The behavior of the principal distributions on a real-analytic surface

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Abstract. The purposes of this paper are: (1) to study the behavior of the principal distributions around an isolated umbilical point on a real-analytic surface of a certain type, which contains all the real-analytic, special Weingarten surfaces; (2) to give one condition such that if a real-analytic surface satisfies the condition, then the index of an isolated umbilical point of the surface is less than or equal to one.

1 Introduction

Let S be a smooth surface in $\mathbf{R}^3$ and $\mathrm{Umb}(S)$ the set of the umbilical points of S and set $\mathrm{Reg}(S) := S \setminus \mathrm{Umb}(S)$. If $\mathrm{Reg}(S) \neq \emptyset$, then there exists *a principal distribution* D_S *on* S, which is a one-dimensional continuous distribution on $\mathrm{Reg}(S)$ such that $\mathsf{D}_S(p)$ is one of the principal directions at $p \in \mathrm{Reg}(S)$. Let p_0 be an isolated umbilical point of S. Then as a quantity in relation to the behavior of D_S around p_0, *the index* $\mathrm{ind}_{p_0}(S)$ *of* p_0 is defined([3, pp. 137]).

For each $l \in \mathbf{N} \cup \{0\}$, let $A_o^{(l)}$ be the set of the real-analytic functions defined on a neighborhood of $(0,0)$ in $\mathbf{R}^2$ such that for each $F \in A_o^{(l)}$, $\dfrac{\partial^{m+n}F}{\partial x^m \partial y^n}(0,0) = 0$ holds for non-negative integers m, n satisfying $0 \leqq m + n < l$. Let F be an element of $A_o^{(2)}$ and G_F the graph of F. If the origin o of $\mathbf{R}^3$ is a non-interior point of $\mathrm{Umb}(\mathsf{G}_F)$, then there exists an element $f_F \in A_o^{(3)}$ satisfying $\mathrm{Reg}(\mathsf{G}_{F-f_F}) = \emptyset$, and there exists a nonzero homogeneous polynomial φ_F of degree k_F satisfying $f_F - \varphi_F \in A_o^{(k_F+1)}$. Let A_o^l be the set of the elements of $A_o^{(l)}$ such that on their graphs, o is an isolated umbilical point, and P_o the set of the homogeneous polynomials contained in A_o^2, and A_{oo}^l the subset of A_o^l such that $\varphi_F \in P_o$ holds for $F \in A_{oo}^l$. The purposes of this paper are

(1) to study the behavior of the principal distributions around o on the graph G_F of $F \in A_{oo}^2$;

(2) to give one condition such that if $F \in A_o^2$ satisfies the condition, then $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$ holds.

For $F \in A_o^2$, there exist two principal distributions $\mathsf{D}_F^{(1)}$, $\mathsf{D}_F^{(2)}$ on G_F which give the principal directions to each point of $\mathrm{Reg}(\mathsf{G}_F)$, and there exists a positive number $\rho_0 > 0$ satisfying $\{0 < x^2 + y^2 < \rho_0^2\} \subset \mathrm{Reg}(\mathsf{G}_F)$. Let $\phi_F^{(i)}$ be a continuous function defined on $B_0 := (0, \rho_0) \times \mathbf{R}$ satisfying

$$\cos \phi_F^{(i)}(\rho, \theta) \frac{\partial}{\partial x} + \sin \phi_F^{(i)}(\rho, \theta) \frac{\partial}{\partial y} \in \mathsf{D}_F^{(i)}(\rho \cos \theta, \rho \sin \theta)$$

for $(\rho, \theta) \in B_0$. In Section 3, we shall prove the following:

Proposition 1.1 *For $F \in A_o^2$ and for $\theta_0 \in \mathbf{R}$,*

(1) *there exists the number $\phi_{F,o}^{(i)}(\theta_0)$ satisfying*

$$\lim_{\rho \to +0} \phi_F^{(i)}(\rho, \theta_0) = \phi_{F,o}^{(i)}(\theta_0);$$

(2) *there exist the numbers $\phi_{F,o}^{(i)}(\theta_0 + 0)$, $\phi_{F,o}^{(i)}(\theta_0 - 0)$ satisfying*

$$\lim_{\theta \to \theta_0 \pm 0} \phi_{F,o}^{(i)}(\theta) = \phi_{F,o}^{(i)}(\theta_0 \pm 0);$$

(3) *there exists the element $\Gamma_{F,o}(\theta_0)$ of $Z_0 := \{n\pi/2\}_{n \in \mathbf{Z}}$ satisfying*

$$\Gamma_{F,o}(\theta_0) = \phi_{F,o}^{(i)}(\theta_0 + 0) - \phi_{F,o}^{(i)}(\theta_0 - 0)$$

for $i = 1, 2$.

For $\theta \in \mathbf{R}$, let $\mathrm{Hess}_{\varphi_F}(\theta)$ be the Hessian of φ_F at $(\cos\theta, \sin\theta)$, and η_{φ_F} a continuous function on $\mathbf{R}$ such that ${}^t(\cos \eta_{\varphi_F}(\theta), \sin \eta_{\varphi_F}(\theta))$ is an eigenvector of $\mathrm{Hess}_{\varphi_F}(\theta)$ for any $\theta \in \mathbf{R}$, and $S_{F,o}$ the set of the numbers at each of which $\mathrm{Hess}_{\varphi_F}$ is scalar. In Section 3, we shall prove the following:

Proposition 1.2 *For $F \in A_o^2$,*

(1) *if $\theta_0 \in \mathbf{R}$ satisfies $\Gamma_{F,o}(\theta_0) \neq 0$, then $\theta_0 \in S_{F,o}$ holds;*

