Vertex algebras and elliptic genus

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This is a report on a joint work with V. Gorbounov, F. Malikov and A. Vaintrob, cf. [MSV], [MS1], [MS2], [GMS].

1. Let X be a smooth complex variety (algebraic or analytic). One can associate with X a canonically defined gerbe (champ of groupoids) $\mathfrak{D}_X^{ch}$ called *the gerbe of chiral differential operators* on X. For an open $U \subset X$, the objects of the groupoid $\mathfrak{D}_X(U)$ are sheaves of vertex algebras over X, called *sheaves of chiral differential operators* on X. The gerbe $\mathfrak{D}_X$ is bounded by the lien isomorphic to the sheaf $\Omega_X^{2,cl}$ of closed two-forms.

The equivalence class

$$c(\mathfrak{D}_X) \in H^2(X, \Omega_X^{2,cl})$$

is equal to

$$2ch_2(\mathcal{T}_X) := c_1^2(\mathcal{T}_X) - 2c_2(\mathcal{T}_X) \tag{1.1}$$

where $c_i(\mathcal{T}_X) \in H^i(X, \Omega_X^{i,cl})$ are the Chern classes.

For example, (1.1) vanishes for products of curves, or for flag spaces G/B.

2. Let E be a vector bundle over X. Generalizing the previous construction, one can define the gerbe $\mathfrak{D}_{\Lambda E}$ of chiral differential operators on the exterior algebra ΛE, also bounded by the sheaf $\Omega_X^{2,cl}$.

It is plausible that its equivalence class is equal to

$$c(\mathfrak{D}_{\Lambda E}) = 2ch_2(\mathcal{T}_X) - 2ch_2(E) \tag{2.1}$$

For example, (2.1) vanishes for $E = \Omega_X^1$ or $\mathcal{T}_X$.

3. Consider the case $E = \Omega_X^1$. Let us denote the gerbe $\mathfrak{D}_{\Lambda E}$ of chiral differential operators on the de Rham algebra of differential forms by $\mathfrak{D}_\Omega$. The groupoid $\mathfrak{D}_\Omega(X)$ is non-empty.

In fact, one can define certain *canonical* object $\mathcal{D}_\Omega$ of $\mathfrak{D}_\Omega(X)$. It was introduced in [MSV], and called *chiral de Rham complex* there. Recently A. Beilinson gave a nice coordinate-free construction of $\mathcal{D}_\Omega$ and some of its generalizations in the language of chiral algebras (in the sense of Beilinson - Drinfeld).

$\mathcal{D}_\Omega$ is a sheaf of conformal vertex superalgebras, $\mathbb{Z}$-graded by *fermionic charge*. For a compact X, the cohomology algebra

$$H^*(X, \mathcal{D}_\Omega) = \oplus_{p=0}^{n} \oplus_{q=-\infty}^{\infty} \oplus_{i=0}^{\infty} H^p(X, \mathcal{D}_{\Omega,i}^q) \tag{3.1}$$

is called the *chiral Hodge cohomology* of X and has some nice properties. Here $\mathcal{D}_{\Omega,i}^q$ denotes the component of conformal weight i and fermionic charge q, $n = \dim(X)$.

The space (3.1) is a conformal vertex superalgebra. It is a $N = 2$ superalgebra if X is Calabi-Yau (i.e. when the canonical bundle is trivial). We have the "Poincaré duality":

$$H^p(X, \mathcal{D}^q_{\Omega,i})^* = H^{n-p}(X, \mathcal{D}^{n-q}_{\Omega,i}) \tag{3.2}$$

As was noted in [BL], if X is Calabi-Yau then the generating series

$$\chi_X(q,y) := y^{n/2} \sum_{a,b,i} (-1)^{a+b} \dim H^a(X, \mathcal{D}^b_{\Omega,i}) y^b q^i \tag{3.3}$$

coincides with the Ochanine — Witten elliptic genus of X.

References

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