Representations of Finite Group Schemes

Eric M. Friedlander; with Julia Pevtsova and Andrei Suslin

0 Introduction

This is a report of work in progress with Julia Pevtsova and Andrei Suslin, based on earlier work which appears in published or preprint form in various papers with Julia and Andrei.

We work over an arbitrary field k of positive characteristic $p > 0$.

Definition 0.1 *A finite group scheme G is a functor*

$$G : (fin\ dim'l\ comm\ k - alg) \to (grps)$$

representable by a finite dimensional (commutative) k-algebra $k[G]$.

We denote by kG the linear dual of $k[G]$, a cocommutative Hopf algebra finite dimensional over k.

Here are a few important examples

Example 0.2
- *G a finite group (constant group scheme), kG the usual group algebra.*
- *G a p-restricted Lie algebra, kG the p-restricted enveloping algebra of this Lie algebra.*
- *$\mathcal{G}$ an algebraic group, $G = \mathcal{G}_{(r)} = ker\{F^r : \mathcal{G} \to \mathcal{G}^{(r)}\}$ the r-th Frobenius kernel of $\mathcal{G}$.*

1 Cohomology

We use $H^\bullet(G, k)$ to denote the even dimensional cohomology of G if $p > 2$, and the entire cohomology algebra of G if $p = 2$. Then $H^\bullet(G, k)$ is a commutative algebra. For example, if G is an elementary abelian p-group of rank $r > 0$ (i.e., $G \simeq \mathbb{Z}/p^{\times r}$), then $\operatorname{Spec} H^\bullet(G, k) = \mathbb{A}^r$.

Theorem 1.1 *The reduced spectrum of the cohomology of the following finite group schemes has been determined.*

- *(Quillen) if G is a finite group, then Spec $H^\bullet(G,k)$ is isomorphic to the colimit indexed by conjugacy classes of elementary abelian subgroups $E < G$ of Spec $H^\bullet(G,k)$.*

- *(Friedlander-Parshall, Jantzen) If g is a finite dimensional, p-restricted Lie algebra, then Spec $H^\bullet(G,k)$ is the subvariety $N_p \subset g$ of p-nilpotent elements.*

- *(Suslin-Friedlander-Bendel) If G is the r-th Frobenius kernel of the algebraic group $\mathcal{G}$, then Spec $H^\bullet(G,k)$ is the variety of r-tuples of p-nilpotent, pairwise commuting elements of Lie$(\mathcal{G})$.*

We are interested in the study of kG-modules. This is a challenging topic, as the following example indicates.

Example 1.2 *If G is the finite group $\mathbb{Z}/p$, then there is exactly one isomorphism class of indecomposable kG-modules for each $i, 1 \leq i \leq p$ and this exhausts the isomorphism classes of indecomposable $\mathbb{Z}/p$-modules. Only the trivial (dimension 1) module is irreducible.*

On the other hand, if G is an elementary abelian p-group of rank ≥ 2 for $p > 2$ or of rank ≥ 3 for $p = 2$, then the representation theory of kG is wild. This means that the indecomposable modules of a given dimension can not be parametrized by finitely many paramters.

The following theorem, extending a theorem of Quillen and Venkov for finite groups, is fundamental to all cohomological considerations.

Theorem 1.3 *(Friedlander-Suslin) Let G be a finite group scheme. Then $H^*(G,k)$ is finitely generated and $Ext^*_{kG}(M,M)$ is a finite $H^*(G,k)$-module.*

Definition 1.4 *We introduce the notation $|G|$ to denote the maximal ideal spectrum of $H^\bullet(G,k)$. For any kG-module M, we denote by*

$$|G|_M \subset |G|$$

the cohomological support variety of M defined as the support of the $H^\bullet(G,k)$-moudle $Ext^*_{kG}(M,M)$.

