CM values and central L-values of elliptic modular forms (II)

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Abstract Let f be a holomorphic cusp form on $\Gamma_0(N)$ and ξ a Hecke character of an imaginary quadratic field K. In this paper, under the condition that N is square free, we prove a formula relating the square of the absolute value of a ξ-twisted sum of values of f at CM-points in K to the central value of Rankin-Selberg L-function attached to f and ξ.

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1 Introduction

1.1

This paper is a continuation of [Mu]. Let φ be a holomorphic cusp form of weight k on $SL_2(\mathbb{Z})$ which is a common eigenfunction of the Hecke operators, and Ω a Hecke character of an imaginary quadratic field K such that $\Omega((\alpha)) = (\alpha/|\alpha|)^k$ for $\alpha \in K^\times$. Let $L(\varphi, \Omega; s)$ be the Rankin-Selberg L-function attached to (φ, Ω) with a functional equation under $s \mapsto 1 - s$, and $P(\varphi, \Omega)$ an Ω-averaged sum of values of φ at certain CM points in K (for the precise definition, see [Mu]). Due to Shimura's fundamental results on critical values of Rankin-Selberg L-functions ([S1],[S2]), we have the equality $L(\varphi, \Omega; 1/2) = c\pi^{k+1}|P(\varphi, \Omega)|^2$ with an algebraic number c. A similar formula for the central value of $L(\varphi, \Omega; s)$ was investigated by Waldspurger ([W]) in a great generality, though the constant of proportionality is not explicit in his work. The main result of [Mu] is a precise description of the constant c under the assumption that the class number of K is odd.

In this paper, we generalize the results of [Mu] for a holomorphic cusp form on $\Gamma_0(N)$ with a character and an algebraic Hecke character of conductor $\mathfrak{f}$, assuming that N is square free (without any assumptions on K and $\mathfrak{f}$). The method of the proof is similar to that of [W], though we need a more precise knowledge of theta lifts used in the proof to determine an explicit value of the constant of proportionality. For related works, see Remark 1.5.

1.2

We now state the main results of this paper in an adelic setting. Let K be an imaginary quadratic field of discriminant D. Denote by ω the quadratic Hecke character of $\mathbb{Q}$ corresponding to $K/\mathbb{Q}$.

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For $z \in K$, we put $\mathrm{Tr}(z) = z + z^\sigma$ and $\mathrm{N}(z) = zz^\sigma$, where σ is the nontrivial automorphism of $K/\mathbb{Q}$. Let $\mathcal{O}_K$ be the integer ring of K. We take and fix an element θ of $\mathcal{O}_K$ such that $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\theta, \mathrm{Im}(\theta) > 0$ and $\mathrm{ord}_p \mathrm{N}(\theta) = 1$ for any prime factor p of D. For a place v of $\mathbb{Q}$, we put $K_v = K \otimes_{\mathbb{Q}} \mathbb{Q}_v$, where $\mathbb{Q}_v$ is the completion of $\mathbb{Q}$ at v. Let p be a finite place of $\mathbb{Q}$ and fix a prime element π_p of $\mathbb{Q}_p$. Let $\mathcal{O}_{K,p} = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Let N be a positive square free integer and χ a Hecke character of $\mathbb{Q}$ of finite order whose conductor M divides N. Denote by χ^* the Dirichlet character modulo N corresponding to χ (cf. 2.1). Let $G = GL_2$ and $G_{\mathbb{A}} = GL_2(\mathbb{Q}_{\mathbb{A}})$ the adelization of G. Let $\mathbf{S}_k(N, \chi)$ be the space of functions on $G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$ which correspond to holomorphic cusp forms on $\Gamma_0(N)$ of weight k and character χ^* (cf. 3.1). Let $f \in \mathbf{S}_k(N, \chi)$ and assume that f is a primitive form with Hecke eigenvalues $\{\lambda_p\ (p \not| N), \lambda_p^{\pm}\ (p|N)\}$ (cf. 3.3).

Let ξ be a Hecke character of K. We denote by ξ_v the restriction of ξ to $K_v^\times$. We assume that ξ satisfies the following two conditions:

$$\xi|_{\mathbb{Q}_{\mathbb{A}}^\times} = \chi. \tag{1.1}$$

$$\xi_\infty(z_\infty) = (z_\infty/|z_\infty|)^k \quad (z_\infty \in \mathbb{C}^\times). \tag{1.2}$$

Note that ξ is of infinite order. For a finite place p of $\mathbb{Q}$, define

$$\alpha_p(\xi) = \mathrm{Min}\{a \in \mathbb{Z}_{\geq 0} \mid \xi_p \text{ is trivial on } (1 + \pi_p^a \mathcal{O}_{K,p})^\times\} \tag{1.3}$$

and put

$$A(\xi) = \prod_{p<\infty} p^{\alpha_p(\xi)}. \tag{1.4}$$

Let $L(f, \xi^{-1}; s)$ be the Rankin-Selberg L-function attached to (f, ξ^{-1}) (for the definition, see 3.4). The L-function $L(f, \xi^{-1}; s)$ is continued to a meromorphic function on $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 1 - s$.

Let ι be an embedding of K into $M_2(\mathbb{Q})$ given by

$$\iota(x + \theta y) = \begin{pmatrix} x & \mathrm{N}(\theta)y \\ -y & x + \mathrm{Tr}(\theta)y \end{pmatrix} \qquad (x, y \in \mathbb{Q}). \tag{1.5}$$

We define a period integral attached to (f, ξ) by

$$\mathcal{P}(f, \xi; g) = \int_{\mathbb{Q}_{\mathbb{A}}^\times K^\times \backslash K_{\mathbb{A}}^\times} \xi^{-1}(z) f(\iota(z)g) d^\times z \qquad (g \in G_{\mathbb{A}}), \tag{1.6}$$

where the measure $d^\times z$ is normalized as in 2.4.

Let $g_0 = (g_{0,v})_v \in G_{\mathbb{A}}$, where

$$g_{0,v} = \begin{cases} \begin{pmatrix} \pi_p^{-\alpha_p(\xi)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } v = p < \infty \text{ and } p \not| N, \\ \begin{pmatrix} \pi_p^{-\alpha_p(\xi)+1} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } v = p < \infty \text{ and } p | N, \\ \begin{pmatrix} 1 & \mathrm{Re}(\theta) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\mathrm{Im}(\theta)} & 0 \\ 0 & \sqrt{\mathrm{Im}(\theta)}^{-1} \end{pmatrix} & \text{if } v = \infty. \end{cases}$$

Note that $g_{0,\infty} \cdot i = \theta$.

Let $S_1(\xi)$ (respectively $S_2(\xi)$) be the set of prime factors p of $M^{-1}N$ such that $\alpha_p(\xi) = 0$ and p is inert (respectively ramifies) in $K/\mathbb{Q}$. We have $S_2(\xi) = S_2^+(f,\xi) \cup S_2^-(f,\xi)$, where

$$S_2^{\pm}(f,\xi) = \{p \in S_2(\xi) \mid \xi_p^{-1}(\Pi_p)\lambda_p^+ = \pm 1\}$$

and Π_p is a prime element of K_p. We are now able to state our main results.

THEOREM 1.1. *Let $f \in \mathbf{S}_k(N,\chi)$ be a primitive form and ξ a Hecke character of K satisfying (1.1) and (1.2).*

(i) We have $\mathcal{P}(f,\xi;g) = 0$ for any $g \in G_{\mathbb{A}}$ if $S_1(\xi) \neq \emptyset$ or $S_2^+(f,\xi) \neq \emptyset$.

(ii) Suppose that $S_1(\xi) = S_2^+(f,\xi) = \emptyset$. Then we have

$$|\mathcal{P}(f,\xi;g_0)|^2 = C(f,\xi)L(f,\xi^{-1};1/2),$$

where the constant of proportionality is given by

$$C(f,\xi) = (4\pi)^{1-k}(k-1)!\,|D|^{-1/2}A(\xi)^{-1}2^{|S_2(\xi)|}\prod_{p|A(\xi)} L_p(\omega;1)^2.$$

COROLLARY 1.2. *Suppose that $S_1(\xi) = S_2^+(f,\xi) = \emptyset$.*

(i) We have $L(f,\xi^{-1};1/2) \geq 0$.

(ii) The central L-value $L(f,\xi^{-1};1/2)$ vanishes if and only if the period integral $\mathcal{P}(f,\xi;g_0)$ vanishes.

REMARK 1.3. If $N = 1$ or $M = N$, we have $S_1(\xi) = S_2(\xi) = \emptyset$. In particular, Theorem 1.1 in the case $N = 1$ implies that the main results of [Mu] hold for any imaginary quadratic field K.

REMARK 1.4. The period integral $\mathcal{P}(f,\xi;g_0)$ can be seen as a ξ^{-1}-average of values of f at certain CM points in K.

REMARK 1.5. In the works of Gross and Zagier ([G] and [GZ]), the central values $L(f, \xi; 1/2)$ or $L'(f, \xi; 1/2)$ are studied in the case where ξ is a ring class character of K (and hence of finite order). For a generalization of their works, we refer to the work of Zhang [Z]. Recently, explicit formulas for the central value of $L(f, \xi; s)$ for a Hecke character ξ of infinite order have been studied by a number of authors, notably Popa [P], Xue [X] and Martin and Whitehouse [MW]. They work in a representation theoretic framework and impose an assumption that D, N and $A(\xi)$ are coprime to each other. On the other hand, we only assume that N is square free in this paper.

REMARK 1.6. The period integrals $\mathcal{P}(f, \xi; g_0)$ appear in explicit formulas for Fourier (or Fourier-Jacobi) expansions of certain theta liftings (see [MS], [MN]).

1.3

The paper is organized as follows. The first two sections are of preliminary nature. In Section 2, we prepare several notation and facts used in later discussions. In Section 3, we recall several facts on automorphic forms on $G = GL_2$. In Section 4, we study local spherical functions on $(G_p, \iota(K_p^\times))$, where ι is the embedding of $K^\times$ into G defined by (1.5). In particular, we prove the vanishing of the spherical functions under certain conditions (Proposition 4.1), which immediately implies the first asssertion of Theorem 1.1. In Section 5, we construct a mapping $\mathcal{L}$ from the space of cusp forms on $G_{\mathbb{A}}$ to the space of those on $G_{\mathbb{A}} \times G_{\mathbb{A}}$ as a theta lifting with a suitably chosen test function. One of the key in the proof of Theorem 1.1 is Theorem 6.1, which says that, for a primitive form $f \in \mathbf{S}_k(N, \chi)$, $\mathcal{L}(f)$ coincides with $(-2i)^k \, \overline{f_N} \otimes f_N$, where f_N is a twist of f (cf. 3.6). This fact is proved by calculating the Fourier expansion of $\mathcal{L}(f)$ (Proposition 6.2). The proof of Proposition 6.2 is reduced to that of a local result (Proposition 6.8), which is proved in Section 7 by a lengthy calculation involving local Whittaker functions. In Section 8, applying a method of Waldspurger in [W], we relate $|\mathcal{P}(f, \xi; g)|^2$ to a product of certain local integrals. The proof of the second assertion of Theorem 1.1 is completed by combining the results in Section 6 and the calculation of the local integrals carried out in Section 9.

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Notation

For a place v of $\mathbb{Q}$, denote by $|\cdot|_v$ the valuation of $\mathbb{Q}_v$. For $a = (a_v)_v \in \mathbb{Q}_{\mathbb{A}}^\times$, put $|a|_{\mathbb{A}} = \prod_v |a_v|_v$. For a linear algebraic group X defined over $\mathbb{Q}$, X_v stands for the group of $\mathbb{Q}_v$-rational points of X. We denote by $X_{\mathbb{A}}$ and $X_{\mathbb{A},f}$ the adelization of X and the finite part of $X_{\mathbb{A}}$, respectively. Let ψ be the additive character of $\mathbb{Q}_{\mathbb{A}}/\mathbb{Q}$ such that $\psi(x_\infty) = \mathbf{e}[x_\infty]$ for $x_\infty \in \mathbb{R}$. Denote by ψ_v

the restriction of ψ to $\mathbb{Q}_v$. A Hecke character of an algebraic number field E is a continuous homomorphism of $E_{\mathbb{A}}^{\times}/E^{\times}$ to $\mathbb{C}^{\times}$.

Throughout the paper, we fix an imaginary quadratic field K of discriminant D. Denote by ω the quadratic Hecke character of $\mathbb{Q}$ corresponding to $K/\mathbb{Q}$. Let p be a finite place of $\mathbb{Q}$. If $K_p/\mathbb{Q}_p$ is ramified, we fix a prime element Π_p of K_p. If p splits in $K/\mathbb{Q}$, we fix an identification between K_p and $\mathbb{Q}_p \oplus \mathbb{Q}_p$, and put $\Pi_{1,p} = (\pi_p, 1), \Pi_{2,p} = (1, \pi_p)$. We put $\mathcal{O}_{K,f} = \prod_{p<\infty} \mathcal{O}_{K,p}$. Denote by h_K and w_K the class number of K and the number of roots of unity in K, respectively. For $X \in M_{mn}(K)$, we put $X^* = {}^t X^{\sigma}$.

For a non-Archimedean local field F, we denote by $\mathcal{O}_F$ and $\mathfrak{p}_F$ the integer ring of F and the maximal ideal of $\mathcal{O}_F$, respectively. For a local field F, we define the local zeta function $\zeta_F(s)$ as follows: If F is a non-Archimedean local field, we put $\zeta_F(s) = (1 - q_F^{-s})^{-1}$, where $q_F = |\mathcal{O}_F/\mathfrak{p}_F|$. If F is an Archimedean local field, we put

$$\zeta_F(s) = \begin{cases} \pi^{-s/2}\Gamma(s/2) & \text{if } F = \mathbb{R}, \\ 2(2\pi)^{-s}\Gamma(s) & \text{if } F = \mathbb{C}, \end{cases}$$

where $\Gamma(s)$ denotes the gamma function.

We write $\mathrm{diag}(a_1, \ldots, a_n)$ for the diagonal matrix of degree n with the (i,i)-component a_i. For $z \in \mathbb{C}$, we put $\mathbf{e}[z] = \exp(2\pi\sqrt{-1}z)$. For a finite-dimensional vector space V over a local field, $\mathcal{S}(V)$ stands for the space of Schwartz-Bruhat functions on V. We put $\delta(P) = 1$ if a condition P is satisfied and $\delta(P) = 0$ otherwise. For a set X, char_X denotes the characteristic function of X. The cardinality of a finite set X is denoted by $|X|$.

