New formulae for solutions of quantum Knizhnik-Zamolodchikov equations on level -4

Hermann Boos [1]
Max-Planck Institut für Mathematik
Vivatsgasse 7, 53111 Bonn, Germany

Vladimir Korepin
C.N. Yang Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794–3840, USA

Feodor Smirnov [2]
LPTHE, Tour 16, 1-er étage, 4, pl. Jussieu
75252, Paris Cedex 05, France

Abstract

We present a new form of solution to the quantum Knizhnik-Zamolodchikov equation [qKZ] on level -4 in a special case corresponding to the Heisenberg XXX spin chain. Our form is equivalent to the integral representation obtained by Jimbo and Miwa in 1996 [7]. An advantage of our form is that it is reduced to the product of single integrals. This fact is deeply connected with a cohomological nature of our formulae. Our approach is also based on the deformation of hyper-elliptic integrals and their main property – deformed Riemann bilinear relation. Jimbo and Miwa also suggested a nice conjecture which relates solution of the qKZ on level -4 to any correlation function of the XXX model. This conjecture together with our form of solution to the qKZ make it possible to prove a conjecture that any correlation function of the XXX model can be expressed in terms of the Riemann ζ-function at odd arguments and rational coefficients suggested in [8, 9]. This issue will be discussed in our next publication.

[1] on leave of absence from the Institute for High Energy Physics, Protvino, 142284, Russia
[2] Membre du CNRS

1 Introduction.

This paper originates from the problem of calculation of correlators in XXX model. Let us remind some history. First non-trivial correlator on three cites was calculated by Takahashi [1]. It happens to be given by $\zeta(3)$ (ζ is Riemann ζ-function). Then tremendous progress was done by Kyoto group (Jimbo, Miwa, Miki, Nakayashiki) who provided general formula for correlators in terms of multiple integrals [2, 7]. Their final results later were confirmed by Bethe anzatz calculations [3, 4]. However, the most interesting in our opinion result consists in relation of certain generalized correlator introduced by Kyoto group with quantum Knizhnik-Zamolodchikov equations (qKZ) [5, 6].

In the special case of level 0 qKZ equations appeared in the paper [12] in study of form factors for integrable models of quantum field theory. Kyoto group found that the correlators are related to the dual case of level -4.

Another source of our inspiration is the papers [8, 9] in which the conjecture (confirmed by explicit calculations in many particular cases) was put forward that the correlators in XXX model can be expressed in terms of values of Riemann ζ-function with odd positive integer arguments. This conjecture is rather courageous because the formula for correlators following from Kyoto group results are given by multiple integrals which at the first glance define some combination of multiple ζ-function. The claim that these multiple ζ-values are expressible in terms of single ones is highly non-trivial. The goal of the present paper is to explain, at least partly, this miracle.

Let us mention also the paper [10] in which a generalization of XXX correlators to inhomogeneous case was considered. It so happens that in inhomogeneous case the statement concerning reducibility of correlators becomes much more transparent. Namely, they are expressed in terms of ψ-functions depending on the inhomogeneity parameters, ζ-values occur in the homogeneous limit. The logical continuation of the ideas of this paper requires the consideration of further generalization of the correlators given by Kyoto group which is related to level -4 qKZ equations.

In the present paper we show the real origin of reducibility of correlators. Namely, we explain that in the most far reaching generalization of XXX correlators i.e. in the Kyoto group generalization the reducibility takes place. To do that we use the relation of level -4 and level 0 qKZ equation. In the latter case the formulae for the solutions is much nicer [12, 11]. The multiple integrals in these solutions are reduced to single ones from the very beginning. Using the deformed Riemann bilinear relation [13] we show that similar fact is valid for level -4 case.

The fact of reducibility of multiple integrals in solution of qKZ on level -4 is the Theorem formulated and proved in the Section 5. Some polynomial coefficients remain undetermined. The technically complicated part of the problem is finding these polynomials. We were able to solve this problem only partly.

The mathematical meaning of reducibility in question is illustrated in the Section 6. The integrals for solutions of qKZ on level -4 can be thought about as some deformations of integrals of differential forms on affine Jacobi variety of hyper-elliptic curve. In the classical case the possibility of reducing the multiple integrals to single ones is explained by the fact that cohomologies of this variety are especially simple [15, 16]. From this point of view one understands why consecutive generalizations are so useful. The cases of homogeneous XXX, inhomogeneous XXX and Kyoto generalizations correspond to q-deformation of different Riemann surfaces. Kyoto generalization corresponds to the case of hyper-elliptic curve in generic position. Inhomogeneous XXX corresponds to the rational curve obtained when the branch points of hyper-elliptic curve coincide pairwise. Finally, the homogeneous case corresponds to the situation when all the branch points come to one point. Obviously, from the point of view of mathematics one has to consider the less degenerate case.

