ON COHEN–MACAULAY MODULES ON SURFACE SINGULARITIES

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Introduction

During the study of Cohen–Macaulay modules on curve singularities (cf. [15, 9, 12, 7]) it was proved that such these singularities split into three classes:

- *Cohen–Macaulay finite*, having only finitely many indecomposable Cohen–Macaulay modules;
- *Cohen–Macaulay tame*, i.e., such that, for each fixed r, the indecomposable Cohen–Macaulay modules of rank r form a finite set of 1-parameter families;
- *Cohen–Macaulay wild*, that can be characterized in two ways:
  - "geometrically" as those having n-parameter families of non-isomorphic indecomposable Cohen–Macaulay modules for arbitrary large n,
  - "algebraically" as such that for every finitely generated algebra Λ there is an exact functor from the category of finite dimensional Λ-modules to the category of Cohen–Macaulay modules over this singularity which maps non-isomorphic modules to non-isomorphic ones and indecomposable to indecomposable.

  The latter "versality" property shows that the study of modules in the wild case is extremely complicated needs indeed essentially new and highly non-trivial approach.

Moreover, it turned out that this "trichotomy" is closely related to the position of a curve singularity in the well-known Arnold's list of singularities having good deformation properties (cf. [1]). Namely:

- a singularity is Cohen–Macaulay finite if and only if it dominates one of the simple plane curves singularities $\mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$;
- a singularity is Cohen–Macaulay tame if and only if it dominates one of the "serial" unimodal singularities T_{pq}.

(Remind that a singularity (X,x) *dominates* (Y,y) if there is a birational surjection $(X,x) \to (Y,y)$.)

In [10, 4] it was proved that a *normal surface singularity* (in characteristic 0) is Cohen–Macaulay finite if and only if it is a *quotient singularity.* In [16] it was shown that a *simple elliptic* singularity is

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always Cohen–Macaulay tame. In both cases a complete description of Cohen–Macaulay modules over such singularities was obtained. Moreover, in [16] Kahn elaborated a general method relating Cohen–Macaulay modules over normal surface singularities to vector bundles over some projective curves (usually singular and even non-reduced).

In this paper we use the Kahn correspondence and the results of [8] to prove the following main theorems:

- every cusp singularity (cf. [14, 17]) is Cohen–Macaulay tame;
- a minimally elliptic singularity (cf. [20]) that is neither simple elliptic nor cusp one is Cohen–Macaulay wild.

As a corollary we get that any *log-canonical* surface singularity (cf. [18]) is Cohen–Macaulay tame (or finite, as quotient singularities are also log-canonical). There is an evidence that any other normal surface singularity is Cohen–Macaulay wild. We also give a description of Cohen–Macaulay modules over cusp singularities and use it to provide a description of Cohen–Macaulay modules over curve singularities T_{pq} (there was no such description in [7], the proof of their tameness was indirect).

1. Generalities

In what follows we use the following

Definitions and Notations 1.1.

- A *surface singularity* means a spectrum $X = \operatorname{Spec} A$, where A is a local henselian noetherian ring of Krull dimension 2. We denote by $\mathfrak{m} = \mathfrak{m}_A$ the maximal ideal of A and by $\mathbf{k} = A/\mathfrak{m}$ the residue field. We always suppose that the field $\mathbf{k}$ is algebraically closed and that Artin's approximation theorem [2] is valid for A (e.g., A is complete, or is an henselization of an affine $\mathbf{k}$-algebra, or is the local ring of a point of a complex analytic variety).
- Such a singularity is called *normal* if the ring A is normal, i.e., integral and integrally closed in its field of fractions Q. In what follows we only consider normal surface singularities. Remind that any normal surface singularity is *isolated*, i.e., $\mathfrak{m}$ is the unique singular point of it.
- A *resolution* of a normal surface singularity is a birational proper map $\pi : \widetilde{X} \to X$, where $\widetilde{X}$ is smooth, which induces an isomorphism $\widetilde{X} \setminus E \xrightarrow{\sim} X \setminus \{\mathfrak{m}\}$, where $E = \pi^{-1}(\mathfrak{m})$. E is called the *exceptional curve* on $\widetilde{X}$ (it is indeed a projective curve over $\mathbf{k}$). Denote by E_i $(i = 1, \dots, s)$ the irreducible components of E. Put also $\check{X} = X \setminus \{\mathfrak{m}\}$.
- A resolution π as above is called *minimal* if it cannot be decomposed as

$$\widetilde{X} \xrightarrow{\pi_1} X' \xrightarrow{\pi_2} X,$$

where X' is also smooth. It is known (cf., e.g., [21]) that π is minimal if and only if neither of the components E_i is a smooth rational curve with the self-intersection index $E_i \cdot E_i = -1$.

- A *cycle* on $\widetilde{X}$ is a divisor of the shape $Z = \sum_{i=1}^{s} k_i E_i$. Such a cycle is called *positive* if all $k_i > 0$. We write $Z \le Z'$, where $Z' = \sum_{i=1}^{s} k'_i E_i$ if $k_i \le k'_i$ for all i.
- The *fundamental cycle*, denoted by Z_0, is the minimal positive cycle (with respect to the above defined order) such that $Z_0 \cdot E_i \le 0$ for all i.
- A positive cycle Z is called a *weak reduction cycle* if the sheaf $\mathcal{O}_Z(-Z)$ is *generically generated* by its global section (i.e., generated outside a finite set) and $\mathrm{H}^1(E, \mathcal{O}_Z(-Z)) = 0$. Here, as usually, we write $\mathcal{O}_Z = \mathcal{O}_{\widetilde{X}}/\mathcal{O}_{\widetilde{X}}(-Z)$ and consider Z as a closed subscheme of $\widetilde{X}$.
- A *reduction cycle* is a weak reduction cycle Z such that the sheaf $\omega_Z^\vee$ is generically generated by global sections, where $\omega_Z = \omega_{\widetilde{X}} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_Z(-Z)$ is the dualizing sheaf for Z and $\mathcal{M}^\vee$, where $\mathcal{M}$ is a coherent sheaf on a scheme S, denotes $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{M}, \mathcal{O}_S)$.
- One calls a surface singularity *rational* if $\mathrm{H}^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ and *minimally elliptic* if it is Gorenstein (i.e., $\mathrm{inj.dim}_A A = 2$) and $\mathrm{H}^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \simeq \mathbf{k}$.
- A *simple elliptic singularity* is such one that E is a smooth curve of genus 1.
- We call E a *cyclic configuration* in the following cases:
  - (i) $s = 1$, E is rational and has a unique singular point which is a simple node;
  - (ii) $s = 2$, $E_1 \simeq E_2 \simeq \mathbb{P}^1$ and they intersect transversally in (exactly) 2 points;
  - (iii) $s > 2$, $E_i \simeq \mathbb{P}^1$ for all i, $E_i \cdot E_{i+1} = 1$, $E_s \cdot E_1 = 1$, and $E_i \cdot E_j = 0$ otherwise.

  For a cyclic configuration we put $E_{s+k} = E_k$ for all integers k.
- A *cusp singularity* is such one that E is a cyclic configuration. Note that both simple elliptic and cusp singularities are minimally elliptic. Moreover, for all of them $Z_0 = E$.
- Denote by $\mathrm{Coh}\, S$ the category of coherent sheaves on a scheme S and by $\mathrm{VB}(S)$ the category of *vector bundles* on S, i.e., locally free coherent sheaves.
- For a surface singularity $X = \mathrm{Spec}\, A$ denote by $\mathrm{CM}(X)$, or by $\mathrm{CM}(A)$, the category of (maximal) Cohen–Macaulay modules over the ring A. If it is normal, Cohen–Macaulay modules coincide with *reflexive* modules, i.e., such A-modules M that $M^{\vee\vee} \simeq M$, where $M^\vee = \mathrm{Hom}_A(M, A)$. We always identify

an A-module M with its "sheafification" $\widetilde{M}$, which is a quasi-coherent sheaf on X.

Remind the main result of Kahn [16, Theorem 1.4] concerning the relations between the vector bundles on a reduction cycle and the Cohen–Macaulay modules over a normal surface singularity.

Theorem 1.2. *Let $\pi : \widetilde{X} \to X$ be a resolution of a normal surface singularity with the exceptional curve E and a reduction cycle Z. Denote by $R_Z : \mathrm{CM}(X) \to \mathrm{VB}(Z)$ the functor which is the composition of the functors*

$$\mathrm{CM}(X) \to \mathrm{VB}(\widetilde{X}) : \ M \mapsto (\pi^* M)^{\vee\vee}$$

and

$$\mathrm{VB}(\widetilde{X}) \to \mathrm{VB}(Z) : \ \mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_Z .$$

This functor maps non-isomorphic objects to non-isomorphic ones, and a vector bundle $\mathcal{V} \in \mathrm{VB}(Z)$ is isomorphic to $R_Z M$ for some M if and only if it is generically generated by global sections and there is an extension of $\mathcal{V}$ to a vector bundle $\mathcal{V}_2$ on $2Z$ such that the exact sequence

$$0 \longrightarrow \mathcal{V}(-Z) \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V} \longrightarrow 0 ,$$

induces a monomorphism $\mathrm{H}^0(E, \mathcal{V}(Z)) \to \mathrm{H}^1(E, \mathcal{V})$.

