Every proper smooth action of a Lie group is equivalent to a real analytic action: A contribution to Hilbert's fifth problem

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MPI / 93-3

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In this paper we prove that if a Lie group G acts on a smooth manifold M by a proper and smooth action then there exists a real analytic structure β on M, compatible with the given smooth structure on M, such that the action of G on M_β is real analytic. By smooth we here mean C^∞, but we could as well considers the C^r case, $r \geq 1$, and the result still holds by essentially the same proof.

We shall now discuss Hilberts fifth problem [6], following the discussion in Montgomery-Zippin [14, Section 2.15]. We quote from [14, p. 70]: "Let us now considers the following questions, the second and third of which are asked by Hilbert:

If a locally compact group G acts effectively on a manifold M (locally-euclidean space) then

1. is G necessarily locally euclidean,
2. if the group G is locally euclidean, is it a Lie group in some appropriate coordinates.
3. If G is a Lie group, can coordinates be chosen in G and M so that the transforming functions are analytic?

The answer to 2. which can be asked, of course, without mentioning transformation groups, is yes and was first solved by the contents of two papers one by Gleason [4], the other by Montgomery and Zippin [13]."

At the time of the appearance of the book [14] by Montgomery and Zippin the answer to 1. was unknown, except in some special cases, and as far as I know the answer to 1. is still an open question, in any case there has not appeared in print any paper containing a complete solution of 1..

Concerning 3. Montgomery and Zippin give the following discussion. "The answer to 3. is no. For example a group of reals can act on $\mathbf{R}^2$ by having fixed $x^2 + y^2 \leq 1$ and slowly rotating the rest of $\mathbf{R}^2$. This can not be analytic since if it where the existence of an open set of fixed points would imply that all points were fixed. The answer to 3. is no even if G is compact as was first shown by Bing [2] by an example of the cyclic group of order two acting on $\mathbf{R}^3$ in a way that could not be differentiable."

More recently it has been shown that the answer to 3. is no also in the case of actions of connected compact Lie groups. For example there exists a compact smoothable 12-dimensional manifold M^{12} which admits an effective, continuous S^1-action, but which admits no nontrivial smooth S^1-action in any differentiable structure, see Bredon [3, Corollary 6.9.6]. Here one can use the powerful theorem of Atiyah and Hirzebruch [1], which says that if a compact connected oriented smooth manifold M, of dimension $4m$ and with $w_2(M) = 0$, admits a non-trivial smooth action of S^1, then $\hat{A}(M) = 0$.

Our theorem shows that the answer to 3. is yes if the action of G on M is proper and smooth. The example given by Montgomery and Zippin with the group of reals acting in $\mathbf{R}^2$ can be chosen to be smooth, but it is not a proper action.

The fact that every smooth manifold has a compatible real analytic structure was first proved by Whitney in [17]. It is also known that if two real analytic manifolds are smoothly diffeomorphic then they are also real analytically diffeomorphic. The proof of this fact requires the use of Grauert's imbedding theorem [5], which says that every real analytic manifold has a real analytic imbedding in some euclidean space. In fact it is known that if M and N are real analytic manifolds then $C^\omega(M,N)$ is dense in $C^r(M,N)$, $0 \leq r \leq \infty$, see e.g. the discussion in [7, Section 2.5]. (Here the function spaces $C^r(M,N)$ and $C^\omega(M,N)$ have the Whitney topology, i.e., the strong topology.)

In the case of smooth manifolds with symmetries there are earlier known results about raising the differentiability in the case when the group of symmetries is a compact Lie group, see Palais [16] and Matumoto-Shiota [12]. Palais proves in [16] that every C^r, $1 \leq r < \infty$, action of a compact Lie group on a compact manifold is C^r-equivalent to a C^∞ action. Matumoto and Shiota use essentially the same method as Palais and their Theorem 1.3 gives the result that every C^r, $1 \leq r \leq \infty$, action on a C^r manifold is C^r-equivalent to a C^ω action, i.e., a real analytic action, on a C^ω manifold. Matumoto and Shiota also show that if M and N are two real analytic H-manifolds, where H is a compact Lie group, such that the number of H-isotropy types occuring in M and N is finite, then they are real analytically H-diffeomorphic if they are C^1 H-diffeomorphic, see Theorem 1.2 is [12].

