COMBINATORIAL IDENTITIES FOR YANGIAN

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Abstract. We prove analogues of Cayley-Hamilton identities and Newton's formulas for the matrix of generators of Yangian of $\mathfrak{gl}_n(\mathbb{C})$.

1. Introduction

In this note we prove the analogues of Cayley-Hamilton theorem and Newton's formulas for the matrix of generators of Yangian of $\mathfrak{gl}_n(\mathbb{C})$. This requires a generalization of the definition of a power of this matrix, which is done in the spirit of the q-version of power sums, introduced in [3], and also used in [6]. They are natural generalizations of powers of matrices in case of quantum groups, since they "remember" the braiding in the defining relations of the quantum group.

All of the three defined in Section 3 generalizations of powers of $T(u)$ are the matrices with coefficients in $Y[[u^{-1}]]$ (or in $Y[[u^{-1}, v_1^{-1} \dots v_m^{-1}]]$). The shifted power $T^l(u|\rho)$ is the most straightforward generalization of a power of an ordinary matrix. The permuted matrix $T^{[l]}(u|\rho)$ can be expressed through the shifted powers $T^l(u|\rho)$ for $l \le m$ (Proposition 1). The matrix $T^{\langle m\rangle}(u|\rho)$, does not enjoy that property, but it satisfies an analogue of Cayley-Hamilton identity with coefficients in Bethe-subalgebra of Yangian (Corollary 1). Both permuted powers lead to the analog of Newton's formula (Corollary 2).

The structure of the note is the following. Section 2 reviews definition of Yangian and notations. Section 3 defines the generalizations of powers of matrix of generators of $T(u)$ and describes their properties. Section 4 summarizes the properties of symmetrizer and antisymmetrizer. Section 5 states and proves the discussed above combinatorial identities.

2. Yangian of $\mathfrak{gl}_n(\mathbb{C})$

Here we review definitions and some properties of Yangians that will be used later ([4], [5]).

Definition 1. The Yangian $Y(n)$ for $\mathfrak{gl}_n(\mathbb{C})$ is a unital associative algebra over $\mathbb{C}$ with countably many generators $\{t_{ij}^{(r)}\}$, $r = 1, 2, ...$, $1 \le i, j \le n$ and the defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)},$$

where $r, s = 0, 1, 2...$ and $t_{ij}^{(0)} = \delta_{ij}$.

The same set of defining relations can be combined into one equation, sometimes called *RTT-relation*. Namely, denote by $T(u) = (t_{ij}(u))_{i,j=1}^n$ the matrix with coefficients $t_{ij}(u)$, which are formal power series of generators of $Y(n)$:

$$t_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} \frac{t_{ij}^{(k)}}{u^k}.$$

For $P = \sum E_{ij} \otimes E_{ji}$, the permutation matrix of $\mathbb{C}^n \otimes \mathbb{C}^n$, define the Yang matrix

$$R(u) = 1 - \frac{P}{u}.$$

It is a rational function with values in $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n$.

We introduce some standard notations. For any vector space V and any element S of $\operatorname{End} V$ we define an element S_k of $\operatorname{End} V^{\otimes m}$ by

$$S_k = 1^{\otimes(k-1)} \otimes S \otimes 1^{\otimes(m-k)}.$$

In particular, we write

$$T_k(u) = \sum_{ij} t_{ij}(u) \otimes (E_{ij})_k \in Y(n) \otimes \operatorname{End}(\mathbb{C}^n)^{\otimes m}.$$

Let S be an element of $\operatorname{End} V \otimes \operatorname{End} V$. Using the abbreviated notation $S = S(1) \otimes S(2)$, we define an element S_{ij} of $\operatorname{End}(\mathbb{C}^n)^{\otimes m}$ by

$$S_{ij} = 1^{\otimes(i-1)} \otimes S(1) \otimes 1^{\otimes(j-i-1)} \otimes S(2) \otimes 1^{\otimes(m-j-i)}.$$

Definition 2. The Yangian $Y(n)$ of $\mathfrak{gl}_n(\mathbb{C})$ is an associative unital algebra over $\mathbb{C}$ with the set of generators $\{t_{ij}^{(k)}\}$ which satisfy the equation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v).$$

Yangian $Y(n)$ is an example of infinite-dimensional quantum group. It has a group-like central element, which is called *quantum determinant* of $Y(n)$.