(2) *for $\theta \in \mathbf{R}$, $\mathrm{ind}_o(\mathsf{G}_F)$ is represented as*

$$\mathrm{ind}_o(\mathsf{G}_F) = \frac{\eta_{\varphi_F}(\theta + 2\pi) - \eta_{\varphi_F}(\theta)}{2\pi} + \frac{1}{2\pi} \sum_{\theta_0 \in S_{F,o} \cap [\theta, \theta + 2\pi)} \Gamma_{F,o}(\theta_0).$$

In Section 4, we shall give one way of computing $\eta_{\varphi_F}(\theta + 2\pi) - \eta_{\varphi_F}(\theta)$. In Section 5, we shall prove the following:

Theorem 1.3 *For $F \in A_{oo}^2$,*

(1) $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$ *hold for $\theta_0 \in S_{F,o}$;*

(2) $\mathrm{ind}_o\left(\mathsf{G}_{\varphi_F}\right) \leqq \mathrm{ind}_o(\mathsf{G}_F) \leqq 1$ *hold.*

Remark For $\varphi_F \in P_o$, $\mathrm{ind}_o\left(\mathsf{G}_{\varphi_F}\right) \in \{1+i-k_F/2\}_{i=0}^{[k_F/2]}$ holds([1]).

We know a conjecture which says that $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$ holds for any $F \in A_o^2$. This is part of Loewner's conjecture([4]). In Section 5, we shall prove the following:

Theorem 1.4 *Let F be an element of A_o^2 satisfying $\Gamma_{F,o}(\theta_0) \leqq \pi$ for any $\theta_0 \in S_{F,o}$. Then $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$ holds.*

Let S be a real-analytic, Weingarten surface and w_S a function of two variables satisfying $w_S(K_S, H_S) \equiv 0$ on S, where K_S and H_S are Gaussian curvature and the mean curvature of S, respectively. In this paper, we suppose that w_S is real-analytic, and according to [2], we call S *special* if w_S satisfies

$$H_S \frac{\partial w_S}{\partial X}(K_S, H_S) + \frac{1}{2}\frac{\partial w_S}{\partial Y}(K_S, H_S) \neq 0 \tag{1.1}$$

on $\mathrm{Umb}(S)$. In Section 6, we shall prove the following:

Theorem 1.5 *Let F be an element of $A_o^{(2)}$ whose graph is a non-totally umbilical, special Weingarten surface with an umbilical point o. Then the following hold*:

(1) $F \in A_{oo}^2$, (2) $\mathrm{ind}_o(\mathsf{G}_F) = \mathrm{ind}_o\left(\mathsf{G}_{\varphi_F}\right) = 1 - k_F/2$.

Remark Since $k_F \geqq 3$, we see that if F is as in Theorem 1.5, then $\mathrm{ind}_o(\mathsf{G}_F) < 0$ holds, which was also obtained in [2].

2 Preliminaries

Let f be a smooth function of two variables and G_f the graph of f. We set $p_f := \dfrac{\partial f}{\partial x}$, $q_f := \dfrac{\partial f}{\partial y}$, and

$$E_f := 1 + p_f^2, \qquad F_f := p_f q_f, \qquad G_f := 1 + q_f^2. \tag{2.1}$$

The first fundamental form of G_f is the symmetric tensor field I_f on G_f of type $(0,2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{I}_f := E_f dx^2 + 2F_f dxdy + G_f dy^2,$$

where

$$dx^2 := dx \otimes dx, \quad dxdy := \frac{1}{2}(dx \otimes dy + dy \otimes dx), \quad dy^2 := dy \otimes dy.$$

We set $r_f := \dfrac{\partial^2 f}{\partial x^2}$, $s_f := \dfrac{\partial^2 f}{\partial x \partial y}$, $t_f := \dfrac{\partial^2 f}{\partial y^2}$, and

$$L_f := \frac{r_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad M_f := \frac{s_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad N_f := \frac{t_f}{\sqrt{\det(\mathrm{I}_f)}}, \tag{2.2}$$

where $\det(\mathrm{I}_f) := E_f G_f - F_f^2$. *The second fundamental form of* G_f is the symmetric tensor field II_f on G_f of type $(0,2)$ represented in terms of the coordinates (x,y) as

$$\mathrm{II}_f := L_f dx^2 + 2M_f dxdy + N_f dy^2.$$

Let $T_{(x_0,y_0)}(\mathsf{G}_f)$ be the tangent plane to G_f at (x_0,y_0) and $U_{(x_0,y_0)}(\mathsf{G}_f)$ the subset of $T_{(x_0,y_0)}(\mathsf{G}_f)$ defined by

$$U_{(x_0,y_0)}(\mathsf{G}_f) := \left\{\mathbf{u} \in T_{(x_0,y_0)}(\mathsf{G}_f) \ ; \ \mathrm{I}_{f,(x_0,y_0)}(\mathbf{u},\mathbf{u}) = 1\right\},$$

and $L_{(x_0,y_0)}(\mathsf{G}_f)$ the set of the one-dimensional subspaces of $T_{(x_0,y_0)}(\mathsf{G}_f)$. Let $\nu_{f,(x_0,y_0)}$ be the function on $U_{(x_0,y_0)}(\mathsf{G}_f)$ defined by

$$\nu_{f,(x_0,y_0)}(\mathbf{u}) := \mathrm{II}_{f,(x_0,y_0)}(\mathbf{u},\mathbf{u})$$

