CHERN CLASSES OF BLOW-UPS

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Abstract. We extend the classical formula of Porteous for blowing-up Chern classes to the case of blow-ups of possibly singular varieties along regularly embedded centers. The proof of this generalization is perhaps conceptually simpler than the standard argument for the nonsingular case, involving Riemann-Roch without denominators. The new approach relies on the explicit computation of an ideal, and a mild generalization of the well-known formula for the normal bundle of a proper transform ([Ful84], B.6.10).

We also discuss alternative, very short proofs of the standard formula in some cases: an approach relying on the theory of Chern-Schwartz-MacPherson classes (working in characteristic 0), and an argument reducing the formula to a straightforward computation of Chern classes for sheaves of differential 1-forms with logarithmic poles (when the center of the blow-up is a complete intersection).

1. Introduction

1.1. A general formula for the Chern classes of the tangent bundle of the blow-up of a nonsingular variety along a nonsingular center was conjectured by J. A. Todd and B. Segre, who established several particular cases ([Tod41], [Seg54]). The formula was eventually proved by I. R. Porteous ([Por60]), using Riemann-Roch. F. Hirzebruch's summary of Porteous' argument in his review of the paper (MR0121813) may be recommend for a sharp and lucid account. For a thorough treatment, detailing the use of Riemann-Roch 'without denominators', the standard reference is §15.4 in [Ful84]. Here is the formula in the notation of the latter reference. For any non-singular variety X, write $c(X)$ for $c(T_X)\cap[X]$, the total Chern class (in the Chow group of X) of the tangent bundle of X. Let $X \subseteq Y$ be nonsingular varieties, and let $\widetilde{Y}$ be the blow-up of Y along X, with exceptional divisor $\widetilde{X}$:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{j} & \widetilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

In this situation, both $\widetilde{X}$ and $\widetilde{Y}$ are nonsingular. Let N be the normal bundle to X in Y, of rank d; identify $\widetilde{X}$ with the projectivization of N, and let $\zeta = c_1(\mathscr{O}_N(1))$. Then (Theorem 15.4 in [Ful84]):

$$c(\widetilde{Y}) - f^*c(Y) = j_*(g^*c(X)\cdot\alpha) \quad ,$$

where

$$\alpha = \frac{1}{\zeta}\left[\sum_{i=0}^{d} g^*c_{d-i}(N) - (1-\zeta)\sum_{i=0}^{d}(1+\zeta)^i g^*c_{d-i}(N)\right] \quad .$$

Proofs of this formula that do not use Riemann-Roch were found by A. T. Lascu and D. B. Scott ([LS75], [LS78]). In [LS78], Lascu and Scott write: *"In this paper we give a simple (and we hope definitive) proof of the result using only simple arguments with vector bundles and some straightforward manipulations."*

1.2. In this paper we go one step beyond the work of Lascu and Scott, and prove 'by simple arguments' a somewhat stronger result than the formula recalled in §1.1. Our general aim is to remove the nonsingularity hypothesis on X and Y; this is what we accomplish, pushing the level of generality to that of any *regular embedding* of schemes $X \subseteq Y$.

As long as X is regularly embedded in Y, the blow-up of Y along X can be regularly embedded into a projective bundle[1] $P(E)$ over Y (see §3.1):

$$\widetilde{Y} \subseteq P(E) \quad .$$

The main result of this paper is the computation of the Chern classes of the normal bundle $N_{\widetilde{Y}}$ of this embedding. In case X and Y (and hence $\widetilde{X}$ and $\widetilde{Y}$) are nonsingular, the Chern classes of $T_{\widetilde{Y}}$ are immediately computed from the classes of $N_{\widetilde{Y}}$ and of $T_{P(E)}$; this recovers the formula recalled in §1.1 (cf. §1.6). The new proof of this formula appears to us simpler than either the approach via Riemann-Roch or the proof found by Lascu and Scott.

1.3. Let Y be a scheme (pure dimensional, separated, of finite type over a field, and admitting a closed embedding into a nonsingular scheme).

To state the result, assume first that X is a complete intersection in Y: that is, X is the zero-scheme of a regular section of a bundle $\hat{N}$ *on* Y of rank equal to the codimension of X. (Thus, the normal bundle N to X in Y is isomorphic to the restriction of $\hat{N}$ to X.) Let $f : \widetilde{Y} \to Y$ be the blow-up of Y along X, and let $\widetilde{X}$ be the exceptional divisor. Let E be a vector bundle on Y, containing $\hat{N}$, and let $\hat{C}$ be the quotient:

$$0 \longrightarrow \hat{N} \longrightarrow E \longrightarrow \hat{C} \longrightarrow 0 \quad .$$

There are natural embeddings $\widetilde{Y} \hookrightarrow P(\hat{N}) \hookrightarrow P(E)$, which are easily seen to be regular (Lemma 2.1); and $\mathscr{O}(\widetilde{X})$ is realized as the restriction of the universal subbundle $\mathscr{O}(-1)$, hence $\mathscr{O}(\widetilde{X}) \subseteq f^*(\hat{N})$. We prove:

Lemma 1.1. *With notation as above, let $N_{\widetilde{Y}}$ be the normal bundle of $\widetilde{Y}$ in $P(E)$. Then there is an exact sequence*

$$0 \longrightarrow f^*(\hat{N})/\mathscr{O}(\widetilde{X}) \longrightarrow N_{\widetilde{Y}} \longrightarrow f^*(\hat{C}) \otimes \mathscr{O}(-\widetilde{X}) \longrightarrow 0 \quad .$$

In particular,

$$c(N_{\widetilde{Y}}) = \frac{c(f^*(\hat{N}))\, c(f^*(\hat{C}) \otimes \mathscr{O}(-\widetilde{X}))}{c(\mathscr{O}(\widetilde{X}))} \quad .$$

[1]As in [Ful84], $P(E)$ denotes the projective bundle of lines in the vector bundle E.

1.4. In general, let $X \hookrightarrow Y$ be a regular embedding, but not necessarily a complete intersection. It is still the case (cf. [Ful84], B.8.2) that X can be expressed as the zero-scheme of a section of a bundle E on Y, and there is an exact sequence of vector bundles *on* X*:*

$$0 \longrightarrow N \longrightarrow E|_X \longrightarrow C \longrightarrow 0$$

where N is the normal bundle of X in Y, $E|_X$ denotes the restriction of E to X, and C is the quotient. The blow-up $\widetilde{Y}$ of Y along X still embeds regularly in $P(E)$ (§3.1). The general result is as follows; parsing the formula requires some considerations, which follow the statement.