Definition 3. Quantum determinant $\operatorname{qdet} T(u)$ is a formal series with coefficients in $Y(n)$, defined by

$$\operatorname{qdet} T(u) = \sum_{\sigma \in S_n} (-1)^{\sigma} t_{1\sigma(1)}(u-n+1) \dots t_{n\sigma(n)}(u).$$

We use the following notations for traces. Let $X \in \operatorname{End}(V)^{\otimes m}$. Then $\operatorname{tr}(X)$ denotes the full trace of X, and $\operatorname{tr}_{\hat{k}}(X) = \operatorname{tr}_{1\dots k-1,k+1\dots m}(X)$ denotes the trace over all tensor components of X, except the k-th component.

3. Powers of $T(u)$ and their properties

3.1. **Definitions of generalized powers.** Consider the matrix of generators of the Yangian

$$T(u) = \sum_{ij} E_{ij} \otimes t_{ij}(u).$$

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be an arbitrary sequence of complex numbers.

Definition 4. (Shifted power of $T(u)$)

$$T^m(u \,|\, \alpha) = T(u-\alpha_1) \dots T(u-\alpha_m) \tag{1}$$

Let $\rho = (0, 1, 2, \dots)$ and let us fix a sequence of complex numbers $v = (v_1, v_2, \dots)$. We use abbreviated notations $R_k = R_{k,k+1}(v_k)$.

Definition 5. (Permuted powers of $T(u)$)

$$T^{[m]}(u|\rho) = \operatorname{tr}_{\hat{1}}(\,T_1(u)T_2(u-1)\dots T_m(u-m+1)R_{m-1}\dots R_1) \tag{2}$$

$$T^{<m>}(u|\rho) = \operatorname{tr}_{\hat{m}}(\,sT_1(u)T_2(u-1)\dots T_m(u-m+1)R_{m-1}\dots R_1) \tag{3}$$

Remark. Occasionally $T^{[m]}(u|-\rho)$ and $T^{<m>}(u|-\rho)$ will be used, which are defined similarly.

3.2. **Properties of the m-th shifted power matrix.** There are two possible interpretations of the "shifted power" $T^m(u|\alpha)$.

For the first one, let $\mu : Y(n) \otimes Y(n) \to Y(n)$ be the multiplication operation in $Y(n)$ and let Δ be the coproduct $Y(n) \to Y(n) \otimes Y(n)$:

$$\Delta(t_{ij}(u)) = \sum_k t_{ik}(u) \otimes t_{kj}(u).$$

For any complex number a define a shift-automomorphism of $Y(n)$ by the formula

$$\tau_a T(u) = T(u-a).$$

Then

$$T^m(u|\alpha) = \mu^{\otimes m} \cdot (\tau_{\alpha_1} \otimes \dots \otimes \tau_{\alpha_m}) \cdot \Delta^{(m)}\, T(u). \tag{4}$$

The second interpretation involves the permutation matrix $P = \sum_{i,j} E_{ij} \otimes E_{ji}$. It defines the action of the group algebra $\mathbb{C}[\mathcal{S}_m]$ of the symmetric group on the tensor product $(\mathbb{C}^n)^{\otimes m}$. Namely, a transposition (k, l) acts as an operator $P_{k,l}$ permuting the k-th and l-th component of $(\mathbb{C}^n)^{\otimes m}$. Then

$$T^m(u|\alpha) = \operatorname{tr}_{\hat{m}}\,(T_m(u-\alpha_1)\dots T_1(u-\alpha_m)(m, \dots 1))\,, \tag{5}$$

where $(m, \dots, 1) = P_{m-1,m}P_{m-2,m-1}\,\dots P_{12}$.

Remark. Also

$$\begin{aligned} T^m(u|\alpha) &= \operatorname{tr}_{\hat{m}}\,((1, \dots, m)\, T_1(u-\alpha_1)\dots T_m(u-\alpha_m)) \\ &= \operatorname{tr}_{\hat{1}}\,(T_1(u-\alpha_1)\dots T_m(u-\alpha_m)\,(m, \dots, 1)) \\ &= \operatorname{tr}_{\hat{1}}\,((1, \dots m)\, T_m(u-\alpha_1)\dots T_1(u-\alpha_m))\,. \end{aligned}$$

3.3. **Propereties of $[m]$-th permuted power matrix.** The matrix $T^{[m]}(u|\rho)$ is a sum of products of some shifted power traces with a shifted power matrix and with rational functions of $\bar{v} = (v_1 \dots v_m)$.

