On certain rigid fibered Calabi-Yau threefolds

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Introduction

In a previous paper [O1], we showed that all the possible fiber space structures on a Calabi-Yau threefold fall into six different classes defined by certain intrinsic properties of the ample cone of the threefold. A special case is the class of fibration of Type II_0. A fibered Calabi-Yau threefold is said to be of Type II_0 if it is an elliptic Calabi-Yau threefold whose base space is a normal, projective, rational surface with only quotient singularities and numerically trivial canonical Weil divisor. Such a base space is a log Enriques surface in the terminology of [Z]. These are classified by the global canonical covering $\pi : \widetilde{W} \longrightarrow W$ ([Ka3]) of the base surface, for which we have either

(1) $\widetilde{W}$ is a smooth abelian surface ; or,

(2) $\widetilde{W}$ is a K3 surface with only rational double points as its singularities.

A fibered Calabi-Yau threefold is called of Type II_0A if it is of Type II_0 and the global canonical cover of the base space is an abelian surface.

The purpose of this paper is to show that, up to isomorphism of fiber spaces, there are only finitely many Calabi-Yau threefolds of Type II_0A. This gives partial, but non-trivial, evidence to support Gross' hope, that the family of elliptic Calabi-Yau threefolds in the biregular sense is finite ([G]).

For the precise statement, we briefly recall Beauville's example.

Beauville's Example ([B], (2.2)). *Let E_ζ be the elliptic curve whose period is the primitive third root of unity ζ in the upper half plane and let $E_\zeta^n / < \zeta >$ be the quotient n-fold of the product manifold E_ζ^n by scalar multiplication by ζ. Then, the blow-up at the 27 singular points of $(E_\zeta^3 / < \zeta >)$ gives a smooth Calabi-Yau threefold X_ϕ and the projection $p_\phi : X_\phi \longrightarrow E_\zeta^2 / < \zeta >$, induced by the projection $p_{12} : E_\zeta^3 \longrightarrow E_\zeta^2$, makes X_ϕ a fibered Calabi-Yau threefold of Type II_0A.*

Our main Theorem is as follows.

Main Theorem ((3.1), (2.7) and (2.4)). *There exist just fourteen different fibered Calabi-Yau threefolds of Type II_0A, up to isomorphism as fiber spaces. Moreover, each of these is obtained from Beauville's example by a composition of flops whose centers are contained in the singular fibers.*

An explicit construction of the representatives is given in (2.4) and (2.7).

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In this paper, by a Calabi-Yau threefold, we simply mean a three-dimensional minimal model ([KMM]) with trivial Cartier canonical class. A priori, we require neither smoothness nor simply-connectedness. However, the main Theorem implies,

Corollary ((3.2)). *Every fibered Calabi-Yau threefold of Type II_0A is smooth, simply-connected and has no non-trivial deformations. In addition, any two such threefolds are birationally equivalent.*

In particular, they are isolated in the family of Calabi-Yau threefolds and have no "mirrors" in the usual sense. Moreover, they have a (birational) canonical model whose first and second Chern classes are zero. (See [SW] for such threefolds.)

For proof of the main Theorem, apart from some standard techniques in minimal model theory for threefolds ([Ka 3,4, KMM, Ko, M, R]), we use the theory of elliptic fibrations developed by [Ka2, N1,2,3], the characterization theorem of abelian varieties by [Ka1] and the characterization theorem of the particular abelian varieties E_ζ^n by [CC].

The organisation of this paper is as follows. After preparing some easy facts on fibered Calabi-Yau threefolds of Type II_0 and special abelian varieties in §1, we study the flop phenomena associated with Beauville's example in §2. Finally, in §3, we apply general theories listed above to prove the main Theorem.

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In addition to the standard notation in [KMM], we employ the following:

Notation.

ζ is the primitive third root of unity in the upper half plane;

$E_\zeta := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta)$ is the elliptic curve with period ζ;

E_ζ^n is the abelian variety defined as n-times direct product of E_ζ;

By Q_0, Q_1 and Q_2, we denote the following three-torsion points in E_ζ, $Q_0 := 0$, $Q_1 := (1-\zeta)/3$, and $Q_2 := -(1-\zeta)/3$;

$\Lambda := \{0, 1, 2\}$ and Λ^n is the n-times direct product of the set Λ;

For $(i_1, ..., i_n) \in \Lambda^n$, we denote by $Q_{i_1 ... i_n}$ the point $(Q_{i_1}, ..., Q_{i_n})$ in E_ζ^n;

By $Q(n)$ we denote the subset $\{Q_{i_1 ... i_n} | (i_1, ..., i_n) \in \Lambda^n\}$ of E_ζ^n;

By $\zeta : E_\zeta^n \longrightarrow E_\zeta^n$, we denote the automorphism of E_ζ^n defined by scalar multiplication by ζ (on the universal covering space $\mathbb{C}^n$) with the origin a point in $Q(n)$;

(Note that the action of ζ does not depend on this choice of the origin.)

