SOME ISOPERIMETRIC INEQUALITIES AND THEIR APPLICATION TO PROBLEMS ON POLYNOMIALS

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Abstract. We prove some sharp inequalities for the product of the Lebesque integral over some power of a positive absolutely continuous and non–decreasing function (or some linear combination of its distinct powers) and the Lebesque integral over the square of its derivative. These inequlities are releted to some problems for polynomials with small Mahler measure. As an application we give a lower bound for the logarithmic height of a non–cyclotomic algebraic number in terms of its degree.

Key words: isoperimetric inequalities, Mahler's measure, linear form in one logarithm.

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1. Introduction.

Let $\mathcal{L}$ be the space of all absolutely continuous and non–decreasing functions G in the interval $[0,1]$ satisfying $G(0)=0$, $G(1)=1$ such that the derivative G' (which, by the Lebesque theorem [16], exists almost everywhere) is in $L^2[0,1]$. For a real number λ set

$$x_\lambda(G)=\int_0^1 G(t)^\lambda dt \int_0^1 G'(t)^2 dt.$$

We begin with the following statement.

Theorem 1. *Given a real number $\lambda > -2$, and $G \in \mathcal{L}$, we have $x_\lambda(G) \geqslant 4/(\lambda+2)^2$ for λ in the range $-2<\lambda<2$, and $x_\lambda(G) > 4/(\lambda+2)^2$ for $\lambda \geqslant 2$. Furthemore, for every $\lambda > -2$ the respective inequality is sharp.*

Corollary 1. *If $G \in \mathcal{L}$ then $x_1(G) \geqslant 4/9$, and this inequality is sharp.*

Corollary 2. *If $G \in \mathcal{L}$ then $x_2(G) > 1/4$, and this inequality is sharp.*

Our purpose here is to investigate which values can be taken by the functional

$$\mathcal{F}_p(G)=\Big(\int_0^1 (x_1(G)-\tau x_2(G))^{p/(p-1)} d\tau\Big)^{(p-1)/p},$$

where $p>1$. In two extreme cases $p \to 1$ and $p \to \infty$, set, taking the limits, $\mathcal{F}_1(G) := x_1(G)$ and $\mathcal{F}_\infty(G) := x_1(G)-(1/2)x_2(G)$.

In the next section we describe a number theoretic problem on small Mahler measure in which the functionals of this type arise. In Section 3 we formulate three isoperimetric problems, and remind two more problems on polynomials in which they arise. In particular, Corollary 1 solves the first isoperimetric problem. In Section 4 we describe the values taken by the linear functional in the first two moments, namely $ax_1(G) + bx_2(G)$. This contains the answer to the second question. In Section 5 a solution to the third problem is given. Some general facts and observations about the behavior of the functional $\mathcal{F}(x_1(G), x_2(G), \ldots, x_m(G))$ are given in Section 6. In Sections 7, 8, and 9 we prove Theorems 1, 2, and 3 stated in Sections 1, 4, and 5, respectively. Some computational results are given in the final Section 10. To simplify the notation we often (especially in the proofs) write $I(f)$ for the integral $\int_0^1 f(t)dt$.

2. A number theoretic problem.

Given a fixed constant $\sigma > 1$, Lehmer [17] asked whether there is a monic polynomial with integer coefficients such that the modulus of the product of its zeroes outside the unit circle is strictly greater than 1, but smaller than or equal to σ. The modulus of the above product (of roots outside the unit circle) is called the *Mahler measure* of the monic polynomial, named after Mahler who investigated this quantity in [19] and later. It is believed since, that there is no such polynomial if σ is sufficiently close to 1. The smallest known value for σ, namely $\sigma_0 = 1.17628082\ldots$, for which such polynomial exists was found by Lehmer himself. Here σ_0 is the only positive root of the polynomial $z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$ which is greater than 1.

In 1857, Kronecker [15] showed that if the Mahler measure of a monic polynomial $P(z)$ with integer coefficients is equal to 1 then the polynomial is a product of cyclotomic polynomials and a power of z. Note that the Mahler measure of the monic polynomial with integer coefficients is never smaller than 1, since the product of all its non–zero roots equals 1 in absolute value. In modern terms, therefore, Lehmer's conjecture is usually reformulated as follows: for every irreducible non–cyclotomic polynomial P, of degree $d \geqslant 2$, we have $M(P) \geqslant \sigma_1$. Here $M(P)$ stands for the Mahler measure of P, and $\sigma_1 > 1$ is either σ_0 or may be a smaller constant. Also, $M(P)$ is a good example of "height", namely certain measure of complexity of a polynomial. See, for instance, a book of Everest and Ward [14] on introduction to the Mahler measure and Chapter 3 of Walschmidt's book [29] on introduction to the theory of heights.

The maximum of the absolute values of roots of P is another example of the height. This is called the house of P. Suppose we are given an integer $d \geqslant 2$ and a positive constant δ. Schinzel and Zassenhaus [26] asked whether there is an irreducible monic polynomial, of degree d, with integer coefficients such that its house is greater than 1, but smaller than or equal to $1 + \delta/d$.

Although these two problems are not yet solved, there are some lower bounds on the minimum taken over the Mahler measures (and over the houses, respectively) of all non–cyclotomic monic integer polynomials of, say, degree d in terms of d. See, for instance, Blanksby [2], Blanksby and Montgomery [3], Stewart [28], Dobrowolski [8]. The best unconditional (depending on d only) both asymptotic (when d is sufficiently large) and uniform (for all d) bounds were obtained using the method of Dobrowolski.