More precisely, let $\lambda = (\lambda_1, \dots \lambda_k)$ be a partition of m. We set

$$\begin{aligned} &a_0 = 0, \\ &a_1 = \lambda_1, \\ &a_2 = \lambda_1 + \lambda_2, \\ &\dots \\ &a_{k-1} = \lambda_1 + \dots + \lambda_{k-1}. \end{aligned}$$

Proposition 1.

$$T^{[m]}(u|\rho) = \sum_{\lambda\vdash m} V(\lambda)\, T^{\lambda_1}(u|\rho)\ \ tr\,(\, T^{\lambda_2}(u-a_1|\rho)\dots T^{\lambda_k}(u-a_{k-1}|\rho)). \tag{6}$$

Here

$$V(\lambda) = (-1)^{(m-k)} \frac{v_{a_{k-1}}\dots v_{a_1}}{v_{m-1}\dots v_1}. \tag{7}$$

Proof. Observe that in the symmetric group $\mathcal{S}_m$ for any r and s such that $m > r \geq s > 1$

$$(r, r+1)(r-1, r), \dots (s, s+1) = (r+1, r, r-1, \dots s).$$

Therefore, the product $R_{m-1}R_{m-2}\dots R_1$ acts on $(\mathbb{C}^n)^{\otimes m}$ as the following element of the group algebra $\mathbb{C}[\mathcal{S}_m]$:

$$\sum_{\lambda\vdash m} V(\lambda)\, \sigma_\lambda,$$

where σ_λ is the product of cycles:

$$\sigma_\lambda = (m, \dots a_{k-1}+1)\dots(a_2, \dots, a_1+1)(a_1, \dots, 1).$$

and $V(\lambda)$ is as in (7). Now

$$\begin{aligned}
&\mathrm{tr}_{\hat{1}}\,(T_1T_2\dots T_m\,\sigma_\lambda) = \\
&= \mathrm{tr}_{\hat{1}} \sum_{i,j} E_{i_1 j_{\sigma_\lambda(1)}} \otimes \dots \otimes E_{i_m j_{\sigma_\lambda(m)}} t_{i_1 j_1}(u)\dots t_{i_m j_m}(u-m+1) \\
&= \sum_{i_1,\bar{j}} E_{i_1 j_{a_1}} \left(t_{i_1 j_1}(u)\dots t_{j_{a_1-1} j_{a_1}}(u-a_1+1)\right)\ \dots\ ((t_{j_m j_{a_{k-1}+1}}(u-a_{k-1})\dots t_{j_{m-1} j_m}(u-m+1)) \\
&= T^{\lambda_1}(u|\rho)\,\mathrm{tr}\, T^{\lambda_2}(u-a_1|\rho)\dots\ \mathrm{tr}\, T^{\lambda_k}(u-a_{k-1}|\rho)).
\end{aligned}$$

By taking the sum over all partitions λ we complete the proof. □

4. Symmetric and Antisymmetric power traces

4.1. **Definition and properties.** Consider the antisymmetirzer and symmetrizer of $(\mathbb{C}^n)^{\otimes m}$, given by

$$A_m = \frac{1}{m!}\sum_{\sigma\in\mathcal{S}_m} (-1)^\sigma \sigma, \qquad S_m = \frac{1}{m!}\sum_{\sigma\in\mathcal{S}_m} \sigma.$$

These operators enjoy the following properties.

Proposition 2. *(a)*

$$A_m^2 = A_m \qquad \text{and} \qquad S_m^2 = S_m.$$

(b) With abbreviated notations $R_{ij} = R_{ij}(v_i - v_j)$, write

$$R(v_1, \dots v_m) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1})...(R_{1,m}\dots R_{1,2}).$$

Then $A_m = \frac{1}{m!}R(u, u-1, \dots u-m+1)$, and $S_m = \frac{1}{m!}R(u, u+1, \dots u+m-1)$.
(c)

$$A_m T_1(u)\dots T_m(u-m+1) = T_m(u-m+1)\dots T_1(u) A_m,$$
$$S_m T_1(u)\dots T_m(u+m-1) = T_m(u+m-1)\dots T_1(u) S_m.$$

(d)

$$A_{m+1} = \frac{1}{m+1} A_m\,(\,(1-mu) + muR_{m,m+1}(u)\,)\,A_m,$$

$$S_{m+1} = \frac{1}{m+1} S_m(\,(1+mu) - muR_{m,m+1}(u)\,)S_m.$$

(e)

$$tr(A_n T_1(u)\dots T_n(u-n+1)) = qdet T(u).$$

Definition 6. Put

$$\tau_k(u) = \mathrm{tr}(A_k T_1(u)\dots T_k(u-k+1)),$$

$$h_k(u) = \mathrm{tr}(S_k T_1(u)\dots T_k(u+k-1)).$$

4.2. **Relation to Bethe subalgebra.** In [1],[2] a commutative subalgebra $B(\mathfrak{gl}_n(\mathbb{C}, Z))$ of the Yangian Y is studied. It is called Bethe subalgebra and its generators are the coefficients of all the series

$$B_k(u, Z) = \mathrm{tr}\,(A_n T_1 \dots T_k Z_{k+1} \dots Z_n),$$

where Z is a matrix of size n by n with complex coefficients. Our elements $\tau_k(u)$ are proportional to $B_k(u, 1)$ with Z being the identity.