For a subset B in a variety V, we put $Aut(V, B) := \{f \in Aut(V) | f(B) = B\}$;

When a group G acts on a set S faithfully, we put $S^G := \{s \in S | g(s) = s$ for some $g \in G - \{1\}\}$ and $S^g := \{s \in S | g(s) = s\}$ for $g \in G - \{1\}$;

For an equivalence relation $\sim$ on a set S, by a minimal complete set of representatives (resp. a complete set of representatives) for the quotient space $S/\sim$, we *mean a subset M of S such that $|M \cap C_s| = 1$ (resp. $M \cap C_s \neq \phi$) for every orbit C_s of the relation;*

By $\sim_F$, we denote an isomorphism as fiber spaces.

§1. Preliminaries.

In this section, we shall note some easy facts on fibered Calabi-Yau threefolds and special abelian varieties, needed in the sequel.

In this paper, by a Calabi-Yau threefold, we simply mean a three-dimensional minimal model([KMM]) whose canonical divisor is Cartier and linearly equivalent to zero. In particular, any Calabi-Yau threefold is assumed to be projective. A proper surjective morphism $\Phi : X \longrightarrow W$ with connected fibers is called a fibered Calabi-Yau threefold of Type II_0 if X is a Calabi-Yau threefold and W is a log Enriques surface ([Z]), that is, a projective normal rational surface with only quotient singularities and numerically trivial canonical Weil divisor K_W. Note that a general fiber of Φ is a smooth elliptic curve and that the canonical class K_W is $\mathbb{Q}$-linearly equivalent to zero. By $I(W)$, we denote the global canonical index of W which is defined as $min\{n \in \mathbb{Z}_{>0} | \mathcal{O}_W(nK_W) \simeq \mathcal{O}_W\}$.

Fibered Calabi-Yau threefolds of TypeII_0 are classified further into the following two sub-classes via the global canonical covering of the base surface W,

$\pi : \widetilde{W} := Spec_{\mathcal{O}_W}(\oplus_{i=0}^{I(W)-1}\mathcal{O}_W(-iK_W)) \longrightarrow W$:

(1) $\widetilde{W}$ is a smooth abelian surface ;

(2) $\widetilde{W}$ is a K3 surface with only rational double points as its singularities.

For the global canonical covering, see [Ka3, Proposition 1.7]. A fibered Calabi-Yau threefold is called of Type II_0A if it is of Type II_0 and the global canonical cover of the base space is an abelian surface.

Lemma (1.1). *Let $\Phi : X \longrightarrow W$ be a fibered Calabi-Yau threefold of Type II_0. Then, for every point $w \in W - Sing(W)$, the scheme theoretic fiber $\Phi^{-1}(w)$ is a smooth elliptic curve. In particular, $\Phi(Sing(X))$ is contained in $Sing(W)$ and Φ is smooth over $W - Sing(W)$.*

Proof. Since K_X and K_W are $\mathbb{Q}$-linearly equivalent to zero, the canonical bundle formula implies that Φ is smooth except over a finite set of points of W (cf.[N1], [O1]). Thus, there is a polydisk Δ^2 around each w in $W - Sing(W)$ such that Φ is smooth over $\Delta^2 - \{w\}$. Now we may apply [N2, Main Theorem] to get the result. q.e.d.

Lemma (1.2). *Let $\Phi : X \longrightarrow W$ be a fibered Calabi-Yau threefold of Type II_0. Then, the global canonical index of W is either 2, 3, 4, 6 or 12.*

Proof. It is sufficient to show that $\mathcal{O}_W(12K_W)$ is isomorphic to $\mathcal{O}_W$. The proof is an easy modification of [N4, Proposition (C.1)]. Let $\mu : V \longrightarrow W$ be a resolution of W such that the exceptional divisor $E := \mu^{-1}(Sing(W))$ is a simple normal crossing divisor. By Lemma (1.1), we find a smooth projective threefold Y, a birational morphism $\nu : Y \longrightarrow X$ and a proper surjective morphism $g : Y \longrightarrow V$ such that

(1) $\Phi \circ \nu = \mu \circ g$ and

(2) $\nu|_{Y^0}$ and $\mu|_{V^0}$ induce an isomorphism

$$(\Phi^0 : X^0 \longrightarrow W^0) \sim_F (g^0 : Y^0 \longrightarrow V^0),$$

where $W^0 := W - Sing(W)$, $V^0 := V - E$, $X^0 := \Phi^{-1}(W^0)$, $Y^0 := g^{-1}(V^0)$, $\Phi^0 := \Phi|X^0$ and $g^0 := g|Y^0$. Note that, by [Ka2, Theorem 20], the J-function, $J : V^0 \longrightarrow \mathbb{C}$ extends uniquely to the morphism, $J : V \longrightarrow \mathbb{P}^1$. Since E is negative definite and $J^*(-\infty)$ is a nef, effective divisor supported in E, we have $J^{-1}(-\infty) = \phi$ by the negative definiteness of E. In particular, $J : V \longrightarrow \mathbb{P}^1$ is a constant map. Thus, the canonical bundle formula ([Ka2, Theorem 20]) implies $(g_*\omega_{Y/V})^{\otimes 12} \simeq \mathcal{O}_V(E')$, where E' is an effective divisor supported in E. Combining this with the isomorphism given by (2), we get, $(\omega_{W^0}^{\otimes 12})^{-1} \simeq ((\Phi^0)_*\omega_{X^0/W^0})^{\otimes 12} \simeq \mathcal{O}_{W^0}$. This implies the result. q.e.d.