In the case of proper actions of a non-compact Lie group G, the question whether the obtained proper real analytic G-manifold M_β is unique up to a real analytic G-equivariant diffeomorphism is a very intriguing question. We are only able to prove that the G-manifold M_β is unique up to a semi-analytic G-equivariant smooth diffeomorphism. (This result is however good enough for some applications that we have in mind.)

The main result of this paper implies that we can use the subanalytic equivariant triangulation theorem for proper real analytic G-manifolds, proved in [8], also in the case of proper smooth actions. In this context our uniqueness result, Theorem 2.1, is also sufficiently strong.

In the case of properly discontinuous actions of a discrete group (i.e., the case of a 0-dimensional Lie group G) the results of this paper where already given in [9], where we also announced Theorem 1.1 and 1.2 of this paper.

This paper was written during my visit to the Max-Planck-Institut für Mathematik, 26.11.-18.12.1992. I wish to thank the Max-Planck-Institut für Mathematik, and its director Professor F. Hirzebruch for the invitation and for providing both inspiring and excellent working conditions.

0. Preliminaries

Let G be a locally compact group acting on a locally compact space X. Recall that the action is said to be proper if $\{g \in G | gA \cap A \neq \phi\}$ is a compact subset of G for every compact subset A of X. This is also equivalent to the fact that $G \times X \to X \times X$, $(g,X) \mapsto$

(gx, x), is a proper map. In particular each isotropy subgroup G_x, $x \in X$, is a compact subgroup of G. When G is a discrete group a proper action is the same thing as the classical notion of a properly discontinuous action. From now on G will denote an arbitrary Lie group.

If M and N are smooth manifolds we denote by $C^\infty(M, N)$ the set of all smooth maps from M to N, and we give $C^\infty(M, N)$ the Whitney topology, i.e., the strong topology. If M and N are real analytic manifolds, then $C^\omega(M, N)$ denotes the set of all real analytic maps from M to N, and $C^\omega(M, N)$ has the induced topology from $C^\infty(M, N)$.

Now let H be a compact Lie group and let M and N be smooth [real analytic] H-manifolds. Then we denote by $C^\infty_H(M, N)$ $[C^\omega_H(M, N)]$ the set of all H-equivariant smooth [real analytic] maps from M to N.

As we already mentioned in the introduction, it is a well-known and deep result that if M and N are real analytic manifolds then $C^\omega(M, N)$ is dense in $C^\infty(M, N)$. We shall use the following equivariant version of this result, due to Matumoto and Shiota [12, Theorem 1.2].

Theorem 01. *Let H be a compact Lie group, and let M and N be real analytic H-manifolds and assume that the number of the orbit types occurring in N is finite. Then $C^\omega_H(M, N)$ is dense in $C^\infty_H(M, N)$.*

□

We shall also use the lemma given below. See [7, Lemma 2.2.8] for a somewhat less general formulation of the same result.

Lemma 0.2. *Suppose that M and N are smooth manifolds and that $\theta \in C^\infty(M, N)$. Let U and V be open subsets of M and N, respectively, such that $\theta(U) \subset V$. Then there exists an open neighborhood $\mathcal{N}$ of $\theta|U$ in $C^\infty(U, V)$ with the following property: If we for each $\eta \in \mathcal{N} \subset C^\infty(U, V)$ define*

$$E(\eta) : M \to N$$

to be the extension of $\eta : U \to V$ given by

$$E(\eta) = \begin{cases} \eta(x), & x \in U \\ \theta(x), & x \in M - U, \end{cases}$$

then we have that $E(\eta) \in C^\infty(M, N)$, and the map

$$E : \mathcal{N} \to C^\infty(M, N)$$

is continuous. (Observe that $E(\theta|U) = \theta$.)