2 Jimbo-Miwa solution to qKZ on level -4.

Consider the R-matrix acting in $\mathbb{C}^2\otimes\mathbb{C}^2$:

$$R(\beta)=R_0(\beta)\overline{R}(\beta) \tag{1}$$

where

$$\overline{R}(\beta)=\frac{\beta+\pi iP}{\beta+\pi i} \tag{2}$$

P is permutation and

$$R_0(\beta)=-\frac{\Gamma\left(\frac{\beta}{2\pi i}\right)\Gamma\left(\frac{1}{2}-\frac{\beta}{2\pi i}\right)}{\Gamma\left(-\frac{\beta}{2\pi i}\right)\Gamma\left(\frac{1}{2}+\frac{\beta}{2\pi i}\right)}$$

The qKZ on level -4 are written for a function $g(\beta_1,\cdots,\beta_{2n})$ which is meromorphic function of β_j and takes values in the tensor product $(\mathbb{C}^2)^{\otimes 2n}$. We write the qKZ equations [5, 6] close to their original form which appeared in study of form factors [11]. Namely, we do not write down indices counting spaces $\mathbb{C}^2$, for example, we imply that $R(\beta_i-\beta_j)$ acts in the tensor product of i-th and j-th spaces. Also we imply that when the "rapidities" β_i, β_j are permuted, corresponding spaces $\mathbb{C}^2$ are permuted as well. With these conventions we can write down the qKZ equations on level -4 as follows:

$$g(\beta_1,\cdots,\beta_{j+1},\beta_j,\cdots,\beta_{2n})=R(\beta_j-\beta_{j+1})\,g(\beta_1,\cdots,\beta_j,\beta_{j+1},\cdots,\beta_{2n}) \tag{3}$$

$$g(\beta_1,\cdots,\beta_{2n-1},\beta_{2n}+2\pi i)=g(\beta_{2n},\beta_1,\cdots,\beta_{2n-1}) \tag{4}$$

For application to correlators we need some particular solution which, according to Jimbo, Miwa [7] can be written in the form:

$$\begin{aligned}&g(\beta_1,\cdots,\beta_{2n})=\\&\quad=\frac{1}{\sum e^{-\beta_j}}\prod_{i<j}\frac{1}{\zeta(\beta_i-\beta_j)}\int_{-\infty}^{\infty}d\alpha_1\cdots\int_{-\infty}^{\infty}d\alpha_{n-1}\prod_{i,j}\varphi(\alpha_i-\beta_j)\\&\quad\times\prod_{i<j}\frac{\sinh(\alpha_i-\alpha_j)}{\alpha_i-\alpha_j-\pi i}\,e^{\sum\alpha_i-\frac{1}{2}\sum\beta_j}D(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n})\end{aligned}$$

where

$$\varphi(\alpha)=\Gamma\left(\frac{1}{4}+\frac{\alpha}{2\pi i}\right)\Gamma\left(\frac{1}{4}-\frac{\alpha}{2\pi i}\right)$$

$$\zeta(\beta)=\sinh\frac{1}{2}(\beta)\exp\left(-\int_0^{\infty}\frac{\sin^2\frac{1}{2}(\beta+\pi i)k\,e^{-\frac{\pi k}{2}}}{k\sinh(\pi k)\cosh\left(\frac{\pi k}{2}\right)}\right)$$

$D(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n})$ is a polynomial taking values in $(\mathbb{C}^2)^{\otimes 2n}$. Due to the symmetry property (3) it is sufficient to give its $\{-\cdots-+\cdots+\}$-component:

$$\begin{aligned} &D(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n})_{-\cdots-+\cdots+} = \\ &=\prod_{r>s}\frac{1}{\alpha_r-\alpha_s-\pi i}\prod_k\prod_{j>k}\left(\alpha_k-\beta_j+\frac{\pi i}{2}\right)\prod_{j<k}\left(\alpha_k-\beta_j-\frac{\pi i}{2}\right) \\ &\times\sum_{l=1}^{n}\Big(2\sum\alpha_k+\beta_l+\pi i(2l-1)\Big)\prod_{j\geq l}\frac{\alpha_j-\beta_j-\frac{\pi i}{2}}{\alpha_j-\beta_{j+1}+\frac{\pi i}{2}} \end{aligned} \tag{5}$$

This formula has one not very pleasant feature: it is not symmetric with respect to $\beta_1,\cdots,\beta_n$, the symmetry takes place only for the integral. Notice that this solution belongs to the invariant with respect to action of $SU(2)$ (singlet) subspace of $(\mathbb{C}^2)^{\otimes 2n}$. However, the main trouble with this formula is in presence of denominators $\alpha_r-\alpha_s-\pi i$ which makes the integrals essentially multi-fold.

3 Smirnov solution to qKZ on level 0.

Originally qKZ equations appeared for level 0 as form factor equations [12]. It is convenient to write them for a covector from $\mathbb{C}^{\otimes 2n}$ denoted by $f(\beta_1,\cdots,\beta_{2n})$:

$$\begin{aligned} &f(\beta_1,\cdots,\beta_{j+1},\beta_j,\cdots,\beta_{2n})=f(\beta_1,\cdots,\beta_j,\beta_{j+1},\cdots,\beta_{2n})R(\beta_{j+1}-\beta_j) \\ &f(\beta_1,\cdots,\beta_{2n-1},\beta_{2n}+2\pi i)=f(\beta_{2n},\beta_1,\cdots,\beta_{2n-1}) \end{aligned} \tag{6}$$