Corollary (1.3). *Let $\Phi : X \longrightarrow W$ be a fibered Calabi-Yau threefold of Type II_0. Assume that the global canonical cover $\widetilde{W}$ of W is a smooth surface. Then, the global canonical index of W is three.*

Proof. Since the set of non-Gorenstein points of W is a non-empty, finite set, the global canonical index is not divisible by two. q.e.d.

Lemma (1.4). *Let A be an n-dimensional abelian variety with an automorphism ζ' written as $(z_1, ..., z_n) \mapsto (\zeta z_1, ..., \zeta z_n)$ for some global coordinates $(z_1, ..., z_n)$ on A. Then, $A = E_\zeta^n$ and $\zeta' = \zeta$ (under an appropriate identification of the origins).*

Proof. This is nothing but a special case of [CC, Proposition (5.7)]. q.e.d.

Lemma (1.5).

(1) $(E_\zeta^n)^\zeta = (E_\zeta^n)^{<\zeta>} = Q(n)$.
(2) $Aut(E_\zeta^n, Q(n))$ *acts transitively on* $Q(n)$.
(3) $Aut(E_\zeta^n, Q(n)) \cap Aut(E_\zeta^n, \{O\})$ *acts transitively on* $Q(n) - \{O\}$.

Proof. direct calculation. q.e.d.

Consider the following set consisting of subsets of $\Lambda^2 = \{0,1,2\}^2$:

(1.6)

$$\begin{aligned}\Omega_{II_0A} :=\{&\phi, \Lambda^2, \{(0,0)\}, \{(0,0)\}^c, \{(0,0),(1,0)\}, \{(0,0),(1,0)\}^c,\\ &\{(0,0),(1,0),(2,0)\}, \{(0,0),(1,0),(0,1)\},\\ &\{(0,0),(1,0),(2,0)\}^c, \{(0,0),(1,0),(0,1)\}^c,\\ &\{(0,0),(1,0),(0,1),(2,0)\}, \{(0,0),(1,0),(0,1),(1,1)\},\\ &\{(0,0),(1,0),(0,1),(2,0)\}^c, \{(0,0),(1,0),(0,1),(1,1)\}^c\},\end{aligned}$$

where $\{\cdot\}^c$ denotes the complement of $\{\cdot\}$ in Λ^2.

For a subset a of Λ^2, we denote by Q_a the corresponding subset of $Q(2)$ under the bijection $\Lambda^2 \leftrightarrow Q(2)$ defined by $(i,j) \leftrightarrow Q_{ij}$ (for example, $Q_{\{(0,0),(1,0)\}} = \{Q_{00}, Q_{10}\}$).

Lemma (1.7). *Let Π be the set of all subsets of $Q(2)$. Then, the subset $\Pi_{min} = \{Q_a | a \in \Omega_{II_0A}\}$ gives a minimal complete set of representatives for the orbit space $\Pi / Aut(E_\zeta^2, Q(2))$.*

Proof. Let us write Π_n for the subset $\{R \in \Pi | |R| = n\}$. Note that $Aut(E_\zeta^2, Q(2))$ acts on each Π_n. If $\{R_1, ..., R_k\}$ is a minimal complete set of representatives for the orbit space $\Pi_n/Aut(E_\zeta^2, Q(2))$, then $\{R_1^c, ..., R_k^c\}$ gives a minimal complete set of representatives for the orbit space $\Pi_{9-n}/Aut(E_\zeta^2, Q(2))$. So we need only consider the cases $n = 0, 1, 2, 3$ and 4. However, when n is 0, 1 or 2, the result is trivial. From Lemma (1.5)(2),(3), we see that $\{\{Q_{00}, Q_{10}, Q_{20}\}, \{Q_{00}, Q_{10}, Q_{01}\}\}$ is a complete set of representatives for the orbit space $\Pi_3/Aut(E_\zeta^2, Q(2))$. Observe that $X_1 + X_2 + X_3 + X_1 \neq X_1$ for $\{X_1, X_2, X_3\} = \{Q_{00}, Q_{10}, Q_{01}\}$ while $Q_{00} + Q_{10} + Q_{20} + Q_{00} = Q_{00}$. Thus this set is minimal because any element g in $Aut(E_\zeta^2, Q(2))$ satisfies $g(P_1 + P_2 + P_3 + P_4) = g(P_1) + g(P_2) + g(P_3) + g(P_4)$. A similar argument together with a concrete calculation shows that $\{\{Q_{00}, Q_{10}, Q_{01}, Q_{20}\}, \{Q_{00}, Q_{10}, Q_{01}, Q_{11}\}\}$ is a minimal complete set of representatives for the orbit space $\Pi_4/Aut(E_\zeta^2, Q(2))$. q.e.d.