In order to formulate these results we shall use the standard notation. Let $d \geqslant 2$ be a positive integer. Since the set of all monic irreducible polynomials of degree d is isomorphic to the set of all algebraic integers of degree d, from now on, as usual, we denote by $M(\alpha)$ the Mahler measure of the minimal polynomial of α. Set also $\overline{|\alpha|} = \max_{1 \leqslant k \leqslant d} |\alpha_k|$ for the house of α. Here $\alpha_1, \dots, \alpha_d$ (with α one of these) are the conjugates of α, namely the roots of its minimal polynomial P of degree d. It is now known that for every positive ε there is a $d(\varepsilon)$ such that if α is a non–cyclotomic algebraic number of degree d, $d \geqslant d(\varepsilon)$, then

$$\log M(\alpha) > \big(\frac{9}{4} - \varepsilon\big)\Big(\frac{\log\log d}{\log d}\Big)^3, \quad \log \overline{|\alpha|} > \big(\frac{64}{\pi^2} - \varepsilon\big)\frac{1}{d}\Big(\frac{\log\log d}{\log d}\Big)^3$$

(see [18] and [9], respectively). Earlier, Dobrowolski [8] proved the first inequality with the constant 1 instead of 9/4. Cantor and Straus [7] and independently Rauch [23] improved the constant to 2. The result of Smyth [27] reduces both problems to the reciprocal algebraic integers, namely those of even degree which have the set of conjugates invariant over the action of the map $z \to z^{-1}$. The result of Louboutin, therefore, implies the second inequality with the constant 9/2.

The proofs of these results involve either Siegel's lemma (see e.g. Bombieri and Vaaler [4] for a modern version of Siegel's lemma) or certain confluent determinants. It was shown in [12] that these constants $1, 2, 9/4, 9/2, 64/\pi^2$ can all be obtained via the values of certain functional $\mathcal{F}$. The latter can be defined on the set of the absolutely continuous functions on $[0,1]$ monotonically decreasing in this interval from 1 to 0.

Let p be a positive number. Set

$$M_p(\alpha) = \Big(\frac{1}{d}\sum_{k=1}^{d} \big|\log|\alpha_k|\big|^p\Big)^{1/p}.$$

This quantity $M_p(\alpha)$ is a height on the multiplicative group of algebraic integers. We can call it the *logarithmic height* of α. The logarithmic height at $p = 1$ is exactly twice the Weil height for reciprocal integers. Indeed, if α is reciprocal then

$$M_1(\alpha) = \frac{2\log M(\alpha)}{d},$$

and

$$M_\infty(\alpha) := \lim_{p\to\infty} M_p(\alpha) = \log\overline{|\alpha|}.$$

Also, for any reciprocal α, the function $p \to M_p(\alpha)$ is non–decreasing (in p). The inequality of the form $\inf dM_p(\alpha) \geqslant f_p$, therefore, lies between the conjecture of Lehmer and the conjecture of Schinzel and Zassenhaus. Here, $p > 1$, f_p is a positive constant which depends only on p and the infimum is taken over every non–cyclotomic reciprocal algebraic integer of degree d.

A contruction of the confluent interpolation determinant which allows us to estimate this infimum from below (in terms of d) for any $p \geqslant 1$ is given in [12]. The results of Section 2 in [12] are given in terms of non–increasing functions. The map $t \to 1 - t$ allows us to reformulate these results in terms of the non–decreasing functions.

3. Three isoperimetric problems.

In this section, all of the infimums are taken over every function in $\mathcal{L}$.

Problem 1. *Find* $\zeta_1 = \inf x_1(G)$.

Problem 2. *Find* $\zeta_\infty = \inf\{x_1(G) - (1/2)x_2(G)\}$.

Problem 3. *Given* $p > 1$, *find*

$$\zeta_p = \inf \Big(\int_0^1 (x_1(G) - \tau x_2(G))^{p/(p-1)} d\tau \Big)^{(p-1)/p}.$$

Note that Corollary 1 settles Problem 1: $\zeta_1 = 4/9$. Our main result in [12] is the inequality

$$dM_p(\alpha) > \Big(\frac{2}{\mathcal{F}_p(G)} - \varepsilon\Big)\Big(\frac{\log\log d}{\log d}\Big)^3$$

(see Theorem 2 in [12]). In particular, for every positive ε there is a $d(\varepsilon)$ such that for every non–cyclotomic algebraic integer α of degree $d \geqslant d(\varepsilon)$

$$\log M(\alpha) > \Big(\frac{1}{x_1(G)} - \varepsilon\Big)\Big(\frac{\log\log d}{\log d}\Big)^3,$$

$$\log \overline{|\alpha|} > \Big(\frac{2}{\mathcal{F}_\infty(G)} - \varepsilon\Big)\frac{1}{d}\Big(\frac{\log\log d}{\log d}\Big)^3.$$

Here, $G \in \mathcal{L}$ is in our disposition. Generally speaking, we can replace $\mathcal{F}_p(G)$, $x_1(G)$, and $\mathcal{F}_\infty(G)$ by ζ_p, ζ_1, and ζ_∞, respectively.

As for the inequality for the Mahler measure, the choice of $G(t) = t$ gives the constant $1/x_1(G) = 1$ obtained by Cantor and Straus [7]. The choice $G(t) = t^2$ gives the constant $1/x_1(G) = 9/4$ obtained by Louboutin [18]. In case of the house, the choices of $G(t) = t$, $G(t) = t^2$, and $G(t) = 1 - \cos(\pi t/2)$ give the constants 6, 45/7, and $64/\pi^2$, respectively.

Problem 2 has an application to estimate of the linear form in one logarithm $\log \alpha$. See the papers of Mignotte [20], Mignotte and Waldschmidt [21], Bugeaud, Mignotte and Normandin [6], Amoroso [1], and also [10], [11]. With the methods used in [10] one can prove that for $\alpha \neq 1$ we have the inequality

$$|\log \alpha| > \exp\Big\{ -\big(\sqrt{2\mathcal{F}_\infty(G)} + o(1)\big) d\sqrt{h(\alpha)\log(1/h(\alpha))}\Big\},$$

where $o(1)$ tends to zero as the Weil height of α, namely $h(\alpha) = (1/d)\log M(\alpha)$, tends to zero. The choice of $G(t) = t$ gives the constant $\sqrt{2/3}$ (as in [6]). A more subtle choice $G(t) = 1 - \cos(\pi t/2)$ gives the constant $\pi/4$.

