# A $C^*$-ANALOGUE OF KAZHDAN'S PROPERTY (T)

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Abstract. This paper deals with a "naive" way of generalization of the Kazhdan's property (T) to $C^*$-algebras. This approach differs from the approach of Connes and Jones, which has already demonstrated its utility. Nevertheless it turned out that our approach is applicable to the following rather subtle question in the theory of $C^*$-Hilbert modules.

We prove that a separable unital $C^*$-algebra $A$ has property MI (*module-infinite*, i. e. any $C^*$-Hilbert module over $A$ is self-dual if and only if it is finitely generated projective) if and only if it has not property ($\mathrm{T_1P}$) (*property* (T) *at one point*, i. e. there exists in $\widehat{A}$ a finite dimensional isolated point).

The commutative case was studied in a previous paper.

## 1. Introduction

There are at least two basic approaches to the definition of Kazhdan's property (T) [14] for topological groups (see [6, 2, 16]). The first one uses the notion of $\varepsilon$-invariant vectors, and property (T) is formulated in terms of existence of a so-called Kazhdan pair for a topological group. More precisely, let $G$ be a topological group, $Q \subset G$ be a compact set, and $\varepsilon > 0$. Then the group $G$ has $T$-property if any unitary representation $(\pi, H)$ of $G$ which has a $(Q, \varepsilon)$-invariant vector also has a non-zero invariant vector (in this case the pair $(Q, \varepsilon)$ is called a Kazhdan pair). The second approach is a reformulation of property (T) in terms of Fell's topology. From this point of view a topological group $G$ has (T)-property if its identity representation $1_G$ is isolated in $\mathcal{R} \cup 1_G$, for every set $\mathcal{R}$ of equivalence classes of unitary representations of $G$ without non-zero invariant vectors.

An analogue of property (T) for $W^*$-algebras was introduced by Connes and Jones in [3, 4]. The key notion of this definition was the notion of a correspondence which plays a role of a representation of a group. Namely, let $A$ and $B$ be von Neumann algebras. A correspondence from $A$ to $B$ is a Hilbert space $H$ which is a left $A$-module and a right $B$-module with commuting normal actions. It is possible to introduce some topology on the set of all correspondences from $A$ to $B$ which is similar to Fell's topology. Let $\mathrm{Id}_A$ be the identity correspondence constructed in [12]. Then $A$ has (T)-property if there is a neighborhood $U$ of $\mathrm{Id}_A$ such that any correspondence in $U$ contains $\mathrm{Id}_A$ as a direct summand. This definition may be reformulated in terms of central and almost central vectors. In [4] was proved that a countable discrete group $G$ with factorial von Neumann algebra $L(G)$ has Kazhdan's property if and only if $L(G)$ has property (T). This is a very natural approach and a number of developments and applications was obtained (in particular, was constructed an example of a homomorphism $\theta$ of a discrete group $Q$ with

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property (T) into $\mathrm{Out}(\lambda(F_\infty)'')$ with trivial obstruction $\mathrm{Ob}\,\theta \in H^3(Q,\mathbb{T})$ but which cannot be lifted to a homomorphism from $Q$ to $\mathrm{Aut}(\lambda(F_\infty)'')$ [4]).

For some $C^*$-algebras an analogue of property (T) was defined in [1] by an adaptation of Connes' definition. On the whole the approach of B. Bekka is close to the first approach for topological groups. Namely, let $A$ be either a unital $C^*$-algebra admitting a tracial state or a finite von Neumann algebra. Then $A$ has Property (T) if there exist a finite subset $F$ of $A$ and $\varepsilon > 0$ such that the following property holds: if a Hilbert $A$-bimodule $H$ contains a unit $(F,\varepsilon)$-central vector, then $H$ has a non-zero central vector. In [1] it is proved that if $G$ is a countable discrete group and $A$ a $C^*$-algebra being a quotient of $C^*_{max}(G)$ such that $C^*_r(G)$ is a quotient of $A$, then $G$ has property (T) if and only if $A$ has Property (T) if and only if $L(G)$ has property (T).

In the present paper we discuss the following "naive" generalization of property (T) to the context of $C^*$-algebras.

**Definition 1.** A unital $C^*$-algebra $A$ has property ($\mathrm{T_1P}$) (*property* (T) (at least) *at one point* of its spectrum) if there exists an isolated point with respect to the Fell's topology in unitary dual $\widehat{A}$ such that the corresponding representation is finite dimensional.

**Remark 2.** As it is known (see e.g. [6, Prop. 14], [2, Theorem 1.2.6]), for locally compact groups the trivial representation is isolated if and only if all finite dimensional representations are isolated, if and only if any finite dimensional representation is isolated. In particular, for $A = C^*(G)$ our definition is exactly the Kazhdan's one, if $G$ is a locally compact group.