We need solution belonging to the singlet subspace. The difference with level -4 case seems to be minor, but the formulae for solutions are much nicer. Many solutions can be written which are counted by
$\{k_1,\cdots,k_{n-1}\}$, with $|k_j|\leq n-1,\ \forall j$:

$$\begin{aligned} &f^{\{k_1,\cdots,k_{n-1}\}}(\beta_1,\cdots,\beta_{2n})=\prod_{i<j}\zeta(\beta_i-\beta_j)\int_{-\infty}^{\infty}d\alpha_1\cdots\int_{-\infty}^{\infty}d\alpha_{n-1}\prod_{i,j}\varphi(\alpha_i-\beta_j) \\ &\times\det|e^{k_i\alpha_j}|_{1\leq i,j\leq n-1}\ \ h(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n}) \end{aligned}$$

where h is skew-symmetric w.r. to α's polynomial. The $\{-\cdots-+\cdots+\}$ component of h is given by

$$\begin{aligned} &h(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n})_{-\cdots-+\cdots+} = \\ &u(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n})\prod_{j=1}^{n}\prod_{j'=n+1}^{2n}\frac{1}{\beta_j-\beta_{j'}+\pi i} \end{aligned}$$

where

$$\begin{aligned} &u(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n})= \\ &=\det(A_i(\alpha_j|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n}))|_{i,j=1,\cdots,n-1} \end{aligned} \tag{7}$$

as for polynomials $A_i(\alpha)$ which depend on β_j as on parameters, it is convenient to write for them generating function:

$$\sum_{i=1}^{n-1}\gamma^{n-i-1}A_i(\alpha|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n})= \tag{8}$$
$$=\frac{\prod_{j=1}^{2n}(\alpha-\beta_j+\frac{\pi i}{2})}{\alpha-\gamma+\pi i}-\frac{\prod_{j=1}^{2n}(\alpha-\beta_j-\frac{\pi i}{2})}{\alpha-\gamma-\pi i}+$$
$$+\frac{\pi i\prod_{j=1}^{n}(\alpha-\beta_j-\frac{\pi i}{2})(\gamma-\beta_{n+j}+\frac{\pi i}{2})}{(\alpha-\gamma)(\alpha-\gamma-\pi i)}+\frac{\pi i\prod_{j=1}^{n}(\gamma-\beta_j-\frac{\pi i}{2})(\alpha-\beta_{n+j}+\frac{\pi i}{2})}{(\alpha-\gamma)(\alpha-\gamma+\pi i)}$$

This expression is manifestly symmetric with respect to two groups of β_j. Important difference with the previous case is that there are no denominators here, effectively the integral is reduced to one-fold ones.

There is an explicit formula expressing h in terms of u:

$$h(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n})=$$
$$=\sum_{\{1,\cdots,2n\}=\{i_1,\cdots,i_n\}\cup\{j_1,\cdots,j_n\}}u(\alpha_1,\cdots\alpha_{n-1}|\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n})$$
$$\times\prod_{p,q=1}^{n}\frac{1}{\beta_{i_p}-\beta_{j_q}}w_{\epsilon_1,\cdots,\epsilon_{2n}}(\beta_1,\cdots\beta_{2n})$$

where $\epsilon_{i_p}=-$, $\epsilon_{j_p}=+$, and the basis $w_{\epsilon_1,\cdots,\epsilon_{2n}}(\beta_1,\cdots\beta_{2n})$ which is described in [11] satisfies important relation:

$$w_{\epsilon_1,\cdots,\epsilon_i,\epsilon_{i+1},\cdots\epsilon_{2n}}(\beta_1,\cdots\beta_i,\beta_{i+1},\cdots\beta_{2n})\overline{R}(\beta_{i+1}-\beta_i)=$$
$$=w_{\epsilon_1,\cdots,\epsilon_{i+1},\epsilon_i,\cdots\epsilon_{2n}}(\beta_1,\cdots\beta_{i+1},\beta_i,\cdots\beta_{2n}) \tag{9}$$

with $\overline{R}$ given by the formula (2). Our main statement is that it is possible to write down similar formula for level -4 case. But before explaining this point we have to remind some properties of deformed hyper-elliptic integrals.

4 Deformed hyper-elliptic integrals.

In this section we follow mostly the paper [13]. The solutions to level 0 qKZ equations are expressed in terms of the following integrals:

$$\langle P\mid p\rangle=\int_{-\infty}^{\infty}\prod_j\varphi(\alpha-\beta_j)\,P(e^{\alpha})\,p(\alpha)\,e^{-(n-1)\alpha}d\alpha \tag{10}$$

where $p(\alpha)$ and $P(e^{\alpha})$ are polynomials which depend respectively on β_j and e^{β_j} as on parameters. For integral to converge we have to require deg$(P)\leq 2n-2$.

We shall intensively use the asymptotic series in α^{-1} for the function $\prod \varphi(\alpha-\beta_j)$. These series (denoted by $\Phi(\alpha)$) can be defined from their main property:

$$\Phi(\alpha+2\pi i)=\Phi(\alpha)\prod_{j=1}^{2n}\frac{\alpha-\beta_j+\frac{\pi i}{2}}{\alpha-\beta_j+\frac{3\pi i}{2}} \tag{11}$$

In the integrals (10) we shall never consider analogues of the differentials of third kind i.e. we shall require:

$$\mathrm{res}_{\alpha=\infty}\left(p(\alpha)\Phi(\alpha)\right)=0 \tag{12}$$

It is easy to see that in this case

$$P(e^{\alpha})=\left(\prod(e^{\alpha}+ie^{\beta_j})-\prod(e^{\alpha}-ie^{\beta_j})\right)e^{-\alpha}\simeq 0 \tag{13}$$

i.e. this polynomial gives zero when substituted into the integral. As for remaining polynomials $p(\alpha)$ one can show that only $2n-2$ give non-trivial result. Indeed, with every polynomial $l(\alpha)$ we can associate an "exact form":

$$l(\alpha+\pi i)\prod\left(\alpha-\beta_j+\frac{\pi i}{2}\right)-l(\alpha-\pi i)\prod\left(\alpha-\beta_j-\frac{\pi i}{2}\right)$$

Being put under the integral (10) this "exact form" gives zero. On the other hand one can reduce degree of any polynomial adding "exact forms" up to $2n-2$.