Theorem 1.2. *With notation as above, let* $N_{\widetilde{Y}}$ *be the normal bundle of* $\widetilde{Y}$ *in* $P(E)$*. Then*

$$c(N_{\widetilde{Y}}) = \frac{c(\mathbb{N})\, c(\mathbb{C} \otimes \mathscr{O}(-\widetilde{X}))}{c(\mathscr{O}(\widetilde{X}))} \quad ,$$

where $\mathbb{N}$, $\mathbb{C}$ *evaluate to the the pull-backs of* N*, resp.* C*.*

1.5. Parsing. The terms $\mathbb{N}$, $\mathbb{C}$ appearing in the statement should be understood as indeterminates with respect to which the right-hand-side can be expanded; the terms in the expansion can then be interpreted (by relating $\mathbb{N}$ to N and $\mathbb{C}$ to C), determining a well-defined operator on the Chow group $A_*(\widetilde{Y})$. The content of the theorem is that this operator equals $c(N_{\widetilde{Y}})$.

Here are the details of this operation. Expanding the expression gives

$$\frac{c(\mathbb{N})\, c(\mathbb{C} \otimes \mathscr{O}(-\widetilde{X}))}{c(\mathscr{O}(\widetilde{X}))} = c(\mathbb{N})\, c(\mathbb{C}) + (\cdots)\, \zeta$$

where $\zeta = c_1(\mathscr{O}(-\widetilde{X}))$, and the term $(\cdots)\,\zeta$ collects monomials $c_i(\mathbb{N})\, c_j(\mathbb{C})\, \zeta^k$ with $k \geq 1$. We prescribe that the first term should act on a class $a \in A_*(\widetilde{Y})$ as the pull-back of E:

$$c(\mathbb{N})\, c(\mathbb{C})\, (a) := f^* c(E) \cap a \quad .$$

As for the remaining terms $c_i(\mathbb{N})\, c_j(\mathbb{C})\, \zeta^k$: if $a \in A_* \widetilde{Y}$, then $\zeta^k \cap a$ is supported on $\widetilde{X}$ for $k \geq 1$, and hence $c_i(g^*N)\, c_j(g^*C) \zeta^k \cap (a)$ makes sense as a class in $A_*(\widetilde{X})$, and determines (by proper push-forward) a class in $A_*(\widetilde{Y})$. We prescribe

$$c_i(\mathbb{N})\, c_j(\mathbb{C})\, \zeta^k(a) := j_*(c_i(g^*N)\, c_j(g^*C) \zeta^k \cap (a)) \quad .$$

In a nutshell, we want to think of $\mathbb{N}$ and $\mathbb{C}$ as pull-backs of make-believe extensions to Y of N and C. If N happens to be the restriction of a bundle $\hat{N}$ (as in the complete intersection case), $\hat{N} \subseteq E$, and $\hat{C} = E/\hat{N}$, then setting $\mathbb{N} = f^*(\hat{N})$ and $\mathbb{C} = f^*(\hat{C})$ leads to the formula presented above. It is a lucky circumstance that the formula can be given a meaning even when N is *not* the restriction of a bundle defined on Y, and an even luckier circumstance that the interpreted formula still computes the Chern class of the normal bundle to the blow-up in the ambient projective bundle.

1.6. In the particular case when X and Y are nonsingular, Theorem 1.2 implies the formula recalled in §1.1. To see this, note that

$$c(T_{P(E)}|_{\widetilde{Y}}) = c(f^*E \otimes \mathscr{O}(1))\, c(f^*T_Y)$$

if Y is nonsingular, by standard facts (for example, see B.5.8 in [Ful84]). Using the same parsing convention as in the statement of Theorem 1.2, this equals

$$c(\mathbb{N} \otimes \mathscr{O}(1))\, c(\mathbb{C} \otimes \mathscr{O}(1))\, c(f^*T_Y) \quad ,$$

and we get (with $\zeta = c_1(\mathscr{O}(1))$):

$$c(\widetilde{Y}) = \frac{c(T_{P(E)}|_{\widetilde{Y}})}{c(N_{\widetilde{Y}})} \cap [\widetilde{Y}] = \frac{(1-\zeta)\, c(\mathbb{N} \otimes \mathscr{O}(1))}{c(\mathbb{N})} \cap f^*c(Y) \quad .$$

This formula still uses the same convention: expand

$$\frac{(1-\zeta)\, c(\mathbb{N} \otimes \mathscr{O}(1))}{c(\mathbb{N})} = 1 + \zeta(\cdots) \quad ;$$

replacing $c_i(\mathbb{N})$ by $g^*c_i(N)$ as explained above and capping against $f^*c(Y)$ gives a class in $A_*(\widetilde{Y})$. It is now easy to check that this recovers on the nose the formula given in §1.1. The push-forward j_* is responsible for the extra factor $-1/\zeta$.

1.7. If Y is singular, but still a local complete intersection in a nonsingular ambient variety M, then it admits a 'virtual tangent bundle' T_Y^{vir} (defined in K-theory as the difference between the restriction of T_M and the normal bundle of Y in M, see B.7.6 in [Ful84]). Thus, Y has well-defined Chern classes $c(Y) := c(T_Y^{\text{vir}})$. In this case X and $\widetilde{Y}$ are also local complete intersections, and it is an easy consequence of Theorem 1.2 that the formula given in §1.1 holds if one uses these virtual Chern classes throughout.

However, Theorem 1.2 is more general than this statement, since it poses no restrictions on how singular Y may be.

There are other notions of 'Chern classes for singular varieties', generalizing the nonsingular case, such as the Chern-Schwartz-MacPherson class c_{SM} mentioned below. It would be valuable to have formulas controlling the behavior of these classes under blow-ups at the level of generality considered in this paper.

1.8. As mentioned above, the proof of Theorem 1.2 appears to us simpler than other approaches to the classical (and less general) formula. Lemma 1.1 is a straightforward exercise; the extension from the complete intersection case to the general case follows from a mild generalization of a standard computational tool, namely B.6.10 in [Ful84]. On the other hand, it is worth noting that this generalization (proved in §4) ultimately relies on the technique known as *deformation to the normal cone;* this is the technical tool behind the proofs found by Lascu and Scott, as well as one of the main approaches to the proof of Riemann-Roch. In fact, the reader may want to compare the 'short' version of the proof given in §4.5, which assumes familiarity with the deformation to the normal cone, and the detailed version given in §4.6. The details in §4.6 are just as demanding as in the paper of Lascu and Scott.

Thus, it may be argued that these proofs of the blowing-up Chern class formula are all different variations on the same theme. Theorem 1.2 is a variation that happens to work under the only requirement that X be regularly embedded in Y.

1.9. The blowing-up Chern class formula has been used for calculations in string theory (see for example [AC99]); however, some of my physicists acquaintances have expressed the opinion that the form recalled here in §1.1 is difficult to apply, and its proof through Riemann-Roch is somewhat obscure. I will close this introduction by giving two short independent proofs of important particular cases, which to my knowledge are not available in the literature. The formulation given in Lemma 1.3 may be more user-friendly than the formula given in §1.1.

This subsection is independent of the rest of the paper, and (unlike the rest) is limited to the case in which X and Y are nonsingular.

