Scalar-flat Kähler metrics with $SU(2)$ symmetry

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Abstract.

We study Kähler metrics of zero scalar curvature in four real dimensions admitting an isometric action of $SU(2)$.

0. Introduction.

A procedure for studying $SU(2)$-invariant anti-self-dual conformal structures on four-manifolds has recently been developed by Hitchin **[H2], [H3],[H4]**. The idea is to use the twistor correspondence of Penrose to asociate to such a four-manifold a complex three-fold (the twistor space) with a holomorphic $SU(2)$ action, and to show that the twistor lines of this space are determined by an isomonodromic family of connections on $\mathbb{P}^1$. In the generic case considered by Hitchin this family in turn is determined by a solution to the sixth Painlevé equation. This is one of a family of second order ODEs collectively known as the Painlevé equations, described for example in **[AC]**. The upshot is that the conformal structure on the four-manifold is specified by a solution of Painlevé VI.

On another front, Pedersen and Poon **[PP]** and Tod **[T]** have produced ansätze which give $SU(2)$-invariant scalar-flat Kähler metrics in four (real) dimensions. The ODEs arising from these ansätze are equivalent to a special case of the *third* Painlevé equation. It is known **[G]** that Kähler metrics of zero scalar curvature (in four real dimensions) are always anti-self-dual.

The purpose of this paper is to apply the twistor methods of Hitchin to attack the general problem of finding scalar-flat Kähler four-manifolds with $SU(2)$ symmetry, and to interpret the results of **[PP]** and **[T]** in this framework. In the special case of diagonal Bianchi IX metrics we are also able to tackle this problem by direct methods. We shall primarily work locally, although in the diagonal case we determine which of our metrics can be completed (even these complete examples are not compact). We assume throughout that the generic orbit of the $SU(2)$ action is three-dimensional.

Our techniques are not well suited to the case when the metrics are Ricci-flat. For Kähler metrics in four real dimensions this condition is equivalent to the metric being locally hyperkähler. However there is no loss in excluding this case because $SU(2)$-invariant hyperkähler four-metrics have been completely classified **[AH]**, **[GP]**.

The layout of the paper is as follows. Section 1 describes the Penrose twistor construction and its application by Hitchin to $SU(2)$-invariant conformal structures. In section 2 we explain how the extra data of a Kähler structure on the four-manifold affects the twistor space, and in section 3 we show how this leads to an isomonodromy problem involving Painlevé III. In section 4 we consider the special case of diagonal Bianchi IX metrics, and in 5 we compare these results with those obtained by direct calculations. Section 6 contains an analysis of which of the diagonal metrics are complete.

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1. Twistor spaces.

Let M be an oriented Riemannian four-manifold with metric g and let Λ^2_+ denote the bundle of selfdual 2-forms. We define Z to be the sphere bundle of Λ^2_+ and let π denote the projection from Z to M. We can identify the fibre of π over a point m of M with the set of complex structures on T_mM compatible with the metric g and with the orientation.

Using the Levi-Civita connection we can split the tangent space to Z at a point z into horizontal and vertical spaces H and V, where H can be identified with the tangent space to M at $\pi(z)$ and V is tangential to the two-sphere fibre of π over $\pi(z)$. Now z represents a complex structure on $T_{\pi(z)}M$ so induces a complex structure J_1 on H. Also we have the standard complex structure on S^2 which defines a complex structure J_2 on V. We can now define an almost complex structure J on Z by setting $J = J_1 \oplus J_2$.

The fundamental theorem of Penrose **[Pe]**, **[AHS]** asserts that this almost complex structure is integrable if and only if the metric on M is anti-self-dual. In this situation, Z is called the *twistor space* of M.

There is a complex four-parameter family of projective lines in Z, each with normal bundle isomorphic to $\mathcal{O}(1)\oplus\mathcal{O}(1)$; these lines are called the *twistor lines*. Moreover Z carries a free antiholomorphic involution τ, called the *real structure*, and there is a real four-parameter family of twistor lines, called the *real twistor lines*, which are preserved by τ. In fact the restriction of τ to each real twistor line is the antipodal map. The real twistor lines are precisely the fibres of the projection π from Z to M.

The above construction depends only on the conformal class of the metric g on M. Moreover, the complex manifold Z, its twistor lines, and the real structure together determine this conformal class.

Hitchin **[H2]**,**[H3]**,**[H4]** studies the case when there is an isometric orientation-preserving $SU(2)$ action on M. This lifts to a holomorphic $SU(2)$ action on Z with associated vector fields X_1, X_2, X_3 satisfying the relations $[X_1, X_2] = -X_3$ etc. These

vector fields generate a local $SL(2,\mathbb{C})$ action on the twistor space. Now, we have a holomorphic section s of K_Z^{-1}, the anticanonical bundle of Z, defined by $s = X_1 \wedge X_2 \wedge X_3$ Assuming that s is not identically zero, it will vanish precisely on some anticanonical divisor E on Z. We have the exact sequence

$$0 \to T\mathbb{P}^1 \to TZ\mid_{\mathbb{P}^1} \to N \to 0$$

where N is the normal bundle of a twistor line $\mathbb{P}^1$ and $T\mathbb{P}^1 \cong \mathcal{O}(2)$. Taking determinants, and using the description of the normal bundle given above, we see that $K_Z^{-1}\mid_{\mathbb{P}^1} \cong \mathcal{O}(4)$ so each twistor line is either included in E or meets E at four points, counted with multiplicity.

The holomorphic action of $SU(2)$ on the twistor space defines a homomorphism χ of vector bundles from the trivial bundle $Z \times \mathfrak{sl}(2,\mathbb{C})$ to the tangent bundle of Z. On $Z - E$ the vector fields X_1, X_2, X_3 are pointwise linearly independent over $\mathbb{C}$ and χ is an isomorphism. The inverse of χ may be viewed as an $\mathfrak{sl}(2,\mathbb{C})$-valued one form Φ on $Z - E$. If we use the local $SL(2,\mathbb{C})$ action to identify a neighbourhood of a point in $Z - E$ with an open set in $SL(2,\mathbb{C})/\Gamma$ for some discrete group Γ, then on this neighbourhood Φ pulls back to the Maurer-Cartan form under the projection $SL(2,\mathbb{C}) \mapsto SL(2,\mathbb{C})/\Gamma$.