Indeed, by Proposition(2) (d) and (a) we obtain that

$$\begin{aligned}
&\mathrm{tr}_{1\dots m+1}(A_{m+1}T_1\dots T_k \otimes 1^{\otimes^{m+1-k}}) \\
&= \frac{1}{m+1}\mathrm{tr}_{1\dots m+1}(A_m\,((1-m(m,m+1))\,A_m T_1 \dots T_k \otimes 1^{\otimes^{m+1-k}}) \\
&= \frac{n}{m+1}\mathrm{tr}_{(1\dots m)}(A_m T_1\dots T_k \otimes 1^{\otimes^{m-k}}) - \frac{1}{m+1}\mathrm{tr}_{(1\dots m+1)}(A_m(m,m+1)A_m\,T_1\dots T_k\otimes 1^{\otimes^{m+1-k}})).
\end{aligned}$$

But by the cyclic property of the trace,

$$\begin{aligned}
&\mathrm{tr}_{(1\dots m+1)}(A_m(m,m+1)A_m\,T_1\dots T_k\otimes 1^{\otimes^{m+1-k}}) \\
&= \mathrm{tr}_{(1\dots m+1)}((A_m\,T_1\dots T_k\otimes 1^{\otimes^{m-k+1}}A_m)(m,m+1)) \\
&= \mathrm{tr}_{(1\dots m)}((A_m\,T_1\dots T_k\otimes 1^{\otimes^{m-k}}A_m) \\
&= \mathrm{tr}_{(1\dots m)}((A_m\,T_1\dots T_k\otimes 1^{\otimes^{m-k}}).
\end{aligned}$$

Therefore,

$$\mathrm{tr}_{(1\dots m+1)}(A_{m+1}T_1\dots T_k\otimes 1^{\otimes^{m+1-k}}) = \frac{(n-1)}{m+1}\,\mathrm{tr}_{(1\dots m)}((A_m\,T_1\dots T_k\otimes 1^{\otimes^{m-k}}). \tag{8}$$

¿From (8) one can show by induction that

$$B_k(u,1) = \mathrm{tr}_{(1\dots n)}(A_n T_1\dots T_k\otimes 1^{\otimes^{n-k}}) = \frac{(n-1)^{n-k}k!}{n!}\,\tau_k(u).$$

5. Combinatorial identities.

5.1. **Cayley-Hamilton theorem.** The classical Cayley-Hamilton theorem states that any matrix with coefficients over $\mathbb{C}$ is annihilated by some polynomial. Here we prove the analogue of this statement for the matrix $T(u)$. The identity involves permuted powers of $T(u)$ instead of ordinary ones, and the coefficients of this identity are commuting elements of Bethe subalgebra $B(\mathfrak{gl}_n(\mathbb{C},1))$.

Recall the notations $R_k = R_{k,k+1}(v_k)$ and

$$T^{<m>}(u|\rho) = \operatorname{tr}_{\hat{m}}(T_1(u)T_2(u-1)\dots T_m(u-m+1)R_{m-1}\dots R_1).$$

Proposition 3. *The matrix $T(u)$ satisfies the following two identities.*

$$\sum_{k=0}^{m-1} a_k\, \tau_k(u)\, T^{\langle m-k\rangle}(u-k|\rho) = m tr_{\hat{m}}(A_m T_1\dots T_m) \tag{9}$$

and

$$\sum_{k=0}^{m-1} b_k\, h_k(u)\, T^{\langle m-k\rangle}(u+k|-\rho) = m tr_{\hat{m}}(S_m T_1\dots T_m), \tag{10}$$

where

$$a_k = v_{m-1}\dots v_{k+1}(1-kv_k),$$
$$b_k = (-1)^{m-k+1} v_{m-1}\dots v_{k+1}(1+kv_k).$$

Proof. By Proposition 2 (d),

$$(1-kv_k)A_k = (k+1)\,A_{k+1} - kv_k\,A_kR_kA_k.$$

In the following we abbreviate $T_l := T_l(u-l+1)$.

