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Homological algebra of noncommutative ’spaces’ I.

by

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Contents

Introduction

The first version of this work came from a lecture course in noncommutative algebraic geometry and related (views on) homological algebra and, in particular, K-theory, which was given at Kansas State University during the Spring of 2006. It was the half of century anniversary of the appearance of Cartan-Eilenberg book and forty nine years after Grothendieck's Tôhoku lectures which based homological algebra on abelian categories.

It seemed timely to look at homological functors from the classical point of view, but, in a very different, much more general, setting: abelian categories were replaced by *right exact* categories. The latter are Grothendieck presites whose covers are strict epimorphisms. The dual structures, *left* exact categories, appear naturally and play a crucial role in a version of K-theory sketched in Chapters V and VI of this work.

It is worth to mention that the starting point of the actual lectures (which is behind the scene in this manuscript) was the homological algebra of exact categories as it is viewed by Keller and Vossieck [KeV]. Besides an optimization of the Quillen's definition of an exact category, they observed that the stable categories of exact categories with enough injective objects have a *suspension* functor and *triangles* whose properties give a 'one-sided' version of Verdier's triangulated category, which they call a *suspended* category. A short exposition of this topic is given in Appendix K.

The next opportunity to look at the subject was preparing my lectures (and especially their extended notes several months later) at the School on Algebraic K-Theory and Applications which took place at the International Center for Theoretical Physics (ICTP) in Trieste during the last two weeks of May of 2007 (see [R2], or [R3]).

A detailed exposition of the homological algebra and K-theory part (– the last three lectures) of [R2] can be found in [R5].

The present text is a considerable refinement of [R5]. It is based on lecture courses on homological algebra and K-theory given at Kansas State University in the Spring of 2009 and in the Fall of 2010.

In Chapter I, we introduce *right exact* (not necessarily additive) categories and sketch their basic properties. In particular, we define *Karoubian* right exact categories and prove the existence of *Karoubian envelopes.* We introduce the notion of a kernel of a morphism in a category with initial objects and study the elementary properties of kernels, which are well known in the abelian case. The properties of kernels are used then for studying right exact categories with initial objects. In the last sections, we look at exact categories. We observe, among other things, that any k-linear right exact category is canonically realized as a subcategory of an exact k-linear category – its *exact envelope.*

Chapter II is dedicated mostly to homological functors on right exact categories, otherwise called *left satellites*, or *universal* ∂^*-functors. Its content might be regarded as a non-abelian and non-additive (that is not necessarily abelian or additive) version of the classical theory of homological functors. We start with preliminaries on trivial morphisms, pointed objects and complexes. Then we introduce ∂^*-functors, and prove the existence of (by producing a formula for) left derived functors of any functor from any right exact category to a category with kernels of morphisms and limits of filtered diagrams. We look at contravariant functoriality of universal ∂^*-functors. One of its applications is replacing the computation of universal ∂^*-functors from a right exact category by computation of universal ∂^*-functors from the category of non-trivial sheaves of sets on it endowed with the canonical right exact structure formed by epimorphisms. The k-linear version of this fact replaces computation of universal k-linear ∂^*-functors from a k-linear right exact category by the computation of the corresponding ∂^*-functors from the canonically associated k-linear Grothendieck category. We consider the dual notion – ∂-functors, and introduce the higher Exts. We establish 'exactness' of ∂^*-functors whose zero component is *right 'semi-exact'* and the target right exact category satisfies an analog of the Grothendieck's (AB5*) property. Then we consider the category of universal ∂^*-functors from a right exact category with values in categories with initial objects and prove that this category has an initial object, which is the ∂^*-functor $Ext^\bullet$. We establish a similar fact in k-linear setting. We show that the initial universal ∂^*-functor $Ext^\bullet$ is also an initial object for the (appropriately defined) category of universal 'exact' functors from a fixed right exact category. We conclude the chapter with a short discussion of relative satellites.

We start Chapter III with studying projective objects of a right exact category and right exact categories with enough projective objects. We observe that projective objects are compatible with the contravariant functoriality of universal ∂^*-functors discussed in Chapter II. In particular, the canonical embedding of a right exact category into the category of non-trivial sheaves of sets on it maps projective objects to projective objects; and if the right exact category has enough projective objects, same holds for the category of sheaves of sets. Projective objects play approximately the same role as in the classical case: higher components of every universal ∂^*-functor annihilate *pointed* (– having morphisms to initial objects) projective objects; and if the right exact category has enough projective objects, then every 'exact' ∂^*-functor which annihilates all projective objects is universal.

Analyzing the structure of universal cohomological functors and results of Chapter II leads to an observation that the information on all universal functors from a given left exact category $(C_X, \mathfrak{I}_X)$ is encoded in a canonical structure of a $\mathbb{Z}_+$-category on the category of non-trivial presheaves of sets on C_X (induced by the functor Ext^1) and the category of *standard triangles* related with conflations of the left exact category $(C_X, \mathfrak{I}_X)$. We apply the obtained structure to producing formulas for satellites of composition of functors.

Then we look at computational aspects of satellites of functors F from left exact categories such that their domain has enough F-acyclic objects. We introduce F-acyclic resolution, cohomology of complexes, and show that if the functor F is weakly left 'exact' and maps inflations with trivial cokernels to isomorphisms, then its satellites are isomorphic to cohomologies of the images of acyclic resolutions, similarly to the classical, abelian, case.

We return to studying the structure of cohomological functors and define *prestable* and *stable* categories of a left exact category. Turning the properties of prestable and stable categories into axioms, we introduce the notions of *presuspended* and *quasi-suspended* categories. We conclude with definition of homology of 'spaces' with coefficients in a right exact category and the homotopy groups of pointed 'spaces'.

Starting from Chapter IV, a noncommutative geometric flavor (introduced by passing in Chapter I) becomes a permanent part of the picture: we interpret svelte right exact categories as dual objects to (noncommutative) *right exact* 'spaces' and 'exact' functors between them as inverse image functors of morphisms of 'spaces'. After introducing and studying certain canonical left exact structures on the category $|Cat|^o$ of 'spaces' represented by categories, we define a family of canonical left exact structures on the category of *right exact 'spaces'* and their 'exact' morphisms (that is morphisms whose inverse image functors are 'exact'). We show that each of these canonical left exact structures has enough injective objects by producing natural inflations from any right exact 'space' into an associated with it injective. We explain the k-linear version of these facts with some ramifications. In particular, we prove the existence of enough injective objects and use this to establish a similar fact for the full subcategory of the category of right exact k-'spaces' formed by 'spaces' represented by Karoubian categories and for the left exact category of *exact k-'spaces'* (that is 'spaces' represented by exact k-linear categories). We conclude with a couple of miscellaneous complements: introducing the *path 'space'* of a right exact 'space' and a short discussion of localizations of right exact 'spaces'.

Chapter V is dedicated to the first applications: the *universal* K-theory of right exact 'spaces'. We define a contravariant functor K_0 from the category of right exact 'spaces' to the category of abelian groups and prove that K_0 is right 'exact' with respect to a certain canonical left exact structure on the category of right exact 'spaces'.

The category of right exact 'spaces' does not have final objects; so that we cannot apply the formalism of cohomological functors developed in Chapters II. A natural way to acquire final objects is to consider the category of right exact 'spaces' over a 'space'. We do this defining the *relative* K_0-functors and their derived functors with respect to a left exact structure on the category of right exact 'spaces' over a right exact 'space'.

Most of the chapter is dedicated to the case of 'spaces' over the 'point' – the category $\mathfrak{Esp}_{\mathfrak{r}}^*$ of right exact 'spaces' over the standard initial object x represented by the category with one morphism. The 'space' x is interpreted as the affine scheme associated with the "field" $\mathbb{F}_1$. So that $\mathfrak{Esp}_{\mathfrak{r}}^*$ can be regarded as the category of right exact 'spaces' over $\mathbb{F}_1$. It is endowed with a canonical left exact structure. The left exact category $\mathfrak{Esp}_{\mathfrak{r}}^*$ is used as a device for producing higher K-theories on other left exact categories. Namely, every functor from a left exact category $(C_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ (having final objects) to $\mathfrak{Esp}_{\mathfrak{r}}^*$ which preserves conflations gives rise to an 'exact' higher K-theory on the left exact category $(C_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$. We apply this consideration to obtain the universal K-theory of ('spaces' represented by) svelte abelian categories and the universal K-theory of k-linear exact categories.

Then we start creating the standard tools of higher K-theory which generalize the corresponding facts of Quillen's K-theory: reduction by resolution and characteristic filtrations and sequences. We conclude the chapter with extending the Quillen's Q-construction to right exact categories with initial objects.

In Chapter VI, we introduce topologizing subcategories of a right exact 'space', their *infinitesimal* neighborhoods, and the (left exact) category of *relative* right exact 'spaces'. We establish some general facts about devissage of higher images of functors and then, as an application, obtain the devissage theorem in higher K-theory.

The next several sections appear under the general title "complementary facts". We start with some examples of kernels and cokernels and simple general constructions and observations, which acquire importance somewhere in the text. Then we spend some effort on expanding standard facts on diagram chasing to right exact categories. Then follow some facts on localizations of exact and (co)suspended categories. In particular, t-structures of (co)suspended categories appear on the scene. Again, a work by Keller and Vossieck, [KV1], suggested the notions. We consider cohomological functors on suspended categories with values in exact categories and prove the existence of a universal cohomological functor. The construction of the universal functor gives, among other consequences, an equivalence between the bicategory of Karoubian suspended svelte categories with triangle functors as 1-morphisms and the bicategory of exact svelte $\mathbb{Z}_+$-categories with enough injective objects whose 1-morphisms are 'exact' functors. We show that if the suspended category is triangulated, then the universal cohomological functor takes values in an abelian category, and our construction recovers the abelianization of triangulated categories by Verdier [Ve2]. It is also observed that the *triangulation* of suspended categories induces an abelianization of the corresponding exact $\mathbb{Z}_+$-categories. We conclude with a discussion of homological dimension and resolutions of suspended categories and exact categories with enough injective objects. These resolutions suggest that the 'right' objects to consider from the very beginning are exact (resp. abelian) and (co)suspended (resp. triangulated) $\mathbb{Z}_+^n$-categories. All the previously discussed facts (including the content of Appendix K) extend easily to this setting. We define the *weak costable* category of a right exact category as the localization of the right exact category at a certain class of arrows related with its projective objects. If the right exact category in question is exact, then its costable category is isomorphic to the costable category in the conventional sense. If a right exact category has enough *pointable* object (in which case all its projective objects are pointable), then its weak costable category is naturally equivalent to the *costable* category of this right exact category introduced in Chapter 4. We study right exact categories of modules over monads and associated stable and costable categories. The general constructions acquire here a concrete shape. We introduce the notion of a *Frobenius* monad. The category of modules over a Frobenius monad is a Frobenius category, hence its stable category is triangulated. We consider the case of modules over an augmented monad, which includes, as special cases, most of standard homological algebra based on complexes and their homotopy and derived categories.

The main purpose of Chapter VII is to extend basic notions and constructions of homological algebra to arbitrary right and left exact categories. This means that we do not require a priori the existence of initial (resp. final) objects in our right (resp. left) exact categories, or in the categories, in which (co)homological functors take their values.

We start with a natural definition of kernels of arrows in an arbitrary category and show that the main properties of kernels summarized in Chapter I hold in the general setting. In order to acquire flexibility, we introduce the notion of a *virtual* kernel, which is a morphism of presheaves of sets. The virtual kernel is a kernel iff this morphism is representable. It turns out that the existence of morphisms with non-trivial virtual kernels (which is a necessary condition of non-triviality of our version of homological algebra) imposes a very precise choice of categories: they should be *virtually semi-complete*, which means, by definition, that each connected component has *pointed objects*, or what is the same, morphisms from constant functors to the identical functor. We introduce ∂^*-functors from a right exact category to an arbitrary category, define *universal* ∂^*-functors (otherwise called *right derived* functors of their zero component) in a standard way (that is by a universal property) and constructively prove their existence (i.e. write a formula) in the case when the target category has pull-backs and limits of filtered diagrams. By duality, we obtain ∂-functors from a left exact category to an arbitrary category and *universal* ∂-functors, otherwise called *left derived* functors. We consider the category of universal ∂^*-functors from a given right exact category taking values in virtually semi-complete categories and establish the existence of an initial object of this category – the functor $Ext^\bullet$, in the case when the right exact category in question is virtually semi-complete. Following the scenario of Chapter III, we define the *stable* category of the category of presheaves of sets associated with a virtually semi-complete left exact category and define prestable and stable categories of a left (or right) exact category. We give a brief account on 'exactness' properties of derived functors and, generalizing and using the corresponding fact of Chapter II, show that, under certain condition on the target right exact category, 'exact' ∂^*-functors are universal. We conclude with a couple of examples-applications: homology of 'spaces' with coefficients in arbitrary right exact category, and the "absolute" higher K-theory of arbitrary right exact 'spaces', which gives rise to absolute K-theories of arbitrary left exact categories over the left exact category of right exact 'spaces'.

Appendix K (where 'K' stands for Bernhard Keller) is dedicated mostly to suspended categories of exact k-linear categories.

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Chapter I

Right Exact Categories.

In Section 1, we introduce right exact categories and different classes of functors between them defined by different 'exactness' properties. Svelte right exact categories are regarded as representatives of *right exact 'spaces'* and functors between right exact categories are interpreted as inverse image functors of morphisms of right exact 'spaces'. In Section 2, we discuss the canonical embedding of a right exact category into the right exact category of sheaves of sets on it and the k-linear version of this embedding. In Section 3, we introduce *Karoubian* (not necessarily additive) categories and *Karoubian right exact* categories. We prove that every category has *Karoubian envelope* and show that same holds for right exact categories. In Section 4, we introduce the notions of kernels and coimages of morphisms in categories with initial objects (dually, the notions of cokernels and images of morphisms in categories with final objects) and study their basic properties which are used through the whole work (and beyond) starting from Section 5, where we discuss shortly right exact categories with initial objects. Every deflation of a right exact category with an initial object has a kernel, which allows to introduce analogs of short exact sequences, which we call (extending Gabriel's terminology) *conflations*. Conflations are interpreted as *extensions*, which prompts definition of *fully exact* subcategories of a right exact category as its full subcategories closed under extensions. In Section 7, we recover the main properties of exact k-linear categories complementing already obtained facts on right exact categories. Section 8 discusses some complements on exact categories.

1. Right exact categories and (right) 'exact' functors.

1.1. Right exact categories. We define a *right exact* category as a pair $(C_X, \mathfrak{E}_X)$, where C_X is a category and $\mathfrak{E}_X$ is a pretopology on C_X whose covers are *strict epimorphisms*; that is, for any element $M \xrightarrow{\mathfrak{s}} L$ of $\mathfrak{E}_X$ (– a cover), the diagram

$$\mathcal{K}_2(\mathfrak{s}) \overset{\mathfrak{s}_1}{\underset{\mathfrak{s}_2}{\rightrightarrows}} M \xrightarrow{\ \mathfrak{s}\ } L$$

is exact. Here $\mathcal{K}_2(\mathfrak{s}) = M \times_L M$ of $M \xrightarrow{\mathfrak{s}} L$ is the *kernel pair* of the morphism $M \xrightarrow{\mathfrak{s}} L$ and $\mathcal{K}_2(\mathfrak{s}) \overset{\mathfrak{s}_1}{\underset{\mathfrak{s}_2}{\rightrightarrows}} M$ the canonical projections – pull-backs of $M \xrightarrow{\mathfrak{s}} L$ along itself.

This exactness of the diagram means precisely that the pretopology $\mathfrak{E}_X$ is subcanonical; i.e. every representable presheaf of sets on C_X is a sheaf on $(C_X, \mathfrak{E}_X)$.

We call the elements of $\mathfrak{E}_X$ *deflations* and assume that all isomorphisms are deflations.

1.2. The coarsest and the finest right exact structures. The coarsest right exact structure on a category C_X is the discrete pretopology: the class of deflations coincides with the class $Iso(C_X)$ of all isomorphisms of the category C_X.

1.2.1. The canonical right exact structure. Right exact structures on a category C_X form a filtered family: if $\{\mathfrak{E}_i \mid i \in I\}$ is a set of right exact structures, then all possible compositions of arrows from $\mathfrak{E}_i,\ i \in I$, form a right exact structure which we denote by $\sup_{i\in I} \mathfrak{E}_i$. This is the coarsest common refinement of all right exact structures $\mathfrak{E}_i,\ i \in I$.

In particular, it follows that the union of all right exact structures on the category C_X is a right exact structure, which we call *canonical* and denote by $\mathfrak{E}^{\mathfrak{s}}_X$.

The canonical right exact structure can be described directly as follows.

1.2.2. Proposition. *The canonical right exact structure $\mathfrak{E}^{\mathfrak{s}}_X$ on the category C_X consists of universally strict epimorphisms; that is morphisms whose arbitrary pull-backs (in particular, themselves) exist and are strict epimorphisms.*

Proof. (i) Let $\mathcal{L}_1 \xrightarrow{\mathfrak{t}_1} \mathcal{L}_2$ and $\mathcal{L}_2 \xrightarrow{\mathfrak{t}_2} \mathcal{L}_3$ be strict epimorphisms whose pull-backs along strict epimorphisms are strict epimorphisms. Then their composition, $\mathcal{L}_1 \xrightarrow{\mathfrak{j}_2\circ\mathfrak{j}_1} \mathcal{L}_3$, is a strict epimorphism.

The kernel pair of the composition $\mathcal{L}_1 \xrightarrow{\mathfrak{t}_2\circ\mathfrak{t}_1} \mathcal{L}_3$ is naturally decomposed into the diagram

$$\begin{array}{ccccc}
\mathcal{K}_2(\mathfrak{t}_2\circ\mathfrak{t}_1) & \xrightarrow{\mathfrak{t}''_1} & \mathcal{K}_{12} & \xrightarrow{\mathfrak{t}''_2} & \mathcal{L}_1 \\
\widetilde{\mathfrak{p}}_1 \downarrow & cart & \mathfrak{p}_1 \downarrow & cart & \downarrow \mathfrak{t}_1 \\
\mathcal{K}_{12} & \xrightarrow{\mathfrak{t}'_1} & \mathcal{K}_2(\mathfrak{t}_2) & \xrightarrow{\mathfrak{t}'_2} & \mathcal{L}_2 \\
\widetilde{\mathfrak{p}}_2 \downarrow & cart & \pi_2 \downarrow & cart & \downarrow \mathfrak{t}_2 \\
\mathcal{L}_1 & \xrightarrow{\mathfrak{t}_1} & \mathcal{L}_2 & \xrightarrow{\mathfrak{t}_2} & \mathcal{L}_3
\end{array} \qquad (1)$$

whose all squares are cartesian.

For any morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{N}$, let $\Lambda^o(\mathfrak{f})$ denote the class of all pairs of arrows $\mathcal{V} \rightrightarrows \mathcal{M}$ which are equalized by the morphism $\mathfrak{f}$.

Let $\mathcal{L}_1 \xrightarrow{\xi} \mathcal{V}$ be a morphism such that $\Lambda^o(\mathfrak{t}_2\circ\mathfrak{t}_1) \subseteq \Lambda^o(\xi)$. In particular, $\Lambda^o(\mathfrak{t}_1) \subseteq \Lambda^o(\xi)$. The latter implies that $\xi = \xi_1\circ\mathfrak{t}_1$ for a uniquely defined morphism $\mathcal{L}_1 \xrightarrow{\xi_1} \mathcal{V}$. The inclusion $\Lambda^o(\mathfrak{t}_2\circ\mathfrak{t}_1) \subseteq \Lambda^o(\xi) = \Lambda^o(\xi_1\circ\mathfrak{t}_1)$ implies (actually, means) that

$$\xi_1\circ\mathfrak{t}_1\circ(\widetilde{\mathfrak{p}}_2\circ\widetilde{\mathfrak{p}}_1) = \xi_1\circ\mathfrak{t}_1\circ(\mathfrak{t}''_2\circ\mathfrak{t}''_1).$$

It follows from the commutativity of the diagram (1) that

$$\xi_1\circ\mathfrak{t}_1\circ(\widetilde{\mathfrak{p}}_2\circ\widetilde{\mathfrak{p}}_1) = (\xi_1\circ\mathfrak{p}_2)\circ(\mathfrak{p}_1\circ\mathfrak{t}''_1) \quad \text{and}$$
$$\xi_1\circ\mathfrak{t}_1\circ(\mathfrak{t}''_2\circ\mathfrak{t}''_1) = (\xi_1\circ\mathfrak{t}'_2)\circ(\mathfrak{p}_1\circ\mathfrak{t}''_1).$$

So that

$$(\xi_1 \circ \mathfrak{p}_2) \circ (\mathfrak{p}_1 \circ \mathfrak{t}_1'') = (\xi_1 \circ \mathfrak{t}_2') \circ (\mathfrak{p}_1 \circ \mathfrak{t}_1''). \tag{2}$$

Since $\mathfrak{p}_1$ and $\mathfrak{t}_1''$ are (strict) epimorphisms, their composition $\mathfrak{p}_1 \circ \mathfrak{t}_1''$ is an epimorphism. Therefore, it follows from the equality (2) that $\xi_1 \circ \mathfrak{p}_2 = \xi_1 \circ \mathfrak{t}_2'$. By hypothesis, $\mathfrak{t}_2$ is a strict epimorphism, that is the cokernel of the pair of arrows $\mathcal{K}_2(\mathfrak{t}_2) \overset{\mathfrak{t}_2'}{\underset{\mathfrak{p}_2}{\rightrightarrows}} \mathcal{L}_2$ (see the lower right square of the diagram (1)). Therefore, $\xi_1 = \xi_2 \circ \mathfrak{t}_2$ for a unique morphism $\mathcal{L}_2 \xrightarrow{\xi_2} \mathcal{V}$.

(ii) Since a pull-back of a composition of morphisms having pull-backs is the composition of pull-backs, it follows from (i) that the composition of universal strict epimorphisms is a strict epimorphism. ■

1.3. Special cases and examples.

1.3.1. Abelian categories and toposes. If C_X is an abelian category or a topos, or a category dual to a topos. Then the canonical right exact structure $\mathfrak{E}^{\mathfrak{s}}_X$ consists of all epimorphisms of the category C_X.

1.3.2. Quasi-abelian categories. A *quasi-abelian category* is an additive category C_X with kernels and cokernels and such that every pullback of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism.

It follows from the definition that if C_X is a quasi-abelian category, then $\mathfrak{E}^{\mathfrak{s}}_X$ consists of all strict epimorphisms.

Notice that abelian categories can be described as quasi-abelian categories in which every epimorphism is strict.

1.3.3. Example. Let C_X be the category Alg_k of associative unital k-algebras. The category Alg_k has arbitrary limits; in particular, it has fiber products. Therefore, the finest right exact structure, $\mathfrak{E}^{\mathfrak{s}}_X$, consists of all strict epimorphisms. The latter are precisely surjective morphisms of algebras.

1.4. Right 'exact' and 'exact' functors. Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories. A functor $C_X \xrightarrow{F} C_Y$ will be called *right 'exact'* (resp. *'exact'*) if it maps deflations to deflations and for any deflation $M \xrightarrow{\mathfrak{e}} N$ of $\mathfrak{E}_X$ and any morphism $\widetilde{N} \xrightarrow{f} N$, the canonical arrow

$$F(\widetilde{N} \times_N M) \longrightarrow F(\widetilde{N}) \times_{F(N)} F(M)$$

is a deflation (resp. an isomorphism). Thus, the functor F is 'exact' if it maps deflations to deflations and preserves pull-backs of deflations.

1.5. Weakly right 'exact' and weakly 'exact' functors. A functor $C_X \xrightarrow{F} C_Y$ is called *weakly right 'exact'* (resp. *weakly 'exact'*) if it maps deflations to deflations and for any arrow $M \longrightarrow N$ of $\mathfrak{E}_X$, the canonical morphism

$$F(M \times_N M) \longrightarrow F(M) \times_{F(N)} F(M)$$

is a deflation (resp. an isomorphism). In particular, weakly 'exact' functors are weakly right 'exact'.

1.5.1. Note. Of coarse, 'exact' (resp. right 'exact') functors are weakly 'exact' (resp. weakly right 'exact'). In the additive (actually, a more general) case, weakly 'exact' functors are 'exact' (see 2.5 and 2.5.2).

1.6. Interpretation: 'spaces' represented by right exact categories. We consider the category $\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}}$ whose objects are pairs $(X, \mathfrak{E}_X)$, where $(C_X, \mathfrak{E}_X)$ is a svelte right exact category. A morphism from $(X, \mathfrak{E}_X)$ to $(Y, \mathfrak{E}_Y)$ is a morphism of 'spaces' $X \xrightarrow{\varphi} Y$ whose inverse image functor $C_Y \xrightarrow{\varphi^*} C_X$ is a right weakly 'exact' functor from $(C_Y, \mathfrak{E}_Y)$ to $(C_X, \mathfrak{E}_X)$. We denote by $\mathfrak{Esp}_{\mathfrak{r}}$ the subcategory of the category $\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}}$ formed by right exact 'spaces' and *'exact' morphisms*, which is the name used for morphisms having 'exact' inverse image functors.

The map which assigns to every 'space' X the pair $(X, Iso(C_X))$ is a full embedding of the category $|Cat|^o$ of 'spaces' into the category $\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}}$. This full embedding is a right adjoint functor to the forgetful functor

$$\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}} \longrightarrow |Cat|^o, \quad (X, \mathfrak{E}_X) \longmapsto X.$$

2. The canonical embedding.

2.0. Preliminaries: functors, (pre)sheaves of sets, and continuous functors.

2.0.1. Notations and conventions.

2.0.1.1. Non-trivial presheaves of sets. Let C_X be a svelte category.

We denote by $\bar{\emptyset}$, or $\bar{\emptyset}_X$, the *trivial* presheaf of sets on C_X; that is the presheaf which maps all objects of C_X to the empty set. The presheaf $\bar{\emptyset}_X$ is the unique initial object of the category $C_X^\wedge$ of presheaves of sets on C_X.

We denote by C_X^* the full subcategory of the category $C_X^\wedge$ formed by all *non-trivial* presheaves of sets; that is we exclude the trivial presheaf $\bar{\emptyset}_X$.

The category $C_X^\wedge$ has a final object, which is the constant presheaf with values in a one-element set. In particular, it is a final object of the category C_X^*. If the category C_X has a final object, y, then $\widehat{y} = C_X(-, y)$ is a final object of the category $C_X^\wedge$.

Notice that the coproduct of any set of non-trivial presheaves of sets is a non-trivial presheaf; and the cokernel of a pair of arrows $\mathcal{F} \rightrightarrows \mathcal{G}$ between non-trivial presheaves is a non-trivial presheaf too (which follows from the corresponding fact for non-empty sets). Therefore, the category C_X^* has colimits of arbitrary small diagrams.

2.0.1.2. Let τ be a (pre)topology on a svelte category C_X. We denote by $(C_X, \tau)^\wedge$ the category of all sheaves of sets on the (pre)site (C_X, τ).

2.0.1.3. We denote by C_{X_τ} the full subcategory of $(C_X, \tau)^\wedge$ generated by non-trivial sheaves; that is $C_{X_\tau} = (C_X, \tau)^\wedge \cap C_X^*$.

If $(C_X, \mathfrak{E}_X)$ is a right exact category, then we usually denote the category of *non-trivial* sheaves of sets on $(C_X, \mathfrak{E}_X)$ by $C_{X_\mathfrak{E}}$ instead of $C_{X_{\mathfrak{E}_X}}$.

2.0.1.4. Conventions. Recall that a category is called *cocomplete* if it has colimits and initial objects. The latter are sometimes interpreted as colimits of the empty diagram. When we say that a category *has colimits*, this is not the same as a *cocomplete* category – only colimits of non-empty diagrams are considered. Thus, the category C_X^* of non-trivial presheaves of sets on a category C_X has colimits, but, it does not have initial objects.

A similar convention in the dual setting: *categories with limits* are categories with limits of non-empty small diagrams. *complete* categories are categories with limits and final objects (interpreted as limits of the empty diagram).

We need a slightly more elaborate version of the canonical extension of functors onto the category of presheaves of sets [GZ, II.1.3], which is as follows.

2.0.2. Proposition. *Let C_X be a svelte category and C_Y a category with colimits.*

(a) The functor of the composition with the Yoneda embedding $C_X \xrightarrow{h_X^*} C_X^*$,

$$\mathcal{H}om(C_X^*, C_Y) \xrightarrow{\widetilde{h}_X^*} \mathcal{H}om(C_X, C_Y), \quad G \longmapsto G \circ h_X^*, \tag{1}$$

has a fully faithful left adjoint,

$$\mathcal{H}om(C_X, C_Y) \xrightarrow{\widetilde{h}_{X!}} \mathcal{H}om(C_X^*, C_Y). \tag{2}$$

The functor $\widetilde{h}_{X!}$ establishes an equivalence between the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y and the full subcategory $\mathfrak{Hom}(C_X^*, C_Y)$ of the category $\mathcal{H}om(C_X^*, C_Y)$ generated by all functors $C_X^* \longrightarrow C_Y$ preserving colimits.

(b) Suppose that the category C_X has an initial object $\mathfrak{x}$. Let $C_X^{\circledast} \stackrel{\mathrm{def}}{=} \widehat{\mathfrak{x}} \backslash C_X^* = \widehat{\mathfrak{x}} \backslash C_X^\wedge$ and $C_X \xrightarrow{h_X^{\circledast}} C_X^{\circledast}$ the canonical fully faithful functor induced by Yoneda embedding.

The following conditions are equivalent:

(b1) There exists a continuous (i.e. having a right adjoint) functor from $C_X^{\circledast}$ to C_Y.

(b2) The category C_Y has initial objects.

If the equivalent conditions (b1), (b2) hold, then the functor

$$\mathcal{H}om(C_X^{\circledast}, C_Y) \xrightarrow{\widetilde{h}_X^{\circledast}} \mathcal{H}om(C_X, C_Y), \quad G \longmapsto G \circ h_X^{\circledast}, \tag{3}$$

establishes an equivalence between the full subcategory $\mathcal{H}om^{\circledast}(C_X, C_Y)$ *of the category* $\mathcal{H}om(C_X, C_Y)$ *generated by functors mapping initial objects to initial objects and the full subcategory* $\mathcal{H}om_{\mathfrak{c}}(C_X^{\circledast}, C_Y)$ *of the category* $\mathcal{H}om(C_X^{\circledast}, C_Y)$ *generated by all continuous functors* $C_X^{\circledast} \longrightarrow C_Y$.

(c) Suppose that the category C_Y *has initial objects (i.e. it is cocomplete). Then the functor of the composition with the Yoneda embedding* $C_X \xrightarrow{h_X} C_X^{\wedge}$

$$\mathcal{H}om(C_X^{\wedge}, C_Y) \xrightarrow{\widetilde{h}_X} \mathcal{H}om(C_X, C_Y), \quad G \longmapsto G \circ h_X,$$

establishes an equivalence between the category $\mathcal{H}om(C_X, C_Y)$ *of functors from* C_X *to* C_Y *and the full subcategory* $\mathcal{H}om_{\mathfrak{c}}(C_X^{\wedge}, C_Y)$ *of the category* $\mathcal{H}om(C_X^{\wedge}, C_Y)$ *generated by all continuous functors* $C_X^{\wedge} \longrightarrow C_Y$.

(c1) If the category C_Y *is cocomplete, then a functor* $C_X^{\wedge} \longrightarrow C_Y$ *is continuous iff it preserves colimits of all small diagrams (including the empty diagram: i.e. it maps initial objects to initial objects).*

(c2) Let C_X *and* C_Y *be categories with final objects and* $\mathcal{H}om(C_X, C_Y)^{\circledast}$ *the full subcategory of the category* $\mathcal{H}om(C_X, C_Y)$ *generated by functors mapping final objects to final objects. The functor* $\mathcal{G} \longmapsto \mathcal{G} \circ h_X$ *induces an equivalence of the category* $\mathcal{H}om(C_X, C_Y)^{\circledast}$ *and the full subcategory*

$$\mathcal{H}om_{\mathfrak{c}}(C_X^{\wedge}, C_Y)^{\circledast} \stackrel{\mathrm{def}}{=} \mathcal{H}om(C_X^{\wedge}, C_Y)^{\circledast} \bigcap \mathcal{H}om_{\mathfrak{c}}(C_X^{\wedge}, C_Y)$$

of the category $\mathcal{H}om(C_X^{\wedge}, C_Y)$ *generated by continuous functors mapping final objects to final objects.*

Proof. (a) For every functor $C_X \xrightarrow{F} C_Y$, we denote by $F^{\diamond}$ the functor $C_X^* \longrightarrow C_Y$, which assigns to every non-trivial presheaf of sets $\mathcal{G}$ on C_X the colimit of the composition of the forgetful functor $h_X^*/\mathcal{G} \longrightarrow C_X$ and the functor $C_X \xrightarrow{F} C_Y$.

The map $F \longmapsto F^{\diamond}$ extends to a functor

$$\mathcal{H}om(C_X, C_Y) \xrightarrow{\widetilde{h}_{X!}} \mathcal{H}om(C_X^*, C_Y)$$

which is left adjoint to the functor (1).

It follows from this definition that the composition of the functor $F^{\diamond}$ with Yoneda embedding coincides with the functor F. This means that one of the adjunction morphisms is an isomorphism, which implies that the functor (1) is a localization its left adjoint, $F \longmapsto F^{\diamond}$, is fully faithful. The functor $F^{\diamond}$ preserves colimits, because every presheaf of sets $\mathcal{G}$ is isomorphic to the colimit of the composition of the forgetful functor $h_X^*/\mathcal{G} \longrightarrow C_X$ with the Yoneda embedding $C_X \xrightarrow{h_X^*} C_X^*$. This also implies that if $C_X^* \xrightarrow{\mathfrak{F}} C_Y$ is a functor preserving colimits, then the natural morphism $(\mathfrak{F} \circ h_X^*)^{\diamond} \longrightarrow \mathfrak{F}$ is an isomorphism.

All together shows that the fully faithful functor $F \longmapsto F^\diamond$ induces an equivalence between the category $\mathcal{H}om(C_X, C_Y)$ and the full subcategory $\mathfrak{Hom}(C_X^*, C_Y)$ of the category $\mathcal{H}om(C_X^*, C_Y)$ generated by all functors $C_X^* \longrightarrow C_Y$ preserving colimits.

(b) The full subcategory $\mathcal{H}om_\mathfrak{c}(C_X^{\circledast}, C_Y)$ of the category $\mathcal{H}om(C_X^*, C_Y)$ generated by continuous functors is contained in the subcategory $\mathfrak{Hom}(C_X^{\circledast}, C_Y)$, because continuous functors preserve colimits.

(b1)⇒(b2). If the category C_X has an initial object, $\mathfrak{x}$, then $(\widehat{\mathfrak{x}} = C_X(-,\mathfrak{x}), id_\mathfrak{x})$ is an initial object of the category $C_X^{\circledast}$. Continuous functors map initial objects to initial objects. So that if there exist continuous functors from $C_X^{\circledast}$ to C_Y, then the category C_Y has initial objects.

(b2)⇒(b1). Let the category C_Y have initial objects, and let $C_X \xrightarrow{F} C_Y$ be a functor which maps initial objects to initial objects. We denote by $C_Y \xrightarrow{F_\diamond} C_X^{\circledast}$ the functor which maps every object $\mathcal{L}$ of the category C_Y to the pair $(C_Y(F(-),\mathcal{L}), \zeta_\mathcal{L})$, where $\zeta_\mathcal{L}$ is the composition of the morphism $\widehat{\mathfrak{x}} = C_X(-,\mathfrak{x}) \xrightarrow{F_{-,\mathfrak{x}}} C_Y(F(-), F(\mathfrak{x}))$ and the map $C_Y(F(-), F(\mathfrak{x})) \longrightarrow C_Y(F(-),\mathcal{L})$ corresponding to the unique morphism $F(\mathfrak{x}) \longrightarrow \mathcal{L}$.

One can see that the functor $F_\diamond$ is a right adjoint to the composition of the forgetful functor $C_X^{\circledast} \xrightarrow{\mathfrak{f}_*} C_X^*$ and the functor $C_X^* \xrightarrow{F^\diamond} C_Y$ constructed in (a) above.

In fact, for every $\mathcal{L} \in ObC_Y$, the object $F^\diamond\mathfrak{f}_*F_\diamond(\mathcal{L})$ is the colimit of the composition of the forgetful functor $h_X/\mathfrak{f}_*F_\diamond(\mathcal{L}) \longrightarrow C_X$ and the functor F. Objects of the category $h_X/\mathfrak{f}_*F_\diamond(\mathcal{L})$ are pairs $(\mathcal{V}, \widehat{\mathcal{V}} \xrightarrow{\xi} \mathfrak{f}_*F_\diamond(\mathcal{L}))$ By Yoneda Lemma, morphisms $\widehat{\mathcal{V}} \xrightarrow{\xi} \mathfrak{f}_*F_\diamond(\mathcal{L})$ are in a natural bijective correspondence with elements of $\mathfrak{f}_*F_\diamond(\mathcal{L})(\mathcal{V}) = C_Y(F(\mathcal{V}),\mathcal{L})$. this correspondence assigns to the category $h_X/\mathfrak{f}_*F_\diamond(\mathcal{L})$ a cone

$$F(V) \xrightarrow{\widehat{\xi}} \mathcal{L}, \quad (\mathcal{V},\xi) \in Obh_X/\mathfrak{f}_*F_\diamond(\mathcal{L}),$$

from the composition the forgetful functor $h_X/\mathfrak{f}_*F_\diamond(\mathcal{L}) \longrightarrow C_X$ and the functor F to $\mathcal{L}$. This cone determines a morphism $F^\diamond\mathfrak{f}_*F_\diamond(\mathcal{L}) \xrightarrow{\epsilon(\mathcal{L})} \mathcal{L}$ which is the value at $\mathcal{L}$ of the adjunction arrow $(F^\diamond\mathfrak{f}_*)F_\diamond \xrightarrow{\epsilon} Id_{C_Y}$.

The other adjunction arrow, $Id_{C_X^{\circledast}} \xrightarrow{\eta} F_\diamond F^\diamond\mathfrak{f}_*$, is defined as follows.

Notice that it suffices to define the values of the adjunction arrow η on representable presheaves, $\widehat{\mathcal{V}} \xrightarrow{\eta(\widehat{\mathcal{V}})} F_\diamond F^\diamond(\mathcal{V}) = C_Y(F(-), F(\mathcal{V}))$. The obvious canonical choice is the morphism corresponding to the identical morphism $F(\mathcal{V}) \longrightarrow F(\mathcal{V})$. This choice, indeed, defines the second adjunction arrow.

This shows that, for every functor $C_X \xrightarrow{F} C_Y$ which maps initial objects to initial objects, the functor $\mathfrak{f}_*F^\diamond$ is continuous. It follows from (a) that the restriction of the functor $F \longmapsto F^\diamond$ to functors preserving initial objects defines an equivalence between the full subcategory $\mathcal{H}om^{\circledast}(C_X, C_Y)$ of the category $\mathcal{H}om(C_X, C_Y)$ generated by functors mapping initial objects to initial objects and the full subcategory $\mathcal{H}om_{\mathfrak{c}}(C_X^{\circledast}, C_Y)$ of the category $\mathcal{H}om(C_X^*, C_Y)$ generated by all continuous functors $C_X^{\circledast} \longrightarrow C_Y$.

The category $\mathcal{H}om^{\circledast}(C_X, C_Y)$ has an initial object – the constant functor, which maps all morphisms of the category C_X to the identical morphism of an initial object of the category C_Y. In particular, the category $\mathcal{H}om_{\mathfrak{c}}(C_X^*, C_Y)$ of continuous functors from C_X^* to C_Y is not empty.

(c) Suppose that the category C_Y has an initial object $\mathfrak{y}$. To every functor $C_X \xrightarrow{F} C_Y$, we assign a functor $C_X^{\wedge} \xrightarrow{F^\star} C_Y$ whose restriction to the subcategory C_X^* of non-trivial functors coincides with the functor $F^\diamond$ defined in (a) and which maps the trivial functor $\bar{\emptyset}$ (– the initial object of the category $C_X^{\wedge}$) to the initial object $\mathfrak{y}$. The functor $F_\star$ which maps every object $\mathcal{L}$ of the category C_Y to the presheaf of sets $C_Y(F(-), \mathcal{L})$ is a right adjoint to the functor $F^\star$. The argument is the same as in the proof of (b2)⇒(b1).

(c1) The fact follows from the argument of (b2)⇒(b1).

(c2) This follows from the fact that Yoneda embedding maps final objects to final objects. Details are left to the reader. ■

2.0.3. Corollary. *Let C_X and C_Y be svelte categories.*

(a) There is an equivalence of categories between the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y and the full subcategory $\mathfrak{Hom}(C_X^*, C_Y^*)$ of the category $\mathcal{H}om(C_X^*, C_Y^*)$ generated by all functors $C_X^* \longrightarrow C_Y^*$ which preserve colimits and map representable presheaves to representable presheaves.

(b) Suppose that the categories C_X and C_Y have initial objects. Then the functor (1) establishes an equivalence between the full subcategory $\mathcal{H}om^{\circledast}(C_X, C_Y)$ of the category $\mathcal{H}om(C_X, C_Y)$ generated by functors mapping initial objects to initial objects and the full subcategory $\mathcal{H}om_{\mathfrak{c}}(C_X^{\circledast}, C_Y^{\circledast})$ of the category $\mathcal{H}om(C_X^{\circledast}, C_Y^{\circledast})$ generated by all continuous functors $C_X^{\circledast} \longrightarrow C_Y^{\circledast}$ which map representable presheaves to representable presheaves.

Proof. (a) The equivalence in question is the functor

$$\mathcal{H}om(C_X, C_Y) \longrightarrow \mathfrak{Hom}(C_X^*, C_Y^*)$$

which assigns to every functor $C_X \xrightarrow{F} C_Y$ a functor $C_X^* \xrightarrow{F^*} C_Y^*$ which preserves colimits and is determined uniquely up to isomorphism by commutativity of the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{h_X^*} & C_X^* \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F^*} \\ C_Y & \xrightarrow{h_Y^*} & C_Y^* \end{array} \qquad (1)$$

This fact follows from (the argument of) 2.0.2(a) applied to the composition of the functor $C_X \xrightarrow{F} C_Y$ with the Yoneda embedding $C_Y \xrightarrow{h_Y^*} C_Y^*$. In the notations of the argument of 2.0.2, the functor F^* coincides with $(F \circ h_X^*)^\diamond$.

(b) If the category C_X has initial objects, then the Yoneda embedding $C_X \xrightarrow{h_X^{\circledast}} C_X^{\circledast}$. maps initial objects to initial objects. So that if C_X and C_Y have initial objects, than the composition $F \longmapsto h_Y^{\circledast} \circ F$ with the Yoneda embedding $C_Y \xrightarrow{h_Y^{\circledast}} C_Y^{\circledast}$ is a functor from $\mathcal{H}om^{\circledast}(C_X, C_Y)$ to $\mathcal{H}om^{\circledast}(C_X, C_Y^{\circledast})$. It follows from (the argument of) 2.0.2(b) that the map $F \longmapsto F^{\circledast} \stackrel{\text{def}}{=} (h_Y^{\circledast} \circ F)^{\diamond} \circ \mathfrak{f}_*$ gives the claimed equivalence. ∎

2.0.4. The categories of functors and subcanonical topologies. Let τ be a pretopology on a svelte category C_X. The sheafification functor $C_X^{\wedge} \longrightarrow (C_X, \tau)^{\wedge}$ maps non-trivial presheaves of sets to non-trivial sheaves of sets. Therefore, it induces an exact localization functor $C_X^* \xrightarrow{\mathfrak{q}_\tau^*} C_{X_\tau}$ which has a right adjoint, $C_{X_\tau} \xrightarrow{\mathfrak{q}_{\tau*}} C_X^*$.

2.0.4.1. For any category C_Y, this pair of adjoint functors induces a pair of adjoint functors

$$\begin{aligned} \mathcal{H}om(C_X^*, C_Y) &\xrightarrow{\widetilde{\mathfrak{q}}_\tau^*} \mathcal{H}om(C_{X_\tau}, C_Y), \quad \mathcal{G} \longmapsto \mathcal{G} \circ \mathfrak{q}_{\tau*}, \\ \mathcal{H}om(C_{X_\tau}, C_Y) &\xrightarrow{\widetilde{\mathfrak{q}}_{\tau*}} \mathcal{H}om(C_X^*, C_Y), \quad \mathcal{F} \longmapsto \mathcal{F} \circ \mathfrak{q}_\tau^*, \end{aligned} \tag{1}$$

with adjunction morphisms induced by the adjunction morphisms for the pair $(\mathfrak{q}_\tau^*, \mathfrak{q}_{\tau*})$. In particular, the adjunction morphism $\widetilde{\mathfrak{q}}_\tau^* \circ \widetilde{\mathfrak{q}}_{\tau*} \longrightarrow Id_{\mathcal{H}om(C_{X_\tau}, C_Y)}$ is an isomorphism, which means that the functor $\mathcal{H}om(C_{X_\tau}, C_Y) \xrightarrow{\widetilde{\mathfrak{q}}_{\tau*}} \mathcal{H}om(C_X^*, C_Y)$ is fully faithful, or, equivalently, $\mathcal{H}om(C_X^*, C_Y) \xrightarrow{\widetilde{\mathfrak{q}}_\tau^*} \mathcal{H}om(C_{X_\tau}, C_Y)$ is a localization functor.

2.0.4.2. It follows that the functor $\mathcal{H}om(C_{X_\tau}, C_Y) \xrightarrow{\widetilde{\mathfrak{q}}_{\tau*}} \mathcal{H}om(C_X^*, C_Y)$ establishes an equivalence between the category $\mathcal{H}om(C_{X_\tau}, C_Y)$ and the full subcategory of the category $\mathcal{H}om(C_X^*, C_Y)$ generated by all functors $C_X^* \xrightarrow{\mathcal{G}} C_Y$ such that the canonical morphism $\mathcal{G} \longrightarrow \mathcal{G} \circ (\mathfrak{q}_{\tau*}\mathfrak{q}_\tau^*)$ is an isomorphism.

2.0.4.3. Suppose that C_Y is a category with colimits. Then it follows from (the proof of) 2.0.2 that the functor

$$\mathcal{H}om(C_X^*, C_Y) \xrightarrow{\widetilde{h}_X^*} \mathcal{H}om(C_X, C_Y), \quad G \longmapsto G \circ h_X^*, \tag{2}$$

has a fully faithful left adjoint

$$\mathcal{H}om(C_X, C_Y) \xrightarrow{\widetilde{h}_{X!}} \mathcal{H}om(C_X^*, C_Y), \quad \mathcal{F} \longmapsto \mathcal{F}^{\diamond}. \tag{3}$$

Taking the composition with the corresponding functors of 2.0.4.1(1), we obtain a pair of adjoint functors

$$\mathcal{H}om(C_X, C_Y) \xrightarrow{{}^\tau \mathfrak{j}_X^*} \mathcal{H}om(C_{X_\tau}, C_Y) \qquad (4)$$

$$\mathcal{H}om(C_{X_\tau}, C_Y) \xrightarrow{{}^\tau \mathfrak{j}_{X*}} \mathcal{H}om(C_X, C_Y) \qquad (5)$$

2.0.4.4. The case of a subcanonical pretopology. One can see that a pretopology τ on the category C_X is *subcanonical* iff the functor 2.0.4.3(4) is fully faithful. It follows from 2.0.4.2 that, in this case, the functor (4) establishes an equivalence between the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y and the full subcategory of the category $\mathcal{H}om(C_{X_\tau}, C_Y)$ generated by all functors $C_{X_\tau} \xrightarrow{\mathcal{F}} C_Y$ such that the canonical morphism

$$(\mathcal{F} \circ \mathfrak{q}_\tau^*)^\diamond \circ \mathfrak{q}_{\tau *} \longrightarrow \mathcal{F}$$

(– the composition of $((\mathcal{F} \circ \mathfrak{q}_\tau^*)^\diamond \longrightarrow \mathcal{F} \circ \mathfrak{q}_\tau^*)\mathfrak{q}_{\tau *}$ and the isomorphism $\mathcal{F} \circ \mathfrak{q}_\tau^* \mathfrak{q}_{\tau *} \xrightarrow{\sim} \mathcal{F}$) is an isomorphism.

2.0.5. Sheafification functors and initial objects. Let C_X be a svelte category with an initial object $\mathfrak{x}$. Notice that the representable presheaf $\widehat{\mathfrak{x}} = C_X(-, \mathfrak{x})$ is a sheaf for any pretopology τ. We denote by $C_{X_\tau^\circledast}$ the category $\widehat{\mathfrak{x}} \backslash C_{X_\tau} = \widehat{\mathfrak{x}} \backslash (C_X, \tau)^\wedge$ and by

$$C_X \xrightarrow{\mathfrak{j}_{X_\tau}^\circledast} \widehat{\mathfrak{x}} \backslash C_{X_\tau} = C_{X_\tau^\circledast} \qquad (6)$$

the composition of the canonical embedding

$$C_X \xrightarrow{\mathfrak{h}_X^\circledast} \mathfrak{x} \backslash C_X^\wedge = \mathfrak{x} \backslash C_X^*$$

and the functor

$$\widehat{\mathfrak{x}} \backslash C_X^* = \widehat{\mathfrak{x}} \backslash C_X^\wedge \xrightarrow{\mathfrak{q}_\tau^*} \widehat{\mathfrak{x}} \backslash (C_X, \tau)^\wedge = \widehat{\mathfrak{x}} \backslash C_{X_\tau}$$

induced by the sheafification functor $C_X^* \xrightarrow{\mathfrak{q}_\tau^*} C_{X_\tau}$.

The functor (6) maps initial objects to initial objects.

2.0.5.1. Subcanonical pretopologies. It follows that *the pretopology τ on the category C_X is subcanonical iff the functor (6) is fully faithful.*

2.0.5.2. Remark. The forgetful functor

$$C_{X_\tau^\circledast} = \widehat{\mathfrak{x}} \backslash (C_X, \tau)^\wedge \xrightarrow{\mathfrak{f}_*} (C_X, \tau)^\wedge$$

has a left adjoint, $(C_X,\tau)^\wedge \xrightarrow{\mathfrak{f}^*} C_{X^\circledast_\tau}$, which maps every sheaf $\mathcal{M}$ to the object $(\mathcal{M}\coprod\mathfrak{x},\mathfrak{x}\longrightarrow\mathcal{M}\coprod\mathfrak{x})$, where $\mathfrak{x}\longrightarrow\mathcal{M}\coprod\mathfrak{x}$ is the coprojection. So that the functor $C_{X^\circledast_\tau} \xrightarrow{\mathfrak{f}_*} (C_X,\tau)^\wedge$ preserves limits. But, it does not preserve colimits.

2.0.6. The categories of k-linear functors. Let C_X and C_Y be svelte k-linear categories. We denote by $\mathcal{H}om_k(C_X,C_Y)$ the full subcategory of the category $\mathcal{H}om(C_X,C_Y)$ of functors from C_X to C_Y generated by k-linear functors.

2.0.6.1. The category of k-linear presheaves of k-modules and k-linear Yoneda embedding. We denote by $\mathcal{M}_k(X)$ the category of k-linear presheaves of k-modules on C_X. In other words,

$$\mathcal{M}_k(X) \stackrel{\mathrm{def}}{=} \mathcal{H}om_k(C_X^{op}, k-mod).$$

We call the canonical embedding

$$C_X \xrightarrow{\mathfrak{h}_X} \mathcal{M}_k(X), \quad \mathcal{L}\longmapsto \widehat{\mathcal{L}} \stackrel{\mathrm{def}}{=} C_X(-,\mathcal{L}),$$

the *k-linear Yoneda embedding.*

2.0.6.2. Proposition. *Let C_X be a svelte k-linear category and C_Y a k-linear category with colimits. The functor*

$$\mathcal{H}om_k(\mathcal{M}_k(X),C_Y) \xrightarrow{\mathfrak{h}_X^*} \mathcal{H}om_k(C_X,C_Y), \quad \mathcal{G}\longmapsto \mathcal{G}\circ\mathfrak{h}_X,$$

has a fully faithful left adjoint,

$$\mathcal{H}om_k(C_X,C_Y) \xrightarrow{\mathfrak{h}_{X!}} \mathcal{H}om_k(\mathcal{M}_k(X),C_Y),$$

which establishes an equivalence between the category $\mathcal{H}om_k(C_X,C_Y)$ and the full subcategory $\mathcal{H}om_k^{\mathfrak{c}}(\mathcal{M}_k(X),C_Y)$ of the category $\mathcal{H}om_k(\mathcal{M}_k(X),C_Y)$ generated by all continuous functors from $\mathcal{M}_k(X)$ to C_Y.

Proof. Fix a k-linear functor $C_X \xrightarrow{\mathcal{G}} C_Y$ and consider the map which assigns to every object $\mathcal{L}$ of the category C_Y the presheaf of k-modules $C_Y(\mathcal{G}(-),\mathcal{L})$. This map defines a k-linear functor $C_Y \xrightarrow{\mathcal{G}_*} \mathcal{M}_k(X)$. If the category C_Y has colimits, then the functor $\mathcal{G}_*$ has a left adjoint, $\mathcal{M}_k(X) \xrightarrow{\mathcal{G}^*} C_Y$, which assigns to every object $\mathcal{V}$ of the category $\mathcal{M}_k(X)$ the colimit of the composition of the forgetful functor $\mathfrak{h}_X/\mathcal{V}\longrightarrow C_X$ with the functor $C_X \xrightarrow{\mathcal{G}} C_Y$. The functor

$$\mathcal{H}om_k(C_X,C_Y) \xrightarrow{\mathfrak{h}_{X!}} \mathcal{H}om_k(\mathcal{M}_k(X),C_Y)$$

assigns to every k-linear functor $C_X \xrightarrow{\mathcal{G}} C_Y$ a left adjoint $\mathcal{M}_k(X) \xrightarrow{\mathcal{G}^\star} C_Y$ to the functor $\mathcal{G}_\star$ and acts accordingly on morphisms of functors. ■

2.0.6.3. Corollary. *Let C_X and C_Y be svelte k-linear categories. There is a natural fully faithful functor*

$$\mathcal{H}om_k(C_X, C_Y) \longrightarrow \mathcal{H}om_k(\mathcal{M}_k(X), \mathcal{M}_k(Y)) \qquad (1)$$

which establishes an equivalence between the category $\mathcal{H}om_k(C_X, C_Y)$ of k-linear functors from C_X to C_Y and the full subcategory $\mathfrak{Hom}_k(\mathcal{M}_k(X), \mathcal{M}_k(Y))$ of the category $\mathcal{H}om_k(\mathcal{M}_k(X), \mathcal{M}_k(Y))$ generated by all continuous functors $\mathcal{M}_k(X) \longrightarrow \mathcal{M}_k(Y)$ which map representable presheaves to representable presheaves.

Proof. The functor (1) assigns to every k-linear functor $C_X \xrightarrow{F} C_Y$ a continuous k-linear functor $\mathcal{M}_k(X) \xrightarrow{F^*} \mathcal{M}_k(Y)$ determined uniquely up to isomorphism by the commutativity of the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\mathfrak{h}_X} & \mathcal{M}_k(X) \\ F \downarrow & & \downarrow F^* \\ C_Y & \xrightarrow{\mathfrak{h}_Y} & \mathcal{M}_k(Y) \end{array} \qquad (2)$$

The existence and uniqueness of F^* follows from 2.0.6.2: in the notations of the argument of 2.0.6.2, $F^* = (\mathfrak{h}_X \circ F)^\star$. ■

2.0.7. The categories of k-linear functors and k-linear sheaves of k-modules. Let τ be a pretopology on a svelte k-linear category C_X. We denote by $Sh_k(X, \tau)$ the category of k-linear sheaves of k-modules on the presite (C_X, τ); that is $Sh_k(X, \tau)$ is the intersection of the category $Sh((C_X, \tau), k-mod)$ of sheaves of k-modules on (C_X, τ) and the category $\mathcal{H}om_k(C_X, k-mod)$ of k-linear functors from C_X to $k-mod$.

The sheafification functor induces an exact functor $\mathcal{M}_k(X) \longrightarrow Sh_k(X, \tau)$ which is a right adjoint to the embedding $Sh_k(X, \tau) \longrightarrow \mathcal{M}_k(X)$.

2.0.7.1. For any k-linear category C_Y, this pair of adjoint functors induces a pair of adjoint functors

$$\begin{aligned} \mathcal{H}om_k(\mathcal{M}_k(X), C_Y) &\xrightarrow{\widetilde{\mathfrak{q}}^*_\tau} \mathcal{H}om(Sh_k(X, \tau), C_Y), \quad \mathcal{G} \longmapsto \mathcal{G} \circ \mathfrak{q}_{\tau *}, \\ \mathcal{H}om(Sh_k(X, \tau), C_Y) &\xrightarrow{\widetilde{\mathfrak{q}}_{\tau *}} \mathcal{H}om(\mathcal{M}_k(X), C_Y), \quad \mathcal{F} \longmapsto \mathcal{F} \circ \mathfrak{q}^*_\tau, \end{aligned} \qquad (1)$$

with adjunction morphisms induced by the adjunction morphisms for the pair $(\mathfrak{q}^*_\tau, \mathfrak{q}_{\tau *})$. Both functors are exact, and it follows that the second functor is fully faithful (see 2.0.4.1).

It establishes an equivalence between the category $\mathcal{H}om_k(Sh_k(X,\tau),C_Y)$ and the full subcategory of the category $\mathcal{H}om(C_X^*,C_Y)$ generated by all functors $\mathcal{M}_(X) \xrightarrow{\mathcal{G}} C_Y$ such that the canonical morphism $\mathcal{G} \longrightarrow \mathcal{G} \circ (\mathfrak{q}_{\tau *}\mathfrak{q}_\tau^*)$ is an isomorphism.

2.0.7.2. Suppose that C_Y is a category with colimits. Then, by 2.0.6.2, the functor

$$\mathcal{H}om_k(\mathcal{M}_k(X),C_Y) \xrightarrow{\mathfrak{h}_X^*} \mathcal{H}om_k(C_X,C_Y), \quad \mathcal{G} \longmapsto \mathcal{G} \circ \mathfrak{h}_X,$$

has a fully faithful left adjoint, $\mathcal{H}om_k(C_X,C_Y) \xrightarrow{\mathfrak{h}_{X!}} \mathcal{H}om_k(\mathcal{M}_k(X),C_Y)$.

Taking the composition with the corresponding functors of 2.0.7.1(1), we obtain a pair of adjoint functors

$$\mathcal{H}om_k(C_X,C_Y) \xrightarrow{{}^\tau\mathfrak{j}_X^*} \mathcal{H}om_k(Sh_k(X,\tau),C_Y) \quad (2)$$

$$\mathcal{H}om_k(Sh_k(X,\tau),C_Y) \xrightarrow{{}^\tau\mathfrak{j}_{X*}} \mathcal{H}om_k(C_X,C_Y) \quad (3)$$

2.0.7.3. The case of a subcanonical pretopology. One can see that a pretopology τ on the category C_X is *subcanonical* iff the functor 2.0.7.2(2) is fully faithful for any k-linear category C_Y. It follows from 2.0.7.2 that, in this case, the functor (2) establishes an equivalence between the category $\mathcal{H}om_k(C_X,C_Y)$ of k-linear functors from C_X to C_Y and the full subcategory of the category $\mathcal{H}om_k(Sh_k(X,\tau),C_Y)$ generated by all functors $Sh_k(X,\tau) \xrightarrow{\mathcal{F}} C_Y$ such that the canonical morphism

$$(\mathcal{F} \circ \mathfrak{q}_\tau^*)^\star \circ \mathfrak{q}_{\tau *} \longrightarrow \mathcal{F}$$

(– the composition of $((\mathcal{F} \circ \mathfrak{q}_\tau^*)^\star \longrightarrow \mathcal{F} \circ \mathfrak{q}_\tau^*)\mathfrak{q}_{\tau *}$ and the isomorphism $\mathcal{F} \circ \mathfrak{q}_\tau^*\mathfrak{q}_{\tau *} \xrightarrow{\sim} \mathcal{F})$ is an isomorphism.

2.1. Proposition. *(a) Let $(C_X,\mathfrak{E}_X)$ be a svelte right exact category. The Yoneda embedding induces an 'exact' fully faithful functor*

$$(C_X,\mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^*} (C_{X_\mathfrak{E}},\mathfrak{E}^\mathfrak{s}_{X_\mathfrak{E}}),$$

from $(C_X,\mathfrak{E}_X)$ to the category $C_{X_\mathfrak{E}}$ of non-trivial sheaves of sets on $(C_X,\mathfrak{E}_X)$ endowed with the canonical (– the finest) right exact structure $\mathfrak{E}^\mathfrak{s}_{X_\mathfrak{E}}$.

(b) Let $(C_X,\mathfrak{E}_X)$ and $(C_Y,\mathfrak{E}_Y)$ be right exact categories and

$$(C_X,\mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y,\mathfrak{E}_Y)$$

a weakly right 'exact' functor. There exists a functor $C_{X_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{E}}}$ *such that the diagram*

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ {\scriptstyle \mathfrak{j}_X^*}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}_Y^*} \\ C_{X_{\mathfrak{E}}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_{\mathfrak{E}}} \end{array}$$

quasi commutes, i.e. $\widetilde{\varphi}^*\mathfrak{j}_X^* \simeq \mathfrak{j}_Y^*\varphi^*$*. The functor* $\widetilde{\varphi}^*$ *is defined uniquely up to isomorphism.*

(c) If the categories C_X *and* C_Y *have initial objects and the functor* φ^* *maps initial objects to initial objects, then the functor* $C_{X_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{E}}}$ *has a right adjoint,* $\widetilde{\varphi}_*$*.*

Proof. (a) By definition, $C_{X_{\mathfrak{E}}}$ is a full subcategory of the category C_X^* of presheaves of sets on C_X whose objects are presheaves $(C_X, \mathcal{E}_X)^{op} \xrightarrow{\mathfrak{F}} Sets$ such that the image

$$\mathfrak{F}(N) \longrightarrow \mathfrak{F}(M) \rightrightarrows \mathfrak{F}(M \times_N M) \qquad (1)$$

of the canonical diagram

$$M \times_N M = \mathcal{K}_2(\mathfrak{s}) \underset{\mathfrak{s}_2}{\overset{\mathfrak{s}_1}{\rightrightarrows}} M \xrightarrow{\mathfrak{s}} N$$

is an exact diagram for any deflation $M \xrightarrow{\mathfrak{s}} N$.

The inclusion functor $C_{X_{\mathfrak{E}}} \longrightarrow C_X^*$ has a left adjoint – the sheafification functor $C_X^* \longrightarrow C_{X_{\mathfrak{E}}}$, which is exact. Since the pretopology $\mathfrak{E}_X$ on C_X is subcanonical (see 1.1), the Yoneda embedding induces an equivalence between the category C_X and a full subcategory of $C_{X_{\mathfrak{E}}}$. It remains to show that the embedding $C_X \xrightarrow{\mathfrak{j}_X^*} C_{X_{\mathfrak{E}}}$ is an 'exact' functor, i.e. it maps deflations to deflations and preserves pull-backs of deflations.

The Yoneda embedding $C_X \longrightarrow C_X^*$ is a left exact functor, and the sheafification functor $C_X^* \longrightarrow C_{X_{\mathfrak{E}}}$ is exact. Therefore, their composition $C_X \xrightarrow{\mathfrak{j}_X^*} C_{X_{\mathfrak{E}}}$ is a left exact functor; in particular, it preserves all pull-backs. Therefore, we need only to show that the functor $\mathfrak{j}_X^*$ maps every deflation to an epimorphism of the category $C_{X_{\mathfrak{E}}}$.

In fact, let $M \longrightarrow N$ be a deflation and $M \times_N M \rightrightarrows M \longrightarrow N$ the associated exact diagram. The Yoneda embedding maps this diagram to the diagram

$$\widehat{M} \times_{\widehat{N}} \widehat{M} \rightrightarrows \widehat{M} \longrightarrow \widehat{N}, \qquad (2)$$

where $\widehat{M} = C_X(-, M)$. For any presheaf of sets $\mathfrak{F}$, the functor $C_X^*(-, \mathfrak{F})$ maps the diagram (2) to a diagram isomorphic to

$$\mathfrak{F}(N) \longrightarrow \mathfrak{F}(M) \rightrightarrows \mathfrak{F}(M \times_N M).$$

which is exact if $\mathfrak{F}$ is a sheaf on $(C_X, \mathfrak{E}_X)$. This shows that for every sheaf $\mathfrak{F}$, the functor $C_{X_\mathfrak{E}}(-, \mathfrak{F})$ maps the diagram (2) to an exact diagram. Therefore the diagram (2) viewed as a diagram in the category of sheaves, is exact.

(b) If the functor φ^* is *weakly right 'exact'*, then the functor

$$C_Y^\wedge \longrightarrow C_X^\wedge, \quad \mathfrak{F} \longmapsto \mathfrak{F} \circ \varphi^*,$$

maps sheaves on the pretopology $(C_Y, \mathfrak{E}_Y)$ to sheaves on $(C_X, \mathfrak{E}_X)$; in particular, it induces a functor $(C_Y, \mathfrak{E}_Y)^\wedge \xrightarrow{\widehat{\varphi}_*} (C_X, \mathfrak{E}_X)^\wedge$.

In fact, for any arrow $M \longrightarrow N$ of $\mathfrak{E}_X$, consider the decomposition

$$\varphi^*(M \prod_N M) \longrightarrow \varphi^*(M) \prod_{\varphi^*(N)} \varphi^*(M) \rightrightarrows \varphi^*(M) \longrightarrow \varphi^*(N) \tag{3}$$

of the diagram

$$\varphi^*(M \prod_N M) \rightrightarrows \varphi^*(M) \longrightarrow \varphi^*(N) \tag{4}$$

Since the functor φ^* is weakly right 'exact', the right and the left arrows of the diagram (3) belong to $\mathfrak{E}_Y$. Therefore, for any sheaf $\mathfrak{F}$ on $(C_Y, \mathfrak{E}_Y)$ the diagram

$$\mathfrak{F}\varphi^*(N) \longrightarrow \mathfrak{F}\varphi^*(M) \rightrightarrows \mathfrak{F}\big(\varphi^*(M) \prod_{\varphi^*(N)} \varphi^*(M)\big)$$

is exact and the morphism

$$\mathfrak{F}\big(\varphi^*(M) \prod_{\varphi^*(N)} \varphi^*(M)\big) \longrightarrow \mathfrak{F}\varphi^*(M \prod_N M)$$

is a monomorphism. Therefore, the diagram

$$\mathfrak{F}\varphi^*(N) \longrightarrow \mathfrak{F}\varphi^*(M) \rightrightarrows \mathfrak{F}\varphi^*(M \prod_N M)$$

is exact. This shows that $\mathfrak{F} \circ \varphi^*$ is a sheaf on the presite $(C_X, \mathfrak{E}_X)$.

(b1) The functor $\widehat{\varphi}_*$ has a left adjoint, $(C_X, \mathfrak{E}_X)^\wedge \xrightarrow{\widehat{\varphi}^*} (C_Y, \mathfrak{E}_Y)^\wedge$.

Notice that this left adjoint maps non-trivial sheaves to non-trivial sheaves. This follows from the fact that there is an adjunction morphism $\mathfrak{F} \longrightarrow \widehat{\varphi}_*\widehat{\varphi}^*(\mathfrak{F})$. So that if $\mathfrak{F}(\mathcal{M}) \neq \emptyset$ for some object $\mathcal{M}$, then $\widehat{\varphi}_*\widehat{\varphi}^*(\mathfrak{F})(\mathcal{M}) \neq \emptyset$.

Therefore, the functor $\widehat{\varphi}^*$ induces a functor $C_{X_\mathfrak{E}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_\mathfrak{E}}$.

It follows that $\widetilde{\varphi}^* \mathfrak{j}^*_X \simeq \mathfrak{j}^*_Y \varphi^*$.

(c) If the categories C_X and C_Y have initial objects and the functor φ^* maps initial objects to initial objects, then the functor $\widehat{\varphi}_*$ maps non-trivial sheaves to non-trivial sheaves. Therefore, it induces a functor $C_{Y_{\mathfrak{e}}} \xrightarrow{\widetilde{\varphi}_*} C_{X_{\mathfrak{e}}}$, which is a right adjoint to the functor $C_{X_{\mathfrak{e}}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{e}}}$. ■

2.1.1. Proposition. *(a) Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with an initial object $\mathfrak{x}$. The Yoneda embedding induces a fully faithful 'exact' functor*

$$(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}^{\circledast}_X} (C_{X_{\mathfrak{e}}}, \mathfrak{E}^{\mathfrak{s}}_{X^{\circledast}_{\mathfrak{e}}}),$$

from $(C_X, \mathfrak{E}_X)$ to the category $C_{X^{\circledast}_{\mathfrak{e}}} \stackrel{\mathrm{def}}{=} \widehat{\mathfrak{x}} \backslash (C_X, \mathfrak{E}_X)^\wedge$ of sheaves of sets on $(C_X, \mathfrak{E}_X)$ over $\widehat{\mathfrak{x}}$ endowed with the canonical (– the finest) right exact structure $\mathfrak{E}^{\mathfrak{s}}_{X^{\circledast}_{\mathfrak{e}}}$.

(b) Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories and

$$(C_X, \mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathfrak{E}_Y)$$

a weakly right 'exact' functor. There exists a functor $C_{X^{\circledast}_{\mathfrak{e}}} \xrightarrow{\widetilde{\varphi}^{\circledast}} C_{Y^{\circledast}_{\mathfrak{e}}}$ such that the diagram

$$\begin{array}{ccc}
C_X & \xrightarrow{\varphi^*} & C_Y \\
{\scriptstyle \mathfrak{j}^{\circledast}_X}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{j}^{\circledast}_Y} \\
C_{X^{\circledast}_{\mathfrak{e}}} & \xrightarrow{\widetilde{\varphi}^{\circledast}} & C_{Y^{\circledast}_{\mathfrak{e}}}
\end{array}$$

quasi commutes, i.e. $\widetilde{\varphi}^{\circledast} \mathfrak{j}^{\circledast}_X \simeq \mathfrak{j}^{\circledast}_Y \varphi^*$. The functor $\widetilde{\varphi}^{\circledast}$ is defined uniquely up to isomorphism.

(c) If the categories C_X and C_Y have initial objects and the functor φ^* maps initial objects to initial objects, then the functor $C_{X^{\circledast}_{\mathfrak{e}}} \xrightarrow{\widetilde{\varphi}^{\circledast}} C_{Y^{\circledast}_{\mathfrak{e}}}$ has a right adjoint, $\widetilde{\varphi}_{\circledast}$.