for $\mathbf{u} \in U_{(x_0,y_0)}(\mathsf{G}_f)$. Suppose that $\nu_{f,(x_0,y_0)}$ attains an extremum at $\mathbf{u}_0 \in U_{(x_0,y_0)}(\mathsf{G}_f)$. Then the extremum $\nu_{f,(x_0,y_0)}(\mathbf{u}_0)$ is called *a principal curvature of* G_f *at* (x_0,y_0) and the element of $L_{(x_0,y_0)}(\mathsf{G}_f)$ determined by $\mathbf{u}_0$ is called *a principal direction of* G_f *at* (x_0,y_0). The set of the principal directions at (x_0,y_0) is denoted by $PD_{(x_0,y_0)}(\mathsf{G}_f)$. *The Weingarten map of* G_f is the tensor field W_f on G_f of type $(1,1)$ satisfying

$$\left[\mathrm{W}_f\left(\frac{\partial}{\partial x}\right), \mathrm{W}_f\left(\frac{\partial}{\partial y}\right)\right] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] W_f,$$

where

$$W_f := \begin{pmatrix} E_f & F_f \\ F_f & G_f \end{pmatrix}^{-1} \begin{pmatrix} L_f & M_f \\ M_f & N_f \end{pmatrix}.$$

By Lagrange's method of indeterminate coefficients, we obtain

Proposition 2.1 *The principal curvatures and the principal directions of* G_f *at* (x_0,y_0) *are given by the eigenvalues and by the one-dimensional eigenspaces of* $\mathrm{W}_{f,(x_0,y_0)}$, *respectively.*

Gaussian curvature $K_f(x_0,y_0)$ and *the mean curvature* $H_f(x_0,y_0)$ *of* G_f *at* (x_0,y_0) are given by the following:

$$K_f(x_0,y_0) := \det\left(\mathrm{W}_{f,(x_0,y_0)}\right), \ H_f(x_0,y_0) := \mathrm{tr}\left(\mathrm{W}_{f,(x_0,y_0)}\right)/2. \tag{2.3}$$

Let PD_f be the symmetric tensor field on G_f of type $(0,2)$ represented in terms of the coordinates (x,y) as

$$\mathrm{PD}_f := \frac{1}{\sqrt{\det(\mathrm{I}_f)}}\big\{A_f dx^2 + B_f dxdy + C_f dy^2\big\},$$

where

$$A_f := F_f L_f - E_f M_f, \quad B_f := G_f L_f - E_f N_f, \quad C_f := G_f M_f - F_f N_f.$$

For $\mathbf{v}_1$, $\mathbf{v}_2 \in T_{(x_0,y_0)}(\mathsf{G}_f)$, the following holds:

$$\frac{1}{2}\sum_{\{i,j\}=\{1,2\}} \mathrm{W}_{f,(x_0,y_0)}(\mathbf{v}_i)\wedge\mathbf{v}_j = \frac{\mathrm{PD}_{f,(x_0,y_0)}(\mathbf{v}_1,\mathbf{v}_2)}{\sqrt{\det\left(\mathrm{I}_{f,(x_0,y_0)}\right)}}\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right)_{(x_0,y_0)}.$$

Then by Proposition 2.1, we obtain

Proposition 2.2 *A tangent vector* $\mathbf{v}_0 \in T_{(x_0,y_0)}(\mathsf{G}_f)$ *belongs to an element of* $PD_{(x_0,y_0)}(\mathsf{G}_f)$ *if and only if* $\mathrm{PD}_{f,(x_0,y_0)}(\mathbf{v}_0,\ \mathbf{v}_0) = 0$ *holds.*

A point (x_0,y_0) is called *an umbilical point of* G_f if $\nu_{f,(x_0,y_0)}$ is constant.

Proposition 2.3 *The following are mutually equivalent*:

(1) $(x_0,y_0)\in \mathrm{Umb}(\mathsf{G}_f)$, (2) $PD_{(x_0,y_0)}(\mathsf{G}_f) = L_{(x_0,y_0)}(\mathsf{G}_f)$,

(3) $A_f(x_0,y_0) = B_f(x_0,y_0) = C_f(x_0,y_0) = 0$.

Let $\mathrm{D}_f, \mathrm{N}_f$ be the symmetric tensor fields on G_f of type $(0,2)$ represented in terms of the coordinates (x,y) as

$$\begin{aligned}
\mathrm{D}_f &:= s_f dx^2 + (t_f - r_f)dxdy - s_f dy^2,\\
\mathrm{N}_f &:= \big(s_f p_f^2 - p_f q_f r_f\big)dx^2 + \big(t_f p_f^2 - r_f q_f^2\big)dxdy + \big(p_f q_f t_f - s_f q_f^2\big)dy^2.
\end{aligned}$$

By (2.1) and by (2.2), we obtain $\det(\mathrm{I}_f)\mathrm{PD}_f = \mathrm{D}_f + \mathrm{N}_f$. For a vector field $\mathbf{V}$ on G_f, we set

$$\begin{aligned}
&\tilde{\mathrm{D}}_f(\mathbf{V}) := \mathrm{D}_f(\mathbf{V},\ \mathbf{V}), \qquad \tilde{\mathrm{N}}_f(\mathbf{V}) := \mathrm{N}_f(\mathbf{V},\ \mathbf{V}),\\
&\widetilde{\mathrm{PD}}_f(\mathbf{V}) := \mathrm{PD}_f(\mathbf{V},\ \mathbf{V}).
\end{aligned}$$

We set

$$\mathrm{grad}_f := \begin{pmatrix} p_f \\ q_f \end{pmatrix}, \quad \mathrm{grad}_f^{\perp} := \begin{pmatrix} -q_f \\ p_f \end{pmatrix}, \quad \mathrm{Hess}_f := \begin{pmatrix} r_f & s_f \\ s_f & t_f \end{pmatrix}.$$

For $\phi \in\mathbf{R}$, we set

$$u_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \qquad \mathbf{U}_\phi := \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y}.$$

Let $\langle\ ,\ \rangle$ be the scalar product in $\mathbf{R}^2$. Then we obtain