For the basis of nontrivial polynomials we take:

$$\begin{aligned} &s_k(\alpha)=A_k(\alpha|\beta_1,\cdots,\beta_n|\beta_{n+1},\cdots,\beta_{2n})\\ &s_{-k}(\alpha)=\alpha^{n-k-1},\qquad k=1,\cdots,n-1 \end{aligned} \tag{14}$$

Define

$$\Delta(f)(\alpha)=f(\alpha+\pi i)-f(\alpha-\pi i)$$

The following skew-symmetric pairing is well defined on polynomials satisfying (12):

$$p\circ q=\mathrm{res}_{\alpha=\infty}\left(p(a)\Phi(\alpha)\Delta^{-1}(q(\alpha)\Phi(\alpha)\right) \tag{15}$$

The polynomials s_a constitute canonical basis with respect to this pairing:

$$s_a\circ s_b=\mathrm{sgn}(a)\delta_{a,-b}$$

For the polynomials of e^{α} one also introduces the pairing:

$$P\circ Q=\int_{-\infty}^{\infty}\frac{P(e^{\alpha})Q(-e^{\alpha})-P(-e^{\alpha})Q(e^{\alpha})}{\prod(e^{2\alpha}+e^{2\beta_j})}d\alpha \tag{16}$$

It is not difficult to give explicit formulae for canonical basis S_a ($|j|=1,\cdots,n-1$) satisfying

$$S_a\circ S_b=\mathrm{sgn}(a)\delta_{a,-b}$$

but we shall not need them. Notice that the structure of the pairing implies that S_{-k} and S_k should be taken as respectively odd and even polynomials. They contain quasi-constants (symmetric functions of e^{β_j} as coefficients). We can take

$$S_{-k} = e^{(2k-1)\alpha}, \quad k = 1, \cdots, n-1$$

as half-basis.

The main property of deformed hyper-elliptic integrals is deformed Riemann bilinear relation:

$$\sum_{k=1}^{n-1} \left(\langle S_k \mid s_a \rangle \langle S_{-k} \mid s_b \rangle - \langle S_k \mid s_b \rangle \langle S_{-k} \mid s_a \rangle \right) = \mathrm{sgn}(a) \delta_{a,-b}$$
$$\sum_{k=1}^{n-1} \left(\langle S_a \mid s_k \rangle \langle S_b \mid s_{-k} \rangle - \langle S_b \mid s_k \rangle \langle S_a \mid s_{-k} \rangle \right) = \mathrm{sgn}(a) \delta_{a,-b}$$

This relation introduces into the game the symplectic group $Sp(2n-2)$. To finish this section let us write two more formulae following from (8). First,

$$c(\alpha_1, \alpha_2) \equiv \sum_{k=1}^{n-1} \left(s_k(\alpha_1) s_{-k}(\alpha_2) - s_k(\alpha_2) s_{-k}(\alpha_1) \right) =$$
$$= \frac{\prod_{j=1}^{2n} (\alpha_1 - \beta_j + \frac{\pi i}{2})}{\alpha_1 - \alpha_2 + \pi i} - \frac{\prod_{j=1}^{2n} (\alpha_1 - \beta_j - \frac{\pi i}{2})}{\alpha_1 - \alpha_2 - \pi i} - \frac{\prod_{j=1}^{2n} (\alpha_2 - \beta_j + \frac{\pi i}{2})}{\alpha_2 - \alpha_1 + \pi i} + \frac{\prod_{j=1}^{2n} (\alpha_2 - \beta_j - \frac{\pi i}{2})}{\alpha_2 - \alpha_1 - \pi i}$$

Second, it is obvious from (8) that for any partition there is a symmetric matrix c_{kl} depending polynomially on rapidities such that

$$A_k(\alpha|\beta_{i_1}, \cdots, \beta_{i_n}|\beta_{j_1}, \cdots, \beta_{j_n}) =$$
$$= s_k(\alpha) + \sum_{l=1}^{n-1} c_{kl}(\beta_{i_1}, \cdots, \beta_{i_n}|\beta_{j_1}, \cdots, \beta_{j_n}) s_{-l}(\alpha) \tag{17}$$

which means that for any partition the polynomials $A_k(\alpha|\beta_{i_1}, \cdots, \beta_{i_n}|\beta_{j_1}, \cdots, \beta_{j_n})$, $s_{-k}(\alpha)$ constitute a canonical basis with respect to the above pairing.

$$A_k \circ A_l = s_{-k} \circ s_{-l} = 0, \quad A_k \circ s_{-l} = \delta_{k,l}$$

5 Level -4 from level 0.

It is obvious from equations (3, 4) and (6, 7) that for any pair of solutions the scalar product

$$f(\beta_1, \cdots, \beta_{2n}) g(\beta_1, \cdots, \beta_{2n})$$

is a quasi-constant. So, if we manage to have a complete set of solutions on level 0 the level -4 solutions are obtained by inverting the square matrix. This is the main idea of our construction. Let us count the solutions on level 0.