Proof. The assertion follows from 2.1 and 2.0.5. Details are left to the reader. ■

2.2. An observation on sheaf epimorphisms. Let $N \in ObC_X$ and $\mathfrak{F} \xrightarrow{\gamma} \widehat{N}$ a morphism of sheaves on $(C_X, \mathfrak{E}_X)$. Regarding γ as a presheaf morphism, we represent it as the composition of the presheaf epimorphism $\mathfrak{F} \longrightarrow Im(\gamma)$ and the embedding $Im(\gamma) \hookrightarrow \widehat{N}$. It follows from the exactness of the sheafification functor that γ is a *sheaf epimorphism* iff the sheafification functor maps the embedding $Im(\gamma) \hookrightarrow \widehat{N}$ to an isomorphism; i.e. $Im(\gamma) \hookrightarrow \widehat{N}$ is a refinement of N in the topology associated with the pretopology $\mathfrak{E}_X$. The latter means that there exists a deflation $M' \xrightarrow{\mathfrak{e}'} N$ such that the image of $\widehat{\mathfrak{e}}'$ is contained in $Im(\gamma)$, i.e. $\widehat{M'} \xrightarrow{\widehat{\mathfrak{e}'}} \widehat{N}$ is the composition of a morphism $\widehat{M'} \xrightarrow{v'} Im(\gamma)$ and the embedding $Im(\gamma) \hookrightarrow \widehat{N}$. Since representable functors are projective objects in C_X^*, the morphism v' factors through the presheaf epimorphism $\mathfrak{F} \longrightarrow Im(\gamma)$. Thus, we obtain a commutative diagram

$$\begin{array}{ccc} \widehat{M'} & \xrightarrow{\widehat{\mathfrak{e}'}} & \widehat{N} \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle id} \\ \mathfrak{F} & \xrightarrow{\gamma} & \widehat{N} \end{array} \qquad (5)$$

2.2.1. Proposition. *Fix a right exact category* $(C_X, \mathfrak{E}_X)$.

(a) Let $\mathcal{L} \xrightarrow{\mathfrak{f}} \mathcal{N}$ *be a morphism of the category* C_X. *Its image* $\widehat{\mathcal{L}} \xrightarrow{\widehat{\mathfrak{f}}} \widehat{\mathcal{N}}$ *is a sheaf epimorphism iff* $\mathfrak{e} = \mathfrak{f} \circ \mathfrak{v}$ *for some deflation* $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N}$ *and an arrow* $\mathcal{M} \xrightarrow{\mathfrak{v}} \mathcal{L}$.

In particular, $\mathcal{L} \xrightarrow{\mathfrak{f}} \mathcal{N}$ *is a strict epimorphism.*

(b) A morphism $\mathfrak{F} \xrightarrow{\gamma} \mathfrak{G}$ *of sheaves of sets on* $(C_X, \mathfrak{E}_X)$ *is an epimorphism iff for any morphism* $\widehat{\mathcal{L}} \xrightarrow{\xi} \mathfrak{G}$, *there exists a commutative diagram*

$$\begin{array}{ccc} \widehat{\mathcal{M}} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{\mathcal{L}} \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathfrak{F} & \xrightarrow{\gamma} & \mathfrak{G} \end{array} \qquad (6)$$

such that $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ *is a deflation.*

Proof. (a) If $\mathfrak{e} = \mathfrak{f} \circ \mathfrak{v}$ for some deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N}$, then $\widehat{\mathfrak{e}} = \widehat{\mathfrak{f}} \circ \widehat{\mathfrak{v}}$, and, by the argument of 2.1(a), $\widehat{\mathfrak{e}}$ is an epimorphism. Therefore, $\widehat{\mathfrak{f}}$ is an epimorphism.

The converse assertion follows from the observation preceding this proposition.

(b) If $\mathfrak{F} \xrightarrow{\gamma} \mathfrak{G}$ is an epimorphism of sheaves, then, for any morphism $\widehat{\mathcal{L}} \xrightarrow{\xi} \mathfrak{G}$, the upper horizontal arrow of the cartesian square

$$\begin{array}{ccc} \widetilde{\mathfrak{F}} & \xrightarrow{\widetilde{\gamma}} & \widehat{\mathcal{L}} \\ {\scriptstyle \xi'}\downarrow & cart & \downarrow{\scriptstyle \xi} \\ \mathfrak{F} & \xrightarrow{\gamma} & \mathfrak{G} \end{array}$$

is an epimorphism. Applying (a) to the epimorphism $\widetilde{\mathfrak{F}} \xrightarrow{\widetilde{\gamma}} \mathcal{L}$, we obtain the claimed commutative diagram (6).

Conversely, suppose that a sheaf morphism $\mathfrak{F} \xrightarrow{\gamma} \mathfrak{G}$ is such that for any morphism $\widehat{\mathcal{L}} \xrightarrow{\xi} \mathfrak{G}$, there exists a commutative diagram (6) whose upper arrow is an epimorphism. Then γ is an epimorphism.

In fact, let $\mathfrak{G} \overset{\psi}{\underset{\phi}{\rightrightarrows}} \mathfrak{H}$ be a pair or sheaf morphisms which equalizes γ, that is $\psi \circ \gamma = \phi \circ \gamma$. It follows from the commutativity of the diagram (6) that $\psi \circ \xi \circ \widehat{\mathfrak{e}} = \phi \circ \xi \circ \widehat{\mathfrak{e}}$. Since $\widehat{\mathfrak{e}}$ is an epimorphism, the latter equality implies that $\psi \circ \xi = \phi \circ \xi$. This shows that the pair of arrows $\mathfrak{G} \overset{\psi}{\underset{\phi}{\rightrightarrows}} \mathfrak{H}$ equalizes any morphism from a representable (pre)sheaf to the sheaf $\mathfrak{G}$. Since any presheaf of sets is a colimit of a diagram of representable presheaves, it follows that $\psi = \phi$, which proves that γ is an epimorphism. ■

2.3. The canonical embedding of a k-linear right exact category. Fix an associative, commutative, unital ring k. Let $(C_X, \mathfrak{E}_X)$ be a svelte k-linear right exact category. We denote by $\mathcal{M}_k(X)$ the category of presheaves of k-modules and by $Sh_k(X, \mathfrak{E}_X)$ the category of sheaves of k-modules. Since the pretopology $\mathfrak{E}_X$ is subcanonical, the Yoneda embedding $C_X \longrightarrow \mathcal{M}_k(X)$ induces a full embedding $C_X \xrightarrow{\mathfrak{j}_X^*} Sh_k(X, \mathfrak{E}_X)$ of the category C_X into the Grothendieck category $Sh_k(X, \mathfrak{E}_X)$ of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$.

2.3.1. Proposition. *(a) For any svelte k-linear right exact category $(C_X, \mathfrak{E}_X)$, the full embedding $C_X \xrightarrow{\mathfrak{j}_X^*} Sh_k(X, \mathfrak{E}_X)$ is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to the Grothendieck category $Sh_k(X, \mathfrak{E}_X)$ of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$.*

(b) Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact k-linear svelte categories and

$$(C_X, \mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathfrak{E}_Y)$$

a right 'exact' k-linear functor. There exists a k-linear functor

$$Sh_k(X, \mathfrak{E}_X) \xrightarrow{\widetilde{\varphi}^*} Sh_k(Y, \mathfrak{E}_Y)$$

such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ {\scriptstyle \mathfrak{j}_X^*}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}_Y^*} \\ Sh_k(X, \mathfrak{E}_X) & \xrightarrow{\widetilde{\varphi}^*} & Sh_k(Y, \mathfrak{E}_Y) \end{array}$$

quasi commutes, i.e. $\widetilde{\varphi}^* \mathfrak{j}_X^* \simeq \mathfrak{j}_Y^* \varphi^*$. The functor $\widetilde{\varphi}^*$ is defined uniquely up to isomorphism and has a right adjoint, $\widetilde{\varphi}_*$.

Proof. The argument is similar to that of 2.1. ■

2.3.2. Note. Even if the functor $(C_X, \mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathfrak{E}_Y)$ is 'exact', the functor

$$Sh_k(X, \mathfrak{E}_X) \xrightarrow{\widetilde{\varphi}^*} Sh_k(Y, \mathfrak{E}_Y)$$

need not to be (left) exact. For instance, let $(C_X, \mathcal{E}_X)$ (resp. $(C_Y, \mathcal{E}_Y)$) be the exact category of projective A-modules (resp. B-modules) of finite type; and let φ^* be the functor $M \longmapsto B \otimes_A M$ corresponding to an algebra morphism $A \longrightarrow B$. Then the category $Sh_k(X, \mathfrak{E}_X)$ is naturally identified with $A - mod$ and the functor $\widetilde{\varphi}^*$ with

$$A - mod \xrightarrow{B\otimes_A} B - mod.$$

Therefore, the functor $\widetilde{\varphi}^*$ is exact iff the algebra morphism $A \longrightarrow B$ turns B into a flat right A-module.

3. Karoubian envelopes of right exact categories.

3.1. Lemma. *Let M be an object of a category C_X and $M \xrightarrow{p} M$ an idempotent (i.e. $p^2 = p$). The following conditions are equivalent:*

(a) The idempotent p splits, i.e. p is the composition of morphisms $M \xrightarrow{\mathfrak{e}} N \xrightarrow{\mathfrak{j}} M$ such that $\mathfrak{e} \circ \mathfrak{j} = id_N$.

(b) There exists a cokernel of the pair $M \overset{id_M}{\underset{p}{\rightrightarrows}} M$.

(c) There exists a kernel of the pair $M \overset{id_M}{\underset{p}{\rightrightarrows}} M$.

If the equivalent conditions above hold, then $Ker(id_M, p) \simeq Coker(id_M, p)$.

Proof. $(b) \Leftarrow (a) \Rightarrow (c)$. If the idempotent $M \xrightarrow{p} M$ is the composition of $M \xrightarrow{\mathfrak{e}} N$ and $N \xrightarrow{\mathfrak{j}} M$ such that $\mathfrak{e} \circ \mathfrak{j} = id_N$, then $M \xrightarrow{\mathfrak{e}} N$ is a cokernel of the pair $M \overset{id_M}{\underset{p}{\rightrightarrows}} M$, because $\mathfrak{e} \circ p = \mathfrak{e} \circ \mathfrak{j} \circ \mathfrak{e} = \mathfrak{e}$ and if $M \xrightarrow{\mathfrak{t}} L$ any morphism such that $\mathfrak{t} \circ p = \mathfrak{t}$, then $\mathfrak{t} = (\mathfrak{t} \circ \mathfrak{j}) \circ \mathfrak{e}$. Since $\mathfrak{e}$ is an epimorphism, there is only one morphism g such that $\mathfrak{t} = g \circ \mathfrak{e}$. This shows that $(a) \Rightarrow (b)$. The implication $(a) \Rightarrow (c)$ follows by duality.

$(b) \Rightarrow (a)$. Let $M \xrightarrow{\mathfrak{e}} N$ be a cokernel of the pair $M \overset{id_M}{\underset{p}{\rightrightarrows}} M$. Since $p \circ p = p$, there exists a unique morphism $N \xrightarrow{\mathfrak{j}} M$ such that $p = \mathfrak{j} \circ \mathfrak{e}$. Since $\mathfrak{e} \circ \mathfrak{j} \circ \mathfrak{e} = \mathfrak{e} \circ p = \mathfrak{e} = id_N \circ \mathfrak{e}$ and $\mathfrak{e}$ is an epimorphism, $\mathfrak{e} \circ \mathfrak{j} = id_N$.

The implication $(c) \Rightarrow (a)$ follows by duality. ■

3.2. Definition. A category C_X is called *Karoubian* if each idempotent in C_X splits. It follows from 3.1 that C_X is a Karoubian category iff for every idempotent $M \xrightarrow{p} M$ in C_X, there exists a kernel (equivalently, a cokernel) of the pair (id_M, p).

3.3. Proposition. *For any category C_X, there exists a Karoubian category C_{X_K} and a fully faithful functor $C_X \xrightarrow{\mathfrak{k}_X^*} C_{X_K}$ such that any functor from C_X to a Karoubian category factors uniquely up to a natural isomorphism through $\mathfrak{k}_X^*$. Every object of C_{X_K} is a retract of an object $\mathfrak{k}_X^*(M)$ for some $M \in ObC_X$.*

Proof. Objects of the category C_{X_K} are pairs (M,p), where M is an object of the category C_X and $M \xrightarrow{p} M$ is an idempotent endomorphism, i.e. $p^2 = p$. Morphisms $(M,p) \longrightarrow (M',p')$ are morphisms $M \xrightarrow{f} M'$ such that $fp = f = p'f$. The composition of $(M,p) \xrightarrow{f} (M',p')$ and $(M',p') \xrightarrow{g} (M'',p'')$ is $(M,p) \xrightarrow{gf} (M'',p'')$. It follows from this definition that $(M,p) \xrightarrow{p} (M,p)$ is the identical morphism. If $(M,p) \xrightarrow{q} (M,p)$ is an idempotent, then it splits into the composition of $(M,p) \xrightarrow{q} (M,q)$ and $(M,q) \xrightarrow{q} (M,p)$. The composition of $(M,q) \xrightarrow{q} (M,p) \xrightarrow{q} (M,q)$ is $(M,q) \xrightarrow{q} (M,q)$, which is the identical morphism. The functor $C_X \xrightarrow{\mathfrak{k}_X^*} C_{X_K}$ assigns to each object M of C_X the pair $(M, id_{_M})$ and to each morphism $M \xrightarrow{g} N$ the morphism $(M, id_{_M}) \xrightarrow{g} (N, id_N)$.

For any functor $C_X \xrightarrow{F} C_Z$ to a Karoubian category C_Z, let $C_{X_K} \xrightarrow{F_K} C_Z$ denote a functor which assigns to every object (M,p) of the category C_{X_K} the kernel of the pair $(id_{F(M)}, F(p))$. It follows that $F_K \circ \mathfrak{K}_X^* \simeq F$. In particular, for any functor $C_X \xrightarrow{F} C_Y$, there exists a natural functor $C_{X_K} \xrightarrow{F_K} C_{X_K}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{F} & C_Y \\ \mathfrak{K}_X^* \downarrow & & \downarrow \mathfrak{K}_Y^* \\ C_{X_K} & \xrightarrow{F_K} & C_{Y_K} \end{array}$$

quasi-commutes. The map $F \longmapsto F_K$ defines a (pseudo) functor from Cat to the category $KCat$ of Karoubian categories, which is a left adjoint to the inclusion functor. This implies, in particular, the universal property of the correspondence $C_X \longmapsto C_{X_K}$.

For every object (M,p) of the category C_{X_K}, the morphism $(M,p) \xrightarrow{p} (M, id_{_M})$ splits; i.e. (M,p) is a retract of $\mathfrak{k}_X^*(M) = (M, id_{_M})$. ■

3.3.1. The category C_{X_K} in 3.3 is called the *Karoubian envelope* of the category C_X.

3.4. Karoubian envelopes of right exact categories.

3.4.1. Definition. We call a right exact category $(C_X, \mathfrak{E}_X)$ *Karoubian*, if the category C_X is Karoubian and any split epimorphism of the category C_X is a deflation.

3.4.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category. Suppose that, for every idempotent $M \xrightarrow{p} M$ in C_X and every morphism $N \xrightarrow{f} M$ such that $f = pf$, there exists a cartesian square*

$$\begin{array}{ccc} N' & \xrightarrow{f'} & M \\ \mathfrak{e}' \downarrow & cart & \downarrow p \\ N & \xrightarrow{f} & M \end{array}$$

Then the Karoubian envelope C_{X_K} of the category C_X has a structure $\mathfrak{E}_{X_K}$ of a right exact Karoubian category such that the canonical functor $C_X \xrightarrow{\mathfrak{k}_X^*} C_{X_K}$ is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_{X_K}, \mathfrak{E}_{X_K})$. The right exact Karoubian category $(C_{X_K}, \mathfrak{E}_{X_K})$ is universal in the following sense: every right exact (resp. weakly right exact) functor from the right exact category $(C_X, \mathfrak{E}_X)$ to a right exact Karoubian category $(C_Y, \mathfrak{E}_Y)$ is uniquely represented as the composition of the canonical exact, hence 'exact', functor from $(C_X, \mathfrak{E}_X)$ to its Karoubian envelope $(C_{X_K}, \mathfrak{E}_{X_K})$ and a right exact (resp. weakly right exact) functor from $(C_{X_K}\mathfrak{E}_{X_K})$ to $(C_Y, \mathfrak{E}_Y)$.

Proof. (a) Let C_X be a category and $M \xrightarrow{\mathfrak{e}} L$ a split epimorphism; i.e. there exists a morphism $L \xrightarrow{\mathfrak{j}} M$ such that $\mathfrak{e} \circ \mathfrak{j} = id_L$. Let $N \xrightarrow{g} L$ be a morphism. Since $\mathfrak{j}$ is a monomorphism, a pullback of $N \xrightarrow{g} L \xleftarrow{\mathfrak{e}} M$ exists iff a pullback of $N \xrightarrow{\mathfrak{j}g} M \xleftarrow{\mathfrak{je}} M$. exists and they are isomorphic to each other. Notice that $p = \mathfrak{je}$ is an idempotent and a morphism $N \xrightarrow{f} M$ factors through $L \xrightarrow{\mathfrak{j}} M$ iff $f = pf$. Thus, we have cartesian squares

$$\begin{array}{ccc} N' & \xrightarrow{f'} & M \\ {\scriptstyle \mathfrak{e}'}\big\downarrow & cart & \big\downarrow{\scriptstyle p} \\ N & \xrightarrow{f} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} N' & \xrightarrow{f'} & M \\ {\scriptstyle \mathfrak{e}'}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{e}} \\ N & \xrightarrow{g} & L \end{array}$$

It follows from the right cartesian square that the morphism $\mathfrak{e}'$ is a split epimorphism, because it is a pullback of a split epimorphism.

(b) Suppose that the condition of 3.4.2 holds, and consider a pair of morphisms $(N, u) \xrightarrow{f} (M, q) \xleftarrow{q} (M, p)$ of the Karoubian envelope C_{X_K}. By definition, $fu = qf = f$ and $qp = pq = q$. By the hypothesis, there exists a pullback $N \times_{f,q} M$. The equality $qf = f$ implies that the projection $N \times_{f,q} M \xrightarrow{q'} N$ splits, i.e. there exists a morphism $N \xrightarrow{\mathfrak{j}'} N \times_{f,q} M$ such that $q'\mathfrak{j}' = id_N$. Set $u' = \mathfrak{j}'q'$. Then

$$\begin{array}{ccc} (N \times_{f,q} M, u') & \xrightarrow{f'} & (M, p) \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ (N, u) & \xrightarrow{f} & (M, q) \end{array}$$

is a cartesian square in C_{X_K}. This shows that split epimorphisms of C_{X_K} are stable under base change. The class of deflations $\mathfrak{E}_{X_K}$ consists of all possible compositions of morphisms of $\mathfrak{k}_X^*(\mathfrak{E}_X)$ and split epimorphisms.

(c) By the universal property of Karoubian envelopes, any functor $C_X \xrightarrow{F} C_Y$ is represented as the composition of the canonical embedding $C_X \longrightarrow C_{X_K}$ and a uniquely determined functor $C_{X_K} \xrightarrow{\widetilde{F}} C_Y$. If F is a (weakly) right 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$, then $\widetilde{F}$ is a (resp. weakly) right 'exact' morphism from the Karoubian envelope $(C_{X_K}, \mathfrak{E}_{X_K})$ of $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. ■

3.4.3. Proposition. *For every svelte right exact category* $(C_X, \mathfrak{E}_X)$, *there exists a "Karoubian envelope" of* $(C_X, \mathfrak{E}_X)$, *which is a Karoubian category* $(C_{X_K}, \mathfrak{E}_{X_K})$ *such that every right exact (resp. weakly right exact) functor from* $(C_X, \mathfrak{E}_X)$ *to a right exact Karoubian category* $(C_Y, \mathfrak{E}_Y)$ *is uniquely represented as the composition of the canonical exact, hence 'exact', functor from* $(C_X, \mathfrak{E}_X)$ *to* $(C_{X_K}, \mathfrak{E}_{X_K})$ *and a right exact (resp. weakly right exact) functor from* $(C_{X_K} \mathfrak{E}_{X_K})$ *to the right exact category* $(C_Y, \mathfrak{E}_Y)$.

Proof. The Karoubian envelope C_{X_K} of the category C_X is naturally equivalent to the smallest Karoubian subcategory C_{X^K} of the category $C_{X_\mathfrak{E}}$ of sheaves of sets on the presite $(C_X, \mathfrak{E}_X)$ containing all representable sheaves. The equivalence is given by the unique functor $C_{X_K} \longrightarrow C_{X^K}$ corresponding by the universal property of Karoubian envelopes to the corestriction to C_{X^K} of the canonical embedding of the category C_X into the category of sheaves $C_{X_\mathfrak{E}}$. Identifying C_{X_K} with C_{X^K}, we take as $\mathfrak{E}_{X_K}$ all possible compositions of the images of deflations and split epimorphisms. It follows that the canonical functor $C_X \longrightarrow C_{X_K}$ is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_{X_K}, \mathfrak{E}_{X_K})$. The universality of the morphism $(C_X, \mathfrak{E}_X) \longrightarrow (C_{X_K}, \mathfrak{E}_{X_K})$ follows from the functoriality (with respect to 'exact' functors) of the canonical embedding of $(C_X, \mathfrak{E}_X)$ into the category $C_{X_\mathfrak{E}}$. ■

3.5. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *and* $(C_Y, \mathfrak{E}_Y)$ *be right exact categories. Suppose that* $\mathfrak{E}_X$ *consists of split deflations. Then a functor* $C_X \xrightarrow{F} C_Y$ *is a weakly right 'exact' functor from* $(C_X, \mathfrak{E}_X)$ *to* $(C_Y, \mathfrak{E}_Y)$ *iff it maps deflations to deflations.*

In particular, every functor $C_X \xrightarrow{F} C_Y$ *is weakly right 'exact', if all split epimorphisms of the category* C_Y *are deflations.*

Proof. Let $M \xrightarrow{\mathfrak{e}} N$ be a split epimorphism in C_X and $N \xrightarrow{\mathfrak{j}} M$ its section. Set $\mathfrak{p} = \mathfrak{j} \circ \mathfrak{e}$. Suppose that $M \times_N M$ exists (which is the case if $\mathfrak{e} \in \mathfrak{E}_X$). Then we have a commutative diagram

$$\begin{array}{ccccc}
M & \overset{\mathfrak{p}}{\underset{id_M}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & N \\
\mathfrak{t} \downarrow & & \downarrow id_M & & \downarrow id_N \\
M \times_L M & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & N
\end{array} \qquad (1)$$

whose left vertical arrow, $\mathfrak{t}$, is uniquely determined. A functor $C_X \xrightarrow{F} C_Y$ maps (1) to the commutative diagram

$$\begin{array}{ccccc} F(M) & \overset{F(\mathfrak{p})}{\underset{id}{\rightrightarrows}} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(N) \\ {\scriptstyle F(\mathfrak{t})}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ F(M \times_L M) & \overset{F(p_1)}{\underset{F(p_2)}{\rightrightarrows}} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(N) \end{array} \qquad (2)$$

whose upper row is an exact diagram (by 3.1). Therefore, the lower row of (2) is an exact diagram. The assertion follows now from the definition of a weakly right 'exact' functor. ■

3.6. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category whose deflations are split. Then every presheaf of sets on $(C_X, \mathfrak{E}_X)$ is a sheaf.*

4. Kernels, cokernels, coimages and images of morphisms.

4.1. Kernels and cokernels of arrows. Let C_X be a category with an initial object x. We define the *kernel* of a morphism $M \xrightarrow{f} N$, (if any) is the upper horizontal arrow in a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \longrightarrow & N \end{array} \qquad (1)$$

when the latter exists.

Cokernels of morphisms are defined dually, via a cocartesian square

$$\begin{array}{ccc} N & \xrightarrow{\mathfrak{c}(f)} & Cok(f) \\ {\scriptstyle f}\uparrow & cocart & \uparrow{\scriptstyle f'} \\ M & \longrightarrow & y \end{array}$$

where y is a final object of C_X.

4.1.0. Note. If C_X is a pointed category (i.e. its initial objects are final), then the notion of the kernel is equivalent to the usual one: the kernel of a morphism $M \xrightarrow{f} N$ (if any) is determined uniquely up to isomorphism by the exactness of the diagram

$$Ker(f) \xrightarrow{\mathfrak{k}(f)} M \overset{f}{\underset{0}{\rightrightarrows}} N.$$

Here $M \xrightarrow{o} N$ denotes the *zero* morphism, i.e. the unique morphism from M to N which factors through a zero object.

Dually, the cokernel of $M \xrightarrow{f} N$ makes the diagram

$$M \overset{f}{\underset{0}{\rightrightarrows}} N \xrightarrow{\mathfrak{c}(f)} Cok(f)$$

exact and is, therefore, determined by this property.

4.1.1. Lemma. *Let C_X be a category with an initial object x.*

(a) Let a morphism $M \xrightarrow{f} N$ of C_X have a kernel. The canonical morphism $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism, if the unique arrow $x \xrightarrow{i_N} N$ is a monomorphism.

(b) If $M \xrightarrow{f} N$ is a monomorphism, then $x \xrightarrow{\mathfrak{i}_M} M$ is the kernel of f.

Proof. (a) By definition of the kernel of f, we have a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{\mathfrak{i}_N} & N \end{array}$$

Pull-backs of monomorphisms are monomorphisms. In particular, $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism if $x \xrightarrow{\mathfrak{i}_N} N$ is a monomorphism.

(b) Suppose that $M \xrightarrow{f} N$ is a monomorphism. If

$$\begin{array}{ccc} L & \xrightarrow{\phi} & x \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \mathfrak{i}_N} \\ M & \xrightarrow{f} & N \end{array}$$

is a commutative square, then f equalizes the pair of arrows $(\psi, \mathfrak{i}_M \circ \phi)$. If f is a monomorphism, the latter implies that $\psi = \mathfrak{i}_M \circ \phi$. Therefore, in this case, the square

$$\begin{array}{ccc} x & \xrightarrow{id_x} & x \\ {\scriptstyle \mathfrak{i}_M}\downarrow & & \downarrow{\scriptstyle \mathfrak{i}_N} \\ M & \xrightarrow{f} & N \end{array}$$

is cartesian. ■

4.1.2. Corollary. *Let C_X be a category with an initial object x. The following conditions on the category C_X are equivalent:*

(a) If $M \xrightarrow{f} N$ *has a kernel, then the canonical arrow* $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ *is a monomorphism.*

(b) The unique arrow $x \xrightarrow{\mathfrak{i}_M} M$ *is a monomorphism for any* $M \in ObC_X$*.*

Proof. $(a) \Rightarrow (b)$. By 4.1.1(b), the unique morphism $x \xrightarrow{\mathfrak{i}_M} M$ is the kernel of the identical morphism $M \xrightarrow{id_M} M$.

The implication $(b) \Rightarrow (a)$ follows from 4.1.1(a). ∎

4.1.3. Note. The converse assertion is not true in general: a morphism might have a trivial kernel without being a monomorphism. It is easy to produce an example in the category of pointed sets.

4.2. Examples and relevant digressions.

4.2.1. Kernels of morphisms of unital k-algebras. Let C_X be the category Alg_k of associative unital k-algebras. The category C_X has an initial object – the k-algebra k. For any k-algebra morphism $A \xrightarrow{\varphi} B$, we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \mathfrak{k}(\varphi)}\uparrow & & \uparrow \\ k \oplus K(\varphi) & \xrightarrow{\epsilon(\varphi)} & k \end{array}$$

where $K(\varphi)$ denote the kernel of the morphism φ in the category of non-unital k-algebras and the morphism $\mathfrak{k}(\varphi)$ is determined by the inclusion $K(\varphi) \longrightarrow A$ and the k-algebra structure $k \longrightarrow A$. This square is cartesian. In fact, if

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \gamma}\uparrow & & \uparrow \\ C & \xrightarrow{\psi} & k \end{array}$$

is a commutative square of k-algebra morphisms, then C is an augmented algebra: $C = k \oplus K(\psi)$. Since the restriction of $\varphi \circ \gamma$ to $K(\psi)$ is zero, it factors uniquely through $K(\varphi)$. Therefore, there is a unique k-algebra morphism $C = k \oplus K(\psi) \xrightarrow{\beta} Ker(\varphi) = k \oplus K(\varphi)$ such that $\gamma = \mathfrak{k}(\varphi) \circ \beta$ and $\psi = \epsilon(\varphi) \circ \beta$.

This shows that each (unital) k-algebra morphism $A \xrightarrow{\varphi} B$ has a canonical kernel $Ker(\varphi)$ equal to the augmented k-algebra corresponding to the ideal $K(\varphi)$.

It follows from the description of the kernel $Ker(\varphi) \xrightarrow{\mathfrak{k}(\varphi)} A$ that it is a monomorphism iff the k-algebra structure $k \longrightarrow A$ is a monomorphism.

Notice that cokernels of morphisms are not defined in Alg_k, because this category does not have final objects.

4.2.2. Kernels and cokernels of maps of sets. Since the only initial object of the category *Sets* is the empty set $\emptyset$ and there are no morphisms from a non-empty set to $\emptyset$, the kernel of any map $X \longrightarrow Y$ is $\emptyset \longrightarrow X$. The cokernel of a map $X \xrightarrow{f} Y$ is the projection $Y \xrightarrow{\mathfrak{c}(f)} Y/f(X)$, where $Y/f(X)$ is the set obtained from Y by the contraction of $f(X)$ into a point. So that $\mathfrak{c}(f)$ is an isomorphism iff either $X = \emptyset$, or $f(X)$ is a one-point set.

4.2.3. Digression: categories with isolated initial objects. Suppose that C_X is a category with initial objects, which are *isolated* in the following sense: every morphism to an initial object is an isomorphism. Then every morphism $M \longrightarrow L$ of C_X has a kernel, but, this kernel is trivial – the unique morphism from an initial object to M.

4.2.3.1. Sets and topological spaces. The category *Sets* and the category $\mathcal{T}op$ of topological spaces are examples of categories with isolated initial objects.

4.2.3.2. Presheaves and sheaves of sets. If C_X is a category with isolated initial objects, then the category $\mathcal{H}om(C_Y, C_X)$ of functors from a svelte category C_Y to the category C_X is a category with isolated initial points.

In particular, the category $C_X^\wedge$ of presheaves of sets on C_X is a category with isolated initial point, which is the presheaf $\bar{\emptyset}$ mapping all objects of C_X to the empty set.

The same $\bar{\emptyset}$ is the isolated initial object in the category $(C_X, \tau)^\wedge$ of sheaves of sets on a (pre)site (C_X, τ).

4.2.3.3. Non-commutative affine schemes. Let $C_X = \mathbf{Aff}_k \stackrel{\text{def}}{=} Alg_k^{op}$ – the category of non-commutative affine k-schemes. The initial object of $\mathbf{Aff}_k$ is the affine k-scheme corresponding to the zero algebra. It is the isolated initial object – the 'empty' affine k-scheme.

4.2.4. Note. The categories with isolated initial objects are useless for constructions which involve kernels. In particular, they should be avoided in the constructions of right derived functors (– universal ∂^*-functors) in Chapter II and related questions (in Chapter III). The following examples are all in the spirit of avoiding isolated initial points.

4.2.5. Digression: categories with pointed objects. An object M of a svelte category C_X is called *pointed*, if there is a cone $M \longrightarrow Id_{C_X}$.

4.2.5.1. Pointed objects in categories with initial objects. If C_X is a category with initial objects, then pointed objects in C_X are precisely those objects which have morphisms to initial objects.

4.2.5.2. Augmented algebras. Thus, pointed objects of the category Alg_k of associative unital algebras are k-algebras which have augmentations.

4.2.5.3. Right exact categories with enough pointed objects. We say that a right exact category $(C_X, \mathfrak{E}_X)$ has *enough* pointed objects, if, for any object L of C_X, there is a deflation $M \xrightarrow{\mathfrak{e}} L$, where M is a pointed object.

4.2.5.4. Augmented algebras. The category Alg_k of associative unital k-algebras endowed with the canonical (that is the finest) right exact structure is an example of a right exact category with enough pointed objects. In fact, if $\mathcal{A}$ is a unital k-algebra and $\mathcal{V}$ a k-submodule of $\mathcal{A}$ generating $\mathcal{A}$, then the embedding of k-modules $\mathcal{V} \longrightarrow \mathcal{A}$ determines a strict epimorphism from the tensor algebra $T_k(\mathcal{V})$ of the k-module $\mathcal{V}$ to the algebra $\mathcal{A}$; and tensor algebras have canonical augmentations.

4.2.6. Non-trivial presheaves of sets. The category C_X^* of non-trivial presheaves of sets has final objects and colimits of arbitrary small diagrams. In particular, the category C_X^* has cokernels of arbitrary morphisms which are computed object-wise (as in 4.2.2).

4.3. Some properties of kernels. Fix a category C_X with an initial object x.

4.3.1. Proposition. *Let* $M \xrightarrow{f} N$ *be a morphism of* C_X *which has a kernel pair,* $M \times_N M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M$. *Then the morphism* f *has a kernel iff* p_1 *has a kernel.*

Proof. Suppose that f has a kernel, i.e. there is a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{\mathfrak{i}_N} & N \end{array} \qquad (1)$$

Then we have the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{\gamma} & M \times_N M & \xrightarrow{p_2} & M \\ {\scriptstyle f'}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{\mathfrak{i}_M} & M & \xrightarrow{f} & N \end{array} \qquad (2)$$

which is due to the commutativity of (1) and the fact that the unique morphism $x \xrightarrow{\mathfrak{i}_N} N$ factors through the morphism $M \xrightarrow{f} N$. The morphism γ is uniquely determined by the equality $p_2 \circ \gamma = \mathfrak{k}(f)$. The fact that the square (1) is cartesian and the equalities $p_2 \circ \gamma = \mathfrak{k}(f)$ and $\mathfrak{i}_N = f \circ \mathfrak{i}_M$ imply that the left square of the diagram (2) is cartesian, i.e. $Ker(f) \xrightarrow{\gamma} M \times_N M$ is the kernel of the morphism p_1.

Conversely, if p_1 has a kernel, then we have a diagram

$$\begin{array}{ccccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_2} & M \\ {\scriptstyle p_1'}\downarrow & cart & {\scriptstyle p_1}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{\mathfrak{i}_M} & M & \xrightarrow{f} & N \end{array}$$

which consists of two cartesian squares. Therefore the square

$$\begin{array}{ccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle p_1'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{\mathfrak{i}_N} & N \end{array}$$

with $\mathfrak{k}(f) = p_2 \circ \mathfrak{k}(p_1)$ is cartesian. ■

4.3.2. Remarks. (a) Needless to say that the picture obtained in (the argument of) 4.3.1 is symmetric, i.e. there is an isomorphism $Ker(p_1) \xrightarrow{\tau_f'} Ker(p_2)$ which is an arrow in the commutative diagram

$$\begin{array}{ccccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_1} & M \\ {\scriptstyle \tau_f'}\downarrow\wr & & {\scriptstyle \tau_f}\downarrow\wr & & \downarrow{\scriptstyle id_M} \\ Ker(p_2) & \xrightarrow{\mathfrak{k}(p_2)} & M \times_N M & \xrightarrow{p_2} & M \end{array}$$

(b) Let a morphism $M \xrightarrow{f} N$ have a kernel pair, $M \times_N M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M$, and a kernel. Then, by 4.3.1, there exists a kernel of p_1, so that we have a morphism $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M$ and the diagonal morphism $M \xrightarrow{\Delta_M} M \times_N M$. Since the right square of the commutative diagram

$$\begin{array}{ccccc} x & \longrightarrow & Ker(p_1) & \xrightarrow{p_1'} & x \\ \downarrow & & {\scriptstyle \mathfrak{k}(p_1)}\downarrow & cart & \downarrow \\ M & \xrightarrow{\Delta_M} & M \times_N M & \xrightarrow{p_1} & M \end{array}$$

is cartesian and compositions of the horizontal arrows are identical morphisms, it follows that its left square is cartesian too. Loosely, one can say that the intersection of $Ker(p_1)$ with the diagonal of $M \times_N M$ is zero. If there exists a coproduct $Ker(p_1) \coprod M$, then the pair of morphisms $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M \xleftarrow{\Delta_M} M$ determine a morphism

$$Ker(p_1) \coprod M \longrightarrow M \times_N M. \tag{3}$$

(b1) If the category C_X is additive, then this morphism is an isomorphism, that is $Ker(f) \coprod M \simeq M \times_N M$. In general, it is rarely the case, as the reader can find out looking at the examples of 4.2.

4.3.3. Proposition. *Let*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle \widetilde{g}}\downarrow & cart & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & N \end{array} \qquad (3)$$

be a cartesian square. Then $Ker(f)$ exists iff $Ker(\widetilde{f})$ exists, and they are naturally isomorphic to each other.

Proof. Suppose that $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ exists, i.e. we have a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{\mathfrak{i}_N} & N \end{array} \qquad (4)$$

Since $x \longrightarrow N$ factors through $N' \xrightarrow{g} N$ and the square (3) is cartesian, there is a unique morphism $Ker(f') \xrightarrow{\gamma} N'$ such that the diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{\gamma} & \widetilde{M} & \xrightarrow{\widetilde{g}} & M \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} & cart & \downarrow{\scriptstyle f} \\ x & \longrightarrow & \widetilde{N} & \xrightarrow{g} & N \end{array} \qquad (5)$$

commutes and $\mathfrak{k}(f) = \widetilde{g} \circ \gamma$. Therefore the left square of (5) is cartesian.

If $Ker(\widetilde{f})$ exists, then we have the diagram

$$\begin{array}{ccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & \widetilde{M} & \xrightarrow{\widetilde{g}} & M \\ {\scriptstyle \widetilde{f}'}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} & cart & \downarrow{\scriptstyle f} \\ x & \longrightarrow & \widetilde{N} & \xrightarrow{g} & N \end{array}$$

whose both squares are cartesian. Therefore, their composition

$$\begin{array}{ccc} Ker(\widetilde{f}) & \xrightarrow{\widetilde{g}\circ\mathfrak{k}(\widetilde{f})} & M \\ {\scriptstyle \widetilde{f}'}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{\mathfrak{i}_N} & N \end{array}$$

is a cartesian square.

It follows that the unique morphism $Ker(\widetilde{f}) \xrightarrow{g'} Ker(f)$ making the diagram

$$\begin{array}{ccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle g'}\downarrow & & {\scriptstyle \widetilde{g}}\downarrow & cart & \downarrow{\scriptstyle g} \\ Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M & \xrightarrow{f} & N \end{array} \qquad (6)$$

commute is an isomorphism. ∎

4.4. The kernel of a composition and related facts. Fix a category C_X with an initial object x.

4.4.1. The kernel of a composition. Let $L \xrightarrow{f} M$ and $M \xrightarrow{g} N$ be morphisms such that there exist kernels of g and $g \circ f$. Then the argument similar to that of 4.3.3 shows that we have a commutative diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\ {\scriptstyle \mathfrak{k}(gf)}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(g)} & cart & \downarrow{\scriptstyle \mathfrak{i}_N} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array} \qquad (1)$$

whose both squares are cartesian and all morphisms are uniquely determined by f, g and the (unique up to isomorphism) choice of the objects $Ker(g)$ and $Ker(gf)$.

Conversely, if there is a commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{u} & Ker(g) & \xrightarrow{g'} & x \\ {\scriptstyle t}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(g)} & & \downarrow{\scriptstyle \mathfrak{i}_N} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

whose left square is cartesian, then its left vertical arrow, $K \xrightarrow{t} L$, is the kernel of the composition $L \xrightarrow{g \circ f} N$.

4.4.2. Remark. It follows from 3.3.3 that the kernel of $L \xrightarrow{f} M$ exists iff the kernel of $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ exists and they are isomorphic to each other. More precisely, we have a commutative diagram

$$\begin{array}{ccccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\ \wr\downarrow & & {\scriptstyle \mathfrak{k}(gf)}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(g)} & cart & \downarrow{\scriptstyle \mathfrak{i}_N} \\ Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

whose left vertical arrow is an isomorphism.

The following observations is useful (and will be used) for analyzing diagrams.

4.4.3. Proposition. *(a) Let* $M \xrightarrow{g} N$ *be a morphism with a trivial kernel. Then a morphism* $L \xrightarrow{f} M$ *has a kernel iff the composition* $g \circ f$ *has a kernel, and these two kernels are naturally isomorphic one to another.*

(b) Let

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \widetilde{M} & \xrightarrow{\phi} & N \end{array}$$

be a commutative square such that the kernels of the arrows f *and* ϕ *exist and the kernel of* g *is trivial. Then the kernel of the composition* $\phi \circ \gamma$ *is isomorphic to the kernel of the morphism* f*, and the left square of the commutative diagram*

$$\begin{array}{ccccccc} Ker(f) & \xrightarrow{\sim} & Ker(\phi\gamma) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M \\ & & {\scriptstyle\widetilde{\gamma}}\big\downarrow & cart & {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ & & Ker(\phi) & \xrightarrow{\mathfrak{k}(\phi)} & \widetilde{M} & \xrightarrow{\phi} & N \end{array}$$

is cartesian.

Proof. (a) Since the kernel of g is trivial, the diagram 4.4.1(1) specializes to the diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\widetilde{f}} & x & \xrightarrow{id_x} & x \\ {\scriptstyle\mathfrak{k}(gf)}\big\downarrow & cart & \big\downarrow{\scriptstyle\mathfrak{k}(g)} & & \big\downarrow{\scriptstyle\mathfrak{i}_N} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

with cartesian squares. The left cartesian square of this diagram is the definition of $Ker(f)$. The assertion follows from 4.4.1.

(b) Since the kernel of g is trivial, it follows from (a) that $Ker(f)$ is naturally isomorphic to the kernel of $g \circ f = \phi \circ \gamma$. The assertion follows now from 4.4.1. ■

4.4.4. Definition. Let C_X be a category with initial objects. A morphism of C_X is called *trivial* if it factors through an initial object.

4.4.5. Proposition. *Let* C_X *be a category with an initial object* x*. Let* $L \xrightarrow{f} M$ *be a strict epimorphism and* $M \xrightarrow{g} N$ *a morphism such that* $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ *exists and is a monomorphism. Then the composition* $g \circ f$ *is a trivial morphism iff* g *is trivial.*

Proof. The morphism $g \circ f$ being trivial means that there is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \gamma\downarrow & & \downarrow g \\ x & \xrightarrow{\mathfrak{i}_N} & N \end{array}$$

By 4.4.3(a), $Ker(g \circ f) \simeq Ker(\gamma) = L$. Thus, we have a commutative diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\ \wr\downarrow & cart & \downarrow \mathfrak{k}(g) & cart & \downarrow \mathfrak{i}_N \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

(cf. 4.4.1). Since f is a strict epimorphism, it follows from the commutativity of the left square that $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ is a strict epimorphism. Since, by hypothesis, $\mathfrak{k}(g)$ is a monomorphism, it is an isomorphism, which implies the triviality of g. ∎

4.4.5.1. Note. The following example shows that the requirement "$Ker(g) \longrightarrow M$ is a monomorphism" in 4.4.5 cannot be omitted.

Let C_X be the category Alg_k of associative unital k-algebras, and let $\mathfrak{m}$ be an ideal of the ring k such that the epimorphism $k \longrightarrow k/\mathfrak{m}$ does not split. Then the identical morphism $k/\mathfrak{m} \longrightarrow k/\mathfrak{m}$ is non-trivial, while its composition with the projection $k \longrightarrow k/\mathfrak{m}$ is a trivial morphism.

4.5. The coimage of a morphism. Let $M \xrightarrow{f} N$ be an arrow which has a kernel, i.e. we have a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ f'\downarrow & cart & \downarrow f \\ x & \xrightarrow{\mathfrak{i}_N} & N \end{array}$$

with which one can associate a pair of arrows $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M$, where 0_f is the composition of the projection f' and the unique morphism $x \xrightarrow{\mathfrak{i}_M} M$. Since $\mathfrak{i}_N = f \circ \mathfrak{i}_M$, the morphism f equalizes the pair $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M$. If the cokernel of this pair of arrows exists, it will be called the *coimage of* f and denoted by $Coim(f)$, or. loosely, $M/Ker(f)$.

Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. Then f is the composition of the canonical strict epimorphism $M \xrightarrow{p_f} Coim(f)$ and a uniquely defined morphism $Coim(f) \xrightarrow{j_f} N$.

4.5.1. Lemma. *Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. There is a natural isomorphism $Ker(f) \xrightarrow{\sim} Ker(p_f)$.*

Proof. The outer square of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(f)}\downarrow & cart & \downarrow & & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \qquad (1)$$

is cartesian. Therefore, its left square is cartesian which implies, by 4.3.3, that $Ker(p_f)$ is isomorphic to $Ker(f')$. But, $Ker(f') \simeq Ker(f)$. ■

4.5.2. Note. By 4.4.1, all squares of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & & \\ {\scriptstyle id}\downarrow & cart & \downarrow & & \\ Ker(j_f p_f) & \xrightarrow{\widetilde{p}_f} & Ker(j_f) & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(f)}\downarrow & cart & \downarrow & cart & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \qquad (2)$$

are cartesian.

If C_X is an additive category and $M \xrightarrow{f} L$ is an arrow of C_X having a kernel and a coimage, then the canonical morphism $Coim(f) \xrightarrow{j_f} L$ is a monomorphism. Quite a few non-additive categories have this property.

4.5.3. Example. Let C_X be the category Alg_k of associative unital k-algebras. Since cokernels of pairs of arrows exist in Alg_k, any algebra morphism has a coimage. It follows from 4.2.1 that the coimage of an algebra morphism $A \xrightarrow{\varphi} B$ is $A/K(\varphi)$, where $K(\varphi)$ is the kernel of ϕ in the usual sense (i.e. in the category of non-unital algebras). The canonical decomposition $\varphi = j_\varphi \circ p_\varphi$ coincides with the standard presentation of φ as the composition of the projection $A \longrightarrow A/K(\varphi)$ and the monomorphism $A/K(\varphi) \longrightarrow B$. In particular, φ is a strict epimorphism of k-algebras iff it is isomorphic to the associated coimage map $A \xrightarrow{p_\varphi} Coim(\varphi) = A/K(\varphi)$.

5. Right exact categories with initial objects.

5.1. Inflations and conflations. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x. We denote by $\mathcal{E}_X$ the class of all sequences of the form $K \xrightarrow{\mathfrak{k}} M \xrightarrow{\mathfrak{e}} N$, where $\mathfrak{e} \in \mathfrak{E}_X$ and $K \xrightarrow{\mathfrak{k}} M$ is a kernel of $\mathfrak{e}$. Expanding the terminology of exact additive categories, we call any such sequence a *conflation*.

Any kernel $K \xrightarrow{\mathfrak{k}} M$ of a deflation will be called an *inflation*. We denote by $\mathfrak{M}_X$ the class of all inflations of the right exact category $(C_X, \mathfrak{E}_X)$.

5.2. A useful observation. Suppose that $L \xrightarrow{f} M$ is a deflation and $M \xrightarrow{g} N$ a morphism having a kernel. Then it follows from 4.4.2 that the canonical morphism $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ is a deflation too, and we have a commutative diagram

$$\begin{array}{ccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) \\ \wr\downarrow & & \mathfrak{k}(gf)\downarrow & cart & \downarrow\mathfrak{k}(g) \\ Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M \end{array}$$

whose rows are conflations.

5.3. The property (†) and 'exactness' of functors.

5.3.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects. Suppose that the right exact category $(C_Y, \mathfrak{E}_Y)$ satisfies the following property:*

(†) if the rows of a commutative diagram

$$\begin{array}{ccccc} \widetilde{L} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{N} \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & M & \longrightarrow & N \end{array}$$

are conflations and its right and left vertical arrows are isomorphisms, then the middle arrow is an isomorphism.

Then the following conditions on a functor $C_X \xrightarrow{F} C_Y$ are equivalent:
(a) F is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.
(b) F maps deflations to deflations and cartesian squares of the form

$$\begin{array}{ccc} Ker(\mathfrak{e}) & \longrightarrow & x \\ \mathfrak{k}(\mathfrak{e})\downarrow & cart & \downarrow \\ M & \xrightarrow{\mathfrak{e}} & N \end{array} \qquad (1)$$

where $M \xrightarrow{\mathfrak{e}} N$ *is a deflation and* x *an initial object, to cartesian squares.*

Proof. The implication (a)⇒(b) follows from definition of an 'exact' functor and does not require any additional conditions.

(b)⇒(a). Let a functor $C_X \xrightarrow{F} C_Y$ satisfy the conditions (b). We need to show that F preserves arbitrary pull-backs of deflations.

Let $M \xrightarrow{\mathfrak{e}} N$ be a deflation and $\widetilde{N} \xrightarrow{f} N$ an arbitrary morphism of C_X. Consider the associated with this data diagram

$$\begin{array}{ccccc} Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{e}})} & \widetilde{M} & \xrightarrow{f'} & M \\ \gamma_{\mathfrak{e}}\downarrow & cart & \widetilde{\mathfrak{e}}\downarrow & cart & \downarrow\mathfrak{e} \\ x & \longrightarrow & \widetilde{N} & \xrightarrow{f} & N \end{array} \qquad (2)$$

with cartesian squares. By hypothesis, the functor $C_X \xrightarrow{F} C_Y$ maps the diagram (2) to the commutative diagram

$$\begin{array}{ccccc} F(Ker(\mathfrak{e})) & \xrightarrow{F(\mathfrak{k}(\widetilde{\mathfrak{e}}))} & F(\widetilde{M}) & \xrightarrow{F(f')} & F(M) \\ F(\gamma_{\mathfrak{e}})\downarrow & cart & F(\widetilde{\mathfrak{e}})\downarrow & & \downarrow F(\mathfrak{e}) \\ F(x) & \longrightarrow & F(\widetilde{N}) & \xrightarrow{F(f)} & F(N) \end{array} \qquad (3)$$

whose left square is cartesian, as well as the outer square

$$\begin{array}{ccc} F(Ker(\mathfrak{e})) & \xrightarrow{F(\mathfrak{k}(\mathfrak{e}))} & F(M) \\ F(\gamma_{\mathfrak{e}})\downarrow & cart & \downarrow F(\mathfrak{e}) \\ F(x) & \longrightarrow & F(N) \end{array} \qquad (3')$$

Since all vertical arrows of (3) are deflations, we can extend the diagram (3) to the commutative diagram

$$\begin{array}{ccccccc} Ker(F(\gamma_{\mathfrak{e}})) & \xrightarrow{\lambda_1} & Ker(F(\widetilde{\mathfrak{e}})) & \xrightarrow{\lambda_2} & Ker(\mathfrak{t}) & \xrightarrow{\lambda_3} & Ker(F(\mathfrak{e})) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F(Ker(\mathfrak{e})) & \xrightarrow{F(\mathfrak{k}(\widetilde{\mathfrak{e}}))} & F(\widetilde{M}) & \xrightarrow{\psi} & \mathfrak{M} & \xrightarrow{\phi} & F(M) \\ F(\gamma_{\mathfrak{e}})\downarrow & cart & F(\widetilde{\mathfrak{e}})\downarrow & & \mathfrak{t}\downarrow & cart & \downarrow F(\mathfrak{e}) \\ F(x) & \longrightarrow & F(\widetilde{N}) & \xrightarrow{id} & F(\widetilde{N}) & \xrightarrow{F(f)} & F(N) \end{array} \qquad (4)$$

whose columns are conflations, right lower square is cartesian, and $\phi \circ \psi = F(f')$. By 4.3.3, the morphisms λ_1 and λ_3 in the upper row of the diagram (4) are isomorphisms. The composition $\lambda_3 \circ \lambda_2 \circ \lambda_1$ is an isomorphism by the same reason, because the square (3') is cartesian. Therefore, $Ker(F(\widetilde{\mathfrak{e}})) \xrightarrow{\lambda_2} Ker(\mathfrak{t})$ is an isomorphism. Since the property (†) holds in $(C_Y, \mathfrak{E}_Y)$, the latter implies that the morphism $F(\widetilde{M}) \xrightarrow{\psi} \mathfrak{M}$ is an isomorphism. This shows that the functor F preserves pull-backs of deflations. ∎

5.3.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects and F a functor $C_X \longrightarrow C_Y$ which maps initial objects to initial objects and deflations to deflations. Suppose that the right exact category $(C_Y, \mathfrak{E}_Y)$ has the property (†) of 5.3.1. Then the functor F is 'exact' iff it maps conflations to conflations.*

5.3.3. Corollary. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be additive k-linear right exact categories and F an additive functor $C_X \longrightarrow C_Y$. Then the functor F is weakly 'exact' iff it is 'exact'.*

Proof. By 4.3.2.1, a k-linear functor $C_X \xrightarrow{F} C_Y$ is a weakly 'exact' iff it maps conflations to conflations. The assertion follows now from 5.3.2. ∎

5.4. Right exact categories with the property (†).

5.4.1. Abelian and additive categories. It is well known (and easy to check) that every abelian category has this property.

Any additive right exact category $(C_Y, \mathfrak{E}_Y)$ has the property (†).

In fact, applying the canonical 'exact' embedding of $(C_Y, \mathfrak{E}_Y)$ to the category $C_{Y_\mathfrak{E}}$ of sheaves of $\mathbb{Z}$-modules on the presite $(C_Y, \mathfrak{E}_Y)$, we reduce the assertion to the case when the category is abelian (with the canonical exact structure).

5.4.2. Groups. One of the simplest non-additive examples of a right exact category with the property (†) is the category of groups with the standard (that is the finest) right exact structure.

5.4.3. An obvious observation: If $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ are right exact categories with initial objects and $C_X \xrightarrow{F} C_Y$ a conservative functor which maps conflations to conflations, then the property (†) holds in $(C_X, \mathfrak{E}_X)$, provided it holds in $(C_Y, \mathfrak{E}_Y)$.

5.4.3.1. Example. Let k be a field and C_X the category Alg_k of k-algebras with strict epimorphisms as deflations. The map which assigns to every k-algebra $(A, k \xrightarrow{\varphi} A)$ the cokernel of φ (more precisely, the cokernel of $\varphi_*(k \xrightarrow{\varphi} A)$) defines a conservative functor $Alg_k \xrightarrow{\bar{\Omega}_k} k - mod$ which maps conflations to conflations.

5.4.3.2. Note. This example is valid for arbitrary unital commutative ring k, if we take as C_X the full subcategory $\mathfrak{Alg}_k$ of the category Alg_k formed by k-algebras $(A, k \xrightarrow{\varphi} A)$ for which the structure morphism φ is a monomorphism.

The following observation is more subtle than the one in 5.4.3 and much more helpful.

5.4.4. Lemma. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects; and let $C_X \xrightarrow{F} C_Y$ be a conservative functor which maps deflations to deflations and cartesian squares*

$$\begin{array}{ccc} Ker(\mathfrak{e}) & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(\mathfrak{e})}\downarrow & cart & \downarrow \\ M & \xrightarrow{\mathfrak{e}} & L \end{array} \qquad (1)$$

where $\mathfrak{e}$ is a deflation and x an initial object, to cartesian squares. Then the property $(\dagger)$ holds in $(C_X, \mathfrak{E}_X)$ provided it holds in $(C_Y, \mathfrak{E}_Y)$.

Proof. Fix an initial object y of the category C_Y. The diagram (1) corresponding to a deflation $M \xrightarrow{\mathfrak{e}} L$ gives rise to a diagram

$$\begin{array}{ccccc} & & Ker(F(\mathfrak{e})) & \longrightarrow & y \\ & & \downarrow & cart & \downarrow \\ Ker(F(\mathfrak{e})) & \longrightarrow & F(Ker(\mathfrak{e})) & \longrightarrow & F(x) \\ {\scriptstyle id}\downarrow\wr & & {\scriptstyle F(\mathfrak{k}(\mathfrak{e}))}\downarrow & cart & \downarrow \\ Ker(F(\mathfrak{e})) & \longrightarrow & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L) \end{array} \qquad (2)$$

whose two right squares are cartesian and two lower rows are conflations. If $\mathcal{M} \xrightarrow{\gamma} M$ is a morphism such that $\mathfrak{e} \circ \gamma$ is a deflation and $Ker(\mathfrak{e} \circ \gamma) = Ker(\mathfrak{e})$, then it follows (from the middle row of the diagram (2)) that this data extends (2) to the diagram

$$\begin{array}{ccccc} & & Ker(F(\mathfrak{e})) & \longrightarrow & y \\ & & \downarrow & cart & \downarrow \\ Ker(F(\mathfrak{e})) & \longrightarrow & F(Ker(\mathfrak{e})) & \longrightarrow & F(x) \\ {\scriptstyle id}\downarrow\wr & & {\scriptstyle F(\mathfrak{k}(\mathfrak{e}))}\downarrow & cart & \downarrow \\ Ker(F(\mathfrak{e})) & \longrightarrow & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L) \\ {\scriptstyle id}\uparrow\wr & & {\scriptstyle F(\gamma)}\uparrow & & \uparrow{\scriptstyle id} \\ Ker(F(\mathfrak{e} \circ \gamma)) & \longrightarrow & F(\mathcal{M}) & \xrightarrow{F(\mathfrak{e} \circ \gamma)} & F(L) \end{array} \qquad (3)$$

whose lower row is also a conflation. If the right exact category $(C_Y, \mathfrak{E}_Y)$ has property $(\dagger)$, then the morphism $F(\gamma)$ is an isomorphism. Since the functor F is, by hypothesis, conservative, this implies that $\mathcal{M} \xrightarrow{\gamma} M$ is an isomorphism. ∎

5.4.5. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and $(C_Y, \mathfrak{E}_Y)$ is a right exact k-linear category. Suppose that there exists a conservative 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{F} (C_Y, \mathfrak{E}_Y)$. Then the property (†) holds in $(C_X, \mathfrak{E}_X)$.*

Proof. The assertion follows from 5.4.4 and the fact that the property (†) holds for k-linear right exact categories (see 5.4.1). ∎

5.4.6. Note. If $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ are right exact categories with initial objects, and the property (†) holds in $(C_Y, \mathfrak{E}_Y)$, then, by 5.3.1, functor $C_X \xrightarrow{F} C_Y$ satisfying the conditions of 5.4.4, are precisely 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.

5.4.7. Example. Let C_X be the right exact category $(Alg_k, \mathfrak{E}^{\mathfrak{s}})$ of associative unital k-algebras with strict epimorphisms as deflations. The forgetful functor $Alg_k \xrightarrow{\mathfrak{f}_*} k - mod$ is conservative, maps deflations to deflations (that is to epimorphisms of k-modules) and, as every functor having a left adjoint, it preserves limits; In particular, $\mathfrak{f}_*$ preserves pull-backs. So that $\mathfrak{f}_*$ is an 'exact' conservative functor from $(Alg_k, \mathfrak{E}^{\mathfrak{s}})$ to the abelian category $k - mod$. By 5.4.4, the right exact category $(Alg_k, \mathfrak{E}^{\mathfrak{s}})$ has the property (†).

5.4.8. A general setting. Let $(C_Y, \mathfrak{E}_Y)$ be a right exact category and $C_X \xrightarrow{\mathfrak{f}_*} C_Y$ a conservative functor having a left adjoint, $\mathfrak{f}^*$, and such that the class $\mathfrak{f}_*^{-1}(\mathfrak{E}_Y)$ consists of *universal* (that is stable under base change) strict epimorphisms: $\mathfrak{f}_*^{-1}(\mathfrak{E}_Y) \in \mathfrak{E}_X^{\mathfrak{s}}$. Then $\mathfrak{f}_*^{-1}(\mathfrak{E}_Y)$ is a right exact structure on the category C_X and the functor $\mathfrak{f}_*$ is a conservative 'exact' functor from $(C_X, \mathfrak{f}_*^{-1}(\mathfrak{E}_Y))$ to $(C_Y, \mathfrak{E}_Y)$.

It follows from 5.4.4 that, if the categories C_X and C_Y have initial objects and the property (†) holds in the right exact category $(C_Y, \mathfrak{E}_Y)$, then it holds in $(C_X, \mathfrak{f}_*^{-1}(\mathfrak{E}_Y))$.

The 'exact' functor $(Alg_k, \mathfrak{E}^{\mathfrak{s}}) \longrightarrow k - mod$ (of example 5.4.7) is a very special case of this setting, as well as its commutative version $(CAlg_k, \mathfrak{E}^{\mathfrak{s}}) \longrightarrow k - mod$.

Another example is the forgetful functor from the category $\mathcal{L}ie_k$ of Lie algebras over k to $k - mod$. Its left adjoint assigns to a k-module V a free Lie k-algebra generated by V.

Both these cases and many others are instances of the categories of *algebras* over an operad and their canonical forgetful functors considered in the next example.

5.4.9. Algebras over operads. Fix a symmetric additive monoidal category $\mathcal{C}^\sim = (\mathcal{C}, \otimes, \mathbf{1}, a, l, r; \beta)$ (here β is a symmetry, $\beta_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$). Let $\mathbf{S}$ denote the category objects of which are sets $[n] = \{1, ..., n\}$, $n \geq 1$, and $[0] = \emptyset$ and morphisms are bijections. Denote by $\mathcal{C}^{\mathbf{S}}$ the category of functors $\mathbf{S}^{op} \longrightarrow \mathcal{C}$. In other words, objects of $\mathcal{C}^{\mathbf{S}}$ are collections $\mathcal{M} = (M(n)|\ n \geq 0)$, where $M(n)$ is an object of $\mathcal{C}$ with an action of the symmetric group S_n.

The category $\mathcal{C}^{\mathbf{S}}$ acts on the category $\mathcal{C}$ by *polynomial functors*:

$$M : V \longmapsto M(V) = \bigoplus_{n \geq 0} M(n) \otimes_{S_n} V^{\otimes n} \qquad (4)$$

The composition of polynomial functors is again a polynomial functor. This defines a tensor product, $\odot$, on $\mathcal{C}^{\mathbf{S}}$ called the *plethism product*. We denote the corresponding monoidal category $(\mathcal{C}^{\mathbf{S}}, \odot, \mathbf{1}_{\mathbf{S}})$ by $\mathcal{C}^{\sim\mathbf{S}}$. Here $\mathbf{1}_{\mathbf{S}}$ is the unit object $\mathbf{1}_{\mathbf{S}}$. One can see that $\mathbf{1}_{\mathbf{S}}(n) = 0$ if $n \neq 1$ and $\mathbf{1}_{\mathbf{S}}(1)$ is the unit object of the category $\mathcal{C}^{\sim}$. Thus we have an action $\mathfrak{C}$ of the monoidal category $\mathcal{C}^{\mathbf{S}}$ on the category $\mathcal{C}$.

Algebras in the monoidal category $\mathcal{C}^{\mathbf{S}}$ are called *operads*, or $\mathcal{C}^{\sim}$*-operads*. For each operad $\mathcal{R}$, the corresponding category of $\mathcal{R}$-modules is usually called *the category of* $\mathcal{R}$*-algebras.* This terminology is, of coarse, due to the same example $Alg_k \xrightarrow{\mathfrak{f}_*} k - mod$.

Fixing any right exact structure on the category $\mathcal{C}$, we obtain the induced structures on the category of algebras over an operad $\mathcal{R}$. By the observation 5.4.8, the obtained this way *right exact* category of $\mathcal{R}$-algebras has the property (†).

5.4.10. Example. A simple special case of 5.4.9, which is not reduced to the category of algebras over an operad is the category $\mathcal{R}\backslash Alg_k$ of k-algebras over a k-algebra $\mathcal{R}$ and the forgetful functor, $\mathfrak{f}_*$, from $\mathcal{R}\backslash Alg_k$ to the abelian category $\mathcal{R} \otimes_k \mathcal{R}^o$-modules (or, what is the same, k-central $\mathcal{R}$-bimodules). Its left adjoint, $\mathfrak{f}^*$ maps every $\mathcal{R} \otimes_k \mathcal{R}^o$-module $\mathcal{M}$ to the tensor algebra $T_{\mathcal{R}}(\mathcal{M})$ with the natural morphism $\mathcal{R} \longrightarrow T_{\mathcal{R}}(\mathcal{M})$ (which identifies the k-algebra $\mathcal{R}$ with the zero component of $T_{\mathcal{R}}(\mathcal{M})$.

5.5. A digression: the property (†) without initial objects. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. Let Σ_X denote the class of morphisms $\mathcal{M} \xrightarrow{\gamma} \mathfrak{M}$ having the following property: there exist a deflation $\mathfrak{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ such that $\mathfrak{e} \circ \gamma$ is also a deflation and a cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\tilde{\mathfrak{e}}} & \mathcal{V} \\ {\scriptstyle \lambda'}\downarrow & cart & \downarrow{\scriptstyle \lambda} \\ \mathcal{M} & \xrightarrow{\mathfrak{e}\circ\gamma} & \mathcal{L} \end{array} \qquad (1)$$

such that the square

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\tilde{\mathfrak{e}}} & \mathcal{V} \\ {\scriptstyle \gamma\circ\lambda'}\downarrow & cart & \downarrow{\scriptstyle \lambda} \\ \mathfrak{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array} \qquad (2)$$

is cartesian too.

5.5.1. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *be a right exact category and* Σ_X *the class of arrows of the category* C_X *defined above.*

(a) The class Σ_X *is stable under pull-backs along deflations.*

(b) The intersection $\Sigma_X \cap \mathfrak{E}_X$ *coincides with the class* $\mathfrak{E}_X^{\circledast}$ *of all deflations* $\mathfrak{e}$ *whose pull-backs contain isomorphisms.*

(c) If the category C_X has initial objects, then Σ_X consists of all morphisms $\mathcal{M} \xrightarrow{\gamma} \mathfrak{M}$ having the following property: there exist a deflation $\mathfrak{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ such that $\mathfrak{e} \circ \gamma$ is also a deflation and the natural morphism $Ker(\mathfrak{e} \circ \gamma) \longrightarrow Ker(\mathfrak{e})$ is an isomorphism.

(d) If the category C_X has initial objects, then the class $\mathfrak{E}_X^{\circledast}$ consists of all deflations with trivial kernels.

Proof. (a) Let $\mathcal{M} \xrightarrow{\gamma} \mathfrak{M}$ be a morphism from Σ_X, $\mathfrak{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ a deflation such that $\mathfrak{e} \circ \gamma$ is a deflation and $\mathcal{V} \xrightarrow{\lambda} \mathcal{L}$ an arrow which give rise to the cartesian squares (1) and (2) above. For any deflation $\mathfrak{N} \xrightarrow{\mathfrak{t}} \mathfrak{M}$, consider the commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{N}} & \xrightarrow{\widetilde{\mathfrak{t}}} & \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\mathfrak{e}}} & \mathcal{V} \\ {\scriptstyle \lambda''}\downarrow & cart & {\scriptstyle \lambda'}\downarrow & cart & \downarrow{\scriptstyle \lambda} \\ \mathcal{N} & \xrightarrow{\mathfrak{t}_1} & \mathcal{M} & \xrightarrow{\mathfrak{e}\circ\gamma} & \mathcal{L} \\ {\scriptstyle \gamma'}\downarrow & cart & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle id_{\mathcal{L}}} \\ \mathfrak{N} & \xrightarrow{\mathfrak{t}} & \mathfrak{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array} \tag{3}$$

with three cartesian squares as indicated. Since the square built of cartesian squares is cartesian, the square

$$\begin{array}{ccc} \widetilde{\mathcal{N}} & \xrightarrow{\widetilde{\mathfrak{e}}\circ\widetilde{\mathfrak{t}}} & \mathcal{V} \\ {\scriptstyle \lambda''}\downarrow & cart & \downarrow{\scriptstyle \lambda} \\ \mathcal{N} & \xrightarrow{\mathfrak{e}\circ\gamma\circ\mathfrak{t}_1} & \mathcal{L} \end{array} \tag{4}$$

is cartesian; and $\mathfrak{e} \circ \gamma \circ \mathfrak{t}_1 = (\mathfrak{e} \circ \mathfrak{t}) \circ \gamma'$.

By a similar reason, since the square (2) is cartesian, the square

$$\begin{array}{ccc} \widetilde{\mathcal{N}} & \xrightarrow{\widetilde{\mathfrak{e}}\circ\widetilde{\mathfrak{t}}} & \mathcal{V} \\ {\scriptstyle \gamma'\circ\lambda''}\downarrow & cart & \downarrow{\scriptstyle \lambda} \\ \mathfrak{N} & \xrightarrow{\mathfrak{e}\circ\mathfrak{t}} & \mathcal{L} \end{array} \tag{5}$$

is cartesian. This shows that the pull-back γ' of the morphism γ belongs to Σ_X.

(b) If $\mathcal{M} \xrightarrow{\gamma} \mathfrak{M}$ is a deflation such that a pull-back, $\widetilde{\gamma}$, of γ along along some morphism $\mathcal{V} \xrightarrow{\lambda} \mathcal{L}$ is an isomorphism, then we obtain the cartesian squares (1) and (2) with $\mathfrak{e} = id_{\mathcal{L}}$. This shows the inclusion $\mathfrak{E}_X^{\circledast} \subseteq \Sigma_X \cap \mathfrak{E}_X$. Conversely, let $\gamma \in \Sigma_X \cap \mathfrak{E}_X$, and let (1) and (2) are cartesian squares. Since γ is a deflation, it can be pulled back along any morphism. Therefore, we can decompose the diagram (1) into two squares:

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \xrightarrow{id} & \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\mathfrak{e}}} & \mathcal{V} \\ {\scriptstyle \lambda'}\downarrow & & {\scriptstyle \gamma\circ\lambda'}\downarrow & cart & \downarrow{\scriptstyle \lambda} \\ \mathcal{M} & \xrightarrow{\gamma} & \mathfrak{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array} \tag{6}$$

Since the right square in (6) is cartesian and the composition of these two squares – the square (1), is cartesian, the left square is cartesian too. This shows that $\gamma \in \mathfrak{E}_X^{\circledast}$; hence the inverse inclusion $\Sigma_X \cap \mathfrak{E}_X \subseteq \mathfrak{E}_X^{\circledast}$.

(c) Suppose that the category C_X has an initial object, x. Let $\mathcal{M} \xrightarrow{\gamma} \mathfrak{M}$ be a morphism from Σ_X and (1) and (2) the cartesian squares acknowledging this fact. Pulling back the deflation $\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\mathfrak{e}}} \mathcal{V}$ along the unique arrow $x \longrightarrow \mathcal{V}$ (see (1) and (2) above), we obtain the diagrams

$$\begin{array}{ccccc} Ker(\widetilde{\mathfrak{e}}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{e}})} & \widetilde{\mathcal{M}} & \xrightarrow{\lambda'} & \mathcal{M} \\ \downarrow & cart & \widetilde{\mathfrak{e}} \downarrow & cart & \downarrow \mathfrak{e}\circ\gamma \\ x & \longrightarrow & \mathcal{V} & \xrightarrow{\lambda} & \mathcal{L} \end{array} \qquad (7)$$

and

$$\begin{array}{ccccc} Ker(\widetilde{\mathfrak{e}}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{e}})} & \widetilde{\mathcal{M}} & \xrightarrow{\gamma\circ\lambda'} & \mathfrak{M} \\ \downarrow & cart & \widetilde{\mathfrak{e}} \downarrow & cart & \downarrow \mathfrak{e} \\ x & \longrightarrow & \mathcal{V} & \xrightarrow{\lambda} & \mathcal{L} \end{array} \qquad (8)$$

built of cartesian squares. It follows from (7) and (8) that $Ker(\widetilde{\mathfrak{e}}) = Ker(\mathfrak{e})$, and we obtain, instead of (1) and (2), the cartesian squares

$$\begin{array}{ccc} Ker(\mathfrak{e}) & \longrightarrow & x \\ \downarrow & cart & \downarrow \\ \mathcal{M} & \xrightarrow{\mathfrak{e}\circ\gamma} & \mathcal{L} \end{array} \quad \text{and} \quad \begin{array}{ccc} Ker(\mathfrak{e}) & \longrightarrow & x \\ \downarrow & cart & \downarrow \\ \mathfrak{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array}$$

So that the object $\mathcal{V}$ in the diagrams (1) and (2) can be replaced by x.