1.9.1. *Complete intersection, nonsingular case.*

Lemma 1.3. *Let $X \subseteq Y$ be nonsingular varieties. If X is a complete intersection of d nonsingular hypersurfaces $Z_1, \dots, Z_d$ meeting transversally in Y, then*

$$(*) \qquad c(T_{\widetilde{Y}}) = \frac{(1+\widetilde{X})(1+f^*Z_1-\widetilde{X})\cdots(1+f^*Z_d-\widetilde{X})}{(1+f^*Z_1)\cdots(1+f^*Z_d)} \cdot f^*c(T_Y) \quad .$$

Proof. By hypothesis, $Z = Z_1 \cup \cdots \cup Z_d$ is a divisor with simple normal crossings in Y, and it is easily checked that the divisor W consisting of the exceptional divisor $\widetilde{X}$ and of the proper transforms W_i of Z_i is a divisor with simple normal crossings in $\widetilde{Y}$. We therefore have bundles of tangent fields with logarithmic zeros (dual to the bundle of differential forms with logarithmic poles) $T_Y(-\log Z)$, resp. $T_{\widetilde{Y}}(-\log W)$ on Y, resp. $\widetilde{Y}$. Comparing sections shows that $T_{\widetilde{Y}}(-\log W) = f^*T_Y(-\log Z)$, and hence

$$c(T_{\widetilde{Y}}(-\log W)) = f^*c(T_Y(-\log Z))$$

by the functoriality of Chern classes. Chern classes of bundles of tangent fields with logarithmic zeros are well-known (see e.g. Lemma 3.8 in [Alu05]); we get

$$\frac{c(T_{\widetilde{Y}})}{(1+\widetilde{X})(1+f^*Z_1-\widetilde{X})\cdots(1+f^*Z_d-\widetilde{X})} = \frac{f^*c(T_Y)}{(1+f^*Z_1)\cdots(1+f^*Z_d)} \quad ,$$

from which (*) follows immediately. □

Expanding formula (*), and keeping in mind that $c_i(N_X)$ is the i-th elementary symmetric function in $Z_1, \dots, Z_d$, one gets precisely the terms in the standard formulation presented in §1.1. In this sense, while Lemma 1.3 has a more limited scope (X has to be a complete intersection), (*) may serve as mnemonics for the classical general formula, and has a completely transparent proof.

Remark 1.4. Lemma 1.3 is a particular case of the following interesting fact. Let $Z := \sum Z_i$ be a divisor with normal crossings and nonsingular components Z_i in a nonsingular variety Y. Say that a blow-up of Y is 'adapted to Z' if its center is the intersection of any collection of the components Z_i. It is easily checked that in this case the exceptional divisor, together with the proper transforms of the components of Z, form a divisor with simple normal crossings in the blow-up. Say that a sequence of blow-ups over Y is 'adapted to Z' if the first blow-up is adapted to Z, the second is adapted to the new normal crossing divisor, etc.

Arguing as in Lemma 1.3, one sees that if $\pi : \widetilde{Y} \to Y$ is any adapted sequence of blow-ups with respect to any divisor Z with simple normal crossings in Y, then $\pi^* c_{\rm SM}(1\!\!1_U) = c_{\rm SM}(1\!\!1_{\widetilde{U}})$, where U is the complement of Z in Y and $\widetilde{U}$ is the complement of $\pi^{-1}(Z)$ in $\widetilde{Y}$, and $c_{\rm SM}$ denotes the 'Chern class for constructible functions' discussed below. (The $c_{\rm SM}$ class of the complement of a normal crossing divisor is computed by the Chern class of the tangent bundle with logarithmic zeros along the divisor, see e.g. [Alu99], §2.) By standard functoriality properties of $c_{\rm SM}$, this formula holds as soon as $\pi : \widetilde{Y} \to Y$ is a proper map dominated by a sequence of adapted blow-ups. See [AM08], §4, for a more extensive discussion, and for an application.

1.9.2. *Characteristic zero, nonsingular case.* Over an algebraically closed field of characteristic zero, the formula of §1.1 admits a very quick proof, without the complete intersection hypothesis of §1.9.1, if one takes for granted the theory of Chern classes for (possibly) singular varieties developed by Robert MacPherson[2] in [Mac74]; see [Ful84], §19.1.7 for a version adapted to the Chow group. According to this theory, there are 'Chern classes' defined for every constructible function on a variety, such that the Chern class of the constant function 1 on a nonsingular variety equals the total Chern class of the tangent bundle. These classes are covariant with respect to a push-forward of constructible functions defined by taking Euler characteristics of fibers. The theory is developed in characteristic 0; the basic covariance property does not extend to positive characteristic (see §5.2 in [Alu06]).

In the case of a blow-up map $f : \widetilde{Y} \to Y$ of a nonsingular variety Y along a codimension d nonsingular subvariety X, the Euler characteristic[3] of the fibers is

$$\chi(f^{-1}(p)) = \begin{cases} 1 & p \notin X \\ d & p \in X \end{cases} \quad ;$$

it follows that

$$f_*(1\!\!1_{\widetilde{Y}}) = 1\!\!1_Y + (d-1) 1\!\!1_X \quad ,$$

where $1\!\!1$ denotes the constant function 1 on the given locus. The covariance of Chern classes proved by MacPherson implies then

$$f_*(c(T_{\widetilde{Y}}) \cap [\widetilde{Y}]) = c(T_Y) \cap [Y] + (d-1)\, i_* c(T_X) \cap [X] \quad , \tag{1}$$

where i is the inclusion $X \hookrightarrow Y$.

On the other hand, it is easy to evaluate the restriction of $c(T_{\widetilde{Y}})$ to $\widetilde{X}$:

$$j^* c(T_{\widetilde{Y}}) = (1 + \widetilde{X})\, c(T_{\widetilde{X}}) = (1 - \zeta)\, c(g^* N_X \otimes \mathscr{O}(1))\, c(g^* T_X) \quad , \tag{2}$$

using the identification $\widetilde{X} \cong P(N_X)$, and with $\zeta = \mathscr{O}(-\widetilde{X})$.

Lemma 1.5. *The class $c(T_{\widetilde{Y}})$ is characterized by formulas (1) and (2).*

[2] These classes are known to coincide, *mutatis mutandis,* with the classes defined earlier by M.-H. Schwartz, see [BS81].

[3] This is the conventional topological Euler characteristic if the ground field is $\mathbb{C}$, and a suitable adaptation over other algebraically closed fields of characteristic zero.

Indeed, *every* class in the Chow group of $\widetilde{Y}$ is characterized by its push-forward to Y and its restriction to $\widetilde{X}$ ([Ful84], Proposition 6.7 (d)). It is now a simple exercise (left to the reader) to check that the formula for $c(\widetilde{Y})$ stated in §1.1 satisfies both (1) and (2), and by Lemma 1.5 this suffices to prove the blowing-up Chern class formula.

1.10. In [LS76], Lascu and Scott propose a simplification of the blow-up formula of §1.1, that is equivalent to the formula given here in Lemma 1.3. However, they obtain this simpler formula as a corollary of their blow-up formula; the proof of Lemma 1.3 given in §1.9.1 is independent (and essentially immediate).