Problem 2 also has an application to the problem of finding an integer polynomial with large number of cyclotomic factors. This problem was first studied by Schinzel [25], see also Pinner and Vaaler [22]. The constant $3\sqrt{2}/\pi$, which was obtained in [13], generally speaking, can be replaced by $3/(4\sqrt{\zeta_\infty})$.

One of the conclusions of Corollaries 1 and 3 is the following: unless there is a new idea, one cannot improve the best known asymptotic constants 9/4 (that of [18]), $64/\pi^2$ (that of [9]), $\pi/4$ (that of [10]), and $3\sqrt{2}/\pi$ (that of [13]), by just replacing the function G by a "better" one. Ideologically similar to this paper is that of Brisebarre [5], where a similar conclusion (although based just on extensive computation near the point of extremum) concerning the irrationality measures of $\log 2$ and $\pi/\sqrt{3}$ was obtained.

4. Linear form in two moments.

Given three real numbers a, b, and c, we are interested whether there is a function $G \in \mathcal{L}$ such that

$$ax_1(G) + bx_2(G) = c.$$

We say that a triple $(a, b, c) \in \mathbb{R}^3$ is *admissible* if there is a function $G \in \mathcal{L}$ which is a solution to the above equation. To describe the admissible triples, note that if (a, b, c) is (is not) admissible then the triple $(\psi a, \psi b, \psi c)$ is (is not) admissible for every real number $\psi \neq 0$. For this reason, it is convenient to describe the admissible triples via points in $2-$ dimensional projective space $\mathbb{P}^2$. We can, therefore, assume that $b = 0$ or $b = \pm 1$ and describe the points $(a : 0 : c) \in \mathbb{P}^2$, $(a : 1 : c) \in \mathbb{P}^2$, $(a : -1 : c) \in \mathbb{P}^2$ with non–negative a so that $(\psi a, \psi b, \psi c) \in \mathbb{R}^3$, where $b \in \{0, -1, 1\}$ and $\psi \in \mathbb{R}$, are the admissible triples.

Theorem 2. *Let $a \geqslant 0$ and let $\psi \neq 0$. The triple $(\psi a, 0, \psi c) \in \mathbb{R}^3$ is admissible if and only if one of the following is true*
(i) $a = 0, c = 0$;
(ii) $a > 0, c \geqslant 4a/9$.
The triple $(\psi a, \psi, \psi c) \in \mathbb{R}^3$ is admissible if and only if one of the following is true
(iii) $a = 0, c > 1/4$;
(iv) $a > 0, c \geqslant (1/16)\big((a+2)\sqrt{a+1} - a^2 \log(\sqrt{1/a} + \sqrt{1+1/a})\big)^2$.
Finally, the triple $(\psi a, -\psi, \psi c) \in \mathbb{R}^3$ is admissible if and only if one of the following is true
(v) $a = 0, c < -1/4$;
(vi) $0 < a < 1$, *and c is any real number;*
(vii) $a \geqslant 1, c \geqslant (1/64)\big(a^2 \arccos(1 - 2/a) - 2(a-2)\sqrt{a-1}\big)^2$.

It is easy to see, by (ii), that the smallest c for which the triple $(1, 0, c)$ is admissible equals $4/9$ (this gives another proof of Corollary 1). Also, Corollary 2 is contained in (iii). On applying (vii) with $\psi = 1/2, a = 2$, we deduce that ζ_∞ is equal to $\pi^2/32$ which gives a solution to the second problem.

Corollary 3. *If* $G \in \mathcal{L}$ *then*

$$\zeta_\infty = \min \int_0^1 \big(G(t) - (1/2)G(t)^2\big)dt \int_0^1 G'(t)^2 dt = \pi^2/32.$$

It is worth pointing out that, by (vi), the linear form $0.99x_1(G) - x_2(G)$ takes any real value (as $G \in \mathcal{L}$), whereas, by (vii), the values of the form $x_1(G) - x_2(G)$ all lie in the interval $[\pi^2/64, \infty)$.

5. Computation of ζ_p.

Recall that

$$\mathcal{F}_p(G) = \Big(\int_0^1 (x_1(G) - \tau x_2(G))^{p/(p-1)} d\tau\Big)^{(p-1)/p},$$

and

$$\zeta_p = \inf \mathcal{F}_p(G),$$

where $p > 1$ and the infimum is taken over every function $G \in \mathcal{L}$.

For every ω in the interval $[0, \pi/2]$ we define the set of functions $\Omega = \{G_\omega(t)\}$ as follows. Denote

$$G_\omega(t) = \frac{1 - \cos(\omega t)}{1 - \cos(\omega)} = \Big(\frac{\sin(\omega t/2)}{\sin(\omega/2)}\Big)^2$$

for every ω in the interval $(0, \pi/2]$, and $G_0(t) = t^2$. (Note that, for every fixed real number t, $G_\omega(t)$ tends to t^2 as $\omega \to 0$.) Clearly, Ω is a subset of $\mathcal{L}$. The proof of the next theorem is the most difficult in this paper. Apparently, the standard methods cannot be applied in the proof. We show that, for every fixed $p > 1$, the infimum ζ_p of the functional $\mathcal{F}_p(G)$, taken over every every function $G \in \mathcal{L}$, is actually attained at some function $G_\omega = G_\omega(t)$.

Theorem 3. *For every fixed* $p > 1$, *we have*

$$\zeta_p = \min \mathcal{F}_p(G_\omega),$$

where the minimum is taken over every ω *in the interval* $[0, \pi/2]$.