□

In the following G denotes an arbitrary Lie group. Let M be a proper smooth G-manifold, and let $x \in M$ and denote $H = G_x$. A smooth H-invariant submanifold S of M is a said to be a smooth slice at x in M if GS is open in M and the map

$$\hat{\imath} : G \times_H S \to GS,\ [g, x] \mapsto gx \tag{1}$$

is a smooth G-equivariant diffeomorphism onto GS. The existence of a smooth slice at each point of a proper smooth G-manifold was proved by Koszul [11], see also Palais [15, Proposition 2.2.2].

In the case when M is a proper real analytic G-manifold and, in addition to the above, S is a real analytic submanifold and $\hat{i}$ in (1) is a real analytic G-equivariant diffeomorphism onto GS, then we say that S is a real analytic slice at x in M. The fact that there exists a real analytic slice at each point of a proper real analytic manifold was proved by Kankaanrinta [10, Theorem 2.5].

Suppose that S is a smooth [real analytic] slice at x in the proper smooth [real analytic] G-manifold M. Let U be an H-invariant open neighborhood of eH in G/H such that there is a real analytic cross section

$$\sigma : U \to G, \tag{2}$$

of $\pi : G \to G/H$, with the property that σ is an H-equivariant map in the sense that

$$\sigma(hu) = h\sigma(u)h^{-1}, \quad \text{for all } h \in H \text{ and } u \in U \tag{3}$$

(That is, σ is H-equivariant when the action of H on G is given by conjugation; $H \times G \to G$, $(h,g) \mapsto hgh^{-1}$.) It is a well-known standard fact that such U and σ exist.

It now follows that

$$\gamma : U \times S \to \sigma(U)S = W, \ (u,x) \mapsto \sigma(u)x \tag{4}$$

is a smooth [real analytic] H-equivariant diffeomorphism onto W, and W is an H-invariant open neighborhood of S in M. We call W a product neighborhood of S in M, and we say that γ in (4) is a presentation of W.

It is always possible, both in the smooth and real analytic case, to find euclidean slices. We say that a smooth [real analytic] slice S at x in M is euclidean if there exists a smooth [real analytic] H-equivariant diffeomorphism

$$\alpha : \mathbf{R}^n(\rho) \xrightarrow{\tilde{=}} S \subset M \tag{5}$$

with $\alpha(0) = x$. Here $\mathbf{R}^n(\rho)$ denotes an orthogonal representation space for H. In this case

$$\hat{\alpha} : G \times_H \mathbf{R}^n(\rho) \to GS, \ [g,a] \mapsto g\alpha(a) \tag{6}$$

is a smooth [real analytic] G-equivariant diffeomorphism onto the tube GS. Furthermore

$$\eta : U \times \mathbf{R}^n(\rho) \to \sigma(U)S = W, \ (u,a) \mapsto \sigma(u)\alpha(a) \tag{7}$$

is a smooth [real analytic] H-equivaraint diffeomorphism onto W. In this case, when S is an euclidean slice, we can moreover find a product neighborhood W of S in M such that the number of H-isotropy types occurring in W is finite. This follows from the fact that we can choose the H-invariant open neighborhood U of eH in G/H to be real analytically and H-equivaraintly diffeomorphic to an orthogonal representation space $\mathbf{R}^k(w)$ for H. Then there exists a smooth [real analytic] H-equivariant diffeomorphism from $\mathbf{R}^k(w) \times \mathbf{R}^n(\rho) = \mathbf{R}^{k+n}(w \oplus \rho)$ onto W, and hence the number of H-isotropy types occurring in W is finite.

The following lemma will be used in the proof of Theorem 1.1, and this lemma is also crucial for the proof of Theorem 2.1.