In [GP07], Hansjörg Geiges and Federica Pasquotto extend the classic blow-up formula of §1.1 to the case of symplectic, complex, and real manifolds; their method follows closely the proof of Lascu and Scott in [LS78], whose algebro-geometric ingredients they transfer to the topological environment.

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2. Proof of Lemma 1.1

2.1. We use notation as in §1.3: X, Y are pure dimensional separated schemes of finite type over a field; Y admits a closed embedding into a nonsingular scheme. We assume that X is a complete intersection in Y of codimension d, given as a the zero-scheme of a regular section of a vector bundle $\hat{N}$ of rank d; $f : \widetilde{Y} \to Y$ is the blow-up of Y along X. An exact sequence

$$0 \longrightarrow \hat{N} \longrightarrow E \longrightarrow \hat{C} \longrightarrow 0 \quad .$$

of vector bundles is given on Y. The embedding $\hat{N} \hookrightarrow E$ gives an embedding of projective bundles

$$\begin{array}{ccc} P(\hat{N}) & \hookrightarrow & P(\hat{E}) \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & Y & \end{array}$$

with normal bundle $\pi^*(\hat{C}) \otimes \mathscr{O}(1)$.

The section of $\hat{N}$ defining X corresponds to a map

$$\mathscr{O} \hookrightarrow \mathscr{N}$$

to the sheaf of sections of $\hat{N}$; dualizing this map gives a surjection

$$\mathscr{N}^\vee \twoheadrightarrow \mathscr{I}$$

onto the ideal sheaf $\mathscr{I}$ of X in Y. Taking Proj of Sym gives an embedding

$$\mathrm{Proj}(\mathrm{Sym}^* \mathscr{I}) \hookrightarrow P(\hat{N}) \quad .$$

Now $\mathrm{Sym}^* \mathscr{I}$ equals the Rees algebra of $\mathscr{I}$, since the embedding of X in Y is regular. Thus, $\mathrm{Proj}(\mathrm{Sym}^* \mathscr{I}) = \widetilde{Y}$, and we have fiberwise linear embeddings

$$\widetilde{Y} \overset{\iota}{\hookrightarrow} P(\hat{N}) \hookrightarrow P(E) \quad .$$

Lemma 2.1. *The embedding ι is regular.*

Proof. The matter is local, so we may assume that $Y = \operatorname{Spec} A$, and that the map $\mathscr{N}^\vee \to \mathscr{O}$ corresponds to a map $A^{\oplus d} \to A$, where d is the codimension of X in Y. By assumption X is regularly embedded in Y, hence the image of this map is an ideal $(a_1, \dots, a_d)$ generated by a regular sequence in A. The blow-up $\widetilde{Y}$ is defined by the equations $a_iT_j - a_jT_i$, $1 \le i < j \le d$, in $Y \times \mathbb{P}^{d-1} = Y \times P(\hat{N})$ ([Ful84], Lemma A.6.1). On the open set U_d of $P(\hat{N})$ defined by $T_d \neq 0$, the ideal of the blow-up is

$$(a_1 - a_dx_1, \dots, a_{d-1} - a_dx_{d-1}) \quad ,$$

where $x_i = T_i/T_d$. Thus $\widetilde{Y} \cap U_d$ is a complete intersection in U_d. The situation is of course analogous on all open charts $U_k = \{T_k \neq 0\}$. The statement follows. □

Note that the tautological line bundle $\mathscr{O}(-1)$ on $P(E)$ restricts to its namesakes on $P(\hat{N})$, on $\widetilde{Y} = \operatorname{Proj}(\operatorname{Sym}^* \mathscr{I})$, and on the exceptional divisor $\widetilde{X} = P(N_XY)$. Further, $\mathscr{O}(\widetilde{X}) \cong \mathscr{O}(-1)|_{\widetilde{Y}}$; this determines an embedding of $\mathscr{O}(\widetilde{X})$ in $\iota^*\pi^*(\hat{N}) = f^*(\hat{N})$.

2.2. At this point we have maps as in the commutative diagram:

$$\begin{array}{ccccc} \widetilde{Y} & \stackrel{\iota}{\hookrightarrow} & P(\hat{N}) & \hookrightarrow & P(E) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} & \swarrow & \\ & & Y & & \end{array}$$

The regular embeddings $\widetilde{Y} \hookrightarrow P(\hat{N}) \hookrightarrow P(E)$ yield an exact sequence of normal bundles

$$0 \longrightarrow N_{\widetilde{Y}}P(\hat{N}) \longrightarrow N_{\widetilde{Y}}P(E) \longrightarrow \iota^*N_{P(\hat{N})}P(E) \longrightarrow 0 \quad .$$

Letting $N_{\widetilde{Y}}$ denote $N_{\widetilde{Y}}P(E)$ as in Lemma 1.1, this is

$$0 \longrightarrow N_{\widetilde{Y}}P(\hat{N}) \longrightarrow N_{\widetilde{Y}} \longrightarrow f^*(\hat{C}) \otimes \mathscr{O}(1) \longrightarrow 0 \quad ,$$

and in order to prove Lemma 1.1 it suffices to verify the following:

Lemma 2.2.

$$N_{\widetilde{Y}}P(\hat{N}) \cong f^*(\hat{N})/\mathscr{O}(\widetilde{X}) \quad .$$

2.3. Proof of Lemma 2.2. Let $\mathscr{K}$ be the kernel of the surjection $\mathscr{N}^\vee \to \mathscr{I}$. Taking Sym, we obtain the exact sequence

$$0 \longrightarrow \mathscr{K} \cdot \operatorname{Sym}^{*-1}\mathscr{N}^\vee \longrightarrow \operatorname{Sym}^*\mathscr{N}^\vee \longrightarrow \oplus_{k\ge 0}\mathscr{I}^k \longrightarrow 0$$

determining the ideal of $\widetilde{Y}$ in $P(\hat{N})$; it follows that the conormal sheaf to $\widetilde{Y}$ in $P(\hat{N})$ is $f^*\mathscr{K} \otimes \mathscr{O}(-1)$.