2 Preliminaries

2.1

Throughout the paper, we fix a positive square free integer N and a Hecke character χ of $\mathbb{Q}$ of finite order whose conductor M divides N. For a place v of $\mathbb{Q}$, χ_v stands for the restriction of χ to $\mathbb{Q}_v^{\times}$. For $n \in \mathbb{Z}$, define

$$\chi^*(n) = \begin{cases} \displaystyle\prod_{p|M} \chi_p^{-1}(n) & \text{if } (n, N) = 1, \\ 0 & \text{if } (n, N) > 1. \end{cases}$$

Then χ^* is a Dirichlet character modulo N.

2.2

Let $Z = \{z1_2 \mid z \neq 0\}$ be the center of $G = GL_2$. For $x \in \mathbb{Q}$ and $y, y' \in \mathbb{Q}^{\times}$, put

$$\mathbf{n}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \overline{\mathbf{n}}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad \mathbf{d}(y, y') = \begin{pmatrix} y & 0 \\ 0 & y' \end{pmatrix} \in G_{\mathbb{Q}}.$$

Let $\mathcal{N}$ be the algebraic subgroup of G with $\mathcal{N}_{\mathbb{Q}} = \{\mathbf{n}(x) \mid x \in \mathbb{Q}\}$. We put $U_f = \prod_{p<\infty} U_p$ and $\mathcal{U}_f = \prod_{p<\infty} \mathcal{U}_p$, where $U_p = GL_2(\mathbb{Z}_p)$ and $\mathcal{U}_p = \{(u_{ij}) \in U_p \mid u_{21} \in N\mathbb{Z}_p\}$ for a finite place p of $\mathbb{Q}$. Note that $\mathcal{U}_p$ is an Iwahori subgroup of G_p if $p|N$ and $\mathcal{U}_p = U_p$ otherwise. We have the decomposition

$$G_p = \begin{cases} \bigcup_{m\in\mathbb{Z}} Z_p \mathcal{N}_p \mathbf{d}_m \mathcal{U}_p & \text{if } p \nmid N, \\ \bigcup_{m\in\mathbb{Z}} Z_p \mathcal{N}_p \mathbf{d}_m \mathcal{U}_p \cup \bigcup_{m\in\mathbb{Z}} Z_p \mathcal{N}_p \mathbf{d}_m w_1^{(p)} \mathcal{U}_p & \text{if } p|N, \end{cases}$$

where

$$\mathbf{d}_m = \mathbf{d}(\pi_p^m, 1),\ w_m^{(p)} = \begin{pmatrix} 0 & -1 \\ \pi_p^m & 0 \end{pmatrix} \qquad (m \in \mathbb{Z}).$$

The real Lie group $G_\infty^+ = \{g \in G_\infty = G(\mathbb{R}) \mid \det g > 0\}$ acts on the upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$g \cdot z = (az+b)(cz+d)^{-1} \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^+, z \in \mathfrak{H} \right)$$

as usual. We put $j(g,z) = (\det g)^{-1/2}(cz+d)$. Let $U_\infty = \mathcal{U}_\infty = \{g \in SL_2(\mathbb{R}) \mid g \cdot i = i\}$.

2.3 Hecke operators

Let p be a finite place of $\mathbb{Q}$. We define a character $\widetilde{\chi}_p$ of $\mathcal{U}_p$ by

$$\widetilde{\chi}_p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} 1 & \text{if } p \nmid N, \\ \chi_p(d) & \text{if } p|N. \end{cases}$$

Note that $\widetilde{\chi}_p$ is trivial unless $p|M$. Let $C(G_p; \widetilde{\chi}_p)$ be the space of continuous function φ on G_p satisfying $\varphi(gu) = \widetilde{\chi}_p(u)\varphi(g)$ $(g \in G_p, u \in \mathcal{U}_p)$. Let $\varphi \in C(G_p; \widetilde{\chi}_p)$ and $g \in G_p$. If $p \nmid N$, we put

$$\mathcal{T}_p\varphi(g) = \varphi(g\,\mathbf{d}(\pi_p^{-1}, 1)) + \sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p} \varphi(g\,\mathbf{n}(a)\mathbf{d}(1, \pi_p^{-1})).$$

Then $\mathcal{T}_p\varphi \in C(G_p; \widetilde{\chi}_p)$. If $p|N$, we put

$$\mathcal{T}_p^+\varphi(g) = \chi_p(\pi_p) \sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p} \varphi(g\mathbf{n}(a)\mathbf{d}(1, \pi_p^{-1})),$$
$$\mathcal{T}_p^-\varphi(g) = \sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p} \varphi(g\,\overline{\mathbf{n}}(\pi_p a)\mathbf{d}(\pi_p^{-1}, 1)).$$

Then $\mathcal{T}_p^\pm\varphi \in C(G_p; \widetilde{\chi}_p)$. Note that the definitions of $\mathcal{T}_p$ and $\mathcal{T}_p^\pm$ do not depend on the choice of π_p.

2.4 Normalization of measures

Let v be a place of $\mathbb{Q}$. Let dx_v (respectively dz_v) be the Haar measure on $\mathbb{Q}_v$ (respectively K_v) self-dual with respect to the pairing $(x_v, x'_v) \mapsto \psi_v(x_v x'_v)$ (respectively $(z_v, z'_v) \mapsto \psi_v(\mathrm{Tr}(z_v^\sigma z'_v))$). Note that $\mathrm{vol}(\mathcal{O}_{K,p}) = |D|_p^{1/2}$ and dz_∞ is twice the usual Lebesgue measure on $K_\infty = \mathbb{C}$. We put $dx = \prod_{v\leq\infty} dx_v$ and $dz = \prod_{v\leq\infty} dz_v$. We normalize the measures $d^\times x_v$ and $d^\times z_v$ on $\mathbb{Q}_v^\times$ and $K_v^\times$ by

$$\begin{aligned} d^\times x_v &= \zeta_{\mathbb{Q}_v}(1)\, |x|_v^{-1} dx_v, \\ d^\times z_v &= \zeta_{K_v}(1)\, |N(z_v)|_v^{-1} dz_v, \end{aligned}$$

respectively. Note that $\mathrm{vol}(\mathcal{O}_{K,p}^\times) = |D|_p^{1/2}$ and $d^\times z_\infty = 2r^{-1} dr dt \quad (z_\infty = r\exp(it))$. We put $d^\times x = \prod_{v\leq\infty} d^\times x_v$ and $d^\times z = \prod_{v\leq\infty} d^\times z_v$. With this normalization, we have

$$\int_{\mathbb{Q}_\mathbb{A}^\times K^\times \backslash K_\mathbb{A}^\times} d^\times z = 2L(\omega; 1) = \frac{4\pi h_K}{w_K |D|^{1/2}}.$$

Let dg_v be the Haar measure on G_v given by

$$\int_{G_v} \varphi(g_v) dg_v = \int_{\mathbb{Q}_v^\times} \int_{\mathbb{Q}_v} \int_{\mathbb{Q}_v^\times} \int_{U_v} |y|_v^{-1} \varphi(z\mathbf{n}(x)\mathbf{d}(y,1)u) du d^\times y dx d^\times z$$

for $\varphi \in L^1(G_v)$. Here du_v is normalized by

$$\mathrm{vol}(U_v) = \begin{cases} 1 & \text{if } v = \infty, \\ [U_p : \mathcal{U}_p] & \text{if } v = p < \infty. \end{cases}$$

We put $dg = \prod_v dg_v$.

2.5 Gauss sum

Let p be a finite place of $\mathbb{Q}$ dividing N. Define the Gauss sum by

$$\begin{aligned} G_p(\chi) &= \chi_p^{-1}(\pi_p) \int_{\mathbb{Z}_p^\times} \psi_p(\pi_p^{-1} x) \chi_p(x) dx \\ &= p^{-1} \chi_p^{-1}(\pi_p) \sum_{a \in (\mathbb{Z}_p - p\mathbb{Z}_p)/p\mathbb{Z}_p} \psi_p(\pi_p^{-1} a) \chi_p(a), \end{aligned}$$

where dx is the Haar measure on $\mathbb{Z}_p$ normalized by $\mathrm{vol}(\mathbb{Z}_p) = 1$. Note that $G_p(\chi)$ does not depend on the choice of π_p. The following is well-known.

Lemma 2.1. *(i) If $p | M^{-1}N$, we have $G_p(\chi) = -p^{-1}\chi_p^{-1}(\pi_p)$.*

(ii) If $p|M$, we have $|G_p(\chi)| = p^{-1/2}, \overline{G_p(\chi)} = \chi(-1) G_p(\chi^{-1})$ and

$$\int_{\mathbb{Z}_p^\times} \psi_p(yx)\chi_p(x) dx = G_p(\chi)\chi_p^{-1}(y) \mathrm{char}_{\pi_p^{-1}\mathbb{Z}_p^\times}(y) \qquad (y \in \mathbb{Q}_p^\times).$$

2.6

Let ξ be a Hecke character of K satisfying (1.1) and (1.2). Note that $\alpha_p(\xi) = 0$ if and only if ξ_p is trivial on $\mathcal{O}_{K,p}^{\times}$, and hence $\alpha_p(\xi) > 0$ for $p|M$. The following is easily verified.

LEMMA 2.2. *For* $0 < m < \alpha_p(\xi)$, *we have*

$$\int_{\pi_p^m \mathbb{Z}_p} \xi_p(1 + y\theta) dy = 0.$$

3 Automorphic forms on $GL(2)$

3.1 Automorphic forms

We henceforth fix a positive integer k satisfying $(-1)^k = \chi_\infty(-1)$. Put $\widetilde{\chi} = \prod_{p<\infty} \widetilde{\chi}_p$ (for the definition of $\widetilde{\chi}_p$, see 2.3). Then $\widetilde{\chi}$ is a character of $\mathcal{U}_f$.

Let $\mathbf{S}_k(N, \chi)$ be the space of smooth functions on $G_{\mathbb{A}}$ satisfying the following conditions:

(i) We have $f(z\gamma g u_f u_\infty) = \chi(z)\widetilde{\chi}(u_f) j(u_\infty, i)^{-k} f(g)$ for $z \in Z_{\mathbb{A}}, \gamma \in G_{\mathbb{Q}}, g \in G_{\mathbb{A}}, u_f \in \mathcal{U}_f, u_\infty \in \mathcal{U}_\infty$.

(ii) For any $g_f \in G_{\mathbb{A},f}$, $\tau \mapsto f_{dm}(\tau; g_f) := j(g_\tau, i)^k f(g_\tau g_f)$ is holomorphic on $\mathfrak{H}$ ($\tau = x + iy \in \mathfrak{H}, g_\tau = \mathbf{n}(x)\mathbf{d}(\sqrt{y}, \sqrt{y}^{-1}) \in G_\infty^+$).

(iii) f is bounded on $G_{\mathbb{A}}$.

Note that $\mathcal{T}_p$ (respectively $\mathcal{T}_p^{\pm}$) defines a linear operator of $\mathbf{S}_k(N, \chi)$ for $p \not| N$ (respectively $p|N$). We write $f_{dm}(\tau)$ for $f_{dm}(\tau; 1)$. Then we easily see that

$$f_{dm}\left(\frac{a\tau + b}{c\tau + d}\right) = \chi^*(d)(c\tau + d)^k f_{dm}(\tau) \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\right),$$

and that $f \mapsto f_{dm}$ defines an isomorphism of $\mathbf{S}_k(N, \chi)$ to the space of holomorphic cusp forms on $\Gamma_0(N)$ of weight k and character χ^*.

3.2 Fourier expansion

For $f \in \mathbf{S}_k(N, \chi)$, we have the Fourier expansion:

$$f(g) = \sum_{a \in \mathbb{Q}^\times} W_f(\mathbf{d}(a, 1)g),$$

where

$$W_f(g) = \int_{\mathbb{Q}\backslash\mathbb{Q}_{\mathbb{A}}} \psi(-x) f(\mathbf{n}(x)g) dx.$$

Let

$$f_{dm}(\tau) = \sum_{n=1}^{\infty} a_n(f_{dm}) \mathbf{e}[n\tau]$$

be the Fourier expansion of f_{dm}. We say that f is *normalized* if $a_1(f_{dm}) = 1$. The following fact is well-known.

LEMMA 3.1. *If f is normalized, we have $W_f(\mathbf{d}(y_\infty, 1)) = \delta(y_\infty > 0)\, y_\infty^{k/2} \exp(-2\pi y_\infty)$ for $y_\infty \in \mathbb{R}^\times$.*

3.3 Primitive forms

We say that $f \in \mathbf{S}_k(N, \chi)$ is a *primitive form* if f_{dm} is a primitive form (cf. [Mi, §4.6]). If f is a primitive form, the following hold:

(i) f is normalized.

(ii) For $p \nmid N$, we have $\mathcal{T}_p f = \lambda_p f$ with $\lambda_p \in \mathbb{C}$, and $\overline{\lambda_p} = \chi_p(\pi_p)\lambda_p$.

(iii) For $p|N$, we have $\mathcal{T}_p^{\pm} f = \lambda_p^{\pm} f$ with $\lambda_p^+, \lambda_p^- \in \mathbb{C}$, and

$$\overline{\lambda_p^+} = \lambda_p^-, \quad \lambda_p^+\lambda_p^- = \begin{cases} 1 & \text{if } p|M^{-1}N, \\ p & \text{if } p|M. \end{cases}$$

If $p|M^{-1}N$, we have $(\lambda_p^{\pm})^2 = \chi_p(\pi_p)^{\pm 1}$.

(iv) For $p|M^{-1}N$, we have

$$f(gw_1^{(p)}) = \epsilon_p f(g) \qquad (g \in G_{\mathbb{A}}),$$

where $\epsilon_p = -\lambda_p^+$ (for the definition of $w_1^{(p)}$, see 2.2).

REMARK 3.2. When $p|N$, we have $\lambda_p^+ = p^{1-k/2} a_p(f_{dm})$.

We say that f is a primitive form with Hecke eigenvalues $\{\lambda_p\ (p \nmid N), \lambda_p^{\pm}\ (p|N)\}$.

3.4 L-functions

Let $f \in \mathbf{S}_k(N, \chi)$ be a primitive form with Hecke eigenvalues $\{\lambda_p\ (p \nmid N), \lambda_p^{\pm}\ (p|N)\}$. When $p \nmid N$, let $t_{1,p}, t_{2,p} \in \mathbb{C}$ be the roots of an equation $X^2 - p^{-1}\chi_p(\pi_p)\lambda_p X + p^{-1}\chi_p(\pi_p) = 0$. Let ξ be a Hecke character of K. The Rankin-Selberg L-function $L(f, \xi; s)$ is defined by

$$L(f, \xi; s) = \prod_{p < \infty} L_p(f, \xi; s).$$

Here the local factor $L_p(f, \xi; s)$ is given as follows: If ξ_p is nontrivial on $\mathcal{O}_{K,p}^\times$, we set $L_p(f, \xi; s) = 1$. Suppose that ξ_p is trivial on $\mathcal{O}_{K,p}^\times$. Then $p \nmid M$.