We prove in the present paper that property ($\mathrm{T_1P}$) is responsible for one very fine property of $C^*$-Hilbert modules over the algebra under consideration.

**Definition 3** ([23])**.** A unital $C^*$-algebra $A$ is called MI (*module-infinite*) if each countably generated Hilbert $A$-module is projective finitely generated if and only if it is self-dual. Let us remark that a projective finitely generated module over a unital algebra is always self-dual.

**Main Theorem.** *Suppose, $A$ is a separable unital $C^*$-algebra. Then $A$ is* MI *if and only if $\widehat{A}$ has no isolated point being a finite-dimensional representation, i. e. $A$ is not* ($\mathrm{T_1P}$)*.*

For commutative algebras this was proved in [23]. The research was motivated by the study of conditional expectations of finite index related to group actions [11, 23, 22] and [18, Sect. 4.5].

**Remark 4.** The property of existence of isolated points can be formulated for $\widehat{A}$ and for $\mathrm{Prim}(A)$. Fortunately, for finite-dimensional representations it does not depend on the space. Indeed, a difference could arise in the case if an isolated point under consideration has several pre-images. But finite-dimensional irreducible representations are equivalent if and only if they have the same kernel (see e.g. Example 5.6.2 and Theorem 5.6.3 in [20]). This will allow us to choice an appropriate setting to simplify argument.

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## 2. Preliminaries and reminding

The necessary information about Hilbert C*-modules can be found in [15] and [18, 17]. Let us remind only some facts and notations for convenience of the reader.

The standard Hilbert module over a $C^*$-algebra $A$ is denoted by $l_2(A)$ or $H_A$.

**Proposition 5** ([18], Proposition 2.5.5). *Consider the set of all sequences $f = (f_i), f_i \in A, i \in \mathbb{N}$, such that the partial sums of the series $\sum f_i^* f_i$ are uniformly bounded. If $A$ is a unital $C^*$-algebra, then this set coincides with $H'_A$, the action of $f$ on $H_A$ is defined by the formula*

$$f(x) = \sum_{i=1}^{\infty} f_i^* x_i,$$

*where $x = (x_i) \in H_A$, and the norm of the functional $f$ is satisfies*

$$\|f\|^2 = \sup_N \left\| \sum_{i=1}^{\infty} f_i^* f_i \right\|.$$

**Proposition 6** ([18], Theorem 5.1.6). *Let $A$ be a $C^*$-algebra. Then the following conditions are equivalent:*

(i) *the Hilbert $C^*$-module $H_A$ is self-dual;*
(ii) *the $C^*$-algebra $A$ is finite-dimensional.*

Let us remind also some notions and notations about spectrum of a $C^*$-algebra $A$ (for more information see [8, 21]). Let $\widehat{A}$ be the space of all equivalence classes of irreducible representations of $A$ and $\mathrm{Prim}(A)$ be the space of primitive ideals of $A$. For $a \in A$ and $I \in \mathrm{Prim}(A)$ we will denote by $\|a + I\|$ the norm of element $a + I$ in the factor algebra $A/I$.

**Theorem 7** (Dauns-Hofmann, [5, 9]). *Let $A$ be a $C^*$-algebra, let $x$ be an element of $A$, and let $f$ be a bounded continuous scalar-valued function on $\mathrm{Prim}(A)$, the space of primitive ideals of $A$ endowed with the hull-kernel topology. Then there exists a unique element $fx$ of $A$ such that*

$$(\widehat{fx})(I) = f(I)\widehat{x}(I), \qquad I \in \mathrm{Prim}(A),$$

*where $\widehat{x}(I) = x + I \in A/I$.*

A point $S \in \mathrm{Prim}(A)$ is called a *separated point* if for any $I \in \mathrm{Prim}(A)$ that is not an accumulation point of $S$ there are disjoint neighborhoods of $S$ and $I$.

**Theorem 8** ([7]). *Suppose $A$ is a $C^*$-algebra Then*

*(i) a point $I \in \mathrm{Prim}(A)$ is separated if and only if for any $x \in A$ the function $I \mapsto \|x+I\|$ is continuous at $I$;*
*(ii) if $A$ is separable, then the set of separated points of $\mathrm{Prim}(A)$ is $G_\delta$ and it is dense in $\mathrm{Prim}(A)$.*

## 3. Some properties of Prim($A$).

Suppose, $A$ is a unital $C^*$-algebra.