Pulling back to $\widetilde{Y}$ the sequence $0 \to \mathscr{K} \to \mathscr{N}^\vee \to \mathscr{O}_Y \to \mathscr{O}_X \to 0$, we get the sequence

$$(*) \qquad 0 \longrightarrow f^*(\mathscr{K}) \longrightarrow f^*(\mathscr{N}^\vee) \longrightarrow \mathscr{O}_{\widetilde{Y}} \longrightarrow \mathscr{O}_{\widetilde{X}} \longrightarrow 0 \quad ,$$

and I claim that replacing $f^*(\mathscr{K})$ by $f^*(\mathscr{K})\otimes\mathscr{O}(-1)$ in this sequence produces an *exact* sequence on $\widetilde{Y}$:

$$(**)\qquad 0\longrightarrow f^*(\mathscr{K})\otimes\mathscr{O}(-1)\longrightarrow f^*(\mathscr{N}^\vee)\longrightarrow \mathscr{O}_{\widetilde{Y}}\longrightarrow \mathscr{O}_{\widetilde{X}}\longrightarrow 0 \quad .$$

Indeed, use again the local presentation obtained in the proof of Lemma 2.1: we start from the exact sequence

$$0\longrightarrow (a_iT_j-a_jT_i)_{1\le i<j\le d}\longrightarrow A^{\oplus d}\longrightarrow A\longrightarrow A/(a_1,\dots,a_d)\longrightarrow 0$$

where T_i is the generator of the i-th factor in the middle, and $A^{\oplus d}\to A$ is defined by $T_i\mapsto a_i$; the fact that the kernel is as stated follows from the fact that $(a_1,\dots,a_d)$ is regular. Pull-back to a representative chart in the blow-up by tensoring by

$$B=\frac{A[x_1,\dots,x_{d-1}]}{(a_1-a_dx_1,\dots,a_{d-1}-a_dx_{d-1})}\quad :$$

this yields the sequence corresponding to (*):

$$0\longrightarrow (a_d(T_i-x_iT_d))_{1\le i<d}\longrightarrow B^{\oplus d}\longrightarrow B\longrightarrow B/(a_d)\longrightarrow 0$$

The morphism $B^{\oplus d}\to B$ is still defined by $T_i\mapsto a_i$, and its kernel is easily checked to be $(T_i-x_iT_d)_{1\le i<d}$ (a_d is a non-zero-divisor in B). We see that this differs from $f^*\mathscr{K}$ by the presence in the latter of the extra factor of a_d. As a_d is a section of $\mathscr{O}(1)$, dividing by a_d corresponds to tensoring by $\mathscr{O}(-1)$, and this concludes the verification that the sequence (**) is exact.

Now rewrite (**) as the exact sequence of locally free sheaves on $\widetilde{Y}$:

$$0\longrightarrow f^*(\mathscr{K})\otimes\mathscr{O}(-1)\longrightarrow f^*(\mathscr{N}^\vee)\longrightarrow \mathscr{O}(-\widetilde{X})\longrightarrow 0 \quad .$$

Dualizing, and using the identification of $f^*(\mathscr{K})\otimes\mathscr{O}(-1)$ with the conormal sheaf to $\widetilde{Y}$ in $P(\hat{N})$, gives the exact sequence of vector bundles on $\widetilde{Y}$:

$$0\longrightarrow \mathscr{O}(\widetilde{X})\longrightarrow f^*(\hat{N})\longrightarrow N_{\widetilde{Y}}P(\hat{N})\longrightarrow 0 \quad ,$$

concluding the proof of Lemma 2.2. □

As noted above, Lemma 1.1 follows from Lemma 2.2. Thus, we have now established Theorem 1.2 under the hypothesis that X is a complete intersection in Y.

3. Proof of Theorem 1.2

3.1. Let $X\hookrightarrow Y$ be a regular embedding. As recalled in §1.4, we can express X as the zero-scheme of a section of a bundle $\rho: E\to Y$, and we have an exact sequence of vector bundles on X:

$$0\longrightarrow N\longrightarrow E|_X\longrightarrow C\longrightarrow 0$$

where N is the normal bundle of X in Y. The blow-up $\widetilde{Y}$ of Y along X embeds in $P(E)$, and this embedding is regular: indeed, this is a local matter, so it reduces to the case considered in Lemma 2.1.

We view this situation as follows. The section of E defining X is an embedding

$$s: Y\hookrightarrow E$$

of Y into the total space of E. Both s and the zero-section $z : Y \hookrightarrow E$ are regular embeddings, with normal bundle E itself; and $s(Y)$, $z(Y)$ meet along X. In other words, we have the fiber square

(†)
$$\begin{array}{ccc} X & \hookrightarrow & s(Y) \\ \downarrow & & \downarrow \\ z(Y) & \hookrightarrow & E \end{array}$$

in which all embeddings are regular.

In particular, this gives an embedding of the normal bundle N to X in $Y = s(Y)$ into the restriction of the normal bundle to $z(Y)$ in E, that is, $E|_X$.

3.2. Now we let $\nu : \widetilde{E} \to E$ be the blow-up along $z(Y)$. The blow-up $\widetilde{Y} = B\ell_X Y$ may be realized as the proper transform of $s(Y)$ in $\widetilde{E}$.

Note that $z(Y)$ is a complete intersection in E: it is the zero-scheme of the 'identity' section $E \to \rho^*(E)$. Thus, we are in the situation of Lemma 1.1, and we can conclude that $\widetilde{E}$ embeds regularly into $P(\rho^*(E))$, with normal bundle

$$N_{\widetilde{E}} P(\rho^*(E)) \cong \frac{\nu^*(E)}{\mathscr{O}(\widetilde{W})} \quad ,$$

where $\widetilde{W}$ denotes the exceptional divisor. Summarizing, we have the commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \hookrightarrow & P(E) \\ \downarrow & & \downarrow \\ \widetilde{E} & \hookrightarrow & P(\rho^*(E)) \end{array}$$

of regular embeddings, where the vertical map on the left is the proper transform of $s(Y)$ (this will be verified to be a regular embedding in Lemma 4.1), and the vertical map on the right is obtained by restricting $P(\rho^*(E))$ to $s(Y)$. It follows that

$$c(N_{P(E)} P(\rho^*(E))) \cdot c(N_{\widetilde{Y}} P(E)) = c(N_{\widetilde{E}} P(\rho^*(E))) \cdot c(N_{\widetilde{Y}} \widetilde{E}) \quad ,$$

(omitting pull-backs for convenience), and hence

$$c(N_{\widetilde{Y}}) = \frac{c(N_{\widetilde{E}} P(\rho^*(E))) \cdot c(N_{\widetilde{Y}} \widetilde{E})}{c(N_{P(E)} P(\rho^*(E)))}$$

where $N_{\widetilde{Y}} = N_{\widetilde{Y}} P(E)$ as in Theorem 1.2. Since $N_{\widetilde{E}} P(\rho^*(E)) \cong \nu^*(E)/\mathscr{O}(\widetilde{W})$ by Lemma 1.1, and $N_{P(E)} P(\rho^*(E))$ is the pull-back of $N_{s(Y)} E$, that is E, and further $\mathscr{O}(\widetilde{W})$ restricts to $\mathscr{O}(\widetilde{X})$ on $\widetilde{Y}$, we can conclude that

$$c(N_{\widetilde{Y}}) = \frac{c(N_{\widetilde{Y}} \widetilde{E})}{c(\mathscr{O}(\widetilde{X}))} \quad ,$$

reducing the computation of $c(N_{\widetilde{Y}}) = c(N_{\widetilde{Y}} P(E))$, which is our objective, to the computation of $c(N_{\widetilde{Y}} \widetilde{E})$.