It reduces the problem of finding ζ_p to computation of minimum of the function $\mathcal{F}_p(G_\omega)$ in one variable in the interval. We give some numerical values of the computation in Section 10. From Corollary 1 and Corollary 3 it follows that Theorem 3 also holds for $p = 1$ (then $\omega = 0$) and for $p = \infty$ (then, respectively, $\omega = \pi/2$.) As p increases from 1 to ∞, the respective ω increases from 0 to $\pi/2$.

6. General observations.

Suppose that the function $\mathcal{F}(x_1, x_2, \ldots, x_m)$ in m variables is continuous in $(0,\infty)^m$. Which values are taken by the functional $\mathcal{F}(G) = \mathcal{F}(x_1(G), \ldots, x_m(G))$, where $G \in \mathcal{L}$? We call the functional continuous if it is a continuous function in $x_1, \ldots, x_m$. Let $V(\mathcal{F})$ denotes the set of the values of $\mathcal{F}(G)$ with $G \in \mathcal{L}$.

Observation 1. *For every continuous functional $\mathcal{F}$ the set $V(\mathcal{F})$ is connected.*

Proof. If $\mathcal{F}(G_1) = c$ and $\mathcal{F}(G_2) = d$, where say $c < d$, then $\mathcal{F}(\mu G_1 + (1-\mu)G_2)$ takes all values in the interval $[c, d]$, as μ runs over the intverval $[0, 1]$, because this function is continuous in μ. Also, $\mu G_1 + (1 - \mu)G_2$ remains in the same class of functions $\mathcal{L}$ as do G_1 and G_2, which completes the proof.

We say that the function $\mathcal{F}(x_1, x_2, \ldots, x_m)$ in m variables is positively homogenius if for every non–zero number y we have

$$\mathcal{F}(x_1 y, x_2 y, \ldots, x_m y) = y^r \mathcal{F}(x_1, x_2, \ldots, x_m)$$

with a positive number r. We call the functional positively homogenius if it is a positively homogenius function in $x_1, \ldots, x_m$.

Observation 2. *Suppose $\mathcal{F}$ is a positively homogenius continuous functional, and $c \in V(\mathcal{F})$, where $c \neq 0$. Then $[c, \infty) \in V(\mathcal{F})$ if $c > 0$ (or $(-\infty, c] \in V(\mathcal{F})$ if $c < 0$, respectively).*

Proof. There is no loss of generality to assume that $c > 0$, as the proof in other case follows by considering $-\mathcal{F}(G)$. Let G be the function at which $\mathcal{F}$ takes the value c. Take two points $t_1 < t_2$ in the interval $(0, 1)$ so that $G(t_1) < G(t_2)$. Suppose ε is a small positive constant and consider the function G_ε defined as follows: $G_\varepsilon(t) = G(t)$ outside the interval $[t_1, t_2]$; $G_\varepsilon(t)$ increases linearly on the interval $[t_1, t_1 + \varepsilon]$ from $G(t_1)$ to $G(t_2)$; $G_\varepsilon(t)$ is equal to $G(t_2)$ on $[t_1 + \varepsilon, t_2]$. Note that $I(G_\varepsilon)$ is greater than $I(G)$ and $I(G'_\varepsilon)$ tends to ∞ as ε tends to zero. It is also clear that if t_2 is sufficiently close to t_1 then $\mathcal{F}(G_\varepsilon) > 0$. By a standard epsilon delta argument, one can easily show that $\mathcal{F}(G_\varepsilon) \to \infty$ as $\varepsilon \to 0$. On the other hand, on applying Observation 1, we see the set $V(\mathcal{F})$ is connected, which proves the observation.

We clearly do need some condition on $\mathcal{F}$ except for continuity in Observation 2 as, for instance, the example of $\mathcal{F}(x_1, x_2, \ldots, x_m) = \sin(x_1 + \ldots + x_m)$ shows.

Observation 3. *Let c be a real number in the interval $(0, 1)$. For every function $G(t) \in \mathcal{L}$ let $G_c(t)$ be the function defined to be 0 on $[0, c]$, and $G\big((t-c)/(1-c)\big)$ on $[c, 1]$. Then $G_c(t) \in \mathcal{L}$, and $x_\lambda(G) = x_\lambda(G_c)$ for every λ.*

Proof. Note that $I(G_c^\lambda) = (1-c)I(G^\lambda)$ and $I(G_c'^2) = I(G'^2)/(1-c)$. Hence

$$x_\lambda(G_c) = I(G_c^\lambda)I(G_c'^2) = (1-c)I(G^\lambda)I(G'^2)/(1-c) = x_\lambda(G),$$

and the result follows (as the first claim is trivial).

The main consequence of this result is the following one. Suppose the function $G(t) \in \mathcal{L}$ is equal to zero in the interval $[0,c]$, and is positive in $(c,1]$. On applying the inverse transformation to that of Observation 3, we can replace G by another function from $\mathcal{L}$ which is positive in $(0,1]$, and which gives the same value of the functional x_λ at it (and so the same value of every functional $\mathcal{F}(x_1, x_2, \ldots, x_m)$) as is its value at G. We can therefore assume that G is always positive in $(0,1]$.

7. Proof of Theorem 1.

Lemma 1. (Causchy–Swarz [24]) *For any two functions f and g in the space $L^2[0,1]$ the inequality $I(|fg|)^2 \leqslant I(f^2)I(g^2)$ holds with equality if and only if there is a constant c such that $f(t) = cg(t)$ almost everywhere in $[0,1]$.*

On applying Lemma 1 to x_λ we deduce that

$$x_\lambda(G) = I(G^\lambda)I(G'^2) \geqslant I(G'G^{\lambda/2})^2.$$

Since

$$I(G'G^{\lambda/2}) = \int_0^1 G^{\lambda/2}dG = \frac{2}{\lambda+2}$$

for $\lambda > -2$ (and equals ∞ for $\lambda \leqslant -2$) we obtain the inequality as required.