We can regard Φ as a flat meromorphic connection on Z, holomorphic on $Z - E$ and with poles on E. Restricting to twistor lines not contained in E gives us a family of flat connections $A(t)$, each meromorphic on $\mathbb{P}^1$ with poles at four points (with multiplicity). Here t is a coordinate on a curve C transverse to the $SU(2)$ orbits in M, and we are considering the real twistor lines over points of C. In the case considered by Hitchin, corresponding to a generic anti-self-dual conformal structure, we do in fact have four simple poles for generic t. Moreover, because each such connection is a restriction of our flat connection Φ on Z, we see that the holonomy of the connections around the poles stays constant as we vary the connection within the family.

We can view the family of connections as defining a family of ordinary differential equations with four regular singularities on $\mathbb{P}^1$. The above statement about the holonomy remaining constant becomes, in this interpretation, the statement that this family

of differential equations is *isomonodromic*. This condition was analysed by Painlevé and his school **[AC]** and shown to be equivalent to the condition that a certain function defined in terms of the coefficients of the connection form $A(t)$ should satisfy the sixth Painlevé equation. Here t is the dependent variable of this differential equation.

Conversely, given a solution of a Painlevé VI equation we can recover the isomonodromic family of connections. This family in turn determines the restriction of the twistor lines to the open set $Z - E$ of the twistor space Z, and this is enough to determine the conformal structure of M.

To summarise, the conformal structure on M is determined by the solution of Painlevé VI.

2. The anticanonical divisor.

Let us now consider the case when M admits a scalar-flat metric g and a complex structure I with respect to which g is Kähler. As noted earlier, g must now be anti-selfdual so we can associate to M a twistor space. The twistor spaces of scalar-flat Kähler surfaces have been studied by Pontecorvo **[Po]**.

Now the complex structure I on M defines a global section of the twistor fibration, and the image of this section is a divisor D on Z intersecting each real twistor line once. Similarly the reverse complex structure $-I$ gives another divisor $\bar{D}$ intersecting each real twistor line at one point, and the two divisors are interchanged by the real structure τ of Z. In the case when g is Kähler with respect to I Pontecorvo shows that the divisor class of $D + \bar{D}$ is given by

$$[D + \bar{D}] = K_Z^{-1/2}$$

In fact, he explicitly constructs a holomorphic section of $K_Z^{-1/2}$ vanishing precisely on $D, \bar{D}$. We now relate the divisor E to Pontecorvo's divisor. As mentioned in the introduction, we are assuming throughout this paper that the generic $SU(2)$ orbit on the four-manifold is three-dimensional. We are interested principally in the local form

of the metric, so in what follows we assume, by restricting to an open set if necessary, that the orbits of $SU(2)$ on M are *all* three-dimensional.

Lemma 2.1

Let M be a four-manifold with an $SU(2)$-invariant scalar-flat Kähler metric g, not locally hyperkähler, where $SU(2)$ acts with three-dimensional orbits. Then the section s of K_Z^{-1} is not identically zero.

Proof

The vector fields X_1, X_2, X_3 on Z span an integrable distribution. If s is identically zero then these vector fields are everywhere linearly dependent so the rank of the distribution is of complex dimension two or less. As we have assumed that the $SU(2)$ orbits on M all have real dimension equal to three we see that the rank of the distribution is in fact precisely equal to two. For each point z in the twistor space we therefore have a complex surface Σ passing through that point, such that X_1, X_2, X_3 are tangential to Σ. These surfaces are just the orbits of the local $SL(2, \mathbb{C})$ action on Z generated by the vector fields X_i.

The members of this family of surfaces define (locally) a family of sections of the twistor fibration, which induces a local hyperhermitian structure on M. That is, we have complex structures I_1, I_2, I_3 multiplying like the quaternions, such that g is hermitian with respect to each of I_1, I_2, I_3; these generate a whole two-sphere of local hermitian structures $I_a = \sum a_i I_i$ where $a = (a_1, a_2, a_3)$ is a unit vector in $\mathbb{R}^3$.

The assumption that the metric is not hyperkähler implies that the space of covariant constant two-forms will have dimension less than three. However this space is a real representation of the isometry group $SU(2)$ and hence will be acted on trivially by this group. Hence the Kähler form Ω and complex structure I on M are preserved by the $SU(2)$ action. It follows that the lift to Z of the $SU(2)$ action preserves D, and so the vector fields generated by this action are tangential to D. We deduce that D is a union of members of the family of surfaces discussed above. It follows that one of the local hyperhermitian structures may be identified with I.

Now there is a unique torsion-free connection, the Obata connection ∇, such that

$$\nabla I_a = 0$$

for each $a \in S^2$.

As explained in **[PS]** , **[PPS]** there is a 1-form ω such that

$$\nabla g = \omega \otimes g$$

and moreover

$$d\Omega_a = \omega \wedge \Omega_a$$

where Ω_a is the two-form defined by g and I_a.

However we have seen that one of the hermitian structures (I_A say) equals I and so is Kähler. Therefore $d\Omega_A = 0$ and as Ω_A is nondegenerate this means that ω is zero. Hence $d\Omega_a = 0$ for all a and all the hermitian structures are Kähler.

The metric on M is therefore locally hyperkähler and we have the required contradiction. □

Theorem 2.2

Let M be a four-manifold with an $SU(2)$-invariant scalar-flat Kähler metric, which is not locally hyperkähler; suppose also that $SU(2)$ acts with three-dimensional orbits. Then the divisor E where s vanishes is equal to $2D + 2\bar{D}$.

Proof

Note first that because s is invariant under the $SU(2)$ action on Z, and because the $SU(2)$ orbits on M are three-dimensional, if s vanishes on a real twistor line then it vanishes everywhere. It follows from Lemma 2.1 that this contradicts the hypotheses of the theorem, so we deduce that E contains no real twistor line and hence E meets each real twistor line at four points counted with multiplicity.

As shown in Lemma 2.1, the $SU(2)$ action on Z preserves the divisor D, and so the holomorphic vector fields X_1, X_2, X_3 are tangential to D, and hence are linearly

dependent over $\mathbb{C}$. It follows that our section s of K_Z^{-1} vanishes on D (and, using a similar argument, on $\tilde{D}$). Note that this argument shows that s vanishes on any $SU(2)$-invariant divisor on Z.

We may now write the anticanonical divisor E where s vanishes as $D + \tilde{D} + F$ for some divisor F. If ρ_1 denotes Pontecorvo's holomorphic section of $K_Z^{-1/2}$ vanishing on $D, \tilde{D}$, then $\rho_2 = s/\rho_1$ is a holomorphic section of $K_Z^{-1/2}$ vanishing on F, so $[F] = K_Z^{-1/2}$. As $E, D, \tilde{D}$ are $SU(2)$-invariant we see that F is also $SU(2)$-invariant.