(i) If p is inert in $K/\mathbb{Q}$, we set

$$L_p(f, \xi; s) = R_p(\xi_p(\pi_p)p^{-2s})^{-1},$$

where

$$R_p(X) = \begin{cases} \prod_{i=1}^2 (1 - pt_{i,p}^2 X) & \text{if } p \nmid N, \\ 1 - p^{-1}(\lambda_p^+)^2 X & \text{if } p|M^{-1}N. \end{cases}$$

(ii) If p ramifies in $K/\mathbb{Q}$, we set

$$L_p(f,\xi;s) = R_p(\xi_p(\Pi_p)p^{-s})^{-1},$$

where

$$R_p(X) = \begin{cases} \prod_{i=1}^2 (1 - p^{1/2} t_{i,p} X) & \text{if } p \not| N, \\ 1 - p^{-1/2}\lambda_p^+ X & \text{if } p | M^{-1}N. \end{cases}$$

(iii) If p splits in $K/\mathbb{Q}$, we set

$$L_p(f,\xi;s) = \prod_{j=1}^{2} R_p(\xi_p(\Pi_{j,p})p^{-s})^{-1},$$

where

$$R_p(X) = \begin{cases} \prod_{i=1}^2 (1 - p^{1/2} t_{i,p} X) & \text{if } p \not| N, \\ 1 - p^{-1/2}\lambda_p^+ X & \text{if } p | M^{-1}N. \end{cases}$$

3.5 Local Whittaker functions

Let p be a finite place of $\mathbb{Q}$. Let $\mathcal{W}_p(\chi_p)$ be the space of functions W on G_p such that $W(z\mathbf{n}(x)gu) = \chi_p(z)\psi_p(x)\widetilde{\chi}_p(u)W(g)$ for $z \in Z_p, x \in \mathbb{Q}_p, g \in G_p, u \in \mathcal{U}_p$.

First suppose that $p \not| N$. For $\lambda \in \mathbb{C}$, let $\mathcal{W}_p(\chi_p;\lambda) = \{W \in \mathcal{W}_p(\chi_p) \mid \mathcal{T}_p W = \lambda W\}$. The following fact is well-known.

LEMMA 3.3. *Assume that $p \not| N$ and let $W \in \mathcal{W}_p(\chi_p;\lambda)$.*

(i) If $W(1_2) = 0$, W is identically equal to zero.

(ii) We have $\dim_{\mathbb{C}} \mathcal{W}_p(\chi_p;\lambda) \leq 1$.

(iii) The support of W is contained in $\bigcup_{n \geq 0} Z_p \mathcal{N}_p \mathbf{d}_n \mathcal{U}_p$ and we have

$$W(\mathbf{d}_n) = \frac{t_1^{n+1} - t_2^{n+1}}{t_1 - t_2} W(1_2) \qquad (n \geq 0),$$

where $t_1, t_2 \in \mathbb{C}$ are the roots of an equation $X^2 - p^{-1}\chi_p(\pi_p)\lambda X + p^{-1}\chi_p(\pi_p) = 0$.

(iv) If $\overline{\lambda} = \chi_p(\pi_p)\lambda$, then $\overline{W(g)} = \chi_p^{-1}(\det g)W(g)$ $(g \in G_p)$.

Next suppose that $p|N$. For $(\lambda^+,\lambda^-) \in \mathbb{C}^2$, let $\mathcal{W}_p(\chi_p;\lambda^+,\lambda^-) = \{W \in \mathcal{W}_p(\chi_p) \mid \mathcal{T}_p^{\pm} W = \lambda^{\pm} W\}$. We can easily verify the following results.

LEMMA 3.4. *Assume that $p|N$ and let $W \in \mathcal{W}_p(\chi_p;\lambda^+,\lambda^-)$.*

(i) If $W(1_2) = 0$, W is identically equal to zero.

(ii) We have $\dim_{\mathbb{C}} \mathcal{W}_p(\chi_p;\lambda^+,\lambda^-) \leq 1$.

(iii) The support of W is contained in $\bigcup_{n\geq 0} Z_p\mathcal{N}_p\mathbf{d}_n\mathcal{U}_p \cup \bigcup_{n\geq 0} Z_p\mathcal{N}_p\mathbf{d}_n w_1^{(p)}\mathcal{U}_p$. *For* $n \geq 0$, *we have*

$$\begin{aligned} W(\mathbf{d}_n) &= (p^{-1}\lambda^+)^n W(1_2), \\ W(\mathbf{d}_n w_1^{(p)}) &= p^{-1}G_p(\chi_p)^{-1}\lambda^- \left(p^{-1}\chi_p(\pi_p)\lambda^-\right)^n W(1_2). \end{aligned}$$

3.6 Global Whittaker functions

For $f \in \mathbf{S}_k(N, \chi)$, set

$$f_N(g) = \chi^{-1}(\det g) f(g w_N), \tag{3.1}$$

where

$$w_N = \prod_{p|N} w_1^{(p)} \qquad \left(w_1^{(p)} = \begin{pmatrix} 0 & -1 \\ \pi_p & 0 \end{pmatrix} \in G_p \right).$$

It is easy to see that $f_N \in \mathbf{S}_k(N, \chi^{-1})$. Lemmas 3.3 and 3.4 implies the following result.

Proposition 3.5. *Assume that* $f \in \mathbf{S}_k(N, \chi)$ *is a primitive form with Hecke eigenvalues* $\{\lambda_p \ (p \nmid N), \lambda_p^{\pm} \ (p|N)\}$.

(i) For $g = (g_v)_v \in G_{\mathbb{A}}$, *we have*

$$W_f(g) = \prod_{v\leq\infty} W_v(g_v).$$

Here W_p *is the element of* $\mathcal{W}_p(\chi_p; \lambda_p)$ *(respectively* $\mathcal{W}_p(\chi_p; \lambda_p^+, \lambda_p^-)$*) with* $W_p(1_2) = 1$ *if* $p \nmid N$ *(respectively* $p|N$*), and*

$$W_\infty(z\mathbf{n}(x)\mathbf{d}(y, 1)u) = \delta(y > 0)\,\chi_\infty(z) y^{k/2}\mathbf{e}[x + iy] j(u, i)^{-k}$$

for $x \in \mathbb{R}, y, z \in \mathbb{R}^\times, u \in \mathcal{U}_\infty$.

(ii) We have

$$W_{f_N}(g) = \prod_{v\leq\infty} W'_v(g_v),$$

where

$$W'_v(g_v) = \chi_v^{-1}(\det g_v) \times \begin{cases} W_v(g_v) & \text{if } v = p \nmid N \text{ or } v = \infty, \\ W_p(g_p w_1^{(p)}) & \text{if } v = p|N. \end{cases}$$

(iii) We have $W'_\infty = W_\infty$. *If* $p \nmid N$, *we have* $W'_p \in \mathcal{W}_p(\chi_p^{-1}; \chi_p(\pi)\lambda_p)$ *and* $W'_p(1_2) = 1$. *If* $p|N$, *we have* $W'_p \in \mathcal{W}_p(\chi_p^{-1}; \lambda_p^-, \lambda_p^+)$ *and* $W'_p(1_2) = p^{-1}G_p(\chi_p)^{-1}\lambda_p^-$.

4 Local spherical functions

4.1

In this section, we study local spherical functions on $(G_p, \iota(K_p^\times))$ and prove Theorem 1.1 (i). Throughout this section, we fix a finite place p of $\mathbb{Q}$ such that K_p is a field, and often suppress the subscript p. We write K and F for K_p and $\mathbb{Q}_p$ respectively. Let $\mathcal{O}_K$ and $\mathcal{O}_F$ be the integer rings of K and F, respectively. Let $\mathcal{B} = \{(u_{ij}) \in GL_2(\mathcal{O}_F) \mid u_{21} \in \mathfrak{p}_F\}$ be an Iwahori subgroup of G. Let ξ be a unitary character trivial on $\mathcal{O}_K^\times$ and write χ for the restriction of ξ to $F^\times$. Denote by $\mathcal{F}_\xi$ the space of functions φ on G satisfying $\varphi(\iota(z)gu) = \xi(z)\varphi(g)$ for $z \in K^\times, g \in G$ and $u \in \mathcal{B}$. For $(\lambda^+, \lambda^-, \epsilon) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^\times$, define $\mathcal{F}_\xi(\lambda^+, \lambda^-; \epsilon) = \{\varphi \in \mathcal{F}_\xi \mid \mathcal{T}^\pm\varphi = \lambda^\pm\varphi, \varphi(gw_1) = \epsilon\varphi(g)\ (g \in G)\}$ (for the definition of $\mathcal{T}^\pm$, see 2.3). In this section, we prove

Proposition 4.1. *Suppose that $|\lambda^\pm| = 1$ and $\lambda^+\lambda^- = 1$.*

(i) If K/F is inert, we have $\mathcal{F}_\xi(\lambda^+, \lambda^-; \epsilon) = \{0\}$.

(ii) If K/F is ramified and $\xi^{-1}(\Pi)\lambda^+ = 1$, we have $\mathcal{F}_\xi(\lambda^+, \lambda^-; \epsilon) = \{0\}$.

Remark 4.2. Let $f \in \mathbf{S}_k(N, \chi)$ be a primitive form with Hecke eigenvalues $\{\lambda_p\ (p \not| N), \lambda_p^\pm\ (p|N)\}$ satisfying $f(gw_1^{(p)}) = \epsilon_p f(g)$ for $p|M^{-1}N$. Note that $|\lambda_p^\pm| = 1$ and $\lambda_p^+\lambda_p^- = 1$ for $p|M^{-1}N$ (cf. 3.3). Then, for $p \in S_1(\xi) \cup S_2(\xi)$, $\mathcal{P}(f, \xi; *)|_{G_p}$ belongs to $\mathcal{F}_{\xi_p}(\lambda_p^+, \lambda_p^-; \epsilon_p)$. Thus Theorem 1.1 (i) directly follows from the above proposition.

4.2

To show Proposition 4.1, we need the following elementary facts (for the proofs, see [MN]).

Lemma 4.3. *We have*

$$G = \begin{cases} \bigsqcup_{m\geq 0} Tg_m\mathcal{B} \sqcup \bigsqcup_{m\geq 0} Tg_m w_1\mathcal{B} & \text{if } K/F \text{ is inert,} \\ \bigsqcup_{m\geq -1} Tg_m\mathcal{B} \sqcup \bigsqcup_{m\geq 0} Tg_m w_1\mathcal{B} & \text{if } K/F \text{ is ramified,} \end{cases}$$

where $T = \iota(K^\times)$ and $g_m = \mathbf{d}(1, \pi^m)$.

Lemma 4.4. *(i) Let $m \geq 0$ and $a \in \mathcal{O}_F$. If $z_a = a - \pi^m\theta \in \mathcal{O}_K^\times$, we have $g_m\mathbf{n}(a)\mathbf{d}(1, \pi^{-1}) \in \iota(\pi^{-1}z_a)g_m w_1\mathcal{B}$.*

(ii) Let $m \geq -1$ and $a \in \mathcal{O}_F$. Then $g_m\overline{\mathbf{n}}(\pi a)\mathbf{d}(\pi^{-1}, 1) \in \iota(\pi^{-1}z'_a)g_{m+1}\mathcal{B}$, where $z'_a = 1 - \pi^{m+1}a\theta \in \mathcal{O}_K^\times$.

(iii) Suppose that K/F is ramified. Then $g_{-1}w_1 \in \iota(\theta)g_{-1}\mathcal{B}$.

4.3 Proof of Proposition 4.1

Assume that there exists a nonzero element φ of $\mathcal{F}_\xi(\lambda^+, \lambda^-; \epsilon)$. Since $\varphi(gw_1^2) = \chi(\pi)\varphi(g)$, we have $\epsilon^2 = \chi(\pi)$ and hence $|\epsilon| = 1$.

First suppose that K/F is inert. By Lemma 4.4 (ii), we obtain

$$\lambda^- \varphi(g_m) = \sum_{a \in \mathcal{O}_F/\mathfrak{p}_F} \xi(\pi^{-1}(1 - \pi^{m+1}a\theta))\varphi(g_{m+1}) = p\chi^{-1}(\pi)\varphi(g_{m+1}) \qquad (m \geq 0).$$

We thus have $\varphi(g_m) = \left(p^{-1}\chi(\pi)\lambda^-\right)^m \varphi(g_0)$ and $\varphi(g_m w_1) = \epsilon \left(p^{-1}\chi(\pi)\lambda^-\right)^m \varphi(g_0)$ for $m \geq 0$. In view of Lemma 4.3, this implies that $\varphi(g_0) \neq 0$. On the other hand, we have

$$\lambda^+ \varphi(g_0) = \chi(\pi) \sum_{a \in \mathcal{O}_F/\mathfrak{p}_F} \xi(\pi^{-1}(a - \theta))\varphi(g_0 w_1) = p\epsilon\varphi(g_0)$$

by Lemma 4.4 (i) and hence $\lambda^+ = p\epsilon$. This contradicts to the assumption $|\lambda^+| = 1$.

Next suppose that K/F is ramified and $\xi^{-1}(\Pi)\lambda^+ = 1$. By an argument similar to the above, we obtain

$$\begin{aligned} \varphi(g_m) &= \left(p^{-1}\chi(\pi)\lambda^-\right)^{m+1} \varphi(g_{-1}) \qquad (m \geq -1), \\ \varphi(g_m w_1) &= \epsilon \left(p^{-1}\chi(\pi)\lambda^-\right)^{m+1} \varphi(g_{-1}) \qquad (m \geq 0), \end{aligned}$$

which implies $\varphi(g_{-1}) \neq 0$. By Lemma 4.4 (iii), we have $\epsilon\varphi(g_{-1}) = \varphi(g_{-1}w_1) = \xi(\theta)\varphi(g_{-1}) = \xi(\Pi)\varphi(g_{-1})$ and hence $\epsilon = \xi(\Pi)$. On the other hand, we have

$$\begin{aligned} \lambda^+ \varphi(g_0) &= \chi(\pi)\varphi(g_{-1}) + (p-1)\varphi(g_0 w_1) \\ &= \left\{\chi(\pi) + (p-1)\epsilon p^{-1}\chi(\pi)\lambda^-\right\} \varphi(g_{-1}). \end{aligned}$$

This implies $\lambda^+ \cdot p^{-1}\chi(\pi)\lambda^- = \chi(\pi) + (p-1)\epsilon p^{-1}\chi(\pi)\lambda^-$. Since $\lambda^-\lambda^+ = 1$, we have $\epsilon = -\lambda^+$ and hence $\xi^{-1}(\Pi)\lambda^+ = -1$, which contradicts to the assumption.