We denote by $\mathrm{VB}^K(Z)$ the full subcategory of $\mathrm{VB}(Z)$ consisting of the vector bundles satisfying the latter Kahn's conditions. Note that obviously $\mathrm{rk}\, R_Z M = \mathrm{rk}\, M$ for any Cohen–Macaulay module M.

For minimally elliptic singularities one can give a simpler description of the latter category (cf. [16, Theorem 2.1]).

Theorem 1.3. *Let $\pi : \widetilde{X} \to X$ be a minimal resolution of a minimal elliptic singularity. Then the fundamental cycle $Z = Z_0$ is a reduction cycle and the category $\mathrm{VB}^K(Z)$ consists of the vector bundles of the form $n\mathcal{O}_Z \oplus \mathcal{G}$, where $\mathcal{G}$ satisfies the following conditions:*

1. *$\mathcal{G}$ is generically generated by global sections,*
2. *$\mathrm{H}^1(E, \mathcal{G}) = 0$,*
3. *$\dim_{\mathbf{k}} \mathrm{H}^0(E, \mathcal{G}(Z)) \le n$.*

In particular, indecomposable objects of $\mathrm{VB}^K(Z)$ are the following:

- *the trivial line bundle $\mathcal{O}_Z$,*
- *$n\mathcal{O}_Z \oplus \mathcal{G}$, where $\mathcal{G}$ is an indecomposable vector bundle satisfying the above conditions (1),(2) and $n = \dim_{\mathbf{k}} \mathrm{H}^0(E, \mathcal{G}(Z))$.*

Remark. In the paper [16] Theorems 1.2 and 1.3 are proved in the "geometric" case, when A is a $\mathbf{k}$-algebra and $\mathrm{char}\, \mathbf{k} = 0$, but one easily verifies that all proofs can be easily extended to the general situation. Those results of Grauert–Riemenschneider [11] as well as of Laufer [20] used in [16] are also valid in this "abstract" context.

2. Cohen–Macaulay modules on cusp singularities

In this section E denotes a cyclic configuration, $\nu : \widetilde{E} \to E$ the normalization of E and E_i ($i = 1, \dots, s$) the irreducible components of $\widetilde{E}$. We put $E_{s+k} = E_k$ for all integers k, $\mathcal{O} = \mathcal{O}_E$, $\widetilde{\mathcal{O}} = \mathcal{O}_{\widetilde{E}}$, $\mathcal{O}_i = \mathcal{O}_{E_i}$, and always identify $\widetilde{\mathcal{O}}$ and $\mathcal{O}_i$ with their images under ν_*. If $s > 2$, denote by p_i ($i = 1, \dots, s$) the unique intersection point of E_i and E_{i+1}. Denote also by $S = \{ p_1, p_2, \dots, p_s \}$ the set of singular point of E and choose their indices in such a way that $\nu^{-1}(p_i) \subset E_i \cup E_{i+1}$. Note that the latter preimage always consists of 2 points, which we denote by p'_i, p''_{i+1} so that $p'_i \in E_i$ and $p''_{i+1} \in E_{i+1}$. Let $\mathcal{J}$ be the *conductor* of $\widetilde{\mathcal{O}}$ in $\mathcal{O}$, i.e., the biggest $\widetilde{\mathcal{O}}$-ideal contained in $\mathcal{O}$. Then $\mathcal{O}_i / \mathcal{J} \mathcal{O}_i = \mathbf{k}(p'_i) \oplus \mathbf{k}(p''_i)$.

Remind the description of vector bundles on a cyclic configuration given in [8]. Call an *s-sequence* any sequence of integers $\mathbf{d} = (d_1, \dots, d_{rs})$ of length multiple of s. Call this sequence *aperiodic* if it cannot be obtained by repetition of a shorter s-sequence. Call an *s-shift* of $\mathbf{d}$ any sequence $\mathbf{d}^k = (d_{ks+1}, \dots, d_{rs}, d_1, \dots, d_{ks})$.

Theorem 2.1. *Indecomposable vector bundles on a cyclic configuration are in one-to-one correspondence with the triples $\mathbf{B} = (\mathbf{d}, m, \lambda)$, where m is a positive integer, $\lambda \in \mathbf{k} \setminus 0$ and $\mathbf{d}$ is a non-periodic s-sequence defined up to an s-shift.*

We denote the vector bundle corresponding to the triple $\mathbf{B}$ by $\mathcal{G}(\mathbf{B})$. The precise construction of this sheaf is the following:

1. Let $\mathbf{d} = (d_1, \dots, d_{rs})$. Put $\mathcal{G}_i = m\mathcal{O}_i(d_i)$ and $\widetilde{\mathcal{G}}(\mathbf{B}) = \bigoplus_{i=1}^{rs} \mathcal{G}_i$. Note that $\mathcal{G}_i / \mathcal{J} \mathcal{G}_i \simeq m\mathbf{k}(p'_i) \oplus m\mathbf{k}(p''_i)$. Put also
$$F(\mathbf{B}) = \widetilde{\mathcal{G}}(\mathbf{B}) / \mathcal{J} \widetilde{\mathcal{G}}(\mathbf{B}) \simeq \bigoplus_{i=1}^{rs} \big(m\mathbf{k}(p'_i) \oplus m\mathbf{k}(p''_i) \big) .$$
As $F(\mathbf{B})$ is a sky-scraper sheaf, we identify it with the vector space of its global sections.
2. Choose a $\mathbf{k}$-bases of this factor $\{ \mathbf{e}'_{ik}, \mathbf{e}''_{ik} \}$ ($k = 1, \dots, m$) in such a way that $\{ \mathbf{e}'_{ik} \}$ form a basis of $m\mathbf{k}(p'_i)$ and $\{ \mathbf{e}''_{ik} \}$ form a basis of $m\mathbf{k}(p''_i)$.
3. Define the elements $\mathbf{e}_{ik} \in F(\mathbf{B})$ ($i = 1, \dots, rs,\ k = 1, \dots, m$) in the following way:
$$\mathbf{e}_{ik} = \begin{cases} \mathbf{e}'_{ik} + \mathbf{e}''_{i+1,k} & \text{if } i \neq rs , \\ \mathbf{e}'_{rs,k} + \lambda \mathbf{e}''_{1k} + \mathbf{e}''_{1,k-1} & \text{if } i = rs,\ k \neq 1 , \\ \mathbf{e}'_{rs,1} + \lambda \mathbf{e}_{11} & \text{if } i = rs,\ k = 1 . \end{cases}$$
Denote by $G(\mathbf{B})$ the subspace of $F(\mathbf{B})$ with the basis
$$\{ \mathbf{e}_{ik} \mid i = 1, \dots, rs,\ k = 1, \dots, m \} .$$

4. Now $\mathcal{G}(\mathbf{B})$ is defined as the preimage of $G(\mathbf{B})$ in $\widetilde{\mathcal{G}}(\mathbf{B})$.
   Note that $\operatorname{rk}\mathcal{G}(\mathbf{B}) = mr$.

To use Kahn's Theorem we have to calculate the cohomologies of the sheaves $\mathcal{G}(\mathbf{B})$. Let $H(\mathbf{B}) = \widetilde{\mathcal{G}}(\mathbf{B})/\mathcal{G}(\mathbf{B}) = F(\mathbf{B})/G(\mathbf{B})$. The exact sequence $0 \to \mathcal{G}(\mathbf{B}) \to \widetilde{\mathcal{G}}(\mathbf{B}) \to H(\mathbf{B}) \to 0$ gives the exact sequence of cohomologies

$$0 \to \mathrm{H}^0(\mathcal{G}(\mathbf{B})) \to \mathrm{H}^0(\widetilde{\mathcal{G}}(\mathbf{B})) \xrightarrow{h(\mathbf{B})} H(\mathbf{B}) \to \mathrm{H}^1(\mathcal{G}(\mathbf{B})) \to \mathrm{H}^1(\widetilde{\mathcal{G}}(\mathbf{B})) \to 0$$

(we identify $H(\mathbf{B})$ with the space of its global sections and omit "E" in the notation of cohomologies). One knows that $\dim_{\mathbf{k}} \mathrm{H}^0(\widetilde{\mathcal{G}}(\mathbf{B})) = m\sum_{i=1}^{rs}(d_i+1)^+$ and $\dim_{\mathbf{k}} \mathrm{H}^1(\widetilde{\mathcal{G}}(\mathbf{B})) = m\sum_{i=1}^{rs}(d_i+1)^-$, where

$$k^+ = \begin{cases} k & \text{if } k > 0\,, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad k^- = \begin{cases} -k & \text{if } k < 0\,, \\ 0 & \text{otherwise}\,, \end{cases},$$

while $\dim_{\mathbf{k}} T = mrs$. So we only have to calculate $\operatorname{rk} h(\mathbf{B})$. Denote by f the natural homomorphism $\mathrm{H}^0(\widetilde{\mathcal{G}}(\mathbf{B})) \to F(\mathbf{B})$. Then $\operatorname{rk} h(\mathbf{B}) = \dim_{\mathbf{k}}(\operatorname{Im} f + G(\mathbf{B})) - \dim_{\mathbf{k}} G(\mathbf{B})$.