Lemma 0.3. *Let M be a proper smooth G-manifold, and let $x \in M$ and denote $G_x = H$. If S is a smooth slice at x in M there exists a product neighborhood W of S in M such that the following holds. There is an open neighborhood $\mathcal{M}$ of $i : S \hookrightarrow W$ in $C^\infty(S,W)$ such that if $j \in \mathcal{M} \cap C^\infty_H(S,W)$ then $j : S \to W$ is a smooth H-equivariant closed imbedding of S into W, and $S' = j(S)$ is a smooth slice in M and $GS' = GS$.*

In the case when S is an euclidean slice we can moreover suppose that the number of H-isotropy types occurring in W is finite, and we can also formulate the lemma in the following form.

Suppose that $\alpha : \mathbf{R}^{(n)}(\rho) \to S \subset M$ is a presentation of a smooth euclidean slice S in M. Then there exists a product neighborhood W of S in M such that the number of H isotropy types occurring in W is finite and such that the following holds. There is an open neighborhood $\mathcal{M}$ of α in $C^\infty_H(\mathbf{R}^n(\rho), W)$ such that if $\alpha' \in \mathcal{M}$ then $\alpha' : \mathbf{R}^n(\rho) \to W$ is a smooth H-equivariant closed imbedding of $\mathbf{R}^n(\rho)$ into W, and $S' = \alpha'(\mathbf{R}^n(\rho))$ is a smooth slice in M and $GS' = GS$.

□

Section 1.

Theorem 1.1. *Let M be a smooth manifold on which a Lie group G acts by a proper and smooth action. Then there exists a real analytic structure β on M, compatible with the given smooth structure, such that the action of G on M_β is real analytic.*

Proof: We define $\mathcal{B}$ to be the family consisting of all pairs (B, β), where B is a non-empty open G-invariant subset of M and β is a real analytic structure on B, compatible with the smooth structure on B, such that the action of G on B_β is real analytic.

Let us first note that $\mathcal{B}$ is a non-empty family. This is seen as follows. Let $x_0 \in M$ and denote $G_{x_0} = K$. By the smooth slice theorem (see Koszul [11], or Palais [15, Proposition 2.2.2]) there exists a smooth slice S_0 at x_0 in M. Furthermore we may suppose that S_0 is such that there exists a smooth K-equivariant diffeomorphism

$$\gamma_0 : \mathbf{R}^k(\sigma) \overset{\approx}{\to} S_0 \subset M, \tag{1}$$

where $\mathbf{R}^k(\sigma)$ denotes an orthogonal representation space for K, and $\gamma_0(0) = x_0$. (We call γ_0 a presentation of S_0.) The map

$$\hat{\gamma}_0 : G \times_K \mathbf{R}^k(\sigma) \to GS_0, \; [g,a] \mapsto g\gamma_0(a) \tag{2}$$

is a smooth G-equivariant diffeomorphism onto the open G-invariant subset $B_0 = GS_0$ of M. Now $G \times_H \mathbf{R}^k(\sigma)$ is a real analytic G-manifold, and we give B_0 the real analytic structure β_0 induced from $G \times_H \mathbf{R}^k(\sigma)$ through $\hat{\gamma}_0$. Since the action of G on $G \times_K \mathbf{R}^k(\sigma)$ is real analytic it follows that the action of G on $(B_0)_{\beta_0}$ is real analytic. Since $\hat{\gamma}_0$ in (2) is a smooth diffeomorphism it follows that the real analytic structure β_0 on B_0 is compatible with the smooth structure on B_0. Thus $(B_0, \beta_0) \in \mathcal{B}$, and we have shown that $\mathcal{B}$ is non-empty.

We define an order in $\mathcal{B}$ by setting

$$(B_1, \beta_1) \leq (B_2, \beta_2)$$

if and only if:

i) $B_1 \subset B_2$

ii) The real analytic structure β_1 on B_1 is the one induced from the real analytic structure β_2 on B_2, i.e., $\beta_1 = \beta_2|B_1$.