Assume now that $F \neq D + \tilde{D}$. From above we then have distinct holomorphic sections ρ_1, ρ_2 of $K_Z^{-1/2}$ with divisors $D + \tilde{D}$ and F respectively. Now as $D, \tilde{D}$ are $SU(2)$-invariant , the image of ρ_1 under the $SU(2)$ action has the same divisor as ρ_1, so must be of the form $\rho_1' = f\rho_1$ where f is a nowhere zero holomorphic function on Z. However any holomorphic function on a twistor space is constant **[H1]**, so ρ_1' must be a constant multiple of ρ_1. Hence the one-dimensional subspace of $H^0(Z, K_Z^{-1/2})$ spanned by ρ_1 is a representation of $SU(2)$ and so is acted on trivially by $SU(2)$. In particular ρ_1 is $SU(2)$-invariant. The same argument shows that ρ_2 is also $SU(2)$-invariant.

Consider now the two-dimensional space of $SU(2)$-invariant sections of $K_Z^{-1/2}$ spanned by ρ_1, ρ_2. As above, two such sections define the same divisor if and only if they agree up to a constant, so we obtain a pencil of distinct $SU(2)$-invariant divisors in the linear system $|\, K_Z^{-1/2}\,|$. But s must vanish on every member of this pencil, so will vanish on some twistor line at more than four points (counted with multiplicity), thus contradicting the hypotheses of the theorem. It follows that our assumption that F and $D + \tilde{D}$ are unequal is false, and the theorem is proved. □

3. The isomonodromic family.

In the light of Theorem 2.2, we know that our flat meromorphic connection Φ has double poles on $D, \tilde{D}$ and is holomorphic elsewhere on Z. We shall next show that restriction to the real twistor lines defines an isomonodromic family of connections, leading to the appearance of the third Painlevé equation.

Let C be a curve in M transverse to the $SU(2)$ orbits, and consider the region U of the twistor space projecting onto C. Let t be a coordinate along C and x a coordinate on the real twistor lines such that $D, \tilde{D}$ meet each twistor line at $0, \infty$ respectively.

The restriction of our connection form to U is given in these coordinates by

$$-A\,dx - B\,dt$$

where A, B are $\mathfrak{sl}(2,\mathbb{C})$-valued functions of x, t.

From our comments above about the poles of Φ we can write A, B as

$$A = A_0 + \frac{A_{-1}}{x} + \frac{A_{-2}}{x^2}$$

and

$$B = B_2 x^2 + B_1 x + B_0 + \frac{B_{-1}}{x} + \frac{B_{-2}}{x^2}$$

where A_i, B_i are functions of t taking values in $\mathfrak{sl}(2,\mathbb{C})$.

We shall assume that there is a range of t where the eigenvalues of A_0 and A_{-2} are nowhere zero, and restrict ourselves to this range in the following calculations.

By rescaling x we can take A_0 and A_{-2} to have the same eigenvalues and hence be conjugate. If the eigenvalues of A_0, A_{-2} are constant in t it is easy to show, using the flatness of $-A\,dx - B\,dt$, that we can gauge B to be zero and A to be constant in t. Excluding this trivial case, and restricting the range of t if necessary, we can choose t so that the eigenvalues of A_0 and A_{-2} are $\frac{1}{2}t, -\frac{1}{2}t$. Moreover, by a choice of gauge we can take A_0 to be diagonal.

We write A_0, A_{-1}, A_{-2} as

$$A_0 = \begin{pmatrix} \frac{1}{2}t & 0 \\ 0 & -\frac{1}{2}t \end{pmatrix}, A_{-1} = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}, A_{-2} = t \begin{pmatrix} u & v \\ w & -u \end{pmatrix} \qquad (1)$$

where

$$u^2 + vw = \frac{1}{4}. \tag{2}$$

We can perform a gauge transformation by a diagonal matrix in $SL(2,\mathbb{C})$ (depending on t) to ensure that

$$B_0 = \begin{pmatrix} t^{-1}p & \delta \\ \epsilon & t^{-1}p \end{pmatrix} \tag{3}$$

for some functions δ, ϵ of t.

We say that A, B are in *canonical form* if (1)-(3) are satisfied.

The flatness condition for the connection is

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + [A, B] = 0. \tag{4}$$

This is the isomonodromy condition for the ordinary differential equation

$$\frac{d\Xi}{dx} = \left(A_0 + \frac{A_{-1}}{x} + \frac{A_{-2}}{x^2}\right)\Xi$$

and it is well known that this condition is equivalent to the Painlevé III equation. We now give details of an argument to show this equivalence.

Equation (4) is equivalent to the relations

$$\begin{aligned}
[A_{-2}, B_{-2}] &= 0 && (5)\\
2B_{-2} + [A_{-2}, B_{-1}] + [A_{-1}, B_{-2}] &= 0 && (6)\\
\frac{dA_{-2}}{dt} + B_{-1} + [A_0, B_{-2}] + [A_{-1}, B_{-1}] + [A_{-2}, B_0] &= 0 && (7)\\
\frac{dA_{-1}}{dt} + [A_0, B_{-1}] + [A_{-1}, B_0] + [A_{-2}, B_1] &= 0 && (8)\\
\frac{dA_0}{dt} - B_1 + [A_0, B_0] + [A_{-1}, B_1] + [A_{-2}, B_2] &= 0 && (9)\\
-2B_2 + [A_0, B_1] + [A_{-1}, B_2] &= 0 && (10)\\
[A_0, B_2] &= 0. && (11)
\end{aligned}$$

Let us now analyse these equations.

The first and last equations imply that $B_{-2} = \psi A_{-2}$ and $B_2 = \phi A_0$, where ϕ and ψ are scalar-valued functions of t. Multiplying (10) by A_0 and (6) by A_{-2} and taking the trace implies that ϕ and ψ (and hence B_2 and B_{-2}) are both zero.

Equations (6) and (10) now show that $B_{-1} = \mu A_{-2}$ and $B_1 = \lambda A_0$, where λ, μ are scalar-valued functions of t.