**Lemma 9.** *Let $U$ be a neighborhood of a point $\rho \in \widehat{A}$. Then there exists a positive element $a \in A$ of norm 1 such that*
(i) $\|\rho(a)\| = 1$,
(ii) $\pi(a) = 0$ *for all* $\pi \in \widehat{A} \setminus U$.
*The same is true in* Prim($A$).

*Proof.* By [8, Lemma 3.3.3] there exists a positive element $x \in A$ such that the set $Z = \{\pi \in \widehat{A} : \|\pi(x)\| > 1\}$ contains $\rho$ and belongs to $U$. Let $\|\rho(x)\| = t$, $1 < t \le \|x\|$, and let $t_1 \in (1, t)$. Let $f$ be a continuous function on $[0, \|x\|]$, equal to 0 on $[0, t_1]$, equal to 1 on $[t, \|x\|]$ and linear on $[t_1, t]$, and let $a = f(x)$. Then $a \ge 0$, $\|a\| = 1$. Besides,

$$\|\rho(a)\| = \|\rho(f(x))\| = \|f(\rho(x))\| = 1$$

and

$$\|\pi(a)\| = \|\pi(f(x))\| = \|f(\pi(x))\| = 0$$

for all $\pi \in \widehat{A} \setminus U$, because $\|\pi(x)\| \le 1 < t_1$. □

**Lemma 10.** *Let $\{S_i\}$ be a sequence of different separated points from* Prim($A$). *Then there exist a subsequence $\{S_{i(j)}\}$ of $\{S_i\}$ and neighborhoods $V_j$ of $S_{i(j)}$, which do not contain the remaining points of the subsequence.*

*Proof.* Suppose, $\{S_i\}$ has no accumulation points among its members. Then each of them has a neighborhood with no other points of the sequence.

Now, suppose that $S_k$ is an accumulation point. Without loss of generality (we can pass to a subsequence) we assume that $S_k$ is a limit point. Let $V_1'$ and $V_1$ be disjoint neighborhoods of $S_k$ and $S_1 =: S_{i(1)}$ (since Prim($A$) is a $T_0$ space, separated points there form a Hausdorff space, in particular, any finite number of them has pairwise disjoint neighborhoods). Then there exist $S_{i(2)} \in V_1'$, $S_{i(2)} \ne S_k$, and their disjoint neighborhoods $W_2' \ni S_k$ and $W_2 \ni S_{i(2)}$. Take $V_2' := W_2' \cap V_1'$ and $V_2 := W_2 \cap V_1'$. And so on. □

## 4. The property DINC

**Definition 11.** A unital C*-algebra is said to be DINC (*divisible infinite for the noncommutative case*) if for any sequence $u_i \in A$ of elements of norm $1 \ge \|u_i\| \ge C > 0$ there exist a subsequence $i(k)$, elements $b_k \in A$ of norm 1 such that
(i) the partial sums of the series $\sum_k b_k^* b_k$ are uniformly bounded, and
(ii) for each $k$

$$\|u_{i(k)} b_k\| \ge C/2. \tag{1}$$

The following result is a generalization of [23, Theorem 32].

**Theorem 12.** *If a unital $C^*$-algebra $A$ is* DINC, *then it is* MI.

*Proof.* We have to prove that if a countable generated Hilbert $A$-module $\mathcal{M}$ is not finitely generated projective, then it is not self-dual. By the Kasparov stabilization theorem [13] (see also [15, 18]) one has $\mathcal{M} \oplus l_2(A) \cong l_2(A)$. Denote by $p_{\mathcal{M}} : l_2(A) \to \mathcal{M} \subset l_2(A)$

the corresponding orthoprojection. Let $p_j : l_2(A) \to E_j \cong A^j$ be the orthoprojection on the first $j$ standard summands of $l_2(A)$ and $q_j$ the orthoprojection on the $j$-th standard summand in such a way that $p_j = q_1 + \cdots + q_j$. Two possibilities can arise: 1) $\|(1 - p_j)p_{\mathcal{M}}\| \to 0$ as $j \to \infty$, and 2) the opposite case.

1) Let us show that in this case $\mathcal{M}$ is finitely generated projective, i.e. this case is impossible under our assumptions. One can argue as in [19]: for a sufficiently large $j$ the operator

$$J(x) = \begin{cases} p_j(x) & \text{if } x \in \mathcal{M}, \\ x & \text{if } x \in \mathcal{M}^\perp \cong l_2(A) \end{cases}$$

is close to identity, hence an isomorphism. It maps $\mathcal{M}$ isomorphically onto a direct summand of $E_j$.