Identifying E_i with $\mathbb{P}^1$ we can suppose that $p_i' = (1:0)$ and $p_i'' = (0:1)$. If $d_i > 0$, a basis $\{g_0, \dots, g_{d_i}\}$ in $\mathrm{H}^0(\mathcal{O}_i(d_i))$ can be chosen in such a way that $g_0(p_i') = 1,\ g_1(p_i'') = 1$, while all other values $g_j(p_i')$ and $g_j(p_i'')$ equal 0, so the basic elements $\mathbf{e}'_{ik}$ and $\mathbf{e}''_{ik}$ can be chosen as the images of g_0 and g_1 respectively. Hence, $\operatorname{Im} f$ is generated by the set $\{\mathbf{e}'_{ik}, \mathbf{e}''_{ik} \mid d_i > 0\} \cup \{\mathbf{e}'_{ik} + \mathbf{e}''_{ik} \mid d_i = 0\}$.

Call a subsequence $\mathbf{p} = (d_{k+1}, \dots, d_{k+l})$, where $0 \le k < rs$ and $1 \le l \le rs$, a *positive part* of $\mathbf{d}$ if $d_{k+j} \ge 0$ for $j = 1, \dots, l$ and either $l = rs$ or both $d_k < 0$ and $d_{k+l+1} < 0$. For such a positive part put $\theta(\mathbf{p}) = l$ if either $l = rs$ or $\mathbf{p} = (0, \dots, 0)$ and $\theta(\mathbf{p}) = l+1$ otherwise. Denote by $\mathcal{P}$ the set of all positive parts of $\mathbf{d}$ and put $\theta(\mathbf{d}) = \sum_{\mathbf{p}\in\mathcal{P}} \theta(\mathbf{p})$. Put also $\delta(\mathbf{d},\lambda) = 1$ if $\mathbf{d} = (0, \dots, 0)$ and $\lambda = 1$ and $\delta(\mathbf{d},\lambda) = 0$ otherwise. The preceding consideration and a rather simple straightforward calculation show that, for $\mathbf{B} = (\mathbf{d}, m, \lambda)$,

$$\operatorname{rk} h(\mathbf{B}) = m\theta(\mathbf{d}) - \delta(\mathbf{d},\lambda)\,,$$

whence the following:

Theorem 2.2. *For the vector bundle $\mathcal{G} = \mathcal{G}(\mathbf{d}, m, \lambda)$,*

$$\dim_{\mathbf{k}} \mathrm{H}^0(E,\mathcal{G}) = m\left(\sum_{i=1}^{r} s(d_i+1)^+ - \theta(\mathbf{d})\right) + \delta(\mathbf{d},\lambda)$$

$$\dim_{\mathbf{k}} \mathrm{H}^1(E,\mathcal{G}) = m\left(\sum_{i=1}^{r} s(d_i+1)^- + rs - \theta(\mathbf{d})\right) + \delta(\mathbf{d},\lambda)\,.$$

We write $\mathbf{d} > 0$ if $d_i \ge 0$ for all $i = 1, \dots, rs$ and at least one of them is strictly positive. Denote by $\mathbf{0}$ the zero sequence $(0, \dots, 0)$ of length s (the unique non-periodic zero s-sequence).

Corollary 2.3. *A vector bundle $\mathcal{G} = \mathcal{G}(\mathbf{d}, m, \lambda)$ satisfies Kahn's conditions* (1),(2) *of* Theorem 1.3 *if and only if either* $\mathbf{d} > 0$ *or* $\mathbf{d} = \mathbf{0}$ *and* $\lambda \neq 1$ *.*

Suppose now that the cycle configuration E is actually the exceptional curve of a cusp singularity $X = \operatorname{Spec} A$. If $s > 1$, E_i are the components of E, so, the self-intersection indices $E_i \cdot E_i$ are well defined. Note that $E_i \cdot E_i \leq -2$ and at least one of them is smaller than -2. Let $b_i = -E \cdot E_i = -E_i \cdot E_i - 2$. If $s = 1$, put $b_1 = -E \cdot E$. Let $\mathbf{b}^r = (b_1, \dots, b_s, \dots, b_1, \dots, b_s)$ (r times). We write $\mathbf{b}$ for $\mathbf{b}^1$ Put also $n(\mathcal{G}) = \dim_{\mathbf{k}} \mathrm{H}^0(E, \mathcal{G}(E))$. Remind that in this case one can choose E itself as a reduction cycle.

Corollary 2.4. 1. *Suppose that a vector bundle $\mathcal{G} = \mathcal{G}(\mathbf{d}, m, \lambda)$ satisfies the Kahn conditions* (1),(2)*. Then*

$$n(\mathcal{G}) = m \left(\sum_{i=1}^{rs} (d_i - b_i + 1)^+ - \theta(\mathbf{d} - \mathbf{b}^r) \right) + \delta(\mathbf{d} - \mathbf{b}^r, \lambda)$$

and $n(\mathcal{G})\mathcal{O} \oplus \mathcal{G} \simeq R_E M(\mathbf{d}, m, \lambda)$, where $M(\mathbf{d}, m, \lambda)$ is a Cohen–Macaulay A-module of rank $rs + n(\mathcal{G})$.

2. *All Cohen–Macaulay modules $M(\mathbf{d}, m, \lambda)$ are indecomposable and every indecomposable Cohen–Macaulay A-module is isomorphic either to A or to one of the modules $M(\mathbf{d}, m, \lambda)$.*

Note that here either $\mathbf{d} > 0$ or $\mathbf{d} = \mathbf{0}$ and $\lambda \neq 1$.

3. Lifting families

Vector bundles on cyclic configurations form natural 1-parametric families. We are going to show that these families can be lifted to 1-parametric families of Cohen–Macaulay modules on cusp singularities. To do it, we first prove analogues of Kahn's results for families of modules and vector bundles. For the sake of simplicity, we suppose from now on that A is a fK-algebra and all other algebras and schemes are $\mathbf{k}$-algebras and $\mathbf{k}$-schemes. Moreover, we only consider $\mathbf{k}$-schemes S such that $\mathbf{k}(x) = \mathbf{k}$ for each closed point $x \in S$ (for instance, schemes of finite type over $\mathbf{k}$). We write $\otimes$, dim, $\times$, etc., instead of $\otimes_{\mathbf{k}}$, $\dim_{\mathbf{k}}$, $\times_{\operatorname{Spec} \mathbf{k}}$, etc.

Definition 3.1. Let S, T be $\mathbf{k}$-schemes, p be the projection of $S \times T$ onto T and g_t, where t is a closed point of T, be the embedding $S \simeq S \times t \to S \times T$.

- A *family of sheaves over S based on T* is, by definition, a coherent sheaf over $S \times T$ flat over T.
- Such a family is called a *family of vector bundles* (respectively, *of (maximal) Cohen–Macaulay sheaves* if it is a locally free sheaf over $S \times T$ (respectively, a sheaf of (maximal) Cohen–Macaulay modules).

- If $\mathcal{M}$ is a family of sheaves over S based on T and t is a closed point of T, denote by $\mathcal{M}(t)$ the sheaf $g_t^*\mathcal{M}$.
- We say that a family $\mathcal{F}$ is *generically generated over* S if there is an open subset $U \subseteq S$ such that $S \setminus U$ is a finite set of closed points and the restriction onto $U \times T$ of the natural homomorphism $\mathrm{p}^* \mathrm{p}_* \mathcal{F} \to \mathcal{F}$ is an epimorphism.