We obtain the equations

$$\frac{dA_{-2}}{dt} + \mu A_{-2} + \mu[A_{-1}, A_{-2}] + [A_{-2}, B_0] = 0 \tag{12}$$

$$\frac{dA_{-1}}{dt} + \mu[A_0, A_{-2}] + [A_{-1}, B_0] + \lambda[A_{-2}, A_0] = 0 \tag{13}$$

$$\frac{dA_0}{dt} - \lambda A_0 + [A_0, B_0] + \lambda[A_{-1}, A_0] = 0. \tag{14}$$

The last equation implies that $\lambda = t^{-1}$ (so $B_1 = t^{-1}A_0$) and that

$$B_0 = \begin{pmatrix} t^{-1}p & t^{-1}q \\ t^{-1}r & -t^{-1}p \end{pmatrix} = t^{-1}A_{-1}.$$

Observe from (12) that

$$\frac{d}{dt}\mathrm{Tr}A_{-2}^2 = -2\mu\mathrm{Tr}A_{-2}^2.$$

But we know that $\mathrm{Tr}A_{-2}^2 = \frac{1}{4}t^2$, so $\mu = -t^{-1}$ and $B_{-1} = -t^{-1}A_{-2}$.

The flatness equation (4) now becomes

$$\begin{aligned} \frac{dQ}{dt} &= 2t[P, R] \\ t\frac{dR}{dt} &= 2[R, Q] \end{aligned}$$

where $P = -t^{-1}A_0$, $Q = -A_{-1}$ and $R = t^{-1}A_{-2}$.

This system of equations (together with the condition $\frac{dP}{dt} = 0$ which is automatic in our case) is a reduction of the self-dual Yang-Mills equations and has been studied by Mason and Woodhouse **[MW]**. They show that it is equivalent (if neither r nor w is identically zero) to the third Painlevé equation

$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{\kappa_1 y^2 + \kappa_2}{t} + \kappa_3 y^3 + \frac{\kappa_4}{y}$$

where y is defined by $y = -\frac{r}{wt}$.

The constant parameters κ_i $(i = 1, 2, 3, 4)$ are given by

$$\kappa_1 = 8\ \mathrm{Tr}(A_{-1}A_{-2}/t) \tag{15}$$

$$\kappa_2 = 4 - 8\ \mathrm{Tr}(A_0 A_{-1}/t) \tag{16}$$

$$\kappa_3 = 4 \tag{17}$$

$$\kappa_4 = -4. \tag{18}$$

Any Painlevé III equation with κ_3 and κ_4 nonzero may be brought to this form by scaling y and t.

If r or w is identically zero we can use the substitution $y = -\frac{q}{vt}$ to obtain Painlevé III, unless q or v is identically zero, when the flatness equation (4) becomes trivial. In the latter case both r, w, q and v are all in fact identically zero.

The next theorem summarises our findings.

Theorem 3.1

Let M be a scalar-flat Kähler, not locally hyperkähler, manifold of real dimension four, admitting an isometric action of $SU(2)$ with three-dimensional orbits. Suppose that the matrices A_0, A_{-2} of (1) have nonzero, nonconstant eigenvalues, and that r, w, q and v are not all identically zero. Then we have an isomonodromic deformation problem leading to the Painlevé III equation. □.

Finally, it is straightforward to show that matrices A_i, B_j $(i = 0, 1, 2; j = -2, -1, 0, 1, 2)$ satisfying equations (5-11) and such that A, B are in canonical form are determined by $y = -\frac{r}{wt}$ up to a gauge transformation.

$$A_i \to \Theta A_i \Theta^{-1} \quad , \quad B_j \to \Theta B_j \Theta^{-1} \tag{19}$$

where Θ is a diagonal $SL(2,\mathbb{C})$-valued matrix, constant in x and t.

Therefore our solution of Painlevé III determines the connexion

$$\Phi = -A\,dx - B\,dt$$

up to constant gauge transformations.

Now, the real twistor lines in U are embedded into Z by a family of maps f_t. Equivalently we have a map from $C \times \mathbb{P}^1$ into Z given by $(t,x) \mapsto f_t(x)$. Restricting, we get a map

$$\mathcal{F} : C \times (\mathbb{P}^1 - \{0,\infty\}) \to Z - E$$

Moreover the restriction of $-Adx - Bdt$ to the domain of $\mathcal{F}$ is the pullback by $\mathcal{F}$ of the restriction of Φ to $Z - E$.

Locally we can identify Φ with the restriction of the Maurer-Cartan form to an open set in $SL(2,\mathbb{C})$. Under this identification $\mathcal{F}$ is a fundamental solution for the equations

$$\frac{\partial \mathcal{F}}{\partial x} = -\mathcal{F}A \tag{20}$$

$$\frac{\partial \mathcal{F}}{\partial t} = -\mathcal{F}B. \tag{21}$$

The general real twistor line in Z is given by

$$x \mapsto \mathcal{F}(x,t)G$$

for some t and some $G \in SU(2)$.

We see that $\Phi = -A\,dx - B\,dt$ determines $\mathcal{F}$ up to premultiplication by a constant matrix in $SL(2,\mathbb{C})$. Also, conjugating Φ by a constant matrix Θ as in (19) just corresponds to postmultiplying $\mathcal{F}$ by Θ^{-1}.

We can view t and G as giving coordinates on the four-manifold M. An element $(\dot{t},\dot{G})$ of the complexified tangent space at (t,G) may be identified with an infinitesimal deformation

$$\frac{\partial \mathcal{F}}{\partial t} G\dot{t} + \mathcal{F}\dot{G}$$

of the twistor line. Here $\dot{t} \in \mathbb{C}$ and $\dot{G} \in \mathfrak{sl}(2, \mathbb{C})$.

We can rewrite this deformation using (20),(21) as

$$-\mathcal{F}BG\dot{t} + \mathcal{F}\dot{G}. \tag{22}$$

The conformal structure is defined by declaring the tangent vector to be null if and only if the projection of the deformation (22) onto the normal bundle of the real twistor line vanishes for some x. This is equivalent to (22) being tangential to the real twistor line for some x. Now the tangent vectors to the real twistor line are multiples of $(\partial\mathcal{F}/\partial x)G = -\mathcal{F}AG$, so we see that $(\dot{t}, \dot{G})$ is null if and only if

$$-B\dot{t} + \dot{G}G^{-1} = -\Lambda A \tag{23}$$

for some x, Λ.

It is clear that conjugating Φ by the constant matrix Θ just corresponds to a reparametrisation of the conformal metric by $G \mapsto \Theta G$, so we have the following result.