The equality holds if almost everywhere in the interval $[0,1]$ we have

$$G'(t) = cG^{\lambda/2}(t)$$

with a constant c. Suppose that G' exists in $(0,1)$. After solving the above diferential equation and substituting $G(0)=0$, $G(1)=1$, we obtain the function $G(t) = t^{2/(2-\lambda)}$ if $-2 < \lambda < 2$. These are clearly not the only functions for which the equality holds (see Observation 3).

We will show now that there is no function $G \in \mathcal{L}$ for which the above differential equation holds almost everywhere if $\lambda \geqslant 2$. Suppose that there is one. By Observation 3, then there is another one, say G, which is positive on $(0,1]$, and for which the functional takes the same value $4/(\lambda+2)^2$. The latter satisfies the differential equation with some (may be different) c. This new c is positive, since $G(t) > 0$ on $(0,1]$. Consider first the case $\lambda = 2$. In the interval $(0,1]$ we define the function $h(t) = \log G(t)$. Then $h'(t) = c$ almost everywhere in $[0,1]$. Consequently,

$$-\log G(\varepsilon) = \log G(1) - \log G(\varepsilon) = \int_\varepsilon^1 h'(t)dt = c(1-\varepsilon).$$

The right–hand side here is bounded above by c, whereas the left–hand side tends to ∞ as $\varepsilon \to 0$, a contradiction. In the case $\lambda > 2$, one can argue similarly, by taking the function $h(t) = G(t)^{1-\lambda/2}$.

All said above implies that in the range $-2 < \lambda < 2$ the inequality is sharp and becomes an equality with, say, the function $G(t) = t^{2/(2-\lambda)}$. This also shows that the inequality in the range $\lambda \geqslant 2$ is strict. We need, therefore, to construct some sequence of functions in $\mathcal{L}$ to show that the strict inequality is sharp. In the case $\lambda = 2$ it suffices to take $G_n(t) = t^n$. Then $x_2(G_n) = n^2/(4n^2-1)$ tends to $1/4$ as $n \to \infty$.

Suppose that $\lambda > 2$. We define $G_n(t)$ to be $n^{(\lambda-4)/(\lambda-2)}t$ in the interval $[0, 1/n]$ and $G_n(t) = (n+1-nt)^{2/(2-\lambda)}$ in the interval $[1/n, 1]$. Then, by an easy computation,

$$I(G_n^\lambda) = \frac{\lambda-2}{(\lambda+2)n} - \frac{\lambda-2}{(\lambda+2)n^{2\lambda/(\lambda-2)}} + \frac{1}{(\lambda+1)n^{(3\lambda-2)/(\lambda-2)}},$$

$$I(G_n'^2) = \frac{4n}{\lambda^2-4} + n^{(\lambda-6)/(\lambda-2)} - \frac{4}{(\lambda^2-4)n^{4/(\lambda-2)}}.$$

So that the product of these, namely $x_\lambda(G_n)$, tends to $4/(\lambda+2)^2$ if $n \to \infty$, as required. This completes the proof.

8. Proof of Theorem 2.

We begin with case $b = 0$. If $a = 0$ then, clearly, $c = 0$. Let $a > 0$. From Theorem 1 and from Observation 2, it follows that $x_1(G)$ takes all values in the interval $[4/9, \infty)$. This proves that if $b = 0$ then only the alternatives for $(a, 0, c)$ to be an admissible triple are (i) and (ii).

Suppose now that $b = 1$. If $a = 0$ then, from Theorem 1 and from Observation 1 again, we obtain that $(0, 1, c)$ is an admissible triple if $c > 1/4$. This gives the alternative (iii). Let $a > 0$. On applying Lemma 1, we deduce that

$$ax_1(G) + x_2(G) = I(aG + G^2)I(G'^2) \geqslant I\big(G'\sqrt{aG+G^2}\big)^2.$$

By a standard computation of the integral,

$$I(G'\sqrt{aG+G^2}) = \int_0^1 \sqrt{aG+G^2}\,dG = \frac{(a+2)\sqrt{a+1} - a^2\log\big(\sqrt{1/a} + \sqrt{1+1/a}\big)}{4}.$$

It suffices to take the square of the last expression to obtain the lower bound for c as stated in (iv).

Let θ be a positive solution of the equation $\cosh\theta = 1 + 2/a$. Then with the function $G(t) = (a/2)(\cosh(\theta t) - 1)$ the inequality becomes an equality. Once again we remark that this choice is not unique. One can apply the inverse transformation to that of Observation 3 to obtain many such functions. On applying Observation 2, we complete the proof in the case $b = 1$.

Finally, let $b=-1$. If $a=0$ then we get the alternative (v) analogously to (iii). We postpone the less trivial case $a<1$ and suppose now that $a\geqslant 1$. On applying Lemma 1, we obtain

$$ax_1(G)-x_2(G)=I(aG-G^2)I(G'^2)\geqslant I\big(G'\sqrt{aG-G^2}\big)^2.$$

Once again, by a standard computation of the integral,

$$I(G'\sqrt{aG-G^2})=\int_0^1\sqrt{aG-G^2}dG=\frac{-2(a-2)\sqrt{a-1}+a^2\arccos(1-2/a)}{8},$$

and by taking a square, we obtain the lower bound as in (vii). With the function, for instance, $G(t)=(a/2)(1-\cos(\theta t))$, where $\theta=\arccos(1-2/a)$, the equality is true throughout. Thus, we have the alternative (vii), by Observation 2.