(d) If some pull-back of a morphism $\mathcal{M} \xrightarrow{\gamma} \mathcal{L}$ can be included into a cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \longrightarrow & \mathcal{M} \\ \mathfrak{s} \downarrow \wr & cart & \downarrow \gamma \\ \widetilde{\mathcal{L}} & \longrightarrow & \mathcal{L} \end{array}$$

whose left arrow is an isomorphism, than, by 4.3.3, $Ker(\gamma) = Ker(\mathfrak{s})$ and $Ker(\mathfrak{s})$ is an initial object. ■

5.5.2. Note. Because of 5.5.1(d), the arrows of $\mathfrak{E}_X^{\circledast}$ are called *deflations with trivial kernel*, even if the category C_X does not have initial objects.

5.5.3. The property (†). We say that a right exact category $(C_X, \mathfrak{E}_X)$ has the property (†) if the class of morphisms Σ_X consists only of isomorphisms.

If follows from 5.5.1(b) that if $(C_X, \mathfrak{E}_X)$ has the property (†), then the class $\mathfrak{E}_X^{\circledast}$ of deflations with trivial kernel also consists of isomorphisms; in particular, $\Sigma_X = \mathfrak{E}_X^{\circledast}$.

5.5.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories and $(C_X, \mathfrak{E}_X) \xrightarrow{F} (C_Y, \mathfrak{E}_Y)$ an 'exact' functor. Then $F(\Sigma_X) \subseteq \Sigma_Y$.*

Proof. The fact that $\mathcal{M} \xrightarrow{\gamma} \mathfrak{M}$ belongs to Σ_X means that there exist cartesian squares

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\mathfrak{e}}} & \mathcal{V} \\ {\scriptstyle \lambda'}\big\downarrow & cart & \big\downarrow{\scriptstyle \lambda} \\ \mathcal{M} & \xrightarrow{\mathfrak{e}\circ\gamma} & \mathcal{L} \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\mathfrak{e}}} & \mathcal{V} \\ {\scriptstyle \gamma\circ\lambda'}\big\downarrow & cart & \big\downarrow{\scriptstyle \lambda} \\ \mathfrak{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array} \qquad (9)$$

in which all horizontal arrows are deflations. Since the functor F is 'exact', it preserves deflations and their pull-backs. In particular, it maps the cartesian squares (9) to cartesian squares whose horizontal arrows are deflations, hence $F(\gamma) \in \Sigma_X$. ■

5.5.5. Corollary. *Suppose that an 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{F} (C_Y, \mathfrak{E}_Y)$ is conservative and the right exact category $(C_Y, \mathfrak{E}_Y)$ has the property (†). Then $(C_X, \mathfrak{E}_X)$ has this property.*

Proof. By 5.5.4, the 'exactness' of F implies that $F(\Sigma_X) \subseteq \Sigma_Y$. The property (†) for $(C_Y, \mathfrak{E}_Y)$ means that Σ_Y coincides with the class $Iso(C_Y)$ of all isomorphisms of the category C_Y. Since F is conservative, this implies that $\Sigma_X = Iso(C_X)$. ■

There is also a more general version of Proposition 5.3.1:

5.5.6. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and $(C_Y, \mathfrak{E}_Y)$ a right exact category with the property (†). Then the following conditions on a functor $C_X \xrightarrow{F} C_Y$ are equivalent:*

(a) F is an exact functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.

(b) F maps deflations to deflations and cartesian squares

$$\begin{array}{ccc} Ker(\mathfrak{e}) & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(\mathfrak{e})}\big\downarrow & cart & \big\downarrow \\ M & \xrightarrow{\mathfrak{e}} & N \end{array}$$

where $\mathfrak{e}$ is a deflation and x an initial object, to cartesian squares.

Proof. We need to show that a functor satisfying the conditions (b) maps any cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{f'} & M \\ {\scriptstyle \widetilde{\mathfrak{e}}}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{e}} \\ \widetilde{N} & \xrightarrow{f} & N \end{array}$$

whose vertical arrows are deflations to a cartesian square. The argument is similar to that of 5.3.1, only instead of the diagram 5.3(4), we take the diagram

$$\begin{array}{ccccccc} F(Ker(\mathfrak{e})) & \xrightarrow{id} & F(Ker(\mathfrak{e})) & \xrightarrow{\lambda_2} & F(Ker(\mathfrak{e})) & \xrightarrow{id} & F(Ker(\mathfrak{e})) \\ {\scriptstyle id}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ F(Ker(\mathfrak{e})) & \xrightarrow{F(\mathfrak{k}(\widetilde{\mathfrak{e}}))} & F(\widetilde{M}) & \xrightarrow{\psi} & \mathfrak{M} & \xrightarrow{\phi} & F(M) \\ {\scriptstyle F(\gamma_{\mathfrak{e}})}\downarrow & cart & {\scriptstyle F(\widetilde{\mathfrak{e}})}\downarrow & & {\scriptstyle \mathfrak{t}}\downarrow & cart & \downarrow{\scriptstyle F(\mathfrak{e})} \\ F(x) & \longrightarrow & F(\widetilde{N}) & \xrightarrow{id} & F(\widetilde{N}) & \xrightarrow{F(f)} & F(N) \end{array} \qquad (10)$$

which is obtained taking the pull-backs of the lower vertical arrows in (10) along the morphisms $F(x \longrightarrow \widetilde{N})$ and $F(x \longrightarrow N)$. Here $\phi \circ \psi = F(f')$. Since the composition of all upper horizontal arrows is an isomorphism, the arrow $F(Ker(\mathfrak{e})) \xrightarrow{\lambda_2} F(Ker(\mathfrak{e}))$ is an isomorphism too. This implies that the arrow $F(\widetilde{M}) \xrightarrow{\psi} \mathfrak{M}$ belongs to Σ_Y.

In fact, since the composition of three lower squares is, by hypothesis (b), a cartesian square, the composition of the two left lower squares is cartesian too. Together with the extreme left lower cartesian square, this gives the criteria that $F(\widetilde{M}) \xrightarrow{\psi} \mathfrak{M}$ belongs to Σ_Y. Since $\Sigma_Y = Iso(C_Y)$, the morphism ψ is an isomorphism, hence the square

$$\begin{array}{ccc} F(\widetilde{M}) & \xrightarrow{F(f')} & F(M) \\ {\scriptstyle F(\widetilde{\mathfrak{e}})}\downarrow & cart & \downarrow{\scriptstyle F(\mathfrak{e})} \\ F(\widetilde{N}) & \xrightarrow{F(f)} & F(N) \end{array}$$

is cartesian. ∎

5.5.7. Examples of deflations with trivial kernel.

5.5.7.1. Non-empty sets. Let C_X be the category $Sets^*$ of non-empty sets and $\mathfrak{E}_X$ the class of all surjective maps – the canonical right exact structure. The category $Sets^*$ does not have initial objects. One can see that a map $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{N}$ has a trivial kernel iff there is an element $y \in \mathcal{N}$ such that $\mathfrak{f}^{-1}(y)$ consists of one element. So that if the set $\mathcal{N}$ has more than one element, then most of deflations to $\mathcal{N}$ having a trivial kernel are not bijective maps.

5.5.7.2. Non-zero non-unital algebras. Let C_X be the category of non-unital non-zero k-algebras and non-unital k-algebra morphisms with the canonical right exact structure: deflations are surjective homomorphisms. Then any deflation with a trivial kernel is an isomorphisms.

6. Fully exact subcategories of a right exact category.

6.1. Definition. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We call a full subcategory $\mathcal{B}$ of C_X a *fully exact* subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, if $\mathcal{B}$ contains the initial object x and is *closed under extensions*; i.e. if objects K and N in a conflation $K \xrightarrow{\mathfrak{k}} M \xrightarrow{\mathfrak{e}} N$ belong to $\mathcal{B}$, then M is an object of $\mathcal{B}$.

In particular, fully exact subcategories of $(C_X, \mathfrak{E}_X)$ are strictly full subcategories.

6.2. Example. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object, x; and let $C_{X_\mathfrak{m}}$ be the full subcategory of C_X generated by all objects L such that the unique arrow $x \longrightarrow L$ is a monomorphism. Notice that if L is an object of $C_{X_\mathfrak{m}}$ and $M \xrightarrow{\mathfrak{e}} L$ a deflation, then M is an object of $C_{X_\mathfrak{m}}$ too. In fact, it follows from the diagram

$$\begin{array}{ccc} Ker(\mathfrak{e}) & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(\mathfrak{e})}\downarrow & cart & \downarrow \\ M & \xrightarrow{\mathfrak{e}} & L \end{array}$$

with cartesian square that $Ker(\mathfrak{e}) \xrightarrow{\mathfrak{k}(\mathfrak{e})} M$ is a monomorphism and $x \longrightarrow Ker(\mathfrak{e})$ is a split monomorphism, hence their composition, $x \longrightarrow M$, is a monomorphism.

In particular, $C_{X_\mathfrak{m}}$ is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$.

6.2.1. If $(C_X, \mathfrak{E}_X)$ is the category Alg_k of associative unital k-algebras with the finest right exact structure, then $C_{X_\mathfrak{m}}$ coincides with its full subcategory $\mathfrak{Alg}_k$ generated by k-algebras A for which the structure morphism $k \longrightarrow A$ is a monomorphism (it already appeared in 5.4.3.2).

6.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and $\mathcal{B}$ its fully exact subcategory. Then the class $\mathfrak{E}_{X,\mathcal{B}}$ of all deflations $M \xrightarrow{\mathfrak{e}} N$ such that M, N, and $Ker(\mathfrak{e})$ are objects of $\mathcal{B}$ is a structure of a right exact category on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \longrightarrow C_X$ is an 'exact' functor $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}}) \longrightarrow (C_X, \mathfrak{E}_X)$.*

Proof. (a) We start with the invariance of $\mathfrak{E}_{X,\mathcal{B}}$ under base change. Let

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{N} \\ {\scriptstyle \widetilde{g}}\downarrow & cart & \downarrow{\scriptstyle g} \\ M & \xrightarrow{\mathfrak{e}} & N \end{array}$$

be a cartesian square such that $\mathfrak{e}$ (hence $\widetilde{\mathfrak{e}}$) is a deflation and the objects M, N, $Ker(\mathfrak{e})$, and $\widetilde{N}$ belong to $\mathcal{B}$. The claim is that the remaining object, $\widetilde{M}$, belongs to $\mathcal{B}$.

In fact, consider the diagram

$$\begin{array}{ccccc} Ker(\widetilde{\mathfrak{e}}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{e}})} & \widetilde{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{N} \\ {\scriptstyle g'}\downarrow & & {\scriptstyle \widetilde{g}}\downarrow & cart & \downarrow{\scriptstyle g} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}(\mathfrak{e})} & M & \xrightarrow{\mathfrak{e}} & N \end{array} \qquad (1)$$

Since its right square is cartesian, it follows from 4.3.3 that the canonical morphism $Ker(\widetilde{\mathfrak{e}}) \xrightarrow{g'} Ker(\mathfrak{e})$ is an isomorphism; i.e. the upper row of the diagram (1) is a conflation whose ends, $Ker(\widetilde{\mathfrak{e}})$ and $\widetilde{N}$, are objects of $\mathcal{B}$. Since $\mathcal{B}$ is fully exact, the middle object, $\widetilde{M}$, belongs to $\mathcal{B}$, which means that the deflation $\widetilde{M} \xrightarrow{\widetilde{\mathfrak{e}}} \widetilde{N}$ belongs to $\mathfrak{E}_{X,\mathcal{B}}$.

(b) The invariance of $\mathfrak{E}_{X,\mathcal{B}}$ under base change implies that it is closed under composition. In fact, let $L \xrightarrow{\mathfrak{s}} M \xrightarrow{\mathfrak{t}} N$ be morphisms of $\mathfrak{E}_{X,\mathcal{B}}$. By 4.4.1, we have a commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{ts}) & \xrightarrow{\widetilde{\mathfrak{s}}} & Ker(\mathfrak{s}) & \xrightarrow{\mathfrak{t}'} & x \\ {\scriptstyle \mathfrak{k}(\mathfrak{ts})}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{s})} & cart & \downarrow{\scriptstyle \mathfrak{i}_N} \\ L & \xrightarrow{\mathfrak{s}} & M & \xrightarrow{\mathfrak{t}} & N \end{array} \qquad (2)$$

whose squares are cartesian. Since $\mathfrak{s}$ belongs to $\mathfrak{E}_{X,\mathcal{B}}$, its kernel $Ker(\mathfrak{s}) \xrightarrow{\mathfrak{k}(\mathfrak{s})} M$ is an arrow of $\mathcal{B}$. Applying (a) to the left cartesian square of (2), we obtain that $Ker(\mathfrak{ts}) \xrightarrow{\mathfrak{k}(\mathfrak{ts})} L$ is an arrow of $\mathcal{B}$, which means that $\mathfrak{ts} \in \mathfrak{E}_{X,\mathcal{B}}$.

(c) Each isomorphism of the category $\mathcal{B}$ belongs to the class $\mathfrak{E}_{X,\mathcal{B}}$, because each isomorphism is a deflation and its kernel is an initial object, and, by hypothesis, initial objects belong to $\mathcal{B}$. ∎

6.4. Remark. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and $\mathcal{B}$ its strictly full subcategory containing x. Let $\mathfrak{E}$ be a right exact structure on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \xrightarrow{\mathfrak{J}} C_X$ maps deflations to deflations and preserves kernels of deflations. Then $\mathfrak{E}$ is contained in $\mathfrak{E}_{X,\mathcal{B}}$. In particular, $\mathfrak{E}$ is contained in $\mathfrak{E}_{X,\mathcal{B}}$ if the inclusion functor is an 'exact' functor from $(\mathcal{B}, \mathfrak{E})$ to $(C_X, \mathfrak{E}_X)$. This shows that if $\mathcal{B}$ is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$, then $\mathfrak{E}_{X,\mathcal{B}}$ is the finest right exact structure on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \longrightarrow C_X$ is an 'exact' functor from $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}})$ to $(C_X, \mathfrak{E}_X)$.

7. Exact k-linear categories and their fully exact subcategories.

7.1. Definition. An *exact* k-linear category is a right exact k-linear category whose class of inflations is stable under arbitrary push-forwards.

We denote by $ExCat_k$ the category whose objects are exact k-linear categories and morphisms 'exact' k-linear functors, that is k-linear functors which map conflations to conflations.

7.1.1. Observations. For any right exact category $(C_X, \mathfrak{E}_X)$, all arrows from an initial object are (kernels of identical morphisms, hence) inflations. So that the category C_X has finite coproducts iff push-forwards of inflations of initial objects exist. Moreover, if these push-forwards are inflations, then coprojections of summands in a finite coproduct are inflations. In particular, any *exact* k-linear category is additive and all its split monomorphisms are inflations, or, equivalently, all its split epimorphisms are deflation.

7.1.2. Note. It is easy to see that our definition of an exact k-linear category is equivalent to the one given by Keller and Vossieck [KeV] (see Appendix K).

7.2. Examples of exact categories.

7.2.1. The smallest exact structure. For any additive k-linear category C_X, let $\mathfrak{E}_X^{spl}$ denote the class of all split epimorphisms. The pair $(C_X, \mathfrak{E}_X^{spl})$ is an exact category. It follows from 7.1.1 that $\mathfrak{E}_X^{spl}$ is the smallest exact structure on C_X.

Let C_X and C_Y be additive k-linear categories. Every k-linear functor $C_X \xrightarrow{F} C_Y$ is an 'exact' functor $(C_X, \mathfrak{E}_X^{spl}) \xrightarrow{F} (C_Y, \mathfrak{E}_Y^{spl})$. The map which assigns to an additive k-linear category C_X the exact category $(C_X, \mathfrak{E}_X^{spl})$ and to a k-linear functor the corresponding 'exact' functor is a full embedding of the category Add_k of additive k-linear categories and k-linear functors to the category $ExCat_k$ of exact k-linear categories and 'exact' k-linear functors. This embedding is a left adjoint to the forgetful functor $ExCat_k \longrightarrow Add_k$.

7.2.2. The category of complexes. Let $\mathbf{C}(\mathcal{A})$ be the category of complexes of an additive k-linear category $\mathcal{A}$. Deflations are morphisms $M^\bullet \xrightarrow{\mathfrak{e}^\bullet} N^\bullet$ such that the morphism $M^n \xrightarrow{\mathfrak{e}^n} N^n$ is split for every $n \in \mathbb{Z}$.

7.2.3. Quasi-abelian categories. A *quasi-abelian* k-linear category is an additive k-linear category C_X with kernels and cokernels and such that every pullback of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism. It follows from definitions that the pair $(C_X, \mathfrak{E}_s)$, where $\mathfrak{E}_s$ is the class of all strict epimorphisms in C_X, is an exact category.

7.2.3.1. 'Exact' functors from a quasi-abelian category. Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be exact k-linear categories. If $(C_X, \mathfrak{E}_X)$ is quasi-abelian, then a k-linear functor $C_X \xrightarrow{F} C_Y$ is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ iff it preserves finite limits and colimits. This follows from the fact that all k-linear functors preserve finite coproducts, which coincide with the products in additive case, and the functors mapping strict epimorphisms of C_X to strict epimorphisms of C_Y are precisely those functors which preserve the cokernels of pairs of morphisms. Dually, the functors which map strict monomorphisms to strict monomorphisms are the functors preserving kernels of pairs of arrows.

In other words, 'exact' functors in this case are precisely exact functors.

7.2.3.2. Abelian categories. Abelian k-linear categories are quasi-abelian k-linear categories in which every epimorphism is strict.

7.2.4. Filtered objects. Let $(C_X, \mathfrak{E}_X)$ be an exact k-linear category. Objects of the *filtered category* $\mathfrak{F}(C_X, \mathfrak{E}_X)$ are sequences of inflations

$$\mathcal{M} = (\ldots \longrightarrow M_n \xrightarrow{\mathfrak{j}_n} M_{n+1} \longrightarrow \ldots)$$

such that $M_n = 0$ for $n \ll 0$ and $\mathfrak{j}_m$ are identical isomorphisms for $m \gg 1$. We denote by $\mathcal{M}_\infty$ the object M_m for $m \gg 1$ and regard $\mathcal{M}$ as a filtration of the object M_∞.

Morphisms of filtered objects are defined in a natural way: a morphism from

$$\mathcal{M} = (\ldots \longrightarrow M_n \xrightarrow{\mathfrak{j}_n} M_{n+1} \longrightarrow \ldots) \quad \text{to} \quad \widetilde{\mathcal{M}} = (\ldots \longrightarrow \widetilde{M}_n \xrightarrow{\widetilde{\mathfrak{j}}_n} \widetilde{M}_{n+1} \longrightarrow \ldots)$$

is a sequence of morphisms $M_n \xrightarrow{g_n} L_n$ for each n making the diagrams

$$\begin{array}{ccc} M_n & \xrightarrow{\mathfrak{j}_n} & M_{n+1} \\ {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle g_{n+1}} \\ \widetilde{M}_n & \xrightarrow{\widetilde{\mathfrak{j}}_n} & \widetilde{M}_{n+1} \end{array}$$

commute for all $n \in \mathbb{Z}$. Deflations are morphisms whose components belong to $\mathfrak{E}_X$.

Since inflations are monomorphisms, every morphism $\mathcal{M} \xrightarrow{(g_n)} \widetilde{\mathcal{M}}$ of filtered objects is determined by a morphism $\mathcal{M}_\infty \xrightarrow{g_\infty} \widetilde{\mathcal{M}}_\infty$, which is the notation for the morphism $M_m \xrightarrow{g_m} \widetilde{M}_m$ when $m \gg 1$. So that a morphism $\mathcal{M} \longrightarrow \widetilde{\mathcal{M}}$ of filtered objects can be viewed as a morphism $\mathcal{M}_\infty \longrightarrow \widetilde{\mathcal{M}}_\infty$ of the category C_X which is compatible with the filtration of these objects.

7.2.4.1. Remarks. (i) Notice that, for any nonzero k-linear exact category $(C_X, \mathfrak{E}_X)$, the category $\mathfrak{F}(C_X, \mathfrak{E}_X)$ is *non-abelian.* This follows from the fact that, for any nonzero object M of the category C_X, the morphism $(0, id_M)$ from the filtered object $(0 \to M)$ to $(M \overset{id_M}{\to} M)$ is a bimorphism (– both mono- and epimorphism), which is not invertible.

(ii) The class of quasi-abelian categories turns out to be much more robust: if $(C_X, \mathfrak{E}_X)$ is a quasi-abelian category, then the filtered category $\mathfrak{F}(C_X, \mathfrak{E}_X)$ is quasi-abelian too.

In particular, the category of filtered objects of an abelian category is quasi-abelian.

7.2.5. The category of Banach spaces. Let C_X be the category of complex Banach spaces. Then C_X is a quasi-abelian category. The canonical (that is the finest) exact structure on this category is formed by morphisms $M \xrightarrow{\mathfrak{e}} N$ of Banach spaces such that the map $\mathfrak{e}$ is surjective.

7.2.6. Categories of functors. Let $(C_X, \mathfrak{E}_X)$ be a right exact k-linear category and C_Z a category. The standard right exact structure on the category $\mathcal{H}om(C_Z, C_X)$ of functors from C_Z to C_X is defined as follows: a functor morphism $F \xrightarrow{\mathfrak{e}} F''$ is a deflation if $F(M) \xrightarrow{\mathfrak{e}(M)} F''(M)$ is a deflation for every object M of the category C_Z. If the right exact category $(C_X, \mathfrak{E}_X)$ is exact, then the right exact category of functors defined this way is an exact k-linear category too.

7.2.7. Examples created via pairs of adjoint functors. Let $(C_X, \mathfrak{E}_X)$ be right exact category and $C_X \xrightarrow{f^*} C_Y$ a functor having a right adjoint functor, $C_Y \xrightarrow{f_*} C_X$. Let $\mathfrak{E}_Y$ be the intersection of $f_*^{-1}(\mathfrak{E}_X)$ with the finest right exact structure, $\mathfrak{E}_Y^{\mathfrak{st}}$, on C_Y. Since the functor f_* preserves limits, in particular, pull-backs, the class $\mathfrak{E}_Y$ is a right exact structure on the category C_Y. It is the finest right exact structure on C_Y such that f_* is an 'exact' functor from $(C_Y, \mathfrak{E}_Y)$ to $(C_X, \mathfrak{E}_X)$.

7.2.7.1. Proposition. *((a) Suppose that the category C_Y has cokernels of pairs of arrows and the functor $C_Y \xrightarrow{f_*} C_X$ preserves cokernels of pairs of arrows and is conservative.*

(a1) $\mathfrak{E}_Y = f_*^{-1}(\mathfrak{E}_X)$.

(a2) Suppose that, in addition, the category C_Y has initial objects and the functor f_* preserves push-forwards and maps initial objects to initial objects. Then the class of inflations of $(C_Y, \mathfrak{E}_Y)$ is stable under push-forwards, if the class of inflations of $(C_X, \mathfrak{E}_X)$ has this property.

(b) Suppose that the categories C_X and C_Y are additive and k-linear and the functors f^*, f_* are k-linear. Let the functor $C_Y \xrightarrow{f_*} C_X$ be exact and conservative. Then $\mathfrak{E}_Y = f_*^{-1}(\mathfrak{E}_X)$ and the right exact structure $\mathfrak{E}_Y$ is exact iff $\mathfrak{E}_X$ is exact.

Proof. (a1) By Beck's theorem, the functor f_* is isomorphic to the forgetful functor from the category $\mathcal{F}_f - mod$ of modules over the monad $\mathcal{F} = (f_* f^*, \mu_f)$ associated with the pair of adjoint functors f_*, f^* and the adjunction morphism $f^* f_* \xrightarrow{\epsilon_f} Id_{C_Y}$ $(\mu_f = f^* \epsilon_f f_*)$. This forgetful functor preserves and reflects universally strict epimorphisms.

(a2) Let $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$ be a conflation in $(C_Y, \mathfrak{E}_Y)$ and $N \xrightarrow{\xi} \mathfrak{N}$ an arbitrary morphism. Let

$$\begin{array}{ccc} f_*(N) & \xrightarrow{f_*(\mathfrak{j})} & f_*(M) \\ {\scriptstyle f_*(\xi)}\downarrow & cocart & \downarrow{\scriptstyle \xi'} \\ f_*(\mathfrak{N}) & \xrightarrow{\widetilde{\mathfrak{j}}} & \widetilde{\mathfrak{M}} \end{array} \qquad (1)$$

be a cocartesian square. It follows from the fact that f_* preserves cocartesian squares and from Beck's theorem (which allows to replace the category C_Y by the category of $\mathcal{F}$-modules and the functor f_* by the forgetful functor $\mathcal{F}-mod \longrightarrow C_X$) that the diagram (1) is the image of a cocartesian square

$$\begin{array}{ccc} N & \xrightarrow{\ \mathfrak{j}\ } & M \\ \xi \downarrow & cocart & \downarrow \zeta \\ \mathfrak{N} & \xrightarrow{\ \mathfrak{i}\ } & \mathfrak{M} \end{array}$$

of the category C_Y.

(b) The assertion follows from (a) and (b). ■

7.2.7.2. Application. Let C_X and C_Y be additive k-linear categories and

$$C_X \xrightarrow{f^*} C_Y \xrightarrow{f_*} C_X$$

a pair of k-linear adjoint functors such that the the functor f_* is conservative and exact. Then the preimage $\mathfrak{E}_Y = f_*^{-1}(\mathfrak{E}_X^{spl})$ of the coarsest exact structure on the category C_X is an exact structure on the category C_Y.

The example 7.2.2 is a special case of this.

7.2.8. Intersection of exact structures. For any family $\{\mathfrak{E}_i \mid i \in J\}$ of right exact structures on a category C_X, their intersection $\mathfrak{E}_J = \bigcap_{i \in J} \mathfrak{E}_i$ is a right exact structure on C_X. If C_X is an additive k-linear category and all the right exact structures $\mathfrak{E}_i$, $i \in J$, are exact, then their intersection $\mathfrak{E}_J$ is exact.

7.3. Proposition. *Any fully exact subcategory $\mathcal{B}$ of an exact k-linear category $(C_X, \mathfrak{E}_X)$ is an exact category with respect to the canonical right exact structure $\mathfrak{E}_{X,\mathcal{B}}$.*

Proof. By 6.3, $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}})$ is a right exact subcategory of $(C_X, \mathfrak{E}_X)$. But, if $(C_X, \mathfrak{E}_X)$ is an exact category, then the notion of a fully exact subcategory of an exact category is self-dual: $\mathcal{B}$ is a fully exact subcategory of the exact category $(C_X, \mathfrak{E}_X)$ iff $\mathcal{B}^{op}$ is a fully exact subcategory of $(C_X^{op}, \mathfrak{M}_X^{op})$. Therefore, the class $\mathfrak{M}_{X,\mathcal{B}}$ of inflations of the right exact category $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}})$ is a structure of a left exact category on $\mathcal{B}$. ■

7.4. Proposition. *A svelte right exact k-linear category $(C_X, \mathfrak{E}_X)$ is exact iff the canonical embedding of $(C_X, \mathfrak{E}_X)$ into the category $Sh_k(X, \mathfrak{E}_X)$ of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$ induces an equivalence of $(C_X, \mathfrak{E}_X)$ and a fully exact subcategory of $Sh_k(X, \mathfrak{E}_X)$.*

Proof. (i) Let $N \in ObC_X$ and $\mathfrak{F} \xrightarrow{\gamma} \widehat{N}$ an epimorphism of sheaves on $(C_X, \mathfrak{E}_X)$. By (the k-linear version of) the argument of 2.2, there is a commutative diagram

$$\begin{array}{ccc} \widehat{M'} & \xrightarrow{\ \widehat{\mathfrak{e}'}\ } & \widehat{N} \\ v \downarrow & & \downarrow id \\ \mathfrak{F} & \xrightarrow{\ \gamma\ } & \widehat{N} \end{array} \qquad (1)$$

such that $M' \xrightarrow{\mathfrak{e}'} N$ is a deflation.

(ii) Suppose that the kernel of γ is representable by an object L, and let $L' \xrightarrow{\mathfrak{j}'} M'$ be the kernel of $M' \xrightarrow{\mathfrak{e}'} N$. Then the diagram (1) extends to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{L'} & \xrightarrow{\widehat{\mathfrak{j}'}} & \widehat{M'} & \xrightarrow{\widehat{\mathfrak{e}'}} & \widehat{N} & \longrightarrow & 0 \\ & & {\scriptstyle \widehat{u}}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \widehat{L} & \xrightarrow{\lambda} & \mathfrak{F} & \xrightarrow{\gamma} & \widehat{N} & \longrightarrow & 0 \end{array} \qquad (2)$$

with exact rows. Since the right exact category $(C_X, \mathfrak{E}_X)$ is exact and $L' \xrightarrow{\mathfrak{j}'} M'$ is an inflation, we have a commutative diagram

$$\begin{array}{ccccc} L' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & N \\ {\scriptstyle u}\downarrow & cocart & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle id} \\ L & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & N \end{array} \qquad (3)$$

whose left square is cocartesian and both rows are conflations.

(iii) The Yoneda functor assigns to the diagram (3) the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{L'} & \xrightarrow{\widehat{\mathfrak{j}'}} & \widehat{M'} & \xrightarrow{\widehat{\mathfrak{e}'}} & \widehat{N} & \longrightarrow & 0 \\ & & {\scriptstyle \widehat{u}}\downarrow & & \downarrow{\scriptstyle \widehat{u}'} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \widehat{L} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0 \end{array} \qquad (\widehat{3})$$

whose rows are exact sequences in the category of sheaves. We claim that the latter implies that the left square of the diagram $(\widehat{3})$ is cocartesian.

In fact, let

$$\begin{array}{ccc} \widehat{L'} & \xrightarrow{\widehat{\mathfrak{j}'}} & \widehat{M'} \\ {\scriptstyle \widehat{u}}\downarrow & cocart & \downarrow{\scriptstyle \nu} \\ \widehat{L} & \xrightarrow{\widetilde{\mathfrak{j}}} & \mathcal{G} \end{array}$$

be a cocartesian square. Applying the argument of (iii) above, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{L'} & \xrightarrow{\widehat{\mathfrak{j}'}} & \widehat{M'} & \xrightarrow{\widehat{\mathfrak{e}'}} & \widehat{N} & \longrightarrow & 0 \\ & & {\scriptstyle \widehat{u}}\downarrow & cocart & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \widehat{L} & \xrightarrow{\widetilde{\mathfrak{j}}} & \mathcal{G} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0 \end{array}$$

with exact rows. Therefore, the canonical morphism $\mathcal{G} \xrightarrow{g} \widehat{M}$ gives rise to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{L} & \xrightarrow{\widetilde{\mathfrak{j}}} & \mathcal{G} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0 \\
 & & {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \widehat{L} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0
\end{array} \qquad (4)$$

of sheaves on $(C_X, \mathfrak{E}_X)$ with exact rows, which shows that $\mathcal{G} \xrightarrow{g} \widehat{M}$ is an isomorphism.

(iv) The commutative diagrams (2) and $\widehat{(3)}$ give rise to the commutative diagram of sheaves

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{L} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0 \\
 & & {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \widehat{L} & \xrightarrow{\lambda} & \mathfrak{F} & \xrightarrow{\gamma} & \widehat{N} & \longrightarrow & 0
\end{array} \qquad (5)$$

with exact rows, which implies that $\widehat{M} \xrightarrow{\mathfrak{t}} \mathfrak{F}$ is an isomorphism.

(v) By (iii) above, $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ is a conflation. Therefore, the isomorphism (5) shows also that the functor $\mathfrak{j}_X^*$ reflects conflations: if $0 \longrightarrow \widehat{\mathcal{L}} \xrightarrow{\widetilde{i}} \widehat{\mathcal{M}} \xrightarrow{\widehat{e}} \widehat{\mathcal{N}} \longrightarrow 0$ is an exact sequence of sheaves on $(C_X, \mathfrak{E}_X)$, then $\mathcal{L} \xrightarrow{i} \mathcal{M} \xrightarrow{e} \mathcal{N}$ is a conflation. ∎

7.4.1. Corollary. *A right exact k-linear category is exact iff it is equivalent to a fully exact subcategory of a k-linear Grothendieck category.*

Proof. By 7.3, any fully exact subcategory of an exact category is exact. By 7.4, any exact k-linear category is equivalent to a fully exact subcategory of the Grothendieck category $Sh_k(X, \mathfrak{E}_X)$ of sheaves of k-modules on the presite $(C_X, \mathfrak{E}_X)$. ∎

7.4.2. Remark. Definition 7.1 of an exact category is (up to a rewording) the one introduced by Keller and Vossieck [KeV]. Quillen's original definition contains some additional axioms. On the other hand, it is easy to show that fully exact subcategories of abelian categories are exact in the sense of Quillen. Therefore, it follows from Proposition 7.4 that the two notions are equivalent.

In particular, 7.4 implies the self-duality of Keller's axioms:

A k-linear right exact category $(C_X, \mathfrak{E}_X)$ is exact iff the class of its inflations, $\mathfrak{M}_X$, forms a left exact structure on the category C_X; that is, besides stability under push-forwards, $\mathfrak{M}_X$ is closed under compositions and contains all isomorphisms of C_X.

7.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte k-linear right exact category. Then there exists an exact category $(C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}})$ and a fully faithful k-linear 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{\gamma_X^*} (C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}})$ which is universal; that is any 'exact' k-linear functor from $(C_X, \mathfrak{E}_X)$ to an exact k-linear category factors uniquely through γ_X^*.*

Proof. We take as $C_{X_{\mathfrak{e}}}$ the smallest fully exact subcategory of the category $C_{X_{\mathfrak{E}}}$ of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$ containing all representable sheaves. Objects of the category $C_{X_{\mathfrak{e}}}$ are sheaves $\mathcal{F}$ such that there exists a finite filtration

$$0 = \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \ \dots \ \longrightarrow \mathcal{F}_n = \mathcal{F}$$

such that $\mathcal{F}_m/\mathcal{F}_{m-1}$ is representable for $1 \leq m \leq n$. By 7.3, the subcategory $C_{X_{\mathfrak{e}}}$, being a fully exact subcategory of an abelian category, is exact.

Let $(C_Y, \mathfrak{E}_Y)$ be an exact k-linear category and $(C_X, \mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathfrak{E}_Y)$ an 'exact' k-linear functor. The functor φ^* extends to a continuous (i.e. having a right adjoint) functor $C_{X_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{E}}}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ {\scriptstyle \mathfrak{j}_X^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{j}_Y^*} \\ C_{X_{\mathfrak{E}}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_{\mathfrak{E}}} \end{array}$$

is quasi-commutative (see 2.1). Since the functor φ^* is 'exact', it preserves pullbacks of deflations. In particular, it preserves kernels of deflations. Therefore, the restriction of φ^* to the Gabriel square, $C_{X^{(2)}}$, of C_X regarded as a subcategory of the exact category $(C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$, preserves conflations, hence it is 'exact'. This implies that the restriction of φ^* to the n-th Gabriel power $C_{X^{(n)}}$, of C_X (in $(C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$) is 'exact' for all n, whence the assertion. ■

7.5.1. The bicategories of exact and right exact k-linear categories. Right exact svelte k-linear categories are objects of a bicategory $\mathfrak{Rer}_k$. Its 1-morphisms are right weakly 'exact' k-linear functors and 2-morphisms are morphisms between those functors.

We denote by $\mathfrak{Exr}_k$ the full subbicategory of $\mathfrak{Rer}_k$ whose objects are exact k-linear categories. It follows from 7.5 that the inclusion functor $\mathfrak{Exr}_k \longrightarrow \mathfrak{Rer}_k$ has a left adjoint (in the bicategorical sense).

7.6. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be an exact category. The Karoubian envelope C_{X_K} has a structure of an exact category, $\mathfrak{E}_K$, whose conflations are direct summands of conflations of $\mathfrak{E}$.*

Proof. Consider the Gabriel-Quillen embedding $C_X \xrightarrow{\mathfrak{j}_X^*} Sh_k(X, \mathfrak{E}_X)$. The category $Sh_k(X, \mathfrak{E}_X)$ is abelian, hence Karoubian. It follows from 3.4.3 that the functor $\mathfrak{j}_X^*$ factors through $C_X \xrightarrow{\mathfrak{K}_X^*} C_{X_K}$, i.e. there exists a canonical morphism $C_{X_K} \longrightarrow Sh_k(X, \mathfrak{E}_X)$ which induces an equivalence between the category C_{X_K} and the full subcategory of $Sh_k(X, \mathfrak{E}_X)$ whose objects are all direct summands of objects of $\mathfrak{j}_X^*(C_X)$ (see the argument of 3.4.3).

Since the subcategory $\mathfrak{j}_X^*(C_X)$ is closed under extensions in $Sh_k(X,\mathfrak{E}_X)$, by 7.4, the image of C_{X_K} in $Sh_k(X,\mathfrak{E}_X)$ has the same property.

In fact, let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $Sh_k(X,\mathfrak{E}_X)$ such that $L \oplus L'$ and $N \oplus N'$ are isomorphic to objects of $\mathfrak{j}_X^*(C_X)$ for some objects L' and N' of $Sh_k(X,\mathfrak{E}_X)$. Since the subcategory $\mathfrak{j}_X^*(C_X)$ is closed under extensions in $Sh_k(X,\mathfrak{E}_X)$ and the sequence

$$0 \longrightarrow L \oplus L' \longrightarrow M \oplus L' \oplus N' \longrightarrow N \oplus N' \longrightarrow 0$$

is exact, the object $M \oplus L' \oplus N'$ is isomorphic to an object of $\mathfrak{j}_X^*(C_X)$. This shows that M is a direct summand of an object of $\mathfrak{j}_X^*(C_X)$ and that any exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $Sh_k(X,\mathfrak{E}_X)$ whose objects belong to the image of C_{X_K} is a direct summand of an image of a conflation. The assertion follows now from 7.3. ∎

7.7. Digression: non-additive exact categories.

7.7.1. Definition. We call a right exact category $(C_X,\mathfrak{E}_X)$ (and the corresponding right exact 'space' $(X,\mathfrak{E}_X)$) an *exact* category (resp. an *exact* 'space'), if the Yoneda embedding induces an equivalence of $(C_X,\mathfrak{E}_X)$ with a fully exact subcategory of the right exact category $(C_{X_{\mathfrak{E}}},\mathfrak{E}^{\mathfrak{st}})$ of sheaves on $(C_X,\mathfrak{E}_X)$.

Let $\mathfrak{Esp}_{\mathfrak{e}}$ denote the full subcategory of the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' generated by exact 'spaces'.

7.7.2. Proposition. *The inclusion functor* $\mathfrak{Esp}_{\mathfrak{e}} \xrightarrow{\mathfrak{J}^*} \mathfrak{Esp}_{\mathfrak{r}}$ *has a right adjoint.*

Proof. This right adjoint, $\mathfrak{J}_*$, assigns to each right exact 'space' $(X,\mathfrak{E}_X)$ the 'space' $(X_{\mathfrak{e}},\mathfrak{E}_{X_{\mathfrak{e}}})$, where $C_{X_{\mathfrak{e}}}$ is the smallest fully exact subcategory of the right exact category of sheaves on $(C_X,\mathfrak{E}_X)$ containing all representable sheaves and endowed with the induced right exact structure. ∎

8. Complements.

8.1. Admissible morphisms. Let $(C_X,\mathcal{E}_X)$ be an exact k-linear category with the class of inflations $\mathfrak{M}_X$ and the class of deflations $\mathfrak{E}_X$. We call arrows of $\mathfrak{M}_X \circ \mathfrak{E}_X$ *admissible.* In general, the class of admissible morphisms is not closed under composition.

8.1.1. Lemma. *Suppose that for any pair of arrows* $L \xrightarrow{\mathfrak{j}} M \xleftarrow{\mathfrak{j}'} \widetilde{L}$ *of* $\mathfrak{M}_X$, *there exists a cartesian square*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\mathfrak{j}''} & \widetilde{L} \\ {\scriptstyle \widetilde{\mathfrak{j}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{j}} \\ L & \xrightarrow{\mathfrak{j}'} & M \end{array} \qquad (1)$$

Then the class of admissible arrows is closed under composition.

Proof. (i) Notice that if (1) is a cartesian square with $\mathfrak{j} \in \mathfrak{M}_X \ni \mathfrak{j}'$, then the remaining two arrows, $\mathfrak{j}''$ and $\widetilde{\mathfrak{j}}$, belong to $\mathfrak{M}_X$ too. In fact, the arrows $\mathfrak{j}''$ and $\widetilde{\mathfrak{j}}$ are (strict) monomorphisms in any category. The Gabriel-Quillen embedding, preserves cartesian squares, maps arrows of $\mathfrak{M}_X$ to monomorphisms, and reflects monomorphisms to arrows of $\mathfrak{M}_X$.

(ii) It suffices to show that $\mathfrak{E}_X \circ \mathfrak{M}_X \subseteq \mathfrak{M}_X \circ \mathfrak{E}_X$. Let $L \xrightarrow{\mathfrak{j}} M$ be a morphism of $\mathfrak{M}_X$ and $M \xrightarrow{\mathfrak{e}} N$ a morphism of $\mathfrak{E}_X$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ker(\widetilde{\mathfrak{e}}) & \xrightarrow{\widetilde{\mathfrak{j}}} & L & \xrightarrow{\widetilde{\mathfrak{e}}} & M' & \longrightarrow & 0 \\ & & {\scriptstyle \mathfrak{j}''}\downarrow & & {\scriptstyle \mathfrak{j}}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}'} & & \\ 0 & \longrightarrow & Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{j}_\mathfrak{e}} & M & \xrightarrow{\mathfrak{e}} & N & \longrightarrow & 0 \end{array} \qquad (2)$$

with exact rows. Its left square is cartesian and formed by arrows of $\mathfrak{M}_X$. The morphism $L \xrightarrow{\widetilde{\mathfrak{e}}} M'$ is a cokernel of $\widetilde{\mathfrak{j}}$; in particular, belongs to $\mathfrak{E}_X$. The existence of the right vertical arrow in (2), $M' \xrightarrow{\mathfrak{j}'} N$, follows from the exactness of the rows. Applying the Gabriel-Quillen embedding, $\mathfrak{j}^*_X$. to the diagram (2), we reduce to the case of an abelian category with the canonical exact structure. One can see that $\mathfrak{j}^*_X(\mathfrak{j}')$ is a monomorphism. Therefore, $\mathfrak{j}'$ is an arrow of $\mathfrak{M}_X$. Thus, we obtain the equality $\mathfrak{e} \circ \mathfrak{j} = \mathfrak{j}' \circ \widetilde{\mathfrak{e}}$, where $\mathfrak{j}' \in \mathfrak{M}_X$ and $\widetilde{\mathfrak{e}} \in \mathfrak{E}_X$. ■

8.1.2. Remarks. (a) If the condition of 8.1.1 holds, then the dual condition holds for deflations. In fact, let $N' \xleftarrow{\mathfrak{e}'} M \xrightarrow{\mathfrak{e}} N$ be a pair of arrows of $\mathfrak{E}_X$. So that we have exact sequences $0 \longrightarrow L' \xrightarrow{\mathfrak{j}'} M \xrightarrow{\mathfrak{e}'} N' \longrightarrow 0$ and $0 \longrightarrow L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N \longrightarrow 0$. By hypothesis (and the part (i) of the argument above), there is a cartesian square

$$\begin{array}{ccc} \widetilde{L} & \xrightarrow{\widetilde{\mathfrak{j}}} & L' \\ {\scriptstyle \mathfrak{j}''}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}'} \\ L & \xrightarrow{\mathfrak{j}} & M \end{array}$$

with all arrows from $\mathfrak{M}_X$. Since $\mathfrak{j} \circ \mathfrak{j}'' \in \mathfrak{M}_X$, there is an exact sequence

$$0 \longrightarrow \widetilde{L} \xrightarrow{\mathfrak{j}\circ\mathfrak{j}''} M \xrightarrow{\mathfrak{e}_1} \widetilde{N} \longrightarrow 0.$$

By the universal properties of cokernels, there exists a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{e}} & N \\ {\scriptstyle \mathfrak{e}'}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\ N' & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{N} \end{array} \qquad (3)$$

with arrows $\mathfrak{e}''$ and $\widetilde{\mathfrak{e}}$ uniquely determined by the equalities $\mathfrak{e}'' \circ \mathfrak{e} = \mathfrak{e}_1 = \widetilde{\mathfrak{e}} \circ \mathfrak{e}'$. Since $\mathfrak{e}_1 \in \mathfrak{E}_X$, it follows, by a property of exact categories, that $\mathfrak{e}''$ and $\widetilde{\mathfrak{e}}$ are arrows of $\mathfrak{E}_X$. It is easy to see that the square (3) is cocartesian.

(b) The assumption of 8.1.1 holds for exact categories associated with quasi-abelian categories (discussed shortly in 8.2 below), because in quasi-abelian categories all fibred products and coproducts exist, $\mathfrak{M}_X$ is the class of all strict monomorphisms, and a pull-back of a strict monomorphism is a strict monomorphism.

8.1.3. Proposition. *Suppose the condition of 8.1.1 holds. Then the class of all admissible morphisms of the exact category $(C_X, \mathcal{E}_X)$ forms the largest abelian exact subcategory, $C_{X_{\mathfrak{a}}(\mathcal{E})}$, of $(C_X, \mathcal{E}_X)$.*

Proof. Let $M \overset{g}{\underset{h}{\rightrightarrows}} N$ be a pair of morphisms of C_X. Their sum is the composition of the arrows

$$M \xrightarrow{\Delta_M} M \oplus M \xrightarrow{g \oplus h} N \oplus N \xrightarrow{+_N} N, \tag{4}$$

where Δ_M is the diagonal morphism and $+_N$ is the *codiagonal* morphism. Since the composition of Δ_M and any of projections $M \oplus M \longrightarrow M$ is the identical morphism, $\Delta_M \in \mathfrak{M}$. Dually, $+_N$ belongs to $\mathfrak{E}_X$. If both g and h are admissible arrows, then $g \oplus h$ is admissible. Therefore, in this case, $g+h$ is the composition of admissible morphisms. Under the condition of 8.1.1, the composition of admissible morphisms is an admissible morphism. The subcategory $C_{X_{\mathfrak{a}}(\mathcal{E})}$ has same objects as C_X. Therefore, since the category C_X is additive, $C_{X_{\mathfrak{a}}(\mathcal{E})}$ is additive too. It is quasi-abelian, because every admissible morphism has a kernel and a cokernel. An admissible arrow is a monomorphism iff it belongs to $\mathfrak{M}_X$. Since all arrows of $\mathfrak{M}_X$ are strict monomorphisms, an inflation is an epimorphism iff it is an isomorphism. Altogether means that $C_{X_{\mathfrak{a}}(\mathcal{E})}$ is an abelian subcategory. The exact structure $\mathcal{E}_X$ induces the canonical exact structure on the subcategory $C_{X_{\mathfrak{a}}(\mathcal{E})}$. It follows that any other abelian exact subcategory of $(C_X, \mathcal{E}_X)$ is formed by admissible arrows, i.e. it is contained in $C_{X_{\mathfrak{a}}(\mathcal{E})}$. ■

8.1.4. Example: the category of torsion-free objects. Let $(C_X, \mathcal{E}_X)$ be an exact k-linear category. Let $\mathcal{T}$ be a full subcategory of C_X such that if $M' \longrightarrow M$ is an inflation and $M \in Ob\mathcal{T}$, then M' is an object of $\mathcal{T}$ too. In particular, the subcategory $\mathcal{T}$ is strictly full. Let $C_{X_{\mathcal{T}}}$ denote the full subcategory of C_X generated by all $\mathcal{T}$*-torsion free* objects; i.e. objects N such that the only inflation $L \longrightarrow N$ with $L \in Ob\mathcal{T}$ is zero.

8.1.4.1. Lemma. *Suppose that for any pair $L' \longrightarrow L \longleftarrow L''$ of inflations of $(C_X, \mathcal{E}_X)$, there exists a pull-back $L' \times_L L''$. Then the subcategory $C_{X_{\mathcal{T}}}$ of $\mathcal{T}$-torsion free objects is closed under extensions. In particular, $C_{X_{\mathcal{T}}}$ is an exact subcategory of $(C_X, \mathcal{E}_X)$.*

Proof. Let $M' \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} M''$ be a conflation with $M' \in ObC_{X_{\mathcal{T}}}$. Let $L \longrightarrow M$ be an inflation with $L \in Ob\mathcal{T}$. Then we have a commutative diagram

$$\begin{array}{ccccc} L' & \xrightarrow{\mathfrak{j}'} & L & \xrightarrow{\mathfrak{e}'} & L'' \\ \downarrow & & \downarrow & & \downarrow \\ M' & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & M'' \end{array} \qquad (1)$$

whose left square is cartesian and the both rows are conflations.

In fact, by 8.1.2(a), all arrows of the left square are inflations. The arrow $\mathfrak{e}'$ is the cokernel of $\mathfrak{j}'$. It follows from the argument of 8.1.3 (or direct application of the Gabriel-Quillen embedding and the corresponding fact for abelian categories) that the remaining (right) vertical arrow is an inflation too. Since $L' \in Ob\mathcal{T}$ and M' is $\mathcal{T}$-torsion free, it follows that $L' = 0$ therefore $\mathfrak{e}'$ is an isomorphism. Therefore, if M'' is also $\mathcal{T}$-torsion free, then $L'' = 0$ which implies that $L = 0$. This shows that if the ends of a conflation are $\mathcal{T}$-torsion free, same holds for the middle. ∎

8.2. Quasi-abelian categories. Recall that a *quasi-abelian category* is an additive category C_X with kernels and cokernels and such that every pullback of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism.

It follows from definitions that the pair $(C_X, \mathcal{E}_s)$, where $\mathcal{E}_s$ is the class of all short exact sequences in C_X, is an exact category.

Every abelian category is quasi-abelian.

8.2.1. Proposition. *Let C_X be a quasi-abelian category. There exist two canonical fully faithful functors $C_{\mathfrak{L}X} \hookleftarrow C_X \hookrightarrow C_{\mathfrak{R}X}$ of C_X into abelian categories which preserve and reflect exactness. Moreover, the category C_X is stable under extensions in these embeddings. The category C_X is closed under taking subobjects in $C_{\mathfrak{L}X}$ and every object of $C_{\mathfrak{L}X}$ is a quotient of an object of C_X. Dually, C_X is closed under taking quotients in $C_{\mathfrak{R}X}$ and every object of $C_{\mathfrak{R}X}$ is a subobject of an object of C_X.*

Proof. See [Sch, 1.2.35, 1.2.31]. ∎

8.2.2. Quasi-abelian categories and torsion pairs. Let C_X be a quasi-abelian category, and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in C_X. That is $\mathcal{T}$ and $\mathcal{F}$ are full subcategories of C_X such that $\mathcal{F} \subseteq \mathcal{T}^{\perp}$ and $C_X = \mathcal{T} \bullet \mathcal{F}$. The latter means that every object M of C_X fits into an exact sequence

$$0 \longrightarrow M' \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} M'' \longrightarrow 0 \qquad (1)$$

with $M' \in \mathcal{T}$ and $M'' \in Ob\mathcal{F}$. Notice that the exact sequence (1) is unique up to isomorphism. In fact, if $N \xrightarrow{f} M$ is a morphism and $N \in Ob\mathcal{T}$, then $\mathfrak{e} \circ f = 0$, hence f factors uniquely through the monomorphism $M' \xrightarrow{\mathfrak{j}} M$.

This implies, in particular, that $\mathcal{T}$ is closed under taking quotients (in C_X) and, dually, $\mathcal{F}$ is closed under taking strict subobjects.

The assignments $M \longmapsto M'$ and $M \longmapsto M''$ in (1) extend to functors $C_X \xrightarrow{\mathfrak{j}_{\mathcal{T}*}} \mathcal{T}$ and $C_X \xrightarrow{\mathfrak{j}_{\mathcal{T}!}} \mathcal{F}$ which are resp. a right and a left adjoint to the inclusion functors $\mathcal{T} \xrightarrow{\mathfrak{j}^*_{\mathcal{T}}} C_X$ and $\mathcal{F} \xrightarrow{\mathfrak{j}^*_{\mathcal{F}}} C_X$. By [GZ, 1], the categories $\mathcal{T}$ and $\mathcal{F}$ have all types of limits and colimits which exist in the category C_X given by the formulas

$$\lim \mathfrak{D} = \mathfrak{j}_{\mathcal{T}*}(\lim(\mathfrak{j}^*_{\mathcal{T}} \circ \mathfrak{D})) \quad \text{and} \quad colim\mathfrak{D} = \mathfrak{j}_{\mathcal{T}*}(colim(\mathfrak{j}^*_{\mathcal{T}} \circ \mathfrak{D})) \tag{2}$$

for any small diagram $\mathcal{D} \xrightarrow{\mathfrak{D}} \mathcal{T}$. In particular, $\mathcal{T}$ has kernels and cokernels given by $Coker_{\mathcal{T}} = Coker_{C_X}$ and $Ker_{\mathcal{T}} = \mathfrak{j}_{\mathcal{T}*}(Ker_{\mathcal{T}})$. Similarly for $\mathcal{F}$.

A torsion pair $(\mathcal{T}, \mathcal{F})$ in C_X is called *tilting* if every object of C_X is a subobject of an object of $\mathcal{T}$. Dually, $(\mathcal{T}, \mathcal{F})$ is called a *cotilting* torsion pair if every object of C_X is a quotient of an object of $\mathcal{F}$.

8.2.2.1. Proposition. *Let C_X be an additive category. The following conditions are equivalent.*

(a) C_X is quasi-abelian.

(b) There exists a tilting torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category C_Y such that $\mathcal{T}$ is equivalent to C_X.

(c) There exists a cotilting torsion pair $(\mathcal{T}', \mathcal{F}')$ in an abelian category C_W such that $\mathcal{F}'$ is equivalent to C_X.

Proof. It follows from 8.2.1 that $C_Y = C_{\mathfrak{R}X}$ and $C_Y = C_{\mathfrak{L}X}$. See details of the proof in [BOVdB, B.3]. ∎

Chapter II

Derived Functors in Right and Left Exact Categories.

After necessary preliminaries on trivial morphisms, pointed objects and complexes gathered in Section 1, we introduce, in Section 2, ∂^*-functors from a right exact category. Universal ∂^*-functors, which can be called otherwise *left derived functors*, or (left) satellites, or homological functors of their zero components, appear in Section 3. We prove the existence of the left satellites for every functor from a right exact category to any category with kernels of morphisms and filtered limits. In Section 4, we look at contravariant functoriality of universal ∂^*-functors. A particular instance is the contravariant functoriality of ∂^*-functors from a right exact category $(C_X, \mathfrak{E}_X)$ with respect to the canonical embedding of $(C_X, \mathfrak{E}_X)$ into the category of non-trivial sheaves on $(C_X, \mathfrak{E}_X)$. It allows to replace the computation of universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$ by computation of universal ∂^*-functors from the category of non-trivial presheaves of sets on $(C_X, \mathfrak{E}_X)$. The k-linear version of this fact replaces computation of universal k-linear ∂^*-functors from a k-linear right exact category by the computation of the corresponding ∂^*-functors from the abelian category k-linear category of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$. In Section 5, we consider the dual notion – ∂-functors, and introduce the higher Exts. In Sections 6 and 7, we establish 'exactness' of ∂^*-functors whose zero component is *weakly right 'exact'* and the target right exact category satisfies an analog of the Grothendieck's (AB5*) property. In Section 8, we consider the category of universal ∂^*-functors from a right exact category with values in categories with initial objects and prove that this category has an initial object, which is the ∂^*-functor $Ext^\bullet$. We establish a similar fact in k-linear setting. In Section 9, we prove that the initial universal ∂^*-functor of Section 8 is also an initial object for the (appropriately defined) category of universal 'exact' functors from a fixed right exact category. We conclude the chapter with a short discussion of relative satellites.

1. Preliminaries: trivial morphisms, pointed objects, and complexes.

Let C_X be a category with initial objects. We call a morphism of C_X *trivial* if it factors through an initial object. It follows that an object M is initial iff id_M is a trivial morphism. If C_X is a pointed category, then the trivial morphisms are usually called *zero morphisms.*

1.1. Trivial compositions and pointed objects. If the composition of arrows $L \xrightarrow{f} M \xrightarrow{g} N$ is trivial, i.e. there is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \xi \downarrow & & \downarrow g \\ x & \xrightarrow{i_N} & N \end{array}$$

where x is an initial object, and the morphism g has a kernel, then f is the composition of the canonical arrow $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ and a morphism $L \xrightarrow{f_g} Ker(g)$ uniquely determined by f and ξ. If the arrow $x \xrightarrow{i_N} N$ is a monomorphism, then the morphism ξ is uniquely determined by f and g ; therefore in this case, the arrow f_g does not depend on ξ.

1.1.1. Pointed objects. In particular, f_g does not depend on ξ, if N is a *pointed* object. The latter means that there exists an arrow $N \longrightarrow x$.

1.2. Complexes. A sequence of arrows

$$\dots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \dots \qquad (1)$$

is called a *complex* if each of its arrows has a kernel and the next arrow factors *uniquely* through this kernel.

1.3. Lemma. *Let each arrow in the sequence*

$$\dots \xrightarrow{f_3} M_3 \xrightarrow{f_2} M_2 \xrightarrow{f_1} M_1 \xrightarrow{f_0} M_0 \qquad (2)$$

of arrows have a kernel and the composition of any two consecutive arrows is trivial. Then

$$\dots \xrightarrow{f_4} M_4 \xrightarrow{f_3} M_3 \xrightarrow{f_2} M_2 \qquad (3)$$

is a complex. If M_0 is a pointed object, then (2) is a complex.

Proof. The composition $M_2 \xrightarrow{f_0 \circ f_1} M_0$ factors through an initial object; in particular, there exist morphisms from M_i to an initial object x of C_X for all $i \geq 2$. Therefore, the unique morphism $x \longrightarrow M_i$ is a (split) monomorphism for all $i \geq 2$. By 1.1, this implies that $Ker(f_i) \xrightarrow{\mathfrak{k}(f_i)} M_{i+1}$ is a monomorphism. Therefore, there exists a unique arrow $M_{i+2} \xrightarrow{f'_{i+1}} Ker(f_i)$ whose composition with $Ker(f_i) \xrightarrow{\mathfrak{k}(f_i)} M_{i+1}$ equals to f_{i+1}.

By the similar reason, if there exists a morphism from M_0 (resp. M_1) to x, then $Ker(f_i) \xrightarrow{\mathfrak{k}(f_i)} M_{i+1}$ is a monomorphism for $i \geq 0$ (resp. for $i \geq 1$). ■

1.4. Corollary. *A sequence of morphisms*

$$\dots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \dots$$

unbounded on the right is a complex iff the composition of any pair of its consecutive arrows is trivial and for every i, there exists a kernel of the morphism f_i.

1.4.1. Example. Let C_X be the category Alg_k of unital associative k-algebras. The algebra k is its initial object, and every morphism of k-algebras has a kernel. Pointed objects of C_X which have a morphism to initial object are precisely augmented k-algebras. If the composition of pairs of consecutive arrows in the sequence

$$\ldots \xrightarrow{f_3} A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_0} A_0$$

is trivial, then it follows from the argument of 1.3 that A_i is an augmented k-algebra for all $i \geq 2$. Any unbounded on the right sequence of algebras with trivial compositions of pairs of consecutive arrows is formed by augmented algebras.

1.5. The categories of complexes. Let C_X be a category with initial objects. For any integer m, we denote by $\mathcal{K}_m(C_X)$ the category whose objects are complexes of the form

$$\ldots \xrightarrow{f_{m+2}} M_{m+2} \xrightarrow{f_{m+1}} M_{m+1} \xrightarrow{f_m} M_m$$

and morphisms are defined as usual. Every finite complex

$$M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_{m+2}} M_{m+2} \xrightarrow{f_{m+1}} M_{m+1} \xrightarrow{f_m} M_m \qquad (3)$$

is identified with an object of $\mathcal{K}_m(C_X)$ by adjoining on the left the infinite sequence of trivial objects and (unique) morphisms from them.

We call an object (3) of the category $\mathcal{K}_m(C_X)$ a *bounded* complex if M_n is an initial object for all $n \gg m$. We denote by $\mathcal{K}_m^b(C_X)$ the full subcategory of $\mathcal{K}_m(C_X)$ generated by bounded complexes.

The categories $\mathcal{K}_m(C_X)$ (resp. $\mathcal{K}_m^b(C_X)$) are naturally isomorphic to each other via obvious *translation* functors.

We denote by $\mathcal{K}(C_X)$ the category whose objects are complexes

$$\ldots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \ldots \qquad (4)$$

which are infinite in both directions. Unless C_X is a pointed category, there are no natural embeddings of the categories $\mathcal{K}_m(C_X)$ into $\mathcal{K}(C_X)$. There is a natural embedding into $\mathcal{K}(C_X)$ of the full subcategory $\mathcal{K}_{m,*}(C_X)$ of $\mathcal{K}_m(C_X)$ generated by all complexes (3) with M_m equal to an initial object.

We say that an object (4) of the category $\mathcal{K}(C_X)$ is a complex bounded on the left (resp. on the right) if M_n is an initial object for all $n \gg 0$ (resp. $n \ll 0$). We denote by $\mathcal{K}^+(C_X)$ (resp. by $\mathcal{K}^-(C_X)$) the full subcategory of $\mathcal{K}(C_X)$ whose objects are complexes bounded on the left (resp. on the right). Finally, we set $\mathcal{K}^b(C_X) = \mathcal{K}^-(C_X) \bigcap \mathcal{K}^+(C_X)$ and call objects of the subcategory $\mathcal{K}^b(C_X)$ *bounded* complexes.

1.6. 'Exact' and strictly 'exact' complexes. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects.

1.6.1. We call a sequence of two arrows $L \xrightarrow{f} M \xrightarrow{g} N$ of the category C_X 'exact', if the arrow $M \xrightarrow{g} N$ has a kernel, and $L \xrightarrow{f} M$ is the composition of $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ and a deflation $L \xrightarrow{f_g} Ker(g)$.

1.6.2. We call an 'exact' sequence of arrows $L \xrightarrow{f} M \xrightarrow{g} N$ *strictly 'exact'*, if the deflation $L \xrightarrow{f_g} Ker(g)$ coincides with its coimage.

1.6.3. A complex is called 'exact' (resp. strictly 'exact'), if any pair of its consecutive arrows forms an 'exact' (resp. strictly 'exact') sequence.

2. ∂^*-Functors.

2.0. Definitions. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x and a category C_Y with an initial object. A ∂^**-functor* from $(C_X, \mathfrak{E}_X)$ to C_Y is a sequence of functors $C_X \xrightarrow{T_i} C_Y$, $i \geq 0$, together with a functorial assignment to every conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ and every $i \geq 0$ a morphism $T_{i+1}(L) \xrightarrow{\mathfrak{d}_i(E)} T_i(N)$, which depends functorially on the conflation E and such that the sequence of arrows

$$\dots \xrightarrow{T_2(\mathfrak{e})} T_2(L) \xrightarrow{\mathfrak{d}_1(E)} T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_1(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is a complex. The morphisms $T_{i+1}(L) \xrightarrow{\mathfrak{d}_i(E)} T_i(N)$, $i \geq 0$, are called *connecting* morphisms.

Taking the trivial conflation $x \longrightarrow x \longrightarrow x$, we obtain that $T_i(x) \xrightarrow{id_{T_i(x)}} T_i(x)$ is a trivial morphism, or, equivalently, $T_i(x)$ is an initial object, for every $i \geq 1$.

Let $T = (T_i, \mathfrak{d}_i | \ i \geq 0)$ and $T' = (T'_i, \mathfrak{d}'_i | \ i \geq 0)$ be a pair of ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to C_Y. A morphism from T to T' is a family $f = (T_i \xrightarrow{f_i} T'_i \mid i \geq 0)$ of functor morphisms such that for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of the exact category C_X and every $i \geq 0$, the diagram

$$\begin{array}{ccc} T_{i+1}(L) & \xrightarrow{\mathfrak{d}_i(E)} & T_i(N) \\ {\scriptstyle f_{i+1}(L)} \downarrow & & \downarrow {\scriptstyle f_i(N)} \\ T'_{i+1}(L) & \xrightarrow{\mathfrak{d}'_i(E)} & T'_i(N) \end{array}$$

commutes. The composition of morphisms is naturally defined. Thus, we have the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$ of ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to C_Y.

2.1. Trivial ∂^*-functors. We call a ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ *trivial* if all T_i are functors with values in initial objects. One can see that trivial ∂^*-functors are precisely initial objects of the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$. Once an initial object y of the category C_Y is fixed, we have a canonical trivial functor whose components equal to the constant functor with value in y – it maps all arrows of C_X to id_y.

2.2. Some natural functorialities. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object and C_Y a category with initial object. If C_Z is another category with an initial object and $C_Y \xrightarrow{F} C_Z$ a functor which maps initial objects to initial objects, then for any ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$, the composition $F \circ T = (F \circ T_i, F\mathfrak{d}_i|\ i \geq 0)$ of T with F is a ∂^*-functor. The map $(F, T) \longmapsto F \circ T$ is functorial in both variables; i.e. it extends to a functor

$$Cat_*(C_Y, C_Z) \times \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \longrightarrow \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Z). \qquad (1)$$

Here Cat_* denotes the subcategory of Cat whose objects are categories with initial objects and morphisms are functors which map initial objects to initial objects.

2.2.1. On the other hand, let $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ be another right exact category with an initial object and Φ a *weakly 'exact'* functor from $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ to $(C_X, \mathfrak{E}_X)$; that is Φ is functor $C_{\mathfrak{X}} \longrightarrow C_X$ which maps conflations to conflations. In particular, it maps initial objects to initial objects (because if x is an initial object of $C_{\mathfrak{X}}$, then $x \longrightarrow M \xrightarrow{id_M} M$ is a conflation; and $\Phi(x \longrightarrow M \xrightarrow{id_M} M)$ being a conflation implies that $\Phi(x)$ is an initial object). For any ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y, the composition $T \circ \Phi = (T_i \circ \Phi, \mathfrak{d}_i\Phi|\ i \geq 0)$ is a ∂^*-functor from $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ to C_Y. The map $(T, \Phi) \longmapsto T \circ \Phi$ extends to a functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \times \mathcal{E}x_*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), (C_X, \mathfrak{E}_X)) \longrightarrow \mathcal{H}om^*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), C_Y), \qquad (2)$$

where $\mathcal{E}x_*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), (C_X, \mathfrak{E}_X))$ denotes the full subcategory of $\mathcal{H}om(C_{\mathfrak{X}}, C_X)$ whose objects are preserving conflations functors $C_{\mathfrak{X}} \longrightarrow C_X$.

3. Universal ∂^*-functors.

3.0. Definition. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x and a category C_Y with an initial object y.

A ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y is called *universal* if, for every ∂^*-functor $T' = (T'_i, \mathfrak{d}'_i|\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y and every functor morphism $T'_0 \xrightarrow{g} T_0$, there exists a unique ∂^*-functor morphism $f = (T'_i \xrightarrow{f_i} T_i \mid i \geq 0)$ from T' to T such that $f_0 = g$.

3.0.1. Notation. We denote by $\partial^*\mathfrak{Un}((C_X, \mathfrak{E}_X), C_Y)$ the full subcategory of the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$ of ∂^*-functors from the right exact category $(C_X, \mathfrak{E}_X)$ to the category C_Y generated by universal ∂^*-functors.

3.1. Interpretation. Consider the functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \qquad (3)$$

which assigns to every ∂^*-functor (resp. every morphism of ∂^*-functors) its zero component. For any functor $C_X \xrightarrow{F} C_Y$, we have a presheaf of sets $\mathcal{H}om(C_X, C_Y)(\Psi^*(-), F)$ on the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$. Suppose that this presheaf is representable by an object (i.e. a ∂^*-functor) $\Psi_*(F)$. Then $\Psi_*(F)$ is a universal ∂^*-functor.

Conversely, if $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ is a universal ∂^*-functor, then $T \simeq \Psi_*(T_0)$.

3.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x; and let C_Y be a category with initial objects, kernels of morphisms, and limits of filtered diagrams. Then, for any functor $C_X \xrightarrow{F} C_Y$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ such that $T_0 = F$.*

In other words, the functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \qquad (3)$$

which assigns to each morphism of ∂^*-functors its zero component has a right adjoint, Ψ_*.

Proof. (a) For an arbitrary functor $C_X \xrightarrow{F} C_Y$, we set $S_-F(L) = \lim Ker(F(\mathfrak{k}(\mathfrak{e})))$, where the limit is taken by the (filtered) system of all deflations $M \xrightarrow{\mathfrak{e}} L$. Since deflations form a pretopology, the map $L \longmapsto S_-F(L)$ extends naturally to a functor $C_X \xrightarrow{S_-F} C_Y$. By the definition of S_-F, for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, there exists a unique morphism $S_-F(L) \xrightarrow{\widetilde{\partial}^0_F(E)} Ker(F(\mathfrak{j}))$. We denote by $\partial^0_F(E)$ the composition of $\widetilde{\partial}^0_F(E)$ and the canonical morphism $Ker(F(j)) \longrightarrow F(N)$.

(b) Notice that the correspondence $F \longmapsto S_-F$ is functorial. Applying the iterations of the functor S_- to F, we obtain a ∂^*-functor $S^\bullet_-(F) = (S^i_-(F) | i \geq 0)$. The claim is that this ∂^*-functor is universal.