Consider the space of skew-symmetric polynomials of variables $\alpha_1, \cdots, \alpha_{n-1}$ with the basis det$\|s_{a_p}(\alpha_q)\|_{p,q}$. The group $Sp(2n-2)$ acts in this space as on the space of skew-symmetric tensors. In this space we define a subspace H_{irrep} of maximal irreducible representation of $Sp(2n-2)$ which is the orbit of this group obtained by action on the polynomial det$\|s_p(\alpha_q)\|_{p,q}$ This is the fundamental representation of maximal dimension:

$$d_{\text{irrep}} = \binom{2n-2}{n-1} - \binom{2n-2}{n-3}$$

The formula (17) implies that for any partition the polynomial

$$\det(A_k(\alpha_l|\beta_{i_1}, \cdots, \beta_{i_n}|\beta_{j_1}, \cdots, \beta_{j_n}))|_{1\le k,l\le n-1} \in H_{\text{irrep}}$$

belongs to the representation of $Sp(2n-2)$ in skew-symmetric tensors of degree $n-1$.

Due to deformed Riemann bilinear relation among the solutions counted by $\{k_1, \cdots, k_{n-1}\}$ only d_{irrep} are linearly independent over the ring of quasi-constants. To obtain them we take the polynomial

$$\det(S_{-k}(e^{\alpha_l}))$$

as basic one and obtain the rest as orbit under the action of Borel subgroup, i.e. by the matrices

$$\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \in Sp(2n-2)$$

where the matrix Z is symmetric. Thus we obtain d_{irrep} linearly independent solutions.

On the other hand the covectors $f(\beta_1, \cdots, \beta_{2n})$ belong to singlet subspace of $(\mathbb{C}^2)^{\otimes 2n}$. The dimension of this subspace equals:

$$d_{\text{sing}} = \binom{2n}{n} - \binom{2n}{n-1}$$

The marvelous identity [14]

$$d_{\text{irrep}} = d_{\text{sing}}$$

shows that we have exactly the same number of solutions as the dimension of space. So, different linear independent solutions can be combined into the square matrix $\mathcal{F}(\beta_1, \cdots, \beta_{2n})$. Now we can find the solutions on level -4 solving the equation:

$$\mathcal{F}(\beta_1, \cdots, \beta_{2n})\mathcal{G}(\beta_1, \cdots, \beta_{2n}) = I$$

So, our goal is to find an efficient way for inverting the matrix $\mathcal{F}$.

Notice that $\mathcal{F}(\beta_1, \cdots, \beta_{2n})$ naturally splits into the product:

$$\mathcal{P}(\beta_1, \cdots, \beta_{2n})\mathcal{H}(\beta_1, \cdots, \beta_{2n})$$

where the multipliers $\mathcal{P}$ and $\mathcal{H}$ carrying respectively transcendental and rational dependence on β_j are defined as follows.

$$\mathcal{P}(\beta_1, \cdots, \beta_{2n}) = P_{\text{irrep}}\widetilde{\mathcal{P}}(\beta_1, \cdots, \beta_{2n})P_{\text{irrep}}$$

where P_{irrep} is the projector on the irreducible representation of $Sp(2n-2)$ discussed above and the matrix $\widetilde{\mathcal{P}}$ acts in the $(n-1)$-th skew-symmetric power of $\mathbb{C}^{2n-2}$, its matrix elements are given by

$$\det(\langle S_{a_k}|s_{b_l}\rangle)|_{1\le k,l\le n-1}$$

The rational in β_j matrix $\mathcal{H}(\beta_1,\cdots,\beta_{2n})$ acts from the singlet subspace of $(\mathbb{C}^2)^{\otimes 2n}$ into the space of maximal irreducible representation of $Sp(2n-2)$ in $\wedge^{n-1}\mathbb{C}^{2n-2}$. In the space $(\mathbb{C}^2)^{\otimes 2n}$ we take the basis $w_{\epsilon_1,\cdots,\epsilon_{2n}}(\beta_1,\cdots\beta_{2n})$ in such a way that the components of covectors in this space are counted by partitions $\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n}$. Different vectors from $\wedge^{n-1}\mathbb{C}^{2n-2}$ are counted by $-(n-1)\le a_1<a_2<\cdots<a_{n-1}\le(n-1)$. In this basis the matrix elements of $\mathcal{H}(\beta_1,\cdots,\beta_{2n})$ are

$$\frac{1}{\prod(\beta_{i_p}-\beta_{j_q})}\det\left(\widetilde{c}_{k,a_l}(\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n})\right)|_{1\le k,l\le n-1}$$

where

$$\widetilde{c}=\begin{pmatrix} I & c\\ 0 & I\end{pmatrix}$$

and $c_{i,j}$ is defined in (17).

Now we want to invert these matrices. Due to deformed Riemann bilinear relation inverting of the transcendental part is trivial:

$$\mathcal{P}(\beta_1,\cdots,\beta_{2n})^{-1}=P_{\text{irrep}}\widetilde{\mathcal{P}}^{\dagger}(\beta_1,\cdots,\beta_{2n})P_{\text{irrep}}$$

where the matrix elements of $\widetilde{\mathcal{P}}^{\dagger}$ are given by

$$\det(\langle S^{\dagger}_{b_k}|s_{a_l}\rangle)|_{1\le k,l\le n-1}$$

with

$$S^{\dagger}_b=\text{sgn}(b)S_{-b}$$

So, surprisingly enough the main difficulty happens to be in inverting of the rational matrix. In this section we give one approach to the problem which proves that the inverse matrix possesses nice properties.