5 Theta lift

5.1 Metaplectic representations

Let $V = M_2(\mathbb{Q})$. For $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$, put $\overline{X} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. For a place v of $\mathbb{Q}$, let $V_v = M_2(\mathbb{Q}_v)$.

For $p < \infty$, $\mathcal{S}'(V_p \times \mathbb{Q}_p^\times)$ denotes the space of locally constant and compactly supported functions on $V_p \times \mathbb{Q}_p^\times$. We denote by $\mathcal{S}'(V_\infty \times \mathbb{Q}_\infty^\times)$ the space of smooth functions Φ on $V_\infty \times \mathbb{Q}_\infty^\times$ such that, for any $t \in \mathbb{Q}_\infty^\times$, $X \mapsto \Phi(X, t)$ is in $\mathcal{S}(V_\infty)$.

Lemma 5.1 ([W]). *Let v be a place of $\mathbb{Q}$. There exists a smooth representation R_v of $G_v \times$*

$G_v \times G_v$ *on* $\mathcal{S}'(V_v \times \mathbb{Q}_v^\times)$ *such that the following holds:*

$$R_v(1, g_1, g_2)\Phi(X,t) = \Phi(g_2^{-1}Xg_1, \det(g_1^{-1}g_2)t) \qquad (g_1, g_2 \in G_v),$$
$$R_v(\mathbf{n}(b), 1, 1)\Phi(X,t) = \psi_v(bt\det X)\Phi(X,t) \qquad (b \in \mathbb{Q}_v),$$
$$R_v(\mathbf{d}(a,d), 1, 1)\Phi(X,t) = \left|\frac{a}{d}\right|_v \Phi(d^{-1}X, adt) \qquad (a, d \in \mathbb{Q}_v^\times),$$
$$R_v\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1, 1\right)\Phi(X,t) = |t|_v^2 \int_{V_v} \psi_v(-t\ \mathrm{Tr}(\overline{Y}X))\Phi(Y,t)dY.$$

Here dY *is the Haar measure on* V_v *self-dual with respect to the pairing* $(X,Y) \mapsto \psi_v(\mathrm{Tr}(\overline{Y}X))$.

Remark 5.2. For $z \in \mathbb{Q}_v^\times$, we have

$$R_v(z,1,1)\Phi(X,t) = R_v(1, z^{-1}, 1)\Phi(X,t) = R_v(1,1,z)\Phi(X,t) = \Phi(z^{-1}X, z^2t).$$

5.2 Intertwining operators

We put

$$\mathcal{I}_v\Phi\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) = |t|_v \int_{\mathbb{Q}_v^2} \psi_v(t(xy' - yx'))\Phi\left(\begin{pmatrix} x' & y' \\ z & w \end{pmatrix}, t\right) dx'dy'$$

for $\Phi \in \mathcal{S}'(V_v \times \mathbb{Q}_v^\times)$. A straightforward calculation shows the following:

Lemma 5.3. *We have*

$$\mathcal{I}_v \circ R_v(g, g_1, 1)\Phi(X,t) = \mathcal{I}_v\Phi(g^{-1}Xg_1, \det(gg_1^{-1})\,t) \qquad (g, g_1 \in G_v),$$
$$\mathcal{I}_v \circ R_v(1,1,\mathbf{n}(b))\Phi(X,t) = \psi_v(bt\det X)\mathcal{I}_v\Phi(X,t) \qquad (b \in \mathbb{Q}_v),$$
$$\mathcal{I}_v \circ R_v(1,1,\mathbf{d}(a,d))\Phi(X,t) = \left|\frac{a}{d}\right|_v \mathcal{I}_v\Phi(d^{-1}X, adt) \qquad (a, d \in \mathbb{Q}_v^\times)$$

for $\Phi \in \mathcal{S}'(V_v \times \mathbb{Q}_v^\times)$.

5.3 Test functions

For a place v of $\mathbb{Q}$, we define a test function $\Phi_{0,v} \in \mathcal{S}'(V_v \times \mathbb{Q}_v^\times)$ as follows. When $v = p < \infty$, we put

$$\Phi_{0,p}(X,t) = \begin{cases} \mathrm{char}_{\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}}(X) \cdot \mathrm{char}_{\mathbb{Z}_p^\times}(t) & \text{if } p \nmid M, \\ \chi_p^{-1}(X_{22})\mathrm{char}_{\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}}(X) \cdot \mathrm{char}_{\mathbb{Z}_p^\times}(t) & \text{if } p | M, \end{cases}$$

where X_{22} is the $(2,2)$-component of X. When $v = \infty$, put

$$\Phi_{0,\infty}(X,t) = \begin{cases} \left((i\,1)X\begin{pmatrix} 1 \\ i \end{pmatrix}\right)^k \exp(-\pi t\,\mathrm{Tr}({}^tXX)) & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Lemma 5.4. *(i) We have* $\mathcal{I}_p\Phi_{0,p} = \Phi_{0,p}$.

(ii) We have $\mathcal{I}_\infty\Phi_{0,\infty}(X,t) = \Phi_{0,\infty}(\mathbf{d}(-1,1)X,t)$.

Proof. This is proved by a direct calculation. □

Lemma 5.5. *(i) For* $u, u_1, u_2 \in \mathcal{U}_p$, *we have* $R_p(u,u_1,u_2)\Phi_{0,p} = \widetilde{\chi}_p(uu_1^{-1}u_2)\Phi_{0,p}$.

(ii) For $u, u_1, u_2 \in \mathcal{U}_\infty$, *we have* $R_\infty(u,u_1,u_2)\Phi_{0,\infty} = j(u,i)^{-k}j(u_1,i)^{-k}j(u_2,i)^k\Phi_{0,\infty}$.

(iii) For $z \in Z_\infty$, *we have* $R_\infty(z,1,1)\Phi_{0,\infty} = z^{-k}\Phi_{0,\infty}$.

Proof. This follows from the definition of R_v and Lemmas 5.3 and 5.4. □

5.4 Theta kernel

Let $\mathcal{S}'(V_\mathbb{A}\times\mathbb{Q}_\mathbb{A}^\times)$ be the restricted tensor product of $\mathcal{S}'(V_v\times\mathbb{Q}_v^\times)$ over v with respect to $\{\Phi_{0,p}\}_{p<\infty}$. Then $R = \bigotimes_{v\le\infty} R_v$ defines a smooth representation of $G_\mathbb{A}\times G_\mathbb{A}\times G_\mathbb{A}$ on $\mathcal{S}'(V_\mathbb{A}\times\mathbb{Q}_\mathbb{A}^\times)$. We put $\mathcal{I} = \bigotimes_v \mathcal{I}_v$. We define a theta kernel by

$$\theta(g,g_1,g_2) = \left|\det(gg_1^{-1}g_2)\right|_\mathbb{A}^{k/2} \sum_{(X,t)\in V\times\mathbb{Q}^\times} R(g,g_1,g_2)\Phi_0(X,t), \tag{5.1}$$

where $\Phi_0 = \otimes_v\Phi_{0,v} \in \mathcal{S}'(V_\mathbb{A}\times\mathbb{Q}_\mathbb{A}^\times)$. By Poisson summation formula and Lemma 5.5, we have the following:

Lemma 5.6. *(i) For* $\gamma,\gamma_1,\gamma_2 \in G_\mathbb{Q}$, $g,g_1,g_2\in G_\mathbb{A}$, $u_f, u_{1,f}, u_{2,f}\in\mathcal{U}_f$ *and* $u_\infty, u_{1,\infty}, u_{2,\infty} \in \mathcal{U}_\infty$, *we have*

$$\begin{aligned}&\theta(\gamma g u_f u_\infty, \gamma_1 g_1 u_{1,f}u_{1,\infty}, \gamma_2 g_2 u_{2,f}u_{2,\infty})\\ &\quad = \widetilde{\chi}(u_f u_{1,f}^{-1}u_{2,f}) j(u_\infty,i)^{-k} j(u_{1,\infty},i)^{-k} j(u_{2,\infty},i)^k \theta(g,g_1,g_2).\end{aligned}$$

(ii) For $z_\infty \in Z_\infty$, *we have* $\theta(z_\infty g, g_1, g_2) = \theta(g,g_1,g_2)$.

(iii) For $z\in Z_\mathbb{A}$, *we have* $\theta(zg,g_1,g_2) = \theta(g,z^{-1}g_1,g_2) = \theta(g,g_1,zg_2)$.

5.5 Theta lift

For $f \in \mathbf{S}_k(N,\chi)$, we define

$$\mathcal{L}f(g_1,g_2) = \int_{Z_\infty^+G_\mathbb{Q}\backslash G_\mathbb{A}} f(g)\overline{\theta(g,g_1,g_2)}dg. \tag{5.2}$$

Since $g\mapsto\theta(g,g_1,g_2)$ is of moderate growth on $Z_\infty^+G_\mathbb{Q}\backslash G_\mathbb{A}$ for (g_1,g_2) in any compact subset of $G_\mathbb{A}\times G_\mathbb{A}$, the integral (5.2) converges absolutely. By Lemma 5.6, we have the following:

Lemma 5.7. *For* $z_i\in Z_\mathbb{A}, \gamma_i\in G_\mathbb{Q}, g_i\in G_\mathbb{A}, u_{i,f}\in\mathcal{U}_f, u_{i,\infty}\in\mathcal{U}_\infty$ $(i=1,2)$, *we have*

$$\begin{aligned}&\mathcal{L}f(z_1\gamma_1g_1u_{1,f}u_{1,\infty}, z_2\gamma_2g_2u_{2,f}u_{2,\infty})\\ &\quad = \chi(z_1z_2^{-1})\widetilde{\chi}(u_{1,f}u_{2,f}^{-1}) j(u_{1,\infty},i)^k j(u_{2,\infty},i)^{-k}\mathcal{L}f(g_1,g_2).\end{aligned}$$

6 The image of the theta lift

6.1

The object of this section is to show the following key result of the paper.

THEOREM 6.1. *Let $f \in \mathbf{S}_k(N,\chi)$ be a primitive form. Then we have*

$$\mathcal{L}f(g_1,g_2) = (-2i)^k\overline{f_N(g_1)}f_N(g_2) \qquad (g_1,g_2 \in G_{\mathbb{A}}).$$

Recall that we have defined $f_N(g) = \chi^{-1}(\det g)f(gw_N)$ (cf. 3.6). For $(m,n) \in \mathbb{Q}^2$, set

$$\mathcal{L}f^{(m,n)}(g_1,g_2) = \int_{(\mathbb{Q}\backslash\mathbb{Q}_{\mathbb{A}})^2} \psi(mx_1 - nx_2)\mathcal{L}f(\mathbf{n}(x_1)g_1,\mathbf{n}(x_2)g_2)dx_1dx_2. \tag{6.1}$$

Since $\mathcal{L}f^{(m,n)}(g_1,g_2) = \mathcal{L}^{(1,1)}(\mathbf{d}(m,1)g_1,\mathbf{d}(n,1)g_2)$ for $(m,n) \in (\mathbb{Q}^\times)^2$, the proof of Theorem 6.1 is reduced to that of the following fact:

PROPOSITION 6.2. *Let $f \in \mathbf{S}_k(N,\chi)$ be a primitive form.*

(i) If $mn = 0$, we have $\mathcal{L}f^{(m,n)}(g_1,g_2) = 0$.

(ii) We have $\mathcal{L}f^{(1,1)}(g_1,g_2) = (-2i)^k\,\overline{W_{f_N}(g_1)}W_{f_N}(g_2)$.

6.2 Intertwining operators

In this section, we recall a definition of a certain intertwining operator introduced in [W]. Let v be a place of $\mathbb{Q}$. Define a linear operator $I_v\colon \mathcal{S}'(V_v \times \mathbb{Q}_v^\times) \to \mathcal{S}'(V_v \times \mathbb{Q}_v^\times)$ by

$$I_v\Phi\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) = |t|_v^{1/2}\int_{\mathbb{Q}_v} \psi_v(tyy')\Phi\left(\begin{pmatrix} x & y' \\ z & w \end{pmatrix}, t\right)dy' \tag{6.2}$$

for $\Phi \in \mathcal{S}'(V_v \times \mathbb{Q}_v^\times)$. A straightforward calculation shows the following:

LEMMA 6.3 ([W]). *Let $\Phi \in \mathcal{S}'(V_v \times \mathbb{Q}_v^\times)$.*

(i) For $b, b' \in \mathbb{Q}_v$, we have

$$\begin{aligned} I_v \circ R_v(1,\mathbf{n}(b),\mathbf{n}(b'))\Phi&\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) \\ &= \psi_v(ty(-bx + b'w + bb'z))\, I_v\Phi\left(\begin{pmatrix} x - b'z & y \\ z & w + bz \end{pmatrix}, t\right). \end{aligned}$$

(ii) For $a, d \in \mathbb{Q}_v^\times$, we have

$$I_v \circ R_v(1,\mathbf{d}(a,d),1)\Phi\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) = \left|\frac{a}{d}\right|_v^{1/2} I_v\Phi\left(\begin{pmatrix} ax & ay \\ az & dw \end{pmatrix}, (ad)^{-1}t\right).$$

(iii) For $a', d' \in \mathbb{Q}_v^\times$, we have

$$I_v \circ R_v(1, 1, \mathbf{d}(a', d'))\Phi \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right) = \left| \frac{a'}{d'} \right|_v^{1/2} I_v\Phi \left(\begin{pmatrix} x/a' & y/d' \\ z/d' & w/d' \end{pmatrix}, a'd't \right).$$

(iv) For $b \in \mathbb{Q}_v$, we have

$$I_v \circ R_v(\mathbf{n}(b), 1, 1)\Phi \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right) = \psi_v(btxw) I_v\Phi \left(\begin{pmatrix} x & y - bz \\ z & w \end{pmatrix}, t \right).$$

(v) For $a, d \in \mathbb{Q}_v^\times$, we have

$$I_v \circ R_v(\mathbf{d}(a, d), 1, 1)\Phi \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right) = \left| \frac{a}{d} \right|_v^{1/2} I_v\Phi \left(\begin{pmatrix} x/d & y/a \\ z/d & w/d \end{pmatrix}, adt \right).$$

(vi) We have

$$\begin{aligned} &I_v \circ R_v (w_0, 1, 1)\, \Phi \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right) \\ &= |t|_v \int_{\mathbb{Q}_v^2} \psi_v(-t(wx_1 + xw_1)) I_v\Phi \left(\begin{pmatrix} x_1 & z \\ -y & w_1 \end{pmatrix}, t \right) dx_1 dw_1. \end{aligned}$$

6.3 Fourier coefficients of $\mathcal{L}f$

Put $I = \bigotimes_v I_v$. By Poisson summation formula, we obtain

$$\theta(g, g_1, g_2) = \theta_0(g, g_1, g_2) + \theta_1(g, g_1, g_2),$$

where

$$\begin{aligned} \theta_0(g, g_1, g_2) &= \left|\det(g g_1^{-1} g_2)\right|_{\mathbb{A}}^{k/2} \sum_{x, w \in \mathbb{Q}, t \in \mathbb{Q}^\times} I \circ R(g, g_1, g_2)\Phi_0 \left(\begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix}, t \right), \\ \theta_1(g, g_1, g_2) &= \left|\det(g g_1^{-1} g_2)\right|_{\mathbb{A}}^{k/2} \sum_{x, w \in \mathbb{Q}, t \in \mathbb{Q}^\times} \sum_{(y,z) \in \mathbb{Q}^2 - \{(0,0)\}} I \circ R(g, g_1, g_2)\Phi_0 \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right). \end{aligned}$$

Lemma 6.4. *(i) We have*

$$\begin{aligned} \theta_0(g, g_1, g_2) &= \left|\det(g g_1^{-1} g_2)\right|_{\mathbb{A}}^{k/2} \sum_{\gamma \in \mathcal{N}_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \mathcal{I} \circ R(\gamma g, g_1, g_2)\Phi_0 \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1 \right), \\ \theta_1(g, g_1, g_2) &= \left|\det(g g_1^{-1} g_2)\right|_{\mathbb{A}}^{k/2} \sum_{x, w \in \mathbb{Q}} \sum_{\gamma \in \mathcal{N}_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} I \circ R(\gamma g, g_1, g_2)\Phi_0 \left(\begin{pmatrix} x & -1 \\ 0 & w \end{pmatrix}, 1 \right). \end{aligned}$$

(ii) For $i = 0, 1$, $g \mapsto \theta_i(g, g_1, g_2)$ is left $Z_\infty^+ G_{\mathbb{Q}}$-invariant.