Lemma 2.4 *For any $\phi \in \mathbf{R}$, the following hold:*

$$\tilde{\mathrm{D}}_f(\mathbf{U}_\phi) = \big\langle \operatorname{Hess}_f u_\phi,\ u_{\phi+\pi/2}\big\rangle,$$
$$\tilde{\mathrm{N}}_f(\mathbf{U}_\phi) = \big\langle \operatorname{grad}_f,\ u_\phi\big\rangle\big\langle \operatorname{grad}_f^\perp,\ \operatorname{Hess}_f u_\phi\big\rangle.$$

We set $\mathbf{Grad}_f := p_f\dfrac{\partial}{\partial x} + q_f\dfrac{\partial}{\partial y}$. By Lemma 2.4, we obtain

Lemma 2.5 $\widetilde{\mathrm{PD}}_f(\mathbf{Grad}_f) = \tilde{\mathrm{D}}_f(\mathbf{Grad}_f)$.

3 The behavior of the principal distributions around an umbilical point

Let $A_o^{\langle 2\rangle}$ be the set of the elements of $A_o^{(2)}$ such that for $F \in A_o^{\langle 2\rangle}$, o is a non-interior point of $\mathrm{Umb}(\mathsf{G}_F)$. For $F \in A_o^{\langle 2\rangle}$, let Φ_F be a real-analytic function on $(-\rho_0,\rho_0)\times\mathbf{R}\times\mathbf{R}$ defined by

$$\Phi_F(\rho,\theta,\phi) := \det\big(\mathrm{I}_{F,(\rho\cos\theta,\rho\sin\theta)}\big)\widetilde{\mathrm{PD}}_{F,(\rho\cos\theta,\rho\sin\theta)}\big(\mathbf{U}_\phi\big).$$

Then by Lemma 2.4 and by $\Phi_{F-f_F}\equiv 0$, we obtain

$$\lim_{\rho\to 0}\frac{\Phi_F(\rho,\theta,\phi)}{\rho^{k_F-2}} = \big\langle \operatorname{Hess}_{\varphi_F}(\theta)u_\phi, u_{\phi+\pi/2}\big\rangle. \tag{3.1}$$

Therefore we obtain

Proposition 3.1 *Let F be an element of $A_o^{\langle 2\rangle}$ satisfying $S_{F,o}=\emptyset$. Then $F\in A_{oo}^2$ holds.*

Suppose $\theta_0\in S_{F,o}$. Then we set

$$\Phi_{F,\theta_0,\phi}(\rho) := \rho^2\Phi_F(\rho,\theta_0,\phi),\quad X_{F,\theta_0} := \{\phi_0\in\mathbf{R}\ ;\ \Phi_{F,\theta_0,\phi_0}\not\equiv 0\}.$$

For $\phi_0\in X_{F,\theta_0}$, let k_{F,θ_0,ϕ_0} be the order of Φ_{F,θ_0,ϕ_0} at 0, and set

$$k_{F,\theta_0} := \min_{\phi_0\in X_{F,\theta_0}} k_{F,\theta_0,\phi_0}.$$

Then there exists a non-scalar matrix $M(\theta_0)$ satisfying

$$\lim_{\rho\to 0}\frac{\Phi_F(\rho,\theta_0,\phi)}{\rho^{k_{F,\theta_0}-2}} = \big\langle M(\theta_0)u_\phi, u_{\phi+\pi/2}\big\rangle. \tag{3.2}$$

Proof of Proposition 1.1 By (3.1) and by (3.2), we obtain (1), and we see that for $\theta_1,\theta_2\in\mathbf{R}$ satisfying $\theta_1<\theta_2$ and $S_{F,o}\cap(\theta_1,\theta_2)=\emptyset$, there exists the element $z^{(i)}(\theta_1,\theta_2)$ of Z_0 satisfying

$$\phi_{F,o}^{(i)}(\theta) = \eta_{\varphi_F}(\theta) + z^{(i)}(\theta_1,\theta_2) \tag{3.3}$$

for any $\theta \in (\theta_1, \theta_2)$. Therefore noticing that the set $S_{F,o}$ is empty or discrete, we obtain (2). By (3.1) and by (3.2), we obtain $\phi^{(2)}_{F,o} - \phi^{(1)}_{F,o} \equiv z_0$ for some $z_0 \in Z_0$. Therefore by (3.3), we obtain (3). □

Proof of Proposition 1.2 By (3.3), we obtain (1). For $\theta \in \mathbf{R}$, the following holds:

$$\mathrm{ind}_o(\mathsf{G}_F) = \frac{\phi^{(i)}_{F,o}(\theta + 2\pi) - \phi^{(i)}_{F,o}(\theta)}{2\pi}. \tag{3.4}$$

By (3.3), we obtain

$$\begin{aligned}\phi^{(i)}_{F,o}(\theta + 2\pi) - \phi^{(i)}_{F,o}(\theta) = \eta_{\varphi_F}(\theta + 2\pi) - \eta_{\varphi_F}(\theta) \\ + \sum_{\theta_0 \in S_{F,o} \cap [\theta, \theta + 2\pi)} \Gamma_{F,o}(\theta_0).\end{aligned} \tag{3.5}$$

From (3.4) and from (3.5), we obtain (2). □

4 Homogeneous polynomials

Let P be the set of the homogeneous polynomials in two variables. For $\varphi \in P$, set $\tilde{\varphi}(\theta) := \varphi(\cos\theta, \sin\theta)$. A number $\theta_0 \in \mathbf{R}$ is called *a root of* φ if $\frac{d\tilde{\varphi}}{d\theta}(\theta_0) = 0$. The set of the roots of φ is denoted by R_φ. The straight line $L(\theta_0) := \{(\rho\cos\theta_0, \rho\sin\theta_0)\}_{\rho \in \mathbf{R}}$ on $\mathbf{R}^2$ determined by $\theta_0 \in R_\varphi$ is called *a root line of* φ.