First, we have to take care of the basis w. What we actually need is a construction of dual basis. This construction can be found in [11] , we do not give it explicitly here, the main properties of the dual basis $w^{\dagger}$ are:

$$\begin{aligned}&\overline{R}(\beta_{i+1}-\beta_i)w^{\dagger}(\beta_1,\cdots,\beta_i,\beta_{i+1},\cdots,\beta_{2n})_{\epsilon_1,\cdots,\epsilon_i,\epsilon_{i+1},\cdots,\epsilon_{2n}}=\\ &=w^{\dagger}(\beta_1,\cdots,\beta_{i+1},\beta_i,\cdots,\beta_{2n})_{\epsilon_1,\cdots,\epsilon_{i+1},\epsilon_i,\cdots,\epsilon_{2n}},\\ &w(\beta_1,\cdots,\beta_{2n})_{\epsilon_1,\cdots,\epsilon_{2n}}w^{\dagger}(\beta_1,\cdots,\beta_{2n})_{\epsilon'_1,\cdots,\epsilon'_{2n}}=\prod\delta_{\epsilon_i,\epsilon'_i}\end{aligned}$$

Consider the operator $\mathcal{H}^*(\beta_1,\cdots,\beta_{2n})$ which coincide with Hermitian conjugation of $\mathcal{H}(\beta_1,\cdots,\beta_{2n})$ for real β_j and then is continued analytically. For the matrix elements of this operator in usual basis for $\wedge^{n-1}\mathbb{C}^{2n-2}$ and the basis $w^{\dagger}$ in $(\mathbb{C}^2)^{\otimes 2n}$ one finds [11]:

$$\frac{1}{\prod(\beta_{i_p}-\beta_{j_q}+i\pi)}\det\left(\widetilde{c}_{k,a_l}(\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n})\right)|_{1\le k,l\le n-1}$$

Let us write the identity:

$$\mathcal{H}^{-1}(\beta_1,\cdots,\beta_{2n}) = \mathcal{H}^*(\beta_1,\cdots,\beta_{2n})\left(\mathcal{H}(\beta_1,\cdots,\beta_{2n})\mathcal{H}^*(\beta_1,\cdots,\beta_{2n})\right)^{-1}$$

The operator $\mathcal{H}\mathcal{H}^*$ is nicer then $\mathcal{H}$ itself because it acts from the space of irreducible representation of $Sp(2n-2)$ to itself. Its matrix elements are:

$$\sum_{\{1,\cdots,2n\}=\{i_1,\cdots,i_n\}\cup\{j_1,\cdots,j_n\}} \frac{1}{\prod(\beta_{i_p}-\beta_{j_q}+i\pi)(\beta_{i_p}-\beta_{j_q})}$$
$$\times \det\left(\widetilde{c}_{k,a_l}(\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n})\right)\det\left(\widetilde{c}_{k,b_l}(\beta_{i_1},\cdots,\beta_{i_n}|\beta_{j_1},\cdots,\beta_{j_n})\right)$$

Unfortunately we could not find a way of efficiently inverting this operator, but we were able to calculate its determinant:

$$\det(\mathcal{H}(\beta_1,\cdots,\beta_{2n})\mathcal{H}^*(\beta_1,\cdots,\beta_{2n})) =$$
$$= Const\left(\prod_{i,j}(\beta_i-\beta_j-\pi i)\right)^{-\left(\binom{2n-4}{n-2}-\binom{2n-4}{n-4}\right)} \tag{18}$$

The proof of this formula is based on two facts. First, one can easily calculate the degree of the determinant as function of β's. Second, the polynomial u satisfies the following recurrence relation [11] :

$$u(\alpha_1,\cdots,\alpha_{n-1}|\beta_1,\cdots,\beta_{n-1},\beta-\frac{\pi i}{2}|\beta_{n+1},\cdots,\beta_{2n-1},\beta+\frac{\pi i}{2}) =$$
$$=\prod_{j=1}^{n-1}(\alpha_j-\beta)\sum_{j=1}^{n-1}(-1)^j\left(\prod_k(\alpha_j-\beta_k+\frac{\pi i}{2})-\prod_k(\alpha_j-\beta_k-\frac{\pi i}{2})\right)$$
$$\times u(\alpha_1,\cdots,\widehat{\alpha_j},\cdots,\alpha_{n-1}|\beta_1,\cdots,\beta_{n-1}|\beta_{n+1},\cdots,\beta_{2n-1}) \tag{19}$$

Using this relations one can calculate the rank of residue of $\mathcal{H}\mathcal{H}^*$ at the point $\beta_i=\beta_j+\pi i$.

Putting all this information together we arrive at the following

Theorem. *The solutions to qKZ equations on level -4 counted by* $\{k_1,\cdots,k_{n-1}\}$, *with* $|k_j|\leq n-1$, $\forall j$ *can be written in the following form:*

$$g^{\{k_1,\cdots,k_{n-1}\}}(\beta_1,\cdots,\beta_{2n}) = \prod_{i<j}\frac{1}{\zeta(\beta_i-\beta_j)}\int_{-\infty}^{\infty}d\alpha_1\cdots\int_{-\infty}^{\infty}d\alpha_{n-1}\prod_{i,j}\varphi(\alpha_i-\beta_j)$$
$$\times \det|e^{k_i\alpha_j}|_{1\leq i,j\leq n-1}\ \widetilde{h}(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n}) \tag{20}$$

where $\widetilde{h}(\alpha_1,\cdots\alpha_{n-1}|\beta_1,\cdots,\beta_{2n})$ *is a polynomial of all its argument, skew-symmetric with respect to* $\alpha_1,\cdots\alpha_{n-1}$

Proof. The only point which remains to be proved is that $\widetilde{h}$ is indeed a polynomial because *a priori* we can be sure only that it is a rational function. The structure of h and the formula for determinant (18) imply that there are no other possible singularities than poles at $\beta_i=\beta_j+\pi i$. By recurrence relation following from (19) the residue of h at $\beta_i = \beta_j + \pi i$ is defined by the same function h for $n \to n-1$. So, the rank of the residue is defined by the dimension of singlet subspace of $\mathbb{C}^{2n-2}$:

$$\binom{2n-2}{n-1} - \binom{2n-2}{n-2}$$

which is the same as the exponent in (18). Now it is clear that in inverse matrix the pole is canceled by zero coming from the determinant. **QED**

As in level 0 case there is a linear dependence between the solutions which is removed by Riemann bilinear relation.