Proof. By Poisson summation formula, $\left|\det(gg_1^{-1}g_2)\right|_{\mathbb{A}}^{-k/2}\theta_0(g,g_1,g_2)$ is equal to

$$\sum_{x,w\in\mathbb{Q},t\in\mathbb{Q}^\times}\mathcal{I}\circ R(g,g_1,g_2)\Phi_0\left(\begin{pmatrix}0 & x\\ 0 & w\end{pmatrix},t\right).$$

By Lemma 5.3, this is equal to

$$\sum_{t\in\mathbb{Q}^\times}\mathcal{I}\circ R(1,1,g_2)\Phi_0\left(\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\det(gg_1^{-1})t\right)+\sum_{\gamma\in\mathcal{N}_\mathbb{Q}\backslash G_\mathbb{Q}}\mathcal{I}\circ R(\gamma g,g_1,g_2)\Phi_0\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},1\right).$$

The first term vanishes, since

$$\mathcal{I}_\infty\circ R_\infty(1,1,g_\infty)\Phi_\infty\left(\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},t\right)=0$$

holds for any $g_\infty\in G_\infty$ and $t\in\mathbb{R}^\times$ in view of Lemmas 5.3, 5.4 and 5.5. Thus the first formula of (i) has been verified. Next observe that

$$\sum_{x,w\in\mathbb{Q}}\sum_{\gamma\in\mathcal{N}_\mathbb{Q}\backslash G_\mathbb{Q}}I\circ R(\gamma g,g_1,g_2)\Phi_0\left(\begin{pmatrix}x & -1\\ 0 & w\end{pmatrix},1\right)=J_1+J_2.$$

where

$$J_1=\sum_{x,w\in\mathbb{Q}}\sum_{a,d\in\mathbb{Q}^\times}I\circ R(\mathbf{d}(a,d)g,g_1,g_2)\Phi_0\left(\begin{pmatrix}x & -1\\ 0 & w\end{pmatrix},1\right),$$

$$J_2=\sum_{x,w\in\mathbb{Q}}\sum_{a,d\in\mathbb{Q}^\times,b\in\mathbb{Q}}I\circ R(w_0\mathbf{d}(a,d)\mathbf{n}(b)g,g_1,g_2)\Phi_0\left(\begin{pmatrix}x & -1\\ 0 & w\end{pmatrix},1\right).$$

By Lemma 6.3 and Poisson summation formula, we have

$$\begin{aligned}J_2&=\sum_{x,w\in\mathbb{Q}}\sum_{a,d\in\mathbb{Q}^\times,b\in\mathbb{Q}}I\circ R(\mathbf{d}(a,d)\mathbf{n}(b)g,g_1,g_2)\Phi_0\left(\begin{pmatrix}x & 0\\ 1 & w\end{pmatrix},1\right)\\&=\sum_{x,w\in\mathbb{Q}}\sum_{a,d\in\mathbb{Q}^\times,b\in\mathbb{Q}}I\circ R(g,g_1,g_2)\Phi_0\left(\begin{pmatrix}d^{-1}x & -d^{-1}b\\ d^{-1} & d^{-1}w\end{pmatrix},ad\right)\\&=\sum_{x,w\in\mathbb{Q},t\in\mathbb{Q}^\times}\sum_{(y,z)\in\mathbb{Q}\times\mathbb{Q}^\times}I\circ R(g,g_1,g_2)\Phi_0\left(\begin{pmatrix}x & y\\ z & w\end{pmatrix},t\right).\end{aligned}$$

On the other hand, we have

$$J_1=\sum_{x,w\in\mathbb{Q},t\in\mathbb{Q}^\times}\sum_{y\in\mathbb{Q}^\times}I\circ R(g,g_1,g_2)\Phi_0\left(\begin{pmatrix}x & y\\ 0 & w\end{pmatrix},t\right),$$

which completes the proof the second formula of (i). The left $G_\mathbb{Q}$-invariance of $g\mapsto\theta_i(g,g_1,g_2)$ follows from (i). The left Z_∞^+-invariance is derived from Lemma 5.5 (iii). □

LEMMA 6.5. *For $f \in \mathbf{S}_k(N,\chi)$, we have*

$$\int_{Z_\infty^+ G_\mathbb{Q}\backslash G_\mathbb{A}} f(g)\overline{\theta_0(g,g_1,g_2)}dg = 0 \qquad (g_1, g_2 \in G_\mathbb{A})$$

and hence

$$\mathcal{L}f(g_1,g_2) = \int_{Z_\infty^+ G_\mathbb{Q}\backslash G_\mathbb{A}} f(g)\overline{\theta_1(g,g_1,g_2)}dg.$$

PROOF. This follows from Lemma 6.4, the cuspidality of f and the fact that

$$\mathcal{I}\circ R(\mathbf{n}(x),1,1)\Phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},1\right) = \mathcal{I}\Phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},1\right)$$

for any $\Phi \in \mathcal{S}'(V_\mathbb{A}\times\mathbb{Q}_\mathbb{A}^\times)$ and $x \in \mathbb{Q}_\mathbb{A}$. □

For $m \in \mathbb{Q}$, we put

$$W_f^m(g) = \int_{\mathbb{Q}\backslash\mathbb{Q}_\mathbb{A}} \psi(-mx)f(\mathbf{n}(x)g)dx.$$

Note that $W_f^0(g) = 0$ and $W_f^m(g) = W_f(\mathbf{d}(m,1)g)$ for $m \in \mathbb{Q}^\times$.

PROPOSITION 6.6. *For $(m,n) \in \mathbb{Q}^2$, we have*

$$\mathcal{L}f^{(m,n)}(g_1,g_2) = |\det g_1^{-1}g_2|_\mathbb{A}^{k/2}\int_{Z_\infty^+\mathcal{N}_\mathbb{A}\backslash G_\mathbb{A}} |\det g|_\mathbb{A}^{k/2} W_f^{mn}(g)\,\overline{I\circ R(g,g_1,g_2)\Phi_0\left(\begin{pmatrix} m & -1 \\ 0 & n \end{pmatrix},1\right)}dg.$$

PROOF. By Lemmas 6.4 and 6.5, $\mathcal{L}f^{(m,n)}(g_1,g_2)$ is equal to

$$|\det(g_1^{-1}g_2)|_\mathbb{A}^{k/2}\int_{(\mathbb{Q}\backslash\mathbb{Q}_\mathbb{A})^2}\int_{Z_\infty^+ G_\mathbb{Q}\backslash G_\mathbb{A}} \psi(mx_1-nx_2)|\det g|_\mathbb{A}^{k/2} f(g)$$
$$\sum_{x,w\in\mathbb{Q}}\sum_{\gamma\in\mathcal{N}_\mathbb{Q}\backslash G_\mathbb{Q}} \overline{I\circ R(\gamma g,\mathbf{n}(x_1)g_1,\mathbf{n}(x_2)g_2)\Phi_0\left(\begin{pmatrix} x & -1 \\ 0 & w \end{pmatrix},1\right)}dgdx_1dx_2.$$

Since

$$\begin{aligned} &I\circ R(\gamma g,\mathbf{n}(x_1)g_1,\mathbf{n}(x_2)g_2)\Phi_0\left(\begin{pmatrix} x & -1 \\ 0 & w \end{pmatrix},1\right) \\ &\quad= \psi(xx_1-wx_2)\,I\circ R(\gamma g,g_1,g_2)\Phi_0\left(\begin{pmatrix} x & -1 \\ 0 & w \end{pmatrix},1\right), \end{aligned}$$

we obtain

$$
\begin{aligned}
&\mathcal{L}f^{(m,n)}(g_1,g_2)\\
&= |\det(g_1^{-1}g_2)|_{\mathbb{A}}^{k/2}\int_{Z_\infty^+G_{\mathbb{Q}}\backslash G_{\mathbb{A}}}|\det g|_{\mathbb{A}}^{k/2}f(g)\sum_{\gamma\in\mathcal{N}_{\mathbb{Q}}\backslash G_{\mathbb{Q}}}\overline{I\circ R(\gamma g,g_1,g_2)\Phi_0\left(\begin{pmatrix} m & -1\\ 0 & n\end{pmatrix},1\right)}dg\\
&= |\det(g_1^{-1}g_2)|_{\mathbb{A}}^{k/2}\int_{Z_\infty^+\mathcal{N}_{\mathbb{A}}\backslash G_{\mathbb{A}}}\int_{\mathbb{Q}\backslash\mathbb{Q}_{\mathbb{A}}}|\det g|_{\mathbb{A}}^{k/2}f(\mathbf{n}(x)g)\\
&\qquad\overline{\psi(mnx)I\circ R(g,g_1,g_2)\Phi_0\left(\begin{pmatrix} m & -1\\ 0 & n\end{pmatrix},1\right)}dxdg\\
&= |\det(g_1^{-1}g_2)|_{\mathbb{A}}^{k/2}\int_{Z_\infty^+\mathcal{N}_{\mathbb{A}}\backslash G_{\mathbb{A}}}|\det g|_{\mathbb{A}}^{k/2}W_f^{mn}(g)\,\overline{I\circ R(g,g_1,g_2)\Phi_0\left(\begin{pmatrix} m & -1\\ 0 & n\end{pmatrix},1\right)}dg.
\end{aligned}
$$

□

The first assertion of Proposition 6.2 immediately follows from this proposition.

6.4

Let $\{\lambda_p\ (p\not|N), \lambda_p^{\pm}\ (p|N)\}$ be the Hecke eigenvalues of f. Recall that $\overline{\lambda_p^{\pm}}=\lambda_p^{\mp}$ if $p|N$, and

$$
\lambda_p^+\lambda_p^- = \begin{cases} 1 & \text{if } p|M^{-1}N,\\ p & \text{if } p|M.\end{cases}
$$

For a place v of $\mathbb{Q}$, let W_v and W_v' be as in Proposition 3.5. Set

$$
\begin{aligned}
&J_v(g_1,g_2)\\
&= \int_{\mathcal{N}_v'\backslash G_v}|\det g|_v^{k/2}W_v(g)\,\overline{I_v\circ R_v(g,g_1,g_2)\Phi_{0,v}\left(\begin{pmatrix} 1 & -1\\ 0 & 1\end{pmatrix},1\right)}dg \qquad (g_1,g_2\in G_v),
\end{aligned}
$$

where

$$
\mathcal{N}_v' = \begin{cases} \mathcal{N}_p & \text{if } v=p<\infty,\\ Z_\infty^+\mathcal{N}_\infty & \text{if } v=\infty.\end{cases}
$$

The following is easily verified.

Lemma 6.7. *For $z_i\in\mathbb{Q}_v^\times, x_i\in\mathbb{Q}_v, g_i\in G_v, u_i\in\mathcal{U}_v\ (i=1,2)$, we have*

$$
\begin{aligned}
&J_v(z_1\mathbf{n}(x_1)g_1u_1, z_2\mathbf{n}(x_2)g_2u_2)\\
&= J_v(g_1,g_2)\cdot\chi_v(z_1z_2^{-1})|z_1z_2^{-1}|_v^k\ \psi_v(-x_1+x_2)\times\begin{cases}\widetilde{\chi}_v(u_1^{-1}u_2) & \text{if } v<\infty,\\ j(u_1,i)^k j(u_2,i)^{-k} & \text{if } v=\infty.\end{cases}
\end{aligned}
$$

In view of Propositions 3.5 and 6.6, we have

$$
\mathcal{L}f^{(1,1)}(g_1,g_2) = |\det g_1^{-1}g_2|_{\mathbb{A}}^{k/2}\prod_{v\le\infty}J_v(g_{1,v},g_{2,v}) \qquad (g_i=(g_{i,v})_v\in G_{\mathbb{A}}, i=1,2).
$$

In order to show Proposition 6.2 (ii), it now suffices to show the following.