Theorem 3.2

The conformal structure is determined by the solution y of Painlevé III, where $y = -\frac{r}{wt}$ and r, w are as in (1). □

4. The diagonal case.

In this section we shall consider the special case when the metric can be put in the form

$$g = h^2 dT^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{24}$$

where $\sigma_1, \sigma_2, \sigma_3$ are invariant one-forms, satisfying the relations $d\sigma_1 = \sigma_2 \wedge \sigma_3$ and cyclically, T is a coordinate orthogonal to the orbits of $SU(2)$, and a, b, c, h are functions of T only. We refer to this as diagonal Bianchi IX form. We can choose the coordinate T so that $h = abc$.

Let us now apply the techniques of the preceding sections to this case. Because the metric is in the diagonal form (24) we have, for each point m in M, a copy of the four-group $V_4 \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ which preserves the metric g and fixes m. The non-identity elements of the group change the signs of two of the one-forms $\sigma_1, \sigma_2, \sigma_3$. This action lifts to a holomorphic effective action of V_4 on the twistor space preserving the real twistor line over m. So we have an injection of V_4 into the group of Möbius transformations of this line.

As before, we choose a coordinate x on the twistor line so that $D, \tilde{D}$ intersect the line at $0, \infty$ respectively.

There are two possibilities; either V_4 fixes the Kähler form or else two of the order two elements change the sign of the Kähler form and the third order two element fixes the form. But the first possibility means that the V_4 action on the twistor line must fix $0, \infty$, and the only Möbius transformations with square equal to the identity which fix these two points are $x \to x$ and $x \to -x$. This contradicts the effectiveness of the action.

So we must have two order two elements interchanging 0 and ∞ , and two elements fixing these two points. This means that on the twistor line the action of the four-group must be by the Möbius transformations

$$\begin{aligned} x &\to x \\ x &\to -x \\ x &\to \frac{K}{x} \\ x &\to -\frac{K}{x} \end{aligned}$$

for some $K \in \mathbb{C}^*$.

The $\mathfrak{sl}(2,\mathbb{C})$-valued one-form Φ transforms by the adjoint representation under the V_4 action. So, restricting to a twistor line, we see that the connection form $-A(x,t)\,dx$ will be conjugated by an element of $SU(2)$ when we apply the Möbius transformations above.

This implies that there exist Θ_1, Θ_2 such that

$$\Theta_1(-A_0 - \frac{A_{-1}}{x} - \frac{A_{-2}}{x^2})\Theta_1^{-1} = A_0 - \frac{A_{-1}}{x} + \frac{A_{-2}}{x^2}$$

and

$$\Theta_2(-A_0 - \frac{A_{-1}}{x} - \frac{A_{-2}}{x^2})\Theta_2^{-1} = K\frac{A_0}{x^2} + \frac{A_{-1}}{x} + K^{-1}A_{-2}.$$

We deduce that $\mathrm{Tr}\,A_0A_{-1}$ and $\mathrm{Tr}\,A_{-1}A_{-2}$ are zero. Referring to our expressions (15)-(18) for the parameters of the Painlevé equation in terms of A_0, A_{-1} and A_{-2}, we have the following result.

Theorem 4.1

The Painlevé III equation arising from a scalar-flat Kähler, not locally hyperkähler metric in diagonal Bianchi IX form has parameters $\kappa_1 = 0, \kappa_2 = 4, \kappa_3 = 4, \kappa_4 = -4$. □.

Remark

Tod [**T**] has found an ansatz to produce scalar-flat Kähler metrics with S^1 action. A special case of this ansatz gives diagonal Bianchi IX solutions. The differential equation produced in this case is a rescaled version of the above Painlevé III equation.

If the metric is of diagonal Bianchi IX form we can find the conformal structure explicitly in terms of Painlevé transcendents.

As in section 3, we parametrise the manifold underlying our metric by (t, G) where $G \in SU(2)$ and t is a coordinate transverse to the $SU(2)$ orbits.

We saw that if $(\dot{t}, \dot{G})$ (where $t \in \mathbb{C}$ and $\dot{G} \in \mathfrak{sl}(2,\mathbb{C})$) is an element of the complexified tangent space, then it is null if and only if

$$-B\dot{l} + \dot{G}G^{-1} = -\Lambda A$$

for some x, Λ. The matrices A, B are as in §3.

Using our explicit expressions for A, B from section 3 we find after some calculation that the conformal class of the (real) metric is represented by

$$2\,dt^2 + \frac{\sigma_1^2}{2\Psi - 1} + \frac{\sigma_2^2}{2\Psi + 1} + \frac{8(\Psi^2 - \frac{1}{4})\sigma_3^2}{(\frac{d\Psi}{dt})^2}$$

where $\Psi = -u$ and u is as in (1) of section 3. We can relate Ψ to the Painlevé transcendent by remarking that the equations (12-14) imply that

$$y = \frac{1}{4}(\frac{\Psi - \frac{1}{2}}{\Psi + \frac{1}{2}})^{-1}\frac{d}{dt}(\frac{\Psi - \frac{1}{2}}{\Psi + \frac{1}{2}})$$

where y is our solution to Painlevé III.

5. A direct approach.

In the case of diagonal Bianchi IX metrics we can also classify the scalar-flat Kähler examples by direct methods. These involve a result used in **[DS]**, which we now recall.

Suppose that we have a Kähler, non-hyperkähler, diagonal Bianchi IX metric expressed in the form

$$g = (abc)^2 dT^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{25}$$

where a, b and c are functions of T, and σ_i are the invariant one-forms satisfying $d\sigma_1 = \sigma_2 \wedge \sigma_3$ etc.

We have an orthonormal coframe for g given by $e_0 = abc\,dt, e_1 = a\sigma_1, e_2 = b\sigma_2, e_3 = c\sigma_3$, and we choose the orientation so that $\Omega^+ = e_0 \wedge e_1 + e_2 \wedge e_3$ and the forms obtained from Ω^+ by cyclically permuting $(1, 2, 3)$ are self-dual.

We make the standard change of variables

$$
\begin{aligned}
w_1 &= bc \\
w_2 &= ac \\
w_3 &= ab
\end{aligned}
$$

and define functions α, β, γ by

$$
\begin{aligned}
w_1' &= w_2 w_3 + \alpha w_1 \\
w_2' &= w_3 w_1 + \beta w_2 \\
w_3' &= w_1 w_2 + \gamma w_3
\end{aligned}
$$

Theorem 5.1 [DS]

If the metric g of (25), with our choice of orientation, is Kähler and non-hyperkähler then one of the following three statements is true.

(i) $\alpha = 0, \beta = \gamma$

(ii) $\beta = 0, \gamma = \alpha$

(iii) $\gamma = 0, \alpha = \beta$.