Let k_φ be the degree of φ. If $R_\varphi = \mathbf{R}$, then k_φ is even and φ is represented by $(x^2 + y^2)^{k_\varphi/2}$ with constant multiplication([1]). Suppose $R_\varphi \neq \mathbf{R}$. Then for any $\theta_0 \in R_\varphi$, there exists a positive integer m satisfying $\frac{d^{m+1}\tilde{\varphi}}{d\theta^{m+1}}(\theta_0) \neq 0$. The minimum of such integers as m is called *the multiplicity of* θ_0 and denoted by $\mu_\varphi(\theta_0)$. A root $\theta_0 \in R_\varphi$ is said to be

(1) *related* if θ_0 satisfies $\tilde{\varphi}(\theta_0) = 0$ or if $\mu_\varphi(\theta_0)$ is odd;

(2) *non-related* if θ_0 satisfies $\tilde{\varphi}(\theta_0) \neq 0$ and if $\mu_\varphi(\theta_0)$ is even.

Suppose that $\theta_0 \in R_\varphi$ is related. Then it is said that *the critical sign of* θ_0 *is positive*(*resp. negative*) if the following holds:

$$\tilde{\varphi}(\theta_0)\frac{d^{\mu_\varphi(\theta_0)+1}\tilde{\varphi}}{d\theta^{\mu_\varphi(\theta_0)+1}}(\theta_0) \leqq 0(\text{resp.} > 0).$$

The critical sign of θ_0 is denoted by $\text{c-sign}_\varphi(\theta_0)$.

For any $\theta, \phi \in \mathbf{R}$, we set

$$\mathrm{Hess}_\varphi(\theta) := \mathrm{Hess}_\varphi(\cos\theta, \sin\theta), \quad d_\varphi(\theta, \phi) := \tilde{\mathrm{D}}_{\varphi,(\cos\theta,\sin\theta)}(\mathbf{U}_\phi),$$

and let η_φ be a continuous function on $\mathbf{R}$ satisfying $d_\varphi(\theta, \eta_\varphi(\theta)) = 0$ for any $\theta \in\mathbf{R}$. A number $\theta_0 \in\mathbf{R}$ is called *a root of φ on Hessian* if $\theta_0 - \eta_\varphi(\theta_0) \in Z_0$ holds. The set of the roots of φ on Hessian is denoted by $R(\mathrm{Hess}_\varphi)$. For any $\theta \in\mathbf{R}$. the following holds:

$$d_\varphi(\theta, \theta) = (k_\varphi - 1)\frac{d\tilde{\varphi}}{d\theta}(\theta). \tag{4.1}$$

Therefore we obtain $R(\mathrm{Hess}_\varphi) \subset R_\varphi$.

Lemma 4.1 *The following hold*:

(1) *The set $R_\varphi \setminus R(\mathrm{Hess}_\varphi)$ consists of the numbers at each of which* Hess_φ *is nonzero scalar*;

(2) *An element $\theta_0 \in R_\varphi \setminus R(\mathrm{Hess}_\varphi)$ is related and satisfies* $\text{c-sign}_\varphi(\theta_0) = -$.

Proof If $\mathrm{Hess}_\varphi(\theta_0)$ is not scalar for $\theta_0 \in R_\varphi$, then by (4.1) we obtain $\theta_0 \in R(\mathrm{Hess}_\varphi)$. Suppose $\tilde{\varphi}(\theta_0) = 0$ for $\theta_0 \in R_\varphi$. Then there exist an integer $n_0 \geqq 2$ and an element $\varphi_0 \in P$ satisfying $\tilde{\varphi}(\theta) = \sin^{n_0}(\theta - \theta_0)\tilde{\varphi}_0(\theta)$ and $\tilde{\varphi}_0(\theta_0) \neq 0$. Then we obtain

$$\begin{aligned}\frac{d_\varphi(\theta, \phi)}{\sin^{n_0-2}(\theta - \theta_0)} =& {}_{n_0}C_2\tilde{\varphi}_0(\theta)\sin 2(\phi - \theta_0) \\ &+ a_{\varphi_0}(\theta, \phi)\sin(\theta - \theta_0) + d_{\varphi_0}(\theta, \phi)\sin^2(\theta - \theta_0),\end{aligned} \tag{4.2}$$

where a_{φ_0} satisfies $\tilde{\varphi}_0(\theta_0)a_{\varphi_0}(\theta_0, \theta_0) > 0$. By (4.2), we obtain $\theta_0 \in R(\mathrm{Hess}_\varphi)$. Suppose that $\mathrm{Hess}_\varphi(\theta_0)$ is the unit matrix for $\theta_0 \in R_\varphi$, and $\theta_0 = 0$. Then we may represent φ as

$$\varphi(x, y) = \frac{1}{k_\varphi(k_\varphi - 1)}x^{k_\varphi} + \frac{1}{2}x^{k_\varphi - 2}y^2 + \sum_{i=3}^{k_\varphi} a_i x^{k_\varphi - i}y^i. \tag{4.3}$$

Therefore $\mathrm{Hess}_\varphi(\theta) = \sum_{i=0}^{k_\varphi - 2} M_i \cos^{k_\varphi - 2 - i}\theta \sin^i\theta$ holds, where M_1 is not diagonal. Then we obtain $0 \notin R(\mathrm{Hess}_\varphi)$. Hence we have proved (1) of Lemma 4.1. For $\theta_0 \in R_\varphi \setminus R(\mathrm{Hess}_\varphi)$, we obtain $\dfrac{d^2\tilde{\varphi}}{d\theta^2}(\theta_0) = k_\varphi(k_\varphi - 2)\tilde{\varphi}(\theta_0)$. Therefore we obtain $\mu_\varphi(\theta_0) = 1$ and $\text{c-sign}_\varphi(\theta_0) = -$. Hence we have proved (2) of Lemma 4.1. □