We are left with the case $0<a<1$. It suffices to show that $ax_1(G)-x_2(G)$ takes a positive and a negative value. Then, by Observations 1 and 2, this expression takes every real value. Let us consider the functions $G_n(t)=nt$ on the interval $[0,1/n]$ and $G_n(t)=1$ on $[1/n,1]$. Then, as $I(G_n)=1-1/(2n)$, $I(G_n^2)=1-2/(3n)$, and $I(G_n'^2)=n$, we obtain that

$$ax_1(G_n)-x_2(G_n)=(a-1)n-\frac{a}{2}+\frac{2}{3}<0$$

for n sufficiently large.

It suffices to prove that this expression also takes a positive value. Let us fix a positive number δ to be chosen later. Set

$$G_n(t)=t\exp\{-n(1-t)^\delta\}.$$

Clearly, $I(G_n'^2)>0$. Also, on replacing $n(1-t)^\delta$ by t, we obtain that

$$I(G_n)=\int_0^1 t\exp\{-n(1-t)^\delta\}dt=\frac{1}{\delta n^{1/\delta}}\int_0^n t^{1/\delta-1}\Big(1-\big(t/n\big)^{1/\delta}\Big)\exp\{-t\}dt.$$

It follows that

$$I(G_n)=\frac{\Gamma(1/\delta)(1+\varepsilon_n)}{\delta n^{1/\delta}},$$

where $\varepsilon_n\to 0$ as $n\to\infty$. Here Γ is the gamma function.

Also,

$$I(G_n^2)=\int_0^1 t^2\exp\{-2n(1-t)^\delta\}dt=\frac{1}{\delta(2n)^{1/\delta}}\int_0^{2n} t^{1/\delta-1}\Big(1-\big(t/(2n)\big)^{1/\delta}\Big)^2\exp\{-t\}dt.$$

Consequently,

$$I(G_n^2)=\frac{\Gamma(1/\delta)(1+\mu_n)}{\delta(2n)^{1/\delta}},$$

where $\mu_n \to 0$ as $n \to \infty$. It is now convenient to take δ so that

$$\delta < \frac{\log 2}{\log(1/4a)} < \frac{1}{2}.$$

Then

$$\frac{I(G_n)}{I(G_n^2)} > 2^{1/\delta - 1} > \frac{2}{a}$$

for n sufficiently large. It follows that $ax_1(G_n) - x_2(G_n) > x_2(G_n) > 0$, and the proof of Theorem 2 is completed.

9. Proof of Theorem 3.

On the set Ω, the quantity $I(G^2)/I(G)$ can only have values in the interval

$$\Big[\frac{3}{5}, \frac{3}{2} - \frac{1}{\pi - 2}\Big].$$

Indeed, the value at the left endpoint $3/5$ is attained at the function $G_0(t)$ whereas the value at the right endpoint

$$\nu = \frac{3}{2} - \frac{1}{\pi - 2} = 0.624030803\ldots$$

is attained at $G_{\pi/2}(t)$. The above claim now follows by continuity and from the direct (but cumbersome) proof that the function $I(G_\omega^2)/I(G_\omega)$ is increasing in ω. We will give the proof of this fact later in Lemma 4.

Set

$$H_p(\varrho) = \Big(\int_0^1 (1 - \varrho\tau)^{p/(p-1)} d\tau\Big)^{(p-1)/p} = \Big(\frac{(p-1)\big(1 - (1-\varrho)^{(2p-1)/(p-1)}\big)}{(2p-1)\varrho}\Big)^{(p-1)/p}.$$

Lemma 2. *For every fixed $p > 1$ the function $(1 - \varrho/2)/H_p(\varrho)$ is decreasing (in ϱ) in the interval $[0, 1]$.*

Proof. Note that

$$\frac{(p-1)2^{p/(p-1)}}{2p-1}\Big(\frac{1 - \varrho/2}{H_p(\varrho)}\Big)^{p/(p-1)} = \frac{(1+t)^\sigma(1-t)}{1 - t^{\sigma+1}},$$

where $\sigma = p/(p-1)$ and $1 - \varrho = t$. We need, therefore, to show that the function on the right–hand side is increasing in t. Its derivative is equal to

$$(1+t)^{\sigma-1}(1 - t^{\sigma+1})^2\Big((\sigma - 1)(1 - t^{\sigma+1}) - (\sigma + 1)(t - t^\sigma)\Big).$$

It is non–negative if

$$1-t^{\sigma+1}-\frac{\sigma+1}{\sigma-1}(t-t^{\sigma})\geqslant 0.$$

The latter function equals 1 at $t=0$ and equals 0 at $t=1$. Its derivative

$$-(\sigma+1)t^{\sigma}-\frac{\sigma+1}{\sigma-1}(1-\sigma t^{\sigma-1})$$

is non–positive on $[0,1]$ if $(\sigma-1)t^{\sigma}-\sigma t^{\sigma-1}+1$ is non–negative. This is the case, because the derivative of the latter $\sigma(\sigma-1)(t^{\sigma}-t^{\sigma-1})$ is non–positive and so it is decreasing from 1 to 0 on the interval $[0,1]$. This completes the proof of Lemma 2.

From now on, we consider the functional

$$\mathcal{F}_p(G)=\Big(\int_0^1(x_1(G)-\tau x_2(G))^{p/(p-1)}d\tau\Big)^{(p-1)/p}=x_1(G)H_p(I(G^2)/I(G)).$$

The functional satisfies the conditions of the observations, provided that $p>1$. Also, for every fixed number ϱ let $\mathcal{F}_p(G,\varrho)=x_1(G)H_p(\varrho)$. Then $\zeta_p=\inf\mathcal{F}_p(G,\varrho)$, where the infimum is taken over every number ϱ in $(0,1)$ and over every function $G\in\mathcal{L}$ such that $\varrho=I(G^2)/I(G)$. We will first show that for every ϱ either inside or outside the interval $[0.6,\nu]$ the values of $\mathcal{F}_p(G,\varrho)$ are greater than or equal to some value of the functional with ϱ in this interval and with the function G in Ω.