If one of these statements holds then the metric is Kähler. The Kähler forms are

(i) $\Omega = w_2 w_3 \, dT \wedge \sigma_1 + w_1 \sigma_2 \wedge \sigma_3$

(ii) $\Omega = w_3 w_1 \, dT \wedge \sigma_2 + w_2 \sigma_3 \wedge \sigma_1$

(iii) $\Omega = w_1 w_2 \, dT \wedge \sigma_3 + w_3 \sigma_1 \wedge \sigma_2$

respectively. □

Hyperkähler structures with an $SU(2)$ action preserving each Kähler form correspond precisely to the case $\alpha = \beta = \gamma = 0$.

In the last section we considered the action of V_4 on the Kähler forms. It is clear from the expressions given in Theorem 5.1 that there will be two elements of V_4 fixing Ω and two elements changing the sign of Ω, in accordance with our previous discussion.

Let us now require that the scalar curvature is zero. Using the expressions of Pedersen-Poon **[PP]** for the connection forms of the metric we can calculate the Riemann curvature tensor and hence the scalar curvature. We find that the latter quantity is given by

$$\frac{-1}{4w_1w_2w_3}(2\alpha' + 2\beta' + 2\gamma' + \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\alpha\gamma). \qquad (26)$$

Comparing with the expressions of (i),(ii),(iii) in the statement of Theorem 5.1, we find that the scalar-flat condition forces α, β, γ to be constant.

We summarise our results as follows.

Theorem 5.2

The general scalar-flat Kähler, non-hyperkähler metric of diagonal Bianchi IX form compatible with our choice of orientation is given (up to permutations of w_1, w_2, w_3) by

$$g = w_1w_2w_3\, dT^2 + \frac{w_2w_3}{w_1}\sigma_1^2 + \frac{w_3w_1}{w_2}\sigma_2^2 + \frac{w_1w_2}{w_3}\sigma_3^2$$

where

$$\begin{aligned} w_1' &= w_2w_3 + \alpha w_1 \\ w_2' &= w_3w_1 + \alpha w_2 \\ w_3' &= w_1w_2 \end{aligned}$$

and α is a nonzero constant.

□

Note

Choosing the opposite orientation just corresponds to reversing the sign of T.

Remark

The metrics of Theorem 5.2 are precisely those arising from the Pedersen-Poon ansatz **[PP]**. So we have shown that this ansatz produces *all* scalar-flat Kähler non-hyperkähler diagonal Bianchi IX metrics compatible with our choice of orientation.

As remarked in [**PP**], the metrics with $w_1 = w_2$ are the $U(2)$-invariant examples of LeBrun [**L**].

Let us compare the metrics of Theorem 5.2 with those which we obtained using the method of isomonodromic deformations in section 4.

As explained in [**PP**], if one makes the substitutions

$$w_1 = e^{\alpha T} F_1, \quad w_2 = e^{\alpha T} F_2, \quad w_3 = F_3,$$

$$F_1 = \frac{\Delta}{2}(\tilde{y} + \frac{1}{\tilde{y}}), \quad F_2 = \frac{\Delta}{2}(\tilde{y} - \frac{1}{\tilde{y}})$$

and

$$\zeta = \alpha^{-1} e^{\alpha T}$$

in the system of equations given in Theorem 5.2, then Δ is constant and $\tilde{y}$ satisfies

$$\frac{d^2\tilde{y}}{d\zeta^2} = \frac{1}{\tilde{y}}\left(\frac{d\tilde{y}}{d\zeta}\right)^2 - \frac{1}{\zeta}\frac{d\tilde{y}}{d\zeta} + \frac{1}{4}\Delta^2(\tilde{y}^3 - \frac{1}{\tilde{y}}).$$

This is the Painlevé III equation with parameters

$$\kappa_1 = \kappa_2 = 0, \kappa_3 = -\kappa_4 = \frac{1}{4}\Delta^2.$$

Of course, this substitution is only valid where $w_1 \neq w_2$, but in fact it follows from the equations that if w_1 and w_2 are not identically equal then they are never equal, so the above procedure is valid except in the special case $w_1 \equiv w_2$ (when we get the metrics of [**L**]).

Now, if we change variables by

$$z = -\frac{1}{2}\Delta^2\zeta^2, \quad Y = 1 + \frac{4}{\tilde{y}^2 + \tilde{y}^{-2} - 2},$$

we arrive at the equation

$$\frac{d^2Y}{dz^2} = \left(\frac{1}{2Y} + \frac{1}{Y-1}\right)\left(\frac{dY}{dz}\right)^2 - \frac{1}{z}\frac{dY}{dz} + \frac{Y}{z},$$

which is Painlevé V with parameters

$$\bar{\kappa_1} = \bar{\kappa_2} = \bar{\kappa_4} = 0, \bar{\kappa_3} = 1.$$

(We are lead to this change of variables by a comparison of the calculations of **[PP]** and **[T]**).

In these variables the metric is

$$\frac{dY}{dz}\left(\frac{dz^2}{2(Y-1)^2} - \frac{z}{Y(Y-1)}\sigma_1^2 - \frac{z}{Y-1}\sigma_2^2\right) + 2Y\left(\frac{dY}{dz}\right)^{-1}\sigma_3^2.$$

Letting

$$z = 2t^2 \quad , \quad Y = \frac{\Psi - \frac{1}{2}}{\Psi + \frac{1}{2}},$$

we find that the metric is

$$t\frac{d\Psi}{dt}\left(2dt^2 + \frac{\sigma_1^2}{2\Psi - 1} + \frac{\sigma_2^2}{2\Psi + 1} + \frac{8(\Psi^2 - \frac{1}{4})\sigma_3^2}{(\frac{d\Psi}{dt})^2}\right)$$

and

$$y = \frac{1}{4}Y^{-1}\frac{dY}{dt} = \frac{1}{4}\left(\frac{\Psi - \frac{1}{2}}{\Psi + \frac{1}{2}}\right)^{-1}\frac{d}{dt}(\frac{\Psi - \frac{1}{2}}{\Psi + \frac{1}{2}})$$

satisfies Painlevé III with parameters.

$$\kappa_1 = 0, \kappa_2 = 4, \kappa_3 = 4, \kappa_4 = -4.$$

This agrees with the expression for the conformal structure we derived by twistor methods in section 4.

6. Completeness analysis.

In the case of diagonal Bianchi IX metrics we can classify the complete scalar-flat Kähler examples. This analysis has already been performed for hyperkähler metrics **[GP],[AH]** so, as in the rest of this paper, we exclude this case.