3.3. The proof of Theorem 1.2 is now complete if we show:

Claim 3.1. *With the notational convention explained in §1.5,*

$$c(N_{\widetilde{Y}}\widetilde{E}) = c(\mathbb{N})\, c(\mathbb{C}\otimes\mathscr{O}(-\widetilde{X})) \quad .$$

This is an instance of a general template, which appears to be independently useful, and which we treat in the next section. Claim 3.1 is the result of applying Theorem 4.2 to the situation of diagram (†). Therefore, the proof of Theorem 4.2 will conclude the proof of Theorem 1.2 (and this paper).

4. The normal bundle of a proper transform

4.1. Let $X\subseteq Y$ and $Y\subseteq Z$ be regular embeddings, and let $\widetilde{Y}$, $\widetilde{Z}$ be the blow-ups along X; $\widetilde{Y}$ may be identified with the proper transform of Y in $\widetilde{Z}$. Then ([Ful84], B.6.10) $\widetilde{Y}$ is regularly embedded in $\widetilde{Z}$, and

$$N_{\widetilde{Y}}\widetilde{Z}\cong f^*(N_YZ)\otimes\mathscr{O}(-\widetilde{X}) \quad ,$$

where $f:\widetilde{Y}\to Y$ is the blow-up map, and $\widetilde{X}$ is the exceptional divisor in $\widetilde{Y}$.

We wish to extend this formula (at the level of Chern classes) to the case in which the center W of the blow-up is not necessarily contained in Y, but $X=W\cap Y$ is still regularly embedded in both W and Y: all embeddings in the diagram

(‡)
$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ W & \hookrightarrow & Z \end{array}$$

are regular. Note that the diagram (†) of §3 is an instance of this situation: take $Y=s(Y)$, $W=z(Y)$, $Z=E$.

4.2. Let $\widetilde{Z}\to Z$ be the blow-up along W. The blow-up $\widetilde{Y}$ of Y along $W\cap Y=X$ embeds in $\widetilde{Z}$ as the proper transform of Y.

Lemma 4.1. *$\widetilde{Y}$ is regularly embedded in $\widetilde{Z}$.*

Proof. This is a local verification, which follows closely the case $X=W$ proved in [Ful84], B.6.10 (from which we already borrowed in the proof of Lemma 2.1). We may assume that $Z=\operatorname{Spec}A$, the ideal of $W\subset Z$ is generated by a regular sequence $(a_1,\dots,a_d)$, and the ideal of $Y\subset Z$ is also generated by a regular sequence $(a_1,\dots,a_e,b_1,\dots,b_\ell)$, with $1\le e\le d$. The blow-up $\widetilde{Z}$ is defined by $a_iT_j-a_jT_i$, $1\le i<j\le d$, in $Z\times\mathbb{P}^{d-1}=Z\times P(\hat{N})$. On the open set defined by $T_d\neq 0$, $\widetilde{Z}$ has coordinate ring

$$\widetilde{A}:=\frac{A[x_1,\dots,x_{d-1}]}{(a_1-a_dx_1,\dots,a_{d-1}-a_dx_{d-1})} \quad ,$$

where $x_i=T_i/T_d$.

At the same time, Y has coordinate ring $A' = A/(a_1, \dots, a_e, b_1, \dots, b_\ell)$; by assumption, the cosets $\overline{a}_{e+1}, \dots, \overline{a}_d \in A'$ of $a_1, \dots, a_d$ form a regular sequence. The coordinate ring of a matching chart for $\widetilde{Y}$ is

$$\widetilde{A}' := \frac{A'[x_{e+1}, \dots, x_{d-1}]}{(\overline{a}_{e+1} - \overline{a}_d x_{e+1}, \dots, \overline{a}_{d-1} - \overline{a}_d x_{d-1})}$$
$$\cong \frac{A[x_{e+1}, \dots, x_{d-1}]}{(b_1, \dots, b_\ell, a_1, \dots, a_e, a_{e+1} - a_d x_{e+1}, \dots, a_{d-1} - a_d x_{d-1})} \quad .$$

On this chart, the inclusion $\widetilde{Y} \subset \widetilde{Z}$ corresponds to the surjection $\widetilde{A} \twoheadrightarrow \widetilde{A}'$ of A-algebras given by $x_1 \mapsto 0, \dots, x_e \mapsto 0, \dots, x_{e+1} \mapsto x_{e+1}, \dots, x_d \mapsto x_d$. The kernel of this surjection is generated by

$$x_1, \dots, x_e, b_1, \dots, b_\ell \quad ,$$

clearly a regular sequence at each point of $\widetilde{Y}$. This verifies that the embedding is regular on this chart, and the situation is identical in the other charts $T_k \neq 0$, $k > e$. (The argument also implies that $\widetilde{Y}$ has empty intersection with the charts $T_k \neq 0$, $k \leq e$.) □

The challenge is to compute $c(N_{\widetilde{Y}}\widetilde{Z})$. At one extreme, $X = W$ and we are in the situation of [Ful84], B.6.10: in this case $c(N_{\widetilde{Y}}\widetilde{Z}) = c(f^*(N_Y Z) \otimes \mathscr{O}(\widetilde{X}))$.

At the other extreme, Y and W intersect properly in Z: that is, $\widetilde{Y}$ equals the total transform of Y in $\widetilde{Z}$; in this case, $c(N_{\widetilde{Y}}\widetilde{Z}) = c(f^*(N_Y Z))$.

The general case lies 'in between' these two special cases.

4.3. In the fiber square (‡):

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ {\scriptstyle i}\downarrow & & \downarrow \\ W & \hookrightarrow & Z \end{array}$$

note that there is an embedding $N_X Y \subset i^* N_W Z$, and therefore an exact sequence

$$0 \longrightarrow N_X Y \longrightarrow i^* N_W Z \longrightarrow C \longrightarrow 0$$

The cokernel C is the *excess normal bundle* of the square (cf. [Ful84], §6.3).

As a useful warm-up, assume that there is a regularly embedded subscheme Z' of Z containing Y and W, and in which Y and W intersect properly:

$$\begin{array}{cccc} X & \hookrightarrow & Y & \\ {\scriptstyle i}\downarrow & & \downarrow & \\ W & \hookrightarrow & Z' & \\ & & & \searrow \\ & & & Z \end{array}$$

Also, assume that all embeddings are regular; and let $h : \widetilde{Z}' \to Z'$ be the blow-up of Z' along W.

Denote by $\mathbb{N}$ the normal bundle of Y in Z', and its pull-backs; and denote by $\mathbb{C}$ the normal bundle of Z' in Z, as well as its pull-backs. Note that

$$c(N_Y Z) = c(\mathbb{N})\, c(\mathbb{C}) \quad ,$$

while the fact that Y and W meet properly in Z' implies that $i^* N_W Z' = N_X Y$, and hence that $\mathbb{C}$ restricts to C on X. By the same token, $\mathbb{N}$ restricts to $N_X W$ on X.