In fact, let $T = (T_i, \mathfrak{d} | i \geq 0)$ be a ∂^*-functor and $T_0 \xrightarrow{\lambda_0} F$ a functor morphism. For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, we have a commutative diagram

$$\begin{array}{ccccccc}
T_1(L) & \xrightarrow{\mathfrak{d}_0(E)} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) & \xrightarrow{T_0(\mathfrak{e})} & T_0(L) \\
 & & \lambda_0(N) \downarrow & & \downarrow \lambda_0(M) & & \downarrow \lambda_0(L) \\
 & & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L)
\end{array} \qquad (4)$$

Since $T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$ is a complex, the morphism $\mathfrak{d}_0(E)$ is the composition of a uniquely defined morphism $T_1(L) \xrightarrow{\widetilde{\mathfrak{d}}_0(E)} Ker(T_0(\mathfrak{j}))$ and the canonical arrow $Ker(T_0(\mathfrak{j})) \longrightarrow T_0(N)$. We denote by $\widetilde{\lambda}_1(E)$ the composition of the morphism $\widetilde{\mathfrak{d}}_0(E)$ and the morphism $Ker(T_0(\mathfrak{j})) \xrightarrow{\lambda_1'} Ker(F(\mathfrak{j})$ uniquely determined by the commutativity of the diagram

$$\begin{array}{ccccc}
Ker(T_0(\mathfrak{j})) & \xrightarrow{\mathfrak{k}(T_0(\mathfrak{j}))} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) \\
\lambda_1' \downarrow & & \lambda_0(N) \downarrow & & \downarrow \lambda_0(M) \\
Ker(F(\mathfrak{j})) & \xrightarrow{\mathfrak{k}(F(\mathfrak{j}))} & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M)
\end{array}$$

Thus, we have a commutative diagram

$$\begin{array}{ccccccc}
T_1(L) & \xrightarrow{\mathfrak{d}_0(E)} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) & \xrightarrow{T_0(\mathfrak{e})} & T_0(L) \\
\widetilde{\lambda}_1(E) \downarrow & & \lambda_0(N) \downarrow & & \downarrow \lambda_0(M) & & \downarrow \lambda_0(L) \\
Ker(F(\mathfrak{j})) & \xrightarrow{\mathfrak{k}(F(\mathfrak{j}))} & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L)
\end{array}$$

with the morphism $\widetilde{\lambda}_1(E)$ uniquely determined by the arrows of the diagram (4). Since the *connecting* morphism $T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N)$ depends on the conflation E functorially, same is true for $\widetilde{\lambda}_1(E)$; that is the morphisms $T_1(L) \xrightarrow{\widetilde{\lambda}_1(E)} Ker(F(\mathfrak{j}))$, where E runs through conflations $N \longrightarrow M \longrightarrow L$ (with fixed L and morphisms of the form (h, g, id_L)), form a cone. This cone defines a unique morphism $T_1(L) \xrightarrow{\lambda_1(L)} S_-F(L)$. It follows from the universality of this construction that $\lambda = (\lambda_1(L)|\ L \in ObC_X)$ is a functor morphism $T_1 \xrightarrow{\lambda_1} S_-F$ such that the diagram

$$\begin{array}{ccccccc}
T_1(L) & \xrightarrow{\mathfrak{d}_0(E)} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) & \xrightarrow{T_0(\mathfrak{e})} & T_0(L) \\
\lambda_1(L) \downarrow & & \lambda_0(N) \downarrow & & \downarrow \lambda_0(M) & & \downarrow \lambda_0(L) \\
S_-F(L) & \xrightarrow{\mathfrak{k}(F(\mathfrak{j}))} & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L)
\end{array}$$

commutes. Iterating this construction, we obtain uniquely defined functor morphisms $T_i \xrightarrow{\lambda_i} S_-^i(F)$ for all $i \geq 1$. ∎

3.3. Remark. Let the assumptions of 3.2 hold. Then we have a pair of adjoint functors

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \xrightarrow{\Psi_*} \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$$

By 3.2, the adjunction morphism $\Psi^*\Psi_* \longrightarrow Id$ is an isomorphism which means that Ψ_* is a fully faithful functor and Ψ^* is a localization functor at a left multiplicative system. The fully faithful functor Ψ^* establishes an equivalence between the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y and the category $\partial^*\mathfrak{Un}((C_X, \mathfrak{E}_X), C_Y)$ of universal ∂^*-functors from the right exact category $(C_X, \mathfrak{E}_X)$ to the category C_Y.

3.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y. Let C_Z be another category with an initial object and F a functor from C_Y to C_Z which preserves initial objects, kernels of morphisms and limits of filtered systems. Then*

(a) If T is a universal ∂^*-functor, then $F \circ T = (F \circ T_i, F\mathfrak{d}_i \mid i \geq 0)$ is universal.

(b) If, in addition, the functor F is fully faithful, then the ∂^*-functor $F \circ T$ is universal iff T is universal.

Proof. (a) Suppose that the ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ is universal. Then it follows from the argument of 3.2 that $T_i \simeq S_-^i(T_0)$ for all $i \geq 0$, where $S_-(G)(L) = \lim Ker(G(\mathfrak{k}(\mathfrak{e})))$ and the limit is taken by the system of all deflations $M \xrightarrow{\mathfrak{e}} L$. Since the functor F preserves kernels of morphisms and filtered limits (that is all types of limits which appear in the construction of $S_-(G)(L)$), the natural morphism

$$F \circ S_-(G)(L) \longrightarrow S_-(F \circ G)(L)$$

is an isomorphism for any functor $C_X \xrightarrow{G} C_Y$ such that $S_-(G)(L) = \lim Ker(G(\mathfrak{k}(\mathfrak{e})))$ exists. Therefore, the natural morphism $F \circ S_-^i(T_0)(L) \longrightarrow S_-^i(F \circ T_0)(L)$ is an isomorphism for all $i \geq 0$ and all $L \in ObC_X$.

(b) Suppose that the functor F is fully faithful and the ∂^*-functor $F \circ T$ is universal. Then

$$\begin{aligned} &F \circ T_{i+1}(L) \simeq S_-(F \circ T_i)(L) = \lim Ker(F \circ T_i(\mathfrak{k}(\mathfrak{e}))) \simeq \\ &\lim F(Ker(T_i(\mathfrak{k}(\mathfrak{e})))) \simeq F(\lim Ker(T_i(\mathfrak{k}(\mathfrak{e})))) = F(S_-(T_i)(L)), \end{aligned}$$

where the isomorphisms are due to compatibility of F with kernels of morphisms and filtered limits. Since all these isomorphisms are natural (i.e. functorial in L), we obtain a functor isomorphism $F \circ T_{i+1} \xrightarrow{\sim} F \circ S_-(T_i)$. Since the functor F is fully faithful, the latter implies an isomorphism $T_{i+1} \xrightarrow{\sim} S_-(T_i)$ for all $i \geq 0$. The assertion follows now from (the argument of) 3.2. ■

3.5. An application. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a category, both with initial objects. Fix an initial object $\mathfrak{y}$ of the category C_Y and consider the "reduced" Yoneda embedding

$$C_Y \xrightarrow{h_Y^{\circledast}} C_{Y^{\circledast}}, \quad \mathcal{M} \longmapsto (\widehat{\mathcal{M}}, \widehat{\mathfrak{y}} \to \widehat{\mathcal{M}}).$$

of the category C_Y into the category $C_Y^{\circledast} \stackrel{\mathrm{def}}{=} \mathfrak{y}\backslash C_Y^{\wedge}$ of non-trivial presheaves of sets on C_Y over $\widehat{\mathfrak{y}} = C_Y(-,\mathfrak{y})$.

3.5.1. Proposition. *Let $(C_X,\mathfrak{E}_X)$ be a svelte right exact category with initial objects and C_Y a category with initial objects. A ∂^*-functor $T = (T_i,\mathfrak{d}_i \mid i \geq 0)$ from $(C_X,\mathfrak{E}_X)$ to C_Y is universal iff the ∂^*-functor $\widehat{T} \stackrel{\mathrm{def}}{=} h_Y^{\circledast} \circ T = (h_Y^{\circledast} \circ T_i, \widehat{\mathfrak{d}}_i \mid i \geq 0)$ from $(C_X,\mathfrak{E}_X)$ to the category $C_{Y^{\circledast}}$ is universal.*

Proof. The "reduced" Yoneda embedding $C_Y \xrightarrow{h_Y^{\circledast}} C_{Y^{\circledast}}$ is a fully faithful functor which preserves all limits and maps initial objects to initial objects. In particular, it satisfies the conditions of 3.4(b). The assertion follows from 3.4. ∎

3.5.2. The existence of derived functors and representability. Let $(C_X,\mathfrak{E}_X)$ be a svelte right exact category. By 3.2, for any functor $C_X \xrightarrow{G} C_Y^*$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i,\mathfrak{d}_i|\ i \geq 0) = \Psi_*(G)$ from $(C_X,\mathfrak{E}_X)$ to C_Y^* whose zero component coincides with G. In particular, for every functor $C_X \xrightarrow{F} C_Y$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i,\mathfrak{d}_i \mid i \geq 0)$ such that $T_0 = h_Y^* \circ F = \widetilde{F}$. It follows from 3.4(b) that there exists a universal ∂^*-functor whose zero component coincides with F iff for all $L \in ObC_X$ and all $i \geq 1$, the presheaves $T_i(L)$ are representable.

4. Contravariant functoriality for universal ∂^*-functors.

4.1. Proposition. *Let $(C_X,\mathfrak{E}_X)$ and $(C_{\mathfrak{X}},\mathfrak{E}_{\mathfrak{X}})$ be right exact categories with initial objects and $C_X \xrightarrow{\Phi} C_{\mathfrak{X}}$ a fully faithful functor which maps conflations to conflations. Let $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$ denote the class of all arrows $\mathfrak{M} \xrightarrow{\mathfrak{t}} \mathfrak{L}$ of $\mathfrak{E}_{\mathfrak{X}}$ such that, for any morphism $\Phi(\mathcal{L}) \xrightarrow{\mathfrak{f}} \mathfrak{L}$, there exists a commutative square*

$$\begin{array}{ccc} \Phi(\mathcal{M}) & \xrightarrow{\mathfrak{f}'} & \mathfrak{M} \\ {\scriptstyle \Phi(\mathfrak{s})}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} \\ \Phi(\mathcal{L}) & \xrightarrow{\mathfrak{f}} & \mathfrak{L} \end{array}$$

where $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{L}$ is a deflation.

(a) The class $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$ is a right exact structure on the category $C_{\mathfrak{X}}$.

(b) If $T = (T_i,\mathfrak{d}_i \mid i \geq 0)$ is a universal ∂^*-functor from $(C_{\mathfrak{X}},\mathfrak{E}_{\mathfrak{X}}^{\Phi})$ to a category C_Y, then $T \circ \Phi = (T_i \circ \Phi, \mathfrak{d}_i\Phi \mid i \geq 0)$ is a universal ∂^*-functor from $(C_X,\mathfrak{E}_X)$ to C_Y.

Proof. (a) Evidently, the class of deflations $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$ is closed under composition and contains all isomorphisms. Let

$$\begin{array}{ccc}\mathfrak{M}_{\mathfrak{t},\gamma} & \xrightarrow{\gamma_{\mathfrak{t}}} & \mathfrak{M}\\ {\scriptstyle \mathfrak{t}'_{\gamma}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}}\\ \mathfrak{L}_{\gamma} & \xrightarrow{\gamma} & \mathfrak{L}\end{array}$$

be a cartesian square whose right vertical arrow belongs to $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$ and $\Phi(\mathcal{L}) \xrightarrow{\mathfrak{f}} \mathfrak{L}_{\gamma}$ a morphism. Then there exists a commutative square

$$\begin{array}{ccc}\Phi(\mathcal{M}) & \xrightarrow{\mathfrak{f}'} & \mathfrak{M}\\ {\scriptstyle \Phi(\mathfrak{s})}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}}\\ \Phi(\mathcal{L}) & \xrightarrow{\gamma\circ\mathfrak{f}} & \mathfrak{L}\end{array}$$

By the universal property of cartesian squares, there is a morphism $\Phi(\mathcal{M}) \xrightarrow{\mathfrak{f}''} \mathfrak{M}_{\mathfrak{t},\gamma}$ uniquely determined by the equalities $\gamma_{\mathfrak{t}} \circ \mathfrak{f}'' = \mathfrak{f}'$ and $\mathfrak{t}'_{\gamma} \circ \mathfrak{f}'' = \gamma \circ \Phi(\mathfrak{s})$. The latter equality means the commutativity of the square

$$\begin{array}{ccc}\Phi(\mathcal{M}) & \xrightarrow{\mathfrak{f}''} & \mathfrak{M}_{\mathfrak{t},\gamma}\\ {\scriptstyle \Phi(\mathfrak{s})}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}'_{\gamma}}\\ \Phi(\mathcal{L}) & \xrightarrow{\mathfrak{f}} & \mathfrak{L}_{\gamma}\end{array}$$

which shows that the pull-back $\mathfrak{M}_{\mathfrak{t},\gamma} \xrightarrow{\mathfrak{t}'_{\gamma}} \mathfrak{L}_{\gamma}$ of the deflation $\mathfrak{M} \xrightarrow{\mathfrak{t}} \mathfrak{L}$ along $\mathfrak{L}_{\gamma} \xrightarrow{\gamma} \mathfrak{L}$ belongs to the class $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$.

(a1) It follows that the functor Φ maps conflation of $(C_X, \mathfrak{E}_X)$ to conflations of the right exact category $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}^{\Phi})$.

(b) Thanks to 3.5.1, we can (and will) assume that the category C_Y has limits of small diagrams. This allows to use the formula

$$\mathcal{S}_{-}F(\mathfrak{L}) = \lim_{\mathfrak{L}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \mathfrak{L}} Ker(F(\mathfrak{k}(\mathfrak{t})))$$

for any functor $C_{\mathfrak{X}} \xrightarrow{F} C_Y$. Here $\mathfrak{L}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \mathfrak{L}$ runs through deflations of $\mathfrak{L}$ from $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$.

It follows from the definition of the right exact structure $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$ that, for any object $\mathcal{L}$ of the category C_X, the images $\Phi(\mathcal{L}_{\mathfrak{s}} \xrightarrow{\mathfrak{s}} \mathcal{L})$ of deflations of $\mathcal{L}$ contain refinements of any deflation $\mathfrak{L}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \Phi(\mathcal{L})$ from $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$. This implies that the canonical morphism

$$\mathcal{S}_{-}F(\Phi(\mathcal{L})) = \lim_{(\mathfrak{L}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \Phi(\mathcal{L})) \in \mathfrak{E}_{\mathfrak{X}}^{\Phi}} Ker(F(\mathfrak{k}(\mathfrak{t}))) \longrightarrow \lim_{(\mathcal{L}_{\mathfrak{s}} \xrightarrow{\mathfrak{s}} \mathcal{L}) \in \mathfrak{E}_X} Ker(F(\mathfrak{k}(\Phi(\mathfrak{s}))))$$

is an isomorphism; or, equivalently, $\mathcal{S}_-F(\Phi(\mathcal{L})) \xrightarrow{\sim} \mathcal{S}_-(F\circ\Phi)(\mathcal{L}))$.

The assertion (b) follows now from 3.2. ∎

4.2. Example-application. Let $(C_X,\mathfrak{E}_X)$ be a right exact category with an initial object $\mathfrak{x}$ and

$$(C_X,\mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^{\circledast}} (C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$$

the canonical embedding of $(C_X,\mathfrak{E}_X)$ into the category $C_{X_{\mathfrak{E}}^{\circledast}} = \widehat{\mathfrak{x}}\backslash(C_X,\mathfrak{E}_X)^\wedge$ of non-trivial sheaves of sets on the presite $(C_X,\mathfrak{E}_X)$ over sheaf $\widehat{\mathfrak{x}}$ endowed with the canonical (that is the finest) right exact structure $\mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}}$. By I.2.1.1, $(C_X,\mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^{\circledast}} (C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$ is a fully faithful 'exact' functor. In particular, the functor $\mathfrak{j}_X^{\circledast}$ maps conflations to conflations.

Moreover, it follows from I.2.2.1(b) that the canonical (that is the finest) right exact structure $\mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}}$ on the category of sheaves of sets $C_{X_{\mathfrak{E}}}$ coincides with the right exact structure coinduced by the embedding $(C_X,\mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^{\circledast}} (C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$; that is, in the notations of 4.1, $\mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}} = \mathfrak{E}_X^{\mathfrak{j}_X^{\circledast}}$.

4.2.1. Proposition. *Suppose that* $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ *is a universal* ∂^**-functor from the right exact category* $(C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$ *to a category* C_Y. *Then* $T\circ\mathfrak{j}_X^{\circledast} = (T_i\circ\mathfrak{j}_X^{\circledast}, \mathfrak{d}_i\mathfrak{j}_X^{\circledast} \mid i\geq 0)$ *is a universal* ∂^**-functor from* $(C_X,\mathfrak{E}_X)$ *to* C_Y.

Proof. The fact follows from the preceding discussion and 4.1. ∎

4.2.2. Proposition. *Let* $(C_X,\mathfrak{E}_X)$ *be a svelte right exact category with initial objects and* C_Y *a category with colimits, kernels of morphisms and limits of filtered diagrams. Then the functor*

$$\begin{aligned} \partial^*\mathfrak{Un}(C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}}), C_Y) &\longrightarrow \partial^*\mathfrak{Un}((C_X,\mathfrak{E}_X), C_Y), \\ T = (T_i,\mathfrak{d}_i \mid i\geq 0) &\longmapsto T\circ\mathfrak{j}_X^{\circledast} = (T_i\circ\mathfrak{j}_X^{\circledast}, \mathfrak{d}_i\mathfrak{j}_X^{\circledast} \mid i\geq 0) \end{aligned}$$

has a fully faithful right adjoint which establishes an equivalence between the category $\partial^*\mathfrak{Un}((C_X,\mathfrak{E}_X), C_Y)$ *of universal* ∂^**-functors from* $(C_X,\mathfrak{E}_X)$ *to* C_Y *and the full subcategory of the category* $\partial^*\mathfrak{Un}((C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}}), C_Y)$ *of universal* ∂^**-functors from the right exact category* $(C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$ *to the category* C_Y, *which is generated by universal* ∂^**-functors* $T = (T_i,\mathfrak{d}_i \mid i\geq 0)$ *such that* $(T_0\circ\mathfrak{j}_X^{\circledast})^\diamond \circ \mathfrak{q}_{X^{\circledast}} \simeq T_0$.

Here $\mathfrak{q}_{X^{\circledast}}$ *is a right adjoint to the "reduced" sheafification functor* $C_{X^{\circledast}} \xrightarrow{\mathfrak{q}_X^{\circledast}} C_{X_{\mathfrak{E}}^{\circledast}}$.

Proof. For any category C_Y, we have a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{H}om^*((X_{\mathfrak{E}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}}), C_Y) & \longrightarrow & \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y), \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}om(C_{X_{\mathfrak{E}}}, C_Y) & \longrightarrow & \mathcal{H}om(C_X, C_Y) \end{array} \qquad (1)$$

whose vertical arrows map ∂^*-functors to their zero components and the horizontal arrows are functors of composition with the canonical embedding $C_X \xrightarrow{\mathfrak{j}^{\circledast}_X} C_{X^{\circledast}_{\mathfrak{E}}}$.

(a) If the category C_Y has kernels of arrows and limits if filtered diagrams, then it follows from 3.2 that the vertical arrows of the diagram (1) are continuous localizations, and their right adjoint functors assign to every functor $C_X \xrightarrow{F} C_Y$ the universal ∂^*-functor $\mathcal{S}^{\bullet}_{-}(F)$, whose zero component is F. Thus, the diagram (1) yields a commutative diagram of functors

$$\begin{array}{ccc} \partial^*\mathfrak{Un}((C_{X^{\circledast}_{\mathfrak{E}}}, \mathfrak{E}^{\mathfrak{s}}_{X^{\circledast}_{\mathfrak{E}}}), C_Y) & \longrightarrow & \partial^*\mathfrak{Un}((C_X, \mathfrak{E}_X), C_Y), \\ \wr\wr\big\downarrow & & \big\downarrow\wr\wr \\ \mathcal{H}om(C_{X^{\circledast}_{\mathfrak{E}}}, C_Y) & \longrightarrow & \mathcal{H}om(C_X, C_Y) \end{array} \qquad (2)$$

whose vertical arrows are category equivalences.

(b) If the category C_Y has colimits, then it follows from I.2.0.4.4 that the lower horizontal arrow of the diagram (2),

$$\mathcal{H}om(C_{X^{\circledast}_{\mathfrak{E}}}, C_Y) \xrightarrow{\widetilde{\mathfrak{j}^{\circledast}_X}} \mathcal{H}om(C_X, C_Y), \quad \mathcal{G} \longmapsto \mathcal{G} \circ \mathfrak{j}^{\circledast}_X,$$

is a localization functor having a (necessarily) fully faithful right adjoint. The latter assigns to every functor $C_X \xrightarrow{\mathcal{F}} C_Y$ the composition of the embedding $C_{X^{\circledast}_{\mathfrak{E}}} \xrightarrow{\mathfrak{q}_{X^{\circledast}}} C_{X^{\circledast}}$ – a right adjoint to the "reduced" sheafification functor $C_{X^{\circledast}} \xrightarrow{\mathfrak{q}^{\circledast}_X} C_{X^{\circledast}_{\mathfrak{E}}}$, the forgetful functor $C_{X^{\circledast}} = \mathfrak{x}\backslash C^*_X \longrightarrow C^*_X$, and the functor $C^*_X \xrightarrow{\mathcal{F}^{\diamond}} C_Y$, which preserves colimits and whose composition with the Yoneda embedding $C_X \xrightarrow{h^*_X} C^*_X$ coincides with $\mathcal{F}$.

(c) The assertion follows from (a) and (b). ∎

4.3. Derived functors via the category of sheaves. Let $(C_X, \mathfrak{E}_X)$ be a svelte category with an initial object $\mathfrak{x}$ and C_Y an arbitrary category with an initial object $\mathfrak{y}$. Proposition 4.2.2 allows to replace computation of derived functors of any functor $C_X \xrightarrow{F} C_Y$ (that is the universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y whose zero component coincides with F) by the computation of derived functors of a canonically associated with F functor $F^{\circledast}$ from the right exact category $(C_{X^{\circledast}_{\mathfrak{E}}}, \mathfrak{E}^{\mathfrak{s}}_{X^{\circledast}_{\mathfrak{E}}})$ to the category $C_{Y^{\circledast}} \stackrel{\mathrm{def}}{=} \widehat{\mathfrak{h}}\backslash C_Y^{\wedge} = \widehat{\mathfrak{h}}\backslash C_Y^{*}$.

In fact, it follows from I.2.0.2 that there is a natural equivalence between the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X and C_Y and the full subcategory $\mathfrak{Hom}_{\mathfrak{c}}(C_X, C_Y)$ of the category $\mathcal{H}om_{\mathfrak{c}}(C_X^{\wedge}, C_Y^{\wedge})$ generated by all continuous (that is having a right adjoint) functors from $C_X^{\wedge}$ to $C_Y^{\wedge}$ which map representable presheaves to representable presheaves. This equivalence assigns to any functor $C_X \xrightarrow{F} C_Y$ the (unique up to isomorphism) continuous functor $C_X^{\wedge} \xrightarrow{F^{\wedge}} C_Y^{\wedge}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{h_X} & C_X^{\wedge} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F^{\wedge}} \\ C_Y & \xrightarrow{h_Y} & C_Y^{\wedge} \end{array} \qquad (1)$$

commutes. The commutative diagram (1) induces the commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{h_X^{\circledast}} & C_{X^{\circledast}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F^{\circledast}} \\ C_Y & \xrightarrow{h_Y^{\circledast}} & C_{Y^{\circledast}} \end{array} \qquad (2)$$

Let $C_{X^{\circledast}_{\mathfrak{E}}} \xrightarrow{F^{\circledast}_{\mathfrak{E}_X}} C_{Y^{\circledast}}$ denote the composition of the functor $F^{\circledast}$ with the inclusion functor $C_{X^{\circledast}_{\mathfrak{E}}} \longrightarrow C_{X^{\circledast}}$. The composition of $F^{\circledast}_{\mathfrak{E}_X}$ with the canonical embedding $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^{\circledast}} (C_{X^{\circledast}_{\mathfrak{E}}}, \mathfrak{E}^{\mathfrak{s}}_{X^{\circledast}_{\mathfrak{E}}})$ is isomorphic to the composition of $C_X \xrightarrow{F} C_Y$ with the "reduced" Yoneda embedding $C_Y \xrightarrow{h_Y^{\circledast}} C_{Y^{\circledast}}$.

By 4.2.1, the universal ∂^*-functor $\mathcal{S}_-^{\bullet}(h_Y^{\circledast} \circ F)$ whose zero component is $h_Y^{\circledast} \circ F$ is isomorphic to the composition $\mathcal{S}_-^{\bullet}(F^{*}_{\mathfrak{E}_X}) \circ \mathfrak{j}_X^{\circledast}$ of the universal ∂^*-functor, whose zero component is the functor $F^{\circledast}_{\mathfrak{E}_X}$ with the canonical embedding $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^{\circledast}} (C_{X^{\circledast}_{\mathfrak{E}}}, \mathfrak{E}^{\mathfrak{s}}_{X^{\circledast}_{\mathfrak{E}}})$.

It follows that the universal ∂^*-functor $(C_X, \mathfrak{E}_X) \xrightarrow{\mathcal{S}_-^{\bullet} F} C_Y$ exists iff the functors $\mathcal{S}_-^{n}(F^{\circledast}_{\mathfrak{E}_X}) \circ \mathfrak{j}_X^{\circledast}$ factor through the "reduced" Yoneda embedding $C_Y \xrightarrow{h_Y^{\circledast}} C_Y^{\circledast}$ for every $n \geq 1$. In this case, $h_X^{\circledast} \circ \mathcal{S}_-^{\bullet} F \simeq \mathcal{S}_-^{\bullet}(F^{\circledast}_{\mathfrak{E}_X}) \circ \mathfrak{j}_X^{\circledast}$.

4.4. Deflations with trivial kernels and derived functors. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We denote by $\mathfrak{E}_X^{\circledast}$ the class of all arrows of $\mathfrak{E}_X$ whose kernel is an initial object.

4.4.1. Proposition. *The class $\mathfrak{E}_X^{\circledast}$ of deflations with trivial kernel is a right exact structure on the category C_X.*

Proof. The class $\mathfrak{E}_X^{\circledast}$ contains all isomorphisms of the category C_X. It is closed under compositions, because, by I.4.4.5, if $Ker(\mathfrak{s})$ is trivial (i.e is an initial object of C_X), then $Ker(\mathfrak{s} \circ \mathfrak{t})$ is naturally isomorphic to $Ker(\mathfrak{t})$. In particular, $Ker(\mathfrak{s} \circ \mathfrak{t})$ is trivial, if both $Ker(\mathfrak{s})$ and $Ker(\mathfrak{t})$ are trivial. Finally, if

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{p_1} & M \\ {\scriptstyle \mathfrak{t}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{s}} \\ N & \xrightarrow{f} & L \end{array}$$

is a cartesian square, then, by I.3.3.3, $Ker(\mathfrak{s}) \simeq Ker(\mathfrak{t})$, which shows that $\mathfrak{E}_X^{\circledast}$ is stable under base change. ■

4.4.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x; and let C_Y be a category with initial objects, kernels of morphisms, and limits of filtered systems. Let $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ be a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y.*

If the functor T_0 maps all arrows of $\mathfrak{E}_X^{\circledast}$ to isomorphisms, then all functors T_i, $i \geq 0$, have this property.

Proof. By the argument of 3.2, the assertion is equivalent to the following one:

If a functor $C_X \xrightarrow{F} C_Y$ maps arrows of $\mathfrak{E}_X^{\circledast}$ to isomorphisms, then its satellite, S_-F, has the same property.

In fact, let $\mathcal{L} \xrightarrow{\mathfrak{s}} L$ be an arrow of $\mathfrak{E}_X^{\circledast}$ and $M \xrightarrow{\mathfrak{e}} L$ an arbitrary deflation. Then we have a commutative diagram

$$\begin{array}{ccccc} Ker(\widetilde{\mathfrak{e}}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{e}})} & \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\mathfrak{e}}} & \mathcal{L} \\ {\scriptstyle \mathfrak{s}_2}\big\downarrow\wr & & {\scriptstyle \mathfrak{s}_1}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{s}} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}(\mathfrak{e})} & M & \xrightarrow{\mathfrak{e}} & L \end{array} \qquad (1)$$

whose vertical arrows belong to $\mathfrak{E}_X^{\circledast}$. By hypothesis, $F(\mathfrak{s}_1)$ is an isomorphism. Therefore, the left square of (1) determines isomorphism $Ker(F(\mathfrak{k}(\widetilde{\mathfrak{e}})) \xrightarrow{\phi(\mathfrak{e})} Ker(F(\mathfrak{k}(\mathfrak{e}))$, which is functorial in $\mathfrak{e}$. So that we obtain an isomorphism

$$\lim Ker(F(\mathfrak{k}(\widetilde{\mathfrak{e}})) \xrightarrow{\sim} \lim Ker(F(\mathfrak{k}(\mathfrak{e})) = S_-F(L),$$

whose composition with the canonical arrow $S_-F(\mathcal{L}) \longrightarrow \lim Ker(F(\mathfrak{k}(\widetilde{\mathfrak{e}}))$ coincides with the morphism $S_-F(\mathcal{L}) \xrightarrow{S_-F(\mathfrak{s})} S_-F(L)$ (see the argument of 3.2).

On the other hand, for any deflation $\mathcal{M}_1 \xrightarrow{\gamma} \mathcal{L}$, there is a commutative diagram

$$\begin{array}{ccccc} Ker(\gamma) & \xrightarrow{\mathfrak{k}(\gamma)} & \mathcal{M}_1 & \xrightarrow{\gamma} & \mathcal{L} \\ \downarrow \wr & & id \downarrow & & \downarrow \mathfrak{s} \\ Ker(\mathfrak{s}\gamma) & \xrightarrow{\mathfrak{k}(\mathfrak{s}\gamma)} & \mathcal{M}_1 & \xrightarrow{\mathfrak{s}\gamma} & L \end{array} \qquad (2)$$

Here the left vertical arrow is an isomorphism, because $Ker(\mathfrak{s})$ is an initial object (see I.4.4.5). The left square of (2) induces an isomorphism

$$Ker(F(\mathfrak{k}(\mathfrak{s}\gamma)) \xrightarrow{\phi(\gamma)} Ker(F(\mathfrak{k}(\gamma)),$$

which is functorial in γ. The latter implies that the composition $\varphi(\gamma)$ of $\phi(\gamma)$ with the unique morphism $S_{-}F(L) \longrightarrow Ker(F(\mathfrak{k}(\mathfrak{s}\gamma))$ defines a cone

$$S_{-}F(L) \xrightarrow{\varphi(\gamma)} Ker(F(\mathfrak{k}(\gamma)),$$

hence a unique morphism $S_{-}F(L) \xrightarrow{\varphi} S_{-}F(\mathcal{L})$. The claim is that φ is the inverse to the morphism $S_{-}F(\mathcal{L}) \xrightarrow{S_{-}F(\mathfrak{s})} S_{-}F(L)$.

We complete (2) to a commutative diagram

$$\begin{array}{ccccc} Ker(\widetilde{\mathfrak{s}\gamma}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{s}\gamma})} & \widetilde{\mathcal{M}}_1 & \xrightarrow{\widetilde{\mathfrak{s}\gamma}} & \mathcal{L} \\ \mathfrak{t}_2 \downarrow \wr & & \mathfrak{t}_1 \downarrow & & \downarrow id \\ Ker(\gamma) & \xrightarrow{\mathfrak{k}(\gamma)} & \mathcal{M}_1 & \xrightarrow{\gamma} & \mathcal{L} \\ \mathfrak{s}_2 \downarrow \wr & & id \downarrow & & \downarrow \mathfrak{s} \\ Ker(\mathfrak{s}\gamma) & \xrightarrow{\mathfrak{k}(\mathfrak{s}\gamma)} & \mathcal{M}_1 & \xrightarrow{\mathfrak{s}\gamma} & L \end{array} \qquad (3)$$

where the square

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_1 & \xrightarrow{\widetilde{\mathfrak{s}\gamma}} & \mathcal{L} \\ \mathfrak{t}_1 \downarrow & cart & \downarrow \mathfrak{s} \\ \mathcal{M}_1 & \xrightarrow{\mathfrak{s}\gamma} & L \end{array}$$

is cartesian. Since $\mathfrak{t}_1 \in \mathfrak{E}_X^{\circledast}$, by hypothesis, $F(\mathfrak{t}_1)$ is an isomorphism. So that the diagram (3) induces isomorphisms

$$KerF(\mathfrak{k}(\widetilde{\mathfrak{s}\gamma})) \xrightarrow{\sim} KerF(\mathfrak{k}(\gamma)) \xrightarrow{\sim} KerF(\mathfrak{k}(\mathfrak{s}\gamma)),$$

which imply isomorphisms of the lower row of the commutative diagram

$$\begin{array}{ccccccc}
S_-F(\mathcal{L}) & \xrightarrow{id} & S_-F(\mathcal{L}) & \xrightarrow{S_-F(\mathfrak{s})} & S_-F(L) & \xleftarrow{id} & S_-F(L) \\
\downarrow & & \downarrow id & & \downarrow \varphi & & \downarrow \\
\lim KerF(\mathfrak{k}(\widetilde{\mathfrak{s}\gamma})) & \xrightarrow{\sim} & \lim KerF(\mathfrak{k}(\gamma)) & \xrightarrow{id} & S_-F(\mathcal{L}) & \xrightarrow{\sim} & \lim KerF(\mathfrak{k}(\mathfrak{s}\gamma))
\end{array}$$

The fact that φ is an isomorphism (or, equivalently, that $S_-F(\mathfrak{s})$ is an isomorphism) follows from the universal property of limits. ■

4.5. Remark about the shape of "triangles". Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with an initial object x and C_Y a category with an initial object y and limits. Then, by the argument of 3.2, we have an endofunctor S_- of the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y, together with a cone $S_- \xrightarrow{\lambda} \mathfrak{y}$, where $\mathfrak{y}$ is the constant functor with the values in the initial object y of the category C_Y. For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of $(C_X, \mathfrak{E}_X)$ and any functor $C_X \xrightarrow{F} C_Y$, we have a commutative diagram

$$\begin{array}{ccccc}
S_-F(L) & \xrightarrow{\lambda(L)} & y & & \\
\mathfrak{d}_0(E) \downarrow & & \downarrow & & \\
F(N) & \xrightarrow{F\mathfrak{j}} & F(M) & \xrightarrow{F\mathfrak{e}} & F(L)
\end{array} \qquad (1)$$

4.6. The case of k-linear categories. Let $(C_X, \mathfrak{E}_X)$ be a k-linear right exact category and C_Y a k-linear category.

4.6.1. Linearity of derived functors. Suppose that the category C_Y has kernels of morphisms (which this time can be understood in usual way) and limits of filtered diagrams. If $C_X \xrightarrow{\mathcal{F}} C_Y$ is a k-linear functor, then, as it follows from the formula for the derived functor $\mathcal{S}_-(\mathcal{F})$ (see the argument of 3.2), the functor $\mathcal{S}_-(\mathcal{F})$ is k-linear too. Therefore, all its iterations, $\mathcal{S}_-^n(\mathcal{F})$, $n \geq 1$, are k-linear functors from C_X to C_Y.

4.6.2. Using k-linear Yoneda embedding. Let C_Y be an arbitrary k-linear category. The Yoneda embedding $C_Y \xrightarrow{h_Y^*} C_Y^*$ is the composition of the full embedding

$$C_Y \xrightarrow{\mathfrak{h}_Y} \mathcal{M}_k(Y), \quad \mathcal{L} \longmapsto \widehat{\mathcal{L}} = C_Y(-, \mathcal{L}), \qquad (1)$$

of the category C_Y into the k-linear Grothendieck category $\mathcal{M}_k(Y)$ of presheaves of k-modules on C_Y and the forgetful functor $\mathcal{M}_k(Y) \longrightarrow C_Y^*$.

Since the category $\mathcal{M}_k(Y)$ has all limits (and colimits), for any functor $C_X \xrightarrow{\mathcal{F}} C_Y$, there exist all derived functors $\mathcal{S}_-^n(\mathfrak{h}_Y \circ \mathcal{F})$, $n \geq 1$, of the composition $C_X \xrightarrow{\mathfrak{h}_Y \circ \mathcal{F}} \mathcal{M}_k(Y)$.

Since the functor $C_Y \xrightarrow{\mathfrak{h}_Y} \mathcal{M}_k(Y)$ preserves limits and is fully faithful, it follows from 3.4 that the universal ∂^*-functor $\mathcal{S}_-^\bullet(\mathcal{F})$ exists iff all functors $\mathcal{S}_-^n(\mathfrak{h}_Y \circ \mathcal{F})$, $n \geq 1$, take values in the subcategory of representable presheaves on C_Y. If this is the case, $\mathcal{S}_-^n(\mathfrak{h}_Y \circ \mathcal{F}) \simeq \mathfrak{h}_Y \circ \mathcal{S}_-^n(\mathcal{F})$ for all $n \geq 1$.

4.6.2.1. Linearity of derived functors in general case. If $C_X \xrightarrow{\mathcal{F}} C_Y$ is a k-linear functor, then the composition $\mathfrak{h}_Y \circ \mathcal{F}$ is a k-linear functor and, as it was observed in 4.6.1, all derived functors $\mathcal{S}_-^n(\mathfrak{h}_Y \circ \mathcal{F})$, $n \geq 1$, are k-linear. If these derived functors take values in the subcategory of representable presheaves, then the isomorphisms $\mathcal{S}_-^n(\mathfrak{h}_Y \circ \mathcal{F}) \simeq \mathfrak{h}_Y \circ \mathcal{S}_-^n(\mathcal{F})$ imply that the functors $\mathcal{S}_-^n(\mathcal{F})$ are k-linear.

4.6.2.2. Notations. We denote by $\mathcal{H}om_k^*((C_X, \mathfrak{E}_X), C_Y)$ the full subcategory of the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$ of the category ∂^*-functors from a k-linear right exact category $(C_X, \mathfrak{E}_X)$ to a k-linear category C_Y generated by k-linear ∂^*-functors.

We denote by $\partial_k^*\mathfrak{Un}((C_X, \mathfrak{E}_X), C_Y)$ the full subcategory of $\mathcal{H}om_k^*(C_X, \mathfrak{E}_X), C_Y)$ generated by universal ∂^*-functors.

4.6.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a k-linear right exact category and C_Y a k-linear category with limits and colimits. The functor*

$$\mathcal{H}om_k^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om_k^*(C_X, C_Y),$$

which assigns to every k-linear ∂^*-functor its zero component is a has a fully faithful right adjoint, Ψ_*. The latter establishes an equivalence between the category $\mathcal{H}om_k^*(C_X, C_Y)$ of k-linear functors from C_X to C_Y and the category $\partial_k^*\mathfrak{Un}((C_X, \mathfrak{E}_X), C_Y)$ of k-linear universal ∂^*-functors from the right exact category $(C_X, \mathfrak{E}_X)$ to the category C_Y.

Proof. The functor

$$\mathcal{H}om_k^*(C_X, C_Y) \xrightarrow{\Psi_*} \mathcal{H}om_k^*((C_X, \mathfrak{E}_X), C_Y)$$

assigns to every k-linear functor $C_X \xrightarrow{F} C_Y$ the universal ∂^*-functor

$$\mathcal{S}_-^\bullet(F) = (\mathcal{S}_-^n(F), \mathfrak{d}_n^F \mid n \geq 0),$$

which, by 4.6.1, is automatically k-linear. The fact that $\Psi^*\Psi_*(F) = \mathcal{S}_-^0(F) = F$ means that the adjunction arrow $\Psi^*\Psi_* \longrightarrow Id_{\mathcal{H}om_k^*(C_X, C_Y)}$ is the identical isomorphism, which implies that the functor Ψ_* is fully faithful; hence it is the composition of a category equivalence

$$\mathcal{H}om_k^*(C_X, C_Y) \xrightarrow{\approx} \partial_k^*\mathfrak{Un}((C_X, \mathfrak{E}_X), C_Y), \quad F \longmapsto \mathcal{S}_-^\bullet(F),$$

and the full embedding of $\partial_k^*\mathfrak{Un}((C_X,\mathfrak{E}_X),C_Y)$ into $\mathcal{H}om_k^*(C_X,C_Y)$. ■

4.6.4. Proposition. *Let $(C_X,\mathfrak{E}_X)$ be a k-linear right exact category and $T = (T_i,\mathfrak{d}_i \mid i\ge 0)$ a k-linear ∂^*-functor from $(C_X,\mathfrak{E}_X)$ to a k-linear category C_Y.*

Let $C_Y \xrightarrow{F} C_Z$ a k-linear functor which preserves limits. Then

(a) If T is a universal ∂^*-functor, then $F\circ T = (F\circ T_i, F\mathfrak{d}_i \mid i\ge 0)$ is universal.

(b) If, in addition, the functor F is fully faithful, then the ∂^*-functor $F\circ T$ is universal iff the ∂^*-functor T is universal.

Proof. The assertion follows from (the argument of) 3.4. ■

4.6.4.1. Corollary. *Let $(C_X,\mathfrak{E}_X)$ be a k-linear right exact category. A k-linear ∂^*-functor $T = (T_i,\mathfrak{d}_i \mid i\ge 0)$ from $(C_X,\mathfrak{E}_X)$ to a k-linear category C_Y is universal iff its composition with the Yoneda embedding $C_Y \xrightarrow{\mathfrak{h}_Y} \mathcal{M}_k(Y)$ of the category C_Y into the category of presheaves of k-modules on C_Y is a universal ∂^*-functor.*

Proof. The fact follows from 4.6.4(b). ■

4.6.5. Proposition. *Let $(C_X,\mathfrak{E}_X)$ be a k-linear right exact category.*

If $T = (T_i,\mathfrak{d}_i \mid i\ge 0)$ is a universal k-linear ∂^*-functor from the abelian category $Sh_k(C_X,\mathfrak{E}_X)$ of sheaves of k-modules on $(C_X,\mathfrak{E}_X)$ to a k-linear category C_Y, then the composition $T\circ\mathfrak{j}_X^* = (T_i\circ\mathfrak{j}_X^*, \mathfrak{d}_i\mathfrak{j}_X^* \mid i\ge 0)$ of the ∂^*-functor T with the canonical 'exact' embedding $(C_X,\mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^*} Sh_k(C_X,\mathfrak{E}_X)$ is a universal ∂^*-functor from $(C_X,\mathfrak{E}_X)$ to C_Y.

Proof. The assertion follows from 4.1 and the fact that the right exact structure on the category $Sh_k(X,\mathfrak{E}_X)$ of sheaves of k-modules on the presite $(C_X,\mathfrak{E}_X)$ by the canonical embedding $C_X \longrightarrow Sh_k(X,\mathfrak{E}_X)$ coincides with the class of all epimorphisms of the category $Sh_k(X,\mathfrak{E}_X)$. ■

There is also a k-linear version of Proposition 4.2.2:

4.6.6. Proposition. *Let $(C_X,\mathfrak{E}_X)$ be a svelte right exact k-linear category and C_Y a k-linear category with limits and colimits. Then the functor*

$$\partial_k^*\mathfrak{Un}(Sh_k(X,\mathfrak{E}_X),C_Y) \longrightarrow \partial_k^*\mathfrak{Un}((C_X,\mathfrak{E}_X),C_Y),$$
$$T = (T_i,\mathfrak{d}_i \mid i\ge 0) \longmapsto T\circ\mathfrak{j}_X^* = (T_i\circ\mathfrak{j}_X^*, \mathfrak{d}_i\mathfrak{j}_X^* \mid i\ge 0)$$

has a fully faithful right adjoint which establishes an equivalence between the category $\partial_k^*\mathfrak{Un}((C_X,\mathfrak{E}_X),C_Y)$ of universal k-linear ∂^*-functors from $(C_X,\mathfrak{E}_X)$ to C_Y and the full subcategory $\partial_k^*\mathfrak{Un}(Sh_k(X,\mathfrak{E}_X),C_Y)$ of universal k-linear ∂^*-functors from the abelian category $Sh_k(X,\mathfrak{E}_X)$ of sheaves of k-modules on the presite $(C_X,\mathfrak{E}_X)$ to the category C_Y, which is generated by universal k-linear ∂^*-functors $T = (T_i,\mathfrak{d}_i \mid i\ge 0)$ such that $(T_0\circ\mathfrak{j}_X^*)^\diamond\circ\mathfrak{q}_{X*} \simeq T_0$. Here $\mathcal{M}_k(X) \xrightarrow{\mathfrak{q}_{X*}} Sh_k(X,\mathfrak{E})$ is the sheafification functor.

Proof. For any category C_Y, we have a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{H}om_k^*(Sh_k(X,\mathfrak{E}_X),C_Y) & \longrightarrow & \mathcal{H}om_k^*((C_X,\mathfrak{E}_X),C_Y), \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}om_k(Sh_k(X,\mathfrak{E}_X),C_Y) & \longrightarrow & \mathcal{H}om_k(C_X,C_Y) \end{array} \qquad (1)$$

whose vertical arrows map ∂^*-functors to their zero components and the horizontal arrows are functors of composition with $C_X \xrightarrow{\mathfrak{j}_{X*}} Sh_k(X,\mathfrak{E}_X)$.

(a) If the category C_Y has kernels of arrows and limits if filtered diagrams, then it follows from 3.2 that the vertical arrows of the diagram (1) are continuous localizations, and their right adjoint functors assign to every functor $C_X \xrightarrow{F} C_Y$ the universal ∂^*-functor $\mathcal{S}^{\bullet}_{-}(F)$, whose zero component is F. Thus the diagram (1) yields a commutative diagram of functors

$$\begin{array}{ccc} \partial_k^*\mathfrak{Un}(Sh_k(X,\mathfrak{E}_X),C_Y) & \longrightarrow & \partial_k^*\mathfrak{Un}((C_X,\mathfrak{E}_X),C_Y), \\ \wr\wr\big\downarrow & & \big\downarrow\wr\wr \\ \mathcal{H}om_k(Sh_k(X,\mathfrak{E}_X),C_Y) & \longrightarrow & \mathcal{H}om_k(C_X,C_Y) \end{array} \qquad (2)$$

whose vertical arrows are category equivalences.

(b) If the category C_Y has colimits, then it follows from I.2.0.4.4 that the lower horizontal arrow of the diagram (2),

$$\mathcal{H}om_k(Sh_k(X,\mathfrak{E}_X),C_Y) \xrightarrow{\widetilde{\mathfrak{j}_X^*}} \mathcal{H}om_k(C_X,C_Y), \quad \mathcal{G} \longmapsto \mathcal{G}\circ\mathfrak{j}_X^*,$$

is a localization functor having a (necessarily) fully faithful right adjoint. The latter assigns to every functor $C_X \xrightarrow{\mathcal{F}} C_Y$ the composition of the embedding $Sh_k(X,\mathfrak{E}_X) \longrightarrow \mathcal{M}_k(X)$ (– a right adjoint to the sheafification functor) and the functor $\mathcal{M}_k(X) \xrightarrow{\mathcal{F}^\star} C_Y$, which preserves colimits and whose composition with the Yoneda embedding $C_X \xrightarrow{\mathfrak{h}_X} \mathcal{M}_k(X)$ coincides with $\mathcal{F}$.

(c) The assertion follows from (a) and (b). ■

5. The dual picture: ∂-functors and universal ∂-functors.

Let $(C_X,\mathfrak{I}_X)$ be a left exact category, which means by definition that $(C_X^{op},\mathfrak{I}_X^{op})$ is a right exact category. A ∂-functor on $(C_X,\mathfrak{I}_X)$ is the data which becomes a ∂^*-functor in the dual right exact category. A ∂-functor on $(C_X,\mathfrak{I}_X)$ is *universal* if its dualization is a universal ∂^*-functor. We leave to the reader the reformulation in the context of ∂-functors of all notions and facts about ∂^*-functors. Below, there are two versions – non-linear and linear, of a fundamental example of a universal ∂-functor.

5.1. Example: $Ext^\bullet(-,L)$. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object. For any $L \in ObC_X$, we have the corresponding representable functor

$$C_X^{op} \xrightarrow{h_X(L)} Sets, \quad M \longmapsto C_X^{op}(L, M) = C_X(M, L).$$

Therefore, by (the dual version of) 3.2, there exists a universal ∂-functor $Ext_X^\bullet(-,L) = (Ext_X^i(L)|\ i \geq 0)$ from the left exact category $(C_X, \mathfrak{E}_X)^{op} = (C_X^{op}, \mathfrak{E}_X^{op})$ to the category $Sets$, whose zero component, $Ext_X^0(-,L)$, coincides with $h_X(L) = C_X(-,L)$.

5.2. The functors $\mathcal{E}xt_X^\bullet(-,\mathcal{L})$. Suppose that the category C_X is k-linear. Then for any $\mathcal{L} \in ObC_X$, the functor $h_X(\mathcal{L})$ factors through the category $k-mod$ (that is through the forgetful functor $k-mod \longrightarrow Sets$). Therefore, by 3.2, there exists a universal ∂-functor $\mathcal{E}xt_X^\bullet(-,\mathcal{L}) = (\mathcal{E}xt_X^i(-,\mathcal{L})|\ i \geq 0)$, whose zero component, $\mathcal{E}xt_X^0(-,\mathcal{L})$, coincides with the presheaf of k-modules $\widehat{\mathcal{L}} = C_X(-,\mathcal{L})$.

6. Universal ∂^*-functors and 'exactness'.

6.1. The properties $(CE5^*)$ **and** $(CE5)$. Let $(C_X, \mathfrak{E}_X)$ be a right exact category.

6.1.1. We say that it satisfies $(CE5^*)$, if the category C_X has limits of filtered diagrams and the limit of a filtered diagram of deflations is a deflation.

Dually, a left exact category $(C_Y, \mathfrak{I}_Y)$ satisfies (CE5), if the category C_Y has colimits of filtered diagrams and colimits of filtered diagrams of inflations are inflations.

6.1.2. In terms of conflations. If $(C_X, \mathfrak{E}_X)$ is a right exact category with initial objects, then the property $(CE5^*)$ is equivalent to the requirement that limits of filtered diagrams of conflations exist and are conflations. In this case, one can formulate an analog of the property (CE5) for right exact categories by replacing 'limits' with 'colimits': colimits of filtered system of conflations exist and are conflations.

6.1.2.1. In particular, if C_X is a svelte abelian category with the canonical exact structure, then the property $(CE5^*)$ is equivalent to the Grothendieck's property $(AB5^*)$.

Dually, for an abelian category, the property (CE5) is equivalent to the Grothendieck's property (AB5) (see [Gr, 1.5]).

6.1.3. The property $(CE5)$ holds for Grothendieck toposes.

6.1.4. Remark. Apparently, the property (CE5) does not make much sense for a right exact category, unless the kernels of deflations form a left exact structure. Therefore, in what follows, we use $(CE5^*)$ for right exact categories and the dual property $(CE5)$ for left exact categories.

6.2. Weakly right 'semi-exact' functors. Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects. A functor $C_X \xrightarrow{F} C_Y$ will be called a *right* *weakly 'semi-exact'* functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$, if it maps initial objects to initial objects and, for any deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$, there exists the kernel $Ker(F(\mathfrak{e}))$ of the morphism $F(\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L})$ and the canonical morphism $F(Ker(\mathfrak{e})) \longrightarrow Ker(F(\mathfrak{e}))$ is a deflation.

6.2.1. The dual notion. The notion of a weakly right 'semi-exact' functor is dualized in an obvious way: a functor $C_{\mathfrak{X}} \xrightarrow{F} C_{\mathfrak{Y}}$ between categories with final objects is called a *weakly left 'semi-exact'* functor from a left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to a left exact category $(C_{\mathfrak{Y}}, \mathfrak{I}_{\mathfrak{Y}})$, if the opposite functor $C_{\mathfrak{X}}^{op} \xrightarrow{F^{op}} C_{\mathfrak{Y}}^{op}$ is a weakly right 'semi-exact' functor from $(C_{\mathfrak{X}}^{op}, \mathfrak{I}_{\mathfrak{X}}^{op})$ to $(C_{\mathfrak{Y}}^{op}, \mathfrak{I}_{\mathfrak{Y}}^{op})$.

6.2.2. Note. Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be exact categories and $C_X \xrightarrow{F} C_Y$ a functor which is both left and right weakly 'semi-exact'. Then, for any deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$, we have a decomposition

$$F(Ker(\mathfrak{e})) \longrightarrow Ker(F(\mathfrak{e})) \xrightarrow{\mathfrak{k}(F(\mathfrak{e}))} F(\mathcal{M}) \xrightarrow{\mathfrak{c}(F(\mathfrak{e}))} Coim(F(\mathfrak{e})) \longrightarrow F(\mathcal{L}) \qquad (1)$$

of the morphism $F(\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L})$, in which the first arrow on the left is a deflation, because F is weakly right 'semi-exact' and $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ is a deflation, and the last arrow is an inflation, because F is weakly left 'semi-exact' and $Ker(\mathfrak{e}) \xrightarrow{\mathfrak{k}(\mathfrak{e})} \mathcal{M}$ is an inflation.

The middle pair of arrows form (the essential part of) a short exact sequence in the usual sense: $F(\mathcal{M}) \xrightarrow{\mathfrak{c}(F(\mathfrak{e}))} Coim(F(\mathfrak{e})$ is the cokernel of $Ker(F(\mathfrak{e})) \xrightarrow{\mathfrak{k}(F(\mathfrak{e}))} F(\mathcal{M})$ and $Ker(F(\mathfrak{e})) \xrightarrow{\mathfrak{k}(F(\mathfrak{e}))} F(\mathcal{M})$ is the kernel of $F(\mathcal{M}) \xrightarrow{\mathfrak{c}(F(\mathfrak{e}))} Coim(F(\mathfrak{e})$. So that if all strict epimorphisms in C_Y are deflations (or all strict monomorphism are inflations), then $F(\mathcal{M}) \xrightarrow{\mathfrak{c}(F(\mathfrak{e}))} Coim(F(\mathfrak{e})$ is a deflation and $Ker(F(\mathfrak{e})) \xrightarrow{\mathfrak{k}(F(\mathfrak{e}))} F(\mathcal{M})$ is an inflation.

6.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects; and let F be a weakly right 'semi-exact' functor $(C_X, \mathfrak{E}_X) \longrightarrow (C_Y, \mathfrak{E}_Y)$ such that S_-F exists. Suppose that $(C_Y, \mathfrak{E}_Y)$ satisfies $(CE5^*)$.*

Then, for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ in $(C_X, \mathfrak{E}_X)$, the sequence

$$S_-F(N) \xrightarrow{S_-F(\mathfrak{j})} S_-F(M) \xrightarrow{S_-F(\mathfrak{e})} S_-F(L) \xrightarrow{\mathfrak{d}_0(E)} F(N) \xrightarrow{F(\mathfrak{j})} F(M) \qquad (1)$$

is 'exact'. In particular, S_-F is a weakly right 'semi-exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.

Proof. (a) Let $(C_X, \mathfrak{E}_X) \xrightarrow{F} (C_Y, \mathfrak{E}_Y)$ be a weakly right 'semi-exact' functor such that its derived functor S_-F exists. The claim is that, for any conflation

$$E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$$

of the right exact category $(C_X, \mathfrak{E}_X)$, the canonical morphism

$$S_-F(L) \xrightarrow{\widetilde{\mathfrak{d}}_0(E)} Ker(F(\mathfrak{j}))$$

is a deflation.

(a1) Let $M \xrightarrow{\mathfrak{e}} L$ and $M' \xrightarrow{\mathfrak{e}'} L$ be deflations of an object L of the category C_X, and let

$$\begin{array}{ccc} M' & \xrightarrow{f} & M \\ \mathfrak{e}' \searrow & & \swarrow \mathfrak{e} \\ & L & \end{array}$$

be a commutative diagram (– a morphism of deflations). This diagram extends to a morphism of the corresponding conflations

$$\begin{array}{ccccc} N' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & L \\ f' \downarrow & cart & \downarrow f & & \downarrow id_L \\ N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L \end{array} \qquad (2)$$

Since $\mathfrak{e}' = \mathfrak{e} \circ f$, it follows from I.3.3.4.1 that the left square of (2) is cartesian.

For an arbitrary functor $C_X \xrightarrow{F} C_Y$, the diagram (2) gives rise to the commutative diagram

$$\begin{array}{ccccccc} Ker(F(\mathfrak{j}')) & \xrightarrow{\mathfrak{k}'} & F(N') & \xrightarrow{F(\mathfrak{j}')} & F(M') & \xrightarrow{F(\mathfrak{e}')} & F(L) \\ \widetilde{\gamma} \downarrow & cart & \gamma \downarrow & & \downarrow id & & \downarrow id \\ Ker(\alpha) & \xrightarrow{\mathfrak{k}(\alpha)} & \mathcal{N} & \xrightarrow{\alpha} & F(M') & \xrightarrow{F(\mathfrak{e}')} & F(L) \\ \widetilde{\phi} \downarrow \wr & & \phi \downarrow & cart & \downarrow F(f) & & \downarrow id \\ Ker(F(\mathfrak{j})) & \xrightarrow{\mathfrak{k}} & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L) \end{array} \qquad (3)$$

where the lower middle square is cartesian, which implies (by I.3.3.3) that the left lower vertical arrow $Ker(\alpha) \xrightarrow{\widetilde{\phi}} Ker(F(\mathfrak{j}))$ is an isomorphism. The morphism $F(N') \xrightarrow{\gamma} \mathcal{N}$ is uniquely determined by the equalities $\phi \circ \gamma = F(f')$ and $F(\mathfrak{j}') = \alpha \circ \gamma$; and the left upper square is cartesian due to the latter equality (see I.3.3.4.1).

(a2) Suppose now that the morphism $M' \xrightarrow{f} M$ in the diagram (2) (and (3)) is a deflation and the functor $C_X \xrightarrow{F} C_Y$ is right 'semi-exact'. Since the left square of the diagram (2) is cartesian, the morphism $F(N') \xrightarrow{\gamma} \mathcal{N}$ in (3) is a deflation. Therefore, since the left upper square in (3) is cartesian, the vertical arrow $Ker(F(\mathfrak{j}')) \xrightarrow{\widetilde{\gamma}} Ker(\alpha)$ is a deflation; or, what is the same, the canonical morphism $Ker(F(\mathfrak{j}')) \longrightarrow Ker(F(\mathfrak{j}))$ (equal to the composition $\widetilde{\phi} \circ \widetilde{\gamma}$) is a deflation.

(a3) Notice that the object $S_-F(L)$ is isomorphic to the limit of $Ker(F(\mathfrak{k}(\mathfrak{e}')))$, where $\mathfrak{e}'$ runs through the (filtered) diagram of *refinements* of the deflation $M \xrightarrow{\mathfrak{e}} L$. That is

$$S_-F(L) = \lim_{\mathfrak{M} \xrightarrow{\mathfrak{t}} M} Ker(F(\mathfrak{k}(\mathfrak{t} \circ \mathfrak{e}))),$$

where $\mathfrak{M} \xrightarrow{\mathfrak{t}} M$ runs through the deflations of M (and morphisms of this diagram are also deflations). Thus, the canonical morphism

$$S_-F(L) \xrightarrow{\widetilde{\mathfrak{d}}_0(E)} Ker(F(\mathfrak{j}))$$

is the limit of a filtered diagram of deflations. Therefore, since, by hypothesis, the limit of filtered diagram of deflations is a deflation, the morphism $S_-F(L) \xrightarrow{\widetilde{\mathfrak{d}}_0(E)} Ker(F(\mathfrak{j}))$ is a deflation.

(b) For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of the right exact category $(C_X, \mathfrak{E}_X)$, the canonical morphism $S_-F(M) \longrightarrow Ker(\mathfrak{d}_0(E))$ is a deflation.

In fact, let

$$\begin{array}{ccccc}
N'' & \xrightarrow{\mathfrak{j}''} & M'' & \xrightarrow{\mathfrak{e}''} & M \\
\mathfrak{k}\downarrow & & \downarrow id & & \downarrow \mathfrak{e} \\
\widetilde{N}' & \xrightarrow{\widetilde{\mathfrak{j}}'} & M'' & \xrightarrow{\widetilde{\mathfrak{e}}'} & L \\
\widetilde{\mathfrak{t}}'\downarrow & cart & \downarrow \mathfrak{t}' & & \downarrow id \\
N' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & L \\
\widetilde{\mathfrak{t}}\downarrow & cart & \downarrow \mathfrak{t} & & \downarrow id \\
N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L
\end{array} \qquad (4)$$

be a commutative diagram whose rows are conflations and the central vertical arrows $M' \xrightarrow{\mathfrak{t}} M$ and $M'' \xrightarrow{\mathfrak{t}'} M'$ are deflations. By I.4.4.5, the two lower left squares of the diagram (4) are cartesian. In particular, the left vertical arrows $N'' \xrightarrow{\widetilde{\mathfrak{t}}'} N'$ and $N' \xrightarrow{\widetilde{\mathfrak{t}}} N$ are deflations. It follows from I.3.3.4.2(b) that the upper two arrows of the left column of the diagram (4) form a conflation; i.e. $N'' \xrightarrow{\mathfrak{k}} N'$ is the kernel of $\widetilde{\mathfrak{t}}'$.

Applying the functor $C_X \xrightarrow{F} C_Y$ to the diagram (4) and taking kernels of the horizontal arrows yields a commutative diagram

$$\begin{array}{ccccccc}
Ker(F(\mathfrak{j}'')) & \longrightarrow & F(N'') & \xrightarrow{F(\mathfrak{j}'')} & F(M'') & \xrightarrow{F(\mathfrak{e}'')} & F(M) \\
\downarrow & cart & {\scriptstyle F(\mathfrak{k})}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle F(\mathfrak{e})} \\
Ker(F(\widetilde{\mathfrak{j}}')) & \longrightarrow & F(\widetilde{N}') & \xrightarrow{F(\widetilde{\mathfrak{j}}')} & F(M'') & \xrightarrow{F(\widetilde{\mathfrak{e}}')} & F(L) \\
\downarrow & & {\scriptstyle F(\widetilde{\mathfrak{t}}')}\downarrow & & \downarrow{\scriptstyle F(\mathfrak{t}')} & & \downarrow{\scriptstyle id} \\
Ker(F(\mathfrak{j}')) & \longrightarrow & F(N') & \xrightarrow{F(\mathfrak{j}')} & F(M') & \xrightarrow{F(\mathfrak{e}')} & F(L) \\
\downarrow & & {\scriptstyle F(\widetilde{\mathfrak{t}})}\downarrow & & \downarrow{\scriptstyle F(\mathfrak{t})} & & \downarrow{\scriptstyle id} \\
Ker(F(\mathfrak{j})) & \longrightarrow & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L)
\end{array} \qquad (5)$$

whose left upper square is cartesian.

Since the functor $C_X \xrightarrow{F} C_Y$ is weakly right 'semi-exact', the diagram (5) is decomposed into the diagram

$$\begin{array}{ccccccc}
Ker(F(\mathfrak{j}'')) & \longrightarrow & F(N'') & & & & \\
{\scriptstyle \gamma_1}\downarrow & cart & {\scriptstyle \gamma_2}\downarrow & & & & \\
Ker(\mathfrak{s}) & \longrightarrow & Ker(F(\widetilde{\mathfrak{t}}')) & \xrightarrow{F(\mathfrak{j}'')} & F(M'') & \xrightarrow{F(\mathfrak{e}'')} & F(M) \\
{\scriptstyle \mathfrak{k}(\mathfrak{s})}\downarrow & cart & {\scriptstyle \mathfrak{k}'}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle F(\mathfrak{e})} \\
Ker(F(\widetilde{\mathfrak{j}}')) & \longrightarrow & F(\widetilde{N}') & \xrightarrow{F(\widetilde{\mathfrak{j}}')} & F(M'') & \xrightarrow{F(\widetilde{\mathfrak{e}}')} & F(L) \\
{\scriptstyle \mathfrak{s}}\downarrow & & {\scriptstyle F(\widetilde{\mathfrak{t}}')}\downarrow & & \downarrow{\scriptstyle F(\mathfrak{t}')} & & \downarrow{\scriptstyle id} \\
Ker(F(\mathfrak{j}')) & \longrightarrow & F(N') & \xrightarrow{F(\mathfrak{j}')} & F(M') & \xrightarrow{F(\mathfrak{e}')} & F(L) \\
\downarrow & & {\scriptstyle F(\widetilde{\mathfrak{t}})}\downarrow & & \downarrow{\scriptstyle F(\mathfrak{t})} & & \downarrow{\scriptstyle id} \\
Ker(F(\mathfrak{j})) & \longrightarrow & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L)
\end{array} \qquad (6)$$

where γ_1, γ_2 are deflations, $\mathfrak{k}'$ is the kernel (morphism) of $F(\widetilde{\mathfrak{t}}')$; $F(\mathfrak{j}') \circ \gamma_2 = F(\mathfrak{j}'')$, and $\mathfrak{k}' \circ \gamma_2 = F(\mathfrak{k})$. It follows that the two upper left squares of (6) are cartesian. The left column of the diagram (6) induces, via passing to limit, the sequence of arrows

$$S_-F(M) \xrightarrow{\widetilde{\gamma}} Ker(\mathfrak{d}_0(E)) \xrightarrow{\mathfrak{k}_0} S_-F(L) \xrightarrow{\sigma} Ker(F(\mathfrak{j})) \xrightarrow{\mathfrak{k}(F(\mathfrak{j}))} F(N)$$

where $\mathfrak{k}(F(\mathfrak{j})) \circ \sigma = \mathfrak{d}_0(E)$, $\mathfrak{k}_0 \circ \widetilde{\gamma} = S_-F(\mathfrak{e})$; σ is a deflation by (a) above, and $\widetilde{\gamma}$ is a deflation by hypothesis, because it is a filtered limit of deflations. ∎

7. 'Exact' ∂^*-functors and universal ∂^*-functors.

Fix right exact categories $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$, both with initial objects.

A ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y is called *'exact'*, if, for every conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ in $(C_X, \mathfrak{E}_X)$, the complex

$$\ldots \xrightarrow{T_2(\mathfrak{e})} T_2(L) \xrightarrow{\mathfrak{d}_1(E)} T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_i(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is 'exact'.

7.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with inital objects. Suppose that $(C_Y, \mathfrak{E}_Y)$ has the property $(CE5^*)$. Let $T = (T_i|\ i \geq 0)$ be a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. If the functor T_0 is weakly right 'exact', then the universal ∂^*-functor T is 'exact'.*

Proof. If T_0 is weakly right 'exact', then, by 6.3, the functor $T_1 \simeq S_-(T_0)$ is weakly right 'exact' and, for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, the sequence

$$T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_1(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is 'exact'. Since $T_{n+1} = S_-(T_n)$, the assertion follows from 6.3 by induction. ■

7.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. For each object L of the category C_X, the ∂-functor $Ext^\bullet_X(-, L) = (Ext^i_X(-, L) \mid i \geq 0)$ is 'exact'.*

Suppose that the category C_X is k-linear. Then, for each $L \in ObC_X$, the ∂-functor $\mathcal{E}xt^\bullet_X(-, L) = (\mathcal{E}xt^i_X(-, L) \mid i \geq 0)$ is 'exact'.

Proof. In fact, the ∂-functor $Ext^\bullet_X(-, L)$ is universal by definition (see 5.1), and the functor $Ext^0_X(-, L) = C_X(-, L)$ is left exact. In particular, it is left 'exact'.

If C_X is a k-linear category, then the universal ∂-functors $\mathcal{E}xt^\bullet_X(-, L)$, $L \in ObC_X$, with the values in the category of k-modules (see 5.2) are 'exact' by a similar reason. ■

8. Universal problems for universal ∂^*- and ∂-functors.

8.1. The categories of homological and cohomological functors.

8.1.0. Morphisms of ∂^*- functors. Let T be a universal ∂^*- functor from a svelte right exact category $(C_X, \mathfrak{E}_X)$ to a category C_Y and T' a universal ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Z. A morphism from T to T' is a pair (F, ϕ), where $C_X \xrightarrow{F} C_Z$ is a functor which preserves filtered limits and ϕ is a ∂^*-functor isomorphism $F \circ T \xrightarrow{\sim} T'$. It follows from this isomorphism and the fact that higher components of ∂^*-functors map initial objects to initial objects that the functor F maps initial objects to initial objects.

If (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$. This defines a category of ∂^*-functors from $(C_X, \mathfrak{E}_X)$ which we denote by $\partial^*\widetilde{\mathfrak{Un}}(C_X, \mathfrak{E}_X)$.

8.1.0.1. Note. The category $\partial^*\widetilde{\mathfrak{Un}}((C_X, \mathfrak{E}_X), C_Y)$ of ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to a category C_Y (defined in 3.0.1) is identified with the subcategory of the category $\partial^*\widetilde{\mathfrak{Un}}(C_X, \mathfrak{E}_X)$ formed by all morphisms of the form (Id_{C_Y}, ϕ).

8.1.1. The category $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$. We denote by $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ the full subcategory of the category $\partial^*\widetilde{\mathfrak{Un}}(C_X, \mathfrak{E}_X)$ generated by universal ∂^*-functors from the right exact category $(C_X, \mathfrak{E}_X)$ whose zero components (hence all components) map initial objects to initial objects.

8.1.2. The category $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$. We denote by $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ the subcategory of $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ whose objects are ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to categories with limits (and initial objects) and morphisms are pairs (F, ϕ) such that the functor F preserves limits.

8.1.3. The categories $\partial\widetilde{\mathfrak{Un}}(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ and $\partial\mathfrak{Un}(X, \mathfrak{E}_X)$. Dually, for a svelte left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ with a final object, we denote by $\partial\widetilde{\mathfrak{Un}}(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal ∂-functors from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to categories with final object. Given two universal ∂-functors T and T' from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to respectively C_Y and C_Z, a morphism from T to T' is a pair (F, ψ), where F is a functor from C_Y to C_Z preserving filtered colimits and ψ is a functor isomorphism $T' \xrightarrow{\sim} F \circ T$ (which implies that the functor F maps final objects to final objects). The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

8.1.3.1. We denote by $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the full subcategory of the category $\partial\widetilde{\mathfrak{Un}}(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ generated by universal ∂-functors from the right exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ whose zero components (hence all components) map final objects to final objects.

8.1.4. The category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$. We denote by $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ whose objects are ∂-functors with values in categories with colimits and final objects and morphisms are pairs (F, ψ) such that the functor F preserves colimits.

8.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with initial objects and $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ a svelte left exact category with final objects. The categories $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$, $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$, $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$, and $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. (a) We start with the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$. Consider the Yoneda embedding

$$C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}^*} C_X^*, \quad M \longmapsto \widehat{M} = C_{\mathfrak{X}}(-, M).$$

We denote by $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ the universal ∂-functor from the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to the category $C_{\mathfrak{X}}^*$ whose zero component coincides with $C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}^*} C_X^*$.

The claim is that $Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ is an initial object of the category $\partial\mathfrak{Un}^\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$.

