 & 0 & 0 \\ q_8 & q_4 & 1 & 0 & 0 \\ q_9 & q_5 & q_2 & 1 & 0 \\ q_{10} & q_6 & q_3 & q_1 & 1 \end{pmatrix}$$

The corresponding matrix ρ is

$$\begin{pmatrix} \lambda_5 & p_7 + p_8q_4 + p_9q_5 + p_{10}q_6 & p_8 + p_9q_2 + p_{10}q_3 & p_9 + p_{10}q_1 & p_{10} \\ 0 & \lambda_4 & p_4 + p_5q_2 + p_6q_3 & p_5 + p_6q_1 & p_6 \\ 0 & 0 & \lambda_3 & p_2 + p_3q_1 & p_3 \\ 0 & 0 & 0 & \lambda_2 & p_1 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

Example 3. *The Jordan box 4×4 with zero eigenvalue*

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_4 & 1 & 0 & 0 \\ q_5 & q_2 & 1 & 0 \\ q_6 & q_3 & q_1 & 1 \end{pmatrix}, \rho = \begin{pmatrix} 0 & p_4 + p_5q_2 + p_6q_3 & p_5 + p_6q_1 & p_6 \\ 0 & 0 & p_2 + p_3q_1 & p_3 \\ 0 & 0 & 0 & p_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 4. *Let $N = 6$, $J =$*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ q_{10} & q_6 & 1 & 0 & 0 & 0 \\ q_{11} & q_7 & 0 & 1 & 0 & 0 \\ q_{12} & q_8 & q_4 & q_2 & 1 & 0 \\ q_{13} & q_9 & q_5 & q_3 & q_1 & 1 \end{pmatrix}$$

the corresponding matrix ρ is

$$\begin{pmatrix} 0 & 0 & p_{10} + p_{12}q_4 + p_{13}q_5 & p_{11} + p_{12}q_2 + p_{13}q_3 & p_{12} + p_{13}q_1 & p_{13} \\ 0 & 0 & p_6 + p_8q_4 + p_9q_5 & p_7 + p_8q_2 + p_9q_3 & p_8 + p_9q_1 & p_9 \\ 0 & 0 & 0 & 0 & p_4 + p_5q_1 & p_5 \\ 0 & 0 & 0 & 0 & p_2 + p_3q_1 & p_3 \\ 0 & 0 & 0 & 0 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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MPIM BONN, PDMI ST.PETERSBURG

E-mail address: mbabich@pdmi.ras.ru, misha.babich@gmail.com