This situation is a combination of the two 'extremes' mentioned at the end of §4.2:

—Since Y and W meet properly in Z', we have

$$c(N_{\widetilde{Y}} \widetilde{Z}') = c(f^* N_Y Z') = c(\mathbb{N}) \quad ;$$

—Since Z' contains the center W of the blow-up, we have

$$c(N_{\widetilde{Z}'} \widetilde{Z}) = c(h^* N_{Z'} Z \otimes \mathscr{O}(1)) = c(\mathbb{C} \otimes \mathscr{O}(1)) \quad ,$$

where $\mathscr{O}(-1)$ stands for the line bundle of the exceptional divisor.

—Therefore,

$$c(N_{\widetilde{Y}} \widetilde{Z}) = c(\mathbb{N})\, c(\mathbb{C} \otimes \mathscr{O}(1))$$

(omitting evident pull-backs).

Our main result is that this formula holds in the general case (even if Z' is not present), provided that it is interpreted appropriately.

4.4. The statement. Summarizing: in general, two bundles are defined on X, namely $N_X W$ and the excess intersection bundle C. In the particular case considered in §4.3, these two bundles extend to bundles $\mathbb{N}$, resp. $\mathbb{C}$ defined on the whole of Y, such that $c(N_Y Z) = c(\mathbb{N}) c(\mathbb{C})$, and we have verified that

$$c(N_{\widetilde{Y}} \widetilde{Z}) = c(\mathbb{N})\, c(\mathbb{C} \otimes \mathscr{O}(\widetilde{X}))$$

where pull-backs via $f : \widetilde{Y} \to Y$ are understood.

Here is how the right-hand-side of this formula may be interpreted as an operator on $A_* \widetilde{Y}$, even when $N_X W$ and C are not assumed to be restrictions of bundles $\mathbb{N}$, $\mathbb{C}$ (cf. §1.5).

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{j} & \widetilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

— Formally expand $c(\mathbb{N})\, c(\mathbb{C} \otimes \mathscr{O}(\widetilde{X}))$:

$$c(\mathbb{N})\, c(\mathbb{C} \otimes \mathscr{O}(\widetilde{X})) = c(\mathbb{N})\, c(\mathbb{C}) + Q(c_i(\mathbb{N}), c_j(\mathbb{C})) \cdot \widetilde{X}$$

for a well-defined polynomial Q in the (formal) variables $c_1(\mathbb{N}), c_2(\mathbb{N}), \dots, c_1(\mathbb{C}), c_2(\mathbb{C}), \dots$;

— For $\alpha \in A_* \widetilde{Y}$, define

$$\boxed{c(\mathbb{N})\, c(\mathbb{C} \otimes \mathscr{O}(\widetilde{X})) \cap \alpha := f^* c(N_Y Z) \cap \alpha + j_* Q(c_i(g^* N_X W), c_j(g^* C)) \cap (\widetilde{X} \cdot \alpha)} \quad .$$

Theorem 4.2. *With notation as above,*

$$c(N_{\widetilde{Y}}\widetilde{Z}) \cap \alpha = c(\mathbb{N})\, c(\mathbb{C} \otimes \mathscr{O}(\widetilde{X})) \cap \alpha$$

for all $\alpha \in A_*\widetilde{Y}$.

In the application to (†) in §3, $N_X W$ equals the normal bundle N of X in (the image via the zero-section of) Y, and C equals the cokernel of the inclusion of N into $E|_X$, as prescribed in §1.4. Thus, Theorem 4.2 does provide the last ingredient in the proof of Theorem 1.2, as pointed out in §3.3. Proving Theorem 4.2 is our last task.

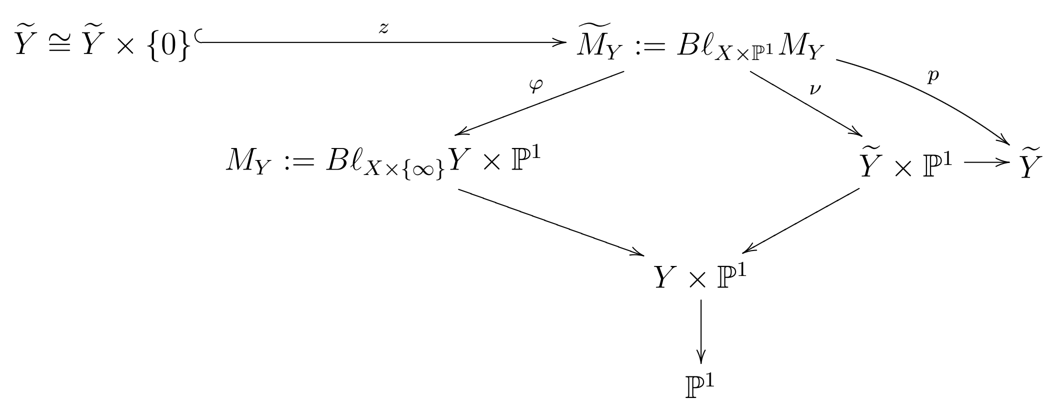
4.5. Proof of Theorem 4.2: short version. The following summary will suffice for the expert. The deformation to the normal cone ([Ful84], Chapter 5) may be used to reduce the general situation (‡) to the 'linearized' situation

$$(\ddagger') \qquad \begin{array}{ccc} X & \hookrightarrow & N_X Y \\ \downarrow & & \downarrow \\ N_X W & \hookrightarrow & N_X Z \end{array}$$

This is covered by the particular case considered in §4.3, by taking Z' to be the (direct) sum of $N_X Y$ and $N_X W$ in $N_X Z$. As shown in §4.3 the formula holds in this case, hence it holds in general.

We end this article by spelling out this argument, for the benefit of readers who may be less familiar with the deformation to the normal cone.

4.6. Proof of Theorem 4.2: long version. The following diagram may be helpful in tracing the argument:



Here M_Y is the deformation of Y to the normal cone (bundle) $N_X Y$. The subscheme $X \times \mathbb{P}^1$ of $Y \times \mathbb{P}^1$ lifts to an isomorphic copy in M_Y, and $\varphi : \widetilde{M_Y} \to M_Y$ is the blow-up along this isomorphic copy. It is easily checked that the inverse image of $X \times \mathbb{P}^1 \subset Y \times \mathbb{P}^1$ in $\widetilde{M_Y}$ is a Cartier divisor, and more precisely it equals the sum of the two exceptional divisors; by the universal property of blow-ups, the map $\widetilde{M_Y} \to \widetilde{Y} \times \mathbb{P}^1$ factors through $\widetilde{Y} \times \mathbb{P}^1$, as indicated in the diagram. In fact, $\nu : \widetilde{M_Y} \to \widetilde{Y} \times \mathbb{P}^1$ is the blow-up along $\widetilde{X} \times \{\infty\}$.

Also, note that the composition

$$p \circ z \quad : \quad \widetilde{Y} \to \widetilde{M_Y} \to \widetilde{Y}$$

is the identity.