2) In this case consider the matrix of the orthogonal projection $p_{\mathcal{M}}$. This is an adjointable (in fact self-adjoint) operator $p_{\mathcal{M}} : l_2(A) \to l_2(A)$. Hence $\|p_j p_{\mathcal{M}}(1 - p_k)\| \to 0$ as $k \to \infty$ for fixed $j$. Indeed, it is sufficient to verify that $\|q_m p_{\mathcal{M}}(1 - p_k)\| \to 0$ for each $m = 1, \dots, j$, i.e.

$$\sup_{\|x\|=1} |\langle e_m, p_{\mathcal{M}}(1 - p_k)x\rangle| = \sup_{\|x\|=1} |\langle (1 - p_k)p_{\mathcal{M}} e_m, x\rangle| = \|(1 - p_k)p_{\mathcal{M}} e_m\| \to 0.$$

The last is evident because $p_{\mathcal{M}} e_m$ is a fixed element of $l_2(A)$. So, we have in this case the following relations:

$$\|(1 - p_j)p_{\mathcal{M}}\| \not\to 0 \quad (j \to \infty), \qquad \|p_j p_{\mathcal{M}}(1 - p_k)\| \to 0 \quad (k \to \infty,\ j \text{ is fixed}).$$

Using these two observations one can choose decompositions $l_2(A) = M_1 \oplus M_2 \oplus \dots$ of the domain and $l_2(A) = N_1 \oplus N_2 \oplus \dots$ of the range of $p_{\mathcal{M}}$ in such a way that

- $M_i$ and $N_i$ are free modules generated by several consequent vectors
$$e_{\mu(1,i)}, \dots, e_{\mu(m_i,i)} \qquad \text{and} \qquad e_{\nu(1,i)}, \dots, e_{\nu(n_i,i)}$$
of the standard base;
- there exist elements $v_i \in M_i$, $\|v_i\| \le 1$, such that the projection of $p_{\mathcal{M}}(v_i)$ onto $N_i$ is not small, more precisely
$$\|(p_{\nu(n_i,i)} - p_{\nu(1,i)-1})p_{\mathcal{M}}(v_i)\| \ge C$$
for some fixed $C > 0$ for any $i$;
- for these elements the projection of $p_{\mathcal{M}}(v_i)$ onto $N_1 \oplus \cdots \oplus N_{i-1} \oplus N_{i+1} \oplus \dots$ is small, more precisely
$$\|(1 - p_{\nu(n_i,i)} + p_{\nu(1,i)-1})p_{\mathcal{M}}(v_i)\| < \frac{\varepsilon}{2^i}$$
for any beforehand fixed sufficiently small $\varepsilon > 0$ and any $i$.

We find these elements by induction over $i$. By the supposition there exists a number $K > 0$ such that for any $j$

$$\|(1 - p_j)p_{\mathcal{M}} w_j\| \ge K \tag{2}$$

for some $w_j$ of norm 1. Evidently, $K \le 1$. Let $\varepsilon = K/4$. Let $x$ be any vector of norm 1 from $\mathcal{M}$. Then up to $\varepsilon/4$ the vector $x$ is in $M_1 = N_1 = E_s$ for some $s = \mu(m_1, 1) = \nu(n_1, 1) = m_1 = n_1$ and the conditions hold with $v_1$ being the projection of $x$ onto $M_1$ and $C = 1 - \varepsilon \ge K - \varepsilon$. Now choose a number $t > \mu(m_1, 1)$ such that $\|p_s p_{\mathcal{M}}(1 - p_t)\| < \varepsilon/16$

and after that a number $d > \nu(n_1, 1)$ such that $\|(1 - p_d)p_{\mathcal{M}}p_t\| < \varepsilon/16$. Choose $w_d$ as in (2) and a number $r > t$ such that $\|(1 - p_r)(1 - p_t)w_d\| < \varepsilon/16$. Then

$$\|(1 - p_d)p_{\mathcal{M}}p_r(1 - p_t)w_d\| \geq \|(1 - p_d)p_{\mathcal{M}}(1 - p_t)w_d\| - \frac{\varepsilon}{16}$$
$$\geq \|(1 - p_d)p_{\mathcal{M}}w_d\| - \frac{\varepsilon}{8} \geq K - \frac{\varepsilon}{8}.$$

Now choose a number $l > d$ such that

$$\|p_l(1 - p_d)p_{\mathcal{M}}p_r(1 - p_t)w_d\| \geq K - \frac{\varepsilon}{4}$$

and

$$\|(1 - p_l)p_{\mathcal{M}}p_r(1 - p_t)w_d\| < \frac{\varepsilon}{8}. \tag{3}$$

Let $\mu(m_2, 2) := r$, $\nu(n_2, 2) := l$, $v_2 := p_r(1 - p_t)w_d$. Then $\|v_2\| \leq 1$ and $v_2 \in M_2$, because $t > \mu(m_1, 1)$. Since $l > d > \nu(n_1, 1)$, one has

$$\|(p_{\nu(n_2,2)} - p_{\nu(n_1,1)})p_{\mathcal{M}}(v_2)\| \geq \|(p_l - p_d)p_r(1 - p_t)w_d\|$$
$$= \|p_l(1 - p_d)p_r(1 - p_t)w_d\| \geq K - \frac{\varepsilon}{4}.$$