§2. Standard models of fibered Calabi-Yau threefolds of Type II_0A.

In this section, we shall construct some fibered Calabi-Yau threefolds of Type II_0A from the abelian threefold E_ζ^3 ((2.1),(2.3)) and classify them up to isomorphism as fiber spaces ((2.7)). The resulting complete representatives will turn out to be all the non-isomorphic fibered Calabi-Yau threefolds of Type II_0A ((4.1)).

Let $p_{12} : E_\zeta^3 \longrightarrow E_\zeta^2$ be the projection to the first two factors. Note that $p_{12} \circ \zeta = \zeta \circ p_{12}$. Let $b : \widetilde{E_\zeta^3} \longrightarrow E_\zeta^3$ be the blow-up of E_ζ^3 at the 27 points $Q(3)(= (E_\zeta^3)^\zeta)$ and let $\widetilde{E_{ijk}}$ be the exceptional divisor corresponding to $Q_{ijk}(\in Q(3))$. The multiplication map ζ induces the automorphism $\tilde{\zeta}$ of $\widetilde{E_\zeta^3}$ with $\tilde{\zeta}|\widetilde{E_{ijk}} = id_{\widetilde{E_{ijk}}}$. For simplicity, we denote by $\tilde{?}$ the proper transform of ? under b (this does not apply to the $\widetilde{E_{ijk}}$).

Definition (2.1). *X_ϕ is the quotient threefold $\widetilde{E_\zeta^3}/ < \tilde{\zeta} >$, B is the quotient surface $E_\zeta^2/ < \zeta >$ and $p_\phi : X_\phi \longrightarrow B$ is the morphism induced by p_{12}.*

Proposition(2.2).

(1) *([B]) X_ϕ is a smooth, simply-connected Calabi-Yau threefold with no deformations, that is, $h^{2,1}(X_\phi) = 0$.*

(2) *$p_\phi : X_\phi \longrightarrow B$ is a fibered Calabi-Yau threefold of Type II_0A whose non-singular fiber is isomorphic to the elliptic curve E_ζ. The map p_ϕ only has singular fibers over the 27 points $\overline{Q_{ij}}(\in B)$ and $(p_\phi)^{-1}(\overline{Q_{ij}}) = l_{ij} \cup E_{ij0} \cup E_{ij1} \cup E_{ij2}$, where $\overline{Q_{ij}}$ is the image of Q_{ij}, $l_{ij}(\simeq \mathbb{P}^1)$ is the image of $\widetilde{\{Q_{ij}\} \times E_\zeta}$ for $(i,j) \in \Lambda^2$ and $E_{ijk}(\simeq \mathbb{P}^2)$ is the image of $\widetilde{E_{ijk}}$ for $(i,j,k) \in \Lambda^3$. Moreover E_{ijk} does not meet $E_{ijk'}$ if $k \neq k'$, and E_{ijk} meets l_{ij} transeversely at one point (see figure 1).*

(3) *$N_{X_\phi|l_{ij}} = \mathcal{O}_{l_{ij}}(-1)^{\oplus 2}$ and $N_{X_\phi|E_{ijk}} = \mathcal{O}_{E_{ijk}}(-3)$.*

(4) *The only isolated rational curve in $(p_\phi)^{-1}(\overline{Q_{ij}})$ is l_{ij}.*

Proof. Since the construction is concrete, everything stated in (2.2) may be checked directly, except possibly for the first equality in (3). But this is also immediate. Define T_i to be the image of $\widetilde{\{Q_i\} \times E_\zeta^2}$. Then $l_{ij} \subset T_i$ and $(l_{ij})^2_{T_i} = -1$ and we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{l_{ij}}(-1) \longrightarrow N_{X_\phi|l_{ij}} \longrightarrow N_{X_\phi|T_i}|l_{ij} \longrightarrow 0.$$

Combining this with the equality $c_1(N_{X_\phi|l_{ij}}) = -2$, we get the result. q.e.d.

By (2.2)(3), we can apply the elementary transformation on X_ϕ along l_{ij} to get another smooth threefold.

Definition (2.3). *If T is a subset of Λ^2, then X_T is the threefold obtained from X_ϕ by the elementary transformation along $\cup_{(i,j)\in T} l_{ij}$ and $p_T : X_T \longrightarrow B$ is the morphism induced by p_ϕ.*

Proposition(2.4).