Lemma 3. *Suppose ϱ is a fixed real number in the interval $[3/5,\nu]$. Then there is an $\omega\in[0,\pi/2]$ such that*

$$\inf x_1(G)=x_1(G_\omega),$$

where the infimum is taken over every function $G\in\mathcal{L}$ such that $\varrho=I(G^2)/I(G)$.

Proof. Fix ϱ in the above interval and take an ω so that $\varrho=I(G_\omega^2)/I(G_\omega)$. Such ω exists, by to be proved later Lemma 4. From the proof of Theorem 2 (vii) on substituting $a=2/(1-\cos\omega)$ we now have that

$$x_1(G)\left(1-\frac{\varrho(1-\cos\omega)}{2}\right)\geqslant\frac{\big(2\omega-\sin(2\omega)\big)^2}{32(1-\cos\omega)^3}$$

with equality if $G=G_\omega$. This shows that for every fixed ϱ the infimum $\inf x_1(G)$ is attained at a function $G_\omega\in\Omega$, as claimed.

Fix a number ϱ, $0<\varrho<1$, and consider all functions $G\in\mathcal{L}$ so that $\varrho=I(G^2)/I(G)$. For every such a ϱ the function $H_p(\varrho)$ is increasing in p. Also, from integral representation of $H_p(\varrho)$ it easily seen that for every fixed p the function $H_p(\varrho)$ is decreasing in ϱ. If $\varrho<3/5$ then for every $G\in\mathcal{L}$, by Corollary 1, we have

$$\mathcal{F}_p(G,\varrho)=x_1(G)H_p(\varrho)\geqslant\frac{4H_p(\varrho)}{9}>\frac{4H_p(3/5)}{9}=\mathcal{F}_p(G_0,3/5).$$

On the other hand, if $\varrho > \nu$ then for every function $G \in \mathcal{L}$

$$\mathcal{F}_p(G, \varrho) = x_1(G)H_p(\varrho) > x_1(G_{\pi/2})H_p(\nu) = \mathcal{F}_p(G_{\pi/2}, \nu).$$

Indeed, if the opposite inequality is true, namely if there is a function $G \in \mathcal{L}$ such that

$$x_1(G)H_p(\varrho) \leqslant x_1(G_{\pi/2})H_p(\nu),$$

then, by Lemma 2, we would have that

$$x_1(G)(1 - \varrho/2) < x_1(G_{\pi/2})(1 - \nu/2).$$

Here, the right–hand side is equal to $x_1(G) - (1/2)x_2(G)$ and the left hand side is equal to $x_1(G_{\pi/2}) - (1/2)x_2(G_{\pi/2}) = \zeta_\infty$, a contradiction to Corollary 3.

We conclude that if ϱ is outside the interval $[0.6, \nu]$, then the infimum of $\mathcal{F}_p(G)$ is not attained. For every ϱ in the interval $[0.6, \nu]$ there is an ω in the interval $[0, \pi/2]$ such that $\varrho = I(G^2)/I(G)$ with $G \in \Omega$, and, vice versa, for every ω in $[0, \pi/2]$ there is a $\varrho = \varrho_\omega$ for which $\varrho_\omega = I(G^2)/I(G)$. By Lemma 3, for every ϱ in the interval $[0.6, \nu]$ the infimum of the functional $\mathcal{F}_p(G, \varrho)$ is actually the minimum and is attained at a certain function G from Ω. Supose now that the infimum of $\mathcal{F}_p(G)$, where $G \in \mathcal{L}$, is attained as the limit of some sequence of functions $\{G_n\}$. We already know that there is no generality to assume that $I(G_n^2)/I(G_n)$ is bounded below by 3/5 and bounded above by ν. Hence there is a subsequence (by abuse of notation, we denote it by $\{G_n\}$ again) such that

$$I(G_n^2)/I(G_n) \to \varrho^*,$$

as $n \to \infty$ with $\varrho^* \in [3/5, \nu]$. As we are interested in the infimum, there is no loss of generality again, to consider the functions such that, e.g., $x_1(G_n) \leqslant 10$. There is a number x_1^*, therefore, so that on a certain subsequence $\{G_n\}$ (here, we do not change the notation once again) $x_1(G_n) \to x_1^*$ as $n \to \infty$. Let ω^* be such that $\varrho^* = I(G_{\omega^*}^2)/I(G_{\omega^*})$. By Lemma 3, we have that

$$x_1(G_{\omega^*}) \leqslant x_1^*.$$

On the other hand, by the definition of x_1^* and ϱ^*, we have the inequality $x_1^* H_p(\varrho^*) \leqslant x_1(G_{\omega^*})H_p(\varrho^*)$. This is only possible if last two inequalities are both equalities, thus

$$\zeta_p = x_1^* H_p(\varrho^*) = x_1(G_{\omega^*})H_p(\varrho^*) = \min \mathcal{F}_p(G_\omega).$$

We now conclude the proof of Theorem 3, by establishing the following statement.

Lemma 4. *The function*

$$I(G_\omega^2)/I(G_\omega)$$

is increasing in ω *in the interval* $[0, \pi/2]$.