Proposition 3.2. *Suppose that T is a Cohen–Macaulay $\mathbf{k}$-scheme. A coherent sheaf $\mathcal{M}$ on $S \times T$ flat over T is a family of (maximal) Cohen–Macaulay modules if and only if $\mathcal{M}(t)$ is a sheaf of (maximal) Cohen–Macaulay modules over S for each closed point $t \in T$.*

Proof. As all these conditions are local, we can suppose that $S = \operatorname{Spec} A$, $T = \operatorname{Spec} B$, where A, B are local noetherian Cohen–Macaulay $\mathbf{k}$-algebras. Denote by $\mathbf{m}$ and $\mathbf{n}$ respectively there maximal ideals and choose a B-sequence $(b_1, b_2, \dots, b_m)$ in $\mathbf{n}$, where $m = \operatorname{Kr.dim} B$. Then $\mathbf{p} = \mathbf{m} \otimes B \oplus A \otimes \mathbf{n}$ is the maximal ideal of the local ring $C = A \otimes B$ and $C/\mathbf{p} = \mathbf{k}$. If $\mathcal{M}$ is a Cohen–Macaulay C-module flat over B, $(b_1, b_2, \dots, b_m)$ is an $\mathcal{M}$-sequence and $M = \mathcal{M}/(b_1, b_2, \dots, b_m)\mathcal{M}$ is a Cohen–Macaulay A-module that has a finite filtration with all factors isomorphic to $\mathcal{M}(\mathbf{n}) = \mathcal{M}/\mathbf{n}\mathcal{M}$. Now $\operatorname{Ext}^i_A(\mathbf{k}, M) = 0$ easily implies $\operatorname{Ext}^i_A(\mathbf{k}, \mathcal{M}(\mathbf{n})) = 0$ for $i < \operatorname{Kr.dim} A$, hence, $\mathcal{M}(\mathbf{n})$ is a Cohen–Macaulay A-module.

On the contrary, suppose that $\mathcal{M}(\mathbf{n})$ is a Cohen–Macaulay A-module and $\mathcal{M}$ is flat over B. Then M is also a Cohen–Macaulay A-module. Moreover, if $b \in B$ is a non-zero-divisor, then it is also a non-zero-divisor on $\mathcal{M}$ and $\mathcal{M}/b\mathcal{M}$ is flat over B/bB. Therefore, $(b_1, b_2, \dots, b_m)$ is an $\mathcal{M}$-sequence. Put $A' = C/(b_1, b_2, \dots, b_m)C$. It is an A-algebra finitely generated as A-module and M is a finitely generated A'-module. As A-module, it has a finite filtration with all factors isomorphic to $\mathcal{M}(\mathbf{n})$, so it is a Cohen–Macaulay module. Then it is also a Cohen–Macaulay A'-module [22, Proposition IV.11], hence, $\mathcal{M}$ is a Cohen–Macaulay C-module. □

Proposition 3.3. *Suppose that T is a smooth $\mathbf{k}$-scheme. A coherent sheaf $\mathcal{M}$ on $S \times T$ flat over T is a family of vector bundles if and only if $\mathcal{M}(t)$ is a vector bundle over S for each closed point $t \in T$.*

We return to Definitions and Notations 1.1 and prove some results analogous to those of Kahn about the relations between Cohen–Macaulay modules and vector bundles. In this case we call families of sheaves on X *families of A-modules.* A family of vector bundles over $\widetilde{X}$ based on a scheme T is called *full* if it is isomorphic to $((\pi \times 1)^*\mathcal{M})^{\vee\vee}$, where $\mathcal{M}$ is a family of Cohen–Macaulay A-modules based on T.

Proposition 3.4. (Cf. [16, Proposition 1.2].) *Suppose that a family $\mathcal{F}$ of vector bundles over $\widetilde{X}$ based on a smooth scheme T satisfies the following conditions:*

1. *$\mathcal{F}$ is generically generated over X.*
2. *The restriction map $\mathrm{p}_*\mathcal{F} \to \mathrm{p}_*\iota^*\mathcal{F}$ is an epimorphism (or, the same, the map $R^1\mathrm{p}^E_*\mathcal{F} \to R^1\mathrm{p}_*\mathcal{F}$ is a monomorphism).*

Then $\mathcal{F}$ is full.

Here ι denotes the embedding $\check{X} \times T \to \widetilde{X} \times T$, $p^E_*\mathcal{F}$ is the "*direct image with support E*", i.e., the kernel of the map $\mathrm{p}_*\mathcal{F} \to \mathrm{p}_*\iota^*\mathcal{F}$ and $R^1\mathrm{p}^E_*$ is the first derived functor of p^E_*.

Proof. Obviously, this claim is local, so one can suppose that $T = \operatorname{Spec} B$. Thus, $\mathrm{p}_*\mathcal{F}$ can be identified with the B-module $\mathrm{H}^0(\widetilde{X} \times T, \mathcal{F})$. Put $\mathcal{M} = (\pi \times 1)_*\mathcal{F}$, $\mathcal{F}' = (\pi \times 1)^*\mathcal{M}/(\text{torsion})$ and consider $\mathcal{F}'$ as a subsheaf of $\mathcal{F}$ in a natural way. Then $\mathcal{M}$ is flat over T, Cohen–Macaulay as so is $\mathcal{F}$, and coherent as π is proper. Moreover, as $\mathcal{F}$ is generically generated over $\widetilde{X}$, $\operatorname{codim} \operatorname{Supp} \mathcal{F}/\mathcal{F}' \geq 2$. But $\widetilde{X} \times T$ is normal, hence, any sheaf of the form $\mathcal{N}^\vee$ is completely defined by its stalks at the points of height 1. In particular, $((\pi \times 1)^*\mathcal{M})^\vee \simeq (\mathcal{F}')^\vee \simeq \mathcal{F}^\vee$, whence $\mathcal{F} \simeq \mathcal{F}^{\vee\vee} \simeq ((\pi \times 1)^*\mathcal{M})^{\vee\vee}$. □

Proposition 3.5. (Cf. [16, Proposition 1.6].) *Suppose that Z is a weak reduction cycle on $\widetilde{X}$, $\mathcal{F}$ is a family of vector bundles on $\widetilde{X}$ based on a smooth scheme T and $\mathcal{V} = \mathcal{F} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_Z$. If $\mathcal{V}$ is generically generated by global sections and the map $\mathrm{p}_*\mathcal{V}(Z) \to R^1\mathrm{p}_*\mathcal{V}$ induced by the exact sequence*

$$0 \longrightarrow \mathcal{V}(-Z) \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V} \longrightarrow 0,$$

where $\mathcal{V}_2 = \mathcal{F} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{2Z}$, is a monomorphism, then $\mathcal{F}$ is full.

Proof. Again one can suppose that T is affine; then one simply has to follow literally the proof of Kahn. □

Proposition 3.6. (Cf. [16, Proposition 1.9].) *Suppose that Z is a weak reduction cycle on $\widetilde{X}$ and a family $\mathcal{V}$ of vector bundles over Z based on a smooth affine scheme T of finite type satisfies the following conditions:*

1. *$\mathcal{V}$ is generically generated over Z.*
2. *There exists an extension of $\mathcal{V}$ to a family of vector bundles $\mathcal{V}_2$ on $2Z$ such that the induced map $\mathrm{p}_*\mathcal{V}(Z) \to R^1\mathrm{p}_*\mathcal{V}$ is injective.*

Then there is a full family of vector bundles $\mathcal{F}$ over $\widetilde{X}$ such that $\mathcal{V} \simeq \mathcal{F} \otimes_{\mathcal{O}_{tX}} \mathcal{O}_Z$. If, moreover, Z is a reduction cycle, for any other family of vector bundles $\mathcal{F}'$ over $\widetilde{X}$ such that $\mathcal{V} \simeq \mathcal{F}' \otimes_{\mathcal{O}_{tX}} \mathcal{O}_Z$ there is an open subset $U \subseteq T$ such that $\mathcal{F}|_U \simeq \mathcal{F}'|_U$.

Proof. As $\mathrm{H}^2(E \times T, \mathcal{N}) = 0$ for each coherent sheaf over $Z \times T$, any family of vector bundles over $2Z$ can be lifted to a family of vector bundles over nZ for every n, hence, to a family of vector bundles over $\widetilde{X}$. (Remind that we suppose that the Artin's Approximation holds.) Proposition 3.5 implies that $\mathcal{F}$ is full.

If Z is a reduction cycle, [16, Proposition 1.9] implies that $\mathcal{F}(g) \simeq \mathcal{F}'(g)$, where g is the generic point of T, whence that last assertion follows. □

For minimally elliptic singularities one gets a simpler version.