Theorem 5.2 gives us a description of the metrics we are interested in. It is convenient to cast the equations of Theorem 5.2 in terms of a, b, c. They become

$$a' = \frac{1}{2}a(b^2 + c^2 - a^2) \tag{27}$$

$$b' = \frac{1}{2}b(c^2 + a^2 - b^2) \tag{28}$$

$$c' = \frac{1}{2}c(a^2 + b^2 - c^2 + 2\alpha). \tag{29}$$

As we are assuming the metric is not hyperkähler we take the constant α to be nonzero.

Consider a solution to (27-29), analytic on a maximal interval (ξ, η). It is clear from the equations that if any one of a, b, c is zero at some point in this open interval, then it is identically zero. As this will not give a metric with three-dimensional orbits we can exclude this case, and hence assume that a, b, c are nowhere zero in (ξ, η). It follows that the metric will be defined for $T \in (\xi, \eta)$, so to decide whether it is complete we need to study the behaviour of a, b, c as T approaches ξ from above and η from below.

As the equations and metric are invariant under sign changes of a, b, c we shall from now on take a, b, c to be nonnegative on (ξ, η).

We record some useful facts about the equations (27-29) in the next lemma.

Lemma 6.1

$$\begin{aligned}
(ab)' &= abc^2 \\
(bc)' &= bc(a^2 + \alpha) \\
(ac)' &= ac(b^2 + \alpha) \\
(a^2 - b^2)' &= (a^2 - b^2)(c^2 - a^2 - b^2) \\
(\frac{a}{b})' &= \frac{a}{b}(b^2 - a^2) \\
(abc)' &= \frac{1}{2}abc(a^2 + b^2 + c^2 + 2\alpha).
\end{aligned}$$

The critical points of the equations (27-29) are the points (a, b, c) satisfying

(i) $a = b, c = 0$ or

(ii) $a = b = 0, c = \sqrt{2\alpha}$ (if $\alpha > 0$).

□

We see that either a is identically equal to b or else a is never equal to b on (ξ, η). In the former case the metrics are those of LeBrun [**L**]. If a is never equal to b then, by the symmetry of the equations, we can without loss of generality take $a > b$.

From the above remarks, we can take $a \geq b$. It follows from (27-29) and Lemma 6.1 that b and ab are increasing; moreover $\frac{a}{b}$ is greater than or equal to 1 and either identically equal to 1 or else strictly decreasing on (ξ, η). In particular, note that if $\frac{a}{b}$ tends to 1 as T tends to ξ, then a is identically equal to b.

We first consider the situation when α is positive.

It follows from Lemma 6.1 that b, ab, ac, bc are increasing on (ξ, η).

Case 1. Suppose that ξ is finite. From Lemma 6.1 we see that abc is increasing on (ξ, η) , so tends to a finite limit as $T \to \xi$. Hence the geodesic distance

$$\int_{\xi}^{T} abc$$

is finite, and to get a complete metric we would have to add a nut (point orbit of $SU(2)$) or bolt (two-dimensional orbit of $SU(2)$) at $T = \xi$. In the former case we would have $a, b, c = 0$ at ξ; in the latter case one of a, b, c would be zero and the other two would attain nonzero finite limits at ξ. Both cases would force at least one of a, b, c to be identically zero, giving a contradiction.

Case 2. Assume instead that the maximal interval is $(-\infty, \eta)$.

As b, ab, bc, ac are increasing they tend to finite nonnegative limits $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ as T tends to $-\infty$.

(i) If the limit λ_1 of b is nonzero then a, c also tend to finite limits, so (a, b, c) tends to a critical point (μ, λ_1, ν) with λ_1 positive. By Lemma 6.1 we must have $\lambda_1 = \mu$

and $\nu = 0$, so $\frac{a}{b}$ tends to 1 as T tends to $-\infty$. From our comments following Lemma 6.1 we deduce that a is identically equal to b. The trajectory is an unstable curve of the critical point $(\mu, \mu, 0)$.

(ii) Let us assume instead that $\lambda_1 = 0$, that is, b tends to 0 as T tends to $-\infty$. Using Lemma 6.1 we now find that $(ac)'$ tends to $\lambda_4 \alpha$ as T tends to $-\infty$. However we know that ac tends to a finite limit as T tends to $-\infty$ so, as α is nonzero, it follows that $\lambda_4 = 0$; that is, ac tends to 0 as T tends to $-\infty$.

If $b^2 + c^2 - a^2$ is negative at some T_0 then it follows from the equations that the derivative of this expression is positive at T_0. Hence $b^2+c^2-a^2$ is negative on $(-\infty, T_0)$ and b, c are increasing and a decreasing on this interval. We deduce that

$$a \to \nu, b \to 0, c \to \mu \text{ as } T \to -\infty$$

where ν is positive and may be ∞, while μ is nonnegative and finite.

Since ac tends to zero as T tends to $-\infty$ we have $\mu = 0$. By considering the limit of a' as T tends to $-\infty$ we find that $\nu = \infty$ and it readily follows from the equations that a becomes infinite at a finite value of T less than T_0, giving a contradiction.

Similar arguments show that if $a^2 + b^2 - c^2 + 2\alpha$ is negative at some point T_0 then c becomes infinite at a finite value of T less than T_0, again giving a contradiction.

We see, therefore, that we need

$$a^2 \leq b^2 + c^2, c^2 \leq a^2 + b^2 + 2\alpha$$

on $(-\infty, \eta)$.

This implies that a, b, c are increasing so tend to finite limits as T tends to $-\infty$. Therefore (a, b, c) tends to a critical point. Moreover, recall that b tends to 0 so this critical point is either $(0, 0, 0)$ or $(0, 0, \sqrt{2\alpha})$. If the critical point is $(0, 0, 0)$ then the equations (27-29) imply that abc decays exponentially fast as T tends to $-\infty$. Hence the geodesic distance

$$\int_{-\infty}^{T} abc$$

to $-\infty$ is finite and to obtain a complete metric we must have a nut at $T = -\infty$. This means that as T tends to $-\infty$ the terms a, b and c all tend to zero like $\frac{1}{4}$ times the square of the geodesic distance, and it is easy to see that this is inconsistent with the equations.

If, on the other hand, the critical point is $(0, 0, \sqrt{2\alpha})$ then for T large and negative we have

$$a \simeq \tau_1 e^{\alpha T}, b \simeq \tau_2 e^{\alpha T}, c \simeq \sqrt{2\alpha}$$

and it easily follows that the metric is incomplete.