In fact, let C_Y be a category with colimits of small diagrams. By I.2.0.2(a), the composition with the Yoneda embedding $C_\mathfrak{X} \xrightarrow{h^*_\mathfrak{X}} C^*_\mathfrak{X}$ is an equivalence between the category $\mathfrak{Hom}(C^*_\mathfrak{X}, C_Y)$ of preserving colimits functors $C^*_\mathfrak{X} \longrightarrow C_Y$ and the category $\mathcal{H}om(C_\mathfrak{X}, C_Y)$ of functors from $C_\mathfrak{X}$ to C_Y. Let $C_\mathfrak{X} \xrightarrow{F} C_Y$ be an arbitrary functor and $C^*_\mathfrak{X} \xrightarrow{F^*} C_Y$ its preserving colimits extension. By definition,

$$S_+F(L) = \mathrm{colim}(Cok(F(M \longrightarrow Cok(\mathfrak{j})),$$

where $L \xrightarrow{\mathfrak{j}} M$ runs through inflations of the object L. Since the functor F^* preserves colimits, it follows from (the dual version of) 3.4(a) that $F^* \circ Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ is a universal ∂-functor whose zero component is $F^* \circ Ext^0_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}} = F^* \circ h^*_\mathfrak{X} = F$. Therefore, by (the dual version of the argument of) 3.2, the universal ∂-functor $F^* \circ Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ is isomorphic to $S^\bullet_+F$. This shows that $Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ is an initial object of the category $\partial\mathfrak{Un}^\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$.

(b) Let $C_{\mathfrak{X}_\mathfrak{s}}$ denote the smallest strictly full subcategory of the category $C^*_\mathfrak{X}$ containing all presheaves $Ext^n_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}(L)$, $L \in ObC_\mathfrak{X}$, $n \geq 0$. The claim is that the corestriction of the ∂-functor $Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ to the subcategory $C_{\mathfrak{X}_\mathfrak{s}}$ is an initial object of the category $\partial\mathfrak{Un}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$.

Indeed, let C_Y be a category with a final object $\mathfrak{y}$ and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a universal ∂-functor from the left exact category $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ to C_Y. Set $C_{Y^o} = C^{op}_Y$ and consider the "reduced" Yoneda functor

$$C^{op}_Y = C_{Y^o} \xrightarrow{h^\circledast_{Y^o}} C^\circledast_{Y^o} \stackrel{\mathrm{def}}{=} \mathfrak{y}^\vee \backslash C^\wedge_{Y^o} \tag{1}$$

(see I.2.0.2(b)). Here $\mathfrak{y}^\vee = C_{Y^o}(-,\mathfrak{y}) = C_Y(\mathfrak{y},-)$. The "reduced" Yoneda embedding preserves limits and maps initial objects to initial objects. So that the opposite functor,

$$C^{op}_{Y^o} = C_Y \xrightarrow{(h^\circledast_{Y^o})^{op}} (C^\circledast_{Y^o})^{op} \stackrel{\mathrm{def}}{=} (\mathfrak{y}^\vee \backslash C^\wedge_{Y^o})^{op}, \tag{2}$$

preserves colimits and maps final objects to final objects.

Therefore, by 3.4, the composition $(h^\circledast_{Y^o})^{op} \circ T$ is a universal ∂-functor from the left exact category $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ to the category $(C^\circledast_{Y^o})^{op}$. By (a) above, the ∂-functor $(h^\circledast_{Y^o})^{op} \circ T$ is the composition of the universal ∂-functor $Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ from the left exact category $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ to $C^*_\mathfrak{X}$ and the unique up to isomorphism functor $C^*_\mathfrak{X} \xrightarrow{G} (C^\circledast_{Y^o})^{op}$ which preserves colimits and satisfies the equation $G \circ h_\mathfrak{X} = (h^\circledast_{Y^o})^{op} \circ T_0$. Since the functor $(h^\circledast_{Y^o})^{op}$ is fully faithful, this implies that the universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ is isomorphic to the composition of the corestriction of $Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ to the subcategory $C_{\mathfrak{X}_\mathfrak{s}}$ and a unique functor $C_{\mathfrak{X}_\mathfrak{s}} \xrightarrow{G_\mathfrak{s}} C_Y$ such that the composition $h^*_{Y^o} \circ G_\mathfrak{s}$ coincides with the restriction of the functor G to the subcategory $C_{\mathfrak{X}_\mathfrak{s}}$.

(c) The assertions about ∂^*-functors are obtained via dualization. Essential details are as follows. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We take the category $C^*_{X^o}$ of non-trivial functors $C_X \longrightarrow Sets$ (interpreted as presheaves of sets on $C_{X^o} = C_X^{op}$) and the dual to the Yoneda functor:

$$C^{op}_{X^o} = C_X \xrightarrow{(h^*_{X^o})^{op}} (C^*_{X^o})^{op}, \quad M \longmapsto C_X(M, -).$$

Let $Ext^\bullet_{(X,\mathfrak{E}_X)}$ denote the universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C^*_{X^o})^{op}$ such that $Ext^0_{(X,\mathfrak{E}_X)} = (h^*_{X^o})^{op}$. Let C_Y be a category with limits and initial objects. By the dual version of I.2.0.2, the composition with the functor $(h^*_{X^o})^{op}$ gives a category equivalence between $\mathcal{H}om(C_X, C_Y)$ and the category $\mathfrak{Hom}^{\mathfrak{c}}(C^*_{X^o}, C_Y)$ of functors $(C^*_{X^o})^{op} \longrightarrow C_Y$ which preserve limits. Let F be a functor $C_X \longrightarrow C_Y$ and $F^{\mathfrak{c}}$ the corresponding functor from $(C^*_{X^o})^{op}$ to C_Y. Since the functor $F^{\mathfrak{c}}$ preserves limits, it follows from 3.4 (a), that the composition $F^{\mathfrak{c}} \circ Ext^\bullet_{(X,\mathfrak{E}_X)}$ is a universal ∂^*-functor. Its zero component, $F^{\mathfrak{c}} \circ Ext^0_{(X,\mathfrak{E}_X)} = F^{\mathfrak{c}} \circ h^*_{X^o}$, coincides with the functor F. Therefore, by 3.2, the universal ∂^*-functor $F^{\mathfrak{c}} \circ Ext^\bullet_{(X,\mathfrak{E}_X)}$ is isomorphic to $S^\bullet_- F$. This shows that $Ext^\bullet_{(X,\mathfrak{E}_X)}$ is an initial object of the category $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$.

(d) It follows from (b) (by duality) that the corestriction of the ∂^*-functor $Ext^\bullet_{(X,\mathfrak{E}_X)}$ to the smallest subcategory of the category $C^*_{X^o}$ containing all representable functors and closed under the endofunctor S_- (that is the full subcategory of $C^*_{X^o}$ generated by the functors $Ext^n_{(X,\mathfrak{E}_X)}(L)$, $L \in ObC_X$, $n \geq 0$) is an initial object of the category $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ of universal ∂^*-functors. ■

8.2.1. The categories $\partial\mathfrak{Un}^{\circledast}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ and $\partial\mathfrak{Un}_{\circledast}(X, \mathfrak{E}_X)$. Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ be a svelte left exact category. We denote by $\partial\mathfrak{Un}^{\circledast}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the subcategory of the category $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ whose objects are ∂-functors with values in categories with colimits having both final and initial objects, and morphisms are pairs (F, ψ) such that the functor F (maps final objects to final objects and) has a right adjoint. The latter implies that the functor F maps initial objects to initial objects. Since any functor having a right adjoint preserve colimits, the category $\partial\mathfrak{Un}^{\circledast}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is a subcategory of the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$.

Dually, given a svelte right exact category $(C_X, \mathfrak{E}_X)$, we denote by $\partial\mathfrak{Un}_{\circledast}(X, \mathfrak{E}_X)$ the subcategory of the category $\partial\mathfrak{Un}(X, \mathfrak{E}_X)$ whose with values in categories with limits having both final and initial objects, and morphisms are pairs (F, ψ) such that the functor F maps initial objects to initial objects and has a left adjoint. The latter implies that the functor F maps final objects to final objects.

8.2.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with initial objects and $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ a svelte left exact category with final objects. The categories $\partial^*\mathfrak{Un}_{\circledast}(X, \mathfrak{E}_X)$ and $\partial\mathfrak{Un}^{\circledast}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. (a) The canonical initial object of the category $\partial\mathfrak{Un}^{\circledast}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is the universal ∂-functor $\mathfrak{Ext}^\bullet_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ from the left exact category $(C_X, \mathfrak{E}_X)$ to the category $C^\wedge_{\mathfrak{X}}$ of presheaves of sets on the category $C_{\mathfrak{X}}$ whose zero component coincides with the Yoneda functor $C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} C^{\wedge}_{\mathfrak{X}}$. The proof of this fact is similar to the argument 8.2(a) and uses the assertion I.2.0.2(c).

(b) By duality, we obtain a canonical initial object of the category $\partial^*\mathfrak{Un}_{\circledast}(X, \mathfrak{E}_X)$. ∎

8.3. The k-linear version. Fix a right exact k-linear additive category $(C_X, \mathfrak{E}_X)$. Let $\partial^*_k\mathfrak{Un}(X, \mathfrak{E}_X)$ denote the category whose objects are universal k-linear ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to k-linear additive categories. Let T be a universal k-linear ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Y and $\widetilde{T}$ a universal k-linear ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Z. A morphism from T to T' is a pair (F, ϕ), where F is a k-linear functor from C_Y to C_Z which preserves limits of filtered diagrams and ϕ is a ∂^*-functor isomorphism $F \circ T \xrightarrow{\sim} T'$. If (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

We denote by $\partial^*_k\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ the subcategory of $\partial^*_k\mathfrak{Un}(X, \mathfrak{E}_X)$ whose objects are k-linear ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to *complete* (i.e. having limits of small diagrams) k-linear categories C_Y and morphisms are pairs (F, ϕ) such that the functor F preserves limits.

Dually, for a left exact additive k-linear category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, we denote by $\partial_k\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal k-linear ∂-functors from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to k-linear additive categories. Given two universal k-linear ∂-functors T and T' from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to respectively C_Y and C_Z, a morphism from T to T' is a pair (F, ψ), where F is a k-linear functor from C_Y to C_Z preserving filtered colimits and ψ is a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

We denote by $\partial_k\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial_k\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ whose objects are k-linear ∂-functors with values in cocomplete categories and morphisms are pairs (F, ψ) such that the functor F preserves colimits.

8.3.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte k-linear right exact category and $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ a svelte k-linear left exact category. The categories $\partial^*_k\mathfrak{Un}(X, \mathfrak{E}_X)$, $\partial^*_k\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$, $\partial_k\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$, and $\partial_k\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. The argument is similar to that of 8.2, except for we replace the category $C^*_{\mathfrak{X}}$ (resp. $C^*_{\mathfrak{X}^o}$) of non-trivial presheaves of sets on $C_{\mathfrak{X}}$ (resp. on $C^{op}_{\mathfrak{X}}$) by the category $\mathcal{M}_k(\mathfrak{X})$ (resp. $\mathcal{M}_k(\mathfrak{X}^o)$) of k-linear presheaves of k-modules on $C_{\mathfrak{X}}$ (resp. on $C^{op}_{\mathfrak{X}} = C_{\mathfrak{X}^o}$).

(a) The initial object of the category $\partial_k\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is the universal k-linear ∂-functor $\mathcal{E}xt^{\bullet}_{X,\mathfrak{I}_X}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to the category $\mathcal{M}_k(\mathfrak{X})$ of k-linear presheaves of k-modules on $C_{\mathfrak{X}}$ whose zero component is the Yoneda embedding $C_{\mathfrak{X}} \longrightarrow \mathcal{M}_k(\mathfrak{X})$, $L \longmapsto C_{\mathfrak{X}}(-, L)$.

(b) The initial object of the category $\partial_k\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is the corestriction of $\mathcal{E}xt^{\bullet}_{X,\mathfrak{I}_X}$ to the smallest additive strictly full subcategory of $\mathcal{M}_k(\mathfrak{X})$ which contains all presheaves $\mathcal{E}xt^n_{X,\mathfrak{I}_X}(L)$, $L \in ObC_{\mathfrak{X}}$, $n \geq 0$.

(c) The universal k-linear ∂^*-functor $\mathcal{E}xt^{\bullet}_{(X,\mathfrak{E}_X)}$ from the right exact k-linear category $(C_X, \mathfrak{E}_X)$ to the category $\mathcal{M}_k(X^o)$ of presheaves of k-modules on $C_{X^o} = C^{op}_X$ is an initial object of the category $\partial^*_k\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$.

(d) The corestriction of the ∂^*-functor $\mathcal{E}xt^{\bullet}_{(X,\mathfrak{E}_X)}$ to the smallest strictly full additive subcategory of $\mathcal{M}_k(X^o)$ spanned by the presheaves $\mathcal{E}xt^n_{(X,\mathfrak{E}_X)}(L)$, $L \in ObC_X$, $n \geq 0$, is an initial object of the category $\partial^*_k\mathfrak{Un}(X, \mathfrak{E}_X)$.

The argument is similar to that of 8.2. Details are left to the reader. ∎

9. Universal problems for universal 'exact' ∂^*- and ∂-functors.

9.0. The category $\partial^*\mathfrak{UEx}(X, \mathfrak{E}_X)$. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with initial objects. Let $\partial^*\mathfrak{UEx}(X, \mathfrak{E}_X)$ denote the category whose objects are universal 'exact' ∂^*-functors $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to right exact categories $(C_Y, \mathfrak{E}_Y)$ satisfying $(CE5^*)$ (see 6.1) such that the functor T_0 maps deflations to deflations. Let T be a universal 'exact' ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ and $\widetilde{T}$ a universal 'exact' ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to $(C_Z, \mathfrak{E}_Z)$. A morphism from T to T' is a pair (F, ϕ), where F is a functor from C_Y to C_Z which preserves filtered limits and conflations, and ϕ is an isomorphism of ∂^*-functors $F \circ T \xrightarrow{\sim} T'$. If (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

9.0.1. The category $\partial^*\mathfrak{UEx}_\mathfrak{c}(X, \mathfrak{E}_X)$. We denote by $\partial^*\mathfrak{UEx}_\mathfrak{c}(X, \mathfrak{E}_X)$ the subcategory of $\partial^*\mathfrak{UEx}(X, \mathfrak{E}_X)$ whose objects are ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to *complete* right exact categories $(C_Y, \mathfrak{E}_Y)$ satisfying $(CE5^*)$ and morphisms are pairs (F, ϕ) such that the functor F preserves limits.

Dually, for a left exact category $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ with a final object, we denote by $\partial\mathfrak{UEx}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ the category whose objects are universal 'exact' ∂-functors $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ to left exact categories satisfying $(CE5)$ such that the functor T_0 maps inflations to inflations. Given two universal 'exact' ∂-functors T and T' from $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ to respectively $(C_Y, \mathfrak{I}_Y)$ and $(C_Z, \mathfrak{I}_Z)$, a morphism from T to T' is a pair (F, ψ), where F is a functor from C_Y to C_Z preserving filtered colimits and conflations and ψ is a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

We denote by $\partial\mathfrak{UEx}^\mathfrak{c}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ the subcategory of $\partial\mathfrak{UEx}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ whose objects are ∂-functors with values in cocomplete left exact categories (with final objects) satisfying $(CE5)$ and morphisms are pairs (F, ψ) such that the functor F preserves colimits.

9.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with initial objects and $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ a svelte left exact category with final objects. The categories $\partial^*\mathfrak{UEx}(X, \mathfrak{E}_X)$, $\partial^*\mathfrak{UEx}_\mathfrak{c}(X, \mathfrak{E}_X)$, $\partial\mathfrak{UEx}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$, and $\partial\mathfrak{UEx}^\mathfrak{c}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ have initial objects.*

Proof. (a) The Yoneda embedding

$$C_\mathfrak{X} \xrightarrow{h^*_\mathfrak{X}} C^*_\mathfrak{X}, \quad L \longmapsto \widehat{L} = C_\mathfrak{X}(-, L)$$

is a fully faithful left exact functor. Therefore, it maps strict monomorphisms (in particular, inflations – arrows of $\mathfrak{I}_\mathfrak{X}$) to strict monomorphisms of $C^*_\mathfrak{X}$; and the latter are universally strict. We denote by $\mathfrak{I}^*_{\mathfrak{X}}$ the coarsest left exact structure on $C^*_{\mathfrak{X}}$ which contains $h^*_{\mathfrak{X}}(\mathfrak{I}_X)$ and is closed with respect to inductive colimits.

Since the functor $h^*_{\mathfrak{X}}$ is left exact, it is a left 'exact' functor from the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to the left exact category $(C^*_{\mathfrak{X}}, \mathfrak{I}^*_{\mathfrak{X}})$. Therefore, by (the dual version of) 7.1, the universal ∂-functor $Ext^\bullet_{X,\mathfrak{I}_X}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $C^*_{\mathfrak{X}}$ whose zero component is the Yoneda embedding $h_{\mathfrak{X}}$ is an 'exact' ∂-functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C^*_{\mathfrak{X}}, \mathfrak{I}^*_{\mathfrak{X}})$.

The claim is that the universal 'exact' ∂-functor $Ext^\bullet_{X,\mathfrak{I}_X}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C^*_{\mathfrak{X}}, \mathfrak{I}^*_{\mathfrak{X}})$ is an initial object of the category $\partial\mathfrak{U}\mathfrak{E}\mathfrak{x}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$.

Let $(C_Z, \mathfrak{I}_Z)$ be a left exact category such that the category C_Z is cocomplete, and let F be a left 'exact' functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C_Z, \mathfrak{I}_Z)$. Then the corresponding functor $C^*_{\mathfrak{X}} \xrightarrow{F^*} C_Z$ is an 'exact' functor from $(C^*_{\mathfrak{X}}, \mathfrak{I}^*_{\mathfrak{X}})$ to $(C_Z, \mathfrak{I}_Z)$.

Since the functor F^* is right exact, it suffices to show that F^* maps inflations to inflations, i.e. $\mathfrak{I}^*_{\mathfrak{X}}$ to $\mathfrak{I}_Z$. The arrows of $\mathfrak{I}^*_{\mathfrak{X}}$ are obtained from the class of (strict) monomorphisms $h_{\mathfrak{X}}(\mathfrak{I}_{\mathfrak{X}})$ via compositions, push-forwards and filtered colimits. The functor F^* preserves all colimits, in particular, it preserves push-forwards and (any functor preserves) compositions. Since $F = F^* \circ h^*_{\mathfrak{X}}$, the class of morphisms $F^*(h^*_{\mathfrak{X}}(\mathfrak{I}_{\mathfrak{X}}))$ coincides with the class of monomorphisms $F(\mathfrak{I}_{\mathfrak{X}})$. Therefore, it follows from the above description of $\mathfrak{I}^*_{\mathfrak{X}}$ (and the fact that F^* preserves colimits) that $F^*(\mathfrak{I}^*_{\mathfrak{X}})$ is contained in $\mathfrak{I}_Z$.

(b) The initial object of the category $\partial\mathfrak{U}\mathfrak{E}\mathfrak{x}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is the corestriction of the universal ∂-functor $\mathfrak{E}xt^\bullet_{X,\mathfrak{I}_X}$ to the smallest strictly full subcategory of $C_{\mathfrak{X}^{\circ}_{\mathfrak{I}}}$ which contains all presheaves $\mathfrak{E}xt^n_{X,\mathfrak{I}_X}(L)$, $L \in ObC_{\mathfrak{X}}$, $n \geq 0$.

(c) The universal ∂^*-functor $\mathfrak{E}xt^\bullet_{(X,\mathfrak{E}_X)}$ from the right exact category $(C_X, \mathfrak{E}_X)$ to the category $C^*_{X^o}$ of presheaves of sets on $C_{X^o} = C_X^{op}$ endowed with the coarsest right exact structure containing the image of $\mathfrak{E}_X$ is an initial object of the category $\partial^*\mathfrak{U}\mathfrak{E}\mathfrak{x}_{\mathfrak{c}}(X, \mathfrak{E}_X)$.

(d) The corestriction of the ∂^*-functor $\mathcal{E}xt^\bullet_{(X,\mathfrak{E}_X)}$ to the smallest strictly full subcategory of $C^*_{X^o}$ spanned by the presheaves $\mathcal{E}xt^n_{(X,\mathfrak{E}_X)}(L)$, $L \in ObC_X$, $n \geq 0$, is an initial object of the category $\partial^*\mathfrak{U}\mathfrak{E}\mathfrak{x}(X, \mathfrak{E}_X)$.

The argument is similar to that of 8.2. Details are left to the reader. ∎

9.2. The k-linear version. Fix a right exact k-linear category $(C_X, \mathfrak{E}_X)$. Let $\partial^*_k\mathfrak{U}\mathfrak{E}\mathfrak{x}(X, \mathfrak{E}_X)$ denote the category whose objects are universal 'exact' k-linear ∂^*-functors $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to right exact k-linear categories $(C_Y, \mathfrak{E}_Y)$ satisfying $(CE5^*)$ such that T_0 maps deflations to deflations. Let T be a universal 'exact' k-linear ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ and $\widetilde{T}$ a universal 'exact' k-linear ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to $(C_Z, \mathfrak{E}_Z)$. A morphism from T to T' is a pair (F, ϕ), where F is a k-linear functor from C_Y to C_Z which preserves filtered limits and conflations, and ϕ is an isomorphism of ∂^*-functors $F \circ T \xrightarrow{\sim} T'$. If (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

We denote by $\partial_k^*\mathfrak{UEx}_\mathfrak{c}(X,\mathfrak{E}_X)$ the subcategory of $\partial_k^*\mathfrak{UEx}(X,\mathfrak{E}_X)$ whose objects are ∂^*-functors from $(C_X,\mathfrak{E}_X)$ to *complete* right exact categories $(C_Y,\mathfrak{E}_Y)$ and morphisms are pairs (F,ϕ) such that the functor F preserves limits.

Dually, for a left exact k-linear category $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$, we denote by $\partial_k\mathfrak{UEx}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category whose objects are universal 'exact' k-linear ∂-functors $T=(T_i,\mathfrak{d}_i \mid i\geq 0)$ from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to k-linear left exact categories satisfying $(CE5)$ such that the functor T_0 maps inflations to inflations. Given two universal 'exact' k-linear ∂-functors T and T' from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to respectively $(C_Y,\mathfrak{I}_Y)$ and $(C_Z,\mathfrak{I}_Z)$, a morphism from T to T' is a pair (F,ψ), where F is a k-linear functor from C_Y to C_Z preserving filtered colimits and conflations and ψ is a functor isomorphism $T'\xrightarrow{\sim}F\circ T$. The composition is defined by

$$(F',\psi')\circ(F,\psi)=(F'\circ F,F'\psi\circ\psi').$$

We denote by $\partial_k\mathfrak{UEx}^\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the subcategory of $\partial_k\mathfrak{UEx}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ whose objects are k-linear ∂-functors with values in cocomplete left exact categories and morphisms are pairs (F,ψ) such that the functor F preserves colimits.

9.2.1. Proposition. *Let $(C_X,\mathfrak{E}_X)$ be a svelte k-linear right exact category and $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ a svelte k-linear left exact category. The defined above categories $\partial_k^*\mathfrak{UEx}(X,\mathfrak{E}_X)$, $\partial_k^*\mathfrak{UEx}_\mathfrak{c}(X,\mathfrak{E}_X)$, $\partial_k\mathfrak{UEx}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$, and $\partial_k\mathfrak{UEx}^\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ have initial objects.*

Proof. The argument is similar to that of 9.1, except for we replace the category $C_\mathfrak{X}^*$ (resp. $C_{\mathfrak{X}^o}^*$) of presheaves of sets on $C_\mathfrak{X}$ (resp. on $C_\mathfrak{X}^{op}$) by the category $\mathcal{M}_k(\mathfrak{X})$ (resp. $\mathcal{M}_k(\mathfrak{X}^o)$) of presheaves of k-modules on $C_\mathfrak{X}$ (resp. on $C_\mathfrak{X}^{op}=C_{\mathfrak{X}^o}$).

(a) For a svelte k-linear left exact category $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$, we denote by $\mathfrak{I}_{\mathfrak{X},k}$ the coarsest left exact structure on the category $\mathcal{M}_k(\mathfrak{X})$ of presheaves of k-modules on $C_\mathfrak{X}$ closed under inductive colimits and such that the Yoneda embedding $C_\mathfrak{X}\xrightarrow{h_{\mathfrak{X},k}}\mathcal{M}_k(\mathfrak{X})$ maps inflations to inflations (i.e. $\mathfrak{I}_\mathfrak{X}$ to $\mathfrak{I}_{\mathfrak{X},k}$) and is a left exact k-linear functor, hence it is a left 'exact' functor from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to $(\mathcal{M}_k(\mathfrak{X}),\mathfrak{I}_{\mathfrak{X},k})$. Therefore, by the k-linear version of 7.1, the universal functor $\mathcal{E}xt^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ whose zero component is the Yoneda embedding $h_{\mathfrak{X},k}$ is 'exact'.

If $(C_Z,\mathfrak{I}_Z)$ be a left exact k-linear category such that the category C_Z is cocomplete and F a left 'exact' k-linear functor from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to $(C_Z,\mathfrak{I}_Z)$, then the corresponding continuous functor $C_\mathfrak{X}^*\xrightarrow{F^*}C_Z$ is an 'exact' functor from $(\mathcal{M}_k,\mathfrak{I}_{\mathfrak{X},k})$ to $(C_Z,\mathfrak{I}_Z)$.

The argument is similar to that of the corresponding part of 9.1.

This implies that the universal k-linear ∂-functor $\mathcal{E}xt^\bullet_{X,\mathfrak{I}_X}$ from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to the left exact category $(\mathcal{M}_k(\mathfrak{X}),\mathfrak{I}_{\mathfrak{X},k})$ is the initial object of the category $\partial_k\mathfrak{UEx}^\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$.

(b) The initial object of the category $\partial_k\mathfrak{UEx}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ is the corestriction of $\mathcal{E}xt^\bullet_{X,\mathfrak{I}_X}$ to the smallest additive strictly full left exact subcategory of $(\mathcal{M}_k(\mathfrak{X}),\mathfrak{I}_{\mathfrak{X},k})$ containing all presheaves $\mathcal{E}xt^n_{X,\mathfrak{I}_X}(L)$, $L\in ObC_\mathfrak{X}$, $n\geq 0$.

(c) It follows from (a) (by duality) that the universal k-linear ∂^*-functor $\mathcal{E}xt^{\bullet}_{(X,\mathfrak{E}_X)}$ from the right exact k-linear category $(C_X, \mathfrak{E}_X)$ to the category $\mathcal{M}_k(X^o)$ of presheaves of k-modules on $C_{X^o} = C_X^{op}$ is an 'exact' universal k-linear ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to the right exact category $(\mathcal{M}_k(X^o), \mathfrak{E}_{\mathfrak{X}_k^o})$, where $\mathfrak{E}_{\mathfrak{X}_k^o}$ is the coarsest right exact structure on $\mathcal{M}_k(X^o)$ such that the Yoneda embedding $C_{X^o} \longrightarrow \mathcal{M}_k(X^o)$ maps $\mathfrak{E}_X$ to $\mathfrak{E}_{\mathfrak{X}_k^o}$. This 'exact' universal k-linear ∂^*-functor is an initial object of the category $\partial_k^*\mathfrak{U}\mathfrak{E}\mathfrak{x}_{\mathfrak{c}}(X, \mathfrak{E}_X)$.

(d) The corestriction of the ∂^*-functor $\mathcal{E}xt^{\bullet}_{(X,\mathfrak{E}_X)}$ to the smallest strictly full additive right exact subcategory of $\mathcal{M}_k(X^o)$ spanned by the presheaves $\mathcal{E}xt^n_{(X,\mathfrak{E}_X)}(L)$, $L \in ObC_X$, $n \geq 0$, is an initial object of the category $\partial_k^*\mathfrak{U}\mathfrak{E}\mathfrak{x}(X, \mathfrak{E}_X)$.

Details are left to the reader. ■

10. Relative satellites.

Fix a right exact category $(C_{\mathfrak{S}}, \mathfrak{E}_{\mathfrak{S}})$. Fix an object $\mathcal{Y}$ of $C_{\mathfrak{S}}$ and denote by $C_{\mathcal{Y}\backslash\mathfrak{S}}$ the category $\mathcal{Y}\backslash C_{\mathfrak{S}}$. Consider the right exact category $(C_{\mathcal{Y}\backslash\mathfrak{S}}, \mathfrak{E}_{\mathcal{Y}\backslash\mathfrak{S}})$, where $\mathfrak{E}_{\mathcal{Y}\backslash\mathfrak{S}}$ denote the right exact structure on $C_{\mathcal{Y}\backslash\mathfrak{S}} = \mathcal{Y}\backslash C_{\mathfrak{S}}$ induced by $\mathfrak{E}_{\mathfrak{S}}$.

10.1. The ∂^*-functor $\mathcal{F}^{\mathcal{Y}}_{\bullet}$. For a functor $C_{\mathfrak{S}} \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$, let $C_{\mathcal{Y}\backslash\mathfrak{S}} \xrightarrow{\mathcal{F}^{\mathcal{Y}}_0} C_{\mathcal{Z}}$ be the composition of the forgetful functor $C_{\mathcal{Y}\backslash\mathfrak{S}} = \mathcal{Y}\backslash C_{\mathfrak{S}} \longrightarrow C_{\mathfrak{S}}$ and the functor $C_{\mathfrak{S}} \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$.

Suppose that the category $C_{\mathcal{Z}}$ has initial objects, kernels of arrows, and limits of filtered diagrams. Then the functor $\mathcal{F}^{\mathcal{Y}}_0$ extends to a (unique up to isomorphism) ∂^*-functor $\mathcal{F}^{\mathcal{Y}}_{\bullet} = (\mathcal{F}^{\mathcal{Y}}_n, \mathfrak{d}^{\mathcal{Y}}_n \mid n \geq 0)$ from the right exact category $(C_{\mathcal{Y}\backslash\mathfrak{S}}, \mathfrak{E}_{\mathcal{Y}\backslash\mathfrak{S}})$ to $C_{\mathcal{Z}}$. If the category $C_{\mathfrak{S}}$ has initial objects and $\mathcal{Y}$ is one of them, then the category $C_{\mathcal{Y}\backslash\mathfrak{S}} = \mathcal{Y}\backslash C_{\mathfrak{S}}$ is isomorphic to the category $C_{\mathfrak{S}}$ and the functor $\mathcal{F}^{\mathcal{Y}}_{\bullet}$ is the composition of this isomorphism and the universal ∂^*-functor $\mathcal{F}_{\bullet}$, where $\mathcal{F}_0 = \mathcal{F}$.

It follows from the definition of satellites that, for every object $(\mathcal{V}, \mathcal{Y} \xrightarrow{\xi_{\mathcal{V}}} \mathcal{V})$ of the category $\mathcal{Y}\backslash C_{\mathfrak{S}}$, we have

$$\mathcal{F}^{\mathcal{Y}}_1(\mathcal{V}, \xi_{\mathcal{V}}) = \mathcal{S}_{\mathcal{Y}}(\mathcal{F})(\mathcal{V}, \xi_{\mathcal{V}}) = \lim Ker(\mathcal{F}(\mathcal{Y} \prod_{\mathfrak{e}, \xi_{\mathcal{V}}} \mathcal{W} \xrightarrow{\mathfrak{p}_{\mathcal{W}}} \mathcal{W})), \qquad (1)$$

where $\mathfrak{p}_{\mathcal{W}}$ is the canonical projection and the limit is taken by the filtered system of deflations $(\mathcal{W}, \xi_{\mathcal{W}}) \xrightarrow{\mathfrak{e}} (\mathcal{V}, \xi_{\mathcal{V}})$. By (the argument of) 3.2, $\mathcal{F}^{\mathcal{Y}}_n = \mathcal{S}^n_{\mathcal{Y}}(\mathcal{F})$ for all $n \geq 0$.

10.2. The ∂^*-functor $\mathcal{F}^{\mathcal{Y},\mathfrak{E}_{\mathfrak{S}}}_{\bullet}$. Let $C_{\mathcal{Z}}$ be a category with final objects and cokernels of arbitrary morphisms. For any functor $C_{\mathfrak{S}} \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$, let $\mathcal{F}_{\mathcal{Y}}$ denote the functor $\mathcal{Y}\backslash C_{\mathfrak{S}} \longrightarrow C_Z$ which assigns to every object $(\mathcal{W}, \mathcal{Y} \xrightarrow{\xi} \mathcal{W})$ the object $Cok(F(\xi))$ and acts correspondingly on morphisms. Notice that the functor $\mathcal{F}_{\mathcal{Y}}$ maps the initial object $(\mathcal{Y}, id_{\mathcal{Y}})$ of the category $\mathcal{Y}\backslash C_{\mathfrak{S}}$ to a final object of the category $C_{\mathcal{Z}}$. If, in addition, the category $C_{\mathcal{Z}}$ has initial objects (e.g. it is pointed), kernels of arrows and limits of filtered systems, then there exists a (unique up to isomorphism) universal ∂^*-functor $\mathcal{F}_\bullet^{\mathcal{Y},\mathfrak{E}_\mathfrak{S}} = (\mathcal{F}_n^{\mathcal{Y},\mathfrak{E}_\mathfrak{S}}, \mathfrak{d}_n^{\mathcal{Y},\mathfrak{E}_\mathfrak{S}} \mid n \geq 0)$ such that $\mathcal{F}_0^{\mathcal{Y},\mathfrak{E}_\mathfrak{S}} = \mathcal{F}_\mathcal{Y}$.

Chapter III

Stable Categories and Homological Functors.

Stable categories were defined (by Keller and Vossieck) for exact categories with enough injective objects via homotopy equivalence of arrows: two morphisms are homotopy equivalent, if their difference factors through an injective object. In this chapter, we define, among other things, the stable category of an arbitrary (non-additive in general) right exact category. Considering that every category has a canonical (the finest) structure of a right exact category, we obtain the notion of the stable category of an arbitrary category.

Section 1 is dedicated to projective objects of a right exact category (and injective objects of a left exact category) and right exact categories with enough projective objects. Our main examples are the right exact categories of modules over monads (and left exact categories of comodules over comonads).

We observe that projective objects are compatible with the contravariant functoriality of universal ∂^*-functors discussed in Section 4 of Chapter II. In particular, the canonical embedding of a right exact category $(C_X, \mathfrak{E}_X)$ into the category of non-trivial sheaves of sets on $(C_X, \mathfrak{E}_X)$ maps projective objects to projective objects; and if the right exact category $(C_X, \mathfrak{E}_X)$ has enough projective objects, same holds for the category of sheaves of sets on $(C_X, \mathfrak{E}_X)$. Similarly, the canonical embedding of a k-linear right exact category $(C_X, \mathfrak{E}_X)$ into the k-linear Grothendieck category $Sh_k(X, \mathfrak{E}_X)$ of k-linear sheaves of k-modules maps projective objects to projective objects. If the right exact k-linear category $(C_X, \mathfrak{E}_X)$ has enough projective objects, then the abelian category $Sh_k(X, \mathfrak{E}_X)$ has enough projective objects too.

In Section 2, we extend (in an obvious way) the notion of a *coeffaceable* functor to right exact categories and prove that a coeffaceable 'exact' ∂^*-functor is universal, if in the target category, all deflations with trivial kernels are isomorphisms. On the other hand, we show that if a right exact category has enough projective objects and all projective objects are pointed, then derived functors of any functor from this right exact category are coeffaceable. In abelian case, these statements recover well known facts from Grothendieck's Tôhoku paper [Gr]. We start Section 3 with observations on the structure of universal ∂-functors related with results of Sections 3, 8 and 9 of Chapter II. These observations produce, for a given left exact category $(C_X, \mathfrak{I}_X)$, a structure of a $\mathbb{Z}_+$-category on the category C_X^* of non-trivial presheaves of sets on C_X (induced by the functor Ext^1) and the category of *standard triangles* related with conflations of the left exact category $(C_X, \mathfrak{I}_X)$. We apply the obtained structure to deriving some formulas for satellites of composition of functors. In Section 4, we look at computational aspects of satellites of functors F from left exact categories such that their domain has enough F-acyclic objects. We introduce

F-acyclic resolution, cohomology of complexes, and show that if the functor F is weakly left 'exact' and maps inflations with trivial cokernels to isomorphisms, then its satellites are isomorphic to cohomologies of the images of acyclic resolutions. The observations of Section 3 lead to definitions, in Section 5, of *prestable* and *stable* categories of a left exact category. Turning the properties of prestable and stable categories into axioms, we introduce, in Section 6, the notions of *presuspended* and *quasi-suspended* categories. In Section 7, we define homology of 'spaces' with coefficients in a right exact category and the homotopy groups of pointed 'spaces'.

1. Projective objects and injective objects.

Fix a right exact category $(C_X, \mathfrak{E}_X)$.

1.1. Lemma. *The following conditions on an object P of C_X are equivalent:*
(a) Every deflation $M \longrightarrow P$ splits.

(b) For every deflation $M \xrightarrow{\mathfrak{e}} N$ and a morphism $P \xrightarrow{f} N$, there exists a morphism $P \xrightarrow{g} M$ such that $f = \mathfrak{e} \circ g$.

Proof. Obviously, $(b) \Rightarrow (a)$: it suffices to take $f = id_P$.
$(a) \Rightarrow (b)$. Since deflations are stable under any base change, there is a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{f'} & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ P & \xrightarrow{f} & N \end{array}$$

whose left vertical arrow is a deflation. By (a), it splits; i.e. there is a morphism $P \xrightarrow{g} \widetilde{M}$ such that $\mathfrak{e}' \circ g = id_P$. Therefore, $\mathfrak{e} \circ (f' \circ g) = (\mathfrak{e} \circ f') \circ g = (f \circ \mathfrak{e}') \circ g = f$. ∎

1.2. Projective objects. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. We call an object P of C_X a *projective* object of $(C_X, \mathfrak{E}_X)$, if it satisfies the equivalent conditions of 1.1. We denote by $\mathcal{P}_{\mathfrak{E}_X}$ the full subcategory of C_X generated by projective objects.

1.2.1. Example. Let $(C_X, \mathfrak{E}_X)$ be a right exact category whose deflations are split. Then every object of C_X is a projective object of $(C_X, \mathfrak{E}_X)$.

1.3. Functorialities and right exact categories with enough projective objects. We say that $(C_X, \mathfrak{E}_X)$ has *enough projective objects* if for every object N of C_X there exists a deflation $P \longrightarrow N$, where P is a projective object.

1.3.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and let $C_Y \xrightarrow{f^*} C_X$ be a functor having a right adjoint, f_*. If the functor f_* maps deflations to deflations, then f^* maps projective objects to projective objects.*

Proof. Let P be a projective object of $(C_Y, \mathfrak{E}_Y)$ and $M \xrightarrow{\mathfrak{e}} f^*(P)$ a deflation. Then, by hypothesis, $f_*(M) \xrightarrow{f_*(\mathfrak{e})} f_*f^*(P)$ is a deflation. Since P is a projective object, there exists an arrow $P \xrightarrow{\mathfrak{t}} f^*(M)$ such that the diagram

$$\begin{array}{ccc} & P & \\ {\scriptstyle \mathfrak{t}}\swarrow & & \searrow{\scriptstyle \eta(P)} \\ f_*(M) & \xrightarrow{f_*(\mathfrak{e})} & f_*f^*(P) \end{array} \qquad (1)$$

commutes (here $\eta(P)$ is an adjunction arrow). Then the composition $f^*(P) \xrightarrow{\mathfrak{t}'} M$ of $f^*(P) \xrightarrow{f^*(\mathfrak{t})} f^*f_*(M)$ and the adjunction morphism $f^*f_*(M) \xrightarrow{\varepsilon(M)} M$ splits the deflation $M \xrightarrow{\mathfrak{e}} f^*(P)$. This follows from the commutativity of the diagram

$$\begin{array}{ccc} & f^*(P) & \\ {\scriptstyle f^*(\mathfrak{t})}\swarrow & & \searrow{\scriptstyle f^*\eta(P)} \\ f^*f_*(M) & \xrightarrow{f^*f_*(\mathfrak{e})} & f^*f_*f^*(P) \\ {\scriptstyle \varepsilon(M)}\downarrow & & \downarrow{\scriptstyle \varepsilon f^*(P)} \\ M & \xrightarrow{\mathfrak{e}} & f^*(P) \end{array} \qquad (2)$$

and the equality $\varepsilon f^* \circ f^*\eta = Id_{f^*}$. ∎

1.3.1.1. Note. Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories and $C_Y \xrightarrow{f^*} C_X$ a functor having a right adjoint, f_*. *The functor f_* maps deflations to deflations iff it is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.*

The part 'if' follows from the definition: a functor is 'exact' if it maps deflations to deflations and preserves pull-backs of deflations. On the other hand, f_*, as any functor having a left adjoint, preserves limits; in particular, it preserves pull-backs. So that if f_* preserves deflations it preserves also their pull-backs.

1.3.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and let $C_Y \xrightarrow{f^*} C_X$ be a functor having a right adjoint, f_*. Suppose that $\mathfrak{E}_Y$ consists of all split epimorphisms of C_Y and the functor f_* maps deflations to deflations (that is split epimorphisms) and reflects deflations (i.e. if $f_*(\mathfrak{t})$ is a split epimorphism, then $\mathfrak{t}$ is a deflation). Then $(C_X, \mathfrak{E}_X)$ has enough projective objects.*

Proof. Since $\mathfrak{E}_Y$ consists of split epimorphisms, all objects of C_Y are projective. Therefore, by 1.3.1, every object of the form $f^*(N)$, $N \in ObC_Y$, is projective. For every object M of C_X the adjunction morphism $f^*f_*(M) \xrightarrow{\varepsilon(M)} M$ is a deflation, because the morphism $f_*\varepsilon(M)$ is a split epimorphism, hence, by hypothesis, it belongs to $\mathfrak{E}_Y$. ∎

1.3.3. Coinduced right exact structures and projective objects. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and $C_X \xrightarrow{\Phi} C_Y$ a functor. Let ${}^{\Phi}\mathfrak{E}_Y$ denotes the class of all universally strict epimorphisms $\mathfrak{M} \xrightarrow{\mathfrak{t}} \mathfrak{L}$ of the category C_Y such that for every morphism $\Phi(\mathcal{L}) \xrightarrow{\mathfrak{f}} \mathfrak{L}$, there exists a commutative square

$$\begin{array}{ccc} \Phi(\mathcal{M}) & \xrightarrow{\mathfrak{f}'} & \mathfrak{M} \\ {\scriptstyle \Phi(\mathfrak{s})}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} \\ \Phi(\mathcal{L}) & \xrightarrow{\mathfrak{f}} & \mathfrak{L} \end{array}$$

where $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{L}$ is a deflation.

1.3.3.1. Proposition. *(a) The class of arrows ${}^{\Phi}\mathfrak{E}_Y$ is a right exact structure on the category C_Y.*

(b) The functor $C_X \xrightarrow{\Phi} C_Y$ maps projective objects of the right exact category $(C_X, \mathfrak{E}_X)$ to projective objects of the right exact category $(C_Y, {}^{\Phi}\mathfrak{E}_Y)$.

Proof. (a) The argument is the same as in II.4.1(a).

(b) Let $\mathcal{P}$ be a projective object of $(C_X, \mathfrak{E}_X)$ and $\Phi(\mathcal{P}) \xrightarrow{\mathfrak{f}} \mathfrak{L}$ an arbitrary morphism. By definition of the class ${}^{\Phi}\mathfrak{E}_Y$, for any morphism $\mathfrak{M} \xrightarrow{\mathfrak{t}} \mathfrak{L}$ from ${}^{\Phi}\mathfrak{E}_Y$, there exists a commutative square

$$\begin{array}{ccc} \Phi(\mathcal{M}) & \xrightarrow{\mathfrak{f}'} & \mathfrak{M} \\ {\scriptstyle \Phi(\mathfrak{s})}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} \\ \Phi(\mathcal{P}) & \xrightarrow{\mathfrak{f}} & \mathfrak{L} \end{array}$$

where $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{P}$ is a deflation. Since $\mathcal{P}$ is a projective object of $(C_X, \mathfrak{E}_X)$, the deflation $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{P}$ splits; that is $\mathfrak{s} \circ \mathfrak{j} = id_{\mathcal{P}}$ for some arrow $\mathcal{P} \xrightarrow{\mathfrak{j}} \mathcal{M}$. Therefore $\mathfrak{t} \circ (\mathfrak{f}' \circ \Phi(\mathfrak{j})) = \mathfrak{f} \circ \Phi(\mathfrak{s}) \circ \Phi(\mathfrak{j}) = \mathfrak{f}$, which shows that $\Phi(\mathcal{P})$ is a projective object of the right exact category $(C_Y, {}^{\Phi}\mathfrak{E}_Y)$. ■

1.3.3.2. Note. If $(C_X, \mathfrak{E}_X)$ is a right exact category and $C_X \xrightarrow{\Phi} C_Y$ is a full functor, which maps deflations to universally strict epimorphisms, then $\Phi(\mathfrak{E}_X) \subseteq {}^{\Phi}\mathfrak{E}_Y$.

In fact, for any deflation $\mathcal{L}_{\mathfrak{u}} \xrightarrow{\mathfrak{u}} \mathcal{L}$ in $(C_X, \mathfrak{E}_X)$ and any morphism $\mathcal{L}_{\xi} \xrightarrow{\xi} \mathcal{L}$, we have the image

$$\Phi\left(\begin{array}{ccc} \mathcal{L}_{\mathfrak{t},\xi} & \xrightarrow{\xi_{\mathfrak{u}}} & \mathcal{L}_{\mathfrak{u}} \\ {\scriptstyle \mathfrak{u}'_{\xi}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{u}} \\ \mathcal{L}_{\xi} & \xrightarrow{\xi} & \mathcal{L} \end{array}\right)$$

of the corresponding cartesian square whose vertical arrows are deflations.

1.3.4. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *be a svelte right exact category with an initial object and*

$$(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^{\circledast}} (C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$$

the canonical embedding of $(C_X, \mathfrak{E}_X)$ *into the right exact category* $(C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$ *(see I.2.1.1).*

(a) The functor $\mathfrak{j}_X^*$ *maps projective objects of* $(C_X, \mathfrak{E}_X)$ *to projective objects of the right exact category* $(C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$.

(b) If the right exact category $(C_X, \mathfrak{E}_X)$ *has enough projective objects, then the right exact category of sheaves* $(C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$ *has enough projective objects too.*

Proof. (a) It follows from I.2.2.1(b) that the canonical right exact structure $\mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}}$ on the category $C_{X_{\mathfrak{E}}^{\circledast}}$ coincides with the right exact structure coinduced by the full embedding $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^*} (C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$. Therefore, by 1.3.3.1(b), the functor $\mathfrak{j}_X^{\circledast}$ maps projective objects of $(C_X, \mathfrak{E}_X)$ to projective objects of the right exact category $(C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$.

(b) For any object $\mathfrak{L}$ of the category $C_{X_{\mathfrak{E}}^{\circledast}}$, consider all morphisms $\mathfrak{j}_X^{\circledast}(\mathcal{P}_\xi) \xrightarrow{\xi} \mathfrak{L}$, where $\mathcal{P}_\xi$ runs through projective objects of the right exact category $(C_X, \mathfrak{E}_X)$. Let

$$\coprod_{\mathfrak{j}_X^{\circledast}(\mathcal{P}_\xi) \xrightarrow{\xi} \mathfrak{L}} \mathfrak{j}_X^{\circledast}(\mathcal{P}_\xi) \xrightarrow{\xi_{\mathfrak{L}}} \mathfrak{L} \tag{1}$$

be the corresponding morphism of the coproduct of the objects $\mathfrak{j}_X^{\circledast}(\mathcal{P}_\xi)$. The coproduct of projective objects is a projective object. If there are enough projective objects in $(C_X, \mathfrak{E}_X)$, then (1) is an epimorphism; hence it is a deflation in the right exact category $(C_{X_{\mathfrak{E}}^{\circledast}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$.
■

1.4. Right exact structure with a given class of projective objects. Let C_X be a category and $\mathfrak{P}$ a class of objects of C_X. Let $\mathfrak{E}(\mathfrak{P})$ denote the class of all arrows $M \xrightarrow{f} L$ of C_X such that the map $C_X(P, M) \xrightarrow{C_X(P,f)} C_X(P, L)$ is surjective for all $P \in \mathfrak{P}$ and, for any morphism $N \xrightarrow{g} L$, there exists a pull-back of f along g.

1.4.1. Lemma. *The class* $\mathfrak{E}(\mathfrak{P})$ *is the class of covers of a Grothendieck pretopology.*

Proof. Obviously, the class $\mathfrak{E}(\mathfrak{P})$ contains all isomorphisms and is closed under compositions. By assumption, for any arrow $M \xrightarrow{\mathfrak{t}} L$ of $\mathfrak{E}(\mathfrak{P})$ and an arbitrary morphism

$N \xrightarrow{g} L$ of C_X, there exists a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{g'} & M \\ {\scriptstyle \widetilde{\mathfrak{t}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}} \\ N & \xrightarrow{g} & L \end{array} \qquad (1)$$

The functor $C_X(P, -)$ preserves cartesian squares for any object P of C_X. In particular, the image

$$\begin{array}{ccc} C_X(P, \widetilde{M}) & \xrightarrow{C_X(P,g')} & C_X(P, M) \\ {\scriptstyle C_X(P,\widetilde{\mathfrak{t}})}\downarrow & cart & \downarrow{\scriptstyle C_X(P,\mathfrak{t})} \\ C_X(P, N) & \xrightarrow{C_X(P,g)} & C_X(P, L) \end{array} \qquad (2)$$

of (1) is a cartesian square. If P belongs to $\mathfrak{P}$, then its right vertical of (2) is surjective, hence its left vertical arrow is surjective too. This shows that the pull-back $\widetilde{M} \xrightarrow{\widetilde{\mathfrak{t}}} N$ of the morphism $\mathfrak{t}$ belongs to $\mathfrak{E}(\mathfrak{P})$. ■

1.4.2. Proposition. *Let C_X be a category. For any class of objects $\mathfrak{P}$ of the category C_X, the class of morphisms $\mathfrak{E}_X^{\mathfrak{st}}(\mathfrak{P}) \stackrel{\mathrm{def}}{=} \mathfrak{E}_X^{\mathfrak{st}} \bigcap \mathfrak{E}(\mathfrak{P})$ is the finest among the right exact structures $\mathfrak{E}_X$ on C_X such that all objects of $\mathfrak{P}$ are projective objects of $(C_X, \mathfrak{E}_X)$.*

Proof. Recall that $\mathfrak{E}_X^{\mathfrak{st}}$ is the finest right exact structure on C_X; it consists of all universal strict epimorphisms of C_X. The intersection of Grothendieck pretopologies is a Grothendieck pretopology. Since it is contained in $\mathfrak{E}_X^{\mathfrak{st}}$, it is a right exact structure. Evidently, any right exact structure $\mathfrak{E}_X$ such that all objects of $\mathfrak{P}$ are projective objects of $(C_X, \mathfrak{E}_X)$, is coarser than $\mathfrak{E}_X^{\mathfrak{st}}(\mathfrak{P})$. ■

1.4.3. The closure of deflations. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. We denote by $\bar{\mathfrak{E}}_X$ the class of all morphisms $M \xrightarrow{\mathfrak{t}} L$ such that $\mathfrak{t} \circ \gamma \in \mathfrak{E}_X$ for some morphism γ and there exist pull-backs of $\mathfrak{t}$ along all arrows to L.

1.4.3.1. Proposition. *The class $\bar{\mathfrak{E}}_X$ is a right exact structure on the category C_X.*

Proof. Obviously, $\mathfrak{E}_X \subseteq \bar{\mathfrak{E}}_X$; in particular, $\bar{\mathfrak{E}}_X$ contains all isomorphisms of the category C_X. It remains to show that $\bar{\mathfrak{E}}_X$ is stable under base change and compositions.

(a) Let $M \xrightarrow{\mathfrak{t}} L$ be an arrow of $\bar{\mathfrak{E}}_X$ and $\mathfrak{M} \xrightarrow{\gamma} M$ an arrow such that $\mathfrak{t} \circ \gamma \in \mathfrak{E}_X$. By hypothesis, for any arrow $\mathcal{L} \xrightarrow{f} L$, there exists a cartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\widetilde{\mathfrak{t}}} & \mathcal{L} \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\mathfrak{t}} & L \end{array} \qquad (3)$$

On the other hand, since $\mathfrak{t}\circ\gamma$ is a deflation and the class $\mathfrak{E}_X$ of deflations is stable under pull-backs, we have a cartesian square

$$\begin{array}{ccc} \widetilde{\mathfrak{M}} & \xrightarrow{\ \widetilde{\mathfrak{t}}_1\ } & \mathcal{L} \\ {\scriptstyle f''}\big\downarrow & cart & \big\downarrow{\scriptstyle f} \\ \mathfrak{M} & \xrightarrow{\ \mathfrak{t}\circ\gamma\ } & L \end{array} \tag{4}$$

whose horizontal arrows are deflations.

Notice that the deflation $\widetilde{\mathfrak{M}} \xrightarrow{\widetilde{\mathfrak{t}}_1} \mathcal{L}$ is the composition of $\mathcal{M} \xrightarrow{\widetilde{\mathfrak{t}}} \mathcal{L}$ and a morphism $\widetilde{\mathfrak{M}} \xrightarrow{\widetilde{\gamma}} \mathfrak{M}$ uniquely determined by the commutativity of the diagram

$$\begin{array}{ccccc} \widetilde{\mathfrak{M}} & \xrightarrow{\ \widetilde{\gamma}\ } & \mathcal{M} & \xrightarrow{\ \widetilde{\mathfrak{t}}\ } & \mathcal{L} \\ {\scriptstyle f''}\big\downarrow & & {\scriptstyle f'}\big\downarrow & cart & \big\downarrow{\scriptstyle f} \\ \mathfrak{M} & \xrightarrow{\ \gamma\ } & M & \xrightarrow{\ \mathfrak{t}\ } & L \end{array} \tag{5}$$

This follows from the fact that the square (4) yields a commutative square

$$\begin{array}{ccc} \widetilde{\mathfrak{M}} & \xrightarrow{\ \widetilde{\mathfrak{t}}_1\ } & \mathcal{L} \\ {\scriptstyle \gamma\circ f''}\big\downarrow & cart & \big\downarrow{\scriptstyle f} \\ M & \xrightarrow{\ \mathfrak{t}\ } & L \end{array}$$

and the square (3) is cartesian.

This shows that an arbitrary pull-back of an arrow of $\bar{\mathfrak{E}}_X$ belongs to $\bar{\mathfrak{E}}_X$.

(b) If $M \xrightarrow{\mathfrak{t}} L$ and $N \xrightarrow{\mathfrak{s}} M$ are arrows from $\bar{\mathfrak{E}}_X$, then $\mathfrak{t}\circ\mathfrak{s} \in \bar{\mathfrak{E}}_X$.

In fact, since both arrows, $\mathfrak{s}$ and $\mathfrak{t}$ have pull-backs along any morphism with the same target, their composition has this property. It remains to show that there exists a morphism ψ to N such that $(\mathfrak{t}\circ\mathfrak{s})\circ\psi$ is a deflation.

Let $\mathfrak{M} \xrightarrow{\gamma} M$ be a morphism such that $\mathfrak{t}\circ\gamma$ is a deflation. Consider the diagram

$$\begin{array}{ccccc} \mathfrak{N} & \xrightarrow{\ \widetilde{\mathfrak{s}}\ } & \mathfrak{M} & & \\ {\scriptstyle \gamma'}\big\downarrow & cart & \big\downarrow{\scriptstyle \gamma} & & \\ N & \xrightarrow{\ \mathfrak{s}\ } & M & \xrightarrow{\ \mathfrak{t}\ } & L \end{array} \tag{6}$$

with cartesian square. Since $\mathfrak{s} \in \bar{\mathfrak{E}}_X$ and, by (a) above, $\bar{\mathfrak{E}}_X$ is stable under base change, the upper horizontal arrow, $\widetilde{M} \xrightarrow{\widetilde{\mathfrak{s}}} \mathfrak{M}$, belongs to $\bar{\mathfrak{E}}_X$. Therefore, there exists a morphism $\widetilde{\mathfrak{N}} \xrightarrow{\lambda} \mathfrak{N}$ such that $\widetilde{\mathfrak{s}}\circ\lambda$ is a deflation. It follows from the diagram (6) that

$$(\mathfrak{s}\circ\mathfrak{t})\circ(\gamma'\circ\lambda) = (\mathfrak{t}\circ\gamma)\circ(\widetilde{\mathfrak{s}}\circ\lambda) \in \mathfrak{E}_X\circ\mathfrak{E}_X = \mathfrak{E}_X.$$

(c) It follows from the argument above that elements of $\bar{\mathfrak{E}}_X$ form a Grothendieck pretopology. Notice that if the composition $\mathfrak{t} \circ \gamma$ is a deflation, hence a strict epimorphism, then $\mathfrak{t}$ is a strict epimorphism. So that $\bar{\mathfrak{E}}_X$ is a right exact structure on C_X. ■

1.4.3.2. Definition. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. We call the refinement $\bar{\mathfrak{E}}_X$ of the right exact structure $\mathfrak{E}_X$ the *closure* of $\mathfrak{E}_X$.

1.4.3.3. Note. Suppose that $(C_X, \mathfrak{E}_X)$ is a Karoubian right exact category. Then the closure of $\mathfrak{E}_X$ consists of all universally strict epimorphisms $M \xrightarrow{\mathfrak{t}} L$ such that $\mathfrak{t} \circ \mathfrak{e} \in \mathfrak{E}_X$ for some $\mathfrak{e} \in \mathfrak{E}_X$.

In fact, let $\mathfrak{M} \xrightarrow{\gamma} M$ be a morphism such that $\mathfrak{t} \circ \gamma \in \mathfrak{E}_X$. Then the upper horizontal arrow of the cartesian square

$$\begin{array}{ccc} \mathfrak{N} & \xrightarrow{\widetilde{\mathfrak{t}}} & \mathfrak{M} \\ \Big\downarrow & cart & \Big\downarrow {\scriptstyle \mathfrak{t} \circ \gamma} \\ M & \xrightarrow{\mathfrak{t}} & L \end{array}$$

splits. Since the right exact category $(C_X, \mathfrak{E}_X)$ is Karoubian, this arrow is a deflation. The left vertical arrow is a deflation, because it is a pull-back of a deflation.

1.4.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category. Then*

(a) The right exact categories $(C_X, \mathfrak{E}_X)$ and $(C_X, \bar{\mathfrak{E}}_X)$ have the same class of projective objects: $\mathcal{P}_{\mathfrak{E}_X} = \mathcal{P}_{\bar{\mathfrak{E}}_X}$.

(b) If there are enough projective objects in $(C_X, \mathfrak{E}_X)$, then $\bar{\mathfrak{E}}_X$ coincides with $\mathfrak{E}(\mathcal{P}_{\mathfrak{E}_X})$.

In particular, $\bar{\mathfrak{E}}_X = \mathfrak{E}(\mathcal{P}_{\mathfrak{E}_X}) = \mathfrak{E}_X^{\mathfrak{st}}(\mathcal{P}_{\mathfrak{E}_X})$; that is $\bar{\mathfrak{E}}_X$ is the finest right exact structure on C_X for which all objects of $\mathcal{P}_{\mathfrak{E}_X}$ are projective objects.

Proof. (a) Let $M \xrightarrow{\mathfrak{t}} L$ be an arrow from $\bar{\mathfrak{E}}_X$ and $P \xrightarrow{g} L$ a morphism with P projective of $(C_X, \mathfrak{E}_X)$. Since $\mathfrak{t} \in \bar{\mathfrak{E}}_X$, there exists a morphism $\mathfrak{M} \xrightarrow{\gamma} M$ such that $\mathfrak{t} \circ \gamma \in \mathfrak{E}_X$. Since P is a projective, there exists an arrow $P \xrightarrow{g'} \mathfrak{M}$ such that $g = (\mathfrak{t} \circ \gamma) \circ g'$. So that $g = \mathfrak{t} \circ (\gamma \circ g')$.

(b) Suppose now that there are enough projective objects. Let $M \xrightarrow{\mathfrak{s}} L$ be an arrow of $\mathfrak{E}(\mathcal{P}_{\mathfrak{E}_X})$. Since there are enough projective objects, there exists a deflation $P \xrightarrow{\mathfrak{e}} L$ with P a projective of $(C_X, \mathfrak{E}_X)$. Since $\mathfrak{s} \in \mathfrak{E}(\mathcal{P}_{\mathfrak{E}_X})$, there exists an arrow $P \xrightarrow{\gamma} M$ such that $\mathfrak{e} = \mathfrak{s} \circ \gamma$. By 1.4.1, the class of arrows $\mathfrak{E}(\mathcal{P}_{\mathfrak{E}_X})$ forms a Grothendieck pretopology. In particular, there exist pull-backs of $\mathfrak{s}$ along all arrows to L. ■

1.4.4.1. Corollary. *For any class $\mathfrak{P}$ of objects of a category C_X, the finest right exact structure $\mathfrak{E}_X = \mathfrak{E}_X^{\mathfrak{st}}(\mathfrak{P})$ on C_X, for which all objects of $\mathfrak{P}$ are projective objects (cf. 1.4.2), is closed: $\mathfrak{E}_X = \bar{\mathfrak{E}}_X$.*

Proof. This follows from 1.4.4(a). ■

1.4.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and let $C_Y \xrightarrow{f^*} C_X$ be a functor having a right adjoint, f_*. Suppose that $(C_Y, \mathfrak{E}_Y)$ has enough projective objects and $\mathfrak{E}_Y$ is closed (that is $\mathfrak{E}_Y = \bar{\mathfrak{E}}_Y$) and there exist pull-backs of the adjunction arrows $N \xrightarrow{\eta_f(N)} f_*f^*(N)$ along morphisms $f_*(\mathfrak{t})$, where $M' \xrightarrow{\mathfrak{t}} f^*(N)$ is a deflation. Then f_* is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ iff the functor f^* maps projective objects to projective objects.*

Proof. The 'exactness' of f_* means precisely that it maps deflations to deflations (see 1.3.1.1). Let $M \xrightarrow{\mathfrak{e}} L$ be an arrow of $\mathfrak{E}_X$. Notice, that there exists a pull-back of $f_*(M \xrightarrow{\mathfrak{e}} L)$ along any arrow $N \xrightarrow{\xi} f_*(L)$.

In fact, to the arrow $N \xrightarrow{\xi} f_*(L)$, there corresponds an arrow $f^*(N) \xrightarrow{\xi^\vee} L$. Since $M \xrightarrow{\mathfrak{e}} L$ is a deflation, there exists a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\zeta} & M \\ {\scriptstyle \widetilde{\mathfrak{e}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ f^*(N) & \xrightarrow{\xi^\vee} & L \end{array}$$

The functor f_* maps it to a cartesian square – the right cartesian square in the diagram

$$\begin{array}{ccccc} \mathfrak{N} & \xrightarrow{\gamma} & f_*(\widetilde{M}) & \xrightarrow{f_*(\zeta)} & f_*(M) \\ {\scriptstyle \mathfrak{p}}\downarrow & cart & f_*(\widetilde{\mathfrak{e}})\downarrow & cart & \downarrow{\scriptstyle f_*(\mathfrak{e})} \\ N & \xrightarrow{\eta_f(N)} & f_*f^*(N) & \xrightarrow{f_*(\xi^\vee)} & f_*(L) \end{array}$$

whose left cartesian square exists by hypothesis. Since the composition $f_*(\xi^\vee) \circ \eta_f(N)$ coincides with $N \xrightarrow{\xi} f_*(L)$, the outer square of the diagram

$$\begin{array}{ccc} \mathfrak{N} & \xrightarrow{f_*(\zeta)\circ\gamma} & f_*(M) \\ {\scriptstyle \mathfrak{p}}\downarrow & cart & \downarrow{\scriptstyle f_*(\mathfrak{e})} \\ N & \xrightarrow{\xi} & f_*(L) \end{array}$$

is a pull-back of $f_*(\mathfrak{e})$ along $N \xrightarrow{\xi} f_*(L)$.

Since $(C_Y, \mathfrak{E}_Y)$ has enough projective objects, there exists a deflation $P \xrightarrow{\mathfrak{t}} f_*(L)$. To the arrow $\mathfrak{t}$ corresponds an arrow $f^*(P) \xrightarrow{\hat{\mathfrak{t}}} L$. Since $f^*(P)$ is a projective, the arrow $\hat{\mathfrak{t}}$ factors through the deflation $M \xrightarrow{\mathfrak{e}} L$. But, this implies that $P \xrightarrow{\mathfrak{t}} f_*(L)$ factors through $f_*(M \xrightarrow{\mathfrak{e}} L)$. So that $f_*(\mathfrak{e}) \circ \gamma = \mathfrak{t}$ for some morphism $P \xrightarrow{\gamma} f_*(M)$. ∎

1.5. Right exact categories of modules over monads. Fix a category C_Y such that the class $\mathfrak{E}_Y^{spl}$ of split epimorphisms of C_Y is stable under base change. Equivalently, for each split epimorphism $M \stackrel{\mathfrak{t}}{\longrightarrow} L$ and for an arbitrary morphism $N \stackrel{f}{\longrightarrow} L$ of the category C_Y, there exists a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \stackrel{f'}{\longrightarrow} & M \\ \widetilde{\mathfrak{t}} \downarrow & cart & \downarrow \mathfrak{t} \\ N & \stackrel{f}{\longrightarrow} & L \end{array}$$

whose left vertical arrow splits, because the right vertical arrow $\mathfrak{t}$ splits. In other words, the pair $(C_Y, \mathfrak{E}_Y^{spl})$ is a right exact category.

Let $\mathcal{F} = (F, \mu)$ be a monad on the category C_Y. Set $C_X = \mathcal{F} - mod = (\mathcal{F}/Y) - mod$ (i.e. $X = \mathbf{Sp}(\mathcal{F}/Y)$ – the spectrum of the monad $\mathcal{F}$) and denote by f_* the forgetful functor $C_X \longrightarrow C_Y$. We set $\mathfrak{E}_X = f_*^{-1}(\mathfrak{E}_Y^{spl})$. Since f_* preserves and reflects limits (in particular, pull-backs), the arrows of $\mathfrak{E}_X$ are covers of a subcanonical pretopology, i.e. $(C_X, \mathfrak{E}_X)$ is a right exact category. The functor f_* has a left adjoint, $V \stackrel{f^*}{\longmapsto} (F(V), \mu(V))$, and all together satisfy the conditions of 1.3.2. Therefore, $(C_X, \mathfrak{E}_X)$ has enough projective objects. Explicitly, it follows from (the argument of) 1.3.2 that objects $f^*(V) = (F(V), \mu(V))$ are projective objects of $(C_X, \mathfrak{E}_X)$ for all $V \in ObC_Y$, and for every $\mathcal{F}$-module $\mathcal{M} = (M, \xi)$, the action $F(M) \stackrel{\xi}{\longrightarrow} M$ can be regarded as a canonical deflation from a projective object:

$$f^* f_*(\mathcal{M}) = (F(M), \mu(M)) \stackrel{\xi}{\longrightarrow} \mathcal{M}.$$

1.5.1. Proposition. *Suppose that $(C_Y, \mathfrak{E}_Y^{spl})$ is a Karoubian right exact category (i.e. C_Y is a Karoubian category and split epimorphisms are stable under base change). Then for every monad $\mathcal{F} = (F, \mu)$ on C_Y, the right exact category $(\mathcal{F} - mod, \mathfrak{E}_X)$, where $\mathfrak{E}_X$ is the induced by $\mathfrak{E}_Y^{spl}$ right exact structure, is Karoubian.*

Proof. (a) The forgetful functor $\mathcal{F} - mod \stackrel{f_*}{\longrightarrow} C_Y$ reflects and preserves limits; in particular, it reflects and preserves pull-backs. Therefore, the stability of split epimorphisms of C_Y under base change implies the same property of split epimorphisms of $\mathcal{F} - mod$.

(b) It remains to show that $\mathcal{F} - mod$ is a Karoubian category. Let $\mathcal{M} = (M, \xi)$ be an $\mathcal{F}$-module and $\mathfrak{p}$ an idempotent $\mathcal{M} \longrightarrow \mathcal{M}$. Since C_Y is a Karoubian category, the idempotent $f_*(\mathcal{M}) = M \stackrel{f_*(\mathfrak{p})}{\longrightarrow} M$ splits. By I.2.1, the latter is equivalent to the existence of the kernel of the pair of arrows $M \underset{f_*(\mathfrak{p})}{\stackrel{id_M}{\rightrightarrows}} M$. Since the forgetful functor f_* reflects and preserves limits, in particular kernels of pairs of arrows, there exists the kernel of pair of arrows $\mathcal{M} \underset{\mathfrak{p}}{\stackrel{id_M}{\rightrightarrows}} \mathcal{M}$; i.e. the idempotent $\mathfrak{p}$ splits. ■

1.5.2. Corollary. *Let $\mathcal{G} = (G, \delta)$ be a comonad on a Karoubian category C_X. Suppose that class $\mathfrak{I}_X^{spl}$ of split monomorphisms in C_X is stable under cobase change (i.e. $\mathfrak{I}_X^{spl}$ is a left exact structure on C_X). Let $\mathfrak{I}_{\mathfrak{Y}}$ be the preimage of $\mathfrak{I}_X^{spl}$ in the category $C_{\mathfrak{Y}} = \mathcal{G} - comod$ of $\mathcal{G}$-comodules. Then $(C_{\mathfrak{Y}}, \mathfrak{I}_{\mathfrak{Y}})$ is a Karoubian left exact category having enough injective objects.*

Proof. The assertion is dual to that of 1.5.1. Further on, we need details which are as follows. Let $C_{\mathfrak{Y}}$ be the category $\mathcal{G} - comod$ of $\mathcal{G}$-comodules with the exact structure induced by the forgetful functor

$$C_{\mathfrak{Y}} = \mathcal{G} - comod \xrightarrow{g^*} C_X.$$

Its right adjoint

$$C_X \xrightarrow{g_*} C_{\mathfrak{Y}} = \mathcal{G} - comod, \quad M \longmapsto (G(M), \delta(M)), \tag{1}$$

maps every object M of the category C_X to an $\mathcal{E}_{\mathfrak{Y}}$-injective object. If the category C_X is Karoubian, then, for every object $\mathcal{M} = (M, M \xrightarrow{\nu} G(M))$, the adjunction morphism

$$\mathcal{M} \xrightarrow{\nu} g_*g^*(\mathcal{M}) = (G(M), \delta(M)) \tag{2}$$

is an inflation of $\mathcal{G}$-comodules (see the argument of the dual assertion 1.5.1). ∎

1.5.3. Corollary. *Under the conditions of 1.5.2, an object $\mathcal{M} = (M, \nu)$ of the category $C_{\mathfrak{Y}}$ of $\mathcal{G}$-comodules is $\mathfrak{I}_{\mathfrak{Y}}$-injective iff the adjunction morphism $\mathcal{M} \xrightarrow{\nu} (G(M), \delta(M))$ splits (as a morphism of $\mathcal{G}$-comodules).*

1.5.4. Proposition. *Suppose that C_X is a Karoubian category whose split epimorphisms (resp. split monomorphisms) are stable under base (resp. cobase) change. Let $\mathcal{F} = (F, \mu)$ be a continuous monad on C_X (i.e. the functor F has a right adjoint) and f_* the forgetful functor $\mathcal{F} - mod \longrightarrow C_X$. Set $C_{\mathfrak{X}} = \mathcal{F} - mod$, $\mathfrak{E}_{\mathfrak{X}} = f_*^{-1}(\mathfrak{E}_X^{spl})$, and $\mathfrak{I}_{\mathfrak{X}} = f_*^{-1}(\mathfrak{I}_X^{spl})$. Then $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ is a right exact category with enough projective objects and $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ is a left exact category with enough injective objects.*

Proof. If the monad $\mathcal{F} = (F, \mu)$ is continuous, i.e. the functor F has a right adjoint, $F^!$, then (and only then) the forgetful functor $\mathcal{F} - mod = C_{\mathfrak{X}} \xrightarrow{f_*} C_X$ has a right adjoint, $f^!$, such that $F^! = f_*f^!$. Thus, we have the comonad $\mathcal{F}^! = (F^!, \delta)$ corresponding to the pair of adjoint functors f_*, $f^!$ and an isomorphism of categories

$$\mathcal{F} - mod \xrightarrow{\Phi} \mathcal{F}^! - comod$$

which assigns to every $\mathcal{F}$-module $(M, F(M) \xrightarrow{\xi} M)$ the $\mathcal{F}^!$-comodule $(M, M \xrightarrow{\widehat{\xi}} F^!(M))$ determined (uniquely up to isomorphism) by adjunction. It follows that the diagram

$$\begin{array}{ccc} \mathcal{F} - mod & \xrightarrow{\Phi} & \mathcal{F}^! - comod \\ f_* \searrow & & \swarrow g^* \\ & C_X & \end{array} \qquad (3)$$

commutes. By 1.5.1, the category $C_{\mathfrak{X}} = \mathcal{F} - mod$ has enough $\mathfrak{E}_{\mathfrak{X}}$-injective objects. By 1.5.2, the category $C_{\mathfrak{Y}} = \mathcal{F}^! - comod$ has enough $\mathfrak{I}_{\mathfrak{Y}}$-injective objects. The functor Φ in (3) is an isomorphism of exact categories, hence the assertion. ∎

2. Coeffaceable functors, universal ∂^*-functors, and projective objects.

2.0. Coeffaceable functors. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a category with an initial object. We call a functor $C_X \xrightarrow{F} C_Y$ *coeffaceable*, or $\mathfrak{E}_X$-*coeffaceable*, if, for any object L of C_X, there exists a deflation $M \xrightarrow{\mathfrak{t}} L$ such that $F(\mathfrak{t})$ is a trivial morphism.