We set $U_\theta(\varepsilon) := (\theta - \varepsilon, \theta + \varepsilon)$ for $\theta \in \mathbf{R}$ and for $\varepsilon > 0$. It is said that *the sign of $\theta_0 \in R(\mathrm{Hess}_\varphi)$ is positive* (*resp. negative*) if there exists a positive number $\varepsilon_0 > 0$ satisfying

$$\{\theta - \eta_\varphi(\theta) - (\theta_0 - \eta_\varphi(\theta_0))\}(\theta - \theta_0) > 0(\text{resp.} < 0)$$

for any $\theta \in U_{\theta_0}(\varepsilon_0) \setminus \{\theta_0\}$. Let $n_{\varphi,+}$(resp. $n_{\varphi,-}$) be the number of the root lines determined by the elements of $R(\mathrm{Hess}_\varphi)$ with positive(resp. negative) sign. Then referring to [1], we obtain

Proposition 4.2 *For any $\theta \in \mathbf{R}$, the following holds:*

$$\frac{\eta_\varphi(\theta+2\pi)-\eta_\varphi(\theta)}{2\pi} = 1 - \frac{n_{\varphi,+}-n_{\varphi,-}}{2}.$$

We shall prove

Proposition 4.3 *For $\theta_0 \in R(\mathrm{Hess}_\varphi)$, θ_0 is related if and only if the sign of θ_0 is positive or negative. In addition, for a related root $\theta_0 \in R(\mathrm{Hess}_\varphi)$,*

(1) *if $\tilde{\varphi}(\theta_0) \neq 0$, then the number*

$$\delta_\varphi(\theta_0) := \frac{d^{\mu_\varphi(\theta_0)+1}\tilde{\varphi}}{d\theta^{\mu_\varphi(\theta_0)+1}}(\theta_0)\frac{\partial d_\varphi}{\partial \phi}(\theta_0,\theta_0)$$

is nonzero and the sign of θ_0 is given by the sign of $\delta_\varphi(\theta_0)$;

(2) *if $\tilde{\varphi}(\theta_0) = 0$, then the sign of θ_0 is positive.*

Proof Suppose that for $\theta_0 \in R(\mathrm{Hess}_\varphi)$, $\mathrm{Hess}_\varphi(\theta_0)$ is not scalar. Then $\dfrac{\partial d_\varphi}{\partial \phi}(\theta_0,\theta_0) \neq 0$ holds. Therefore by the implicit function theorem, we see that η_φ is infinitely differentiable and satisfies

$$\left.\frac{d^m}{d\theta^m}(\theta-\eta_\varphi)\right|_{\theta=\theta_0} = (k_\varphi - 1)\frac{d^{m+1}\tilde{\varphi}}{d\theta^{m+1}}(\theta_0)\Big/\frac{\partial d_\varphi}{\partial \phi}(\theta_0,\theta_0) \tag{4.4}$$

for $m = 1,\ldots,\mu_\varphi(\theta_0)$. Therefore θ_0 is related if and only if the sign of θ_0 is positive or negative. By (4.4), we obtain (1) of Proposition 4.3.

Suppose $\tilde{\varphi}(\theta_0) = 0$ for $\theta_0 \in R(\mathrm{Hess}_\varphi)$. Then by (4.2), we obtain (2) of Proposition 4.3.

Hence we have proved Proposition 4.3. □

If $\theta_0 \in R(\mathrm{Hess}_\varphi)$ is related, then the sign of θ_0 is denoted by $\mathrm{sign}_\varphi(\theta_0)$.

Proposition 4.4 *Let θ_0 be a related root of φ satisfying* $\text{c-sign}_\varphi(\theta_0) = +$. *Then $\theta_0 \in R(\mathrm{Hess}_\varphi)$ and $\mathrm{sign}_\varphi(\theta_0) = +$ hold.*

Proof Suppose that a related root θ_0 satisfies $\text{c-sign}_\varphi(\theta_0) = +$. Then from Lemma 4.1, we obtain $\theta_0 \in R(\mathrm{Hess}_\varphi)$. If $\tilde{\varphi}(\theta_0) \neq 0$, then $\delta_\varphi(\theta_0) > 0$ holds([1]). Therefore we obtain $\mathrm{sign}_\varphi(\theta_0) = +$. If $\tilde{\varphi}(\theta_0) = 0$, then Proposition 4.3 says $\mathrm{sign}_\varphi(\theta_0) = +$. Hence we obtain Proposition 4.4. □

An element $\varphi \in P$ is called *harmonic* if $\mathrm{tr}(\mathrm{Hess}_\varphi) \equiv 0$ holds. If φ is harmonic, then we obtain $\tilde{\varphi}(\theta) = c\cos k_\varphi(\theta-\alpha)$, where $c, \alpha \in \mathbf{R}\setminus\{0\}$. Then we immediately obtain

Proposition 4.5 *For a nonzero harmonic element $\varphi \in P$,*

(1) *the number of the root lines of φ is equal to k_φ;*

(2) *any root $\theta_0 \in R_\varphi$ is related and satisfies* $\text{c-sign}_\varphi(\theta_0) = +$;

(3) *for any $\theta \in \mathbf{R}$,* $\mathrm{Hess}_\varphi(\theta)$ *is not scalar.*