Since $\|p_s p_{\mathcal{M}}(1 - p_t)\| < \varepsilon/16$, one has

$$\varepsilon/16 > \|p_{\nu(n_1,1)}p_{\mathcal{M}}(1 - p_t)p_r w_d\| = \|p_{\nu(n_1,1)}p_{\mathcal{M}}p_r(1 - p_t)w_d\| = \|p_{\nu(n_1,1)}p_{\mathcal{M}}v_2\|.$$

Together with (3) this gives the last necessary property. Proceeding in such a way we obtain the desired decompositions and elements with

$$C = K - \varepsilon \geq \frac{3}{4} \cdot K.$$

Denote $u_i := p_{\mathcal{M}}(v_i)$. Let $z_i$ be the orthoprojection of $u_i$ onto $N_i$. Then $C \leq \|u_i\| \leq 1$, $\|z_i - u_i\| < \varepsilon/2^i$, $\|z_i\| \leq (1 + \varepsilon/2^i)$.

According to Definition 11 let us choose elements $b_k$ for $\langle z_k, z_k\rangle$ (for the sake of notational brevity we assume that we do not need to pass to a subsequence the second time). Then the formula

$$\beta(x) = \sum_k b_k^* \langle z_k, x\rangle \tag{4}$$

defines an $A$-functional on $l_2(A)$. By Proposition 5 to verify this, it is sufficient to prove that partial sums of the series

$$\sum_i \beta(e_i)\beta(e_i)^*$$

are uniformly bounded. If $e_i \in N_m$, then

$$\beta(e_i) = \sum_{k \neq m} b_k^* \langle z_k, e_i\rangle + b_m^* \langle z_m, e_i\rangle = b_m^* \langle z_m, e_i\rangle.$$

Hence,

$$\begin{aligned}\sum_i \beta(e_i)\beta(e_i)^* &= \sum_m \sum_{e_i\in N_m} \left(b_m^*\langle z_m, e_i\rangle\right)\left(b_m^*\langle z_m, e_i\rangle\right)^* \\ &= \sum_m b_m^* \left(\sum_{e_i\in N_m} \langle z_m, e_i\rangle\langle z_m, e_i\rangle^*\right) b_m \\ &= \sum_m b_m^*\langle z_m, z_m\rangle b_m \le (1+\varepsilon)^2 \sum_m b_m^* b_m.\end{aligned}$$

Let us show that there is no adjointable functional $\alpha$ on $l_2(A)$ such that $\alpha|_{\mathcal{M}} = \beta|_{\mathcal{M}}$, and hence, $\beta|_{\mathcal{M}}$ is a non-adjointable functional on $\mathcal{M}$ and $\mathcal{M}$ is not a self-dual module. Indeed, suppose, there exists an element $a = (a_1, a_2, \dots) \in \mathcal{M} \subset l_2(A)$ such that $\alpha(x) := \sum_i a_i x^i = \beta(x)$ for any $x \in \mathcal{M}$. Then

$$\alpha(z_j) \to 0, \quad [\alpha(u_j) - \alpha(z_j)] \to 0, \quad \alpha(u_j) = \beta(u_j),$$
$$[\beta(u_j) - \beta(z_j)] \to 0, \quad \beta(z_j) = b_j^*\langle z_j, z_j\rangle \not\to 0.$$

We obtain a contradiction. □

## 5. A description of spectra of $MI$-algebras

The following theorem is proved in [10] in much more generality. We present here a more elementary argument to make the present text more self-contained.

**Theorem 13.** *Let a $C^*$-algebra $A$ be an irreducible subalgebra of $B(H)$. Then the following properties are equivalent.*

- *$H$ is infinite-dimensional;*
- *there exists a sequence $\alpha_i \in A$ such that*
  *(1) $\|\alpha_i\| = 1$;*
  *(2) $0 \le \alpha_i \le 1$;*
  *(3) $\sum_{i=1}^n \alpha_i \le 1$ for any $n$.*

*Proof.* If $H$ is finite-dimensional, evidently a sequence with the mentioned properties does not exist.

Now let $H$ be infinite-dimensional. If there exists an element $\alpha \ge 0$ of $A$ with infinite spectrum, then we can construct the desired $\alpha_i$ with the help of functional calculus on the spectrum of $\alpha$, i.e. to find them inside the commutative $C^*$-algebra generated by $\alpha$.