2.1. Projective deflations. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. We call a deflation $\mathcal{P} \longrightarrow \mathcal{L}$ *projective* if it factors through any other deflation $\mathcal{M} \longrightarrow \mathcal{L}$.

One can see that an object $\mathcal{M}$ is projective iff $id_{\mathcal{M}}$ is a projective deflation. And any deflation $\mathcal{M} \longrightarrow \mathcal{L}$ in which the object $\mathcal{M}$ is projective, is a projective deflation.

2.1.1. Coeffaceable functors and projective objects. If a functor $C_X \xrightarrow{F} C_Y$ is $\mathfrak{E}_X$-coeffaceable, then the morphism $F(\mathfrak{t})$ is trivial for any projective deflation $\mathfrak{t}$, and F maps every projective object of $(C_X, \mathfrak{E}_X)$ to an initial object of C_Y.

In fact, a projective deflation $M \xrightarrow{\mathfrak{t}} L$ factors through any other deflation of L; and, by hypothesis, there exists a deflation $M \xrightarrow{\mathfrak{e}} L$ such that $F(\mathfrak{e})$ is trivial. Therefore, the morphism $F(\mathfrak{t})$ is trivial. An object M is projective iff id_M is a projective deflation; and the triviality of $F(id_M)$ means precisely that $F(M)$ is an initial object.

So that if the right exact category $(C_X, \mathfrak{E}_X)$ has enough projective deflations (resp. enough projective objects), then a functor $C_X \xrightarrow{F} C_Y$ is $\mathfrak{E}_X$-coeffaceable iff $F(\mathfrak{e})$ is trivial for any projective deflation $\mathfrak{e}$ (resp. $F(M)$ is an initial object for every projective object M).

2.2. Universal ∂^*-functors and pointed projective objects. Let C_Z be a category with initial objects. We call an object M of C_Z *pointed* if there are morphisms from M to initial objects, or, equivalently, a unique morphism from an initial object to M is splittable.

2.2.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y. Then $T_i(P)$ is an initial object for any pointed projective object P and for all $i \geq 1$.*

Proof. Let F denote the functor T_i, $i \geq 0$. By II.3.3.2, $T_{i+1}(P) \simeq S_-(F)(P)$. Let x be an initial object of C_X and P a projective object of $(C_X, \mathfrak{E}_X)$ such that there exists a morphism $P \longrightarrow x$. Then $T_{i+1}(P) \simeq S_-(F)(P)$ is an initial object.

In fact, consider the conflation $x \xrightarrow{i_P} P \xrightarrow{id_P} P$. If there exists a morphism $P \longrightarrow x$, then the unique arrow $x \xrightarrow{i_P} P$ is a split monomorphism. Therefore $F(i_P)$ is a (split) monomorphism. By II.1.1, the latter implies that $Ker(F(i_P))$ is an initial object. Since the object P is projective, any deflation $M \xrightarrow{\mathfrak{e}} P$ is split; i.e. there exists a morphism of deflations $(P \xrightarrow{id_P} P) \xrightarrow{\mathfrak{u}} (M \xrightarrow{\mathfrak{e}} P)$. This implies that the canonical morphism $S_-(F)(P) \longrightarrow Ker(F(\mathfrak{k}(\mathfrak{e}))$ factors through the morphism $Ker(F(i_P)) \longrightarrow Ker(F(\mathfrak{k}(\mathfrak{e}))$ determined by the morphism of deflations $\mathfrak{u}$. Since $Ker(F(i_P)) = y$ is an initial object of the category C_Y, it follows that the morphism $Ker(F(i_P)) \longrightarrow Ker(F(\mathfrak{k}(\mathfrak{e}))$ is unique (in particular, it does not depend on the choice of the section $P \xrightarrow{\mathfrak{u}} M$). Therefore, the canonical morphism $S_-(F)(P) \longrightarrow Ker(F(i_P)) = y$ is an isomorphism. ∎

2.2.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y. Suppose that $(C_X, \mathfrak{E}_X)$ has enough projective objects and projective objects of $(C_X, \mathfrak{E}_X)$ are pointed objects. Then the functors T_i are coeffaceable for all $i \geq 1$.*

Proof. The assertion follows from 2.2.1 and 2.1. ∎

2.2.3. Note. If a right exact category $(C_X, \mathfrak{E}_X)$ has enough pointed objects, then each of its projective objects is pointed. In fact, "enough pointed objects" implies that, for any projective P, there exists a deflation $M \longrightarrow P$ with M a pointed object. This deflation splits, because P is a projective object; so that there exists an arrow $P \longrightarrow M$, hence is an arrow from P to an initial object.

2.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Z, \mathfrak{E}_Z)$ be right exact categories with initial objects; and let $C_Z \xrightarrow{f^*} C_X$ be a functor having a right adjoint f_*. Suppose that the functor f^* maps deflations of the form $N \longrightarrow f_*(M)$ to deflations and the adjunction arrow $f^*f_*(M) \xrightarrow{\epsilon(M)} M$ is a deflation for all M (which is the case if any morphism $\mathfrak{t}$ of C_X such that $f_*(\mathfrak{t})$ is a split epimorphism belongs to $\mathfrak{E}_X$). If the right exact category $(C_Z, \mathfrak{E}_Z)$ has enough pointed objects, then each projective of $(C_X, \mathfrak{E}_X)$ is a pointed object.*

If, in addition, f_* maps deflations to deflations, then $(C_X, \mathfrak{E}_X)$ has enough projective objects.

Proof. (a) Let M be an object of C_X. Since $(C_Z, \mathfrak{E}_Z)$ has enough pointed objects, there exists a deflation $\widetilde{P} \xrightarrow{\mathfrak{t}} f_*(M)$, where $\widetilde{P}$ is a *pointed object*. By hypothesis, the morphisms

$$f^*(\widetilde{P}) \xrightarrow{f^*(\mathfrak{t})} f^*f_*(M) \xrightarrow{\epsilon(M)} M \qquad (1)$$

are deflations. Since the functor f^* has a right adjoint, if z is an initial object of the category C_Z, then $f^*(z)$ is an initial object of the category C_X. Therefore, the functor f^* maps pointed objects to pointed objects. In particular, the object $f^*(\widetilde{P})$ in (1) is pointed. Altogether shows that the right exact category $(C_X, \mathfrak{E}_X)$ has enough pointed objects. Therefore, by 2.2.3, every projective object of $(C_X, \mathfrak{E}_X)$ is pointed.

(b) If, in addition, the functor f_* maps deflations to deflations, then, by 1.3.1, its left adjoint f^* maps projective objects to projective objects. So that if the object $\widetilde{P}$ in the argument above is a projective, then the composition of the arrows (1) is a deflation with a projective domain. This shows that the right exact category $(C_X, \mathfrak{E}_X)$ has enough projective objects. ■

2.4. Note. The conditions of 2.3 can be replaced by the requirement that if $N \longrightarrow f_*(M)$ is a deflation, then the corresponding morphism $f^*(N) \longrightarrow M$ is a deflation. This requirement follows from the conditions of 2.3, because the morphism $f^*(N) \longrightarrow M$ corresponding to $N \xrightarrow{\mathfrak{t}} f_*(M)$ is the composition of $f^*(\mathfrak{t})$ and the adjunction arrow $f^*f_*(M) \xrightarrow{\epsilon(M)} M$.

2.5. Example. Let $(C_X, \mathfrak{E}_X)$ be the category Alg_k of associative k-algebras endowed with the canonical (that is the finest) right exact structure. This means that class $\mathfrak{E}_X$ of deflations coincides with the class of all are strict epimorphisms of k-algebras. Let $(C_Y, \mathfrak{E}_Y)$ be the category of k-modules with the canonical exact structure, and f_* the forgetful functor $Alg_k \longrightarrow k - mod$. Its left adjoint, f^* preserves (all colimits; in particular, it preserves) strict epimorphisms, and the functor f_* preserves and reflects deflations; i.e. a k-algebra morphism t is a strict epimorphism iff $f_*(t)$ is an epimorphism. In particular, the adjunction arrow $f^*f_*(A) \longrightarrow A$ is a strict epimorphism for all A. By 2.3, $(C_X, \mathfrak{E}_X)$ has enough projective objects and each projective has a morphism to the initial object k; that is each projective has a structure of an augmented k-algebra.

2.6. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects; and let $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ be an 'exact' ∂^*-functor from $(C_X, \mathcal{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.*

Suppose that $\mathfrak{E}_Y^{\circledast} = Iso(C_Y)$ and the functors T_i are $\mathfrak{E}_X$-coeffaceable for $i \geq 1$. Then T is a universal ∂^*-functor.

Proof. Let $T' = (T'_i, \mathfrak{d}'_i |\ i \geq 0)$ be another ∂^*-functor from $(C_X, \mathcal{E}_X)$ to C_Y and f_0 a functor morphism $T'_0 \longrightarrow T_0$. Fix an object L of C_X. Let $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$ be a conflation such that $T_1(\mathfrak{e})$ factors through the initial object y of C_Y. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
T'_1(M) & \xrightarrow{T'_1(\mathfrak{e})} & T'_1(L) & \xrightarrow{\mathfrak{d}'} & T'_0(N) & \xrightarrow{T'_0(\mathfrak{j})} & T'_0(M) & \xrightarrow{T'_0(\mathfrak{e})} & T'_0(L) \\
 & & & & {\scriptstyle f_0(N)}\downarrow & & \downarrow{\scriptstyle f_0(M)} & & \downarrow{\scriptstyle f_0(L)} \\
T_1(M) & \xrightarrow{T_1(\mathfrak{e})} & T_1(L) & \xrightarrow{\mathfrak{d}} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) & \xrightarrow{T_0(\mathfrak{e})} & T_0(L)
\end{array} \qquad (1)$$

Since the lower row of the diagram (1) is an 'exact' sequence and the morphism

$$T_1(M) \xrightarrow{T_1(\mathfrak{e})} T_1(L)$$

factors through the initial object y of the category C_Y, the sequence

$$y \longrightarrow T_1(L) \xrightarrow{\mathfrak{d}} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M) \qquad (2)$$

is 'exact'. Since, by hypothesis, $\mathfrak{E}_Y^{\circledast} = Iso(C_Y)$, it follows from the 'exactness' of (2) that the canonical morphism from $T_1(L)$ to the kernel of $T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$ is an isomorphism. Therefore, there exists a unique morphism $T_1'(L) \xrightarrow{f_1(L)} T_1(L)$ such that the diagram

$$\begin{array}{ccccc} T_1'(L) & \xrightarrow{\mathfrak{d}'} & T_0'(N) & \xrightarrow{T_0'(\mathfrak{j})} & T_0'(M) \\ {\scriptstyle f_1(L)} \downarrow & & {\scriptstyle f_0(N)} \downarrow & & \downarrow {\scriptstyle f_0(M)} \\ T_1(L) & \xrightarrow{\mathfrak{d}} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) \end{array}$$

commutes. By a standard argument, it follows from the uniqueness of $f_1(L)$ and the fact that the family of the deflations of L is filtered (since pull-backs of deflations are deflations) that the morphism $f_1(L)$ does not depend on a choice of the conflation and the family $f_1 = (f_1(L) \mid L \in ObC_X)$ is a functor morphism $T_1' \longrightarrow T_1$ compatible with the connecting morphisms $\mathfrak{d}_0$, $\mathfrak{d}_0'$. ■

2.6.1. Note. If a right exact category $(C_X, \mathfrak{E}_X)$ has enough projective objects and (enough pointed objects, so that) each projective is a pointed object, then, by 2.2.2, for any universal ∂^*-functor T, the functors T_i are $\mathfrak{E}_X$-coeffaceable for $i \geq 1$.

2.7. A refinement: acyclic objects. Let $(C_X, \mathfrak{E}_X)$ be a right exact category, C_Y a category with initial objects, and F a functor from C_X to C_Y. An object M of the category C_X is called F-*acyclic*, if the higher images (– satellites) of the composition of F with the Yoneda embedding $C_Y \xrightarrow{h_Y^*} C_Y^*$ map M to initial elements; that is $S_-^i(h_Y^* \circ F)(M)$ is an initial object of the category C_Y for all $i \geq 1$.

2.7.1. Remarks. (a) It follows from 2.2.2 that every pointed projective is F-acyclic for all functors F from the category C_X.

(b) If $(C_X, \mathfrak{E}_X)$ is a right exact category and F is a functor $C_X \longrightarrow C_Y$ such that there are enough F-acyclic objects, then, evidently, all higher images (satellites) of F are coeffaceable.

2.7.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$, and $(C_Z, \mathfrak{E}_Z)$ be right exact categories with initial objects. Let $T = (T_i, \mathfrak{d}_i | \ i \geq 0)$ is a universal 'exact' ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ such that there are enough T_0-acyclic objects.*

Suppose that $\mathfrak{E}_Z^{\circledast} = Iso(C_Z)$. *Then, for any functor* F *from* $(C_Y, \mathfrak{E}_Y)$ *to* $(C_Z, \mathfrak{E}_Z)$ *which respects conflations, the composition* $F \circ T = (F \circ T_i, F\mathfrak{d}_i|\ i \geq 0)$ *is a universal 'exact'* ∂^**-functor.*

Proof. Since T is a universal ∂^*-functor and there are enough T_0-acyclic objects, the functors T_i, $i \geq 1$, map all T_0-acyclic objects to initial objects of the category C_Y. If a functor $C_Y \xrightarrow{F} C_Z$ preserves conflations, it maps initial objects to initial objects and the composition and its composition $F \circ T = (F \circ T_i, F\mathfrak{d}_i|\ i \geq 0)$ with the 'exact' ∂^*-functor T is an 'exact' ∂^*-functor. Since there are enough T_0-acyclic objects and the functors $F \circ T_i$, $i \geq 1$, map them to initial objects, all these functors are coeffaceable. Therefore, by 2.6, the ∂^*-functor $F \circ T$ is universal. ∎

2.8. A remark about (co)effaceable functors. Let C_X be a category with initial objects and $\mathcal{B}$ its subcategory. We say that an object M of C_X is *right* (resp. *left*) *orthogonal* to $\mathcal{B}$ if for every $N \in Ob\mathcal{B}$, there are only trivial morphisms, or no morphisms, from N to M (resp. from M to N). We denote by $\mathcal{B}^{\perp}$ (resp. ${}^{\perp}\mathcal{B}$) the full subcategory of C_X generated by objects right (resp. left) orthogonal to $\mathcal{B}$.

Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and let y be an initial object of the category C_Y. The category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y has an initial object, which is the constant functor with values in y. Let $\mathfrak{Rex}((C_X, \mathfrak{E}_X), (C_Y, \mathfrak{E}_Y))$ be the full subcategory of $\mathcal{H}om(C_X, C_Y)$ whose objects are right 'exact' functors. And let $\mathcal{E}ff^o((C_X, \mathfrak{E}_X), C_Y)$ denote the full subcategory of $\mathcal{H}om(C_X, C_Y)$ generated by coeffaceable functors from $(C_X, \mathfrak{E}_X)$ to C_Y.

2.8.1. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *be a svelte right exact category with enough projective objects and* $(C_Y, \mathfrak{E}_Y)$ *a right exact category with initial objects. Suppose that* C_Y *is a category with kernels of morphisms and the morphisms from the initial objects of* C_Y *are monomorphisms. Then* $\mathcal{E}ff^o((C_X, \mathfrak{E}_X), C_Y)$ *is right orthogonal to the subcategory generated by all functors* $C_X \longrightarrow C_Y$ *which map deflations to strict epimorphisms.*

Proof. Let $F \in Ob\mathcal{E}ff^o((C_X, \mathfrak{E}_X), C_Y)$; and let $G \xrightarrow{\phi} F$ be a functor morphism, where G is a functor which maps deflations to strict epimorphisms. Since $(C_X, \mathfrak{E}_X)$ has enough projective objects, for each object L of C_X, there exists a deflation $P \xrightarrow{\mathfrak{e}} L$ such that P is a projective object. Then we have a commutative diagram

$$\begin{array}{ccc} G(P) & \xrightarrow{\phi(P)} & F(P) \\ {\scriptstyle G(\mathfrak{e})}\downarrow & & \downarrow{\scriptstyle F(\mathfrak{e})} \\ G(L) & \xrightarrow{\phi(L)} & F(L) \end{array} \qquad (1)$$

Since P is a projective object of $(C_X, \mathfrak{E}_X)$ and the functor F is coeffaceable, $F(P)$ is an initial object of the category C_Y. Therefore, the square (1) decomposes into

$$\begin{array}{ccccc} G(P) & \xrightarrow{\gamma} & Ker(\phi(L)) & \xrightarrow{\lambda} & F(P) \\ & & {\scriptstyle \mathfrak{k}(\phi(L))}\downarrow & cart & \downarrow {\scriptstyle F(\mathfrak{e})} \\ & & G(L) & \xrightarrow{\phi(L)} & F(L) \end{array} \qquad (2)$$

where the morphism $G(P) \xrightarrow{\gamma} Ker(\phi(L))$ is uniquely determined by the equalities $G(\mathfrak{e}) = \mathfrak{k}(\phi(\mathfrak{e})) \circ \gamma$ and $\phi(P) = \lambda \circ \gamma$. By hypothesis, $G(\mathfrak{e})$ is a strict epimorphism. Therefore, it follows from the equality $G(\mathfrak{e}) = \mathfrak{k}(\phi(\mathfrak{e})) \circ \gamma$ that

$$Ker(\phi(L)) \xrightarrow{\mathfrak{k}(\phi(L))} G(L)$$

is a strict epimorphism.

On the other hand, by I.4.1.2, the condition that the morphisms from initial objects are monomorphisms means precisely that, for any morphism $M \xrightarrow{f} N$, the kernel morphism $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism. Therefore, $Ker(\phi(L)) \xrightarrow{\mathfrak{k}(\phi(L))} G(L)$, being a strict epimorphism and a monomorphism, is an isomorphism. Therefore, it follows from the cartesian square in (2) that $G(L) \xrightarrow{\phi(L)} F(L)$ factors through the initial object $F(P)$; i.e. it is a trivial morphism. ■

2.8.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact pointed category with enough projective objects, and let $(C_Y, \mathfrak{E}_Y)$ be the category of pointed sets with the canonical exact structure. Then a functor $C_X \xrightarrow{F} C_Y$ is coeffaceable iff it is a right orthogonal to the subcategory $\mathfrak{Ex}((C_X, \mathfrak{E}_X), (C_Y, \mathfrak{E}_Y))$ of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.*

Proof. The fact that coeffaceable functors to C_Y are right orthogonal to 'exact' functors follows from 2.8.1, because 'exact' functors map deflations to deflations, and deflations in C_Y are strict epimorphisms.

Conversely, let a functor $C_X \xrightarrow{F} C_Y$ be right orthogonal to all right 'exact' functors from $(C_X, \mathfrak{E}_X)$ to C_Y. Notice that for any projective object P of $(C_X, \mathfrak{E}_X)$, the functor $\check{P} = C_X(P, -)$ is 'exact'. By the (dual version of) Yoneda lemma, $Hom(\check{P}, F) \simeq F(P)$. By hypothesis, $Hom(\check{P}, F)$ consists of the trivial morphism. So that $F(P)$ is trivial for all projective objects P of $(C_X, \mathfrak{E}_X)$. Since $(C_X, \mathfrak{E}_X)$ has enough projective objects, this means precisely that F is a coeffaceable functor. ■

The k-linear version of 2.8.2 is as follows.

2.8.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact k-linear category with enough projective objects. A k-linear functor $C_X \xrightarrow{F} k-mod$ is coeffaceable iff it is right* *orthogonal to the subcategory* $\mathfrak{Ex}_k((C_X,\mathfrak{E}_X), k-mod)$ *of 'exact' k-linear functors from* $(C_X,\mathfrak{E}_X)$ *to the abelian category* $k-mod$.

Proof. The argument is similar to that of 2.8.2. ■

2.9. Digression: sheafification functor and weakly coeffaceable presheaves.

2.9.0. The Heller functor and sheafification. Let $\mathcal{PS}(X,C_{\mathfrak{Z}})$ denote the category of presheaves on the category C_X with values in the category $C_{\mathfrak{Z}}$. We denote by $Sh((X,\mathfrak{E}_X),C_{\mathfrak{Z}})$ the category of sheaves on the right exact category $(C_X,\mathfrak{E}_X)$ with values in the category $C_{\mathfrak{Z}}$. We assume that the category $C_{\mathfrak{Z}}$ has filtered colimits which commute with kernels of pairs of arrows. Let $\mathcal{H}_X$ denote the endofunctor of $\mathcal{PS}(X,C_{\mathfrak{Z}})$, which assigns to every presheaf $C_X^{op}\xrightarrow{F}C_{\mathfrak{Z}}$ the presheaf $\mathcal{H}_X(F)$ defined by

$$\mathcal{H}_X(F)(N)=\operatorname{colim}(Ker(F(M)\rightrightarrows F(M\times_N M))) \qquad (1)$$

where colimit is taken by the diagram $\mathfrak{E}_X(N)$ of deflations $M\longrightarrow N$.

The correspondence $F\longmapsto\mathcal{H}_X(F)$ is functorial in F; i.e. it extends to an endofunctor

$$\mathcal{PS}(X,C_{\mathfrak{Z}})\xrightarrow{\mathcal{H}_X}\mathcal{PS}(X,C_{\mathfrak{Z}})$$

The morphisms

$$F(N)\longrightarrow Ker(F(M)\rightrightarrows F(M\times_N M))$$

determine a morphism $F(N)\xrightarrow{\tau_F(N)}\mathcal{H}_X(F)(N)$ for every $N\in ObC_X$ which is functorial in N; i.e. it defines a functor morphism $F\xrightarrow{\tau_F}\mathcal{H}_X(F)$. The function $F\longmapsto\tau_F$ is a functor morphism from the identical functor to the endofunctor $\mathcal{H}_X$.

2.9.0.1. Note. The endofunctor $\mathcal{PS}(X,C_{\mathfrak{Z}})\xrightarrow{\mathcal{H}_X}\mathcal{PS}(X,C_{\mathfrak{Z}})$ is a special case of the *Heller* endofunctor on presheaves associated with a Grothendieck (pre)topology.

2.9.0.2. Monopresheaves and the Heller functor. A presheaf F on $(C_X,\mathfrak{E}_X)$ is called a *monopresheaf* if for every deflation $M\xrightarrow{\mathfrak{e}}N$, the morphism $F(N)\xrightarrow{F(\mathfrak{e})}F(M)$ is a monomorphism. There are the following facts:

(a) A presheaf of F is a monopresheaf (resp. a sheaf) iff the canonical morphism $F\xrightarrow{\tau_F}\mathcal{H}_X(F)$ is a monomorphism (resp. an isomorphism).

(b) The functor $\mathcal{H}_X$ maps presheaves to monopresheaves and monopresheaves to sheaves.

It follows from (a) and (b) that the functor $\mathcal{H}_X^2$ maps presheaves to sheaves and its corestriction to the subcategory $Sh((X,\mathfrak{E}_X),C_{\mathfrak{Z}})$ of sheaves is isomorphic to the sheafification functor $\mathfrak{t}_X^*$. Or, what is the same, $\mathcal{H}_X^2 \simeq \mathfrak{t}_{X*}\mathfrak{t}_X^*$.

Another consequence of (a) and (b) is that the kernel of $\mathfrak{t}_X^*$ coincides with the kernel of the functor $\mathcal{H}_X$.

2.9.1. Weakly coeffaceable presheaves. Let now $C_{\mathfrak{Z}}$ be the category $k-mod$ of modules over a commutative unital ring k. Recall that $\mathcal{M}_k(X)$ denotes the category of k-linear presheaves of k-modules on the category C_X and $Sh_k(X,\mathfrak{E}_X)$ the category of k-linear sheaves of k-modules on the presite $(C_X,\mathfrak{E}_X)$.

We call a presheaf F *weakly coeffaceable* if, for every pair (N,ξ), where $N \in ObC_X$ and ξ is an element of $F(N)$, there exists a deflation $M \xrightarrow{\mathfrak{e}} N$ such that $F(\mathfrak{e})(\xi) = 0$. Equivalently, for any object N of C_X and any morphism $\widehat{N} \xrightarrow{\widetilde{\xi}} F$, there exists a deflation $M \xrightarrow{\mathfrak{e}} N$ such that the composition of $\widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{N}$ and $\widetilde{\xi}$ equals to zero.

It follows from the formula (1) that a presheaf F belongs to the kernel of $\mathcal{H}_X$ iff it is weakly coeffaceable.

Thus, objects of the kernel $\mathcal{S}_{\mathfrak{E}_X}$ of the sheafification functor $\mathcal{M}_k(X) \xrightarrow{\mathfrak{t}_X^*} Sh_k(X)$ are precisely weakly coeffaceable presheaves. Since the functor $\mathfrak{t}_X^*$ is a flat localization, $\mathcal{S}_{\mathfrak{E}_X}$ is a Serre subcategory of the category $\mathcal{M}_k(X)$, and the category of sheaves $Sh_k(X)$ is equivalent to the quotient category $\mathcal{M}_k(X)/\mathcal{S}_{\mathfrak{E}_X}$.

3. The structure of universal ∂-functors to cocomplete categories.

3.1. Observations. Let $(C_X,\mathfrak{I}_X)$ be a svelte left exact category with a final object x and C_Y a category with a final object y and arbitrary colimits. Then, by the (dual version of the) argument of II.3.3.2, we have an endofunctor S_+ of the category $\mathcal{H}om(C_X,C_Y)$ of functors from C_X to C_Y, together with a cone $\mathfrak{y} \xrightarrow{\lambda} S_+$, where $\mathfrak{y}$ is the constant functor with the values in the final object y of the category C_Y. For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of $(C_X,\mathfrak{I}_X)$ and any functor $C_X \xrightarrow{F} C_Y$, we have a commutative diagram

$$\begin{array}{ccccc}
F(N) & \xrightarrow{F\mathfrak{j}} & F(M) & \xrightarrow{F\mathfrak{e}} & F(L) \\
 & & \big\downarrow & & \big\downarrow \mathfrak{d}_0(E) \\
 & & y & \xrightarrow{\lambda(N)} & S_+F(N)
\end{array} \qquad (1)$$

Here $S_+F(N) = \mathrm{colim}(Cok(F(M' \xrightarrow{\mathfrak{e}'} N)))$, where the colimit is taken by the diagram of all conflations $N \xrightarrow{\mathfrak{j}'} M' \xrightarrow{\mathfrak{e}'} L$ (see the argument of II.3.3.2). By I.2.0.2(a), the functor of composition with the Yoneda embedding $C_X \xrightarrow{h_X^*} C_X^*$ of C_X into the category C_X^* of non-trivial presheaves of sets on C_X,

$$\mathcal{H}om(C_X^*, C_Y) \xrightarrow{\circ h_X^*} \mathcal{H}om(C_X, C_Y), \quad G \longmapsto G \circ h_X^*,$$

establishes an equivalence between the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y and the full subcategory $\mathfrak{Hom}(C_X^*, C_Y)$ of the category $\mathcal{H}om(C_X^*, C_Y)$ generated by all functors $C_X^* \longrightarrow C_Y$ preserving colimits. Let $F^\diamond$ denote a (determined uniquely up to isomorphism) functor from $\mathfrak{Hom}(C_X^*, C_Y)$ corresponding to F, i.e. $F = F^\diamond \circ h_X^*$.

Since the functor $F^\diamond$ preserves colimits, the formula for $S_+F(N)$ can be rewritten as follows:

$$\begin{aligned} S_+F(N) &= \operatorname{colim}(Cok(F(M' \xrightarrow{\mathfrak{e}'} N))) = \operatorname{colim}(Cok(F^\diamond(\widehat{M}' \xrightarrow{\widehat{\mathfrak{e}'}} \widehat{N}))) \\ &= F^\diamond(\operatorname{colim}(Cok(\widehat{M}' \xrightarrow{\widehat{\mathfrak{e}'}} \widehat{N}))) = F^\diamond \circ S_+h_X^*(N) = F^\diamond \circ Ext^1_{X,\mathfrak{I}_X}(N). \end{aligned} \tag{2}$$

where colimit is taken by the diagram of all conflations $N \xrightarrow{\mathfrak{j}'} M' \xrightarrow{\mathfrak{e}'} L$ with the fixed object N, or, what is the same, by the diagram of all inflations $N \xrightarrow{\mathfrak{j}'} M'$ of N.

3.2. A structure of a $\mathbb{Z}_+$-category on C_X^*. We denote by $\widehat{\Theta}_X^*$ the preserving colimits functor $C_X^* \longrightarrow C_X^*$ corresponding to $Ext^1_{X,\mathfrak{I}_X}$; that is, in the notations of 3.1, $\widehat{\Theta}_X^* = (Ext^1_{X,\mathfrak{I}_X})^\diamond$. Thus, a left exact structure $\mathfrak{I}_X$ on the category C_X determines a canonical structure of a $\mathbb{Z}_+$-category on the category C_X^*.

3.3. Standard triangles. Taking as F the Yoneda functor h_X (and setting $\widehat{N} = h_X(N)$), we obtain from the diagram (1) the diagram

$$\begin{array}{ccccc} \widehat{N} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{L} \\ & & \big\downarrow & & \big\downarrow \mathfrak{d}_0(E) \\ & & \widehat{x} & \xrightarrow{\lambda(\widehat{N})} & \widehat{\Theta}_X^*(\widehat{N}) \end{array} \tag{1}$$

3.3.1. Definition. We call the subdiagram

$$\widehat{N} \xrightarrow{\widehat{\mathfrak{j}}} \widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{L} \xrightarrow{\mathfrak{d}_0(E)} \widehat{\Theta}_X^*(\widehat{N})$$

of the diagram (1) a *standard triangle.*

3.3.2. Note. If C_X is a pointed category, then the presheaf $\widehat{x} = C_X(-, x)$ is both a final and an initial object of the category C_X^*. In particular, the morphism

$$\widehat{x} \xrightarrow{\lambda(\widehat{N})} \widehat{\Theta}_X^*(\widehat{N})$$

in (1) is unique; hence it is not a part of the data.

3.4. Triangles in the category of presheaves. A *triangle* is any diagram in C_X^* of the form

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \widehat{\Theta}_X^*(\mathcal{N}),$$

which is isomorphic to a standard triangle. It follows that, for every triangle, the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \downarrow & & \downarrow \mathfrak{d} \\ \widehat{x} & \xrightarrow{\lambda(\mathcal{N})} & \widehat{\Theta}_X^*(\mathcal{N}) \end{array}$$

commutes. Triangles form a category $\mathfrak{Tr}_{X^*}$: morphisms from

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \widehat{\Theta}_X^*(\mathcal{N})$$

to

$$\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \Theta_X^*(\mathcal{N}')$$

are given by commutative diagrams

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \widehat{\Theta}_X^*(\mathcal{N}) \\ f \downarrow & & \downarrow g & & \downarrow h & & \downarrow \widehat{\Theta}_X^*(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \widehat{\Theta}_X^*(\mathcal{N}') \end{array}$$

The composition is obvious.

3.5. Prestable category of presheaves. Thus, the left exact structure $\mathfrak{I}_X$ on the category C_X produces the data $(C_X^*, \mathfrak{I}_X^*, \widehat{\Theta}_X^*, \mathfrak{Tr}_{X^*})$, where $\mathfrak{I}_X^*$ is the coarsest left exact structure on C_X^* which is closed under filtered colimits and makes the Yoneda embedding $C_X \xrightarrow{h_X^*} C_X^*$ an 'exact' functor from $(C_X, \mathfrak{I}_X)$ to $(C_X^*, \mathfrak{I}_X^*)$ (see the argument of II.9.1), $\widehat{\Theta}_X^*$ a continuous endofunctor of C_X^* corresponding to Ext_X^1, $\mathfrak{Tr}_{X^*}$ the category of triangles on the category of presheaves. We call the data $(C_X^*, \mathfrak{I}_X^*, \widehat{\Theta}_X^*, \mathfrak{Tr}_{X^*})$ the *prestable category of presheaves* on the left exact category $(C_X, \mathfrak{I}_X)$.

Notice that already $(C_X^*, \widehat{\Theta}_X^*, \mathfrak{Tr}_{X^*})$ contains all the information about the universal ∂-functor $Ext_X^\bullet = (Ext_X^i, \mathfrak{d}_i \mid i \geq 0)$, and, therefore, due to the universality of $Ext_X^\bullet$, all the information about all universal ∂-functors from the left exact category $(C_X, \mathfrak{I}_X)$ to cocomplete categories. In fact, the universal ∂-functor $Ext_X^\bullet$ is of the form $(\widehat{\Theta}_X^{*n} \circ h_X, \widehat{\Theta}_X^{*n}(\mathfrak{d}_0) | n \geq 0)$; and for any functor F from C_X to a category C_Y with colimits and final objects, the universal ∂-functor $(T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathfrak{I}_X)$ to C_Y with $T_0 = F$ is isomorphic to

$$F^\diamond \circ Ext^\bullet_X = (F^\diamond \widehat{\Theta}^{*n}_X, F^\diamond \widehat{\Theta}^{*n}_X(\mathfrak{d}_0) \mid n \geq 0) \circ h^*_X. \tag{1}$$

Here $\mathfrak{d}_0 = (\mathfrak{d}_0(E))$, where E runs through conflations, and $\mathfrak{d}_0(E)$ is a connecting morphism in the diagram 3.3(1).

3.5.1. A "presuspended" structure on the category $C^\wedge_X$. Following (the argument of) I.2.0.2(c), for any presheaf of sets $\mathcal{G}$ on C_X, we consider the presheaf of sets

$$\widehat{\theta}_{X*}(\mathcal{G})(-) = C^\wedge_X(\mathfrak{Ext}^1_{X,\mathfrak{I}_X}(-), \mathcal{G}), \tag{1}$$

where $\mathfrak{Ext}^1_{X,\mathfrak{I}_X} = \mathcal{S}_+ h_X$ – the derived functor of the Yoneda embedding $C_X \xrightarrow{h_X} C^\wedge_X$ (see II.8.2.2). The map $\mathcal{G} \longmapsto \widehat{\theta}_{X*}(\mathcal{G})$ extends to an endofunctor $C^\wedge_X \xrightarrow{\widehat{\theta}_{X*}} C^\wedge_X$. It follows from the definition of $\widehat{\theta}_{X*}$ (and the Yoneda's formula) that

$$C^\wedge_X(\mathfrak{Ext}^1_{X,\mathfrak{I}_X}(-), \mathcal{G}) = \widehat{\theta}_{X*}(\mathcal{G})(-) \simeq C^\wedge_X(-, \widehat{\theta}_{X*}(\mathcal{G})). \tag{2}$$

Let $\widehat{\theta}^*_X$ denote the continuous functor $C^\wedge_X \longrightarrow C^\wedge_X$ corresponding to $\mathfrak{Ext}^1_{X,\mathfrak{I}_X}$. It follows from the definition of the functor $\widehat{\theta}_{X*}$ (and 7.0.2(c)) that

$$C^\wedge_X(\widehat{\theta}^*_X(-), \mathcal{G}) \simeq C^\wedge_X(-, \widehat{\theta}_{X*}(\mathcal{G})),$$

that is the functor $\widehat{\theta}_{X*}$ is a right adjoint to $\widehat{\theta}^*_X$.

The triangles are the same as in 3.4. Namely, a triangle is a diagram

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \widehat{\theta}^*_X(\mathcal{N}),$$

which is isomorphic to a standard triangle, that is a diagram

$$\widehat{N} \xrightarrow{\ \widehat{\mathfrak{j}}\ } \widehat{M} \xrightarrow{\ \widehat{\mathfrak{e}}\ } \widehat{L} \xrightarrow{\ \mathfrak{d}_0(E)\ } \widehat{\theta}^*_X(\widehat{N}) = \widehat{\Theta}^*_X(\widehat{N})$$

for a conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$.

3.6. Functorialities.

3.6.1. Preliminary remarks. By I.2.0.2, there is a natural equivalence between the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X and C_Y and the full subcategory $\mathfrak{Hom}(C^*_X, C^*_Y)$ of the category $\mathcal{H}om_{\mathfrak{c}}(C^*_X, C^*_Y)$ generated by all those functors from C^*_X

to C_Y^* which map representable presheaves to representable presheaves and preserve colimits. This equivalence assigns to any functor $C_X \xrightarrow{F} C_Y$ the (unique up to isomorphism) continuous functor $C_X^* \xrightarrow{F^*} C_Y^*$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{h_X^*} & C_X^* \\ F \downarrow & & \downarrow F^* \\ C_Y & \xrightarrow{h_Y^*} & C_Y^* \end{array}$$

commutes.

3.6.1.1. Lemma. *Let $(C_X, \mathfrak{I}_X)$ be a svelte left exact category with final objects, C_Y a svelte category with final objects, and C_Z a category with colimits. Then, for any pair of functors $C_X \xrightarrow{F} C_Y \xrightarrow{G} C_Z$, the continuous functor*

$$C_X^* \xrightarrow{(G\circ F)^\diamond} C_Z,$$

which is determined (uniquely up to isomorphism) by the equality $(G \circ F)^\diamond \circ h_X^* = G \circ F$, is isomorphic to the composition $G^\diamond \circ F^*$.

Proof. By definition, $G^\diamond$ is a preserving colimits functor $C_Y^* \longrightarrow C_Z$ such that $G^\diamond \circ h_Y^* = G$. The composition of preserving colimits functors preserves colimits; so that the functor $G^\diamond \circ F^*$ preserves colimits. By definition of the functor F^*, we have:

$$(G^\diamond \circ F^*) \circ h_X^* = G^\diamond \circ (h_Y^* \circ F) = G \circ F,$$

whence the assertion. ■

3.6.2. Proposition. *Let $(C_X, \mathfrak{I}_X)$ be a svelte left exact category with final objects, C_Y a svelte category with final objects, and C_Z a category with colimits. Then, for any pair of functors $C_X \xrightarrow{F} C_Y \xrightarrow{G} C_Z$, we have the following formula:*

$$S_+^\bullet(G \circ F) = G^\diamond \circ F^* \circ Ext^\bullet_{X,\mathfrak{I}_X} = G^\diamond \circ F^* \circ (\widehat{\Theta}_X^{*n}, F^*\widehat{\Theta}_X^{*n}(\mathfrak{d}_0) \mid n \geq 0) \circ h_X^*. \qquad (1)$$

Proof. The fact follows from 3.6.1.1 and the formula (1) in 3.5. ■

3.6.3. Remarks. (a) Suppose that both categories C_Y and C_Z in 3.6.2 have colimits. Then there is a canonical morphism

$$G^\diamond \circ F^* \longrightarrow G \circ F^\diamond, \qquad (2)$$

which is due to the equalities $G^\diamond \circ F^* \circ h_X^* = G \circ F = G \circ F^\diamond \circ h_X^*$ and the fact that, for any presheaf of sets $\mathfrak{F}$,

$$G^\diamond \circ F^*(\mathfrak{F}) = \operatorname{colim} G \circ F \circ (h_X^*/\mathfrak{F} \longrightarrow C_X) \quad \text{and} \quad G \circ F^*(\mathfrak{F}) = G(\operatorname{colim} F \circ (h_X/\mathfrak{F} \longrightarrow C_X)),$$

where $h_X^*/\mathfrak{F} \longrightarrow C_X$ is the forgetful functor.

(b) Since the the category C_Z in 3.6.2 has colimits, the ∂-functor $S_+^\bullet(G \circ F)$ is well defined. Suppose that the ∂-functor $S_+^\bullet(F)$ exists. Then there is a natural morphism

$$S_+^\bullet(G \circ F) = G^\diamond \circ F^* \circ Ext^\bullet_{X,\mathfrak{I}_X} \longrightarrow G \circ F^\diamond \circ Ext^\bullet_{X,\mathfrak{I}_X} = G \circ S_+^\bullet(F), \tag{3}$$

This follows from the universality of the ∂-functor $S_+^\bullet(G \circ F)$ and the fact that its zero component coincides with the zero component of the ∂-functor $G \circ S_+^\bullet(F)$.

3.6.4. Functoriality of prestable category of presheaves. Let $(C_X, \mathfrak{I}_X)$ and $(C_Y, \mathfrak{I}_Y)$ be svelte left exact categories with final objects and $C_X \xrightarrow{F} C_Y$ a functor which preserves conflations; that is F is a *weakly 'exact'* functor from $(C_X, \mathfrak{I}_X)$ to $(C_Y, \mathfrak{I}_Y)$. Then the corresponding (preserving colimits) functor $C_X^* \xrightarrow{F^*} C_Y^*$ (cf. 3.6.1) is a weakly 'exact' functor from the left exact category $(C_X^*, \mathfrak{I}_X^*)$ of presheaves on $(C_X, \mathfrak{I}_X)$ to the left exact category $(C_Y^*, \mathfrak{I}_Y^*)$ of presheaves on $(C_Y, \mathfrak{I}_Y)$.

Since F is a weakly 'exact' functor from $(C_X, \mathfrak{I}_X)$ to $(C_Y, \mathfrak{I}_Y)$, its composition with ∂-functors from $(C_Y, \mathfrak{I}_Y)$ produces ∂-functors (see II.2.1). In particular, $Ext^\bullet_{Y,\mathfrak{I}_Y} \circ F$ is a ∂-functor. On the other hand, $S_+^\bullet(h_Y^* \circ F)$ is a universal ∂-functor having the same zero component as $Ext^\bullet_{Y,\mathfrak{I}_Y} \circ F$. Therefore, there exists a unique morphism

$$S_+^\bullet(h_Y^* \circ F) \longrightarrow Ext^\bullet_{Y,\mathfrak{I}_Y} \circ F \tag{4}$$

of ∂-functors. Notice that

$$Ext^\bullet_{Y,\mathfrak{I}_Y} \circ F = (\widehat{\Theta}_Y^\bullet, \widehat{\Theta}_Y^\bullet(\mathfrak{d}_0)) \circ h_Y^* \circ F = (\widehat{\Theta}_Y^\bullet, \widehat{\Theta}_Y^\bullet(\mathfrak{d}_Y)) \circ F^* \circ h_X^*.$$

On the other hand,

$$S_+^\bullet(h_Y^* \circ F) = S_+^\bullet(F^* \circ h_X^*) = F^* \circ (\widehat{\Theta}_X^\bullet, \widehat{\Theta}_X^\bullet(\mathfrak{d}_X)) \circ h_X^*.$$

Since functors F^*, $\widehat{\Theta}_X$, $\widehat{\Theta}_Y$ and all their compositions are continuous, the morphism (4) induces (and is determined by) a morphism

$$F^* \circ (\widehat{\Theta}_X^\bullet, \widehat{\Theta}_X^\bullet(\mathfrak{d}_X)) \longrightarrow (\widehat{\Theta}_Y^\bullet, \widehat{\Theta}_Y^\bullet(\mathfrak{d}_Y)) \circ F^*. \tag{5}$$

In particular, there is a canonical morphism $F^* \circ \widehat{\Theta}_X \longrightarrow \widehat{\Theta}_Y \circ F^*$.

3.6.5. Proposition. *Let* $(C_X, \mathfrak{I}_X)$ *and* $(C_Y, \mathfrak{I}_Y)$ *be svelte left exact categories with final objects and* $(C_X, \mathfrak{I}_X) \xrightarrow{F} (C_Y, \mathfrak{I}_Y)$ *a weakly 'exact' functor. Suppose that the left exact category* $(C_X, \mathfrak{I}_X)$ *has enough pointed injective objects and all morphisms of* $\mathfrak{I}_Y$ *with trivial cokernel are isomorphisms. Then the morphisms (4) and (5) are isomorphisms. In particular, the canonical morphism* $F^* \circ \widehat{\Theta}_X \longrightarrow \widehat{\Theta}_Y \circ F^*$ *is an isomorphism.*

Proof. In fact, since the ∂-functor $Ext^\bullet_Y$ is 'exact' and F is a weakly 'exact' functor from $(C_X, \mathfrak{I}_X)$ to $(C_Y, \mathfrak{I}_Y)$, their composition, $Ext^\bullet_Y \circ F$ is an 'exact' ∂-functor from $(C_X, \mathfrak{I}_X)$ to $C^*_Y, \mathfrak{I}^*_Y)$. Since the left exact category $(C_X, \mathfrak{I}_X)$ has enough pointed injective objects, by (the dual version of) 3.6.1, the 'exact' functor $Ext^\bullet_{Y,\mathfrak{I}_Y} \circ F$ is universal. Therefore, since the zero component of the ∂-functor morphism (4) is an isomorphism, the morphism (4) is an isomorphism. ■

3.6.6. Corollary. *Let* $(C_X, \mathfrak{I}_X)$ *and* $(C_Y, \mathfrak{I}_Y)$ *be svelte left exact categories with final objects and* $(C_X, \mathfrak{I}_X) \xrightarrow{F} (C_Y, \mathfrak{I}_Y)$ *a weakly 'exact' functor. Suppose that the left exact category* $(C_X, \mathfrak{I}_X)$ *has enough pointed injective objects and all morphisms of* $\mathfrak{I}_Y$ *with trivial cokernel are isomorphisms. Then, for any functor* G *from the category* C_Y *to a cocomplete category, the natural morphism*

$$S^\bullet_+(G \circ F) \longrightarrow S^\bullet_+(G) \circ F$$

(which is due to universality of the ∂*-functor on the left and the fact that the zero components of both* ∂*-functors coincide) is an isomorphism.*

Proof. In fact, $S^\bullet_+(G) \circ F = G^* \circ Ext^\bullet_{Y,\mathfrak{I}_Y} \circ F$ and, by 3.6.5, under the hypothesis, the natural morphism $F^* \circ Ext^\bullet_{X,\mathfrak{I}_X} \longrightarrow Ext^\bullet_{Y,\mathfrak{I}_Y} \circ F$ is an isomorphism. Therefore,

$$S^\bullet_+(G) \circ F = G^* \circ Ext^\bullet_{Y,\mathfrak{I}_Y} \circ F \simeq G^* \circ (F^* \circ Ext^\bullet_{X,\mathfrak{I}_X}) = G^* \circ F^* \circ Ext^\bullet_{X,\mathfrak{I}_X}.$$

But, by 3.6.2, $G^* \circ F^* \circ Ext^\bullet_{X,\mathfrak{I}_X} \simeq S^\bullet_+(G \circ F)$. ■

3.6.7. An application. Let F_1 and F_2 be weakly 'exact' functors from a left exact category $(C_X, \mathfrak{I}_X)$ to a left exact category $(C_Y, \mathfrak{I}_Y)$; and let $F_1 \xrightarrow{\rho} F_2$ a functor morphism. Let G be a functor from a the category C_Y to a cocomplete category $C_{\mathfrak{Z}}$. We have a (quasi-)commutative diagram

$$\begin{array}{ccc}
S^\bullet_+(G \circ F_1) & \xrightarrow{S^\bullet_+(G(\rho))} & S^\bullet_+(G \circ F_2) \\
\downarrow & & \downarrow \\
S^\bullet_+(G) \circ F_1 & \xrightarrow{S^\bullet_+(G)(\rho)} & S^\bullet_+(G) \circ F_2
\end{array}$$

of ∂-functor morphisms. Suppose that the left exact category $(C_X, \mathfrak{I}_X)$ has enough injective objects and morphisms of $\mathfrak{I}_Y$ with a trivial cokernel are isomorphisms. Then, by 3.6.6, the vertical arrows are isomorphisms. Therefore,

$$G \circ F_1 \xrightarrow{G(\rho)} G \circ F_2$$

is an isomorphism iff

$$S_+^\bullet(G) \circ F_1 \xrightarrow{S_+^\bullet(G)(\rho)} S_+^\bullet(G) \circ F_2$$

is an isomorphism.

3.7. The stable category of presheaves of sets on a left exact category. Given a left exact category $(C_X, \mathfrak{I}_X)$, we denote by $C_{X_{\mathfrak{s}}^*}$ the quotient category $\Sigma_{\widehat{\Theta}_X^*}^{-1} C_X^*$, where $\Sigma_{\widehat{\Theta}_X^*}$ denote the class of all arrows $\mathfrak{t}$ of C_X^* such that $\widehat{\Theta}_X^*(\mathfrak{t})$ is an isomorphism. The endofunctor $\widehat{\Theta}_X^*$ induces a conservative endofunctor $\Theta_{X_{\mathfrak{s}}^*}$ of the category $C_{X_{\mathfrak{s}}^*}$.

We denote by $\mathfrak{Tr}_{X_{\mathfrak{s}}^*}$ the category of all diagrams of the form

$$\mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \Theta_{X_{\mathfrak{s}}^*}(\mathcal{L})$$

in the category $C_{X_{\mathfrak{s}}^*}$, which are isomorphic to the images of (standard) triangles. The objects of the category $\mathfrak{Tr}_{X_{\mathfrak{s}}^*}$ will be also called *triangles*.

We call the triple $(C_{X_{\mathfrak{s}}^*}, \Theta_{X_{\mathfrak{s}}^*}, \mathfrak{Tr}_{X_{\mathfrak{s}}^*})$ the *stable category of presheaves of sets* on the left exact category $(C_X, \mathfrak{I}_X)$.

4. Acyclic objects and resolutions.

4.1. Left 'exact' functors with enough acyclic objects. Let $(C_X, \mathfrak{I}_X)$ and $(C_Y, \mathfrak{I}_Y)$ be left exact categories with final objects and $C_X \xrightarrow{F} C_Y$ a functor.

We assume that *the morphisms of $\mathfrak{I}_Y$ with a trivial cokernel are isomorphisms.*

4.1.0. Let $E = (L \longrightarrow M_0 \longrightarrow L_1)$ be a conflation in $(C_X, \mathfrak{I}_X)$ such that M_0 is an F-acyclic object. Then the corresponding long sequence splits into shorter sequences

$$\begin{array}{ll} F(L) \longrightarrow F(M_0) \longrightarrow F(L_1) \xrightarrow{\mathfrak{d}_0(E)} S_+F(L) \longrightarrow S_+F(M_0) = y \\ S_+^nF(M_0) = y \longrightarrow S_+^nF(L_1) \xrightarrow{\mathfrak{d}_n(E)} S_+^{n+1}F(L) \longrightarrow y = S_+^{n+1}F(M_0) \end{array} \qquad (1)$$

for $n \geq 1$. Here y is a final object of the category C_Y.

If the functor F is weakly left 'exact' and the left exact category $(C_Y, \mathfrak{I}_Y)$ satisfies the property (C5), then, by the dual version of II.6.3, all sequences (1) are 'exact'. So that, in this case, the morphism $F(L_1) \xrightarrow{\mathfrak{d}_0(E)} S_+F(L)$ is the cokernel of $F(M_0) \longrightarrow F(L_1)$ and the connecting morphisms

$$S_+^n F(L_1) \xrightarrow{\mathfrak{d}_n(E)} S_+^{n+1}F(L)$$

are isomorphisms for $n \geq 1$.

Let $(C_X, \mathfrak{I}_X)$ and $(C_Y, \mathfrak{I}_Y)$ be left exact categories with final objects and $C_X \xrightarrow{F} C_Y$ a weakly left 'exact' functor such that there are enough F-acyclic objects. Then, for any object L of the category C_X, we can construct a sequence of morphisms

$$L = L_0 \longrightarrow M_0 \longrightarrow L_1 \longrightarrow M_1 \longrightarrow L_2 \longrightarrow \dots \longrightarrow L_i \longrightarrow M_i \longrightarrow L_{i+1} \longrightarrow \dots \qquad (2)$$

where $E_i = (L_i \longrightarrow M_i \longrightarrow L_{i+1})$ are conflations for all $i \geq 0$ and all M_i are F-acyclic objects. Therefore, we have 'exact' sequences

$$F(L_n) \longrightarrow F(M_n) \longrightarrow F(L_{n+1}) \xrightarrow{\mathfrak{d}_0(E_n)} S_+F(L_n) \longrightarrow y \qquad (3)$$

and isomorphisms

$$S_+F(L_{n+1}) \xrightarrow{\mathfrak{d}_1(E_n)} S_+^2 F(L_n) \qquad (4)$$

for $n \geq 0$ (which are due to the assumption that morphisms of $\mathfrak{I}_Y$ with trivial cokernel are isomorphisms). The isomorphisms (4) yield canonical isomorphisms

$$S_+F(L_n) \xrightarrow{\sim} S_+^{n+1}F(L_0) = S_+^{n+1}F(L), \quad n \geq 1. \qquad (5)$$

Thus, for $n \geq 2$, the satellites $S_+^n F(L)$ can be obtained from (5) – as the first satellites of the functor F at the objects L_{n-1}; and $S_+F(L)$ is determined by the exact sequence

$$F(L) \longrightarrow F(M_0) \longrightarrow F(L_1) \xrightarrow{\mathfrak{d}_0(E_0)} S_+F(L) \longrightarrow y \qquad (6)$$

which is the sequence (3) above for $n = 0$.

4.1.1. Proposition. *Let $(C_X, \mathfrak{I}_X)$, $(C_Y, \mathfrak{I}_Y)$, $(C_Z, \mathfrak{I}_Z)$ be svelte left exact categories; and let $(C_Z, \mathfrak{I}_Z)$ satisfy (C5). Let $(C_X, \mathfrak{I}_X) \xrightarrow{F} (C_Y, \mathfrak{I}_Y)$ be a weakly 'exact' functor, and $(C_Y, \mathfrak{I}_Y) \xrightarrow{G} (C_Z, \mathfrak{I}_Z)$ a weakly left 'exact' functor. Suppose that the category C_X has enough F-acyclic objects M such that $F(M)$ is a G-acyclic object and that every morphisms of $\mathfrak{I}_Z$ with trivial cokernel is an isomorphism. Then the natural morphism $S_+^\bullet(G \circ F) \longrightarrow (S_+^\bullet G) \circ F$ is an isomorphism.*

Proof. Let L be an object of C_X and

$$L = L_0 \xrightarrow{\mathfrak{j}_0} M_0 \xrightarrow{\mathfrak{e}_0} L_1 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{e}_1} L_2 \longrightarrow \ldots \longrightarrow L_i \xrightarrow{\mathfrak{j}_i} M_i \xrightarrow{\mathfrak{e}_i} L_{i+1} \longrightarrow \ldots \qquad (7)$$

a sequence of morphisms such that $E_i = (L_i \xrightarrow{\mathfrak{j}_i} M_i \xrightarrow{\mathfrak{e}_i} L_{i+1})$ is a conflation for every $i \geq 0$ and all M_i are F-acyclic objects such that $F(M_i)$ are G-acyclic objects. Since the functor F is weakly 'exact', it maps (7) to a sequence with the similar properties with respect to the functor G: $F(E_i) = (F(L_i) \xrightarrow{F(\mathfrak{j}_i)} F(M_i) \xrightarrow{F(\mathfrak{e}_i)} F(L_{i+1}))$ is a conflation for every $i \geq 0$ and all $F(M_i)$ are G-acyclic objects. Therefore, the satellites of G at the object $F(L)$ are computed via formulas following from the (3) and (5) above.

Namely, $S_+^n G(F(L))$ is the cokernel of the morphism

$$G(F(M_{n-1})) \xrightarrow{GF(\mathfrak{e}_{n-1})} G(F(L_n)), \quad n \geq 1.$$

But, the same formulas compute the satellites $S_+^n(G \circ F)(L)$. ■

4.1.2. Corollary. *Let the conditions of 4.1.1 hold and, in addition, the composition $G \circ F$ of the functors is a weakly 'exact' functor from $(C_X, \mathfrak{I}_X)$ to $(C_Z, \mathfrak{I}_Z)$. Then the object $F(M)$ is G-acyclic for all objects M of the category C_X.*

Proof. It follows from the isomorphism

$$S_+^\bullet(G \circ F) \xrightarrow{\sim} (S_+^\bullet G) \circ F \qquad (8)$$

established in 4.1.1 that every F-acyclic object M such that $F(M)$ is a G-acyclic object is an $G \circ F$-acyclic object. So that the left exact category $(C_X, \mathfrak{I}_X)$ has enough $G \circ F$-acyclic objects. Since the functor $G \circ F$ is weakly 'exact', this implies that $S_+^n(G \circ F)(M)$ is an initial object for $n \geq 1$. But, then the isomorphism (8) says that $S_+^n G(F(M))$ is the initial object for all $n \geq 1$ and all objects M of the category C_X, hence the assertion. ■

4.1.3. Note. The conditions of 4.1.2 appear in the most basic context of noncommutative (and commutative) algebraic geometry. Namely, a locally affine 'space' over a 'space' Z is given by a continuous ("global sections") morphism $\mathfrak{X} \xrightarrow{f} Z$ and a locally affine cover, which is a (weakly) flat morphism $U \xrightarrow{u} \mathfrak{X}$ whose inverse image functor $C_X \xrightarrow{u^*} C_U$ is (weakly) exact (which means that it preserves kernels of coreflexive pairs of arrows) and conservative, and such that the composition $U \xrightarrow{f \circ u} Z$ is affine. In particular, its direct image functor $C_U \xrightarrow{f_* u_*} C_X$ is exact. If the locally affine 'space' is *semi-separated*, then the direct image functor $C_U \xrightarrow{u_*} C_{\mathfrak{X}}$ is exact too. For every object L of the category $C_{\mathfrak{X}}$, the adjunction morphism $L \longrightarrow u_* u^*(L)$ is an inflation for a natural left exact structure on $C_{\mathfrak{X}}$ associated with this picture. We can use this fact to construct a canonical sequence

(2) with all M_n being of the form $u_*u^*(L_n)$; hence all of them are f_*-acyclic objects. So that we compute the satellites of the "global sections" functor f_* using this sequence.

4.1.4. Proposition. *Let F_1 and F_2 be weakly 'exact' functors from a left exact category $(C_X, \mathfrak{I}_X)$ to a left exact category $(C_Y, \mathfrak{I}_Y)$; and let $F_1 \xrightarrow{\rho} F_2$ a functor morphism.*

Let $(C_Z, \mathfrak{I}_Z)$ be a left exact category satisfying (C5) and $(C_Y, \mathfrak{I}_Y) \xrightarrow{G} (C_Z, \mathfrak{I}_Z)$ a weakly left 'exact' functor such that $G \circ F_1 \xrightarrow{G\rho} G \circ F_2$ is an epimorphism.

If the category $(C_X, \mathfrak{I}_X)$ has enough pointed injective objects, then the morphism

$$S_+^\bullet(G) \circ F_1 \xrightarrow{S_+^\bullet(G)(\rho)} S_+^\bullet(G) \circ F_2$$

is an epimorphism.

Proof. (a) Let L be an object of C_X and

$$L = L_0 \xrightarrow{\mathfrak{j}_0} M_0 \xrightarrow{\mathfrak{e}_0} L_1 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{e}_1} L_2 \longrightarrow \ldots \longrightarrow L_i \xrightarrow{\mathfrak{j}_i} M_i \xrightarrow{\mathfrak{e}_i} L_{i+1} \longrightarrow \ldots \qquad (7)$$

a sequence of morphisms such that $E_i = (L_i \xrightarrow{\mathfrak{j}_i} M_i \xrightarrow{\mathfrak{j}_i} L_{i+1})$ is a conflation for every $i \geq 0$ and all M_i are injective objects. We have commutative diagrams

$$\begin{array}{ccccccccc}
G \circ F_1(L) & \longrightarrow & G \circ F_1(M_0) & \longrightarrow & G \circ F_1(L_1) & \xrightarrow{\mathfrak{d}_0(E_0)} & S_+(G) \circ F_1(L) & \longrightarrow y \\
{\scriptstyle G\rho(L)} \downarrow & & \downarrow {\scriptstyle G\rho(M_0)} & & \downarrow {\scriptstyle G\rho(L_1)} & & \downarrow {\scriptstyle S_+G(\rho(L))} & \\
G \circ F_2(L) & \longrightarrow & G \circ F_2(M_0) & \longrightarrow & G \circ F_2(L_1) & \xrightarrow{\mathfrak{d}_0(E_0)} & S_+(G) \circ F_2(L) & \longrightarrow y
\end{array} \qquad (8)$$

with 'exact' rows. Therefore, the extreme right vertical arrow, $S_+G\rho(L)$, is an epimorphism, if the previous vertical arrow, $G \circ F(L_1) \xrightarrow{G\rho(L_1)} G \circ F_2(L_1)$, is an epimorphism.

Since, by hypothesis, $G \circ F_1 \xrightarrow{G\rho} G \circ F_2$ is an epimorphism, this shows that

$$S_+G \circ F_1 \xrightarrow{S_+G\rho} S_+G \circ F_2$$

is an epimorphism.

(b) It follows from the commutative square

$$\begin{array}{ccc}
S_+G \circ F_1(L_n) & \xrightarrow{\sim} & S_+^{n+1}G \circ F_1(L) \\
{\scriptstyle S_+G(\rho(L_n))} \downarrow & & \downarrow {\scriptstyle S_+^{n+1}(G)(\rho(L))} \\
S_+G \circ F_2(L_n) & \xrightarrow{\sim} & S_+^{n+1}G \circ F_2(L)
\end{array}$$

that $S_+^{n+1}(G)(\rho(L))$ is an epimorphism for all $n \geq 1$. ■

4.1.5. Remarks. (a) In 4.1.4, the requirement of having enough pointed injective objects in $(C_X, \mathfrak{I}_X)$ can be replaced by demanding the existence of enough objects M of the category C_X which are F_i-acyclic and $F_i(M)$ are G-acyclic, $i = 1, 2$.

By 4.1.1, this condition implies that the natural morphisms

$$S_+^\bullet(G \circ F_i) \longrightarrow (S_+^\bullet G) \circ F_i$$

are isomorphisms.

(b) Let $\mathfrak{S}$ be any class of arrows of the category C_Z which is closed under composition and has the property: if both horizontal arrows in the commutative square

$$\begin{array}{ccc} M & \longrightarrow & L \\ \big\downarrow & & \big\downarrow \\ \mathfrak{M} & \longrightarrow & \mathfrak{L} \end{array}$$

in the category C_Z are deflations and the left vertical arrow belongs to $\mathfrak{S}$, then the right vertical arrow belongs to $\mathfrak{S}$ too. The class of epimorphisms, the class of strict epimorphisms and natural classes of deflations satisfy this condition.

The argument of 4.1.4 shows that if $G\rho(N) \in \mathfrak{S}$ for all $N \in ObC_Y$, then $S_+^i(G\rho)(N)$ belongs to the class $\mathfrak{S}$ for all $N \in ObC_Y$ and $i \geq 0$.

4.2. Satellites and resolutions.

4.2.1. The image of a morphism. The notion of the *image* of a morphism is dual to the notion of the coimage discussed in I.4.5. Below, we provide the direct definition leaving the dualization of the assertions of Section I.4.5 to the reader.

Let C_X be a category with a final object x. Let $M \xrightarrow{f} N$ be an arrow which has a cokernel, that is we have a cocartesian square

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \mathfrak{i}_M}\big\downarrow & cocart & \big\downarrow {\scriptstyle \mathfrak{c}(f)} \\ x & \xrightarrow{f_x} & \mathcal{C}(f) \end{array} \qquad (1)$$

which gives rise to a pair of arrows $N \overset{\mathfrak{c}(f)}{\underset{0_f}{\rightrightarrows}} \mathcal{C}(f)$, where 0_f is the composition of the unique morphism $N \xrightarrow{\mathfrak{i}_N} x$ and the morphism $x \xrightarrow{f_x} \mathcal{C}(f)$. If the kernel of this pair of arrows exists, it is called the *image* of f and denoted by $\mathfrak{Im}(f)$.

Notice that the morphism $M \xrightarrow{f} N$ equalizes the pair $N \overset{\mathfrak{c}(f)}{\underset{0_f}{\rightrightarrows}} \mathcal{C}(f)$. Therefore, if there exists the image of $M \xrightarrow{f} N$, then it is the composition of the canonical strict monomorphism $\mathfrak{Im}(f) \xrightarrow{\mathfrak{i}_f} N$ and a uniquely defined morphism $M \xrightarrow{\bar{f}} \mathfrak{Im}(f)$.

4.2.1.1. Lemma. *Let* $M \xrightarrow{f} N$ *be a morphism which has cokernel and image. There is a natural isomorphism* $\mathcal{C}(f) \xrightarrow{\sim} \mathcal{C}(\mathfrak{i}_f)$ *between the cokernel of the morphism* $M \xrightarrow{f} N$ *and the cokernel of the canonical embedding* $\mathfrak{Im}(f) \xrightarrow{\mathfrak{i}_f} N$.

Proof. This fact is the dual version of I.4.5.1. ∎

4.2.2. The cohomology of a complex. Let $(C_X, \mathfrak{I}_X)$ be a left exact category with a final object x; and let

$$\mathcal{M}_\bullet = (L = M_{-1} \xrightarrow{g_{-1}} M_0 \xrightarrow{g_0} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} \ldots) \tag{2}$$

be a complex; that is, for every $n \geq -1$, the composition of the arrows

$$M_n \xrightarrow{g_n} M_{n+1} \xrightarrow{g_{n+1}} M_{n+2}$$

uniquely factors through the final object x; or, equivalently, g_{n+1} factors uniquely through the cokernel $M_{n+1} \xrightarrow{\mathfrak{c}(g_n)} \mathcal{C}(g_n)$ of the arrow g_n. Since there is a uniquely defined morphism from the final object x to $\mathcal{C}(g_n)$ (and the unique morphism from M_{n+1} to x), the latter condition determines a trivial morphism $M_{n+1} \xrightarrow{0} \mathcal{C}(g_n)$. All this data is incorporated in the commutative diagram

$$\begin{array}{ccccc} M_n & \xrightarrow{g_n} & M_{n+1} & \xrightarrow{g_{n+1}} & M_{n+2} \\ \downarrow & cocart & \downarrow \mathfrak{c}(g_n) & & \downarrow id \\ x & \longrightarrow & \mathcal{C}(g_n) & \xrightarrow{\gamma_{n+1}} & M_{n+2} \end{array} \tag{3}$$

Assuming that the category C_X has kernels of pairs of arrows (in particular, images of morphisms having cokernels), we can associate with the diagram (3) the diagram

$$M_n \xrightarrow{\bar{g}_n} \mathfrak{Im}(g_n) \xrightarrow{\mathfrak{i}_{g_n}} M_{n+1} \overset{\mathfrak{c}(g_n)}{\underset{0_{g_n}}{\rightrightarrows}} \mathcal{C}(g_n) \xrightarrow{\gamma_{n+1}} M_{n+2}. \tag{4}$$

Let $\mathcal{Z}(g_{n+1})$ denote the kernel of the composition

$$M_{n+1} \overset{g_{n+1}}{\underset{0_{g_{n+1}}}{\rightrightarrows}} M_{n+2} \tag{5}$$

of the pair $M_{n+1} \overset{\mathfrak{c}(g_n)}{\underset{0_{g_n}}{\rightrightarrows}} \mathcal{C}(g_n)$ and the following it morphism $\mathcal{C}(g_n) \xrightarrow{\gamma_{n+1}} M_{n+2}$.

Since $\mathfrak{Im}(g_n) \xrightarrow{\mathfrak{i}_{g_n}} M_{n+1}$ equalizes the pair of arrows (5), it factors through its kernel $\mathcal{Z}(g_{n+1}) \xrightarrow{\mathfrak{k}_{n+1}} M_{n+1}$ of the pair (5).

We set $H^0(\mathcal{M}_\bullet)$ equal to $\mathcal{Z}(g_0.)$ For $n \geq 1$, we denote by $H^n(\mathcal{M}_\bullet)$ the cokernel of the unique morphism

$$\mathfrak{Im}(g_n) \xrightarrow{\mathfrak{i}_{g_n}} \mathcal{Z}(g_{n+1}).$$

For each $n \geq 0$, we call $H^n(\mathcal{M}_\bullet)$ the n^{th} *cohomology object of the complex* $\mathcal{M}_\bullet$.

4.2.3. Resolutions. Let $(C_X, \mathfrak{I}_X)$ be a left exact category with a final object x. Let L be an object of the category C_X and

$$L = L_0 \xrightarrow{\mathfrak{j}_0} M_0 \xrightarrow{\mathfrak{c}_0} L_1 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{c}_1} L_2 \longrightarrow \dots \longrightarrow L_n \xrightarrow{\mathfrak{j}_n} M_n \xrightarrow{\mathfrak{c}_n} L_{n+1} \longrightarrow \dots \qquad (6)$$

a sequence of arrows such that $E_n = (L_n \longrightarrow M_n \longrightarrow L_{n+1})$ are conflations for $n \geq 0$.