The problem of direct inverting the matrix $\mathcal{H}$ seems to be too complicated. So, we need alternative ways to define the polynomials $\widetilde{h}$. First, let us reformulate the original definition. We can present $\widetilde{h}$ in the following form:

$$\begin{aligned}
&\widetilde{h}(\alpha_1, \cdots, \alpha_{n-1}|\beta_1, \cdots, \beta_{2n}) = \\
&= \sum_{\{1,\cdots,2n\}=\{i_1,\cdots,i_n\}\cup\{j_1,\cdots,j_n\}} v(\alpha_1, \cdots \alpha_{n-1}|\beta_{i_1}, \cdots, \beta_{i_n}|\beta_{j_1}, \cdots, \beta_{j_n}) \\
&\times \prod_{p,q=1}^{n} \frac{\beta_{i_p} - \beta_{j_q} + \pi i}{\beta_{i_p} - \beta_{j_q}}\, w^{\dagger}_{\epsilon_1,\cdots,\epsilon_{2n}}(\beta_1, \cdots \beta_{2n})
\end{aligned}$$

Then the function v is subject to two requirements. The first follows from the fact that $\widetilde{h}$ must belong to singlet subspace. Using the transformation of the basis $w^{\dagger}$ under the action of $su(2)$ described in [11] one finds the equations:

$$\begin{aligned}
&\sum_{p=1}^{n+1} v(\alpha_1, \cdots, \alpha_{n-1}|\beta_{i_1}, \cdots, \beta_{i_{n-1}}, \beta_{j_p}|\beta_{j_1}, \cdots, \widehat{\beta_{j_p}}, \cdots, \beta_{j_{n+1}}) \\
&\times \prod_{q\neq p} \frac{\beta_{j_p} - \beta_{j_q} - \pi i}{\beta_{j_p} - \beta_{j_q}} = 0
\end{aligned} \tag{21}$$

The second equation is equivalent to the fact that $\widetilde{h}$ is obtained by inverting the matrix $\mathcal{H}$:

$$\begin{aligned}
&\sum_{\{1,\cdots,2n\}=\{i_1,\cdots,i_n\}\cup\{j_1,\cdots,j_n\}} v(\alpha_1, \cdots, \alpha_{n-1}|\beta_{j_1}, \cdots, \beta_{j_n}|\beta_{i_1}, \cdots, \beta_{i_n}) \\
&\times u(\alpha'_1, \cdots, \alpha'_{n-1}|\beta_{i_1}, \cdots, \beta_{i_n}|\beta_{j_1}, \cdots, \beta_{j_n}) \prod_{p,q=1}^{n} \frac{1}{\beta_{i_p} - \beta_{j_q}} = \\
&= c(\alpha_1, \cdots, \alpha_{n-1}|\alpha'_1, \cdots, \alpha'_{n-1})
\end{aligned} \tag{22}$$

where $c(\alpha_1, \cdots \alpha_{n-1}|\alpha'_1, \cdots \alpha'_{n-1})$ is the "intersection form". Essential part of this "intersection form" is det $\left|c(\alpha_i, \alpha'_j)\right|$, but some additional terms should be added in order that $c(\alpha_1, \cdots \alpha_{n-1}|\alpha'_1, \cdots \alpha'_{n-1})$ belongs to H_{irrep} with respect to both sets $\alpha_1, \cdots \alpha_{n-1}$ and $\alpha'_1, \cdots \alpha'_{n-1}$. Introduce Grassmann variables ξ_j, η_j ($j = 1, \cdots, n-1$):

$$\xi_i\xi_j = -\xi_j\xi_j, \quad \xi_i\eta_j = -\eta_j\xi_j, \quad \eta_i\eta_j = -\eta_j\eta_i$$

Let

$$C = \sum_{i,j} c(\alpha_i, \alpha'_j)\xi_i\eta_j,$$
$$S = \sum_{i<j} c(\alpha_i, \alpha_j)\xi_i\xi_j,$$
$$S' = \sum_{i<j} c(\alpha'_i, \alpha'_j)\eta_i\eta_j,$$

Then

$$c(\alpha_1, \cdots \alpha_{n-1}|\alpha'_1, \cdots \alpha'_{n-1})\xi_1 \cdots \xi_{n-1}\eta_1 \cdots \eta_{n-1} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} c_k (SS')^k C^{n-1-2k}$$

where the coefficients c_k are

$$c_0 = 1, \quad c_k = \frac{(n-1)!}{k!(k+1)!(n-2k-1)!}$$

If we consider $v(\alpha_1, \cdots \alpha_{n-1}|\beta_{i_1}, \cdots, \beta_{i_n}|\beta_{j_1}, \cdots, \beta_{j_n})$ for different partitions as $\binom{2n}{n}$ independent unknowns then we have sufficient number of linear equations: $\binom{2n}{n} - \binom{2n}{n-1}$ from (22) and $\binom{2n}{n-1}$ from (21).