So, suppose, that any positive element of $A$ has a finite spectrum. In this situation any projection $p \in A$ can be decomposed into a sum of 2 non-trivial projections $p = p_0 + p_1$ supposing that the image of $p$ has dimension at least 2. Indeed, let $h_0$ and $h_1$ be two unit vectors from the image of $p$, which are orthogonal to each other. By Kadison transitivity theorem there exists a self-adjoint $\alpha \in A$ such that $\alpha(h_0) = 0$ and $\alpha(h_1) = h_1$. Then $p\alpha^*\alpha p = p\alpha^2 p$ has 0 and 1 as eigenvalues with eigenvectors in $\operatorname{Im} p$. Hence, its spectral decomposition over $\operatorname{Im} p$ is non-trivial, while the projections are in $A$ since the spectrum is finite.

Now we can repeat this taking of decomposition $p = p_0 + p_1$ infinitely many times starting from some spectral projection and obtain a sequence of mutually orthogonal projections, which can be taken as $\alpha_i$. □

We need the following lemma for the proof of the next theorem.

**Lemma 14.** *Suppose, $\widehat{A}$ has an isolated point being a finite-dimensional representation. Then $A$ is not* MI.

*Proof.* Let $M \in \mathrm{Prim}(A)$ be the kernel of this isolated representation $\rho$. Then $M$ is a maximal ideal. Let

$$\chi(I) = \begin{cases} 1, & I = M, \\ 0, & I \neq M \end{cases}$$

be the characteristic function of $\{M\}$. The set $\{M\}$ is open and closed, therefore the function $\chi$ is continuous. Let us consider the finite-dimensional matrix $C^*$-algebra $M_n = \rho(A) \cong A/M$. By the Dauns-Hofmann theorem (Theorem 7) for any $a \in A$ there exists a unique element $(\chi a) \in A$ such that $\chi\widehat{a} = \widehat{\chi a}$. Here

$$\widehat{a}(I) = a + I, \quad I \in \mathrm{Prim}(A).$$

Suppose,

$$A_M = \{a \in A : \widehat{a}(I) = 0 \quad \text{for all} \quad I \neq M\}$$

and $\alpha$ is the composition

$$A \longrightarrow A/M \xrightarrow{\cong} M_n.$$

Then by the Dauns-Hofmann theorem the restriction of $\alpha$ to $A_M$ is an isomorphism. Also, $A_M$ is a direct summand in $A$, because the map $a \mapsto \chi a$ is a projection. Thus $A$ is not MI (see Proposition 6). □

*Proof of the Main Theorem.* In one direction this is just Lemma 14.

Now, suppose, $\widehat{A}$ has no isolated point being a finite-dimensional representation. We will show that $A$ is MI by demonstrating that $A$ is DINC. In accordance with Remark 4 we will work with $\mathrm{Prim}(A)$.

We take any sequence $u_i \in A$ of norm $1 \geq \|u_i\| \geq C$. The functions $f_i(I) = \|u_i + I\|$ are lower semi-continuous on $\mathrm{Prim}(A)$, therefore the sets $G_i = f_i^{-1}(2C/3, \infty)$ are open. Let us choose separated points $S_i$ from $G_i$, so $1 \geq \|u_i + S_i\| \geq 2C/3$. Passing to a subsequence if necessary, we can assume that either (**I**) $S_i = S$ for all $i$ or (**II**) all $S_i$'s are different.

In the case (**I**) we can suppose that $S$ is an isolated point being a kernel of infinite-dimensional representation $\rho : A \to B(H)$. Indeed, in the opposite case obviously there exists a sequence $T_i$ of distinct separated points such that $T_i \to S$. Passing from $S$ to $T_i$, we obtain the case **(II)**.

So, let us study the case (**I**) with this supposition on $S$ to be isolated. The set $\{S\}$ is open, therefore by Lemma 9 there exists a positive element $a \in A$ of norm 1 such that $\|a + S\| = 1$ and $\|a + I\| = 0$ for all primitive ideals $I \neq S$. By Theorem 13 there exists a sequence $a_i \in A$ such that

(1) $\|a_i + S\| = 1$;
(2) $0 \leq a_i + S \leq 1 + S$;
(3) $\sum_{i=1}^{n} a_i + S \leq 1 + S$ for any $n$.