(1) *X_T is a smooth, simply-connected Calabi-Yau threefold with no deformations.*

(2) *$p_T : X_T \longrightarrow B$ is a fibered Calabi-Yau threefold of Type II_0A whose non-singular fiber is isomorphic to the elliptic curve E_ζ. The map p_T only has singular fibers over the 27 points $\overline{Q_{ij}}(\in B)$, $(p_T)^{-1}(\overline{Q_{ij}}) = l_{ij} \cup E_{ij0} \cup E_{ij1} \cup E_{ij2}$, if $(i,j) \notin T$, and $(p_T)^{-1}(\overline{Q_{ij}}) = l'_{ij} \cup E'_{ij0} \cup E'_{ij1} \cup E'_{ij2}$, if $(i,j) \in T$, where $l'_{ij}(\simeq \mathbb{P}^1)$ is the proper transform of l_{ij} and $E'_{ijk}(\simeq \mathbb{F}_1)$ is the proper transform of E_{ijk}, for $(i,j) \in T$. Moreover, for $(i,j) \in T$, any two of E'_{ij0}, E'_{ij1} and E'_{ij2} meet transeversely along l'_{ij}, l'_{ij} is the negative section of E'_{ijk} and $N_{X_T|l'_{ij}} = \mathcal{O}_{l'_{ij}}(-1)^{\oplus 2}$ (see figure 2).*

(3) *The only isolated rational curve in $(p_T)^{-1}(\overline{Q_{ij}})$ is either l_{ij}, if $(i,j) \notin T$, or l'_{ij}, if $(i,j) \in T$.*

Proof. Everything except for the projectivity of X_T is obvious by the construction. Unfortunately, the projectivity of a threefold is not necessarily preserved under elementary transformations (cf.[O2, Theorem1] for an odd counter example). However, in our case, the next Lemma (2.5) will guarantee the projectivity. In fact, once (2.5) is proved, the divisor L_T constructed in (2.5) gives a birational contraction $f := \Phi_{|nL_T|} : X_\phi \longrightarrow Y := Im\Phi_{|nL_T|}(X_\phi) \subset \mathbb{P}^{dim|nL_T|}$, for a large n, such that $Exc(f) = \cup_{(i,j)\in T} l_{ij}$. This implies the projectivity of X_T because $\sum_{(i,j)\in T} -E'_{ij0}$ is relatively ample for the induced birational contraction $f' : X_T \longrightarrow Y \subset \mathbb{P}^{dim|nL_T|}$. q.e.d. up to (2.5).

Lemma(2.5). *There is a divisor L_T on X_ϕ such that*

(1) *L_T is nef and big, and*

(2) *$\{C \subset X_\phi | C.L_T = 0,\ C$ is an irreducble curve$\} = \{l_{ij} | (i,j) \in T\}$.*

Proof. Consider the smooth surface R_k in X_ϕ defined as the image of $\widetilde{E_\zeta^2 \times \{Q_k\}}$. It is clear that R_k is a section of p_ϕ over $B^0 := B - \cup_{(i,j)\in\Lambda^2}\overline{Q}_{ij}$ and that $R_k \cap (p_\phi)^{-1}(\overline{Q}_{ij}) = m_{ijk}$, where $m_{ijk} := R_k \cap E_{ijk}(\simeq \mathbb{P}^1)$. Note that $(m_{ijk})^2_{R_k} = -3$ and $(m_{ijk})^2_{E_{ijk}} = 1$. Since $p_\phi|R_k : R_k \longrightarrow B$ is a resolution of B with $Exc(p_\phi|R_k) = \cup_{(i,j)\in\Lambda^2} m_{ijk}$ and $3K_B \sim 0$, we can apply the adjunction formula to get $3K_{R_k} = \sum_{(i,j)\in\Lambda^2} -m_{ijk}$. Let $H := p_\phi^* H_B$ be the pull-back of a sufficiently ample divisor H_B on B. Consider the divisor $L_T := H + 4R_1 + 4R_2 + 4R_3 + E_T$, where $E_T := \sum_{(i,j,k)\in(\Lambda^2 - T)\times\Lambda} E_{ijk}$. First, we prove

Claim(2.6).

(1) $L_T.l > 0$, *for any irreducible curve* $l \subset R_k$.
(2) $L_T.l > 0$, *for any irreducible curve* $l \subset E_{ijk}$ *with* $(i,j,k) \in (\Lambda^2 - T) \times \Lambda$.

Proof of (2.6). Using the equality $R_k|_{R_k} = K_{R_k} \equiv -(1/3)\sum_{(i,j)\in\Lambda^2} m_{ijk}$, we get $L_T|R_k \equiv H|R_k - \sum_{(i,j)\in\Lambda^2} a_{ijk}m_{ijk}$, where a_{ijk} are positive rational numbers independent of the choice of H_B. Thus $L_T|R_k$ is ample on R_k since H_B is sufficiently ample. So, statement (1) follows. Using the equality $L_T.l = ((H|E_{ijk} + h).l)_{E_{ijk}}$, where $(i,j,k) \in (\Lambda^2 - T) \times \Lambda$ and $h \in |\mathcal{O}_{E_{ijk}}(1)|$, we get the result (2). q.e.d. for (2.6).