Proposition 6.8. *Let v be a place of $\mathbb{Q}$. Then, for $g_1, g_2 \in G_v$, we have*

$$J_v(g_1,g_2) = |\det g_1 g_2^{-1}|_v^{k/2}\,\overline{W_v'(g_1)}W_v'(g_2) \times \begin{cases} 1 & \text{if } v<\infty, \\ (-2i)^k & \text{if } v=\infty. \end{cases}$$

7 Proof of Proposition 6.8

7.1

In the subsections 7.1–7.4, we consider the case where $v = p < \infty$, and often suppress the subscript p. We write $F, \mathcal{O}_F$ and $\mathfrak{p}_F$ for $\mathbb{Q}_p, \mathbb{Z}_p$ and $p\mathbb{Z}_p$, respectively. Recall that $\mathbf{d}_m = \mathbf{d}(\pi^m, 1)$ $(m \in \mathbb{Z})$ and

$$w_1 = \begin{pmatrix} 0 & -1 \\ \pi & 0 \end{pmatrix}.$$

For $m \in \mathbb{Z}$, we write τ_m and τ_m' for the characteristic functions of $\pi^m\mathcal{O}_F$ and $\pi^m\mathcal{O}_F^\times$, respectively. Observe that we have the following integral formula: For $\varphi \in L^1(\mathcal{N}\backslash G/\mathcal{U})$,

$$\int_{\mathcal{N}\backslash G} \varphi(g)dg = \begin{cases} \sum_{i,j\in\mathbb{Z}} p^j \varphi(\pi^{-i}\mathbf{d}_j) & \text{if } p \not| N, \\ \sum_{i,j\in\mathbb{Z}} p^j \varphi(\pi^{-i}\mathbf{d}_j) + \sum_{i,j\in\mathbb{Z}} p^j \varphi(\pi^{-i}\mathbf{d}_j w_1) & \text{if } p|N. \end{cases} \tag{7.1}$$

Recall that

$$W'(g) = \chi^{-1}(\det g) \times \begin{cases} W(g) & \text{if } p \not| N, \\ W(gw_1) & \text{if } p|N. \end{cases}$$

7.2

In this subsection, we suppose $p \not| N$. Note that $\overline{W'(g)} = \chi(\det g)\overline{W(g)} = W(g)$ by Lemma 3.3. To prove Proposition 6.8 in this case, it suffices to show

$$J(\mathbf{d}_m, \mathbf{d}_n) = p^{k(-m+n)/2}\chi(\pi)^{-n}W(\mathbf{d}_m)W(\mathbf{d}_n) \qquad (m,n\in\mathbb{Z}) \tag{7.2}$$

in view of Lemma 6.7. We may (and do) assume that $m,n \geq 0$, since both sides of (7.2) vanish unless $m,n\geq 0$. Recall that $\overline{I\Phi_0(X,t)} = \Phi_0(X,t)$ and $W(\mathbf{d}_m) = (t_1^{m+1} - t_2^{m+1})/(t_1 - t_2)$, where $t_1, t_2 \in \mathbb{C}$ satisfy $t_1 + t_2 = p^{-1}\chi(\pi)\lambda, t_1t_2 = p^{-1}\chi(\pi)$. By (7.1), $J(\mathbf{d}_m, \mathbf{d}_n)$ is equal to

$$\begin{aligned} &\sum_{i\in\mathbb{Z}}\sum_{j=0}^{\infty} p^{j-k(-2i+j)/2} W(\pi^{-i}\mathbf{d}_j)\, \overline{I\circ R(\pi^{-i}\mathbf{d}_j, \mathbf{d}_m, \mathbf{d}_n)\Phi_0\left(\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}, 1\right)} \\ &= p^{-(m+n)/2}\sum_{i\in\mathbb{Z}}\sum_{j=0}^{\infty} p^{ki+(1-k)j/2}\chi(\pi)^{-i}W(\mathbf{d}_j)\Phi_0\left(\begin{pmatrix} \pi^{m-n+i} & -\pi^{m+i-j} \\ 0 & \pi^i\end{pmatrix}, \pi^{-2i+j-m+n}\right) \\ &= p^{-n+k(-m+n)/2}\sum_{0\leq i\leq n,\, i\geq n-m,\, 2i+m-n\geq 0} p^i\chi(\pi)^{-i}W(\mathbf{d}_{2i+m-n+1}). \end{aligned}$$

If $m \geq n$,

$$J(\mathbf{d}_m, \mathbf{d}_n) = p^{-n+(-m+n)k/2} \sum_{0 \leq i \leq n} p^i \chi(\pi)^{-i} \frac{t_1^{2i+m-n+1} - t_2^{2i+m-n+1}}{t_1 - t_2}$$
$$= p^{k(-m+n)/2} \chi(\pi)^{-n} \frac{(t_1^{m+1} - t_2^{m+1})(t_1^{n+1} - t_2^{n+1})}{(t_1 - t_2)^2},$$

which proves (7.2) in this case. We can verify (7.2) similarly in the case $m < n$.

7.3

In this subsection, we suppose $p | M^{-1}N$. We then have

$$W(gw_1) = \epsilon W(g) \qquad (g \in G) \tag{7.3}$$

with $\epsilon = -\lambda^+$. Recall that Φ_0 is the characteristic function of $\begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix} \times \mathcal{O}_F^\times$. Let Φ_0' denote the characteristic function of $\begin{pmatrix} \mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix} \times \pi^{-1}\mathcal{O}_F^\times$. A straightforward calculation shows the following:

Lemma 7.1.
$$R(w_1, 1, 1)\Phi_0 = R(1, w_1^{-1}, 1)\Phi_0 = R(1, 1, w_1)\Phi_0 = \Phi_0'.$$

Lemma 7.2. *We have*

$$J(g_1 w_1, g_2) = p^{-k/2} \epsilon J(g_1, g_2), \quad J(g_1, g_2 w_1) = p^{k/2} \chi^{-1}(\pi) \epsilon J(g_1, g_2) \qquad (g_1, g_2 \in G).$$

Proof. By using the fact $w_1 = -\pi w_1^{-1}$ and Lemma 7.1, we have

$$J(g_1 w_1, g_2) = \chi(\pi) p^{-k} J(g_1 w_1^{-1}, g_2)$$
$$= \chi(\pi) p^{-k} \int_{\mathcal{N} \backslash G} |\det g|^{k/2} W(g) \overline{I \circ R(gw_1, g_1, g_2)\Phi_0 \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, 1 \right)} dg.$$

Changing the variable g into gw_1^{-1} in the integral and using (7.3), we get $J(g_1 w_1, g_2) = p^{-k/2} \epsilon J(g_1, g_2)$ as required. The second formula is proved in a similar manner. □

In view of Lemma 7.2, to prove Proposition 6.8 in this case, it is sufficient to show the following:

Lemma 7.3. *For $m, n \geq 0$, we have*

$$J(\mathbf{d}_m, \mathbf{d}_n) = p^{(-m+n)k/2} \overline{W'(\mathbf{d}_m)} W'(\mathbf{d}_n). \tag{7.4}$$

PROOF. By (7.1), we have

$$
\begin{aligned}
&J(\mathbf{d}_m, \mathbf{d}_n)\\
&= \sum_{i,j\in\mathbb{Z}} p^{ki+(1-k)j/2-(m+n)/2}\chi(\pi)^{-i}W(\mathbf{d}_j)\overline{I\Phi_0\left(\begin{pmatrix}\pi^{i+m-n} & -\pi^{i-j+m}\\ 0 & \pi^i\end{pmatrix}, \pi^{-2i+j-m+n}\right)}\\
&+ \sum_{i,j\in\mathbb{Z}} p^{ki+(1-k)j/2-(m+n)/2-k/2}\epsilon\chi(\pi)^{-i}W(\mathbf{d}_j)\overline{I\Phi_0'\left(\begin{pmatrix}\pi^{i+m-n} & -\pi^{i-j+m}\\ 0 & \pi^i\end{pmatrix}, \pi^{-2i+j-m+n}\right)}.
\end{aligned}
$$

Since $I\Phi_0 = \Phi_0$ and $I\Phi_0' = p^{1/2}\mathrm{char}_{\pi M_2(\mathcal{O}_F)\times\pi^{-1}\mathcal{O}_F^\times}$, $J(\mathbf{d}_m, \mathbf{d}_n)$ is equal to

$$
\begin{aligned}
p^{(-m+n)k/2-n} \times \Bigg\{ &\sum_{0\le i\le n,\ i+m-n\ge 0} p^i\chi(\pi)^{-i}(p^{-1}\lambda^+)^{2i+m-n}\\
&+ \sum_{1\le i\le n,\ i+m-n\ge 1} p^i\epsilon\chi(\pi)^{-i}(p^{-1}\lambda^+)^{2i+m-n-1}\Bigg\}.
\end{aligned}
$$

Using $\epsilon = -\lambda^+$ and $(\lambda^+)^2 = \chi(\pi)$, we have

$$
\begin{aligned}
&J(\mathbf{d}_m, \mathbf{d}_n)\\
&= p^{(-m+n)k/2-n}\left\{\sum_{0\le i\le n, i+m-n\ge 0} p^{-i-m+n} - \sum_{1\le i\le n, i+m-n\ge 1} p^{-i-m+n+1}\right\}(\lambda^+)^{m-n}\\
&= p^{k(-m+n)/2-m-n}(\lambda^+)^{m-n}.
\end{aligned}
$$

On the other hand, by Lemma 3.4, the right-hand side of (7.4) is equal to

$$
p^{(-m+n)k/2-m-n}\chi(\pi)^{m-n}(\overline{\lambda^+})^m(\lambda^+)^n = p^{k(-m+n)/2-m-n}(\lambda^+)^{m-n},
$$

which completes the proof of Lemma 7.3. □

7.4

In this subsection, we suppose that $p|M$. Recall that

$$
\Phi_0\left(\begin{pmatrix} x & y\\ z & w\end{pmatrix}, t\right) = \chi^{-1}(w)\tau_0(x)\tau_0(y)\tau_1(z)\tau_0'(w)\tau_0'(t).
$$

The following is proved by a straightforward calculation.

Lemma 7.4. *For* $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in V$ *and* $t \in F^\times$*, we have*

$$\begin{aligned}
I\Phi_0(X,t) &= \Phi_0(X,t),\\
I \circ R(w_1,1,1)\Phi_0(X,t) &= p^{1/2}\,\overline{G(\chi)}\,\chi(tx)\,\tau_0'(x)\tau_1(y)\tau_1(z)\tau_1(w)\tau_{-1}'(t),\\
I \circ R(1,w_1,1)\Phi_0(X,t) &= p^{1/2}\chi^{-1}(-z)\,\tau_0(x)\tau_0(y)\tau_0'(z)\tau_0(w)\tau_1'(t),\\
I \circ R(w_1,w_1,1)\Phi_0(X,t) &= \chi^{-1}(y)\,\tau_0(x)\tau_0'(y)\tau_1(z)\tau_0(w)\tau_0'(t),\\
I \circ R(1,1,w_1)\Phi_0(X,t) &= p^{1/2}\overline{G(\chi)}\,\chi(ty)\,\tau_1(x)\tau_0'(y)\tau_1(z)\tau_1(w)\tau_{-1}'(t),\\
I \circ R(w_1,1,w_1)\Phi_0(X,t) &= \overline{G(\chi)}\,\chi(tz)\,\tau_1(x)\tau_1(y)\tau_1'(z)\tau_1(w)\tau_{-2}'(t),\\
I \circ R(1,w_1,w_1)\Phi_0(X,t) &= \chi^{-1}(x)\,\tau_0'(x)\tau_0(y)\tau_1(z)\tau_0(w)\tau_0'(t),\\
I \circ R(w_1,w_1,w_1)\Phi_0(X,t) &= p^{1/2}\,\overline{G(\chi)}\,\chi(tw)\,\tau_1(x)\tau_1(y)\tau_1(z)\tau_0'(w)\tau_{-1}'(t).
\end{aligned}$$

Lemma 7.5. *For* $m,n \geq 0$*, we have*

$$\begin{aligned}
J(\mathbf{d}_m,\mathbf{d}_n) &= p^{(-m+n)k/2}(p^{-1}\lambda^+)^m(p^{-1}\lambda^-)^n,\\
J(\mathbf{d}_m w_1,\mathbf{d}_n) &= p^{(-m+n-1)k/2}\chi(-1)G(\chi)^{-1}\chi(\pi)^m(p^{-1}\lambda^-)^{m+n+1},\\
J(\mathbf{d}_m,\mathbf{d}_n w_1) &= p^{(-m+n+1)k/2+1}\chi(-1)G(\chi)\chi(\pi)^{-n}(p^{-1}\lambda^+)^{m+n+1},\\
J(\mathbf{d}_m w_1,\mathbf{d}_n w_1) &= p^{(-m+n)k/2}\chi(\pi)^{m-n}(p^{-1}\lambda^-)^m(p^{-1}\lambda^+)^n.
\end{aligned}$$

Proof. In view of (7.1), we have

$$\begin{aligned}
J(g_1,g_2) = &\sum_{i\in\mathbb{Z}}\sum_{j\geq 0} p^{ki+(-1-k)j/2}\chi(\pi)^{-i}(\lambda^+)^j\\
&\overline{\times\, I \circ R(1,g_1,g_2)\Phi_0\left(\begin{pmatrix} \pi^i & -\pi^{i-j} \\ 0 & \pi^i \end{pmatrix}, \pi^{-2i+j}\right)}\\
&+ \sum_{i\in\mathbb{Z}}\sum_{j\geq 0} p^{ki+(-1-k)j/2-k/2-1}\chi(\pi)^{-i+j}G(\chi)^{-1}(\lambda^-)^{j+1}\\
&\overline{\times\, I \circ R(w_0,g_1,g_2)\Phi_0\left(\begin{pmatrix} \pi^i & -\pi^{i-j} \\ 0 & \pi^i \end{pmatrix}, \pi^{-2i+j}\right)}.
\end{aligned}$$

Then, by Lemma 7.1 and using the fact that $\lambda^+\lambda^- = p$ and $\overline{\lambda^+} = \lambda^-$, we obtain

$$\begin{aligned}
J(\mathbf{d}_m, \mathbf{d}_n) &= \sum_{i\in\mathbb{Z}}\sum_{j\geq 0} p^{ki+(-1-k)j/2-(m+n)/2}\chi(\pi)^{-i}(\lambda^+)^j \\
&\qquad \times \overline{I\Phi_0\left(\begin{pmatrix} \pi^{i+m-n} & -\pi^{i-j+m} \\ 0 & \pi^i \end{pmatrix}, \pi^{-2i+j-m+n}\right)} \\
&\quad + \sum_{i\in\mathbb{Z}}\sum_{j\geq 0} p^{ki+(-1-k)j/2-k/2-1-(m+n)/2}\chi(\pi)^{-i+j}G(\chi)^{-1}(\lambda^-)^{j+1} \\
&\qquad \times \overline{I\circ R(w_1,1,1)\Phi_0\left(\begin{pmatrix} \pi^{i+m-n} & -\pi^{i-j+m} \\ 0 & \pi^i \end{pmatrix}, \pi^{-2i+j-m+n}\right)} \\
&= \delta(m\geq n)p^{(-k-2)m/2+kn/2}(\lambda^+)^{m-n} + \delta(m<n)p^{-km/2+(k-2)n/2}(\lambda^-)^{-m+n} \\
&= p^{(-m+n)k/2}(p^{-1}\lambda^+)^m(p^{-1}\lambda^-)^n,
\end{aligned}$$

which proves the first formula of the lemma. The other ones are proved in a similar manner. □

We now prove Prop 6.8 in the case $p|M$. It suffices to show

$$J(\mathbf{d}_m w_1^{\epsilon}, \mathbf{d}_n w_1^{\epsilon'}) = p^{(-m-\epsilon+n+\epsilon')k/2}\overline{W'(\mathbf{d}_m w_1^{\epsilon})}W'(\mathbf{d}_n w_1^{\epsilon'}) \tag{7.5}$$

for $m, n \geq 0$ and $\epsilon, \epsilon' \in \{0, 1\}$. We prove (7.5) only in the case $\epsilon = \epsilon' = 0$, since the proofs are similar in the other cases. In this case, the right-hand side of (7.5) is equal to

$$\begin{aligned}
&p^{(-m+n)k/2}\chi(\pi)^{m-n}\overline{W(\mathbf{d}_m w_1)}W(\mathbf{d}_n w_1) \\
&= p^{(-m+n)k/2}\chi(\pi)^{m-n}p^{-2}|G(\chi)|^{-2}|\lambda^-|^2\overline{(p^{-1}\chi(\pi)\lambda^+)^m}(p^{-1}\chi(\pi)\lambda^+)^n \\
&= p^{(-m+n)k/2}(p^{-1}\lambda^-)^m(p^{-1}\lambda^+)^n,
\end{aligned}$$

which implies (7.5) by Lemma 7.5.