5 Proof of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3 Let F be an element of A^2_{oo}. Then we obtain $S_{F,o} = R_{\varphi_F} \setminus R(\mathrm{Hess}_{\varphi_F})$. We set

$$\Psi_F(\rho,\theta) := \tilde{\mathrm{D}}_{F,(\rho\cos\theta,\rho\sin\theta)}(\mathbf{Grad}_F(\rho\cos\theta,\rho\sin\theta))/\rho^k,$$

where

$$k := \begin{cases} 3k_F - 4, & \text{if } F - f_F \equiv 0, \\ k_F, & \text{if } F - f_F \not\equiv 0. \end{cases}$$

Then by (4.3) and by the implicit function theorem, we obtain

Lemma 5.1 *For $\theta_0 \in S_{F,o}$, there exist a neighborhood V_{θ_0} of $(0,\theta_0)$ in $\mathbf{R}^2$ and a real-analytic curve C_{θ_0} in V_{θ_0} through $(0,\theta_0)$ satisfying*

(1) $C_{\theta_0} = \{(\rho,\theta) \in V_{\theta_0}\ ;\ \Psi_F(\rho,\theta) = 0\}$;

(2) C_{θ_0} *is not tangent to the θ-axis at $(0,\theta_0)$.*

Suppose $\theta_0 \in S_{F,o}$. Let ψ_F be a continuous function on B_0 such that for $(\rho,\theta) \in B_0$, $\mathrm{grad}_F(\rho\cos\theta,\rho\sin\theta)$ is represented by $u_{\psi_F(\rho,\theta)}$ with constant multiplication. Noticing Lemma 2.5, we suppose $\phi^{(1)}_F = \psi_F$ on $B_0 \cap C_{\theta_0}$. Since $\theta_0 \in R_{\varphi_F} \setminus R(\mathrm{Hess}_{\varphi_F})$, ψ_F may be extended to $\overline{B_0 \cap V_{\theta_0}}$. Let ε_0 be a positive number satisfying $\{0\} \times U_{\theta_0}(\varepsilon_0) \subset V_{\theta_0}$. We set $\psi_{F,o}(\theta) := \psi_F(0,\theta)$ and $\chi_{F,o}(\theta) := \phi^{(1)}_{F,o}(\theta) - \psi_{F,o}(\theta)$ for $\theta \in \overline{U_{\theta_0}(\varepsilon_0)}$. Let α be a continuous function on $U_{\theta_0}(\varepsilon_0)$ satisfying $\alpha(\theta)(\theta-\theta_0) > 0$ for any $\theta \in U_{\theta_0}(\varepsilon_0) \setminus \{\theta_0\}$. Then we obtain

$$\Gamma_{F,o}(\theta_0) = \begin{cases} \pi/2, & \text{if } \alpha\chi_{F,o} > 0 \text{ on } U_{\theta_0}(\varepsilon_0) \setminus \{\theta_0\}, \\ -\pi/2, & \text{if } \alpha\chi_{F,o} < 0 \text{ on } U_{\theta_0}(\varepsilon_0) \setminus \{\theta_0\}, \\ 0, & \text{if } \chi_{F,o}(\theta_0+\varepsilon_0)\chi_{F,o}(\theta_0-\varepsilon_0) > 0. \end{cases}$$

Therefore we obtain (1) of Theorem 1.3.

By Lemma 2.4, we obtain

$$\begin{aligned} &\tilde{\mathrm{N}}_{\varphi_F,(\rho\cos\theta,\rho\sin\theta)}(\mathbf{U}_\phi) \\ &= \rho^{3k_{\varphi_F}-4}\frac{\det(\mathrm{Hess}_{\varphi_F}(\theta))}{k_{\varphi_F}-1}\langle \mathrm{grad}_{\varphi_F}(\cos\theta,\sin\theta)u_\theta, u_\phi\rangle \sin(\phi-\theta). \end{aligned} \tag{5.1}$$

Therefore we may suppose $\phi^{(1)}_{\varphi_F,\rho}(\theta_0) = \psi_{\varphi_F,o}(\theta_0) = \theta_0$, where $\phi^{(1)}_{\varphi_F,\rho}(\theta) := \phi^{(1)}_{\varphi_F}(\rho,\theta)$. By (2) of Lemma 4.1, we obtain $\{\psi_{\varphi_F,o}(\theta) - \theta\}(\theta-\theta_0) > 0$. By (5.1), we obtain $\dfrac{\partial\widetilde{\mathrm{PD}}_{\varphi_F}(\mathbf{U}_\phi)}{\partial\phi} \neq 0$ at $(\rho\cos\theta_0,\rho\sin\theta_0)$. Therefore we see that $\phi^{(1)}_{\varphi_F,\rho}$ is differentiable at θ_0, and we obtain

$$\frac{d}{d\theta}(\theta - \phi^{(1)}_{\varphi_F,\rho})\Big|_{\theta=\theta_0} = \frac{d^2\tilde{\varphi}_F}{d\theta^2}(\theta_0)\Big/ k^3_{\varphi_F}(k_{\varphi_F}-1)\rho^{2k_{\varphi_F}-2}\tilde{\varphi}_F(\theta_0)^3.$$

Then by (2) of Lemma 4.1, we obtain $\{\theta - \phi^{(1)}_{\varphi_F,\rho}(\theta)\}(\theta-\theta_0) > 0$. Therefore we obtain $\alpha\chi_{F,o} < 0$ on $U_{\theta_0}(\varepsilon_0) \setminus \{\theta_0\}$, and $\Gamma_{\varphi_F,o}(\theta_0) = -\pi/2$. By Proposition 1.2 and by (1) of Theorem 1.3, we obtain $\mathrm{ind}_o\left(\mathsf{G}_{\varphi_F}\right) \leqq \mathrm{ind}_o(\mathsf{G}_F)$.