Proof of (2.5)(1). By (2.6), L_T is nef for some H_B. Hence L_T is nef and big for $2H_B$. q.e.d.

Proof of (2.5)(2). Since the inclusion $\supset$ is clear, we show the other inclusion $\subset$. Assume $L_T.l = 0$. Then by (2.6) we get $H.l = R_k.l = E_T.l = 0$. Thus l is contained in a fiber over $\overline{Q}_{ij}$ for some $(i,j) \in T$. If $l \neq l_{ij}$, then l is contained in E_{ijk} for some k. But this implies $l.R_k = (l.R_k|E_{ijk})_{E_{ijk}} > 0$. Thus $l = l_{ij}$ for some $(i,j) \in T$. q.e.d.

Proposition (2.7). $\{p_T : X_T \longrightarrow B | T \in \Omega_{II_0A}\}$ *is a minimal complete set of representatives for the orbit space* $\{p_T : X_T \longrightarrow B | T \in \Lambda^2\}/\sim_F$.

Proof. We can construct a surjective map $Aut(E_\zeta^2, Q(2)) \longrightarrow Aut(B)$ by using Galois theory and the minimality of E_ζ^2. Thus (2.7) follows from (1.7). q.e.d.

§3. Classification of fibered Calabi-Yau threefolds of Type II_0A.

In this section, we shall prove the main theorem.

Theorem (3.1). *Let* $\Phi : X \longrightarrow W$ *be a fibered Calabi-Yau threefold of Type* II_0A. *Then, there exists a unique* $T \in \Omega_{II_0A}$ *((1.6)) such that*
$(\Phi : X \longrightarrow W) \sim_F (p_T : X_T \longrightarrow B)$.

Combining (3.1) with (2.2), (2.4) and (2.7), we get:

Corollary (3.2).

(1) *Up to isomorphism as fiber spaces, there are fourteen fibered Calabi-Yau threefolds of Type* II_0A.
(2) *Every Calabi-Yau threefold with a Type* II_0A *fibration is smooth, simply-connected and has no non-trivial deformations. Moreover any two such threefolds are birationally equivalent.*

In what follows, $\Phi : X \longrightarrow W$ is assumed to be a fibered Calabi-Yau threefold of Type II_0A. Let us write $\pi : \widetilde{W} \longrightarrow W$ for the global canonical covering of W.

Lemma (3.3). $\widetilde{W} = E_\zeta^2$ *and* $W = E_\zeta^2 / < \zeta >$, *under an appropriate identification of the origins.*

Proof. Let $< g >$ $(\cong \mathbb{Z}_3)$ be the Galois group of π. Take a point O in $\widetilde{W}^{<g>}$ as the origin of $\widetilde{W}$ and choose global coordinates (w_1, w_2) of $\widetilde{W}$ such that $(0,0) = O$.

Since $dim\widetilde{W}^{<g>} = 0$ and $g^*\omega = \zeta^i\omega$, where $i = 1$ or 2 and ω is a non-zero holomorphic two-form on $\widetilde{W}$, an easy coordinate calculation implies that g is a scalar multiplication by either ζ or ζ^2. Thus, (1.4) gives the result. q.e.d.

Since π is étale over $W^0 := W - \cup_{(i,j)\in\Lambda^2}\pi(Q_{ij})$ and Φ is smooth over W^0, we have the following commutative diagram:

(3.4). *(Figure 3), where* $\phi : Z \longrightarrow E^2_\zeta$ *is a relatively minimal model of the induced morphism* $\Phi' : X \times_W E^2_\zeta \longrightarrow \widetilde{E^2_\zeta}$ *and* $\beta : X \times_W E^2_\zeta - - \rightarrow Z$ *is a birational map such that* $\beta : X \times_W E^2_\zeta - (\Phi')^{-1}(\cup_{(i,j)\in\Lambda^2}Q_{ij}) \simeq Z - (\phi)^{-1}(\cup_{(i,j)\in\Lambda^2}Q_{ij})$.

Thus, the same argument as in the proofs of (1.1) and (1.2) implies that all the fibers of ϕ are isomorphic smooth elliptic curves. In particular, Z is smooth and $\phi : Z \longrightarrow E^2_\zeta$ is the unique relatively minimal model of Φ'. Thus $< \widetilde{\zeta} >:= Gal(\mathbb{C}(Z)/\mathbb{C}(X)) \cong Gal(\mathbb{C}(E^2_\zeta)/\mathbb{C}(W)) =< \zeta >$ acts on $\phi : Z \longrightarrow E^2_\zeta$ holomorphically. Let $\overline{\phi} : Z/ < \widetilde{\zeta} > \longrightarrow W$ be the incuced morphism. The original fiber space $\Phi : X \longrightarrow W$ is recovered as one of the minimal models of $Z/ < \widetilde{\zeta} >$ for which $\Phi : X \longrightarrow W$ agrees with $\overline{\phi} : Z/ < \widetilde{\zeta} > \longrightarrow W$ over $W - Sing(W)$. Hence the following Lemma (3.5) together with (2.2), (2.4) and (2.7) implies (3.1) by virture of the flop theorem for minimal threefolds ([Ka4], [Ko, proof of Theorem 4.9]).