The invariant $M \mapsto |G|_M$ is a useful invariant of M with many good properties. For elementary abelian p-groups and for Frobenius kernels, this invariant has a purely representation-theoretic description due to Carlson-Avrunin-Scott and Friedlander-Parshall, Suslin-Friedlander-Bendel. We shall see below a representation-theoretic description for all finite group schemes.

We remark that Nakano and others have made numerous computations of $|G|_M$ and used these computations to study orbit closures in nilpotent varieties.

2 π-points

The following construction and consideration of π-points has been investigated in two papers by Friedlander-Pevtsova.

Definition 2.1 *A π-point of G is a left flat map of K-algebras for some field extension K/k*

$$\alpha_K : K[t]/t^p \to KG,$$

which factors through some commutative subgroup $C_K \subset G_K$.

We say that β_L specializes to α_K if for all finite dimensional kG-modules M whenever $\alpha_K^*(M_K)$ is free then $\beta_L^*(M_L)$ is free.

We write $\beta_L \sim \alpha_K$ if β_L specializes to α_K and α_K specializes to β_L. We write the equivalence class of a π-point α_K as $[\alpha_K]$.

Theorem 2.2 *(Friedlander-Pevtsova) Let G be a finite group scheme. There is an isomorphism of schemes*

$$\Pi(G) \stackrel{\sim}{\to} Proj H^\bullet(G, k)$$

where $\Pi(G)$ has a purely representation-theoretic description. In particular, the points of $\Pi(G)$ are the equivalence classes of π-points of G.

The closed subsets of $\Pi(G)$ are subsets (indexed by finite dimensional kG-modules M) of the form $\Pi(G)_M$ consisting of those equivalence classes $[\alpha_K]$ such that $\alpha_K^*(M_K)$ is not free.

This theorem has as an interesting consequence the identification of the lattice of thick, $\otimes$-closed subcategories of the stable module category of G with the lattice of thick, $\otimes$-closed subcategories of the full subcategory of the derived category of coherent sheaves on $Proj H^\bullet(G, k)$ consisting of perfect complexes.

3 Generic and Maximal Jordan types

We seek further, more refined invariants of kG-modules. In "most" examples of a finite group scheme G and a finite dimensional kG-module M, the Jordan types $\alpha_K^*(M_K)$ and $\beta_L^*(M_L)$ are different even if $\alpha_K \sim \beta_L$. This is not at all surprising in view of the definition of the equivalence relation on π-points.

Nonetheless, we have the following result which provides us with new invariants. These results are very new, and may take a different form when presented in final form.

Theorem 3.1 *(Friedlander-Pevtsova-Suslin) Let G be a finite group scheme, $[\alpha_K] \in \Pi(G)$ a generic point. For any finite dimensional kG-module M, the Jordan type of $\alpha_K^*(M)$ does not depend on the representative α_K of $[\alpha_K]$.*

For a given kG-module M, we say that $\alpha_K^*(M_K)$ is maximal if there is no π-point β_L such that the Jordan type of $\beta_L^*(M_L)$ is greater (in the sense of partitions) than the Jordan type of $\alpha_K^*(M_K)$.

The following theorem provides a refinement of the support variety $\Pi(G)_M$; observe that $\Pi(G)_M = \Pi(G)$ whenever the dimension of M is not divisible by p, so this is indeeed a quite crude invariant.

Theorem 3.2 *Let G be a finite group scheme, M a finite dimensional kG-module. Assume that $\alpha_K^*(M_K)$ is maximal. Then for any π-point β_L which specializes to α_K, $\beta_L^*(M_L)$ has the same Jordan type as $\alpha_K^*(M_K)$.*

In particular, the subset $\Gamma(G)_M \subset \Pi(G)_M$ of those equivalence classes of π-points α_K such that $\alpha_K^*(M_K)$ is NOT maximal is a closed subset of $\Pi(G)_M$ which equals $\Pi(G)_M$ if and only if the maximal Jordan type of M is projective.

We fail, for lack of time, to give numerous explicit examples.

We refer the reader to

- MathSciNet
- www.math.northwestern.edu/~eric/preprints

for references.