Now suppose that $\mathcal{T}$ is a tower in $\mathcal{B}$, i.e., if (B_1, β_1), $(B_2, \beta_2) \in \mathcal{T}$ then either $(B_1, \beta_1) \leq (B_2, \beta_2)$ or $(B_2, \beta_2) \leq (B_1, \beta_1)$. Let $\mathcal{T}_1$ denote the family of all B occurring as the first coordinate of a pair in $\mathcal{T}$, and let $\mathcal{T}_2$ be the family of all β occurring as the second coordinate of a pair in $\mathcal{T}$. With this notation we have that

$$B^* = \bigcup_{B \in \mathcal{T}_1} B$$

is a non-empty open G-invariant subset of M, and it is also immediate that

$$\beta^* = \bigcup_{\beta \in \mathcal{T}_2} \beta$$

is a real analytic structure on B^* such that the action of G on $B^*_{\beta^*}$ is real analytic and β^* is compatible with the smooth structure on B^*. Thus $(B^*, \beta^*) \in \mathcal{B}$ and

$$(B, \beta) \leq (B^*, \beta^*), \text{ for all } (B, \beta) \in \mathcal{T},$$

i.e., (B^*, β^*) is an upper bound for $\mathcal{T}$. Thus we have by Zorn's maximality principle that there exists a maximal element (B, β) in $\mathcal{B}$. We claim that $B = M$.

Suppose the contrary and assume that $B \subsetneq M$. If B is closed in M then we could find an open G-invariant tube $B_0 = GS_0$, as in the beginning of the proof, such that $B_0 \cap B = \phi$ and B_0 has a real analytic structure β_0, which is compatible with smooth structure on B_0, and the action of G on $(B_0)_{\beta_0}$ is real analytic. Then $(B \cup B_0, \beta \cup \beta_0) \in \mathcal{B}$ and $(B, \beta) < (B \cup B_0, \beta \cup \beta_0)$, which contradicts the fact that (B, β) is a maximal element in $\mathcal{B}$. Thus B is not closed in M, and we have that $\overline{B} - B \neq \phi$.

Choose $x \in \overline{B} - B$, and denote $G_x = H$. Let S be a smooth euclidean slice at x in M, and let

$$\alpha : \mathbf{R}^n(\rho) \to S$$

be a smooth H-equivariant diffeomorphism with $\alpha(0) = x$. Here $\mathbf{R}^n(\rho)$ denotes an orthogonal representation space for H. Let W be an H-invariant product neighborhood of S in W, such that the number of H-isotropy types occuring in W is finite and such that Lemma 0.3 holds for W. We let

$$\eta : U \times \mathbf{R}^n(\rho) \to \sigma(U)S = W$$

be a smooth H-equivariant diffeomorphism, where $\sigma : U \to G$ is a real analytic H-equivariant cross section as in (2) and (3) in Section 0.

Since $x \in \overline{B} - B$ and GS is an open neighborhood of x in M it follows that $B \cap GS \neq \phi$. Since $G(B \cap S) = B \cap GS$ it follows that $B \cap S \neq \phi$. Thus $B \cap S$ is a non-empty open H-invariant subset of S, and hence

$$U = \gamma^{-1}(B \cap S)$$

is a non-empty open H-invariant subset of $\mathbf{R}^n(\rho)$. (We may note that $0 \in \overline{U} - U$.) We also have that

$$U = \gamma^{-1}(B \cap W).$$

Since U is an open subset of $\mathbf{R}^n(\rho)$ it has a real analytic structure induced from $\mathbf{R}^n$, and since the action of H on $\mathbf{R}^n(\rho)$ is linear it follows that the action of H on U is real analytic.