Lemma (3.5).

(1) $Z = E^3_\zeta$.

(2) $\widetilde{\zeta}$ *acts on* Z *as scalar multiplication by* ζ, *for an appropriate choice of the origin.*

(3) $(\phi : Z \longrightarrow E^2_\zeta) \sim_F (p_{12} : E^3_\zeta \longrightarrow E^2_\zeta)$.

Proof. First, we shall prove:

Claim (3.6). *Z is an abelian threefold.*

Proof of (3.6). We have already shown that Z is smooth. Since $\widetilde{\pi}$ is étale over $X - Sing(X)$, the isomorphism $\omega_{X-Sing(X)} \simeq \mathcal{O}_X$ implies $\omega_{Z^0} \simeq \mathcal{O}_{Z^0}$, where $Z^0 := Z - \cup_{(i,j)\in\Lambda^2}(\phi)^{-1}(Q_{ij})$. Thus $\mathcal{O}_Z(K_Z) \simeq \mathcal{O}_Z$, because $dim\cup_{(i,j)\in\Lambda^2}(\phi)^{-1}(Q_{ij}) = 1$. Put $\mathcal{H} := (R^1\phi_*\mathbb{C}_Z)\otimes\mathcal{O}_{E^2_\zeta}$. Since $\omega_{Z/E^2_\zeta} \simeq F^1(\mathcal{H}) \simeq (R^1\phi_*\mathcal{O}_Z)^{-1}$ by [N3, Theorem (3.7)], we get $(R^1\phi_*\mathcal{O}_Z) \simeq \mathcal{O}_{E^2_\zeta}$. Thus, from the Leray spectral sequence, we get the exact sequence

$$0 \longrightarrow H^1(\mathcal{O}_{E^2_\zeta}) \longrightarrow H^1(\mathcal{O}_Z) \longrightarrow H^0(\mathcal{O}_{E^2_\zeta}) \longrightarrow H^2(\mathcal{O}_{E^2_\zeta}) \longrightarrow H^2(\mathcal{O}_Z).$$

Note that $H^2(\mathcal{O}_{E^2_\zeta}) \longrightarrow H^2(\mathcal{O}_Z)$ is injective because the natural composition $H^{0,2}_{\overline{\partial}}(E^2_\zeta) \longrightarrow H^{0,2}_{\overline{\partial}}(Z) \longrightarrow H^{0,2}_{\overline{\partial}}(D)$ is injective for any smooth, ample divisor D on Z. This exact sequence gives $h^1(\mathcal{O}_Z) = 3$. Now, we can apply [Ka1, Main Theorem] to get the result. q.e.d. for (3.6).

We now return to the proof of (3.5). Note that for each $(i,j) \in \Lambda^2$, the smooth elliptic curve $C_{ij} := (\phi)^{-1}(Q_{ij})$ is stable under the action of $\widetilde{\zeta}$. The induced action $\widetilde{\zeta}|C_{ij}$ is then one of the following:

(1) $\widetilde{\zeta}|C_{ij} = id.$;

(2) $C_{ij} \simeq E_\zeta$ and $\widetilde{\zeta}|C_{ij} : z \mapsto \zeta z$;
(3) $C_{ij} \simeq E_\zeta$ and $\widetilde{\zeta}|C_{ij} : z \mapsto \zeta^2 z$;
(4) $\widetilde{\zeta}|C_{ij}$ is a translation of order 3.

Suppose (4) occurs for every $(i,j) \in \Lambda^2$. Then $Z/ < \widetilde{\zeta} >$ would be a smooth threefold with $3K_{Z/<\widetilde{\zeta}>} \sim 0$ but $K_{Z/<\widetilde{\zeta}>} \not\sim 0$. But this is absurd because $K_X \sim 0$. Hence either (1), (2) or (3) occurs for some $(i,j) \in \Lambda^2$. Take a fixed point O of $\widetilde{\zeta}|C_{ij}$ and regard it as the origin of Z. Considering a local section around O and using appropriate global coordinates (x_1, x_2, x_3) on Z with $(0,0,0) = O$, we get the following (global) descriptions of the action of $\widetilde{\zeta}$ on Z corresponding to the previous possibilities (1), (2), (3):

(1) $\widetilde{\zeta} : (x_1, x_2, x_3) \mapsto (\zeta x_1, \zeta x_2, x_3)$;
(2) $\widetilde{\zeta} : (x_1, x_2, x_3) \mapsto (\zeta x_1, \zeta x_2, \zeta^2 x_3)$;
(3) $\widetilde{\zeta} : (x_1, x_2, x_3) \mapsto (\zeta x_1, \zeta x_2, \zeta x_3)$.