Then

$$(1-kv_k)\tau_k(u)T^{\langle m-k\rangle}(u-k|\rho) = (1-kv_k)\operatorname{tr}_{\hat{m}}(A_k\,T_1\dots T_m\,R_{m-1}\dots R_{k+1})$$
$$= (k+1)\operatorname{tr}_{\hat{m}}\,(A_{k+1}T_1\dots T_mR_{m-1}\dots R_{k+1}) - k\,v_k\operatorname{tr}_{\hat{m}}\,(A_kR_kA_k\,T_1\dots T_mR_{m-1}\dots R_{k+1}).$$

But the operator A_k commutes with $T_{k+1}\dots T_mR_{m-1}\dots R_{k+1}$, so by the cyclic property of trace and by Proposition 2 (a),

$$\operatorname{tr}_{\hat{m}}(A_kR_kA_k\,T_1\dots T_mR_{m-1}\dots R_{k+1}) = \operatorname{tr}_{\hat{m}}(A_k\,T_1\dots T_mR_{m-1}\dots R_{k+1}R_k).$$

Thus,

$$(1-kv_k)\tau_k(u)T^{<m-k>}(u-k|\rho) = (k+1)I_{k+1} - k\,v_k\,I_k$$

with

$$I_k = \operatorname{tr}_{\hat{m}}(A_k\,T_1\dots T_mR_{m-1}\dots R_{k+1}R_k).$$

Put $\tau_0(u) = 1$. Since $I_1 = T^{<m>}(u|\rho)$, and $I_m = \operatorname{tr}_{\hat{m}}(A_mT_1\dots T_m)$, we get (9). The second formula is proved similarly. □

Corollary 1. *(Cayley-Hamilton theorem)*

$$\sum_{k=0}^{n} a_k\,\tau_k(u)\,T^{<n-k>}(u-k|\rho) = 0$$

Remark. Observe that the last term in the sum is $\tau_n(u) = qdet(T(u))$.

5.2. **Newton's formulas.** ¿From the Proposition 3 we deduce Newton's formulas as well.

Definition 7. For $\alpha = (\alpha_1, \alpha_2, \dots)$ the permuted m-th power sum is the full trace of the permuted power matrix:

$$p_m(u|\alpha) = \mathrm{tr}_{(1\dots m)}(T_1(u-\alpha_1)T_2(u-\alpha_2)\dots T_m(u-\alpha_m)R_{m-1}\dots R_1).$$

Of course, $p_m(u|\rho) = \mathrm{tr}\, T^{<m>} = \mathrm{tr}\, T^{[m]}(u|\rho)$.

¿From Proposition 1, the permuted power trace $p_m(u|\rho)$ is a linear combination of "ordinary" (but shifted) power traces:

$$p_m(u|\rho) = \sum_{\lambda\vdash m} V(\lambda)\, \mathrm{tr}\, T^{\lambda_1}(u|\rho)\ \mathrm{tr}\,(\,T^{\lambda_2}(u-a_1|\rho))\dots \mathrm{tr}(T^{\lambda_k}(u-a_{k-1}|\rho)), \tag{11}$$

with $V(\lambda)$ as in (7). From Proposition 3 we get the following corollary:

Corollary 2. *(Newton's formulas)*

$$\sum_{k=0}^{n} a_k\, \tau_k(u)\, p_{n-k}(u-k|\rho) = 0$$

And recursive formulae:

$$\tau_m(u) = \sum_{k=0}^{m-1} a(k)\, \tau_k(u)\, p_{m-k}(u-k|\rho),$$

$$h_m(u) = \sum_{k=0}^{m-1} a(k)\, h_k(u)\, p_{m-k}(u+k|-\rho).$$

References

[1] M. Nazrov, G. Olshanski, Bethe subalgebras in twisted Yangians, Comm. Math. Phys, **178** (2) (1996), 483–506.

[2] A. ,N. Kirillov, N. Reshetikhin, Yangians, Bethe anzatz and combinatorics, Lett. Math. Phys, **12** (1986), 199–208.

[3] J.-M. Maillet, Lax equations and quantum groups, Phys. Lett. **B 245** (no.3-4) (1990),480–486.

[4] A. Molev, Yangians and their applications, Handbook of algebra, **3** (2003), 907–959.

[5] A. Molev, M. Nazrov, G. Olshankii, Yangians and Classical Lie algebras, Russian Math. Surveys **51**(2) (1996), 205–282.

[6] A. Isaev, O. Ogievetsky, P. Pyatov, Generalized Cayley-Hamilton-Newton identities, Czechoslovak J. Phys. **48** (11) (1998), 1369–1374.