To recapitulate, if $\alpha > 0$ there are no complete metrics except those arising from unstable curves of $(\mu, \mu, 0)$ for positive μ. Such metrics all have a identically equal to b.

Let us now consider the situation when α is negative.

Case 1. We suppose first that ξ is finite.

As above, b and ab are increasing so tend to finite limits λ_1, λ_2 as T tends to ξ.

(i) First assume $\lambda_1 > 0$. We deduce that a tends to a finite positive limit also, as T tends to ξ.

If $\lambda_1^2 + \alpha \geq 0$ then ac is increasing so ac, and hence c, tend to finite limits at ξ. This contradicts the fact that ξ is a singularity.

If $\lambda_1^2 + \alpha < 0$ then ac is decreasing for T near ξ so ac tends to a limit , possibly ∞, as T tends to ξ. As ξ is a singularity this limit is in fact ∞ and c tends to ∞ at ξ. Therefore abc is increasing near ξ so tends to a finite limit at ξ. It follows that the geodesic distance

$$\int_{\xi}^{T} abc$$

is finite, so the metric is incomplete.

(ii) Suppose on the other hand that $\lambda_1 = 0$, that is, b tends to 0 as T tends to ξ. It follows that $(ac)' \simeq ac\alpha$ near ξ and ac tends to a finite limit at ξ. Hence abc

tends to 0 at ξ and again the geodesic distance is finite. To obtain a complete metric we must have a nut or bolt at $T = \xi$, and as in the case of positive α this leads to a contradiction.

Case 2. Let us consider, therefore, the case when the trajectory is defined on $(-\infty, \eta)$.

Again b, ab tend to finite limits λ_1, λ_2 as T tends to $-\infty$.

(i) First suppose that $\lambda_1 > 0$, so that a tends to a finite positive limit.

Arguing as before we see that if $\lambda_1^2 + \alpha \geq 0$ then c also tends to a finite limit. So (a, b, c) tends to a critical point which must be $(\mu, \mu, 0)$ for some $\mu > 0$. As before this implies that a is identically equal to b.

If, on the other hand, $\lambda_1^2 + \alpha < 0$ then we find that either (a, b, c) tends to a critical point (in which case $a \equiv b$) or else c tends to ∞. The latter case implies that c becomes infinite at a finite value of T less than η, giving a contradiction.

(ii) Suppose instead that $\lambda_1 = 0$, so b tends to 0 as T tends to $-\infty$. As $(ac)' \simeq ac\alpha$ and α is negative we see that ac tends to ∞ as T tends to $-\infty$.

If $a^2 + b^2 - c^2 < 0$ at T_0 then its derivative is positive at T_0 and so $a^2 + b^2 - c^2 < 0$ on $(-\infty, T_0)$. It follows that a, b are increasing and c is decreasing on $(-\infty, T_0)$. Combined with the fact that ac tends to ∞ as T tends to $-\infty$, this implies that

$$a \to \mu, b \to 0, c \to \infty \text{ as } T \to -\infty$$

where μ is finite. It easily follows that our solution cannot be finite on all of $(-\infty, T_0)$, giving a contradiction.

Similarly, we find that if $a^2 - b^2 - c^2 + 2\alpha > 0$ at any point T_0, then the solution is not finite on all of $(-\infty, T_0)$.

The remaining case to consider is when

$$c^2 \leq a^2 + b^2, a^2 \leq b^2 + c^2 - 2\alpha \tag{30}$$

on $(-\infty, \eta)$. Since b tends to 0 and ac tends to ∞ we deduce from these inequalities that a and c both tend to ∞ as T tends to $-\infty$.

It follows that abc is increasing for T large and negative so tends to a finite limit L as T tends to $-\infty$. Moreover $\frac{(abc)'}{abc}$ tends to ∞. If $L > 0$ then $(abc)'$ tends to ∞ as T tends to $-\infty$, which gives a contradiction. So we have

$$abc \to 0, \frac{(abc)'}{abc} \to \infty \text{ as } T \to -\infty.$$

It easily follows by considering $\log(abc)$ that abc decays exponentially fast as T tends to $-\infty$. Therefore the geodesic distance to $T = -\infty$ is finite, and, since a, c become infinite at $T = -\infty$, the metric is incomplete.

We summarise our results in the final theorem.

Theorem 6.2

The only scalar-flat Kähler, non-hyperkähler, diagonal Bianchi IX metrics which are complete have a^2 identically equal to b^2 (and hence have $U(2)$ symmetry). □.

In fact our discussion shows that all the complete metrics arise from the unstable curves of points $(\mu, \mu, 0)$ for $\mu > 0$. It follows from (27-29) that in this situation we have

$$a \simeq \mu,\ b \simeq \mu,\ c \simeq ke^{(\mu^2+\alpha)T}$$

as T tends to $-\infty$, for some constant k.

Taking

$$v = \frac{k\mu^2}{\mu^2 + \alpha} e^{(\mu^2+\alpha)T}$$

as a new coordinate, we find that the metric is asymptotically

$$dv^2 + \mu^2(\sigma_1^2 + \sigma_2^2) + (1 + \frac{\alpha}{\mu^2})^2 v^2 \sigma_3^2$$

as $v \to 0$.

This metric can be completed (by adding a bolt) precisely when $1 + \frac{\alpha}{\mu^2}$ equals $\frac{1}{2}n$ for some positive integer n.

If we put $r = 2\sqrt{ab}$ then the metric corresponding to a solution of (27-29) with $a \equiv b$ becomes

$$\left(1 + \frac{8\alpha}{r^2} - \frac{16(\mu^4 + 2\alpha\mu^2)}{r^4}\right)^{-1} dr^2 + \frac{1}{4}r^2\left(\sigma_1^2 + \sigma_2^2 + \left(1 + \frac{8\alpha}{r^2} - \frac{16(\mu^4 + 2\alpha\mu^2)}{r^4}\right)\sigma_3^2\right).$$

As explained above, we obtain the complete examples by setting

$$1 + \frac{\alpha}{\mu^2} = \frac{1}{2}n$$

and the resulting metrics are

$$\left(1 + \frac{4\mu^2(n-2)}{r^2} - \frac{16\mu^4(n-1)}{r^4}\right)^{-1} dr^2 + \frac{1}{4}r^2\left(\sigma_1^2 + \sigma_2^2 + \left(1 + \frac{4\mu^2(n-2)}{r^2} - \frac{16\mu^4(n-1)}{r^4}\right)\sigma_3^2\right)$$

where $\mu \in \mathbb{R}$.

These are the complete $U(2)$-invariant metrics found by LeBrun [**L**].

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