Claim(3.7). *Neither (1) nor (2) occurs.*

Proof of (3.7). Assume (1) occurs. Then, by [R, Theorem (4.1)], $Z/ < \widetilde{\zeta} >$ has worse singularities than canonical singularities. Thus $\kappa(Z') = -\infty$ for a resolution of $Z/ < \widetilde{\zeta} >$. But, this is impossible. Assume (2) occurs. Then, by [R, Theorem (4.1)], $Z/ < \widetilde{\zeta} >$ has only $\mathbb{Q}$-factorial terminal singularities of index 3 with $K_{Z/<\widetilde{\zeta}>} \equiv 0$. Thus, both X and $Z/ < \widetilde{\zeta} >$ are minimal models of $\mathbb{C}(X)$. But this is absurd. q.e.d. for (3.7).

Thus (3) occurs. Now we may apply (1.4) to get (3.5)(1),(2). In order to get (3.5)(3), it is sufficient to show that ϕ has a section. Choose global coordinates (z_1, z_2, z_3) on $Z = E_\zeta^3$ such that z_i gives a global coordinate on the i-th factor E_ζ of E_ζ^3 and $(0,0,0) = O$. Using the fact $C_{ij}^{\widetilde{\zeta}} \subset (E_\zeta^3)^\zeta = Q(3)$ and (1.5), we can find an element $h \in Aut(E_\zeta^3, \{O\}) \cap Aut(E_\zeta^3, Q(3))$ such that $h(P) = Q_{100}$ for a point P in $C_{ij}^{\widetilde{\zeta}}$ different from O. Put $\varphi = \phi \circ h^{-1}$ and take Q_{ij} as the origin O of E_ζ^2. Since $\varphi(O) = O$, φ may be written as follows:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix},$$

where (w_1, w_2) are global coordinates on E_ζ^2 with $O = (0,0)$.

Since $Q_{100} = ((1-\zeta)/3, 0, 0) \in (\varphi)^{-1}(O)$, we get $a_{11} = a_{21} = 0$ and $(\varphi)^{-1}(O) = E_\zeta \times \{(0,0)\}$. Thus $\{0\} \times E_\zeta^2$ gives a section of φ. Hence $h(\{0\} \times E_\zeta^2)$ gives a section of ϕ. q.e.d. for (3.5).

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figure 1

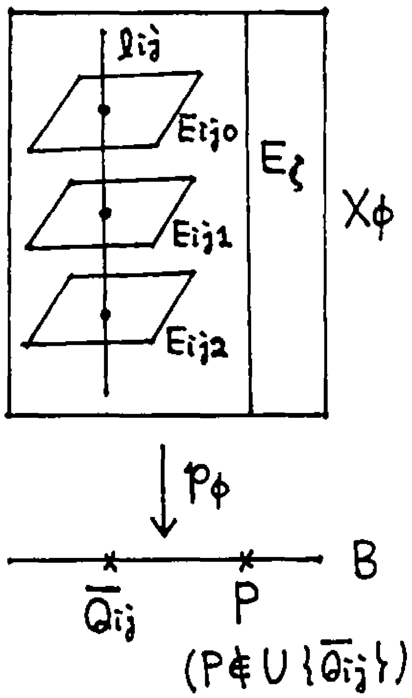

figure 2

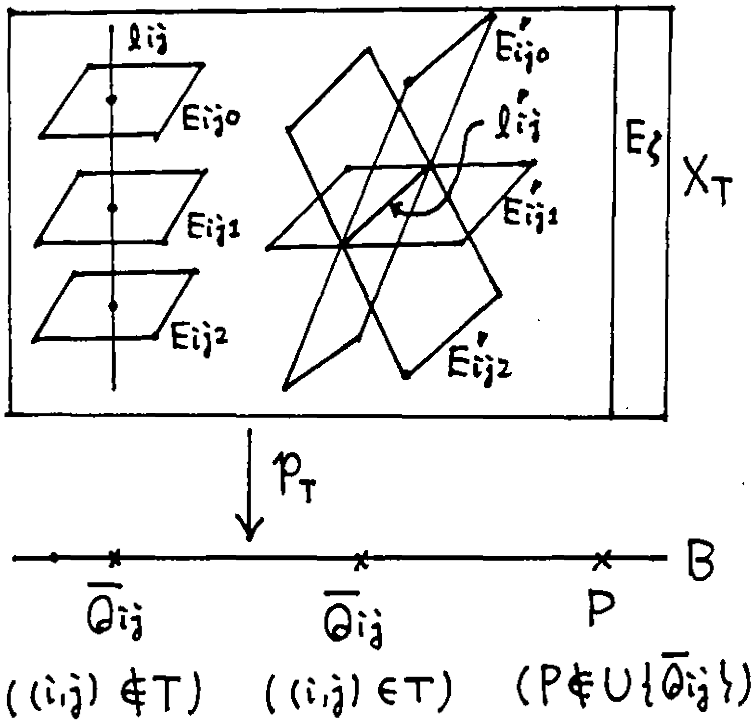

figure 3