7.5

To prove Proposition 6.8 in the case $v = \infty$, it is sufficient to show the following result in view of Lemma 3.1, Proposition 3.5 (iii) and Lemma 6.7:

Lemma 7.6. *For $a_1, a_2 \in \mathbb{R}^\times$, we have*

$$J_\infty(\mathbf{d}(a_1, 1), \mathbf{d}(a_2, 1)) = \begin{cases} (-2i)^k\, a_1^k\, \mathbf{e}[i(a_1 + a_2)] & \text{if } a_1, a_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We write $\mathcal{J}(a_1, a_2)$ for $J_\infty(\mathbf{d}(a_1, 1), \mathbf{d}(a_2, 1))$. Then

$$\mathcal{J}(a_1, a_2) = \int_0^\infty y^{k-1/2}\exp(-2\pi y)\sqrt{|a_1 a_2|}\,\overline{I\Phi_0\left(\begin{pmatrix} a_1 a_2^{-1} & -a_1 y^{-1} \\ 0 & 1 \end{pmatrix}, a_1^{-1}a_2 y\right)}\,d^\times y.$$

Since $I\Phi_0(X,t)=0$ if $t<0$, we see that $\mathcal{J}(a_1,a_2)=0$ if $a_1a_2<0$. Assume that $a_1a_2>0$. Since

$$\begin{aligned}
&I\Phi_0\left(\begin{pmatrix} a_1a_2^{-1} & -a_1y^{-1} \\ 0 & 1 \end{pmatrix}, a_1^{-1}a_2y\right) \\
&= \sqrt{a_1^{-1}a_2y}\int_{\mathbb{R}} \mathbf{e}\left[-a_2x+\frac{i}{2}a_1^{-1}a_2y\left(a_1^2a_2^{-2}+x^2+1\right)\right]\left(-x+i\frac{a_1+a_2}{a_2}\right)^k dx,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\mathcal{J}(a_1,a_2) \\
&= |a_2|\int_{\mathbb{R}}\mathbf{e}[a_2x]\left(-x-i\frac{a_1+a_2}{a_2}\right)^k\int_0^\infty y^k\exp(-\pi y(2+a_1a_2^{-1}+a_1^{-1}a_2+a_1^{-1}a_2x^2))d^\times y\,dx \\
&= |a_2|\Gamma(k)\pi^{-k}\left(\frac{a_1}{a_2}\right)^k\int_{\mathbb{R}}\mathbf{e}[a_2x]\left(-x-i\frac{a_1+a_2}{a_2}\right)^k\left(x^2+\left(\frac{a_1+a_2}{a_2}\right)^2\right)^{-k}dx \\
&= (k-1)!\,\pi^{-k}|a_2|\left(\frac{a_1}{a_2}\right)^k\int_{\mathbb{R}}\mathbf{e}[a_2x]\left(-x+i\frac{a_1+a_2}{a_2}\right)^{-k}dx.
\end{aligned}$$

By the residue theorem, the last integral vanishes if $a_2<0$. Suppose that $a_1,a_2>0$. Then the last integral is equal to

$$\begin{aligned}
&2\pi i(-1)^k\,\mathrm{Res}_{x=a_2^{-1}(a_1+a_2)i}\mathbf{e}[a_2x]\left(x-i\frac{a_1+a_2}{a_2}\right)^{-k} \\
&= 2\pi i(-1)^k\mathbf{e}[i(a_1+a_2)]\frac{1}{(k-1)!}(2\pi i a_2)^{k-1},
\end{aligned}$$

which implies $\mathcal{J}(a_1,a_2)=(-2i)^ka_1^k\mathbf{e}[(a_1+a_2)i]$. □

8 Periods of automorphic forms

8.1

The object of this section is to show that, if f is a primitive form, $|\mathcal{P}(f,\xi;g)|^2$ is expressed as an integral involving the Whittaker function attached to f.

8.2 Waldspurger's formula

We first recall several results due to Waldspurger ([W]) on the period integral of $\mathcal{L}(f)$ defined by

$$\mathcal{Q}(f,\xi;g_1,g_2)=\int_{(\mathbb{Q}_\mathbb{A}^\times K^\times\backslash K_\mathbb{A}^\times)^2}\xi^\sigma(z_1^{-1}z_2)\mathcal{L}(f)(\iota(z_1)g_1,\iota(z_2)g_2)d^\times z_1d^\times z_2 \qquad (g_1,g_2\in G_\mathbb{A}),$$

where $\xi^\sigma(z)=\xi(z^\sigma)$ for $z\in K_\mathbb{A}^\times$. For $g=\mathbf{n}(x)\mathbf{d}(y_1,y_2)u\in G_\mathbb{A}$ $(x\in\mathbb{Q}_\mathbb{A}, y_1,y_2\in\mathbb{Q}_\mathbb{A}^\times, u\in U_fU_\infty)$, we put $a(g)=|y_1y_2^{-1}|_\mathbb{A}$. For $g\in G_\mathbb{A}, z\in K_\mathbb{A}^\times, \Phi\in\mathcal{S}'(V_\mathbb{A}\times\mathbb{Q}_\mathbb{A}^\times)$ and $s\in\mathbb{C}$, set

$$(8.1)\qquad I(g,z;\Phi;s)=\sum_{\gamma\in B_\mathbb{Q}\backslash G_\mathbb{Q}}a(\gamma g)^{s-1/2}\sum_{x\in K,t\in\mathbb{Q}^\times}R(\gamma g,1,\iota(z))\Phi(\iota(x),t),$$

where B is the group of upper triangular matrices in G. It is easily seen that $(g,z) \mapsto I(g,z;\Phi;s)$ is left $G_{\mathbb{Q}} \times K^\times$-invariant, and that $I(g,\alpha z;\Phi;s) = I(\alpha g, z;\Phi;s)$ for $\alpha \in \mathbb{Q}_{\mathbb{A}}^\times$. Set

$$\mathcal{Q}(f,\xi;g_1,g_2;s) = \left|\det(g_1^{-1}g_2)\right|_{\mathbb{A}}^{k/2} \int_{\mathbb{Q}_{\mathbb{A}}^\times K^\times \backslash K_{\mathbb{A}}^\times} \int_{Z_\infty^+ G_{\mathbb{Q}} \backslash G_{\mathbb{A}}} f(g)|\operatorname{N}(z)\det g|_{\mathbb{A}}^{k/2}$$
$$\xi^\sigma(z)\overline{I(g,z;s;R(1,g_1,g_2)\Phi_0)}dgd^\times z.$$

Choose $\eta \in V$ such that $V = \iota(K) + \iota(K)\eta$ and $\iota(x)\eta = \eta\iota(x^\sigma)$ for $x \in K$.

Proposition 8.1 ([W], pages 197–198). *(i) The series* (8.1) *is absolutely convergent if* $\operatorname{Re}(s)$ *is sufficiently large. For* $(g,z) \in G_{\mathbb{A}} \times K_{\mathbb{A}}^\times$ *and* $\Phi \in \mathcal{S}'(V_{\mathbb{A}} \times \mathbb{Q}_{\mathbb{A}}^\times)$*,* $s \mapsto I(g,z;\Phi;s)$ *is continued to a meromorphic function of* s *on* $\mathbb{C}$*, and holomorphic at* $s = 1/2$*.*

(ii) We have

$$L(\omega;1)I(g,z;\Phi;1/2) = \int_{\mathbb{Q}_{\mathbb{A}}^\times K^\times \backslash K_{\mathbb{A}}^\times} \sum_{x,y\in K,\, t\in\mathbb{Q}^\times} (R(g,1,\iota(z))\Phi)\left(\iota(x) + \iota(z_1^\sigma z_1^{-1}y)\,\eta, t\right)d^\times z_1.$$

(iii) For fixed $g_1, g_2 \in G_{\mathbb{A}}$*,* $s \mapsto \mathcal{Q}(f,\xi;g_1,g_2;\overline{s})$ *is continued to a meromorphic function of* s *on* $\mathbb{C}$*, and holomorphic at* $s = 1/2$*. We have*

$$\mathcal{Q}(f,\xi;g_1,g_2) = L(\omega;1)\mathcal{Q}(f,\xi;g_1,g_2;1/2).$$

(iv) We have

$$\mathcal{Q}(f,\xi;g_1,g_2;\overline{s}) = \left|\det(g_1^{-1}g_2)\right|_{\mathbb{A}}^{k/2} \int_{\mathbb{R}_+^\times \backslash K_{\mathbb{A}}^\times} \int_{Z_{\mathbb{A}}\mathcal{N}_{\mathbb{A}} \backslash G_{\mathbb{A}}} W_f(g)|\operatorname{N}(z)^{-1}\det g|_{\mathbb{A}}^{k/2}$$
$$\xi^\sigma(z^{-1})a(g)^{s-1/2}\ \overline{R(g,g_1,g_2)\Phi_0(\iota(z),N(z)^{-1})}\ dgd^\times z.$$

8.3

We henceforth assume that $f \in \mathbf{S}_k(N,\chi)$ is a primitive form. By Theorem 6.1, we have

$$\mathcal{L}f(g_1,g_2) = (-2i)^k\overline{f_N(g_1)}f_N(g_2) \qquad (g_1,g_2 \in G_{\mathbb{A}}),$$

where $f_N(g) = \chi^{-1}(\det g)f(gw_N)$ and $w_N = \prod_{p|N} w_1^{(p)}$. It follows that

$$|\mathcal{P}(f,\xi;g)|^2 = (-2i)^{-k}\mathcal{Q}(f,\xi;gw_N^{-1},gw_N^{-1}) \tag{8.2}$$

for $g \in G_{\mathbb{A}}$. By Proposition 8.1 (iii), we obtain the following.

Proposition 8.2. *For a primitive form* $f \in \mathbf{S}_k(N,\chi)$*, we have*

$$|\mathcal{P}(f,\xi;g)|^2 = (-2i)^{-k}L(\omega;1)\mathcal{Q}(f,\xi;gw_N^{-1},gw_N^{-1};1/2).$$

8.4

Let W_v $(v \le \infty)$ be as in Proposition 3.5. By Proposition 8.1 (iv), we obtain

$$\mathcal{Q}(f,\xi;g,g;\overline{s}) = \prod_v \mathcal{Q}_v(W_v,\xi_v;g_v,g_v;\overline{s}) \qquad (g=(g_v)_v \in G_{\mathbb{A}}),$$

where

$$\mathcal{Q}_v(W_v,\xi_v;g_v,g_v;\overline{s}) = \int_{C_v\backslash K_v^\times}\int_{Z_v\mathcal{N}_v\backslash G_v} W_v(g')a(g')^{s-1/2}|\,\mathrm{N}(z)^{-1}\det g'|_v^{k/2}\xi_v^\sigma(z^{-1}) \overline{R(g',g_v,g_v)\Phi_{0,v}(\iota(z),N(z)^{-1})}dg'd^\times z,$$

where $C_v=\{1\}$ if $v<\infty$ and $C_\infty=\mathbb{R}_+^\times$.

9 Proof of Theorem 1.1 (ii)

9.1

In this section, we assume that $S_1(\xi)=S_2^+(f,\xi)=\emptyset$. Note that $\xi_p^{-1}(\Pi_p)\lambda_p^+=-1$ if $p|M^{-1}N, p|D$ and $\alpha_p(\xi)=0$. For each place v of $\mathbb{Q}$, we write $Q_v(s)$ for $\mathcal{Q}_v(W_v,\xi_v;g_{1,v},g_{1,v};\overline{s})$, where

$$g_{1,v}=(g_0w_N^{-1})_v=\begin{cases}\mathbf{d}(\pi^{-\alpha_p(\xi)},1) & \text{if } v=p<\infty, p\nmid N,\\ \mathbf{d}(\pi^{-\alpha_p(\xi)+1},1)w_1^{(p)} & \text{if } v=p<\infty, p|N,\\ g_{0,\infty} & \text{if } v=\infty.\end{cases}$$

In this section, we prove the following results:

Proposition 9.1. *Suppose that $p\nmid N$. Then we have*

$$Q_p(s)=L_p(f,\xi^{-1};s)\times|D|_p^{1/2}p^{-\alpha_p(\xi)}\begin{cases}L_p(\omega;2s)^{-1} & \text{if } \alpha_p(\xi)=0,\\ L_p(\omega;1) & \text{if } \alpha_p(\xi)>0.\end{cases}$$

Proposition 9.2. *Suppose that $p|N$. Then we have*

$$Q_p(s)=L_p(f,\xi^{-1};s)\times|D|_p^{1/2}p^{-\alpha_p(\xi)}L_p(\omega;1)Y_p(s)\,,$$

where $Y_p(s)$ is given by

$$Y_p(s)=\begin{cases}1-\xi^{-1}(\Pi_p)\lambda_p^+p^{s-1/2} & \text{if } \alpha_p(\xi)=0 \text{ and } p \text{ ramifies in } K/\mathbb{Q},\\ L_p(\omega;1)^{-2} & \text{if } \alpha_p(\xi)=0 \text{ and } p \text{ splits in } K/\mathbb{Q},\\ 1 & \text{otherwise.}\end{cases}$$

Proposition 9.3. *We have*

$$Q_\infty(s)=(-2i)^k(4\pi)^{-(s+k-3/2)}\Gamma(s+k-1/2).$$

We first prove Theorem 1.1 (ii) assuming the above results. We have

$$\begin{aligned}&\mathcal{Q}(f,\xi;g_0w_N^{-1},g_0w_N^{-1};\overline{s})=\prod_v Q_v(s)\\&=L(f,\xi^{-1};s)\,|D|^{-1/2}A(\xi)^{-1}(-2i)^k(4\pi)^{-s-k+3/2}\Gamma(s+k-1/2)\\&\quad\times L(\omega;2s)^{-1}\prod_{p\nmid N,\,p|A(\xi)}L_p(\omega;1)^2\prod_{p|N}\left(L_p(\omega;1)L_p(\omega;2s)Y_p(s)\right).\end{aligned}$$

Observe that

$$Y_p(1/2)=\begin{cases}2 & \text{if } \alpha_p(\xi)=0 \text{ and } p \text{ ramifies in } K/\mathbb{Q},\\ L_p(\omega;1)^{-2} & \text{if } \alpha_p(\xi)=0 \text{ and } p \text{ splits in } K/\mathbb{Q},\\ 1 & \text{otherwise.}\end{cases}$$

Then, by Proposition 8.2, we obtain

$$\begin{aligned}&|\mathcal{P}(f,\xi;g_0)|^2=(-2i)^{-k}L(\omega;1)\mathcal{Q}(f,\xi;g_0w_N^{-1},g_0w_N^{-1};1/2)\\&=(4\pi)^{1-k}(k-1)!\,|D|^{-1/2}A(\xi)^{-1}2^{|S_2(\xi)|}\prod_{p|A(\xi)}L_p(\omega;1)^2\times L(f,\xi^{-1};1/2),\end{aligned}$$

which completes the proof of Theorem 1.1 (ii).