For $\theta \in \mathbf{R}$, set $n_s := \sharp(S_{F,o} \cap [\theta, \theta+2\pi))$, and let $\{\theta_n\}_{n=0}^{n_s}$ be the subset of $S_{F,o}$ satisfying $\theta_{n-1} < \theta_n$ and $(\theta_{n-1}, \theta_n) \cap S_{F,o} = \emptyset$ for $n = 1, \dots, n_s$. Then by Lemma 4.1, we see that for each $n \in \{1, \dots, n_s\}$, the number of the related roots in $R_{\varphi_F} \cap (\theta_{n-1}, \theta_n)$ with positive critical sign is more than the number of the related roots in $R_{\varphi_F} \cap (\theta_{n-1}, \theta_n)$ with negative critical sign. Therefore by Proposition 4.2 and by Proposition 4.4, we obtain

$$\frac{\eta_{\varphi_F}(\theta+2\pi) - \eta_{\varphi_F}(\theta)}{2\pi} \leqq 1 - n_s/4.$$

By Proposition 1.2 and by (1) of Theorem 1.3, we obtain $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$.

Hence we have proved (2) of Theorem 1.3. $\square$

Proof of Theorem 1.4 Let $S^{(0)}_{F,o}$ be the set of the elements of $S_{F,o}$ at each of which $\mathrm{Hess}_{\varphi_F}$ is zero, and set $S^{(1)}_{F,o} := S_{F,o} \setminus S^{(0)}_{F,o}$. For $\theta \in \mathbf{R}$ and for $i \in \{0,1\}$, set $n^{(i)}_s := \sharp\left(S^{(i)}_{F,o} \cap [\theta, \theta+2\pi)\right)$. Then by Proposition 4.2 and by Proposition 4.4, we obtain

$$\frac{\eta_\varphi(\theta+2\pi) - \eta_\varphi(\theta)}{2\pi} \leqq 1 - n^{(0)}_s/2 - n^{(1)}_s/4. \tag{5.2}$$

If $\Gamma_{F,o}(\theta_0) \leqq \pi$ holds for $\theta_0 \in S^{(0)}_{F,o}$, then by Proposition 1.2, (1) of Theorem 1.3 and by (5.2), we obtain $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$. Hence we have proved Theorem 1.4. $\square$

6 Special Weingarten surfaces

We shall prove

Proposition 6.1 *Let F be an element of $A^{\langle 2\rangle}_o$ whose graph is a special Weingarten surface. Then φ_F is harmonic.*

To prove Proposition 6.1, we need lemmas.

For $F \in A^{\langle 2\rangle}_o$, we set $c_F := H_F(0,0)$. Then $K_F(0,0) = c_F^2$ holds. We represent $K_F - c_F^2$ and $H_F - c_F$ as

$$K_F - c_F^2 := \sum_{i \geqq 1} K^{(i)}_F, \qquad H_F - c_F := \sum_{i \geqq 1} H^{(i)}_F,$$

where $K^{(i)}_F$ and $H^{(i)}_F$ are elements of P of degree i. Noticing (2.3) and that K_{F-f_F} and H_{F-f_F} are constant, we obtain

Lemma 6.2 *For $F \in A^{\langle 2\rangle}_o$,*

(1) *if* $c_F = 0$, *then the following holds*:

$$K_F^{(i)} = \begin{cases} 0, & \text{if } i \in \{1, \ldots, 2k_F - 5\}, \\ \det(\mathrm{Hess}_{\varphi_F}), & \text{if } i = 2k_F - 4, \end{cases}$$

(2) *if* $c_F \neq 0$, *then the following holds*:

$$K_F^{(i)} = \begin{cases} 0, & \text{if } i \in \{1, \ldots, k_F - 3\}, \\ c_F \operatorname{tr}(\mathrm{Hess}_{\varphi_F}), & \text{if } i = k_F - 2, \end{cases}$$

(3) *the following holds*:

$$H_F^{(i)} = \begin{cases} 0, & \text{if } i \in \{1, \ldots, k_F - 3\}, \\ \operatorname{tr}(\mathrm{Hess}_{\varphi_F})/2, & \text{if } i = k_F - 2. \end{cases}$$

Let w be an element of $A_o^{(1)}$ satisfying

$$c_F \frac{\partial w}{\partial X}(0,0) + \frac{1}{2}\frac{\partial w}{\partial Y}(0,0) \neq 0,$$

and set

$$\Delta_{F,w}(x,y) := w(K_F(x,y) - c_F^2, H_F(x,y) - c_F).$$

We represent $\Delta_{F,w}$ as $\Delta_{F,w} := \sum_{i \geqq 1} \Delta_{F,w}^{(i)}$, where $\Delta_{F,w}^{(i)}$ is an element of P of degree i. By Lemma 6.2, we obtain

Lemma 6.3 *The following holds*:

$$\Delta_{F,w}^{(i)} = \begin{cases} 0, & \text{if } i \in \{1, \ldots, k_F - 3\}, \\ \left(c_F \frac{\partial w}{\partial X}(0,0) + \frac{1}{2}\frac{\partial w}{\partial Y}(0,0) \right) \operatorname{tr}(\mathrm{Hess}_{\varphi_F}), & \text{if } i = k_F - 2. \end{cases}$$

Proof of Proposition 6.1 If the graph of $F \in A_o^{\langle 2 \rangle}$ is a special Weingarten surface, then by (1.1) and by Lemma 6.3, we see that φ_F is harmonic. □

Proof of Theorem 1.5 Since φ_F is harmonic, from Proposition 3.1 and from (3) of Proposition 4.5, we obtain (1) of Theorem 1.5. By Proposition 1.2, Proposition 4.2, Proposition 4.4, and by Proposition 4.5, we obtain (2) of Theorem 1.5. □

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