We shall construct a smooth slice S' at x in M, with $S' \subset W$, and a smooth H-equivariant diffeomorphism

$$\beta : \mathbf{R}^n(\rho) \to S'$$

with $\beta(0) = x$, such that $\beta^{-1}(B \cap S') = U$ and

$$\beta| : U \to B_\beta \cap S' \hookrightarrow B_\beta \cap W$$

is a real analytic map into $B_\beta \cap W$. Here $B_\beta \cap W$ is an open subset of the real analytic manifold B_β and hence $B_\beta \cap W$ has the induced real analytic structure from B_β. Since the action of G on B_β is real analytic, and W is H-invariant, it follows that the action of H on $B_\beta \cap W$ is real analytic.

By Lemma 0.2 there exists an open neighborhood $\mathcal{N}$ of $\alpha| : U \to B \cap W$ in $C^\infty(U, B \cap W)$ such that we obtain a continuous map

(3) $$E : \mathcal{N} \to C^\infty(\mathbf{R}^n(\rho), M)$$

by defining $E(\mu) : \mathbf{R}^n(\rho) \to M$, for each $\mu \in \mathcal{N}$, by

$$E(\mu)(x) = \begin{cases} \mu(x), & \text{for every } x \in U \\ \alpha(x), & \text{for every } x \in \mathbf{R}^n(\rho) - U \end{cases}$$

Since $E(\alpha|U) = \alpha$ and E is continuous there exists an open neighborhood $\mathcal{N}_1$ of $\alpha|U$ in $\mathcal{N}$ such that $E(\mathcal{N}_1) \subset \mathcal{M}$. Since $\gamma|U \in C^\infty_H(U, B_\beta \cap W)$ and the number of H-isotropy types occurring in $B_\beta \cap W$ is finite, there exists by Theorem 0.1, an $\mu \in \mathcal{N}_0 \cap C^w_H(U, B_\beta \cap W)$. It follows directly from the above definition of $E(\mu)$ that

$$\beta = E(\mu) : \mathbf{R}^n(\rho) \to M$$

is H-equivariant. Hence

$$\beta \in \mathcal{M} \cap C^\infty_H(\mathbf{R}^n(\rho), M),$$

and by applying Lemma 0.3 we have that

$$\beta : \mathbf{R}^n(\rho) \to M$$

is a smooth H-equivariant closed imbedding such that $\beta(\mathbf{R}^n(\rho)) = S'$ a slice at x in M, and $GS' = GS$. Furthermore

$$\beta|U = \mu : U \to B_\beta \cap W \subset B_\beta$$

is real analytic.

No

$$(4) \qquad \hat{\beta} : G \times_H \mathbf{R}^n(\rho) \to GS' = GS$$

is a smooth G-equivariant diffeomorphism onto GS, and

$$(5) \qquad \hat{\beta}| : G \times_H U \to B_\beta$$

is a real analytic G-equivariant diffeomorphism onto $B_\beta \cap GS$.

Let us denote $B_1 = GS$ and give B_1 the real analytic structure β_1 it obtains from $G \times_H \mathbf{R}^n(\rho)$ through $\hat{\beta}$. Then the action of G on B_1 is real analytic, and since $\hat{\beta}$ in (4) is a diffeomorphism it follows that β_1 is compatible with the smooth structure on B_1. Since the map $\hat{\beta}|$ in (5) is a real analytic G-equivariant diffeomorphism onto the open subset $B \cap B_1$ of B_β it follows that the real analytic structure on $B \cap B_1$ induced from $(B_1)_{\beta_1}$ equals the real analytic structure on $B \cap B_1$ induced from B_β. Hence it follows that $(B \cup B_1, \beta \cup \beta_1) \in \mathcal{B}$, but this contradicts the maximality of (B, β). Thus $B = M$.

□

Section 2.

Using methods that are essentially the same as the ones we used in the proof of Theorem 1.1 we are able to prove Theorem 1.2 below. In this case we need to use slice theorem in the real analytic case, see Kankaanrinta [10, Theorem 2.5].

Theorem 2.1. *Let M and N be proper real analytic G-manifolds, and suppose that there exists a smooth G-diffeomorphism $f : M \to N$. Then there exists a semi-analytic and smooth G-diffeomorphism $f^* : M \to N$.*

□

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