With M_W, resp. M_Z obtained similarly from $W \times \mathbb{P}^1$, resp. $Z \times \mathbb{P}^1$ by blowing up along $X \times \{\infty\}$, we have inclusions

$$\begin{array}{ccc} X \times \mathbb{P}^1 & \hookrightarrow & M_Y \\ \downarrow & & \downarrow \\ M_W & \hookrightarrow & M_Z \end{array}$$

Over all $t \neq \infty$, this diagram specializes to (‡); over ∞, the diagram formed by the exceptional divisors:

(‡″)
$$\begin{array}{ccc} X & \hookrightarrow & P(N_X Y \oplus 1) \\ \downarrow & & \downarrow \\ P(N_X W \oplus 1) & \hookrightarrow & P(N_X Z \oplus 1) \end{array}$$

is the projective completion of the 'linearized' diagram (‡′). At ∞ we also find copies of $\widetilde{Y}$, $\widetilde{W}$, $\widetilde{Z}$, meeting the corresponding projective completions along their exceptional divisors. The scheme-theoretic intersection of M_Y and M_W is the lift of $X \times \mathbb{P}^1$ mentioned above. This locus is disjoint from the copy of $\widetilde{Y}$ at $\{\infty\}$. Further, φ restricts to an isomorphism of the proper transform via ν of $\widetilde{Y} \times \{\infty\}$ (which is isomorphic to $\widetilde{Y}$ as $\widetilde{X}$ is a divisor in $\widetilde{Y}$) with this copy of $\widetilde{Y}$ at ∞ in M_Y.

Blow-up M_Z along M_W; the proper transform of M_Y agrees with the blow-up of the latter along $X \times \mathbb{P}^1$, so it is the variety $\widetilde{M_Y}$ appearing in the larger diagram. Over any $t \neq \infty$ (and in particular for $t = 0$), the blow-ups reproduce the situation considered in §4.2.

We have to verify that $c(N_{\widetilde{Y}}\widetilde{Z}) \cap \alpha = c(\mathbb{N})c(\mathbb{C} \otimes \mathscr{O}(\widetilde{X})) \cap \alpha$ for all $\alpha \in A_*\widetilde{Y}$. Letting $\Gamma \cdot \alpha := j_* Q(c_i(g^* N_X W), c_j(g^* C)) \cap \widetilde{X} \cdot \alpha$ as in the definition preceding the statement of Theorem 4.2, the task is to show that

$$(c(N_{\widetilde{Y}}\widetilde{Z}) - c(f^* N_Y Z)) \cap \alpha = \Gamma \cdot \alpha$$

for all $\alpha \in A_*\widetilde{Y}$, and we have verified that this holds in the situation considered in §4.3. We let Γ^* be the operator defined in the same way as Γ on $A_*(\widetilde{M_Y})$, and observe that Γ^* restricts to Γ over all $t \neq \infty$, and to the analogous operator for the linearized version (‡″).

By linearity, we may assume that $\alpha = [V]$, where $V \subset \widetilde{Y}$ is a subvariety of $\widetilde{Y}$. Since $N_{\widetilde{Y}}\widetilde{Z}$, resp. $f^* N_Y Z$ may be realized as pull-backs via z of $N_{\widetilde{M_Y}}\widetilde{M_Z}$, resp. $\varphi^* N_{M_Y} M_Z$, the projection formula gives

$$(*) \quad (c(N_{\widetilde{Y}}\widetilde{Z}) - c(f^* N_Y Z)) \cap [V] = p_* \Big((c(N_{\widetilde{M_Y}}\widetilde{M_Z}) - c(\varphi^* N_{M_Y} M_Z)) \cap ([V \times \{0\}]) \Big) \quad .$$

The proper transform of $V \times \mathbb{P}^1 \subset \widetilde{Y} \times \mathbb{P}^1$ in $\widetilde{M_Y}$ is the blow-up M_V along $(\widetilde{X} \cap V) \times \{\infty\}$; the fiber of M_V over $\{0\}$ is precisely the variety $V \times \{0\}$ appearing in (*). This is rationally equivalent to the fiber of M_V over $\{\infty\}$, that is

$$P(N_{\widetilde{X} \cap V} V \oplus 1) \cup B\ell_{\widetilde{X} \cap V} V \quad .$$

Thus,

$$(c(N_{\widetilde{Y}}\widetilde{Z}) - c(f^*N_Y Z)) \cap [V] \\ = p_*\Big((c(N_{\widetilde{M}_Y}\widetilde{M}_Z) - c(\varphi^* N_{M_Y} M_Z)) \cap ([P(N_{\widetilde{X}\cap V}V \oplus 1)] + [B\ell_{\widetilde{X}\cap V}V])\Big) \quad .$$

As noted earlier, φ restricts to an isomorphism from $B\ell_{\widetilde{X}\cap V}V \cong V$ to $V \subset \widetilde{Y} \subset M_Y$. The target V is disjoint from the center $X \times \mathbb{P}^1$ of the blow-up φ, therefore

$$(c(N_{\widetilde{M}_Y}\widetilde{M}_Z) - c(\varphi^* N_{M_Y} M_Z)) \cap [B\ell_{\widetilde{X}\cap V}V] = 0 \quad ,$$

and hence

$$(c(N_{\widetilde{Y}}\widetilde{Z}) - c(f^*N_Y Z)) \cap [V] = p_*\Big((c(N_{\widetilde{M}_Y}\widetilde{M}_Z) - c(\varphi^* N_{M_Y} M_Z)) \cap [P(N_{\widetilde{X}\cap V}V \oplus 1)]\Big) \ .$$

Now we are squarely in the blow-up over the linearized diagram (‡″). This situation is contemplated by the case considered in §4.3: use $P(N_X Y \oplus N_X W \oplus 1)$ for Z'. Therefore, the theorem holds in this case, giving

$$(c(N_{\widetilde{Y}}\widetilde{Z}) - c(f^*N_Y Z)) \cap [V] = p_*\left(\Gamma^* \cdot [P(N_{\widetilde{X}\cap V}V \oplus 1)]\right) \quad .$$

Next we essentially run through the construction in reverse. Since Γ^* is supported on the exceptional divisor of φ, $\Gamma^* \cdot [B\ell_{\widetilde{X}\cap V}V] = 0$, hence

$$(c(N_{\widetilde{Y}}\widetilde{Z}) - c(f^*N_Y Z)) \cap [V] = p_*\left(\Gamma^* \cap ([P(N_{\widetilde{X}\cap V}V \oplus 1)] + [B\ell_{\widetilde{X}\cap V}V])\right) \quad ;$$

since $[P(N_{\widetilde{X}\cap V}V \oplus 1)] + [B\ell_{\widetilde{X}\cap V}V] = [V \times \{0\}]$ in M_V,

$$(c(N_{\widetilde{Y}}\widetilde{Z}) - c(f^*N_Y Z)) \cap [V] = p_*\left(\Gamma^* \cdot [V \times \{0\}]\right) \quad ;$$

and since Γ^* restricts to Γ on fibers over $t \neq \infty$, the projection formula gives

$$(c(N_{\widetilde{Y}}\widetilde{Z}) - c(f^*N_Y Z)) \cap [V] = \Gamma \cdot [V]$$

as claimed. This concludes the proof of Theorem 4.2. □

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