Proposition 3.7. (Cf. [16, Theorem 2.1].) *Let $\pi : \widetilde{X} \to X$ be the minimal resolution of a minimally elliptic singularity and Z be the fundamental cycle on $\widetilde{X}$. Suppose that a family of vector bundles $\mathcal{G}$ on Z based on a smooth affine scheme T of finite type satisfies the following conditions:*

1. *$\mathcal{G}$ is generically generated over Z.*
2. *$R^1\mathrm{p}_*(E, \mathcal{G}) = 0$.*
3. *$\mathrm{p}_*(E, \mathcal{G}(Z)) = \mathcal{P}$ is locally free over T.*

Then there is a full family of vector bundles $\mathcal{F}$ over $\widetilde{X}$ such that $\mathcal{F} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_Z \simeq \mathcal{G} \oplus \mathcal{O}_Z \otimes \mathcal{P}$. Moreover, if $\mathcal{F}'$ is another family with this property, there is an open subset $U \subseteq T$ such that $\mathcal{F}|_U \simeq \mathcal{F}'|_U$.

Proof. Again we can use the arguments of the corresponding proof from [16, Page 151] to show that the family $\mathcal{V} = \mathcal{G} \oplus \mathcal{O}_Z \otimes \mathcal{P}$ satisfies the conditions of Proposition 3.6. □

For cusp singularities we can now precise Corollary 2.4 in the following way. (We use the notations of Section 2.)

Corollary 3.8. *Let $X = \operatorname{Spec} A$ be a cusp singularity. For each non-periodic s-sequence $\mathbf{d} \geq 0$ there is a family of Cohen–Macaulay A-modules $\mathcal{M}(\mathbf{d})$ based on $\mathbf{k} \setminus \{0, 1\}$ if $\mathbf{d} = \mathbf{0}$ or $\mathbf{d} = \mathbf{b}$ and on $\mathbf{k} \setminus 0$ otherwise such that $M(\mathbf{d}, m, \lambda) \simeq \mathcal{M} \otimes_{\mathcal{O}_T} L(m, \lambda)$ for every $m \in \mathbb{N}$ and every $\lambda \in T$, where $L(m, \lambda)$ denotes the unique indecomposable sheaf on T with $\operatorname{Supp} L(m, \lambda) = \{\lambda\}$ and $\dim L(m, \lambda)_\lambda = m$.*

Proof. Consider the family of vector bundles over $\widetilde{E}$: $\widetilde{\mathcal{G}} = \bigoplus_{i=1}^r s\mathcal{O}_i \otimes \mathcal{O}_T$. Define $\mathcal{G}(\mathbf{d})$ as the preimage in $\widetilde{\mathcal{G}}$ of the sub-bundle in $\widetilde{\mathcal{G}}/\mathcal{J}\widetilde{\mathcal{G}}$ generated by the elements $\mathbf{e}'_i + \mathbf{e}''_{i+1}$ for $i \neq rs$ and the elements $\mathbf{e}'_{rs} + t\mathbf{e}_1$, where t denotes the coordinate on T and $\mathbf{e}'_i$, $\mathbf{e}''_i$ are defined as on page 5. It is easy to see that $\mathcal{G}(\mathbf{d}) \otimes_{\mathcal{O}_T} L(m, \lambda) \simeq \mathcal{G}(\mathbf{d}, m, \lambda)$. Corollaries 2.3 and 2.4 shows that such a family satisfies the conditions of Proposition 3.7. Hence, it gives a full family of vector bundles $\mathcal{F}$ over $\widetilde{X}$ such that $(\mathcal{F} \otimes_{\mathcal{O}_T} L(m, \lambda)) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_E \simeq mn(\mathbf{d})\mathcal{O}_E \oplus \mathcal{G}(\mathbf{d}, m\lambda)$, where $n(\mathbf{d}) = \sum_{i=1}^{rs}(d_i - b_i + 1) - \theta(\mathbf{d} - \mathbf{b}^r)$. Therefore, we obtain a family $\mathcal{M}(\mathbf{d})$ of Cohen–Macaulay A-modules just as we need. □

Note that among $\mathcal{M}(\mathbf{d}) \otimes_{\mathcal{O}_T} L(m, d)$ one obtains all indecomposable Cohen–Macaulay A-modules except $M(\mathbf{0}, m, 1)$ and $M(\mathbf{b}, m, 1)$.

4. Cohen–Macaulay tame and wild singularities

We provide now some formal definitions of "tame" and "wild" singularities (with respect to the classification of Cohen–Macaulay modules), analogous to the usual ones from the representation theory of algebras. Again we consider the case of $\mathbf{k}$-algebras.

Definition 4.1. Let $X = \operatorname{Spec} A$ be a surface singularity. It is said to be *Cohen–Macaulay tame* if it is not Cohen–Macaulay finite and for every r there is a finite set $\mathfrak{M}_r = \{ \mathcal{M}_r^\alpha \}$ of families of Cohen–Macaulay A-modules such that:

- each $\mathcal{M}_r^\alpha$ is based on a smooth curve C_r^α;
- each indecomposable Cohen–Macaulay A-module of rank r is isomorphic to $\mathcal{M}_r^\alpha(t)$ for some index α and some closed point $t \in C_r^\alpha$.

Note that in the second condition one can replace "each indecomposable module" by "each indecomposable module except a finite set of them."

Example 4.2. 1. Corollary 3.8 shows immediately that every cusp singularity is Cohen–Macaulay tame.
2. Just in the same way, the results of [16] show that every simple elliptic singularity is also Cohen–Macaulay tame (the corresponding families are based either on $\operatorname{Pic}^0 E$ or on $\operatorname{Pic}^0 E \setminus \{ 1 \}$).

Remark 4.3. Denote by $\operatorname{Pic}^0 E$ the subgroup of the Picard group of a (singular) curve E consisting of the classes of all line bundles $\mathcal{L}$ such that the degree of the restriction of $\mathcal{L}$ onto every irreducible component of E equals 0. One can easily see that $\operatorname{Pic}^0 E \simeq \mathbf{k} \setminus \{ 0 \}$ for every cusp singularity. So both examples above have a lot of common features. Nevertheless, there is one that provides an essential difference. Indeed, Kahn's description shows that there is a positive integer N such that for each fixed r the set $\mathfrak{M}_r$ has at most Nr elements. On the contrary, for cusp singularities $\operatorname{card} \mathfrak{M}_r$ grows exponentially when $r \to \infty$. They say that simple elliptic singularities are *Cohen–Macaulay tame of bounded growth*, while cusp singularities are *Cohen–Macaulay tame of unbounded growth.*

Definition 4.4. Let $X = \operatorname{Spec} A$ be a surface singularity. It is said to be *Cohen–Macaulay wild* if for every finitely generated (not necessarily commutative) $\mathbf{k}$-algebra Λ there is an exact functor $F : \operatorname{mod} \Lambda \to \operatorname{CM}(A)$ such that:

- if L is indecomposable, so is FL;
- if $L \not\simeq L'$, also $FL \not\simeq FL'$.

Remark 4.5. It is well known (cf. [6]) that to prove that X is Cohen–Macaulay wild one only has to find such a functor F for one of the following algebras:

(i) $\mathbf{k}\langle x, y\rangle$, the free non-commutative algebra with 2 generators;
(ii) $\mathbf{k}[x, y]$, the polynomial algebra with 2 generators;
(iii) $\mathbf{k}[[x, y]]$, the power series algebra with 2 generators.

In case (ii) such a functor is always given by a family of Cohen–Macaulay modules based on $\mathbb{A}^2$. This family is then called *strict.* In view of (iii), it is even enough to construct a strict family of Cohen–Macaulay modules based on any open subset $U \subseteq \mathbb{A}^2$.

Obviously, if X is Cohen–Macaulay wild it has families $\mathcal{M}$ of Cohen–Macaulay modules based on arbitrary algebraic variety T such that all modules $\mathcal{M}(t)$ with $t \in T$ are indecomposable and pairwise non-isomorphic.

Proposition 4.6. *Suppose that a minimally elliptic singularity is neither simple elliptic nor cusp. Then it is Cohen–Macaulay wild.*

Proof. Remind [8] that E is vector bundle wild if E is neither smooth rational or elliptic curve, nor cyclic or linear configuration, where a *linear configuration* is a curve E such that all its components E_i ($i = 1, \dots, s$) are smooth rational, $E_i \cap E_j = 0$ if $j \neq i \pm 1$, while E_i and E_{i+1} intersect transversally in exactly one point for $i = 1, \dots, s-1$. The latter case is impossible as X is not rational. So, if X is neither simple elliptic nor cusp, the fundamental cycle Z is vector bundle wild, i.e., there is a strict family of vector bundles $\mathcal{G}$ over Z based on $\mathbb{A}^2$. By Serre's Theorems [13, Theorems II.5.17 and III.5.2], there is an integer n such that $\mathcal{G}(n)$ is generated by global sections and $R^1\mathrm{p}_*\mathcal{G}(n) = 0$. As $\mathcal{G}(n)$ is obviously also strict, one can suppose that $\mathcal{G}$ itself has these properties. Moreover, $\mathrm{p}_*\mathcal{G}$ is coherent, hence, there is an open affine subset $U \subseteq \mathbb{A}^2$ such that $\mathrm{p}_*\mathcal{G}|U$ is locally free. Replace $\mathbb{A}^2$ by U (cf. Remark 4.5). By Proposition 3.7, there is a full family $\mathcal{F}$ of vector bundles over $\widetilde{X}$ based on U such that $\mathcal{F} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_Z \simeq \mathcal{G} \oplus \mathcal{O}_Z \otimes \mathrm{p}_*\mathcal{G}$. Let $\mathcal{F} = (\pi^*\mathcal{M})^{\vee\vee}$, where $\mathcal{M}$ is a family of Cohen–Macaulay modules. As $\mathcal{G}$ is strict, Kahn's theorem (Theorem 1.3) imply that $\mathcal{M}$ is also strict, hence, X is Cohen–Macaulay wild. $\square$

Consider one procedure that allows to obtain new Cohen–Macaulay tame singularities.