Consider an irreducible representation $\rho$ of $A$ on $H$ with kernel $S$. Now we choose $x_i, y \in H$ of norm 1 such that

$$\|\rho(a_i)x_i\| > \|\rho(a_i)\| - \varepsilon = 1 - \varepsilon, \qquad \|\rho(a)y\| > \|\rho(a)\| - \varepsilon > 1 - \varepsilon,$$

where $(1-\varepsilon)^2 > 1/\sqrt{2}$. By the Kadison transitivity theorem there exist unitary operators $\rho(c_i) \in \rho(A)$, $\|c_i\| = 1$ (cf. [8, Theorem 2.8.3.]), such that

$$y = \rho(c_i)\left(\frac{\rho(a_i)x_i}{\|\rho(a_i)x_i\|}\right).$$

Hence,

$$\|a(c_i a_i) + S\| \geq \|\rho(a c_i a_i)x_i\| = \|\rho(a)y\|\|\rho(a_i)x_i\| > (\|\rho(a_i)\| - \varepsilon)(\|\rho(a)\| - \varepsilon) > 1/\sqrt{2}.$$

Put $v_i = a c_i a_i$ and $b_i = v_i^* v_i$. We claim that

(1) $\|b_i + S\| > 1/2$;
(2) $0 \leq b_i + S \leq 1 + S$;
(3) $\sum_{i=1}^n b_i + S \leq 1 + S$ for any $n$;
(4) $\|b_i + I\| = 0$ for all primitive ideals $I \neq S$.

The third property follows from the estimation

$$\sum_{i=1}^n b_i + S = \sum_{i=1}^n a_i^* c_i^* a^* a c_i a_i + S \leq \sum_{i=1}^n a_i^* a_i + S \leq 1 + S.$$

The others properties above are clear. From $\|b_i\| = \sup\{\|b_i + I\| : I \in \mathrm{Prim}(A)\}$ it follows that $\|b_i\| = \|b_i + S\| > 1/2$ for all $i$ and $\|\sum_{i=1}^n b_i\| \leq 1$ for all $n$. Therefore

$$u_i b_i + I = \begin{cases} u_i b_i + S, & I = S, \\ 0, & I \neq S, \end{cases}$$

and $\|u_i b_i\| = \|u_i b_i + S\|$. Now we choose $\xi_i, \eta_i \in H$ of norm 1 such that

$$\|\rho(b_i)\xi_i\| > \|\rho(b_i)\| - \varepsilon > 1/2 - \varepsilon, \qquad \|\rho(u_i)\eta_i\| > \|\rho(u_i)\| - \varepsilon \geq C/2 - \varepsilon,$$

where $(1/2 - \varepsilon)(C/2 - \varepsilon) > C/4$. By the Kadison transitivity theorem there exist unitary operators $\rho(c_i) \in \rho(A)$, $\|c_i\| = 1$, such that

$$\eta_i = \rho(c_i)\left(\frac{\rho(b_i)\xi_i}{\|\rho(b_i)\xi_i\|}\right).$$

Hence,

$$\|u_i(c_i b_i)\| \geq \|\rho(u_i c_i b_i)\xi_i\| = \|\rho(u_i)\eta_i\|\|\rho(b_i)\xi_i\| > (\|\rho(b_i)\| - \varepsilon)(\|\rho(u_i)\| - \varepsilon) > C/4.$$

Thus $c_i b_i$ can serve as a sequence for $u_i$ denoted by $b_i$ in Definition 11. Therefore $A$ is DINC and by Theorem 12 it is MI.

Lemma 10 there are neighborhoods $V_i$ of $S_i$ such that $S_j \notin V_i$ if $j \neq i$ (for the sake of notational brevity we do not pass to a subsequence). Now by Lemma 9 we can find a sequence $y_n$ of positive elements of $A$ such that

$$\|y_n\| = 1, \qquad \|y_n + S_n\| = 1, \quad \|y_n + S_m\| = 0 \qquad \text{for all} \qquad m \neq n. \tag{5}$$

Now we will pass from the sequence $y_n$ to another sequence $z_n$ with the following properties:

**(a):** the partial sums of $\sum_k z_k^* z_k$ are uniformly bounded by 1,
**(b):** $\|z_i + S_j\| = 0$ for all $j \neq i$,
**(c):** $\|z_i + S_i\| = 1$ for any $i$.

For this purpose let us define $z_1 := y_1$ and by induction

$$z_{i+1} := y_{i+1} \left( 1 - \sum_{k=1}^{i} z_k^* z_k \right)^{1/2}. \tag{6}$$

From point (a) for $z_i$ it follows that $z_{i+1}$ is well-defined. The point (b) is clear. The point (a) follows from the estimation:

$$\sum_{k=1}^{i+1} z_k^* z_k = \sum_{k=1}^{i} z_k^* z_k + \left( 1 - \sum_{k=1}^{i} z_k^* z_k \right)^{1/2} (y_{i+1})^2 \left( 1 - \sum_{k=1}^{i} z_k^* z_k \right)^{1/2} \leq 1.$$

Besides, by (b)

$$\left( 1 - \sum_{k=1}^{i-1} z_k^* z_k \right)^{1/2} + S_i = 1.$$

Hence,

$$\|z_i + S_i\| = \|y_i + S_i\| = 1$$

and (c) holds as well.