Proof of the lemma. By a standard computation of integrals, we find that

$$\frac{I(G_\omega^2)}{I(G_\omega)} = \frac{6\omega - 8\sin\omega + \sin(2\omega)}{4(1-\cos\omega)(\omega - \sin\omega)}.$$

To check that this formula is correct at $\omega = 0$, one may use the Taylor expansions of the involved trigonometric functions. The right–hand side here is equal to

$$\frac{3}{2(1-\cos\omega)} - \frac{\sin\omega}{2(\omega - \sin\omega)}.$$

The derivative of this expression is equal to

$$\frac{-3\sin\omega}{2(1-\cos\omega)^2} + \frac{\sin\omega - \omega\cos\omega}{2(\omega-\sin\omega)^2}.$$

It is positive if

$$(\sin\omega - \omega\cos\omega)(1-\cos\omega)^2 - 3\sin\omega(\omega-\sin\omega)^2 > 0$$

for $\omega \in (0, \pi/2]$. On dividing by the positive function $2\sin(\omega/2)$, we obtain the function

$$2(\sin\omega - \omega\cos\omega)\big(\sin(\omega/2)\big)^3 - 3(\omega - \sin\omega)^2\cos(\omega/2).$$

It is equal to zero at zero, so it suffices to show that its derivative, namely

$$\sin(\omega/2)\Big(\frac{9}{2} + \frac{3}{2}\omega^2 - 8\omega\sin\omega - \frac{5}{4}\omega\sin(2\omega) - \frac{9}{2}\cos(2\omega)\Big),$$

is positive on $(0, \pi/2]$.

We consider the Taylor series of the function

$$\phi(\omega) = \frac{9}{2} + \frac{3}{2}\omega^2 - 8\omega\sin\omega - \frac{5}{4}\omega\sin(2\omega) - \frac{9}{2}\cos(2\omega) := \sum_{k=0}^{\infty} a_k\omega^k$$

at $\omega = 0$, and show that the sign of ϕ in the interval $(0,3)$ is the same as that of the first non–zero coefficient. Using the expansions $\sin z = \sum_{k=1}^{\infty}(-1)^{k-1}z^{2k-1}/(2k-1)!$, and $\cos z = \sum_{k=1}^{\infty}(-1)^{k-1}z^{2k-2}/(2k-2)!$, we see that a_k with k odd are all equal to zero. Also, a_0 and a_2 are both equal to zero. For the remaining coefficients we have the formulae

$$a_{2k} = (-1)^{k-1}\Big(\frac{9}{k}4^{k-1} - 8 - \frac{5}{2}4^{k-1}\Big)/(2k-1)!.$$

Here, $k \geqslant 2$. One can now easily compute more coefficients like $a_4 = 0$, $a_6 = 0$, $a_8 = 1/210$, $a_{10} = -13/25200$, $a_{12} = 43/1663200$. The sign of a_{2k} is $(-1)^k$.

The function ϕ can, therefore, be written as follows

$$\phi(\omega) = \sum_{k=0}^{\infty} a_k\omega^k = \frac{\omega^8}{210} - \frac{13\omega^{10}}{25200} + \frac{43\omega^{12}}{1663200} + \sum_{k=7}^{\infty} a_{2k}\omega^{2k}.$$

Note that

$$4^{k-1} < (5/2)4^{k-1} + 8 - (9/k)4^{k-1} < 4^k$$

if $k \geqslant 6$. Since $0 < \omega < 3$, we deduce the following inequality for the absolute values in the range $k \geqslant 6$:

$$|a_{2k}\omega^{2k}| = (-1)^k a_{2k}\omega^{2k} > \frac{4^{k-1}\omega^{2k}}{(2k-1)!} > \frac{4^{k+1}\omega^{2k+2}}{(2k+1)!} > |a_{2k+2}\omega^{2k+2}|,$$

because $12 \times 13 = 156 > 144 = 16 \times 9$. This inequality is also true for $k = 4$, as $1/210 > (9 \times 13)/25200$, and for $k = 5$, as $13/25200 > (9 \times 43)/1663200$. The function $\phi(\omega)$ is, therefore, positive in the interval $(0, 3)$. The latter contains the interval $(0, \pi/2)$, thus the proof of Lemma 4 is completed.

10. Computations.

Here, we give the table of computations of the minimum of the functional $\mathcal{F}_p$, by Theorem 3. The values presented in the table are those of $2/\zeta_p$, which is more convenient for the main application: the numerical value in estimate of $M_p(\alpha)$ from below (see Section 2). All 10 given decimal digits for the value of ζ_p are true (as well as are those for the extremal value of the respective $\omega = \omega_p$). The obtained function is rapidly increasing from 4.5 to $64/\pi^2 = 6.4845557\ldots$. Recall that the extremal function can be obtained by substituting ω into the formula $G_\omega(t) = \big(1 - \cos(\omega t)\big)/\big(1 - \cos\omega\big)$.

$p :=$	$\omega :=$	$2/\zeta_p :=$
1.00	0.0000000000	4.5000000000
1.0001	0.0258179014	4.5039161853
1.01	0.2562351948	4.6871099991
1.05	0.5563990344	5.0888506114
1.10	0.7602978811	5.3866751418
1.20	1.0038538044	5.7246332387
1.30	1.1433469232	5.9086733046
1.40	1.2304592813	6.0225832793
1.50	1.2890206221	6.0994671085
1.60	1.3307724922	6.1546640810
1.70	1.3619264506	6.1961422045
1.80	1.3860135013	6.2284183470
1.90	1.4051699438	6.2542332844
2.00	1.4207572438	6.2753424528
3.00	1.4936805684	6.3755271804
4.00	1.5189484290	6.4108669595
5.00	1.5317510186	6.4289105418
6.00	1.5394848289	6.4398572661
7.00	1.5446620536	6.4472055338
8.00	1.5483703914	6.4524790707

9.00	1.5511572347	6.4564477950
10.0	1.5533280681	6.4595426331
20.0	1.5624996028	6.4726509701
50.0	1.5675745301	6.4799275275
99.9	1.5691993484	6.4822607841

From the table we see that, for instance,

$$\Big(\frac{1}{d}\sum_{k=1}^{d}\big|\log|\alpha_k|\big|^p\Big)^{1/p} > \frac{6}{d}\Big(\frac{\log\log d}{\log d}\Big)^3$$

if $p \geqslant 1.4$.

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