Definition 4.7. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ be normal surface singularities, $X \to Y$ be a finite morphism given by an embedding $B \subseteq A$. Call it:

- a *split extension* if $A \simeq B \oplus A/B$ as B-module;
- a *weakly separable extension* if the epimorphism of A-bimodules $A \boxtimes A \to A$ induced by the natural homomorphism $A \otimes_B A \to A$ splits.

Here and further on, for every B-module N we put $A \boxtimes N = (A \otimes_B N)^{\vee\vee}$ (as A-module).

Proposition 4.8. (We keep the notations of the latter definition.)

1. *If $X \to Y$ is a split extension and X is Cohen–Macaulay tame, so is also Y.*
2. *If $X \to Y$ is a weakly separable extension and Y is Cohen–Macaulay tame, so is also X.*

Proof. 1. Suppose that this extension is split. If N is any Cohen–Macaulay B-module, then it is a direct summand of $A \otimes_B N$, hence, of $A \boxtimes N$ as of B-module. Suppose that A is Cohen–Macaulay tame. For every family $\mathcal{M}_r^\alpha$ from Definition 4.1 let t_r^α be the general point of the curve C_r^α and $\mathcal{M}_r^\alpha(t_r^\alpha) = \bigoplus_\beta \tilde{M}_r^{\alpha\beta}$, where $\tilde{M}_r^{\alpha\beta}$ are indecomposable $B \otimes \mathbf{k}(t_r^\alpha)$-modules. There is an open subset $U_r^\alpha \subseteq C_r^\alpha$ and a decomposition $\mathcal{M}_r^\alpha|_U \simeq \bigoplus_\beta \mathcal{N}_r^{\alpha\beta}$ (as of family of B-modules) such that $\mathcal{N}_r^{\alpha\beta}(x)$ is indecomposable for all $x \in U_r^\alpha$. Certainly, for every integer m there is only a finite set of $\mathcal{N}_r^{\alpha\beta}$ which are of rank m.

Let N be any indecomposable Cohen–Macaulay B-module. Then it is a direct summand of an indecomposable Cohen–Macaulay A-module M. If $\operatorname{rk} M = r$, $M \simeq \mathcal{M}_r^\alpha(x)$ for some α and some $x \in C_r^\alpha$. If $x \in U_r^\alpha$, $N \simeq \mathcal{N}_r^{\alpha\beta}(x)$ for some β, while there are only finitely many B-direct summands of $\mathcal{M}_r^\alpha(x)$ with $x \notin U_\alpha^r$. Therefore, B is Cohen–Macaulay tame.

2. Suppose that this extension is weakly separable. If M is any Cohen–Macaulay A-module, one easily verifies that $(A \boxtimes M)^\vee \simeq ((A \boxtimes A) \otimes_A M)^\vee$. Hence, $M^\vee$ is isomorphic to a direct summand of $(A \boxtimes M)^\vee$ and M is isomorphic to a direct summand of $A \boxtimes M$. So, there is an indecomposable Cohen–Macaulay B-module N such that M is isomorphic to a direct summand of $A \boxtimes N$. Now the same considerations as in the part 1 above show that if B is Cohen–Macaulay tame, so is A. □

The most interesting example of split and weakly separable extensions are obtained from actions of finite groups.

Proposition 4.9. *Let G be a finite group of automorphisms of the algebra A, $B = A^G$ be its subalgebra of G-invariants.*

1. *If* $\operatorname{char} \mathbf{k}$ *does not divide* $\operatorname{card} G$, *the extension $B \subseteq A$ is split.*
2. *If every prime ideal $\mathfrak{p} \subset A$ is unramified, the extension $B \subseteq A$ is weakly separable.*

Remind that an ideal $\mathfrak{p}$ is unramified *if* $\mathfrak{p}A_\mathfrak{p} = (\mathfrak{p} \cap B)A_\mathfrak{p}$.

In the situation of this proposition we say that the singularity $Y = \operatorname{Spec} B$ is a *quotient* of the singularity $X = \operatorname{Spec} A$ by the group G.

Proof. 1. We get a splitting $f : A \to B$ putting $f(a) = \frac{1}{n} \sum_{\sigma \in G} \sigma(a)$, where $n = \operatorname{card} G$.

2. As $(A \otimes_B A)^\vee \simeq \operatorname{Hom}_B(A, A)$, the second claim follows from the isomorphism $\operatorname{Hom}_B(A, A) \simeq A[G]$ (cf. [3]). □

Call a surface singularity Y *elliptic-quotient* if it is a quotient of either a simple elliptic or a cusp singularity by a group G such that $\operatorname{char}\mathbf{k}$ does not divide $\operatorname{card} G$. If $\operatorname{char}\mathbf{k}=0$, elliptic-quotient singularities coincide with those *log-canonical* ones that are not quotient singularities (cf. [18]).

Corollary 4.10. *Every elliptic-quotient surface singularity is Cohen–Macaulay tame.*

The preceding consideration gives rise to a general conjecture concerning Cohen–Macaulay tame and wild singularities.

Conjecture 4.11. *If a surface singularity is neither quotient nor elliptic-quotient, it is Cohen–Macaulay wild.*

5. Curve singularities T_{pq}

Important example of cusp singularities for $\operatorname{char}\mathbf{k}=0$ are the "serial" unimodal singularities [1] $\mathrm{T}_{pqr}:\ A=\mathbf{k}[[x,y,z]]/(x^p+y^q+z^r+\alpha xyz)$, where $r\le p\le q$ and $1/p+1/q+1/r<1$ (in this case all values of α lead to isomorphic algebras). Note that if $1/p+1/q+1/r=1$, the corresponding singularity is simple elliptic except some special values of α. If $r=2$, Cohen–Macaulay modules over this singularity are closely related to those over the *curve singularity* T_{pq}. Indeed, in this case the singularity T_{pq2} can be rewritten in the form: $A=\mathbf{k}[[x,y,z]]/(z^2+x^p+y^q+\beta x^2y^2)$ for $\beta=-\alpha^2/4$. Therefore, one can use the *Knörrer correspondence* (cf. [19, 23]) described in the following proposition to relate Cohen–Macaulay modules over A and over $A/(z^2)=\mathbf{k}[[x,y]]/(x^p+y^q+\beta x^2y^2)$.

Proposition 5.1. *Let $f\in\mathbf{k}[[x_1,x_2,\dots,x_n]]$ be a non-invertible element, $A=\mathbf{k}[[x_0,x_1,\dots,x_n]]/(x_0^2+f)$ and $A'=\mathbf{k}[[x_1,x_2,\dots,x_n]]/(f)$. For every Cohen–Macaulay module M over A let $\operatorname{rest}M=M/zM$ and M^σ be the A-module that coincide with M as a group, while the action of the elements of A is given by the rule: $a\cdot_{M^\sigma}v=a^\sigma\cdot_M v$, where $g(x_0,x_1,\dots,x_n)^\sigma=g(-x_0,x_1,x_2,\dots,x_n)$. Then:*

- *every indecomposable Cohen–Macaulay A'-module is a direct summand of $\operatorname{rest}M$ for an indecomposable Cohen–Macaulay A-module M and $M\simeq A$ if and only if $\operatorname{rest}M\simeq A'$;*
- *if M is an indecomposable Cohen–Macaulay A-module such that $M\not\simeq M^\sigma$, $\operatorname{rest}M$ is also indecomposable;*
- *if $M\not\simeq A$ is an indecomposable Cohen–Macaulay A-module such that $M\simeq M^\sigma$, then $\operatorname{rest}M$ is a direct sum of two non-isomorphic indecomposable Cohen–Macaulay A'-modules, which we denote by rest_1M and rest_2M.*
- *if M,N are non-isomorphic indecomposable Cohen–Macaulay A-modules and $M\not\simeq N^\sigma$, then the indecomposable A'-modules obtained from M and from N as described above are also non-isomorphic.*

Obviously, always $\operatorname{rest} M \simeq \operatorname{rest}(M^\sigma)'$ and $A^\sigma \simeq A$.