Notice that the objects L_n and M_n are pointed for all $n \geq 1$; that is their morphisms to final objects are split epimorphisms; in particular, they are strict epimorphisms. Since a push-forward of a strict epimorphism is a strict epimorphism and the squares

$$\begin{array}{ccc} L_n & \xrightarrow{\mathfrak{j}_n} & M_n \\ \big\downarrow & cocart & \big\downarrow \mathfrak{c}_n \\ x & \longrightarrow & L_{n+1} \end{array}$$

are cocartesian, the morphisms $M_n \longrightarrow L_{n+1}$ in (6) are strict epimorphisms for $n \geq 1$. This implies that the squares

$$\begin{array}{ccc} M_n & \xrightarrow{g_n} & M_{n+1} \\ \big\downarrow & cocart & \big\downarrow \\ x & \longrightarrow & L_{n+2} \end{array}$$

are cocartesian for $n \geq 1$. Therefore, we can reconstruct (up to isomorphism) the part

$$M_2 \xrightarrow{\mathfrak{c}_2} L_3 \xrightarrow{\mathfrak{j}_3} M_3 \xrightarrow{\mathfrak{c}_3} \dots \xrightarrow{\mathfrak{c}_{n-1}} L_n \xrightarrow{\mathfrak{j}_n} M_n \xrightarrow{\mathfrak{c}_n} L_{n+1} \xrightarrow{\mathfrak{j}_{n+1}} \dots$$

of the sequence (6) from the associated with it complex

$$\mathcal{M}_\bullet = (M_0 \xrightarrow{g_0} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} \dots) \qquad (7)$$

We have a canonical morphism $\mathcal{C}(g_0) \longrightarrow L_2$ from the cokernel of $M_0 \xrightarrow{g_0} M_1$ to L_2, which is due to the commutativity of the diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{g_0} & M_1 \\ \big\downarrow & & \big\downarrow \\ x & \longrightarrow & L_2 \end{array} \qquad (8)$$

It follows from the commutative diagram

$$\begin{array}{ccccc}
L_0 & \longrightarrow & x & & \\
{\scriptstyle \mathfrak{j}_0}\downarrow & \mathit{cocart} & \downarrow & & \\
M_0 & \xrightarrow{\mathfrak{c}_0} & L_1 & \xrightarrow{\mathfrak{j}_1} & M_1 \\
\downarrow & \mathit{cocart} & \downarrow & \mathit{cocart} & \downarrow \\
x & \longrightarrow & \mathcal{C}(\mathfrak{c}_0) & \longrightarrow & \mathcal{C}(g_0) \\
 & & \downarrow & \mathit{cocart} & \downarrow \\
 & & x & \longrightarrow & L_2
\end{array} \tag{9}$$

built of cocartesian squares and the stability of epimorphism under push-forwards, that the canonical morphism $\mathcal{C}(g_0) \longrightarrow L_2$ is an isomorphism (that is the square (8) is cocartesian), if the unique morphism $L_0 \longrightarrow x$ is an epimorphism.

In fact, if $L_0 \longrightarrow x$ is an epimorphism, then the morphism $x \longrightarrow \mathcal{C}(\mathfrak{c}_0)$, being a push-forward of $L_0 \longrightarrow x$, is an epimorphism. On the other hand, $x \longrightarrow \mathcal{C}(\mathfrak{c}_0)$ is a split (hence a strict) monomorphism, which means that it is an isomorphism. Since $\mathcal{C}(\mathfrak{c}_0)$ is a final object, it follows from the right upper cocartesian square in (9) that the canonical morphism $\mathcal{C}(g_0) \longrightarrow L_2$ is an isomorphism.

4.2.3.1. Note. It follows from the lower right cartesian square of the diagram (9) and the fact that $\mathcal{C}(\mathfrak{c}_0) \longrightarrow x$ is a split (hence strict) epimorphism that the canonical morphism $\mathcal{C}(g_0) \longrightarrow L_2$ is always a strict epimorphism.

This follows also from the already observed fact that $M_1 \xrightarrow{\mathfrak{c}_0} L_2$ is a push-forward of the split epimorphism $L_1 = \mathcal{C}(\mathfrak{j}_0) \longrightarrow x$, hence it is a strict epimorphism. Since $M_1 \xrightarrow{\mathfrak{c}_0} L_2$ is the composition of $M_1 \xrightarrow{\mathfrak{c}(g_0)} \mathcal{C}(g_0)$ and the morphism $\mathcal{C}(g_0) \longrightarrow L_2$, the latter is a strict epimorphism too.

4.2.3.2. The sequence with images. Since each morphism $M_n \xrightarrow{g_n} M_{n+1}$ of the complex

$$\mathcal{M}_\bullet = (M_0 \xrightarrow{g_0} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} \ldots) \tag{7}$$

associated with

$$L = L_0 \xrightarrow{\mathfrak{j}_0} M_0 \xrightarrow{\mathfrak{c}_0} L_1 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{c}_1} L_2 \longrightarrow \ldots \longrightarrow L_n \xrightarrow{\mathfrak{j}_n} M_n \xrightarrow{\mathfrak{c}_n} L_{n+1} \longrightarrow \ldots \tag{6}$$

is the composition of $M_n \xrightarrow{\mathfrak{c}_n} L_{n+1}$ and an inflation (in particular, a monomorphism) $L_{n+1} \xrightarrow{\mathfrak{j}_{n+1}} M_{n+1}$, the image of $L_n \xrightarrow{\mathfrak{j}_n} M_n$, which, by definition, is the kernel of the pair of arrows $M_n \overset{\mathfrak{c}_n}{\underset{0_{\mathfrak{j}_n}}{\rightrightarrows}} \mathcal{C}(\mathfrak{j}_n) = L_{n+1}$ (see 4.2.2), is isomorphic to the kernel of the pair $M_n \overset{g_n}{\underset{0_{g_n}}{\rightrightarrows}} M_{n+1}$, which we denote (in 4.2.2) by $\mathcal{Z}(g_n) \xrightarrow{\mathfrak{i}_{g_n}} M_n$.

Therefore, we have a commutative diagram

$$\begin{array}{ccccccccccc}
L_0 & \xrightarrow{\mathfrak{j}_0} & M_0 & \xrightarrow{\mathfrak{c}_0} & L_1 & \xrightarrow{\mathfrak{j}_1} & M_1 & \xrightarrow{\mathfrak{c}_1} & L_2 & \xrightarrow{\mathfrak{j}_2} & \dots \\
\downarrow & & id \downarrow & & \downarrow & & id \downarrow & & \downarrow & & \\
\mathcal{Z}(g_0) & \xrightarrow{\mathfrak{k}_{g_0}} & M_0 & \xrightarrow{\mathfrak{c}_n} & \mathcal{Z}(g_1) & \xrightarrow{\mathfrak{k}_{g_1}} & M_1 & \xrightarrow{\mathfrak{c}_n} & \mathcal{Z}(g_2) & \xrightarrow{\mathfrak{k}_{g_2}} & \dots
\end{array} \tag{10}$$

whose vertical arrows $L_m \longrightarrow \mathcal{Z}(g_m),\ m \geq 0$, are inflations with trivial cokernels and the generic part (that is for $n \geq 2$) can be included into the diagram

$$\begin{array}{ccccccccc}
\dots \xrightarrow{\mathfrak{c}_{n-1}} & \mathcal{C}(g_{n-2}) & \xrightarrow{\gamma_{n-1}} & M_n & \xrightarrow{\mathfrak{c}(g_{n-1})} & \mathcal{C}(g_{n-1}) & \xrightarrow{\mathfrak{j}_{n+1}} \dots \\
& \downarrow & & id \downarrow & & \downarrow \wr & \\
\dots \xrightarrow{\mathfrak{c}_{n-1}} & L_n & \xrightarrow{\mathfrak{j}_n} & M_n & \xrightarrow{\mathfrak{c}_n} & L_{n+1} & \xrightarrow{\mathfrak{j}_{n+1}} \dots \\
& \downarrow & & id \downarrow & & \downarrow & \\
\dots \xrightarrow{\mathfrak{c}_{n-1}} & \mathcal{Z}(g_n) & \xrightarrow{\mathfrak{k}_{g_n}} & M_n & \xrightarrow{\mathfrak{c}_n} & \mathcal{Z}(g_{n+1}) & \xrightarrow{\mathfrak{k}_{g_{n+1}}} \dots
\end{array} \tag{10'}$$

where the vertical arrow $\mathcal{C}(g_{n-2}) \longrightarrow L_n$ is an isomorphism for $n \geq 3$ and, by 4.2.3.1, the arrow $\mathcal{C}(g_0) \longrightarrow L_2$ is a strict epimorphism. Notice that the canonical (strict) monomorphisms $\mathfrak{Im}(g_n) \xrightarrow{\mathfrak{i}_{g_n}} \mathcal{Z}(g_{n+1})$ are isomorphisms here.

4.2.3.3. Resolutions of objects. We call the complex

$$\mathcal{M}_\bullet = (M_0 \xrightarrow{g_0} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} \dots) \tag{7}$$

associated with the sequence

$$L = L_0 \xrightarrow{\mathfrak{j}_0} M_0 \xrightarrow{\mathfrak{c}_0} L_1 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{c}_1} L_2 \longrightarrow \dots \longrightarrow L_n \xrightarrow{\mathfrak{j}_n} M_n \xrightarrow{\mathfrak{c}_n} L_{n+1} \longrightarrow \dots \tag{6}$$

a *resolution* of the object L.

We call the resolution (7) *injective* if all objects M_n are pointed injective objects of $(C_X, \mathfrak{I}_X)$ and F*-acyclic* for some functor F from C_X to a category with final objects, if all M_n are F-acyclic objects.

4.2.3.4. A short summary. It follows from the preceding discussion that the tail

$$M_2 \xrightarrow{\mathfrak{c}_2} L_3 \xrightarrow{\mathfrak{j}_3} M_3 \xrightarrow{\mathfrak{c}_3} \dots \xrightarrow{\mathfrak{c}_{n-1}} L_n \xrightarrow{\mathfrak{j}_n} M_n \xrightarrow{\mathfrak{c}_n} L_{n+1} \xrightarrow{\mathfrak{j}_{n+1}} \dots$$

of the sequence (6) is reconstructed uniquely up to isomorphism from the associated resolution (7) and the beginning

$$L = L_0 \xrightarrow{\mathfrak{j}_0} M_0 \xrightarrow{\mathfrak{c}_0} L_1 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{c}_1} L_2$$

is reconstructed uniquely up to inflations with trivial cokernels (see diagram (10) for the precise statement). In particular, if in the left exact category $(C_X, \mathfrak{I}_X)$, every inflation with a trivial cokernel is an isomorphism, then the whole sequence (6) is reconstructed uniquely up to isomorphism from the resolution (7) of the object L.

4.2.4. Proposition. *Let $(C_X, \mathfrak{I}_X)$ and $(C_Y, \mathfrak{I}_Y)$ be svelte left exact categories with final objects and $(C_X, \mathfrak{I}_X) \xrightarrow{F} (C_Y, \mathfrak{I}_Y)$ a weakly left 'exact' functor. Suppose that*
(a) the functor F maps inflations to inflations isomorphic to their images,
(b) the left exact category $(C_X, \mathfrak{I}_X)$ has enough F-acyclic objects,
(c) every arrow of $\mathfrak{I}_Y$ with trivial cokernel is an isomorphism.
Then, for any $L \in ObC_X$, the satellites of F at L are isomorphic to the cohomology of the complex $F(\mathcal{M}_\bullet)$, where $\mathcal{M}_\bullet$ is any F-acyclic resolution of the object L.

Proof. Since the left exact category $(C_X, \mathfrak{I}_X)$ has enough F-acyclic objects, we can construct, for any object L of the category C_X, the sequence

$$L = L_0 \xrightarrow{\mathfrak{j}_0} M_0 \xrightarrow{\mathfrak{e}_0} L_1 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{e}_1} L_2 \longrightarrow \ldots \longrightarrow L_n \xrightarrow{\mathfrak{j}_n} M_n \xrightarrow{\mathfrak{e}_n} L_{n+1} \longrightarrow \ldots \qquad (6)$$

such that $L_n \xrightarrow{\mathfrak{j}_n} M_n \xrightarrow{\mathfrak{e}_n} L_{n+1}$ is a conflation for every $n \geq 0$ and all objects M_n are F-acyclic. So that the corresponding resolution

$$\mathcal{M}_\bullet = (M_0 \xrightarrow{g_0} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} \ldots) \qquad (7)$$

of the object L is F-acyclic. Since the functor $C_X \xrightarrow{F} C_Y$ is weakly left 'exact', its satellites can be computed via canonical isomorphisms

$$S_+F(L_n) \xrightarrow{\sim} S_+^{n+1}F(L_0) = S_+^{n+1}F(L), \quad n \geq 1. \qquad (\geq 2)$$

for $n \geq 2$ and via the 'exact' sequence

$$F(L) \xrightarrow{F(\mathfrak{j}_0)} F(M_0) \xrightarrow{F(\mathfrak{e}_0)} F(L_1) \xrightarrow{\mathfrak{d}_0(E_0)} S_+F(L) \longrightarrow y \qquad (0,1)$$

for $n = 0$ and $n = 1$ (see Subsection 4.1).

(a) Consider the commutative diagrams

$$\begin{array}{ccccccc} F(L_n) & \xrightarrow{F(\mathfrak{j}_n)} & F(M_n) & \xrightarrow{F(\mathfrak{e}_n)} & F(L_{n+1}) & \xrightarrow{F(\mathfrak{j}_{n+1})} & F(M_{n+1}) \\ \downarrow & & id \downarrow & & \uparrow \gamma_n & & \\ \mathfrak{Im}(F(\mathfrak{j}_n)) & \xrightarrow{\mathfrak{i}_{F(\mathfrak{j}_n)}} & F(M_n) & \xrightarrow{\mathfrak{c}(F(\mathfrak{j}_n))} & \mathcal{C}(F(\mathfrak{j}_n)) & & \end{array} \qquad (11)$$

for $n \geq 0$ (with $L_0 = L$). By definition, the image $\mathfrak{Jn}(F(\mathfrak{j}_n)) \xrightarrow{\mathfrak{i}_{F(\mathfrak{j}_n)}} F(M_n)$ of the morphism $F(\mathfrak{j}_n)$ is the kernel of the pair of arrows

$$F(M_n) \underset{0}{\overset{\mathfrak{c}(F(\mathfrak{j}_n))}{\rightrightarrows}} \mathcal{C}(F(\mathfrak{j}_n)). \tag{12}$$

Since the functor F is weakly left 'exact', the arrows γ_n and $F(L_{n+1}) \xrightarrow{F(\mathfrak{j}_{n+1})} F(M_{n+1})$ in the diagram (11) are inflations, in particular monomorphisms. Therefore, the kernel of the pair (12) is naturally isomorphic to the kernel of the composition of (12) with these arrows, which is $F(M_n \underset{0_{g_n}}{\overset{g_n}{\rightrightarrows}} M_{n+1})$ or $F(M_n) \underset{0}{\overset{F(g_n)}{\rightrightarrows}} F(M_{n+1})$. So that the natural monomorphism from the image $\mathfrak{Jm}(F(\mathfrak{j}_n))$ of the morphism $F(\mathfrak{j}_n)$ to the kernel $\mathcal{Z}(F(g_n))$ of the pair $F(M_n) \underset{0}{\overset{F(g_n)}{\rightrightarrows}} F(M_{n+1})$ is an isomorphism.

By hypothesis, the canonical monomorphism $F(L_n) \longrightarrow \mathfrak{Jn}(F(\mathfrak{j}_n))$ is an isomorphism; so that $F(L_n) \longrightarrow \mathcal{Z}(F(g_n))$ is an isomorphism for all $n \geq 0$.

(b) By definition, $\mathcal{Z}(F(g_0))$ is the zero cohomology object $H^0(F(\mathcal{M}_\bullet))$ of the complex $F(\mathcal{M}_\bullet)$. So that the isomorphism $F(L_0) \longrightarrow \mathcal{Z}(F(g_0))$ is an isomorphism between the zero satellite $F(L) = F(L_0)$ of the functor F at the object L and the zero cohomology of the F-acyclic resolution $\mathcal{M}_\bullet$ of the object L:

$$S^0_+F(L) \stackrel{\text{def}}{=} F(L) \simeq \mathcal{Z}(F(g_0)) \stackrel{\text{def}}{=} H^0(F(\mathcal{M}_\bullet)).$$

(c) Since the sequence

$$F(L_n) \xrightarrow{F(\mathfrak{j}_n)} F(M_n) \xrightarrow{F(\mathfrak{j}_n)} F(L_{n+1}) \xrightarrow{\mathfrak{d}_0(E_n)} S_+F(L_n) \longrightarrow y$$

is 'exact', the first satellite $S_+F(L_n)$ is isomorphic to the cokernel of the morphism $F(M_n) \xrightarrow{F(\mathfrak{e}_n)} F(L_{n+1})$, or, what is the same (according to (a) above), the cokernel of the canonical morphism

$$F(M_n) \xrightarrow{\widehat{g}_n} \mathcal{Z}(F(g_{n+1})).$$

By 4.2.1.1, this cokernel is isomorphic to the cokernel of the embedding of the image of this morphism to $\mathcal{Z}(F(g_{n+1}))$, which is, by definition, the n plus first cohomology object $H^{n+1}(F(\mathcal{M}_\bullet))$ of the complex $F(\mathcal{M}_\bullet)$:

$$S_+F(L_n) \xrightarrow{\sim} H^{n+1}(F(\mathcal{M}_\bullet)).$$

In combination with the isomorphism (≥ 2) above, it gives the claimed isomorphism

$$S_+^n F(L) \xrightarrow{\sim} H^n(F(\mathcal{M}_\bullet)).$$

for $n \geq 2$; hence the assertion. ■

4.2.5. Corollary. *Let $(C_X, \mathfrak{I}_X)$ and $(C_Y, \mathfrak{I}_Y)$ be svelte left exact categories with final objects and $(C_X, \mathfrak{I}_X) \xrightarrow{F} (C_Y, \mathfrak{I}_Y)$ a weakly left 'exact' functor. Suppose that all inflations of $(C_Y, \mathfrak{I}_Y)$ with trivial cokernels are isomorphisms and $(C_X, \mathfrak{I}_X)$ has enough F-acyclic objects. Then, for every object L of the category C_X and for all $n \geq 0$,*

$$S_+^n F(L) \xrightarrow{\sim} H^n(F(\mathcal{M}_\bullet)),$$

where $\mathcal{M}_\bullet$ is any F-acyclic resolution of L.

4.2.6. Remark. Let $(C_X, \mathfrak{I}_X)$ and $(C_Y, \mathfrak{I}_Y)$ be svelte left exact categories with final objects and $(C_X, \mathfrak{I}_X) \xrightarrow{F} (C_Y, \mathfrak{I}_Y)$ a weakly left 'exact' functor. Suppose that all arrows of $\mathfrak{I}_Y$ with trivial cokernel are isomorphisms. Let

$$\mathcal{M}_\bullet = (M_0 \xrightarrow{g_0} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} \ldots)$$

be an F-acyclic resolution of an object L of the category C_X. It follows from 4.1 and 4.2.3 that, for $n \geq 3$, the object $S_+^n F(L)$ is isomorphic to the cokernel of the natural morphism

$$F(M_{n-1}) \longrightarrow F(\mathcal{C}(g_{n-2})),$$

where $\mathcal{C}(g_{n-2})$ is the cokernel of $M_{n-2} \xrightarrow{g_{n-2}} M_{n-1}$.

So that, for $n \geq 3$, the satellites $S_+^n F(L)$ are reconstructed from an F-acyclic resolution of the object L without additional hypothesis. Moreover, if the unique morphism of L to a final object of C_X is an epimorphism, then we can reconstruct $S_+^2 F(L)$ the same way: it is isomorphic to the cokernel of the canonical morphism $F(M_0) \longrightarrow F(\mathcal{C}(g_0))$.

5. Prestable and stable categories of a left exact category.

Consider the full subcategory $C_{X_\mathfrak{p}}$ of the category C_X^* whose objects are $\widehat{\Theta}_X^{*n}(\mathcal{M})$, where $\mathcal{M}$ runs through representable presheaves and n through nonnegative integers. We denote by $\theta_{X_\mathfrak{p}}$ the endofunctor $C_{X_\mathfrak{p}} \longrightarrow C_{X_\mathfrak{p}}$ induced by $\widehat{\Theta}_X^*$. It follows that $C_{X_\mathfrak{p}}$ is the smallest $\widehat{\Theta}_X^*$-stable strictly full subcategory of the category C_X^* containing all presheaves $\widehat{M} = C_X(-, M)$, $M \in ObC_X$.

5.1. Triangles. We call the diagram

$$\widehat{N} \xrightarrow{\widehat{\mathfrak{j}}} \widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{L} \xrightarrow{\mathfrak{d}_0(E)} \widehat{\Theta}_X^*(\widehat{N}), \tag{1}$$

where $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ is a conflation in $(C_X, \mathfrak{I}_X)$, a *standard triangle.*

A *triangle* is any diagram in $C_{X_\mathfrak{p}}$ of the form

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{X_\mathfrak{p}}(\mathcal{N}), \tag{2}$$

which is isomorphic to a standard triangle. It follows that, for every triangle, the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \big\downarrow & & \big\downarrow \mathfrak{d} \\ \widehat{x} & \xrightarrow{\lambda(\mathcal{N})} & \theta_{X_\mathfrak{p}}(\mathcal{N}) \end{array}$$

commutes. Triangles form a category $\mathfrak{Tr}_{X_\mathfrak{p}}$: morphisms from

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{X_\mathfrak{p}}(\mathcal{N})$$

to

$$\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{X_\mathfrak{p}}(\mathcal{N}')$$

are given by commutative diagrams

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{X_\mathfrak{p}}(\mathcal{N}) \\ f \big\downarrow & & \big\downarrow g & & \big\downarrow h & & \big\downarrow \theta_{X_\mathfrak{p}}(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{X_\mathfrak{p}}(\mathcal{N}') \end{array}$$

The composition is obvious.

5.2. The prestable category of a left exact category. Thus, we have obtained a data $(C_{X_\mathfrak{p}}, (\theta_{X_\mathfrak{p}}, \lambda), \mathfrak{Tr}_{X_\mathfrak{p}})$. We call this data the *prestable* category of the left exact category $(C_X, \mathfrak{I}_X)$.

5.3. The stable category of a left exact category with final objects. Let $(C_X, \mathfrak{I}_X)$ be a left exact category with a final object x and $(C_{X_\mathfrak{p}}, \theta_{X_\mathfrak{p}}, \lambda; \mathfrak{Tr}_{X_\mathfrak{p}})$ the associated with $(C_X, \mathfrak{I}_X)$ presuspended category. Let $\Sigma = \Sigma_{\theta_{X_\mathfrak{p}}}$ be the class of all arrows $\mathfrak{t}$ of $C_{X_\mathfrak{p}}$ such that $\theta_{X_\mathfrak{p}}(\mathfrak{t})$ is an isomorphism.

We call the quotient category $C_{X_\mathfrak{s}} = \Sigma^{-1}C_{X_\mathfrak{p}}$ the *stable* category of the left exact category $(C_X, \mathfrak{I}_X)$. The endofunctor $\theta_{X_\mathfrak{p}}$ determines a conservative endofunctor $\theta_{X_\mathfrak{s}}$ of the stable category $C_{X_\mathfrak{s}}$. The localization functor $C_{X_\mathfrak{p}} \xrightarrow{\mathfrak{q}_\Sigma^*} C_{X_\mathfrak{s}}$ maps final objects to final objects. Let $\lambda_\mathfrak{s}$ denote the image $\widetilde{x} = \mathfrak{q}_\Sigma^*(\widehat{x}) \longrightarrow \theta_{X_\mathfrak{s}}$ of the cone $\widehat{x} \xrightarrow{\lambda} \theta_{X_\mathfrak{p}}$.

Finally, we denote by $\mathfrak{Tr}_{X_\mathfrak{s}}$ the strictly full subcategory of the category of diagrams of the form $\mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow \theta_{X_\mathfrak{s}}(\mathcal{N})$ generated by $\mathfrak{q}^*_\Sigma(\mathfrak{Tr}_{X_\mathfrak{s}})$.

The data $(C_{X_\mathfrak{s}}, \theta_{X_\mathfrak{s}}, \lambda_\mathfrak{s}; \mathfrak{Tr}_{X_\mathfrak{s}})$ will be called the *stable* category of the left exact category $(C_X, \mathfrak{I}_X)$.

5.4. Dual notions. If $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ is a right exact category with an initial object, one obtains, dualizing the definitions of 5.2 and 5.3, the notions of the *precostable* and *costable* category of $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$.

6. Presuspended and quasi-suspended categories.

Fix a category $C_\mathfrak{X}$ with a final object x and a functor $C_\mathfrak{X} \xrightarrow{\widetilde{\theta}_\mathfrak{X}} x\backslash C_\mathfrak{X}$, or, what is the same, a pair $(\theta_\mathfrak{X}, \lambda)$, where $\theta_\mathfrak{X}$ is an endofunctor $C_\mathfrak{X} \longrightarrow C_\mathfrak{X}$ and λ is a cone $x \longrightarrow \theta_\mathfrak{X}$. We denote by $\widetilde{\mathfrak{Tr}}_\mathfrak{X}$ the category whose objects are all diagrams of the form

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_\mathfrak{X}(\mathcal{N})$$

such that the square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \downarrow & & \downarrow \mathfrak{d} \\ x & \xrightarrow{\lambda(\mathcal{N})} & \theta_\mathfrak{X}(\mathcal{N}) \end{array}$$

commutes. Morphisms from

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_\mathfrak{X}(\mathcal{N})$$

to

$$\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_\mathfrak{X}(\mathcal{N}')$$

are triples of arrows $(\mathcal{N} \xrightarrow{f} \mathcal{N}', \mathcal{M} \xrightarrow{g} \mathcal{M}', \mathcal{L} \xrightarrow{h} \mathcal{L}')$ making the diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_\mathfrak{X}(\mathcal{N}) \\ f \downarrow & & \downarrow g & & \downarrow h & & \downarrow \theta_\mathfrak{X}(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_\mathfrak{X}(\mathcal{N}') \end{array}$$

commute. The composition of morphisms is natural.

6.1. Definition. A *presuspended* category is a triple $(C_\mathfrak{X}, \widetilde{\theta}_\mathfrak{X}, \mathfrak{Tr}_\mathfrak{X})$, where $C_\mathfrak{X}$ and $\widetilde{\theta}_\mathfrak{X} = (\theta_\mathfrak{X}, \lambda)$ are as above and $\mathfrak{Tr}_\mathfrak{X}$ is a strictly full subcategory of the category $\widetilde{\mathfrak{Tr}}_\mathfrak{X}$, whose objects are called *triangles*, which satisfies the following conditions:

(PS1) Let $C_{\mathfrak{X}_0}$ denote the full subcategory of $C_{\mathfrak{X}}$ generated by objects $\mathcal{N}$ such that there exists a triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$. For every $\mathcal{N} \in ObC_{\mathfrak{X}_0}$, the diagram

$$\mathcal{N} \xrightarrow{id_{\mathcal{N}}} \mathcal{N} \longrightarrow x \xrightarrow{\lambda(\mathcal{N})} \theta_{\mathfrak{X}}(\mathcal{N})$$

is a triangle.

(PS2) For any triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$ and any morphism $\mathcal{N} \xrightarrow{f} \mathcal{N}'$ with $\mathcal{N}' \in ObC_{\mathfrak{X}_0}$, there is a triangle $\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$ such that f extends to a morphism of triangles

$$(\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})) \xrightarrow{(f,g,h)} (\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')).$$

(PS3) For any pair of triangles

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N}) \quad \text{and} \quad \mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$$

and any commutative square

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} \\ f\downarrow & & \downarrow g \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' \end{array}$$

there exists a morphism $\mathcal{L} \xrightarrow{h} \mathcal{L}'$ such that (f, g, h) is a morphism of triangles, i.e. the diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ f\downarrow & & \downarrow g & & \downarrow h & & \downarrow \theta_{\mathfrak{X}}(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{\mathfrak{X}}(\mathcal{N}') \end{array}$$

commutes.

(PS4) For any pair of triangles

$$\mathcal{N} \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{L} \xrightarrow{w} \theta_{\mathfrak{X}}(\mathcal{N}) \quad \text{and} \quad \mathcal{M} \xrightarrow{x} \mathcal{M}' \xrightarrow{s} \widetilde{\mathcal{M}} \xrightarrow{r} \theta_{\mathfrak{X}}(M),$$

there exists a commutative diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{u} & \mathcal{M} & \xrightarrow{v} & \mathcal{L} & \xrightarrow{w} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ id\downarrow & & x\downarrow & & \downarrow y & & \downarrow id \\ \mathcal{N} & \xrightarrow{u'} & \mathcal{M}' & \xrightarrow{v'} & \mathcal{L}' & \xrightarrow{w'} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ & & s\downarrow & & \downarrow t & & \downarrow \theta_{\mathfrak{X}}(u) \\ & & \widetilde{\mathcal{M}} & \xrightarrow{id} & \widetilde{\mathcal{M}} & \xrightarrow{r} & \theta_{\mathfrak{X}}(\mathcal{M}) \\ & & r\downarrow & & \downarrow r' & & \\ & & \theta_{\mathfrak{X}}(\mathcal{M}) & \xrightarrow{\theta_{\mathfrak{X}}(v)} & \theta_{\mathfrak{X}}(\mathcal{L}) & & \end{array} \tag{2}$$

whose two upper rows and two central columns are triangles.

(PS5) For every triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$, the sequence

$$\dots \longrightarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(\mathcal{N}), -) \longrightarrow C_{\mathfrak{X}}(\mathcal{L}, -) \longrightarrow C_{\mathfrak{X}}(\mathcal{M}, -) \longrightarrow C_{\mathfrak{X}}(\mathcal{N}, -)$$

is exact.

6.1.1. Remarks. (a) If $C_{\mathfrak{X}}$ is an additive category, then three of the axioms above coincide with the corresponding Verdier's axioms of triangulated category (under condition that $C_{\mathfrak{X}_0} = C_{\mathfrak{X}}$). Namely, (PS1) coincides with the first half of the axiom (TRI), the axiom (PS3) coincides with the axiom (TRIII), and (PS4) with (TRIV) (see [Ve2, Ch.II]).

(b) It follows from (PS4) that if $\mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow \theta_{\mathfrak{X}}(\mathcal{N})$ is a triangle, then all three objects, $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{L}$, belong to the subcategory $C_{\mathfrak{X}_0}$.

(c) The axiom (PS2) can be regarded as a base change property, and axiom (PS4) expresses the stability of triangles under composition. So that the axioms (PS1), (PS2) and (PS4) say that triangles form a 'pretopology' on the subcategory $C_{\mathfrak{X}_0}$. The axiom (PS5) says that this pretopology is *subcanonical*: the representable presheaves are sheaves.

These interpretations (as well as the axioms themselves) come from the main examples: prestable and stable categories of a left exact category.

6.2. The category of presuspended categories. Let $\mathfrak{T}_+C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda_{\mathfrak{X}}; Tr_{\mathfrak{X}})$ and $\mathfrak{T}_+C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, \lambda_{\mathfrak{Y}}; Tr_{\mathfrak{Y}})$ be presuspended categories. A *triangle* functor from $\mathfrak{T}_+C_{\mathfrak{X}}$ to $\mathfrak{T}_+C_{\mathfrak{Y}}$ is a pair (F, ϕ), where F is a functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{Y}}$ which maps initial object to an initial object and ϕ is a functor isomorphism $F \circ \theta_{\mathfrak{X}} \longrightarrow \theta_{\mathfrak{Y}} \circ F$ such that for every triangle $\mathcal{N} \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{L} \xrightarrow{w} \theta_{\mathfrak{X}}(\mathcal{N})$ of $\mathfrak{T}_+C_{\mathfrak{X}}$, the sequence

$$F(\mathcal{N}) \xrightarrow{F(u)} F(\mathcal{M}) \xrightarrow{F(v)} F(\mathcal{L}) \xrightarrow{\phi(\mathcal{N})F(w)} \theta_{\mathfrak{Y}}(F(\mathcal{N}))$$

is a triangle of $\mathfrak{T}_+C_{\mathfrak{Y}}$. The composition of triangle functors is defined naturally:

$$(G, \psi) \circ (F, \phi) = (G \circ F, \psi F \circ G\phi).$$

Let (F, ϕ) and (F', ϕ') be triangle functors from $\mathfrak{T}_-C_{\mathfrak{X}}$ to $\mathfrak{T}_-C_{\mathfrak{Y}}$. A morphism from (F, ϕ) to (F', ϕ') is given by a functor morphism $F \xrightarrow{\lambda} F'$ such that the diagram

$$\begin{array}{ccc}
\theta_{\mathfrak{Y}} \circ F & \xrightarrow{\phi} & F \circ \theta_{\mathfrak{X}} \\
{\scriptstyle \theta_{\mathfrak{Y}}\lambda}\downarrow & & \downarrow{\scriptstyle \lambda\theta_{\mathfrak{X}}} \\
\theta_{\mathfrak{Y}} \circ F' & \xrightarrow{\phi'} & F' \circ \theta_{\mathfrak{X}}
\end{array}$$

commutes. The composition is the compsition of the functor morphisms.

Altogether gives the definition of a bicategory $\mathfrak{PCat}$ formed by svelte presuspended categories, triangle functors as 1-morphisms and morphisms between them as 2-morphisms.

As usual, the term "category $\mathfrak{PCat}$" means that we forget about 2-morphisms.

Dualizing (i.e. inverting all arrows in the constructions above), we obtain the bicategory $\mathfrak{P}^o\mathfrak{Cat}$ formed by *precosuspended* svelte categories as objects, triangular functors as 1-morphisms, and morphisms between them as 2-morphisms.

6.3. Quasi-suspended categories.

We call a presuspended category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ *quasi-suspended* if the functor $\theta_{\mathfrak{X}}$ is conservative. We denote by $\mathfrak{SCat}$ the full subcategory of the category $\mathfrak{PCat}$ whose objects are quasi-suspended svelte categories.

Let $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ be a presuspended category and $\Sigma = \Sigma_{\theta_{\mathfrak{X}}}$ the class of all arrows s of the category $C_{\mathfrak{X}}$ such that $\theta_{\mathfrak{X}}(s)$ is an isomorphism. Let $\Theta_{\mathfrak{X}}$ denote the endofunctor of the quotient category $\Sigma^{-1}C_{\mathfrak{X}}$ uniquely determined by the equality $\Theta_{\mathfrak{X}} \circ \mathfrak{q}^*_{\Sigma} = \mathfrak{q}^*_{\Sigma} \circ \theta_{\mathfrak{X}}$, where $\mathfrak{q}^*_{\Sigma}$ denotes the localization functor $C_{\mathfrak{X}} \longrightarrow \Sigma^{-1}C_{\mathfrak{X}}$. Notice that the functor $\mathfrak{q}^*_{\Sigma}$ maps final objects to final objects. Let $\widetilde{\lambda}$ denote the morphism $\mathfrak{q}^*_{\Sigma}(x) \longrightarrow \Theta_{\mathfrak{X}}$ induced by $x \xrightarrow{\lambda} \theta_{\mathfrak{X}}$ (that is by $\mathfrak{q}^*_{\Sigma}(x) \xrightarrow{\mathfrak{q}^*_{\Sigma}(\lambda)} \mathfrak{q}^*_{\Sigma} \circ \theta_{\mathfrak{X}} = \Theta_{\mathfrak{X}} \circ \mathfrak{q}^*_{\Sigma}$) and $\widetilde{\mathfrak{Tr}}_{\mathfrak{X}}$ the essential image of $\mathfrak{Tr}_{\mathfrak{X}}$. Then the data $(\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{\mathfrak{X}})$ is a quasi-suspended category.

The constructed above map

$$(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}}) \longmapsto (\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{\mathfrak{X}})$$

extends to a functor $\mathfrak{PCat} \xrightarrow{\mathfrak{J}^*} \mathfrak{SCat}$ which is a left adjoint to the inclusion functor $\mathfrak{SCat} \xrightarrow{\mathfrak{J}_*} \mathfrak{PCat}$. The natural triangle (localization) functors

$$(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}}) \xrightarrow{\mathfrak{q}^*_{\Sigma}} (\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{\mathfrak{X}})$$

form an adjunction arrow $Id_{\mathfrak{PCat}} \longrightarrow \mathfrak{J}_*\mathfrak{J}^*$. The other adjunction arrow is identical.

6.4. The stable category of a left exact category with final objects.

Let $(C_X, \mathfrak{I}_X)$ be a left exact category with a final object x and $(C_{X_{\mathfrak{p}}}, \theta_{X_{\mathfrak{p}}}, \lambda; \mathfrak{Tr}_{X_{\mathfrak{p}}})$ the associated with $(C_X, \mathfrak{I}_X)$ presuspended category. We call the category $\Sigma^{-1}C_{X_{\mathfrak{p}}}$ the *stable* category of the left exact category $(C_X, \mathfrak{I}_X)$. The corresponding quasi-suspended category $(\Sigma^{-1}C_{X_{\mathfrak{p}}}, \Theta_{X_{\mathfrak{p}}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{X_{\mathfrak{p}}})$ will be called the *stable quasi-suspended* category of $(C_X, \mathfrak{I}_X)$.

6.4.1. Proposition. *Let $(C_X, \mathfrak{I}_X)$ be a left exact category with final objects. Suppose that $(C_X, \mathfrak{I}_X)$ has enough pointed (i.e. having a morphism from a final object) injective* *objects. Then the stable quasi-suspended category of* $(C_X, \mathfrak{I}_X)$ *is naturally equivalent to its weak stable category.*

Proof. It is easy to see that the natural functor $C_X \longrightarrow \Sigma^{-1}C_X$ factors through the weak stable category of $(C_X, \mathfrak{I}_X)$. The claim is that the corresponding (unique) functor from the weak stable category of $(C_X, \mathfrak{I}_X)$ to $\Sigma^{-1}C_X$ is a category equivalence. ■

7. Homology and homotopy of 'spaces'.

7.0. Right exact structure on the category of functors. Let C_X be a svelte category and $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ a svelte right exact category. We denote by $C_{\mathfrak{H}(\mathcal{Z},X)}$ the category $\mathcal{H}om(C_X, C_{\mathcal{Z}})$ of functors $C_X \longrightarrow C_{\mathcal{Z}}$.

7.0.1. Lemma. *The class* $\mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)}$ *of all functor morphisms* $\mathcal{F} \xrightarrow{\mathfrak{t}} \mathcal{G}$ *such that* $\mathcal{F}(L) \xrightarrow{\mathfrak{t}(L)} \mathcal{G}(L)$ *is a deflation for every* $L \in ObC_X$ *is a right exact structure on the category* $C_{\mathfrak{H}(\mathcal{Z},X)}$.

Proof. Evidently, the class $\mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)}$ is closed under composition and contains all isomorphisms of the category $C_{\mathfrak{H}(\mathcal{Z},X)}$. It is stable under pull-backs, because if the morphism $\mathfrak{t}$ in the diagram $\mathcal{F} \xrightarrow{\mathfrak{t}} \mathcal{G} \xleftarrow{\mathfrak{f}} \mathcal{G}'$ belongs to the class $\mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)}$, then, for every $L \in ObC_X$, there exist a cartesian square

$$\begin{array}{ccc} \mathcal{F}'(L) & \xrightarrow{\mathfrak{t}'(L)} & \mathcal{G}'(L) \\ {\scriptstyle \widetilde{\mathfrak{f}}(L)} \downarrow & cart & \downarrow {\scriptstyle \mathfrak{f}(L)} \\ \mathcal{F}(L) & \xrightarrow{\mathfrak{t}(L)} & \mathcal{G}(L) \end{array}$$

whose upper horizontal arrow is a deflations too. These cartesian squares determine the functor $\mathcal{F}'$ and a morphism $\mathcal{F}' \xrightarrow{\mathfrak{t}'} \mathcal{G}'$ from $\mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)}$, which is a pull-back of $\mathfrak{t}$. ■

7.0.2. Note. A similar argument shows that any Grothendieck pretopology τ on a category $C_{\mathcal{Z}}$ naturally induces a Grothedieck pretopology on the category $C_{\mathfrak{H}(\mathcal{Z},X)}$ of functors from C_X to $C_{\mathcal{Z}}$.

7.1. Homology of 'spaces' with coefficients in a right exact category. Let C_X be a svelte category and $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ a svelte right exact category with colimits and initial objects. We define the zero *homology object* of a 'space' X with coefficients in $C_X \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$ by setting $H_0(X, \mathcal{F}) = \mathrm{colim}\mathcal{F}$. The higher homology groups, $H_n(X, \mathcal{F})$, $n \geq 1$, are values at $\mathcal{F}$ of satellites of the functor $C_{\mathfrak{H}(\mathcal{Z},X)} \xrightarrow{H_0(X,-)} C_{\mathcal{Z}}$ with respect to the (object-wise) right exact structure $\mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)}$ induced by $\mathfrak{E}_{\mathcal{Z}}$ (cf. 7.0.1).

If the category $C_{\mathcal{Z}}$ has limits of filtered diagrams, then, since the category $(C_{\mathfrak{H}(\mathcal{Z},X)}$ of functors from C_X to $C_{\mathcal{Z}}$ inherits this property, there exists a universal ∂^*-functor

$$H_\bullet(X,-) = (H_n(X,-),\ \mathfrak{d}_n \mid n \geq 0)$$

from the right exact category *of coefficients* $(C_{\mathfrak{H}(\mathcal{Z},X)}, \mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)})$ to $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$.

7.1.1. Proposition. *Suppose that the right exact category $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ satisfies $(CE5^*)$. Then the universal ∂^*-functor $H_\bullet(X,-)$ is 'exact'.*

Proof. Let $\mathfrak{J}_*$ denote the canonical embedding of the category $C_{\mathcal{Z}}$ into the category $C_{\mathfrak{H}(\mathcal{Z},X)} = \mathcal{H}om(C_X, C_{\mathcal{Z}})$ which assigns to every object M of the category $C_{\mathcal{Z}}$ the constant functor mapping all arrows of C_X to id_M. The functor $\mathfrak{J}_*$ has a left adjoint, $\mathfrak{J}^*$, which assigns to every functor $C_X \longrightarrow C_{\mathcal{Z}}$ its colimit and to every functor morphism the corresponding morphism of colimits. The composition $\mathfrak{J}^*\mathfrak{J}_*$ is (isomorphic to) the identical functor; i.e. $\mathfrak{J}^*$ is a (continuous) localization functor. The functor $\mathfrak{J}_*$ is exact, hence 'exact', for any category C_X. Since $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ satisfies $(CE5^*)$, it follows that the functor $\mathfrak{J}^*$ maps deflations to deflations. Besides it is right exact, because it has a right adjoint. Therefore, $\mathfrak{J}^*$ is a right 'exact' functor. The assertion follows now from II.6.3. ∎

7.1.2. Proposition. *Suppose that the right exact category $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ is closed (that is $\mathfrak{E}_{\mathcal{Z}}$ is closed; cf. 1.4.3.2), has enough projective objects and pointed objects, and has small limits. Then, for any svelte category C_X, the right exact category of functors $(C_{\mathfrak{H}(\mathcal{Z},X)}, \mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)})$ has enough projective objects, and all its projective objects are pointed.*

Proof. For every $M \in ObC_X$, the functor

$$C_{\mathfrak{H}(\mathcal{Z},X)} \xrightarrow{\Phi_M} C_{\mathcal{Z}}, \quad \mathcal{F} \longmapsto \mathcal{F}(M),$$

preserves limits (and colimits) of small diagrams. Since the category $C_{\mathcal{Z}}$ has limits and for every $L \in ObC_{\mathcal{Z}}$, there exists a functor $C_X \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$ such that there is a morphism $L \longrightarrow \mathcal{F}(M)$ (e.g. $\mathcal{F}$ is the constant functor with the value L), by Freud's Theorem, the functor Φ_M has a left adjoint, Φ_M^*. Since the functor Φ_M is 'exact', in particular, it maps deflations to deflations, by 1.3.1, its left adjoint, Φ_M^* takes projective objects to projective objects.

Fix a set Ξ of objects of $C_{\mathcal{Z}}$ such that every object of $C_{\mathcal{Z}}$ is isomorphic to some object of Ξ. Given a functor $C_X \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$, we choose for each $M \in \Xi$ a deflation $P_M \xrightarrow{\mathfrak{t}_M} \mathcal{F}(M)$ with P_M a projective object. By hypothesis, the category $C_{\mathcal{Z}}$ has enough pointed objects, hence all its projective objects are pointed (cf. 2.2.3). We fix an initial object z of the category $C_{\mathcal{Z}}$ and, for each $M \in \Xi$, an *augmentation* morphism $P_M \xrightarrow{\lambda_M} z$. Notice that since the functor Φ_M^* maps initial objects to initial objects (as any functor having a right adjoint does), it maps augmentation morphisms λ_M to augmentation morphisms $\Phi_M^*(\lambda_M)$.

The morphisms $\Phi_M^*(P_M) \xrightarrow{\widehat{\mathfrak{t}}_M} \mathcal{F}$ corresponding by adjunction to the morphisms $P_M \xrightarrow{\mathfrak{t}_M} \mathcal{F}(M) = \Phi_M(\mathcal{F})$ determine a morphism

$$\coprod_{M \in \Xi} \Phi_M^*(P_M) \xrightarrow{\ \mathfrak{t}\ } \mathcal{F}. \tag{1}$$

and the morphisms $\Phi_M^*(\lambda_M)$ determine a morphism

$$\coprod_{M \in \Xi} \Phi_M^*(P_M) \xrightarrow{\ \lambda\ } \Phi_M^*(z)$$

to the initial object; so that $\coprod_{M \in \Xi} \Phi_M^*(P_M)$ is a pointed object.

Notice that a coproduct of projective objects is a projective; in particular, $\coprod_{M \in \Xi} \Phi_M^*(P_M)$ is a projective object of the right exact category $(C_{\mathfrak{H}(\mathcal{Z},X)}, \mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)})$.

It remains to show that (1) is a deflation. In fact, for any $L \in \Xi$, the composition of the morphisms

$$P_L \xrightarrow{\eta_L(P_L)} \Phi_L \circ \Phi_L^*(P_L) \xrightarrow{\Phi_L(\pi_L)} \Phi_L\big(\coprod_{M \in \Xi} \Phi_M^*(P_M)\big) \xrightarrow{\Phi_L(\mathfrak{t})} \Phi_L(\mathcal{F}) = \mathcal{F}(L),$$

where η_L is the adjunction morphism and π_L the coprojection

$$\Phi_L^*(P_L) \longrightarrow \coprod_{M \in \Xi} \Phi_M^*(P_M),$$

coincides with the deflation $P_L \xrightarrow{\mathfrak{t}_L} \mathcal{F}(L)$. By hypothesis, the class of deflations $\mathfrak{E}_{\mathcal{Z}}$ is closed. Therefore, $\Phi_L(\mathfrak{t})$ is a deflation for each $L \in \Xi$. Since every object of the category $C_{\mathcal{Z}}$ is isomorphic to an object from Ξ, it follows that $\Phi_L(\mathfrak{t})$ is a deflation for each object of $C_{\mathcal{Z}}$. But, this means, precisely, that $\mathfrak{t}$ is a deflation of the right exact category of functors from C_X to $C_{\mathcal{Z}}$. ■

7.1.2.1. Consequences. Thus, if the conditions of 7.1.2 hold, the homology of any 'space' X with coefficients in an arbitrary functor from C_X to $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ can be computed via projective resolutions.

7.1.3. Note. There is a natural equivalence between the category of local systems of abelian groups on the classifying topological space $\mathcal{B}(X)$ of a category C_X and the morphism inverting functors from C_X to $\mathbb{Z} - mod$. If $\mathcal{F}$ is a morphism inverting functor $C_X \longrightarrow \mathbb{Z} - mod$ and $\mathcal{L}_{\mathcal{F}}$ the corresponding local system, then the homology groups

$H_n(X,\mathcal{F})$ are naturally isomorphic to the homology groups $H_n(\mathcal{B}(X),\mathfrak{L}_{\mathcal{F}})$ of the classifying space $\mathcal{B}(X)$ with coefficients in the local coefficient system $\mathcal{L}_{\mathcal{F}}$ (cf. [Q, Section 1].

7.2. The 'space' of paths of a 'space'. Let $\mathcal{P}\mathfrak{a}_*$ be the functor from Cat to the category of diagrams of sets of the form $A \rightrightarrows B$ which assigns to each category C_X the diagram $HomC_X \overset{\mathfrak{s}}{\underset{\mathfrak{t}}{\rightrightarrows}} ObC_X$, where $\mathfrak{s}$ maps an arrow to its source and $\mathfrak{t}$ to its target. The functor $\mathcal{P}\mathfrak{a}_*$ has a left adjoint, $\mathcal{P}\mathfrak{a}^*$, which assigns to each diagram $T = (T_1 \rightrightarrows T_0)$ the category $\mathcal{P}\mathfrak{a}^*(T)$ of paths of T. The adjunction morphism $\mathcal{P}\mathfrak{a}^*\mathcal{P}\mathfrak{a}_*(C_X) \xrightarrow{\varepsilon(X)^*} C_X$ is a functor which is identical on objects and mapping each path of arrows

$$M_1 \longrightarrow M_2 \longrightarrow \ \dots \ \longrightarrow M_n$$

to its composition $M_1 \longrightarrow M_n$.

We denote by $\mathfrak{Pa}(X)$ the 'space' represented by the category $\mathcal{P}\mathfrak{a}^*\mathcal{P}\mathfrak{a}_*(C_X)$ and call it the *'space' of paths* of the 'space' X. The map $X \longmapsto \mathfrak{Pa}(X)$ extends to an endofunctor, $\mathfrak{Pa}$, of the category $|Cat|^o$. The adjunction morphism $\mathcal{P}\mathfrak{a}^*\mathcal{P}\mathfrak{a}_*(C_X) \xrightarrow{\varepsilon(X)^*} C_X$ is interpreted as an inverse image functor of a morphism of 'spaces' $X \xrightarrow{\varepsilon(X)} \mathfrak{Pa}(X)$. The morphisms $\varepsilon = (\varepsilon(X) \mid X \in Ob|Cat|^o)$ form a functor morphism $Id_{|Cat|^o} \longrightarrow \mathfrak{Pa}$.

7.2.1. The 'space' of paths and the loop 'space' of a pointed 'space'. Consider the pointed category $|Cat|^o/x$ associated with the category of 'spaces' $|Cat|^o$; Here x is the initial object of $|Cat|^o$ represented by the category with one (identical) morphism. By C1.5, a choice of a pseudo-functor

$$|Cat|^o \longrightarrow Cat^{op}, \quad X \longmapsto C_X,\ f \longmapsto f^*;\ (gf)^* \xrightarrow{\mathfrak{c}_{f,g}} f^*g^*,$$

induces an equivalence between the category $|Cat|^o/x$ and the category $|Cat|^o_x$ whose objects are pairs $(X,\mathfrak{O}_X)$, where $\mathfrak{O}_X \in ObC_X$; morphisms from $(X,\mathfrak{O}_X)$ to $(Y,\mathfrak{O}_Y)$ are pairs (f,ϕ), where f is a morphism of 'spaces' $X \longrightarrow Y$ and ϕ is an isomorphism $f^*(\mathfrak{O}_Y) \longrightarrow \mathfrak{O}_X$. The composition of $(X,\mathfrak{O}_X) \xrightarrow{(f,\phi)} (Y,\mathfrak{O}_Y) \xrightarrow{(g,\psi)} (Z,\mathfrak{O}_Z)$ is the morphism $(g\circ f, \phi\circ f^*(\psi)\circ \mathfrak{c}_{f,g})$.

The endofunctor $\mathfrak{Pa}$ of $|Cat|^o$ induces an endofunctor $\mathfrak{Pa}_x$ of $|Cat|^o_x$ which assigns to each pointed 'space' $(X,\mathfrak{O}_X)$ the pointed 'space' $(\mathfrak{Pa}(X),\mathfrak{O}_X)$ of paths of $(X,\mathfrak{O}_X)$. It follows that the canonical morphism $X \xrightarrow{\varepsilon(X)} \mathfrak{Pa}(X)$ is a morphism of pointed 'spaces' $(X,\mathfrak{O}_X) \longrightarrow \mathfrak{Pa}_x(X,\mathfrak{O}_X) = (\mathfrak{Pa}(X),\mathfrak{O}_X)$.

It follows from C1.5.1 that the category representing the cokernel of the canonical morphism $(X,\mathfrak{O}_X) \xrightarrow{\varepsilon(x)} \mathfrak{Pa}_x(X,\mathfrak{O}_X)$ is the subcategory of the category $C_{\mathfrak{Pa}(X)}$ whose objects are isomorphic to $\mathfrak{O}_X$ and morphisms are paths of arrows $M_1 \longrightarrow \ \dots \ \longrightarrow M_n$

whose composition is an isomorphism. This category is equivalent to its full subcategory $C_{\Omega(X,\mathfrak{O}_X)}$ of $C_{\mathfrak{Pa}(X)}$ which has one object, $\mathfrak{O}_X$.

We call the 'space' $\Omega(X,\mathfrak{O}_X)$ represented by the latter category the *loop 'space'* of the pointed 'space' $(X,\mathfrak{O}_X)$.

7.2.2. Left exact structures on the category of pointed 'spaces'. Let $\mathfrak{E}_x^{spl}$ be the class of all split epimorphisms of diagrams $A \rightrightarrows B$. By 1.5.4, the class $\mathcal{P}\mathfrak{a}_*^{-1}(\mathfrak{E}_x^{spl})$ is a right exact structure on the category of svelte pointed categories. This right exact structure determines a left exact structure $\mathfrak{J}_0$ on the category $|Cat|_x^o$ of pointed 'spaces', so that $(|Cat|_x^o,\mathfrak{J}_0)$ is a Karoubian left exact category. Each path 'space' $(\mathfrak{Pa}(X),\mathfrak{O}_X)$ is an injective object of $(|Cat|_x^o,\mathfrak{J}_0)$, and the canonical morphism $(X,\mathfrak{O}_X)\xrightarrow{\varepsilon(X)}(\mathfrak{Pa}(X),\mathfrak{O}_X)$ belongs to $\mathfrak{J}_0$. The fact that every epimorphism of diagrams of the form $A \rightrightarrows B$ splits implies that the class $\mathfrak{J}_0$ consists of all morphisms $(X,\mathfrak{O}_X)\xrightarrow{\mathfrak{j}}(Y,\mathfrak{O}_Y)$ of the pointed 'spaces' such that the image of $\mathfrak{j}^*$ is naturally equivalent to the category C_X.

7.3. The first homotopy group of a pointed 'space'. Given a svelte category C_X, we denote by $C_{\mathcal{G}(X)}$ the groupoid obtained from C_X by localization at $Hom(C_X)$. The map $\mathcal{G}$ which assigns to each object $(X,\mathfrak{O}_X)$ of the category $|Cat|_x^o$ the pair $(\mathcal{G}(X),\mathfrak{O}_X)$ (we identify objects of $C_{\mathcal{G}(X)}$ with objects of C_X) is naturally extended to a functor from $|Cat|_x^o$ to its full subcategory $\mathcal{G}\mathfrak{r}_x^o$ generated by objects $(Y,\mathfrak{O}_Y)$ such that C_Y is a groupoid. This functor is a left adjoint to the inclusion functor $\mathcal{G}\mathfrak{r}_x^o \longrightarrow |Cat|_x^o$.

7.3.1. Definition. The *fundamental group* $\pi_1(X,\mathfrak{O}_X)$ of the pointed 'space' $(X,\mathfrak{O}_X)$ is the group $C_{\mathcal{G}(X)}(\mathfrak{O}_X,\mathfrak{O}_X)$ of the automorphisms of the object $\mathfrak{O}_X$ of the groupoid $C_{\mathcal{G}(X)}$ associated with the category C_X. (see [GZ, II.6.2]).

7.3.2. Note. By [Q, Proposition 1], $\pi_1(X,\mathfrak{O}_X)$ is isomorphic to the fundamental group $\pi_1(\mathcal{B}(X),\mathfrak{O}_X)$ of the pointed classifying space $(\mathcal{B}(X),\mathfrak{O}_X)$ of the category C_X.

7.4. Higher homotopy groups of a pointed 'space'. The map which assigns to every pointed 'space; $(X,\mathfrak{O}_X)$ its fundamental group $\pi_1(X,\mathfrak{O}_X)$ is a functor from $(|Cat|_x^o)^{op}$ to the category *Groups* of groups. The functor π_1 maps every inflation to an epimorphism and every conflation $(X,\mathfrak{O}_X)\longrightarrow(Y,\mathfrak{O}_Y)\longrightarrow(Z,\mathfrak{O}_Z)$ to an exact sequence of groups $\pi_1(Z,\mathfrak{O}_Z)\longrightarrow\pi_1(Y,\mathfrak{O}_Y)\longrightarrow\pi_1(X,\mathfrak{O}_X)$. Therefore, by II.7.1, the universal ∂^*-functor $\pi_\bullet=(\pi_n,\mathfrak{d}_n \mid n\geq 1)$ from $(|Cat|_x^o,\mathfrak{J}_0)^{op}$ to *Groups* is 'exact'. We call $\pi_n(X,\mathfrak{O}_X)$ the *n-th homotopy group* of the pointed 'space' $(X,\mathfrak{O}_X)$.

7.4.1. Proposition. *For any pointed 'space'* $(X,\mathfrak{O}_X)$ *and any* $n\geq 1$, *there is a natural isomorphism* $\pi_{n+1}(X,\mathfrak{O}_X)\simeq\pi_n(\Omega(X,\mathfrak{O}_X))$.

Proof. This follows from the long exact sequence

$$\begin{array}{ccccccc}
\ldots \longrightarrow & \pi_{n+1}(\mathfrak{Pa}(X,\mathfrak{O}_X)) & & \pi_n(\mathfrak{Pa}(X,\mathfrak{O}_X)) & \longrightarrow \ldots \\
& \big\downarrow & & \big\uparrow & \\
& \pi_{n+1}(X,\mathfrak{O}_X) & \longrightarrow & \pi_n(\Omega(X,\mathfrak{O}_X)) &
\end{array}$$

corresponding to the (functorial) conflation $(X, \mathfrak{O}_X) \longrightarrow \mathfrak{Pa}(X, \mathfrak{O}_X) \longrightarrow \Omega(X, \mathfrak{O}_X)$ of pointed 'spaces' and the fact that, by the dual version of 2.2, the 'space' $(\mathfrak{Pa}(X), \mathfrak{O}_X)$ is an injective object of the pointed left exact category $(|Cat|^o_x, \mathfrak{I}_0)^{op}$, hence $\pi_n(\mathfrak{Pa}(X), \mathfrak{O}_X) = 0$ for $n \geq 1$. ■

Chapter IV

Left Exact Categories of 'Spaces'.

After short preliminaries on the category $|Cat|^o$ of 'spaces' represented by svelte categories (existence and construction of limits and colimits, facts about localizations) gathered in Section 1, we start, in Section 2, with observation that the category $|Cat|^o$ with the finest left exact structure (formed by all strict monomorphisms of 'spaces') has enough injective objects and all these injective objects are easy to describe. They are pointed, which opens the possibility to use injective resolution for computing satellites of left 'exact' functors from $|Cat|^o$ (as it is explained in Chapter III). From the finest left exact structure, we pass to a much coarser left exact structure on $|Cat|^o$, which we call *canonical*. This canonical left exact structure has the same class of injective objects and still there are enough of them. In Section 3, we mostly study left exact structures on $|Cat|^o$ formed by localizations. They enter naturally into picture, because with any left exact structure on $|Cat|^o$, there is a canonically associated left exact structure formed by localizations. In Section 4, we define left exact structures on the category of *k-'spaces'*, which, by definition, are 'spaces' represented by k-linear categories. We show that the *canonical* left exact structure on the category of k-'spaces' (induced by the canonical left exact structure on $|Cat|^o$) has enough injective objects. In Section 5, we extend the canonical left exact structure and left exact structures formed by localizations to the category of *right exact 'spaces'* and their 'exact' morphisms (that is morphisms whose inverse image functors are 'exact'). Again, we show that the canonical left exact structure has enough injective objects by producing natural inflations of each right exact 'space' into an associated with it injective. In Section 6, we study the left exact category of right exact k-'spaces', that is right exact 'spaces' represented by k-linear categories. We prove the existence of enough injective objects and use this to establish similar fact for the full subcategory of the category of right exact k-'spaces' formed by 'spaces' represented by Karoubian categories. In Section 7, we consider the left exact category formed by *exact k-'spaces'*, that is 'spaces' represented by exact k-linear categories, and establish, for every exact k-'space', the existence of a canonical inflation into an injective object.

We conclude with a couple of miscellaneous complements: introducing of the *path 'space'* of a right exact 'space' and a short discussion on localizations of right exact 'spaces'.

1. Preliminaries on the category $|Cat|^o$ of 'spaces'.

1.1. Initial objects of $|Cat|^o$. The 'space' $\mathfrak{x}_0$ represented by the category with one (identical) morphism is an initial object of $|Cat|^o$. A morphism $X \xrightarrow{f} Y$ in $|Cat|^o$ with an inverse image functor $C_Y \xrightarrow{f^*} C_X$ is an isomorphism iff f^* is a category equivalence.

In particular, $X \in Ob|Cat|^o$ is an initial object of $|Cat|^o$ iff the category C_X is a connected groupoid; i.e. all arrows of C_X are invertible and there are arrows between any two objects.

Notice that, for any 'space' X, the set $|Cat|^o(X, \mathfrak{x}_0)$ of morphisms $X \to \mathfrak{x}_0$ is isomorphic to the set $|X|$ of isomorphism classes of objects of the category C_X.

The category $|Cat|^o$ has no "real" final objects: its unique final object is the 'space' represented by the empty category.

1.2. Proposition. *The category $|Cat|^o$ has small limits and colimits.*

Proof. (a) Let $\{X_i \mid i \in J\}$ be a set of objects of $|Cat|^o$. Then $X^J = \prod_{i \in J} X_i$ and $X_J = \coprod_{i \in J} X_i$ are defined by

$$C_{X^J} = \coprod_{i \in J} C_{X_i} \quad \text{and} \quad C_{X_J} = \prod_{i \in J} C_{X_i}.$$

(b) Every pair of arrows, $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, in $|Cat|^o$ has a cokernel.

Let $C_Y \overset{f^*}{\underset{g^*}{\rightrightarrows}} C_X$ be inverse image functors of respectively f and g. Let C_Z denote the category whose objects are pairs (x, ϕ), where $x \in ObC_Y$ and ϕ is an isomorphism $f^*(x) \overset{\sim}{\to} g^*(x)$. A morphism from (x, ϕ) to (y, ψ) is an arrow $x \overset{\xi}{\longrightarrow} y$ such that the diagram

$$\begin{array}{ccc} f^*(x) & \xrightarrow{f^*(\xi)} & f^*(y) \\ \phi \downarrow & & \downarrow \psi \\ g^*(x) & \xrightarrow{g^*(\xi)} & g^*(y) \end{array}$$

commutes. Denote by $\mathfrak{h}^*$ the forgetful functor $C_Z \longrightarrow C_Y$, $(x, \phi) \longmapsto x$. Let $Y \overset{w}{\longrightarrow} W$ be a morphism in $|Cat|^o$ with an inverse image functor w^* such that $w \circ f = w \circ g$. This means that there exists an isomorphism $f^* \circ w^* \overset{\psi}{\longrightarrow} g^* \circ w^*$. The pair (w^*, ψ) defines a functor $\gamma^*_{w^*,\psi} : C_W \longrightarrow C_Z$, $b \longmapsto (w^*(b), \psi(b))$. A different choice, w_1^*, of the inverse image functor of w and an isomorphism $\psi_1 : w_1^* \circ f^* \overset{\sim}{\longrightarrow} w_1^* \circ g^*$ produces a functor $\gamma^*_{w_1^*,\psi_1}$ isomorphic to $\gamma^*_{w^*,\psi}$. This shows that the morphism $Y \longrightarrow Z$ having the inverse image $\mathfrak{h}^*$ is the cokernel of the pair (f, g). The existence of cokernels and (small) coproducts is equivalent to the existence of arbitrary (small) colimits.

(c) Every pair of arrows, $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, in $|Cat|^o$ has a kernel.

Let $C_Y \overset{f^*}{\underset{g^*}{\rightrightarrows}} C_X$ be inverse image functors of resp. f and g. Denote by $\mathfrak{D}_{f^*,g^*}$ the diagram scheme defined as follows:

$$Ob\mathfrak{D}_{f^*,g^*} = ObC_Y \coprod ObC_X \quad \text{and} \quad Hom\mathfrak{D}_{f^*,g^*} = HomC_X \coprod \Sigma_{f^*,g^*},$$

where

$$\Sigma_{f^*,g^*} = \{f^*(x) \overset{s_x}{\to} x,\ x \overset{t_x}{\to} g^*(x) \mid x \in ObC_Y\}.$$

Consider the category $\mathcal{P}a\mathfrak{D}_{f^*,g^*}$ of paths of the diagram $\mathfrak{D}_{f^*,g^*}$ together with the natural embeddings $HomC_X \overset{\tau}{\longrightarrow} Hom\mathcal{P}a\mathfrak{D}_{f^*,g^*} \longleftarrow \Sigma_{f^*,g^*}$ which define the corresponding diagrams. We denote by $\mathcal{P}\mathfrak{D}_{f^*,g^*}$ the quotient of the category $\mathcal{P}a\mathfrak{D}_{f^*,g^*}$ by the minimal equivalence relation such that

$$\begin{gathered}\tau(\alpha \circ \beta) \sim \tau(\alpha) \circ \tau(\beta) \quad \text{and} \quad \tau(id_x) \sim id_{\tau(x)} \\ \alpha \circ s_x \sim s_y \circ f^*(\alpha), \quad g^*(\alpha) \circ t_x \sim t_y \circ \alpha\end{gathered}$$

for all composable arrows $x \overset{\alpha}{\longrightarrow} y,\ y \overset{\beta}{\longrightarrow} z$ and for all $x \in ObC_X$.

Finally, we denote by C_W the quotient category $\Sigma^{-1}_{f^*,g^*}\mathcal{P}\mathfrak{D}_{f^*,g^*}$. It follows from the construction that the object W of the category $|Cat|^o$ defined this way is the kernel of the pair (f, g). Details are left to the reader. ∎

1.3. Proposition. *Suppose that* $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ *is a pair of continuous morphisms and the category* C_X *has small limits and a final object. Then the kernel* $\mathcal{K}(f,g) \overset{\mathfrak{k}}{\longrightarrow} X$, *of the pair* (f, g) *is a continuous morphism.*

Proof. (a) Fix direct image functors f_* and g_* of morphisms resp. f and g. Let $C_{\mathcal{K}(f_*,g_*)}$ denote the kernel of the pair of functors $C_X \overset{f_*}{\underset{g_*}{\rightrightarrows}} C_Y$. The objects of the category $C_{\mathcal{K}(f_*,g_*)}$ are pairs (L, ϕ), where $L \in ObC_X$ and ϕ is an isomorphism $f_*(L) \overset{\sim}{\longrightarrow} g_*(L)$. Morphisms from (L, ϕ) to (L', ϕ') are those arrows $L \overset{\xi}{\longrightarrow} L'$ for which the square

$$\begin{array}{ccc} f_*(L) & \overset{f_*(\xi)}{\longrightarrow} & f_*(L') \\ \phi \downarrow \wr & & \wr \downarrow \phi' \\ g_*(L) & \overset{g_*(\xi)}{\longrightarrow} & g_*(L') \end{array}$$

commutes. The composition is defined in a standard way.

The functor $C_{\mathcal{K}(f_*,g_*)} \xrightarrow{\mathfrak{k}_*} C_X$, which maps any morphism $(L,\phi) \xrightarrow{\xi} (L',\phi')$ of the category $C_{\mathcal{K}(f_*,g_*)}$ to the morphism $L \xrightarrow{\xi} L'$, equalizes the pair (f_*,g_*) in a pseudo-functorial way: there is an isomorphism $f_* \circ \mathfrak{k}_* \xrightarrow{\varphi} g_* \circ \mathfrak{k}_*$ which assigns to every object (L,ϕ) of the category $C_{\mathcal{K}(f_*,g_*)}$ the isomorphism

$$f_* \circ \mathfrak{k}_*(L,\phi) = f_*(L) \xrightarrow{\phi} g_*(L) = g_* \circ \mathfrak{k}_*(L,\phi).$$

The claim is that $\mathfrak{k}_*$ is a direct image functor of the kernel of the pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$.

In fact, since both functors f_* and g_* preserve arbitrary small limits, the functor $\mathfrak{k}_*$ has the same property. By hypothesis, the category C_X has small limits. Therefore, by Freud's Theorem, the functor $\mathfrak{k}_*$ has a left adjoint iff for every object M of the category C_X, there exists a morphism $M \longrightarrow \mathfrak{k}_*(L,\phi) = L$ for some object (L,ϕ) of the category $C_{\mathcal{K}(f_*,g_*)}$. By hypothesis, the category C_X has a final object, x. Both functors f_* and g_* map it to final objects of the category C_Y, because each of them has a left adjoint. Therefore, there exists a unique isomorphism $f_*(x) \xrightarrow{\phi_x} g_*(x)$. It is easy to see that (x,ϕ_x) is a final object of the category $C_{\mathcal{K}(f_*,g_*)}$, which the functor $\mathfrak{k}_*$ maps to the final object x. So that the functor $\mathfrak{k}_*$ has a left adjoint, $\mathfrak{k}^*$.

(b) The isomorphism of functors $f_*\circ\mathfrak{k}_* \simeq g_*\circ\mathfrak{k}_*$ induces an isomorphism $\mathfrak{k}^*\circ g^* \simeq \mathfrak{k}^*\circ f^*$. The claim is that the functor $\mathfrak{k}^*$ is universal; that is if $C_X \xrightarrow{\gamma^*} C_Z$ is a functor such that $\gamma^* \circ g^* \simeq \gamma^* \circ f^*$, then $\gamma^* = \bar{\gamma}^* \circ \mathfrak{k}^*$ for a functor $\bar{\gamma}^*$ defined uniquely up to isomorphism.