Our main goal is to describe efficiently the polynomials $\widetilde{h}$. The way to approach this problem will be discussed in another paper. To finish the present paper we would like to give an intuitive idea about reasons behind the possibility of rewriting original Jimbo-Miwa formula in the form without denominators.

6 Cohomological meaning of new formula.

Consider the "classical" limit:

$$\beta_j = \frac{1}{\hbar}x_j, \quad \alpha = \frac{1}{\hbar}z \qquad \hbar \to 0$$

In this limit

$$\langle P \mid p \rangle = \int_{-\infty}^{\infty} \prod_j \varphi(\alpha - \beta_j)\, P(e^\alpha)\, p(\alpha)\, e^{-(n-1)\alpha} d\alpha \to \int_\gamma \frac{p(z)}{w} dz$$

where the hyper-elliptic surface X is defined by

$$w^2 = \prod (z - x_j),$$

There are two points ($\infty^\pm$) on the curve lying above the point $z = \infty$. The genus equals $n-1$. The contour γ is defined by P. In particular,

$$S_{-k} \leftrightarrow b_k, \quad S_k \leftrightarrow a_k$$

The polynomials

$$\widetilde{s}_a = \lim_{\hbar\to 0} s_a$$

describe canonical basis of differentials. Namely for

$$\sigma_a = \frac{\widetilde{s}_a(z)}{w} dz$$

one has

$$\sigma_a \circ \sigma_b = \sum_{\infty^\pm} \mathrm{res}\left(\sigma_a d^{-1}(\sigma_b)\right) = \mathrm{sgn}(a)\delta_{a,-b}$$

The differentials σ_k are of first kind, σ_{-k} - of second, there is also the third kind differential

$$\sigma_0 = \frac{z^{n-1}}{w} dz$$

Consider the Jacobi variety of X:

$$J = \mathbb{C}^{2n-2}/(\mathbb{Z}^{n-1} + B\mathbb{Z}^{n-1})$$

where B is the matrix of B-periods of normalized holomorphic differentials:

$$B_{ij} = \int\limits_{b_i} \omega_j$$

We define Riemann theta-function

$$\theta(\zeta), \qquad \text{for} \quad \zeta \in \mathbb{C}^{2n-2}$$

Consider the divisor $\{P_1, \cdots, P_g\}$ where P_j are points on the Riemann surface: $P_j = \{z_j, w_j\}$. Abel transformation is defined as follows

$$\{P_1, \cdots, P_{n-1}\} \to \zeta = \sum \int^{P_j} \omega$$

defines map

$$\mathrm{Symm}(X^{n-1}) \to J$$

which is not one to one. However, if we consider non-compact varieties

$$J - (\Theta_- \cup \Theta_+)$$

where

$$\Theta_\pm = \{\zeta | \theta(\zeta + \rho_\pm) = 0\}, \ \rho_\pm = \int^{\infty^\pm} \omega$$

and

$$\mathrm{Symm}(X^{n-1}) - D$$

where

$$D = \{\{P_1, \cdots, P_{n-1}\} | P_j = \infty^{\pm}, P_i = \sigma(P_j)\}$$

they are isomorphic. The integrand of the Jimbo-Miwa formula gives in the classical limit a differential form on

$$\mathrm{Symm}(X^{n-1}) - D$$

which is isomorphic to

$$f(\zeta) d\zeta_1 \cdots d\zeta_{n-1}$$

with $f(\zeta)$ meromorphic on J with poles on $\Theta_{\pm}$. The question arises concerning cohomologies. They are described by the following theorem conjectured in [15] and proved in [16].

Theorem (A. Nakayashiki) *The dimension of cohomologies space is*

$$\binom{2n-1}{n-1} - \binom{2n-1}{n-3}$$

in terms of Symm$(X^{n-1}) - D$ *it is realized as follows.* $2n-1$ *differential of 1-st, 2-ond, 3-d kind* σ_a*. Let*

$$\widetilde{\sigma}_a = \sum_{k=1}^{n-1} \sigma_a(P_k)$$

The cohomologies are realized as follows:

$$\{\widetilde{\sigma}_{a_1} \wedge \cdots \wedge \widetilde{\sigma}_{a_{n-1}}\} / \{\omega \wedge \widetilde{\sigma}_{a_1} \wedge \cdots \wedge \widetilde{\sigma}_{a_{n-3}}\}$$

where

$$\omega = \sum_{k=1}^{n-1} \widetilde{\sigma}_k \wedge \widetilde{\sigma}_{-k}$$

This is the reason why classical limit of Jimbo-Miwa formula can be reduced to one-fold integrals. We suppose that similar interpretation is possible in deformed case.

7 Conclusion

In this communication the problem to reduce the Jimbo-Miwa solution of the qKZ on level -4 to one-fold integrals is solved only partially. As we have shown it is connected with the cohomological origin of our formulae. In our next paper we shall give an explicit form of the polynomials $\widetilde{h}$ from the formula (20). We shall also discuss the relation of the above solution to the correlation functions of the XXX model. In order to treat this problem properly we need to carry out an accurate analysis of singularities which appear in intermediate stage. The main task is to prove that final result for the correlation functions is really finite. The explicit form of this result is in agreement with the ansatz from the paper [10]. The conjecture [8, 9] about the structure of the correlation functions in the homogeneous limit through the Riemann ζ-function at odd argument follows from the form of the correlation functions in the inhomogeneous case which in it's turn follows from the solution to the qKZ on level -4.

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