So, to describe the Cohen–Macaulay modules over the curve singularity T_{pq} we have to know when $M^\sigma \simeq M$ for a Cohen–Macaulay module M over the surface singularity T_{pqr}. The automorphism $g \to g^\sigma$ induces an automorphism of the minimal resolution $\widetilde{X}$, hence, an automorphism of the exceptional curve E and of the category of vector bundles over E. We denote all these automorphisms by σ too. The following result is immediate.

Proposition 5.2. *$R_E M^\sigma \simeq (R_E M)^\sigma$ for every Cohen–Macaulay A-module M.*

From the description of the minimal resolutions of T_{pqr} given, for instance, in [17, 20], one can deduce the following shape of the exceptional curve E for the minimal resolution of T_{pq2} and for the action of the automorphism σ on it.

Proposition 5.3. (We use the notations from Section 2.)

1. *If $p = 3$ (hence, $q \geq 7$), then $s = q - 6$. If $q > 7$, E has 1 component E_1 with self-intersection -3 and $s - 1$ components $E_2, \dots, E_s$ with self-intersection -2; $E_1^\sigma = E_1$ and $E_i^\sigma = E_{s+2-i}$. If $q = 7$, E is irreducible with one node and $E \cdot E = -1$. In both cases $\sigma(p_1'') = p_1'$.*
2. *If $p = 4$ (hence, $q \geq 5$), then $s = q - 4$. If $q > 5$, E has 1 component E_1 with $E_1 \cdot E_1 = -4$ and $s - 1$ components $E_2, \dots, E_s$ with $E_i \cdot E_i = -2$. If $q = 5$, E is irreducible with one node and $E \cdot E = -2$. The action of σ is the same as in the previous case.*
3. *If $p \geq 5$, then $s = p + q - 8$. Put $t = p - 3$. E has 2 components E_1 and E_t with $E_i \cdot E_i = -3$ and $s - 2$ components with $E_i \cdot E_i = -2$ (no one if $p = q = 5$). $E_i^\sigma = E_{t+1-i}$ if $1 \leq s \leq t$ and $E_i^\sigma = E_{s+t+1-i}$ if $t < i \leq s$; $\sigma(p_1'') = p_t'$.*

Given an s-sequence $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$, denote by $\mathbf{d}^\sigma$ the sequence $(d_1', d_2', \dots, d_{rs}')$ obtained from $\mathbf{d}$ by the following procedure (we keep the notations of Proposition 5.3):

- if $p = 3$ or $p = 4$, then $d_1' = d_1$ and $d_i' = d_{rs+2-i}$ for $1 < i \leq rs$;
- if $p \geq 5$, then $d_i' = d_{t+1-i}$ for $1 \leq i \leq t$ and $d_i' = d_{rs+t+1-i}$ for $t < i \leq rs$.

Call an s-sequence $\mathbf{d}$ *σ-symmetric* if $\mathbf{d}^\sigma = \mathbf{d}^k$, an s-shift of $\mathbf{d}$ (cf. page 5). Then the description of Cohen–Macaulay modules from Section 2 implies the following results.

Corollary 5.4. *Let $M = M(\mathbf{d}, m, \lambda)$ be an indecomposable Cohen–Macaulay module over the surface singularity T_{pq2} (cf. Corollary 2.4). Then $M^\sigma \simeq M(\mathbf{d}^\sigma, m, 1/\lambda)$. In particular, $M \simeq M^\sigma$ if and only if the sequence $\mathbf{d}$ is σ-symmetric and $\lambda = \pm 1$.*

Corollary 5.5. *The indecomposable Cohen–Macaulay modules over the curve singularity* $A' = \mathbf{k}[[x,y]]/(x^p + y^q + \beta x^2y^2)$ *of type* T_{pq} $(q \ge p,\ 1/p + 1/q < 1/2\,)$ *are the following:*

- $N(\mathbf{d},m,\lambda) = \operatorname{rest} M(\mathbf{d},m,\lambda)$ *, where either the sequence* $\mathbf{d}$ *is not* σ*-symmetric or* $\lambda \ne \pm 1$ *;*
- $N_i(\mathbf{d},m,\pm 1) = \operatorname{rest}_i M(\mathbf{d},m,\pm 1)$ $(i = 1,2\,)$*, where* $\mathbf{d}$ *is* σ*-symmetric and either* $\mathbf{d} \ne \mathbf{0}$ *or* $\lambda = -1$ *;*
- *regular module* A' *.*

The only isomorphisms between so defined Cohen–Macaulay modules are $N(\mathbf{d},m,\lambda) \simeq N(\mathbf{d}^\sigma,m,1/\lambda)$ *.*

Note that the modules $M(\mathbf{d},m,\lambda)$ form rational families based on $\mathbf{k} \setminus \{\,0\,\}$ if $\mathbf{d}$ is not σ-symmetric and on $\mathbf{k} \setminus \{\,0,1,-1\,\}$ if $\mathbf{d}$ is σ-symmetric.

One can consider higher-dimensional hypersurface singularities of type T_{pqr} given by the equation $x^p + y^q + z^r + \alpha xyz + \sum_{i=1}^k t_i^2 = 0$ for some k (maybe, 0). Due to the Knörrer correspondence, all of them are Cohen–Macaulay tame.

Conjecture 5.6. *If an isolated hypersurface singularity is neither simple nor of type* T_{pqr} *, it is Cohen–Macaulay wild.*

If Conjectures 4.11 and 5.6 are true, they imply finite/tame/wild trichotomy and an explicit description both of tame surface and hypersurface singularities and of Cohen–Macaulay modules in tame case.

6. Auslander–Reiten quivers

In this section we calculate Auslander–Reiten quivers for Cohen–Macaulay modules and related vector bundles. Remind that an Auslander–Reiten sequence in a category of modules or of vector bundles is a non-split exact sequence

$$0 \longrightarrow M' \xrightarrow{g} N \xrightarrow{f} M \longrightarrow 0\,,$$

where M, M' are indecomposable, such that any homomorphism $N' \to M$ that is not a split epimorphism factors through f (equivalently, any homomorphism $M' \to N'$ that is not a split monomorphism factors through g). In [5] they proved that for any indecomposable vector bundle $\mathcal{G}$ on a projective curve E there is an Auslander–Reiten sequence

$$0 \longrightarrow \omega_E \otimes_{\mathcal{O}_E} \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0\,.$$

In particular, if E is a cyclic configuration, then, using the notation of Section 2, $\omega_E \simeq \mathcal{O}_E \simeq \mathcal{G}(\mathbf{0},1,1)$ and if $\mathcal{G} = \mathcal{G}(\mathbf{d},m,\lambda)$, there is an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0\,,$$

where

$$\mathcal{E} \otimes_{\mathcal{O}_E} \mathcal{G}(\mathbf{d}, m, \lambda) \simeq \begin{cases} \mathcal{G}(\mathbf{d}, 2, \lambda) & \text{if } m = 1\,, \\ \mathcal{G}(\mathbf{d}, m+1, \lambda) \oplus \mathcal{G}(\mathbf{d}, m-1, \lambda) & \text{if } m > 1\,, \end{cases}$$

that is easily recognized as Auslander–Reiten sequence (if $\operatorname{char} \mathbf{k} = 0$ it follows immediately from [5], and the combinatoric description of $\mathrm{VB}(X)$ in [8] does not depend on the characteristic). In particular, the Auslander–Reiten quiver of the category of vector bundles over a cyclic configuration is a disjoint union of *simple tubes*

$$\mathcal{T}(\mathbf{d}, \lambda): \quad \mathcal{G}(\mathbf{d}, 1, \lambda) \leftrightarrows \mathcal{G}(\mathbf{d}, 2, \lambda) \leftrightarrows \mathcal{G}(\mathbf{d}, 3, \lambda) \leftrightarrows \dots$$

where $\mathbf{d}$ runs through non-periodic s-sequences and $\lambda \in \mathbf{k} \setminus \{0\}$.

Passing from vector bundles to Cohen–Macaulay modules, we use the following result of Kahn:

Proposition 6.1. [16, Theorem 3.1] *If X is a minimally elliptic singularity, Z is its fundamental cycle and $0 \to M \to N \to M \to 0$ is an Auslander–Reiten sequence in $\mathrm{CM}(X)$, then the exact sequence $0 \to R_Z M \to R_Z N \to R_Z M \to 0$ is the direct sum of an Auslander–Reiten sequence $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{G} \to 0$ and a split sequence $0 \to n\mathcal{O}_Z \to 2n\mathcal{O}_Z \to n\mathcal{O}_Z \to 0$.*

(Note that, as minimally elliptic singularity is Gorenstein, any Auslander–Reiten sequence sequence in $\mathrm{CM}(X)$ is of the form $0 \to M \to N \to M \to 0$, cf. [5, Theorem I.3.1].)