9.2 Proof of Proposition 9.1

In this and the next subsections, we often suppress the subscript p from the notation. We write F and K for $\mathbb{Q}_p$ and K_p respectively. Let $\mathcal{O}_F$ and $\mathcal{O}_K$ be the integer rings of F and K, respectively. We denote by τ_n and τ'_n the characteristic functions of $\pi^n\mathcal{O}_F$ and $\pi^n\mathcal{O}_F^\times$, respectively. We write simply α for $\alpha(\xi)$ if there is no fear of confusion.

We suppose that $p\nmid N$. Recall that $g_0=\mathbf{d}(\pi^{-\alpha},1)$ and $W(\mathbf{d}(\pi^n,1))=(t_1^{n+1}-t_2^{n+1})/(t_1-t_2)$, where $t_1,t_2\in\mathbb{C}$ satisfy $t_1+t_2=p^{-1}\chi(\pi)\lambda$ and $t_1t_2=p^{-1}\chi(\pi)$. We also recall that Φ_0 is the characteristic function of $M_2(\mathcal{O}_F)\times\mathcal{O}_F^\times$. Then we have

$$\begin{aligned}&Q(s)\\&=\int_{K^\times}\int_{F^\times}W(\mathbf{d}(a,1))|\,\mathrm{N}(z)^{-1}a|^{k/2}|a|^{s-1/2}\xi^\sigma(z^{-1})\\&\qquad\overline{R(\mathbf{d}(a,1),g_0,g_0)\Phi_0(\iota(z),\mathrm{N}(z)^{-1}))}\,|a|^{-1}d^\times a\,d^\times z\\&=\int_{K^\times}\int_{F^\times}W(\mathbf{d}(a,1))|\,\mathrm{N}(z)^{-1}a|^{k/2}|a|^{s-1/2}\xi^\sigma(z^{-1})\overline{\Phi_0(g_0^{-1}\iota(z)g_0,\mathrm{N}(z)^{-1}a))}d^\times a\,d^\times z\\&=\int_{K^\times}W(\mathbf{d}(\mathrm{N}(z),1))\xi^\sigma(z^{-1})|\,\mathrm{N}(z)|^{s-1/2}\varphi_0(\iota_0(z))d^\times z,\end{aligned}$$

where φ_0 is the characteristic function of $M_2(\mathcal{O}_F)$ and

$$\iota_0(z)=g_0^{-1}\iota(z)g_0=\begin{pmatrix}x & \pi^\alpha\,\mathrm{N}(\theta)y\\ -\pi^{-\alpha}y & x+\mathrm{Tr}(\theta)y\end{pmatrix}\qquad(z=x+\theta y).$$

Suppose that K/F is inert. Note that $\mathrm{vol}(\mathcal{O}_K^\times) = |D|^{1/2} = 1$. Then

$$Q(s) = \sum_{n=0}^{\infty} W(\mathbf{d}_{2n})\xi(\pi)^{-n}p^{-2n(s-1/2)}J_n(\xi),$$

where

$$J_n(\xi) = \int_{\mathcal{O}_K^\times} \xi^\sigma(z^{-1})\varphi_0(\pi^n\iota_0(z))d^\times z.$$

If $\alpha = 0$, we have $J_n(\xi) = 1$ for $n \geq 0$ and hence

$$\begin{aligned} Q(s) &= \sum_{n=0}^{\infty}(\xi^{-1}(\pi)p^{-2s+1})^n \ \frac{t_1^{2n+1} - t_2^{2n+1}}{t_1 - t_2} \\ &= (1 + \chi(\pi)\xi^{-1}(\pi)p^{-2s}) \prod_{i=1}^{2}(1 - \xi^{-1}(\pi)t_i^2 p^{-2s+1})^{-1} \\ &= L_p(\omega; 2s)^{-1}L_p(f, \xi^{-1}; s). \end{aligned}$$

Suppose that $\alpha > 0$. Since $\mathcal{O}_K^\times = (\mathcal{O}_F^\times + \theta\mathcal{O}_F) \cup (\mathfrak{p}_F + \theta\mathcal{O}_F^\times)$, $J_n(\xi)$ is equal to

$$\frac{1}{1-p^{-2}}\left\{\int_{\mathcal{O}_F^\times}\int_{\mathcal{O}_F} \xi(x+\theta y)\tau_{\alpha-n}(y)dydx + \int_{\mathfrak{p}_F}\int_{\mathcal{O}_F^\times} \xi(x+\theta y)\tau_{\alpha-n}(y)dydx\right\},$$

where dx and dy are normalized such that $\mathrm{vol}(\mathcal{O}_F) = 1$. We have

$$\begin{aligned} J_0(\xi) &= \frac{1}{1-p^{-2}}\int_{\mathcal{O}_F^\times}\int_{\pi^\alpha\mathcal{O}_F} \xi(x+\theta y)dydx = \frac{1}{1+p^{-1}}\int_{\pi^\alpha\mathcal{O}_F} \xi(1+\theta y)dy \\ &= \frac{p^{-\alpha}}{1+p^{-1}} = L_p(\omega; 1)p^{-\alpha}. \end{aligned}$$

For $n \geq \alpha$, we have

$$J_n(\xi) = \int_{\mathcal{O}_K^\times} \xi(z)d^\times z = 0,$$

since ξ is nontrivial on $\mathcal{O}_K^\times$. For $0 < n < \alpha$, we have

$$J_n(\xi) = \frac{1}{1-p^{-2}}\int_{\mathcal{O}_F^\times}\int_{\pi^{\alpha-n}\mathcal{O}_F} \xi(x+\theta y)dydx = \frac{1}{1+p^{-1}}\int_{\pi^{\alpha-n}\mathcal{O}_F} \xi(1+\theta y)dy = 0$$

by Lemma 2.2. It follows that $Q_p(s) = p^{-\alpha}L_p(\omega; 1)$, which completes the proof of Proposition 9.1 in the inert case. The proofs in the other cases are similar (though more complicated) and omitted.

9.3 Proof of Proposition 9.2

In this subsection, we suppose that $p|N$. Recall that $g_1 = \mathbf{d}(\pi^{-\alpha+1}, 1)w_1$ and

$$\Phi_0(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t) = \begin{cases} \tau_0(x)\tau_0(y)\tau_1(z)\tau_0(w)\tau_0'(t) & \text{if } p|M^{-1}N, \\ \chi^{-1}(w)\tau_0(x)\tau_0(y)\tau_1(z)\tau_0'(w)\tau_0'(t) & \text{if } p|M. \end{cases}$$

Put $\Phi_0' = R(w_1, 1, 1)\Phi_0$. A straightforward calculation shows

$$\Phi_0'\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) = \begin{cases} \tau_1(x)\tau_0(y)\tau_1(z)\tau_1(w)\tau_{-1}'(t) & \text{if } p|M^{-1}N, \\ \overline{G(\chi)}\chi(tx)\tau_0'(x)\tau_0(y)\tau_1(z)\tau_1(w)\tau_{-1}'(t) & \text{if } p|M. \end{cases}$$

By (7.1), we have

$$Q(s) = \sum_{n=0}^{\infty} W(\mathbf{d}_n)p^{-n(s-1/2)}J_n(\xi) + \sum_{n=0}^{\infty} W(\mathbf{d}_n w_1)p^{-(n-1)(s-1/2)}J_n'(\xi),$$

where

$$J_n(\xi) = \int_{K^\times} |\pi^n \operatorname{N}(z)^{-1}|^{k/2}\xi^\sigma(z^{-1})\overline{\Phi_0(\iota_1(z), \pi^n \operatorname{N}(z)^{-1})}d^\times z,$$
$$J_n'(\xi) = \int_{K^\times} |\pi^{n+1} \operatorname{N}(z)^{-1}|^{k/2}\xi^\sigma(z^{-1})\overline{\Phi_0'(\iota_1(z), \pi^n \operatorname{N}(z)^{-1})}d^\times z$$

and

$$\iota_1(z) = g_1^{-1}\iota(z)g_1 = \begin{pmatrix} x + \operatorname{Tr}(\theta)y & \pi^{-\alpha}y \\ -\pi^\alpha \operatorname{N}(\theta)y & x \end{pmatrix} \qquad (z = x + \theta y).$$

Assume that $p|M^{-1}N$ and K/F is ramified. Put

$$\varphi_0(X) = \tau_0(x)\tau_0(y)\tau_1(z)\tau_0(w),\ \varphi_0'(X) = \tau_1(x)\tau_0(y)\tau_1(z)\tau_1(w) \quad \left(X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}\right).$$

Then we have

$$J_n(\xi) = \int_{\Pi^n\mathcal{O}_K^\times} \xi^\sigma(z^{-1})\varphi_0(\iota_1(z))d^\times z,\ J_n'(\xi) = \int_{\Pi^{n+1}\mathcal{O}_K^\times} \xi^\sigma(z^{-1})\varphi_0'(\iota_1(z))d^\times z.$$

First suppose that $\alpha = 0$. We see that $\varphi_0(\iota_1(z)) = 1$ for $z \in \mathcal{O}_K = \mathcal{O}_F + \theta\mathcal{O}_F$ and $\varphi_0'(\iota_1(z)) = 1$ for $z \in \Pi\mathcal{O}_K = \mathfrak{p}_F + \theta\mathcal{O}_F$, since $\operatorname{Tr}(\theta), \operatorname{N}(\theta) \in \mathfrak{p}_F$. This implies that

$$J_n(\xi) = |D|^{1/2}\xi(\Pi)^{-n},\ J_n'(\xi) = |D|^{1/2}\xi(\Pi)^{-n-1} \qquad (n \geq 0).$$

We thus have

$$\begin{aligned} Q(s) &= |D|^{1/2}\left\{\sum_{n=0}^{\infty}(p^{-1}\lambda^+)^n p^{-n(s-1/2)}\xi(\Pi)^{-n} + \sum_{n=0}^{\infty}(-\lambda^+)(p^{-1}\lambda^+)^n p^{-(n-1)(s-1/2)}\xi(\Pi)^{-n-1}\right\} \\ &= |D|^{1/2}(1 - \xi(\Pi)^{-1}\lambda^+ p^{s-1/2})(1 - \xi(\Pi)^{-1}\lambda^+ p^{-s-1/2})^{-1} \\ &= |D|^{1/2}L_p(f, \xi^{-1}; s)Y_p(s). \end{aligned}$$

Next suppose that $\alpha > 0$. Then we have

$$\begin{aligned} J_0(\xi) &= \frac{|D|^{1/2}}{1 - p^{-1}}\int_{\mathcal{O}_F^\times}\int_{\mathfrak{p}_F^\alpha} \xi^\sigma(x + \theta y)^{-1}dydx = |D|^{1/2}\int_{\mathfrak{p}_F^\alpha} \xi^\sigma(1 + \theta y)^{-1}dy \\ &= |D|^{1/2}p^{-\alpha}. \end{aligned}$$

Suppose that $1 \leq m < \alpha$. Then we have

$$\begin{aligned} J_{2m}(\xi) &= \frac{|D|^{1/2}\xi(\pi^{-m})}{1-p^{-1}} \int_{\mathcal{O}_F^\times} \int_{\mathfrak{p}_F^{\alpha-m}} \xi^\sigma(x+\theta y)^{-1} dy dx \\ &= |D|^{1/2}\xi(\pi^{-m}) \int_{\mathfrak{p}_F^{\alpha-m}} \xi^\sigma(1+\theta y)^{-1} dy \\ &= 0 \end{aligned}$$

by Lemma 2.2. We also have $J_{2m+1}(\xi) = 0$, since $\varphi_0(\iota_1(z)) = 0$ for $z \in \Pi^{2m+1}\mathcal{O}_K^\times = \mathfrak{p}_F^{m+1} + \theta\pi^m\mathcal{O}_F^\times$. If $m \geq \alpha$, we have

$$J_{2m}(\xi) = \int_{\Pi^{2m}\mathcal{O}_K^\times} \xi^\sigma(z)^{-1} d^\times z = 0, \quad J_{2m+1}(\xi) = \int_{\Pi^{2m+1}\mathcal{O}_K^\times} \xi^\sigma(z)^{-1} d^\times z = 0.$$

A similar argument shows that $J'_n(\xi) = 0$ for $n \geq 0$. We thus have

$$Q(s) = J_0(\xi) = |D|^{1/2} p^{-\alpha},$$

which completes the proof of Proposition 9.2 in the case where $p | M^{-1}N$ and K/F is ramified. The proofs in the other cases are similar and omitted.

9.4 Proof of Proposition 9.3

We have

$$Q_\infty(s) = \int_0^\infty \int_{\mathbb{C}^1} W_\infty(\mathbf{d}(y,1)) y^{s+k/2-3/2} z^k \overline{R(\mathbf{d}(y,1), g_0, g_0)\Phi_{0,\infty}(\iota(z),1)} d^\times z d^\times y,$$

where $\mathbb{C}^1 = \{z \in \mathbb{C}^\times \mid z\overline{z} = 1\}$. Observe that, for $y > 0$ and $z = u + iv \in \mathbb{C}^1$,

$$\begin{aligned} R(\mathbf{d}(y,1), g_0, g_0)\Phi_{0,\infty}(\iota(z),1) &= y\Phi_{0,\infty}\left(g_0^{-1}\iota(z)g_0, y\right) \\ = y\Phi_{0,\infty}\left(\begin{pmatrix} u & v \\ -v & u \end{pmatrix}, y\right) &= (2i)^k y z^k \exp(-2\pi y). \end{aligned}$$

We thus have

$$\begin{aligned} Q_\infty(s) &= (-2i)^k \mathrm{vol}(\mathbb{C}^1) \int_0^\infty y^{s+k-1/2} \exp(-4\pi y) d^\times y \\ &= (-2i)^k (4\pi)^{-(s-3/2+k)} \Gamma(s+k-1/2), \end{aligned}$$

which completes the proof of the proposition.

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