Now let $S_i = \ker(\rho_i)$ and $H_i$ be the space of the irreducible representation $\rho_i$. We can choose $\xi_i, \eta_i \in H_i$ of norm 1 such that into $H_i$:

$$\|\rho_i(z_i)\xi_i\| > \|\rho_i(z_i)\| - \varepsilon = 1 - \varepsilon, \qquad \|\rho_i(u_i)\eta_i\| > \|\rho_i(u_i)\| - \varepsilon \geq 2C/3 - \varepsilon,$$

where $(1-\varepsilon)(2C/3 - \varepsilon) \geq C/2$. By the Kadison transitivity theorem there exist unitary operators $\rho_i(c_i) \in \rho_i(A)$, $\|c_i\| = 1$, such that

$$\eta_i = \rho_i(c_i) \left( \frac{\rho_i(z_i)\xi_i}{\|\rho_i(z_i)\xi_i\|} \right).$$

Consequently,

$$\begin{aligned} \|u_i(c_i z_i)\| \geq \|\rho_i(u_i(c_i z_i))\| \geq \|\rho_i(u_i c_i z_i)\xi_i\| \\ = \|\rho_i(u_i)\eta_i\| \|\rho_i(z_i)\xi_i\| > (1-\varepsilon)(\|\rho_i(u_i)\| - \varepsilon) \geq C/2. \end{aligned}$$

Thus $b_i := c_i z_i$ is a sequence for $u_i$ as in Definition 11. Therefore $A$ is DINC and by Theorem 12 it is MI. $\square$

Let us consider two examples.

**Example 15.** Let $K(H)$ be the $C^*$-algebra of compact operators in an infinite-dimensional separable Hilbert space $H$. For $x, y, z \in H$ suppose $\theta_{x,y}(z) = x\langle y, z\rangle$, so $\theta_{x,y} \in K(H)$. Let $A = \widetilde{K(H)}$ be the algebra $K(H)$ with an adjoint unit. Then $\mathrm{Prim}(A) = \{0, K(H)\}$ and all open sets in $\mathrm{Prim}(A)$ are $\emptyset, \{0\}, \{0, K(H)\}$. Thus $\{0\}$ is an isolated point corresponding to the identity (infinite-dimensional irreducible) representation in $H$. Suppose $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $H$ and $p_i$ are projections from $H$ onto $\mathrm{span}\{e_i\}$. Then

(1) $\|p_i\| = 1$;
(2) $0 \leq p_i \leq 1$;
(3) $\sum_{i=1}^{n} p_i \leq 1$ for any $n$;
(4) $\|p_i + K(H)\| = 0$ for any $i$.

Now we take any sequence $u_i \in A$ of norm $1 \geq \|u_i\| \geq C$ and choose $x_i \in H$ of norm 1 such that

$$\|u_i x_i\| > \|u_i\| - \varepsilon \geq C - \varepsilon > C/2.$$

Put $c_i = \theta_{x_i, e_i}$. Then $c_i(e_i) = x_i$ and

$$\|u_i(c_i p_i)\| \geq \|u_i c_i p_i e_i\| = \|u_i x_i\| > C/2.$$

Thus $c_i p_i$ can serve as a sequence for $u_i$ denoted by $b_i$ in Definition 11. Therefore $A$ is DINC and by Theorem 12 it is MI.

**Example 16.** Let $\mathbb{A}$ be the Toeplitz algebra and $H^2$ be the Hardy space. The identity representation of $\mathbb{A}$ in $H^2$ is irreducible (cf. [20, Theorem 3.5.5]). We claim that any non zero ideal $I \subset \mathbb{A}$ contains $K(H^2)$. Indeed, let $u \in I$, $u \neq 0$. Then there exists $x \in H^2$ of norm 1 such that $u(x) \neq 0$. For any $y \in H^2$ put

$$p = \theta_{y,y}, \qquad v = \frac{\theta_{y,\,u(x)}}{\|u(x)\|^2}.$$

Then $vu(x) = y$ and $p = vu\theta_{x,x}u^*v^* \in I$. Therefore $K(H^2) \subset I$. Thus the zero ideal is an unique isolated point in $\mathrm{Prim}(\mathbb{A})$. Similar to the previous example one can demonstrate that the Toeplitz algebra $\mathbb{A}$ is DINC and by Theorem 12 it is MI.

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