Corollary 6.2. *Auslander–Reiten sequence of Cohen–Macaulay modules over a cusp singularity are the following:*

$0 \to M(\mathbf{d}, 1, \lambda) \to M(\mathbf{d}, 2, \lambda) \to M(\mathbf{d}, 1, \lambda) \to 0$ *if* $\mathbf{d} \neq \mathbf{b}$ *or* $\lambda \neq 1$;

$0 \to M(\mathbf{b}, 1, 1) \to A \oplus M(\mathbf{b}, 2, 1) \to M(\mathbf{b}, 1, 1) \to 0$;

$0 \to M(\mathbf{d}, m, \lambda) \to M(\mathbf{d}, m+1, \lambda) \oplus M(\mathbf{d}, m-1, \lambda) \to M(\mathbf{d}, m, \lambda) \to 0$ *if* $m > 1$.

In particular, Auslander–Reiten quiver of the category $\mathrm{CM}(X)$, where X is a cusp singularity, is a disjoint union of simple tubes

$$\mathfrak{T}(\mathbf{d}, \lambda): \quad M(\mathbf{d}, 1, \lambda) \leftrightarrows M(\mathbf{d}, 2, \lambda) \leftrightarrows M(\mathbf{d}, 3, \lambda) \leftrightarrows \dots$$

where $\mathbf{d} \neq \mathbf{b}$ or $\lambda \neq 1$, and one "special" tube

$$\mathfrak{T}(\mathbf{b}, 1): \quad A \leftrightarrows M(\mathbf{b}, 1, 1) \leftrightarrows M(\mathbf{b}, 2, 1) \leftrightarrows M(\mathbf{b}, 3, 1) \leftrightarrows \dots$$

enlarged by the regular module A that is both projective and ext-injective in this category. Remind that in $M(\mathbf{d}, m, \lambda)$ always either $\mathbf{d} > 0$ or $\lambda \neq 1$.

At last, passing from the singularities T_{pq2} to T_{pq}, one only has to note that Knörrer correspondence maps an Auslander–Reiten sequence $0 \to M \to N \to M \to 0$ either to an Auslander–Reiten sequence $0 \to \operatorname{rest} M \to \operatorname{rest} N \to \operatorname{rest} M \to 0$ (if $M^\sigma \not\simeq M$) or to the direct sum of Auslander–Reiten sequences $0 \to \operatorname{rest}_2 M \to N' \to \operatorname{rest}_1 M$ and $0 \to \operatorname{rest}_1 M \to N'' \to \operatorname{rest}_2 M \to 0$ if $M \simeq M^\sigma$ (cf. [19, Corollary 2.10]). In particular, considering the Auslander–Reiten sequence $0 \to M(\mathbf{b},1,1) \to A \oplus M(\mathbf{b},2,1) \to M(\mathbf{b},1,1) \to 0$, we see that A' must be in the middle term of the Auslander–Reiten sequence starting either from $N_1(\mathbf{b},1,1)$ or from $N_2(\mathbf{b},1,1)$. We choose the second possibility (it only depends on the notation). We also choose the notations in such a way that $N_i(\mathbf{d}, m+1, \pm 1)$ were a direct summand of the middle term of the Auslander–Reiten sequence starting from $N_i(\mathbf{d}, m, \pm 1)$ ($i = 1, 2$). Then the Auslander–Reiten quiver is in this case a disjoint union of simple tubes

$$\widetilde{\mathfrak{T}}(\mathbf{d},\lambda): \quad N(\mathbf{d},1,\lambda) \leftrightarrows N(\mathbf{d},2,\lambda) \leftrightarrows N(\mathbf{d},3,\lambda) \leftrightarrows \dots$$

for non-σ-symmetric $\mathbf{d}$ or for $\lambda \neq 1$; regular tubes $\widetilde{\mathfrak{T}}(\mathbf{d},\pm 1)$ of period 2 for σ-symmetric $\mathbf{d}$ ($\mathbf{d} \neq \mathbf{b}$ or $\lambda = -1$):

$$\begin{tikzcd}
N_1(\mathbf{d},1,\pm 1) \arrow[r] & N_1(\mathbf{d},2,\pm 1) \arrow[r] \arrow[dl] & N_1(\mathbf{d},3,\pm 1) \arrow[r] \arrow[dl] & \cdots \arrow[dl] \\
N_2(\mathbf{d},1,\pm 1) \arrow[r] & N_2(\mathbf{d},2,\pm 1) \arrow[r] \arrow[ul] & N_2(\mathbf{d},3,\pm 1) \arrow[r] \arrow[ul] & \cdots \arrow[ul]
\end{tikzcd}$$

and one "special" tube $\widetilde{\mathfrak{T}}(\mathbf{b},1)$ of period 2 enlarged by the regular module A':

$$\begin{tikzcd}
 & N_1(\mathbf{b},1,1) \arrow[r] & N_1(\mathbf{b},2,1) \arrow[r] \arrow[dl] & N_1(\mathbf{b},3,1) \arrow[r] \arrow[dl] & \cdots \arrow[dl] \\
A' \arrow[ur] & & & & \\
 & N_2(\mathbf{b},1,1) \arrow[r] \arrow[ul] & N_2(\mathbf{b},2,1) \arrow[r] \arrow[uul] & N_2(\mathbf{b},3,1) \arrow[r] \arrow[uul] & \cdots \arrow[uul]
\end{tikzcd}$$

References

[1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko. *Singularities of Differentiable Maps.* Vol. I. Birkhäuser, Boston, 1985.

[2] M. Artin. Algebraic approximation of structures over complete local rings. Publ. math. IHES, 36 (1969), 23–58.

[3] M. Auslander. On the purity of the brunch loci. Amer. J. Math. 84 (1962), 116–125.

[4] M. Auslander. Rational singularities and almost split sequences. Trans. Amer. Math. Soc. 293 (1986), 511-531

[5] M. Auslander and I. Reiten. Almost split sequences in dimension two. Adv. Math. 66 (1987), 88–118.

[6] Y. A. Drozd. Tame and wild matrix problems. In: *Representations and Quadratic Forms*, Institute of mathematics, Kiev, 1979, 39–74. (English translation in: Amer. Math. Soc. Translations, 128 (1986), 31–55.)

[7] Y. A. Drozd and G.-M.Greuel. Cohen–Macaulay module type. Compositio Math. 89 (1993), 315–338.

[8] Y. A. Drozd and G.-M.Greuel. On the classification of vector bundles on projective curves. Max-Planck-Institut für Mathematik Preprint Series 1999 (130).

[9] Y. A. Drozd and A. V. Roiter. Commutative rings with a finite number of integral indecomposable representations, Izvestia Acad. Sci. USSR, 31 (1967) 783–798.

[10] H. Esnault. Reflexive modules on quotient surface singularities. J. Reine Angew, Math. 362 (1985), 63–71.

[11] H. Grauert and O. Riemenschneider. Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen. Invent. math. 11 (1970), 263-292

[12] G.-M. Greuel and H. Knörrer. Einfache Kurvensingularitäten und torsionfreie Moduln. Math. Ann. 270 (1985), 417–425.

[13] R. Hartshorn. *Algebraic Geometry.* Springer, New York, 1977.

[14] F. Hirzebruch. The Hilbert modular group, resolution of the singularities at the cusps and related problems. Seminaire Bourbaki 1970/ 71, Exposé 396.

[15] H. Jacobinski. Sur les ordres commutatifs avec un nombre fini de réseaux indecomposables. Acta Math. 118 (1976), 1–31.

[16] C. Kahn. Reflexive modules on minimally elliptic singularities. Math.Ann. 285 (1989), 141-160.

[17] U. Karras. Dissertation, Universität Bonn, 1973.

[18] Y. Kawamata. Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. Ann. Math. 127 (1988), 93–163.

[19] H. Knörrer. Maximal Cohen–Macaulay modules on hypersurface singularities, I. Invent. Math. 88 (1987), 153–164.

[20] H. Laufer. On minimally elliptic singularities. Amer. J. Math. 99 (1975), 1257–1295.

[21] J. Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. Publ. Math. IHES 36 (1969), 195–279.

[22] J.-P. Serre. *Algèbre locale, multiplicités.* Lecture Notes in Math. 11. Springer, Berlin, 1975.

[23] Y. Yoshino. *Cohen–Macaulay Modules over Cohen–Macaulay Rings.* Cambridge University Press, Cambridge, 1990.

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