In fact, consider a commutative diagram

$$\begin{array}{ccccccc}
C_Y & \overset{f^*}{\underset{g^*}{\rightrightarrows}} & C_X & \xrightarrow{\kappa^*} & C_W & \xrightarrow{\mathfrak{k}_1^*} & C_{\mathcal{K}(f_*,g_*)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_Y^{\wedge} & \overset{\widehat{f^*}}{\underset{\widehat{g^*}}{\rightrightarrows}} & C_X^{\wedge} & \xrightarrow{\widehat{\kappa^*}} & C_W^{\wedge} & \xrightarrow{\widehat{\mathfrak{k}_1^*}} & C_{\mathcal{K}(f_*,g_*)}^{\wedge}
\end{array} \qquad (1)$$

where κ^* is an inverse image functor of the kernel of the pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, $\mathfrak{k}_1^*$ is defined uniquely up to isomorphism by the equality $\mathfrak{k}_1^* \circ \kappa^* = \mathfrak{k}^*$, all vertical arrows are Yoneda embeddings and all lower horizontal arrows are functors having a right adjoint. It follows from the fact that $\widehat{\kappa}$ is the cokernel of the pair of functors $(\widehat{f}^*, \widehat{g}^*)$ that $C_W^{\wedge} \xrightarrow{\widehat{\kappa_*}} C_X^{\wedge}$ is the kernel of the pair of functors $C^{\wedge}_X \overset{\widehat{f_*}}{\underset{\widehat{g_*}}{\rightrightarrows}} C^{\wedge}_Y$. Therefore, the right square of the diagram

$$\begin{array}{ccccc} C_Y & \overset{f_*}{\underset{g_*}{\longleftleftarrows}} & C_X & \overset{\mathfrak{k}_*}{\longleftarrow} & C_{\mathcal{K}(f_*,g_*)} \\ \big\downarrow & & \big\downarrow & cart & \big\downarrow \\ C^{\wedge}_Y & \overset{\widehat{f_*}}{\underset{\widehat{g_*}}{\longleftleftarrows}} & C^{\wedge}_X & \overset{\widehat{\kappa_*}}{\longleftarrow} & C^{\wedge}_W \end{array}$$

is cartesian, which implies that the functor $\widehat{\mathfrak{k}}_1^*$ in the diagram (1) is a category equivalence. Therefore the functor $\mathfrak{k}_1^*$ is a category equivalence. ∎

1.4. Push-forwards. Since push-forwards play a special role in the case of left exact categories, for the reader convenience, we describe them separately.

Let $Z \overset{g}{\longleftarrow} X \overset{f}{\longrightarrow} Y$ be morphisms of 'spaces'. Let $\mathfrak{X}$ denote the 'space' $Z \coprod_{f,g} Y$. The category $C_{\mathfrak{X}}$ is $C_Z \prod_{f^*,g^*} C_Y$. Recall that objects of $C_Z \prod_{f^*,g^*} C_Y$ are triples $(L, M; \phi)$, where $L \in ObC_Z$, $M \in ObC_Y$, and ϕ is an isomorphism $f^*(L) \overset{\sim}{\longrightarrow} g^*(M)$. A morphism $(L, M; \phi) \longrightarrow (L', M'; \phi')$ is given by a pair of arrows, $L \overset{\alpha}{\longrightarrow} L'$ and $M \overset{\beta}{\longrightarrow} M'$, such that the diagram

$$\begin{array}{ccc} f^*(L) & \overset{f^*(\alpha)}{\longrightarrow} & f^*(L') \\ \phi \big\downarrow \wr & & \wr \big\downarrow \phi' \\ g^*(M) & \overset{g^*(\beta)}{\longrightarrow} & g^*(M') \end{array} \qquad (1)$$

commutes. The composition of morphisms is defined naturally.

The (canonical) inverse image $C_{\mathfrak{X}} \overset{\widetilde{g}^*}{\longrightarrow} C_Z$ of the coprojection $Z \overset{\widetilde{q}}{\longrightarrow} \mathfrak{X}$ maps each object $(L, M; \phi)$ to L and each morphism $(L, M; \phi) \overset{(\mathfrak{s},\mathfrak{t})}{\longrightarrow} (L', M'; \phi')$ to $L \overset{\mathfrak{s}}{\longrightarrow} L'$.

1.5. Localizations. The following proposition is a refinement of [R1, 1.4.1].

1.5.1. Proposition. *Let $Z \overset{f}{\longleftarrow} X \overset{q}{\longrightarrow} Y$ be morphisms of 'spaces' such that q (i.e. its inverse image functor $C_Y \overset{q^*}{\longrightarrow} C_X$) is a localization. Then*

(a) The canonical morphism $Z \overset{\widetilde{q}}{\longrightarrow} Z \coprod_{f,q} Y$ is a localization.

(b) If q is a continuous localization, then $\widetilde{q}$ is a continuous localization.

(c) If $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s)$ *is invertible*$\}$ *is a left (resp. right) multiplicative system, then* $\Sigma_{\widetilde{q^*}}$ *has the same property.*

Proof. It follows that the class $\Sigma_{\widetilde{q^*}}$ consists of all morphisms

$$(L, M; \phi) \xrightarrow{(\mathfrak{s},\mathfrak{t})} (L', M'; \phi')$$

such that $\mathfrak{s}$ is an isomorphism, hence $\mathfrak{t} \in \Sigma_{q^*}$.

(a) Since q^* is a localization functor, for any $L \in ObC_Z$, there exists $M \in ObC_Y$ such that there is an isomorphism $f^*(L) \xrightarrow{\phi} q^*(M)$. The map $L \longmapsto (L, M; \phi)$ (– a choice for each L of an object M and isomorphism ϕ) extends uniquely to a functor $C_Z \longrightarrow \Sigma_{\widetilde{q^*}}^{-1} C_{\mathfrak{X}}$ which is quasi-inverse to the canonical functor

$$\Sigma_{\widetilde{q^*}}^{-1} C_{\mathfrak{X}} \longrightarrow C_Z, \quad (L, M; \phi) \longmapsto L.$$

(b) Suppose that q is a continuous localization; i.e. the localization functor q^* has a right adjoint, q_*. Fix adjunction arrows $Id_{C_Y} \xrightarrow{\eta} q_* q^*$ and $q^* q_* \xrightarrow{\epsilon} Id_{C_X}$. Since q^* is a localization, ϵ is an isomorphism. Therefore, we have a functor $C_Z \xrightarrow{\widetilde{q_*}} C_{\mathfrak{X}}$ which maps any object L of C_Z to the object $(L, q_* f^*(L); \epsilon f^*(L))$ of the category $C_{\mathfrak{X}}$ and any morphism $L \xrightarrow{\xi} L'$ to the morphism $(\xi, q_* f^*(\xi))$ of $C_{\mathfrak{X}}$.

The functor $\widetilde{q}_*$ is a right adjoint to the projection $\widetilde{q}^*$. The adjunction morphism $Id_{C_{\mathfrak{X}}} \longrightarrow \widetilde{q}_* \widetilde{q}^*$ assigns to each object $(L, M; \phi)$ of the category $C_{\mathfrak{X}}$ the morphism

$$(L, M; \phi) \xrightarrow{(id_L, \widehat{\phi})} (L, q_* f^*(L); \epsilon f^*(L)),$$

where $M \xrightarrow{\widehat{\phi}} q_* f^*(L)$ denote the morphism conjugate to $q^*(M) \xrightarrow{\phi^{-1}} f^*(L)$. The adjunction arrow $\widetilde{q}^* \widetilde{q}_* \longrightarrow Id_{C_Z}$ is the identical morphism. The latter implies that $\widetilde{q}^*$ is a localization functor.

(c) Suppose that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s)$ is invertible$\}$ is a left multiplicative system. Let $(L, M; \phi) \xrightarrow{(\mathfrak{s},\mathfrak{t})} (L', M'; \phi')$ be a morphism of $\Sigma_{\widetilde{q^*}}$ (that is $L \xrightarrow{\mathfrak{s}} L'$ is an isomorphism) and $(L, M; \phi) \xrightarrow{(\xi,\gamma)} (L'', M''; \phi'')$ an arbitrary morphism of $C_{\mathfrak{X}}$. The claim is that there exists a commutative diagram

$$\begin{array}{ccc} (L, M; \phi) & \xrightarrow{(\xi,\gamma)} & (L'', M''; \phi'') \\ {\scriptstyle (\mathfrak{s},\mathfrak{t})} \downarrow & & \downarrow {\scriptstyle (\mathfrak{s}',\mathfrak{t}')} \\ (L', M'; \phi') & \xrightarrow{(\xi',\gamma')} & (\widetilde{L}, \widetilde{M}; \widetilde{\phi}) \end{array} \qquad (1)$$

in $C_{\mathfrak{X}}$ whose right vertical arrow belongs to $\Sigma_{\widetilde{q^*}}$.

In fact, since $M \xrightarrow{\mathfrak{t}} M'$ belongs to Σ_{q^*} and Σ_{q^*} is a left multiplicative system, there exists a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & M'' \\ \mathfrak{t} \downarrow & & \downarrow \mathfrak{t}' \\ M' & \xrightarrow{\gamma'} & \widetilde{M} \end{array}$$

in C_Y such that $\mathfrak{t}' \in \Sigma_{q^*}$. Setting $\widetilde{L} = L''$, $\mathfrak{s}' = id_{L''}$, and $\widetilde{\phi} = q^*(\mathfrak{t}) \circ \phi''$, we obtain the required commutative diagram (1).

(c1) Suppose that $(L, M; \phi) \xrightarrow{(\mathfrak{s},\mathfrak{t})} (L', M'; \phi')$ is a morphism of $\Sigma_{\widetilde{q^*}}$ which equalizes a pair of arrows $(L', M'; \phi') \underset{(\xi,\gamma)}{\overset{(\alpha,\beta)}{\rightrightarrows}} (L'', M''; \phi'')$. Then there exists a morphism $(L'', M''; \phi'') \xrightarrow{(\mathfrak{s}',\mathfrak{t}')} (\widetilde{L}, \widetilde{M}; \widetilde{\phi})$ of $\Sigma_{\widetilde{q^*}}$ which equalizes this pair of arrows.

In fact, since $\mathfrak{s}$ is an isomorphism, the equality $(\alpha, \beta) \circ (\mathfrak{s}, \mathfrak{t}) = (\xi, \gamma) \circ (\mathfrak{s}, \mathfrak{t})$ implies that $\alpha = \xi$. Since Σ_{q^*} is a left multiplicative system, the equality $\beta \circ \mathfrak{t} = \gamma \circ \mathfrak{t}$ (and the fact that $\mathfrak{t} \in \Sigma_{q^*}$) implies the existence of a morphism $M'' \xrightarrow{\mathfrak{t}'} \widetilde{M}$ in Σ_{q^*} such that $\mathfrak{t}' \circ \beta = \mathfrak{t}' \circ \gamma$. Taking $\widetilde{L} = L''$, $\mathfrak{s}' = id_{L''}$, and $\widetilde{\phi} = q^*(\mathfrak{t}') \circ \phi''$, we obtain the required morphism of $\Sigma_{\widetilde{q^*}}$.

(c') Suppose that Σ_{q^*} is stable under the base change. Then $\Sigma_{\widetilde{q^*}}$ has the same property.

In fact, let a morphism $(L', M'; \phi') \xrightarrow{(\mathfrak{s},\mathfrak{t})} (L, M; \phi)$ of $C_{\mathfrak{X}}$ belong to $\Sigma_{\widetilde{q^*}}$, and let $(L'', M''; \phi'') \xrightarrow{(\xi,\gamma)} (L, M; \phi)$ be an arbitrary morphism of $C_{\mathfrak{X}}$. Then there exists a commutative diagram

$$\begin{array}{ccc} (\widetilde{L}, \widetilde{M}; \widetilde{\phi}) & \xrightarrow{(\xi',\gamma')} & (L', M'; \phi') \\ (\mathfrak{s}', \mathfrak{t}') \downarrow & & \downarrow (\mathfrak{s}, \mathfrak{t}) \\ (L'', M''; \phi'') & \xrightarrow{(\xi,\gamma)} & (L, M; \phi) \end{array} \tag{2}$$

in $C_{\mathfrak{X}}$ whose left vertical arrow belongs to $\Sigma_{\widetilde{q^*}}$.

Since $M' \xrightarrow{\mathfrak{t}} M$ belongs to Σ_{q^*} and Σ_{q^*} is a left multiplicative system, there exists a commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\gamma'} & M' \\ \mathfrak{t}' \downarrow & & \downarrow \mathfrak{t} \\ M'' & \xrightarrow{\gamma} & \widetilde{M} \end{array}$$

in C_Y such that $\mathfrak{t}' \in \Sigma_{q^*}$. Setting $\widetilde{L} = L''$, $\mathfrak{s}' = id_{L''}$, and $\widetilde{\phi} = q^*(\mathfrak{t}')^{-1} \circ \phi''$, we obtain a morphism $(\widetilde{L}, \widetilde{M}; \widetilde{\phi}) \xrightarrow{(\mathfrak{s}', \mathfrak{t}')} (L'', M''; \phi'')$ which belongs to $\Sigma_{\widetilde{q^*}}$. Set $\xi' = \mathfrak{s}^{-1} \circ \xi$. The claim is that the pair (ξ', γ') is a morphism from $(\widetilde{L}, \widetilde{M}; \widetilde{\phi})$ to $(L', M'; \phi')$ which makes the diagram (2) commute. By definition, (ξ', γ') being a morphism from $(\widetilde{L}, \widetilde{M}; \widetilde{\phi})$ to $(L', M'; \phi')$ means the commutativity of the diagram

$$\begin{array}{ccc} f^*(\widetilde{L}) & \xrightarrow{f^*(\xi')} & f^*(L') \\ \widetilde{\phi} \downarrow \wr & & \wr \downarrow \phi' \\ q^*(\widetilde{M}) & \xrightarrow{q^*(\gamma')} & q^*(M') \end{array}$$

which amounts to the equalities

$$q^*(\gamma') \circ q^*(\mathfrak{t}')^{-1} \circ \phi'' = q^*(\gamma') \circ \widetilde{\phi} = \phi' \circ f^*(\xi') = \phi' \circ f^*(\mathfrak{s})^{-1} \circ f^*(\xi). \tag{3}$$

It follows from the equality $\mathfrak{t} \circ \gamma' = \gamma \circ \mathfrak{t}'$ that $q^*(\gamma') \circ q^*(\mathfrak{t}')^{-1} = q^*(\mathfrak{t})^{-1} \circ q^*(\gamma)$. On the other hand, the fact that $(\mathfrak{s}, \mathfrak{t})$ is a morphism from $(L', M'; \phi')$ to $(L, M; \phi)$ means that $q^*(\mathfrak{t}) \circ \phi' = \phi \circ f^*(\mathfrak{s})$, or, equivalently, $\phi' \circ f^*(\mathfrak{s})^{-1} = q^*(\mathfrak{t})^{-1} \circ \phi$. Therefore, (3) is equivalent to the equality $q^*(\mathfrak{t})^{-1} \circ q^*(\gamma) \circ \phi'' = q^*(\mathfrak{t})^{-1} \circ \phi \circ f^*(\xi)$, or $q^*(\gamma) \circ \phi'' = \phi \circ f^*(\xi)$. The latter equality expresses the fact that (ξ, γ) is a morphism from $(L'', M''; \phi'')$ to $(L, M; \phi)$; hence (3) holds. The commutativity of the diagram (2) follows directly from the definition of the morphism (ξ', γ').

(c") Let Σ_{q^*} have the property:

(#) if an arrow $M' \xrightarrow{\mathfrak{t}} M$ belongs to Σ_{q^*} and equalizes a pair of arrows $M'' \rightrightarrows M'$, then there exists a morphism $M'' \xrightarrow{\mathfrak{t}'} M'$ in Σ_{q^*} which equalizes this pair of arrows.

Then $\Sigma_{\widetilde{q^*}}$ has the same property; that is if $(L', M'; \phi') \xrightarrow{(\mathfrak{s}, \mathfrak{t})} (L, M; \phi)$ is a morphism of $\Sigma_{\widetilde{q^*}}$ which equalizes a pair of arrows $(L'', M''; \phi'') \underset{(\xi,\gamma)}{\overset{(\alpha,\beta)}{\rightrightarrows}} (L', M'; \phi')$, then there exists a morphism $(\widetilde{L}, \widetilde{M}; \widetilde{\phi}) \xrightarrow{(\mathfrak{s}', \mathfrak{t}')} (L'', M''; \phi'')$ of $\Sigma_{\widetilde{q^*}}$ which equalizes this pair of arrows.

In fact, since $\mathfrak{s}$ is an isomorphism, the equality $(\mathfrak{s}, \mathfrak{t}) \circ (\alpha, \beta) = (\mathfrak{s}, \mathfrak{t}) \circ (\xi, \gamma)$ implies that $\alpha = \xi$. Since Σ_{q^*} is a right multiplicative system, the equality $\mathfrak{t} \circ \beta = \mathfrak{t} \circ \gamma$ (and the fact that $\mathfrak{t} \in \Sigma_{q^*}$) implies the existence of a morphism $M'' \xrightarrow{\mathfrak{t}'} \widetilde{M}$ in Σ_{q^*} such that $\mathfrak{t}' \circ \beta = \mathfrak{t}' \circ \gamma$. Taking $\widetilde{L} = L''$, $\mathfrak{s}' = id_{L''}$, and $\widetilde{\phi} = q^*(\mathfrak{t}')^{-1} \circ \phi''$, we obtain an object $(\widetilde{L}, \widetilde{M}; \widetilde{\phi})$ and a morphism $(\widetilde{L}, \widetilde{M}; \widetilde{\phi}) \xrightarrow{(\mathfrak{s}', \mathfrak{t}')} (L'', M''; \phi'')$ which belongs to $\Sigma_{\widetilde{q^*}}$ and equalizes the pair of arrows $(L'', M''; \phi'') \underset{(\xi,\gamma)}{\overset{(\alpha,\beta)}{\rightrightarrows}} (L', M'; \phi')$.

If follows from (c') and (c") above that $\Sigma_{\widetilde{q}^*}$ is a right multiplicative system if Σ_{q^*} is a right multiplicative system. ∎

1.5.2. Corollary. *Let* $Z \overset{f}{\longleftarrow} X \overset{q}{\longrightarrow} Y$ *be morphisms of 'spaces' such that* q *is a localization, and let* $Z \overset{\widetilde{q}}{\longrightarrow} Z\coprod_{f,q} Y$ *be a canonical morphism. Suppose the category* C_Y *has finite limits (resp. finite colimits). Then* $\widetilde{q}^*$ *is a left (resp. right) exact localization, if the localization* q^* *is left (resp. right) exact.*

Proof. By 1.5.1(a), $\widetilde{q}^*$ is a localization functor.

Suppose that the category C_Y has finite limits and the localization functor $C_Y \overset{q^*}{\longrightarrow} C_X$ is left exact. Then it follows from [GZ, I.3.4] that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s) \text{ is invertible}\}$ is a right multiplicative system. The latter implies, by 1.5.1(c), that $\Sigma_{\widetilde{q}^*}$ is a right multiplicative system. Therefore, by [GZ, I.3.1], the localization functor $\widetilde{q}^*$ is left exact. ∎

The following assertion is a refinement of [R1, 1.4.2].

1.5.3. Proposition. *Let* $X \overset{p}{\longleftarrow} Z \overset{q}{\longrightarrow} Y$ *be morphisms of 'spaces' such that* p^* *and* q^* *are localization functors. Then the square*

$$\begin{array}{ccc} Z & \overset{q}{\longrightarrow} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ X & \overset{q_1}{\longrightarrow} & X\coprod_{p,q} Y \end{array}$$

is cartesian. In particular, every morphism whose inverse image functor is a localization is a strict monomorphism.

Proof. Let $X \overset{u}{\longleftarrow} W \overset{v}{\longrightarrow} Y$ be morphisms of 'spaces' such that $q_1 \circ u = p_1 \circ v$. In other words, there exists an isomorphism $u^* \circ q_1^* \overset{\psi}{\longrightarrow} v^* \circ p_1^*$. Let $M \overset{s}{\longrightarrow} M'$ be any morphism of Σ_{q^*}. Since p^* is a localization functor, there exists $L \in ObC_X$ and an isomorphism $p^*(L) \overset{\phi}{\longrightarrow} q^*(M)$. We have a morphism $(L, M; \phi) \overset{(id_L, s)}{\longrightarrow} (L, M'; \phi')$ of the category $C_{\mathfrak{X}}$, where $\phi' = q^*(s)\phi$ and $\mathfrak{X}$ denotes the 'space' $X\coprod_{p,q} Y$ represented by the category $C_{\mathfrak{X}} = C_X \coprod_{p^*,q^*} C_Y$. By the definition of the canonical functors q_1^* and p_1^*, we have $q_1^*(id_L, s) = id_L$ and $p_1^*(id_L, s) = s$. Therefore, $v^*(s) = v^* \circ p_1^*(id_L, s)$ and $u^* \circ q_1^*(id_L, s) = u^*(id_L) = id_{u^*(L)}$. Since there is an isomorphism, $u^* \circ q_1^* \overset{\psi}{\longrightarrow} v^* \circ p_1^*$, we have a commutative diagram

$$\begin{array}{ccc} u^*(L) & \xrightarrow{\psi(L,M;\phi)} & v^*(M) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle v^*(s)} \\ u^*(L) & \xrightarrow{\psi(L,M';\phi')} & v^*(M') \end{array}$$

whose horizontal arrows are isomorphisms, hence $v^*(s)$ is an isomorphism. Thus, v^* maps arrows of Σ_{q^*} to isomorphisms. Since q^* is a localization, there exists a unique functor $C_Y \xrightarrow{\widetilde{v}^*} C_W$ such that $v^* = \widetilde{v}^* \circ q^*$; that is the morphism v is uniquely represented as the composition $q \circ w$. Similarly, the morphism u is represented as the composition $p \circ \widetilde{u}$ for a unique $\widetilde{u}$. The equality $q_1 \circ u = p_1 \circ v$ can be now rewritten as $(q_1 \circ p) \circ \widetilde{u} = (p_1 \circ q) \circ \widetilde{v} = (q_1 \circ p) \circ \widetilde{v}$, which means that $\widetilde{u}^* \circ (q_1 \circ p)^* \simeq \widetilde{v}^* \circ (q_1 \circ p)^*$. By 1.5.1(a), the functors q_1^* and p_1^* are localizations, hence $(q_1 \circ p)^* = (p_1 \circ q)^* \simeq q^* \circ p_1^*$ is a localization. Therefore the isomorphism $\widetilde{u}^* \circ (q_1 \circ p)^* \simeq \widetilde{v}^* \circ (q_1 \circ p)^*$ implies (is equivalent to) that $\widetilde{v}^*$ is isomorphic to $\widetilde{u}^*$, that is $\widetilde{u} = \widetilde{v}$. ∎

2. Two canonical left exact structures on the category of 'spaces'.

2.1. The finest left exact structure on the category of 'spaces'. By 1.2, the category of 'spaces' $|Cat|^o$ has all small limits (and colimits); in particular, it has arbitrary pull-backs. Therefore, its finest left exact structure on $|Cat|^o$ coincides with the class $\mathfrak{I}^{\mathfrak{st}}_{\mathfrak{Sp}}$ of all strict monomorphisms of 'spaces'.

2.1.1. Note. Let $X \xrightarrow{f} Y$ be a morphism in $|Cat|^o$ and $C_Y \xrightarrow{\mathfrak{f}^*} C_X$ its inverse image functor. The functor $\mathfrak{f}^*$ is the composition of the full functor $C_Y \xrightarrow{\bar{\mathfrak{f}}^*} C_{\mathfrak{f}(X)} \stackrel{\mathrm{def}}{=} \mathfrak{f}^*(C_Y)$ and the inclusion functor $C_{\mathfrak{f}(X)} \xrightarrow{\mathfrak{j}^*_{\mathfrak{f}}} C_X$ regarded as inverse image functors of morphisms of 'spaces', respectively $\mathfrak{f}(X) \xrightarrow{\bar{\mathfrak{f}}} Y$ and $X \xrightarrow{\mathfrak{j}_{\mathfrak{f}}} \mathfrak{f}(X)$. It is easy to see that the morphism $\bar{\mathfrak{f}}$ is a strict monomorphism; and, by general nonsense, the morphism $\mathfrak{j}_{\mathfrak{f}}$ is a strict monomorphism, because the composition $\bar{\mathfrak{f}} \circ \mathfrak{j}_{\mathfrak{f}}$ is a strict monomorphism. So that the morphism $X \xrightarrow{f} Y$ is a strict monomorphism iff $\mathfrak{j}_{\mathfrak{f}}$ is a strict monomorphism. The morphism $\mathfrak{j}_{\mathfrak{f}}$ is an epimorphism (hence an isomorphism) iff its inverse image functor is full.

2.2. Proposition. *The left exact category* $(|Cat|^o, \mathfrak{I}^{\mathfrak{st}}_{\mathfrak{Sp}})$ *has enough injective objects.*

Proof. (a) *Two elementary injective objects.* The smallest injective is the standard initial object, $\mathfrak{x}_0$, represented by the category with one (hence identical) morphism. The second *elementary* injective, $\mathfrak{x}_1$ is represented by the category with two objects and three arrows. Since $\mathfrak{x}_0$ is an initial object of the category of 'spaces', it is injective, because any morphism to $\mathfrak{x}_0$ splits. Any inverse image functor of a monomorphism $\mathfrak{x}_1 \longrightarrow X$ is a surjective functor which, obviously, splits; so that $\mathfrak{x}_1$ is also an injective object.

(b) Let C_X be a svelte category and $C_{\mathfrak{X}}$ its small subcategory such that the inclusion functor $C_{\mathfrak{X}} \hookrightarrow C_X$ is a category equivalence. For every arrow α of the category $C_{\mathfrak{X}}$, we have a functor $C_{\mathfrak{x}_1} \xrightarrow{\mathfrak{i}^*_\alpha} C_X$ which maps the non-identical arrow of $C_{\mathfrak{x}_1}$ to α. The functor $\mathfrak{i}^*_\alpha$ is an inverse image functor of a morphism $\mathfrak{X} \xrightarrow{\mathfrak{i}_\alpha} \mathfrak{x}_1$. The set of morphisms $\{\mathfrak{i}_\alpha \mid \alpha \in HomC_{\mathfrak{X}}\}$ determines a morphism

$$\mathfrak{X} \xrightarrow{\mathfrak{i}_{\mathfrak{X}}} \prod_{\alpha \in HomC_{\mathfrak{X}}} \mathfrak{x}_1 \qquad (1)$$

whose inverse image functor is surjective. Therefore, (1) is a strict monomorphism, that is a inflation of the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{st}}_{\mathfrak{Sp}})$, to the injective object $\prod_{\alpha \in HomC_{\mathfrak{X}}} \mathfrak{x}_1$.

(b1) One can replace (1) by a more economic embedding

$$\mathfrak{X} \xrightarrow{\mathfrak{i}_{\mathfrak{X}}} \big(\prod_{M \in ObC_{\mathfrak{X}}} \mathfrak{x}_0\big) \times \big(\prod_{\alpha \in Hom^*C_{\mathfrak{X}}} \mathfrak{x}_1\big), \qquad (2)$$

where $Hom^*C_{\mathfrak{X}}$ denotes the set of all non-identical arrows of the category $C_{\mathfrak{X}}$. ■

2.3. The canonical left exact structure. We denote by $\mathfrak{J}^{\mathfrak{s}}$ the class of all morphisms of 'spaces' $X \xrightarrow{\mathfrak{f}} Y$ such that every morphism of the category C_X is isomorphic to the inverse image of a morphism of the category C_Y. Since inverse image functors are defined uniquely up to isomorphism, this definition does not depend on a choice of an inverse image functor of the morphism $\mathfrak{f}$.

2.3.1. Proposition. *The class $\mathfrak{J}^{\mathfrak{s}}$ is the smallest left exact structure on $|Cat|^o$ containing all morphisms with surjective inverse image functors.*

Proof. Evidently, the class $\mathfrak{J}^{\mathfrak{s}}$ contains all isomorphisms (because their inverse image functors are equivalences of categories) and is closed under composition. It remains to show that $\mathfrak{J}^{\mathfrak{s}}$ is stable under pull-backs.

Let $X \xrightarrow{f} Y$ be a morphism from $\mathfrak{J}^{\mathfrak{s}}$ and $X \xrightarrow{g} Z$ an arbitrary morphism. Let $\mathfrak{X}$ denote the 'space' $Z \coprod_{f,g} Y$. The category $C_{\mathfrak{X}}$ is $C_Z \prod_{f^*,g^*} C_Y$. Recall that objects of $C_Z \prod_{f^*,g^*} C_Y$ are triples $(L, M; \phi)$, where $L \in ObC_Z$, $M \in ObC_Y$, and ϕ is an isomorphism $f^*(L) \xrightarrow{\sim} g^*(M)$. A morphism $(L, M; \phi) \longrightarrow (L', M'; \phi')$ is given by a pair of arrows, $L \xrightarrow{\alpha} L'$ and $M \xrightarrow{\beta} M'$, such that the diagram

$$\begin{array}{ccc} f^*(L) & \xrightarrow{f^*(\alpha)} & f^*(L') \\ \phi \downarrow \wr & & \wr \downarrow \phi' \\ g^*(M) & \xrightarrow{g^*(\beta)} & g^*(M') \end{array} \qquad (1)$$

commutes. The composition of morphisms is defined naturally.

The (canonical) inverse image $C_{\mathfrak{X}} \xrightarrow{\widetilde{g}^*} C_Z$ of the coprojection $Z \xrightarrow{\widetilde{q}} \mathfrak{X}$ maps each object $(L, M; \phi)$ to L and each morphism $(L, M; \phi) \xrightarrow{(\mathfrak{s},\mathfrak{t})} (L', M'; \phi')$ to $L \xrightarrow{\mathfrak{s}} L'$.

If $L \xrightarrow{\alpha} L'$ is an arbitrary arrow of the category C_Z, then, $g^*(\beta)$ is isomorphic to a morphism $f^*(\alpha)$ for some $M \xrightarrow{\alpha} M'$ of the category C_Y. So that we have a diagram (1) which can be interpreted as a morphism $(L, M; \phi) \xrightarrow{(\alpha,\beta)} (L', M'; \phi')$. By definition of the canonical inverse functor of the coprojection $\widetilde{g}$, we have $\widetilde{g}^*(\alpha, \beta) = \alpha$. So that $\widetilde{g}^*$ is a full functor. In particular, $\widetilde{g}$ belongs to $\mathfrak{J}^{\mathfrak{s}}$. ■

2.3.2. Proposition. *The left exact category $(|Cat|^o, \mathfrak{J}^{\mathfrak{s}})$ has enough injective objects and its injective objects coincide with the injective objects for the finest left exact structure on $|Cat|^o$.*

Proof. Evidently, injective object of a left exact category is injective objects for any coarser left exact structure. In particular, all injective objects of the left exact category

$(|Cat|^o, \mathfrak{I}^{\mathfrak{st}}_{\mathfrak{Sp}})$ are injective objects of $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$. Notice that the inverse image of the canonical embedding (1) (or (2)) in the argument of 2.2 has surjective inverse image functor; i.e. it belongs to the class of inflations $\mathfrak{I}^{\mathfrak{s}}$. So that for every 'space', there exists a morphism from $\mathfrak{I}^{\mathfrak{s}}$ to a product of a set of copies of the injective object $\mathfrak{x}_1$. In particular, every injective object of the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$ is a retract of a product of copies of $\mathfrak{x}_1$. ■

2.4. Remarks and observations.

2.4.1. Both the class $\mathfrak{I}^{\mathfrak{st}}_{\mathfrak{Sp}}$ of strict monomorphisms of 'spaces' and the class $\mathfrak{I}^{\mathfrak{s}}$ are self-dual: if $X \xrightarrow{\mathfrak{j}} Y$ belongs to $\mathfrak{I}^{\mathfrak{st}}_{\mathfrak{Sp}}$ (resp. to $\mathfrak{I}^{\mathfrak{s}}$), then same holds for the morphism $X^o \xrightarrow{\mathfrak{j}^o} Y^o$ of dual 'spaces'.

2.4.2. It follows from 1.5.3 that every morphism of 'spaces' whose inverse image functor is a localization is a strict monomorphism.

Morphisms whose inverse image functors are localizations at left or right multiplicative systems belong to the class $\mathfrak{I}^{\mathfrak{s}}$.

2.4.3. The class of morphisms $\mathfrak{I}^{\mathfrak{s}}$ has the following property: if the composition $X \xrightarrow{f \circ g} Y$ belongs to $\mathfrak{I}^{\mathfrak{s}}$, then $g \in \mathfrak{I}^{\mathfrak{s}}$. In fact, $(f \circ g)^* \simeq g^* \circ f^*$. So that if every morphism of the category C_X is isomorphic to an arrow from the image of $(f \circ g)^*$, then, with more reason this holds for g^*. Taking the canonical decomposition $f = p_f \circ f_c$, where p_f is a localization and f_c is a conservative morphism (that is f_c^* is a conservative functor), we obtain that for any morphism f from $\mathfrak{I}^{\mathfrak{s}}$, its conservative component, f_c, belongs to $\mathfrak{I}^{\mathfrak{s}}$.

2.4.4. In connection with the last remark, notice that

The class $\mathfrak{I}^{\mathfrak{s}}_{\mathfrak{c}}$ of all conservative morphisms from $\mathfrak{I}^{\mathfrak{s}}$ forms a left exact structure.

In fact, all isomorphisms are conservative and composition of conservative morphisms is conservative. It remains to verify that push-forwards of conservative morphisms are conservative. Let

$$\begin{array}{ccc} X & \xrightarrow{\mathfrak{f}} & Y \\ g \big\downarrow & cocart & \big\downarrow p_g \\ Z & \xrightarrow{\widetilde{\mathfrak{f}}} & \mathfrak{Y} \end{array}$$

be a push-forward of a conservative morphism $X \xrightarrow{\mathfrak{f}} Y$ along an arbitrary morphism $X \xrightarrow{g} Z$. It is represented by the cartesian (in pseudo-functorial sense) square of inverse image functors

$$\begin{array}{ccc} C_{\mathfrak{Y}} & \xrightarrow{\widetilde{\mathfrak{f}}^*} & C_Z \\ p_g^* \big\downarrow & cart & \big\downarrow p_g^* \\ C_Y & \xrightarrow{\mathfrak{f}^*} & C_X \end{array}$$

where $C_{\mathfrak{Y}}$ is the category whose objects are triples $(L, M; \phi)$, where ϕ is an isomorphism $\mathfrak{f}^*(L) \xrightarrow{\sim} g^*(M)$, and morphisms from $(L, M; \phi)$ to $(L', M'; \phi')$ are given by pairs of morphisms $L \xrightarrow{\xi_1} L'$, $M \xrightarrow{\xi_2} M'$ such that the diagram

$$\begin{array}{ccc} \mathfrak{f}^*(L) & \xrightarrow{\mathfrak{f}^*(\xi_1)} & \mathfrak{f}^*(L') \\ \phi \big\downarrow \wr & & \wr \big\downarrow \phi' \\ g^*(M) & \xrightarrow{g^*(\xi_2)} & g^*(M') \end{array} \qquad (1)$$

commutes. The functor $\widetilde{\mathfrak{f}}^*$ maps a morphism (ξ_1, ξ_2) to $M \xrightarrow{\xi_2} M'$. It follows from the diagram (1) that if $\widetilde{\mathfrak{f}}^*(\xi_1, \xi_2) = \xi_2$ is an isomorphism, then $\mathfrak{f}^*(\xi_1)$ is an isomorphism. Since, by hypothesis, the functor $\mathfrak{f}^*$ is conservative, the latter means that ξ_1 is an isomorphism. So that the functor $\widetilde{\mathfrak{f}}^*$ is conservative too.

2.5. The left exact category of Karoubian 'spaces'. Let $|KCat|^o$ denote the full subcategory of $|Cat|^o$ generated by 'spaces' represented by Karoubian svelte categories, that is svelte categories in which all idempotents split (see I.3.2). We call the objects of the subcategory $|KCat|^o$ *Karoubian 'spaces'.*

2.5.1. Proposition. *The inclusion functor* $|KCat|^o \xrightarrow{\mathfrak{K}^*} |Cat|^o$ *has a canonical right adjoint,* $|Cat|^o \xrightarrow{\mathfrak{K}_*} |KCat|^o$.

Proof. The functor $|Cat|^o \xrightarrow{\mathfrak{K}_*} |KCat|^o$. assigns to each 'space' X the 'space' X_K represented by the Karoubian envelope, C_{X_K} of the category C_X (see I.3.3.1 and I.3.3). The canonical full embedding $C_X \xrightarrow{\mathfrak{k}_X^*} C_{X_K}$ of the category C_X into its Karoubian envelope is the inverse image functor of (the value at X of) an adjunction morphism $\mathfrak{K}^*\mathfrak{K}_*(X) = X_K \xrightarrow{\mathfrak{k}_X^*} X$. The adjunction morphism $Id_{|KCat|^o} \longrightarrow \mathfrak{K}_*\mathfrak{K}^*$ is a natural isomorphism (see the argument of I.3.3). ∎

2.5.2. The canonical left exact structure on $|KCat|^o$. The left exact structure $\mathfrak{I}^{\mathfrak{s}}$ on the category of 'spaces' $|Cat|^o$ induces a left exact structure on $|KCat|^o$, which we denote by $\mathfrak{I}^{\mathfrak{Ka}}$.

2.5.3. Proposition. *The left exact category* $(|KCat|^o, \mathfrak{I}^{\mathfrak{Ka}})$ *has enough injective objects.*

Proof. The inclusion functor $|KCat|^o \xrightarrow{\mathfrak{K}^*} |Cat|^o$ preserves small colimits and maps $\mathfrak{I}^{\mathfrak{Ka}}$ to $\mathfrak{I}^{\mathfrak{s}}$. In particular, $\mathfrak{K}^*$ is an 'exact' functor from $(|KCat|^o, \mathfrak{I}^{\mathfrak{Ka}})$ to $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$.

(a) The fact that $\mathfrak{K}^*$ maps inflations to inflations implies that its right adjoint, $\mathfrak{K}_*$ maps injective objects to injective objects. The claim is that there are enough injective objects of this form.

(b) Notice that if X is a Karoubian 'space' and $\mathfrak{K}^*(X) \xrightarrow{\mathfrak{f}} Y$ is an inflation (– a morphism from $\mathfrak{I}^{\mathfrak{s}}$), then the adjoint morphism $X \xrightarrow{\widehat{\mathfrak{f}}} \mathfrak{K}_*(Y)$ is an inflation.

In fact, $\widehat{\mathfrak{f}}$ is a morphism from the 'space' X to the Karoubian envelope Y_K of the 'space' Y; and the inflation $X \xrightarrow{\mathfrak{f}} Y$ is the composition of $X \xrightarrow{\widehat{\mathfrak{f}}} Y_K$ and the canonical morphism $Y_K \xrightarrow{\mathfrak{k}_Y} Y$ (corresponding to the full embedding C_Y into its Karoubian envelope C_{Y_K}). Therefore, by 2.4.3, the morphism $X \xrightarrow{\widehat{\mathfrak{f}}} Y_K$ belongs to $\mathfrak{I}^{\mathfrak{s}}$. Both X and Y_K are Karoubian 'spaces'; so that $\widehat{\mathfrak{f}} \in \mathfrak{I}^{\mathfrak{K}\mathfrak{a}}$.

(c) Since the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$ has enough injective objects, for every Karoubian 'space' X, we have an inflation (actually, a canonical inflation) $\mathfrak{K}^*(X) \xrightarrow{\mathfrak{f}} \mathfrak{X}$ to an injective object $\mathfrak{X}$ of the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$. By (b) above, the adjoint morphism $X \xrightarrow{\widehat{\mathfrak{f}}} \mathfrak{K}_*(\mathfrak{X}) = \mathfrak{X}_K$ is an inflation; and by (a), $\mathfrak{K}_*(\mathfrak{X}) = \mathfrak{X}_K$ is an injective object of the left exact category $(|KCat|^o, \mathfrak{I}^{\mathfrak{K}\mathfrak{a}})$. ■

3. Left exact structures formed by localizations and related constructions.

Let $\mathfrak{L}$ denote the class of all localizations of 'spaces' (i.e. morphisms whose inverse image functors are localizations). We denote by $\mathfrak{L}_\ell$ (resp. $\mathfrak{L}_\mathfrak{r}$) the class of localizations $X \xrightarrow{q} Y$ of 'spaces' such that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s) \text{ is invertible}\}$ is a left (resp. right) multiplicative system. We denote by $\mathfrak{L}_\mathfrak{e}$ the intersection of $\mathfrak{L}_\ell$ and $\mathfrak{L}_\mathfrak{r}$ (i.e. the class of localizations q such that Σ_{q^*} is a multiplicative system) and by $\mathfrak{L}^\mathfrak{c}$ the class of continuous (i.e. having a direct image functor) localizations of 'spaces'. Finally, we set $\mathfrak{L}^\mathfrak{c}_\mathfrak{e} = \mathfrak{L}^\mathfrak{c} \cap \mathfrak{L}_\mathfrak{e}$; i.e. $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}$ is the class of continuous localizations $X \xrightarrow{q} Y$ such that Σ_{q^*} is a multiplicative system.

3.1. Proposition. *Each of the classes of morphisms* $\mathfrak{L}$, $\mathfrak{L}_\ell$, $\mathfrak{L}_\mathfrak{r}$, $\mathfrak{L}_\mathfrak{e}$, $\mathfrak{L}^\mathfrak{c}$, *and* $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}$ *are structures of a left exact category on the category* $|Cat|^o$ *of 'spaces'.*

Proof. It is immediate that each of these classes is closed under composition and contains all isomorphisms of the category $|Cat|^o$. It follows from 1.5.1 that each of the classes is stable under cobase change. In other words, the arrows of each class can be regarded as cocovers of a copretopology. It remains to show that these copretopologies are subcanonical. Since $\mathfrak{L}$ is the finest copretopology, it suffices to show that $\mathfrak{L}$ is subcanonical.

The copretopology $\mathfrak{L}$ being subcanonical means precisely that for any localization $X \xrightarrow{q} Y$, the square

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q_1} \\ Y & \xrightarrow{q_2} & Y \coprod_{q,q} Y \end{array}$$

is cartesian. But, this follows from 1.5.3. ■

3.2. Observation. Each object of the left exact category $(|Cat|^o, \mathfrak{L}^\mathfrak{c})$ is injective.

In fact, a 'space' X is an injective object of $(|Cat|^o, \mathfrak{L}^\mathfrak{c})$ iff each morphism $X \xrightarrow{q} Y$ is split; i.e. there is a morphism $Y \xrightarrow{\mathfrak{t}} X$ such that $\mathfrak{t} \circ q = id_X$. Since the morphism q is continuous, it has a direct image functor, q_*, which is fully faithful, because q^* is a localization functor. The latter means precisely that the adjunction arrow $q^* q_* \longrightarrow Id_{C_X}$ is an isomorphism. Therefore, the morphism $Y \xrightarrow{\mathfrak{t}} X$ whose inverse image functor coincides with q_* satisfies the equality $\mathfrak{t} \circ q = id_X$.

3.3. The left exact structures $\mathfrak{I}^\mathfrak{s}_\ell$ and $\mathfrak{I}^\mathfrak{s}_\mathfrak{r}$. We denote by $\mathfrak{I}^\mathfrak{s}_\ell$ (resp. by $\mathfrak{I}^\mathfrak{s}_\mathfrak{r}$) the class of all morphisms $X \xrightarrow{f} Y$ from $\mathfrak{I}^\mathfrak{s}$ such that $\Sigma_{f^*} \stackrel{\mathrm{def}}{=} \{s \in HomC_Y \mid f^*(s) \text{ is invertible}\}$ is a left (resp. right) multiplicative system. We denote by $\mathfrak{I}^\mathfrak{s}_\mathfrak{e}$ the intersection of the classes $\mathfrak{I}^\mathfrak{s}_\ell$ and $\mathfrak{I}^\mathfrak{s}_\mathfrak{r}$.

3.3.1. Proposition. *(a) The classes of morphisms $\mathfrak{I}^\mathfrak{s}_\ell$ and $\mathfrak{I}^\mathfrak{s}_\mathfrak{r}$ (hence their intersection $\mathfrak{I}^\mathfrak{s}_\mathfrak{e}$) are left exact structures on the category $|Cat|^o$.*

(b) The left exact category $(|Cat|^o, \mathfrak{I}^\mathfrak{s}_\mathfrak{e})$ has enough injective objects and its injective objects coincide with injective objects for the finest left exact structure on $|Cat|^o$.

Proof. (a) The class $\mathfrak{I}^\mathfrak{s}_\ell$ is formed by compositions $\mathfrak{q} \circ \mathfrak{f}$, where $\mathfrak{f}$ is an arbitrary morphism from the class $\mathfrak{I}^\mathfrak{s}_\mathfrak{c}$ of conservative morphisms from $\mathfrak{I}^\mathfrak{s}$ and $\mathfrak{q}$ is any morphism (composable with $\mathfrak{f}$) from from the class $\mathfrak{L}_\ell$ of morphisms whose inverse image functors are localizations at left multiplicative systems. It follows that if $\gamma = \mathfrak{f} \circ \mathfrak{q}$ for some $\mathfrak{q} \in \mathfrak{L}_\ell$ and $\mathfrak{f} \in \mathfrak{I}^\mathfrak{s}_\mathfrak{c}$ having a surjective inverse image functor, then the class Σ_{γ^*} of arrows mapped to isomorphisms is a left multiplicative system. Since every morphism of $\mathfrak{I}^\mathfrak{s}$ is a composition of an isomorphism and a morphism with a surjective inverse image functor, it follows that the class $\mathfrak{I}^\mathfrak{s}_\ell$ is closed under composition. Since both $\mathfrak{I}^\mathfrak{s}_\mathfrak{c}$ and $\mathfrak{L}_\ell$ are stable under push-forwards, same holds for the class $\mathfrak{I}^\mathfrak{s}_\ell = \mathfrak{L}_\ell \circ \mathfrak{I}^\mathfrak{s}_\mathfrak{c}$.

(b) All morphisms from $\mathfrak{I}^\mathfrak{s}$ to a product of copies of the injective 'space' $\mathfrak{x}_1$ belong to the class $\mathfrak{I}^\mathfrak{s}_\mathfrak{e}$, hence the assertion. ■

3.3.2. Note. The fact that the left exact category $(|Cat|^o, \mathfrak{I}^\mathfrak{s}_\mathfrak{e})$ has enough injective objects and its injective objects coincide with injective objects for the finest left exact structure on $|Cat|^o$ implies a similar assertion for any left exact structure on $|Cat|^o$, which is finer than $\mathfrak{I}^\mathfrak{s}_\mathfrak{e}$. In particular, same assertion holds for $(|Cat|^o, \mathfrak{I}^\mathfrak{s}_\mathfrak{r})$ and $(|Cat|^o, \mathfrak{I}^\mathfrak{s}_\ell)$.

3.4. Relative 'spaces'. The category $|Cat|^o$ has canonical initial object represented by the category with one object and one morphism, but does not have final objects (since we do not allow empty categories). In particular, the notion of the cokernel of a morphism is not defined in $|Cat|^o$. So that we cannot apply to $|Cat|^o$ the theory of derived functors (satellites) sketched in Chapter II. The category of *relative* 'spaces' (i.e. 'spaces' over a given 'space') has both final objects and cokernels of arbitrary morphisms.

Fix a 'space' S. The category $|Cat|^o/S$ has a natural final object – (S, id_S), and cokernels of morphisms. The cokernel of a morphism $(X, g) \xrightarrow{f} (Y, h)$ of 'spaces' over S is the pair $(Y \coprod_{f,g} S, \widetilde{h})$, where $Y \coprod_{f,g} S \xrightarrow{\widetilde{h}} S$ is the unique arrow determined by the commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \big\downarrow & & \big\downarrow h \\ S & \xrightarrow{id_S} & S \end{array}$$

The canonical inverse image functor $\widetilde{h}^*$ of the morphism $\widetilde{h}$ maps every object M of the category C_S to the object $(h^*(M), M; f^*h^*(M) \xrightarrow{\sim} g^*(M))$ of the category $C_Y \coprod_{f^*,g^*} C_S$ representing the 'space' $Y \coprod_{f,g} S$.

3.4.1. Lemma. *Let C_X be a category and V its object. Any left exact structure $\mathfrak{I}_X$ on C_X induces a left exact structure, $\mathfrak{I}_X/V$ on the category C_X/V.*

Proof. By the definition of $\mathfrak{I}_X/V$, a morphism $(L, \xi) \xrightarrow{f} (L', \xi')$ of C_X/V belongs to $\mathfrak{I}_X/V$ iff the morphism $L \xrightarrow{f} L'$ belongs to $\mathfrak{I}_X$. We leave to the reader the verifying that $\mathfrak{I}_X/V$ is a left exact structure on C_X/V. ∎

In particular, each left exact structure constructed above induces a left exact structure on the category $|Cat|^o/S$.

4. Left exact structures on the category of k-'spaces'.

Fix a commutative associative unital ring k. Recall that *k-'spaces'* are 'spaces' represented by k-linear categories. They are objects of the category $|Cat_k|^o$ whose arrows $X \longrightarrow Y$ are represented by isomorphism classes of k-linear functors $C_Y \longrightarrow C_X$.

4.1. Cokernels in $|Cat_k|^o$. The category $|Cat_k|^o$ is pointed: its zero object is represented by the zero category. Every morphism $X \xrightarrow{f} Y$ of $|Cat_k|^o$ has a canonical cokernel $Y \xrightarrow{\mathfrak{c}} Cok(f)$, where $C_{Cok(f)}$ is the subcategory $Ker(f^*)$ of C_Y (– the full subcategory generated by all objects L such that $f^*(L) = 0$) and $\mathfrak{c}^*$ is the inclusion functor $Ker(f^*) \longrightarrow C_Y$.

4.2. Proposition. *The category $|Cat_k|^o$ has small colimits and products.*

Proof. (a) The category $|Cat_k|^o$ has small coproducts and products.

Let $\{X_i \mid i \in J\}$ be a set of objects of $|Cat_k|^o$. The coproduct $X_J = \coprod_{i\in J} X_i$ is defined by $C_{X_J} = \prod_{i\in J} C_{X_i}$. The product $X^J = \prod_{i\in J} X_i$ is defined by $C_{X^J} = \coprod_{i\in J} C_{X_i}$, where $\coprod_{i\in J} C_{X_i}$ is the full subcategory of $C_{X_J} = \prod_{i\in J} C_{X_i}$ generated by all objects $(M_i \mid i \in J)$ such that $M_i \neq 0$ only for a finite number of $i \in J$.

(b) Every pair of arrows, $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, in $|Cat_k|^o$ has a cokernel.

Let $C_Y \overset{f^*}{\underset{g^*}{\rightrightarrows}} C_X$ be inverse image functors of respectively f and g. The cokernel $\mathcal{C}(f,g)$ is represented, like in non-additive case, by the kernel of the pair (f^*, g^*) of their respective inverse image functors: objects of $C_{\mathcal{C}(f,g)}$ are pairs (L,ϕ), where ϕ is an isomorphisms $f^*(L) \xrightarrow{\sim} g^*(L)$ and morphisms $(L,\phi) \longrightarrow (L',\phi')$ are given by morphisms $L \xrightarrow{\xi} L'$ such that the diagram

$$\begin{array}{ccc} f^*(L) & \xrightarrow{f^*(\xi)} & f^*(L') \\ \phi \downarrow & & \downarrow \phi' \\ g^*(L) & \xrightarrow{g^*(\xi)} & g^*(L') \end{array}$$

commutes. Since the categories C_X and C_Y are k-linear and additive and functors are k-linear (in particular, they are additive), the category $C_{\mathcal{C}(f,g)}$ is additive an k-linear and the canonical functor $C_{\mathcal{C}(f,g)} \longrightarrow C_Y$, which maps a morphism $(L,\phi) \xrightarrow{\xi} (L',\phi')$ to $L \xrightarrow{\xi} L'$ is k-linear. ∎

4.3. Proposition. *Suppose that $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is a pair of continuous morphisms of k-'spaces', and let the category C_X have small limits. Then the kernel $\mathcal{K}(f,g) \xrightarrow{\mathfrak{k}} X$, of the pair (f,g) is a continuous morphism.*

Proof. The 'space' $\mathcal{K}(f,g)$ is represented by the kernel of the pair $C_X \overset{f_*}{\underset{g_*}{\rightrightarrows}} C_Y$ of the morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$. The direct image functor of the kernel morphism $\mathcal{K}(f,g) \xrightarrow{\mathfrak{k}} X$ maps every morphism $(L,\phi) \xrightarrow{\xi} (L',\phi')$ of the category $C_{\mathcal{K}(f,g)}$ to the morphism $L \xrightarrow{\xi} L'$ of the category C_X. The fact follows from (the argument of) 1.3. ∎

4.3.1. A construction of the kernel in general case. Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be an arbitrary pair of morphisms of k-'spaces' with the inverse image functors $C_Y \overset{f^*}{\underset{g^*}{\rightrightarrows}} C_X$.

Then we have a (quasi-)commutative diagram

$$\begin{array}{ccccccc} C_Y & \overset{f^*}{\underset{g^*}{\rightrightarrows}} & C_X & \xrightarrow{\kappa^*} & C_W & \dashrightarrow & C_{\mathcal{K}(f_*,g_*)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_Y^\wedge & \overset{\widehat{f^*}}{\underset{\widehat{g^*}}{\rightrightarrows}} & C_X^\wedge & \xrightarrow{\widehat{\kappa^*}} & C_W^\wedge & \xrightarrow{\widehat{\mathfrak{k}_1^*}} & C_{\mathcal{K}(\widehat{f_*},\widehat{g_*})} \end{array} \qquad (1)$$

where C_W is the image of the composition of the Yoneda embedding and the inverse image functor $C_X^\wedge \xrightarrow{\mathfrak{k}^*(\widehat{f,g})} C_{\mathcal{K}(\widehat{f_*},\widehat{g_*})}$. It follows from the universality of this functor (due to the fact that, by 4.3, it is an inverse image functor of the kernel of the pair of morphisms $(\widehat{f},\widehat{g})$) that the functor $C_W^\wedge \xrightarrow{\widehat{\mathfrak{k}_1^*}} C_{\mathcal{K}(\widehat{f_*},\widehat{g_*})}$ in the diagram (1) is a category equivalence.

It follows from this construction that any morphism $Z \longrightarrow X$ which equalizes the pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ and such that C_Z is a k-linear category with small colimits factors uniquely through the morphism $W \xrightarrow{\kappa} X$. One can also see that $W \xrightarrow{\kappa} X$ is isomorphic to the kernel of the pair (f,g) provided this kernel exists. Actually, it does exist, and the proof of existence is a k-linear version of the argument 1.2(c). Since we do not need this fact, we omit the argument.

4.4. Left exact structures on the category of k-spaces.

4.4.1. The finest left exact structure. We denote the finest left exact structure on $|Cat_k|^o$ by $\mathfrak{I}_k^{\mathfrak{st}}$. Since the category $|Cat_k|^o$ has all colimits, in particular all push-forwards, the class $\mathfrak{I}_k^{\mathfrak{st}}$ consists of all strict monomorphisms of k-'spaces'.

4.4.2. The left exact structure $\mathfrak{I}_k^{\mathfrak{s}}$. We denote by $\mathfrak{I}_k^{\mathfrak{s}}$ the k-linear version of the left exact structure $\mathfrak{I}^{\mathfrak{s}}$: it is formed by morphisms $X \xrightarrow{\mathfrak{f}} Y$ of k-'spaces' such that every morphism of the category C_X is isomorphic to a morphism $\mathfrak{f}^*(\alpha)$ for some $\alpha \in HomC_Y$.

4.4.3. Proposition. *The class of morphisms $\mathfrak{I}_k^{\mathfrak{s}}$ is a left exact structure on the category of k-'spaces'.*

Proof. Evidently, the class $\mathfrak{I}_k^{\mathfrak{s}}$ is closed under composition and contains all isomorphisms. It follows also that every morphism from $\mathfrak{I}_k^{\mathfrak{s}}$ is a strict monomorphism. It remains to show that $\mathfrak{I}_k^{\mathfrak{s}}$ is stable under push-forwards. The argument is the same as for the left exact structure $\mathfrak{I}^{\mathfrak{s}}$ on $|Cat|^o$ (cf. 2.3.1). ∎

4.4.4. Proposition. *The left exact category $(|Cat_k|^o, \mathfrak{I}_k^{\mathfrak{s}})$ has enough injective objects.*

Proof. (a) *The k-'space'* $\mathfrak{x}_k$. We denote by $\mathfrak{x}_k$ the k-'space' represented by the k-linear category with two objects, a and b such that

$$C_{\mathfrak{x}_k}(a,b) = k, \quad C_{\mathfrak{x}_k}(b,a) = 0, \quad C_{\mathfrak{x}_k}(a,a) = k, \quad \text{and} \quad C_{\mathfrak{x}_k}(b,b) = k.$$

Every inflation $\mathfrak{x}_k \xrightarrow{\mathfrak{f}} X$ has a surjective inverse image functor $C_X \xrightarrow{\mathfrak{f}^*} C_{\mathfrak{x}_k}$; that is there exist objects L, M of the category C_X such that $\mathfrak{f}^*(L) = a$, $\mathfrak{f}^*(M) = b$, and the functor $\mathfrak{f}^*$ maps $C_X(L,M)$ onto $C_{\mathfrak{x}_k}(a,b) = k$. Therefore, since k is a projective k-module, the map $C_Y(L,M) \xrightarrow{\mathfrak{f}^*_{L,M}} C_{\mathfrak{x}_k}(a,b)$ splits. This splitting determines a functor $C_{\mathfrak{x}_k} \xrightarrow{\gamma^*} C_X$ such that $\mathfrak{f}^* \circ \gamma^*$ is the identical functor.

(b) *Inflations into injective objects.* Let X be a k-'space' represented by a small k-linear category C_X. For each arrow $L \xrightarrow{\alpha} M$, of the category C_X, consider the functor

$$C_{\mathfrak{x}_k} \xrightarrow{\gamma^*_\alpha} C_X$$

which maps a to L, b to M and each element $\lambda \in k = C_{\mathfrak{x}_k}(a,b)$ to the morphism $\lambda \cdot \alpha$. The functors $\{\gamma^*_\alpha \mid \alpha \in HomC_X\}$ define a functor

$$\coprod_{\alpha \in HomC_X} C_{\mathfrak{x}_k} \xrightarrow{\gamma^*} C_X$$

which is (by construction) surjective. Therefore, γ^* is an inverse image functor of an inflation $X \xrightarrow{\gamma} \prod_{\alpha \in HomC_X} \mathfrak{x}_k$.

Since any product of injective objects is an injective object, this proves the assertion. ■

4.4.5. Elementary injective objects. The following k-'spaces' are all retracts of the injective k-'space' $\mathfrak{x}_k$, hence all of them are injective objects.

(i) The k-'space' $\mathfrak{x}^1_k$ represented by the category with one object, x, whose endomorphism ring is $k \cdot id_x$. There are two (obvious) inflations from $\mathfrak{x}^1_k$ to $\mathfrak{x}_k$.

(ii) For each idempotent e of the ring k, there is an injective 'space' $\mathfrak{x}^{1,e}_k$ represented by category with one object whose endomorphism ring is $k \cdot e$.

(iii) We have also an injective k-space $\mathfrak{x}^e_k$ represented by the category with two objects, a and b, defined by

$$C_{\mathfrak{x}^e_k}(a,b) = k \cdot e, \quad C_{\mathfrak{x}^e_k}(b,a) = 0, \quad C_{\mathfrak{x}^e_k}(a,a) = k, \quad \text{and} \quad C_{\mathfrak{x}^e_k}(b,b) = k.$$

(iv) Finally, we obtain more elementary injective objects by taking in the last example the algebra of endomorphisms of a (resp. b) equal to $k \cdot e_a$ (resp. $k \cdot e_b$), where e_a and e_b are idempotents of the ring k such that $e_a \cdot e = e = e_b \cdot e$.

4.4.5.1. Note. If the ring k does not have non-trivial idempotents, then each of the four examples either produces $\mathfrak{r}_k^1$, or reproduces $\mathfrak{r}_k$.

4.4.6. Left exact structures formed by localizations of k-'spaces'. Each of the left exact structures $\mathfrak{L}$, $\mathfrak{L}_\ell$, $\mathfrak{L}_\mathfrak{r}$, $\mathfrak{L}_\mathfrak{e}$, $\mathfrak{L}^\mathfrak{c}$, and $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}$ on the category $|Cat|^o$ of 'spaces' (see 2) induces a left exact structure on the category $|Cat_k|^o$ of k-spaces. Thus, we have left exact structures $\mathfrak{L}(k)$, $\mathfrak{L}_\ell(k)$, $\mathfrak{L}_\mathfrak{r}(k)$, $\mathfrak{L}_\mathfrak{e}(k)$, $\mathfrak{L}^\mathfrak{c}(k)$, and $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}(k)$ on $|Cat_k|^o$.

4.5. Additivization. Let Cat_k denote the category whose objects are svelte k-linear categories and morphisms k-linear functors. We denote by Add_k the full subcategory of Cat_k generated by additive k-linear categories. Let $\widetilde{\mathfrak{J}}_*$ denote the inclusion functor $Add_k \longrightarrow Cat_k$. The functor $\widetilde{\mathfrak{J}}_*$ is right adjoint to the *additivization* functor $\widetilde{\mathfrak{J}}^*$ which maps every k-linear category C_X to the smallest full additive subcategory $C_{X_\mathfrak{a}}$ of the category $\mathcal{M}_k(X)$ of presheaves of k-modules on C_X containing all representable presheaves.

Let $|Add_k|^o$ denote the full subcategory of the category $|Cat_k|^o$ of k-'spaces' generated by the 'spaces' represented by *additive* k-linear categories. The pair of adjoint functors

$$Add_k \xrightarrow{\widetilde{\mathfrak{J}}_*} Cat_k \xrightarrow{\widetilde{\mathfrak{J}}^*} Add_k$$

induces a pair of adjoint functors

$$|Add_k|^o \xrightarrow{\mathfrak{J}^*} |Cat_k|^o \xrightarrow{\mathfrak{J}_*} |Add_k|^o.$$

Only here the inclusion functor $|Add_k|^o \xrightarrow{\mathfrak{J}^*} |Cat_k|^o$ is left adjoint to the additivization functor. The functor $Add_k \xrightarrow{\widetilde{\mathfrak{J}}_*} Cat_k$ preserves small limits and colimits; hence the functor $|Add_k|^o \xrightarrow{\mathfrak{J}^*} |Cat_k|^o$ does the same.

The left exact structure $\mathfrak{I}_k^\mathfrak{s}$ induces a left exact structure $\mathfrak{I}_k^+$ on $|Add_k|^o$. Since (by definition of $\mathfrak{I}_k^+$) the functor $\mathfrak{J}^*$ maps inflations to inflations, its right adjoint $\mathfrak{J}_*$ maps injective objects to injective objects. The argument similar of that of 2.5.3(a) and (b) deduces from the fact that $(|Cat_k|^o, \mathfrak{I}_k^\mathfrak{s})$ has enough injective objects (4.4.4) that the left exact category $(|Add_k|^o, \mathfrak{I}_k^+)$ has enough injective objects.

4.6. Karoubianization. Let $|KCat_k|^o$ denote the full subcategory of $|Cat_k|^o$ generated by 'spaces' represented by Karoubian svelte k-linear categories. We call the objects of the subcategory $|KCat|^o$ *Karoubian k-'spaces'*.

4.6.1. Proposition. *The inclusion functor* $|KCat_k|^o \xrightarrow{\mathfrak{K}_k^*} |Cat_k|^o$ *has a canonical right adjoint,* $|Cat|^o \xrightarrow{\mathfrak{K}_{k*}} |KCat_k|^o$.

Proof. The argument is similar to the proof of 2.5.1. ∎

4.6.2. The canonical left exact structure on $|KCat_k|^o$. The left exact structure $\mathfrak{I}_k^{\mathfrak{s}}$ on the category of 'spaces' $|Cat|^o$ induces a left exact structure on $|KCat|^o$, which we denote by $\mathfrak{I}_k^{\mathfrak{Ka}}$.

4.6.3. Proposition. *The left exact category $(|KCat|^o, \mathfrak{I}_k^{\mathfrak{Ka}})$ has enough injective objects.*

Proof. The argument is the same as in 2.5.3. ∎

5. Left exact structures on the category of right (or left) exact 'spaces'.

A *right exact 'space'* is a pair $(X, \mathfrak{E}_X)$, where X is a 'space' and $\mathfrak{E}_X$ is a right exact structure on the category C_X. We denote by $\mathfrak{Esp}_\mathfrak{r}$ the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ and morphisms from $(X, \mathfrak{E}_X)$ to $(Y, \mathfrak{E}_Y)$ are given by morphisms $X \xrightarrow{f} Y$ of 'spaces' whose inverse image functor, f^*, is *'exact'*; i.e. f^* maps deflations to deflations and preserves pull-backs of deflations.

Dually, a *left exact 'space'* is a pair $(Y, \mathfrak{I}_Y)$, where $(C_Y, \mathfrak{I}_Y)$ is a left exact category. We denote by $\mathfrak{Esp}_\ell$ the category whose objects are left exact 'spaces' $(Y, \mathfrak{I}_Y)$ and morphisms $(Y, \mathfrak{I}_Y) \longrightarrow (Z, \mathfrak{I}_Z)$ are given by morphisms $Y \longrightarrow Z$ whose inverse image functors are *'coexact'*, which means that they preserve arbitrary push-forwards of inflations.

5.1. Note. The categories $\mathfrak{Esp}_\mathfrak{r}$ and $\mathfrak{Esp}_\ell$ are naturally isomorphic to each other: the isomorphism is given by the dualization functor $(X, \mathfrak{E}_X) \longmapsto (X^o, \mathfrak{E}_X^{op})$. Therefore every assertion about the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces' translates into an assertion about the category $\mathfrak{Esp}_\ell$ of left exact 'spaces' and vice versa.

5.2. Proposition. *The category $\mathfrak{Esp}_\mathfrak{r}$ has fibred coproducts.*

Proof. Let $(X, \mathfrak{E}_X) \xleftarrow{f} (Z, \mathfrak{E}_Z) \xrightarrow{g} (Y, \mathfrak{E}_Y)$ be morphisms of $\mathfrak{Esp}_\mathfrak{r}$; and let $\mathfrak{X}$ denote the 'space' $X \coprod_{f,g} Y$, i.e. $C_\mathfrak{X} = C_X \prod_{f^*,g^*} C_Y$. Let $\mathfrak{E}_\mathfrak{X}$ denote the class of all morphisms $(L, M; \phi) \xrightarrow{(\xi,\gamma)} (L', M'; \phi')$ of $C_\mathfrak{X}$ such that $L \xrightarrow{\xi} L'$ belongs to $\mathfrak{E}_X$ and $M \xrightarrow{\gamma} M'$ is an arrow of $\mathfrak{E}_Y$. The claim is that $\mathfrak{E}_\mathfrak{X}$ is a right exact structure on $C_\mathfrak{X}$ and $(\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ is a coproduct $(X, \mathfrak{E}_X) \coprod_{f,g} (Y, \mathfrak{E}_Y)$ of right exact 'spaces'.

It is immediate that $\mathfrak{E}_\mathfrak{X}$ contains all isomorphisms and is closed under composition. Let $(L, M; \phi) \xrightarrow{(\xi,\gamma)} (L', M'; \phi')$ be a morphism of $\mathfrak{E}_\mathfrak{X}$, and let $(L'', M''; \phi'') \xrightarrow{(\alpha,\beta)} (L', M'; \phi')$ be an arbitrary morphism of $C_\mathfrak{X}$. Since the inverse image functors f^* and g^* preserve corresponding deflations and their pull-backs and ξ, γ are deflations, the isomorphisms ϕ, ϕ', and ϕ'' induce an isomorphism $f^*(\widetilde{L}) \xrightarrow{\widetilde{\phi}} g^*(\widetilde{M})$, where $\widetilde{L} = L \coprod_{\xi,\alpha} L''$ and $\widetilde{M} = M \coprod_{\gamma,\beta} M''$. It is easy to see that the square

$$\begin{array}{ccc} (\widetilde{L}, \widetilde{M}; \widetilde{\phi}) & \xrightarrow{(\alpha',\beta')} & (L, M; \phi) \\ {\scriptstyle (\widetilde{\xi},\widetilde{\gamma})}\downarrow & & \downarrow{\scriptstyle (\xi,\gamma)} \\ (L'', M''; \phi'') & \xrightarrow{(\alpha,\beta)} & (L', M'; \phi') \end{array} \qquad (1)$$

is cartesian, $\widetilde{\xi} \in \mathfrak{E}_X$, and $\widetilde{\gamma} \in \mathfrak{E}_Y$. Therefore, $(\widetilde{\xi}, \widetilde{\gamma}) \in \mathfrak{E}_{\mathfrak{X}}$.

If $(\alpha, \beta) = (\xi, \gamma)$, then the square (1) is cocartesian, because the squares

$$\begin{array}{ccc} \widetilde{L} & \xrightarrow{\xi'} & L \\ {\scriptstyle \widetilde{\xi}}\downarrow & & \downarrow{\scriptstyle \xi} \\ L & \xrightarrow{\xi} & L' \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{M} & \xrightarrow{\gamma'} & M \\ {\scriptstyle \widetilde{\gamma}}\downarrow & & \downarrow{\scriptstyle \gamma} \\ M & \xrightarrow{\gamma} & M' \end{array}$$

are (both cartesian and) cocartesian. Altogether shows that the arrows of $\mathfrak{E}_{\mathfrak{X}}$ are covers of a subcanonical pretopology; i.e. $\mathfrak{E}_{\mathfrak{X}}$ is a structure of a right exact category on $C_{\mathfrak{X}}$. ■

5.3. The left exact structure $\mathfrak{J}^{\mathfrak{es}}$ on the category of right exact 'spaces'. We denote by $\mathfrak{J}^{\mathfrak{es}}$ the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (Y, \mathfrak{E}_Y)$ of right exact 'spaces' such that each arrow of C_X is isomorphic to the image of some arrow of C_Y (that is $X \xrightarrow{\mathfrak{f}} Y$ belongs to $\mathfrak{J}^{\mathfrak{s}}$) and each arrow of $\mathfrak{E}_X$ is isomorphic to an arrow $\mathfrak{f}^*(\mathfrak{e})$ for some $\mathfrak{e} \in \mathfrak{E}_Y$.

The latter condition implies that $\mathfrak{E}_X$ is the smallest right exact structure on C_X containing $\mathfrak{f}^*(\mathfrak{E}_Y)$.

5.3.1. Proposition. *The class $\mathfrak{J}^{\mathfrak{es}}$ is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces'.*

Proof. The class $\mathfrak{J}^{\mathfrak{es}}$ contains, obviously, all isomorphisms, and it is easy to see that it is closed under composition. It remains to show that $\mathfrak{J}^{\mathfrak{es}}$ is stable under cobase change and its arrows are cocovers of a subcanonical copretopology.

Let $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{J}^{\mathfrak{es}}$ and $(X, \mathfrak{E}_X) \xrightarrow{f} (Z, \mathfrak{E}_Z)$ an arbitrary morphism. The claim is that the canonical morphism $Z \xrightarrow{\widetilde{q}} Z \coprod_{f,q} Y$ belongs to $\mathfrak{J}^{\mathfrak{es}}$.

Consider the corresponding cartesian (in pseudo-categorical sense) square of right exact categories:

$$\begin{array}{ccc} (C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}) & \xrightarrow{p^*} & (C_Y, \mathfrak{E}_Y) \\ {\scriptstyle \widetilde{q}^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ (C_Z, \mathfrak{E}_Z) & \xrightarrow{f^*} & (C_X, \mathfrak{E}_X) \end{array} \qquad (2)$$

where $\mathfrak{X} = Z \coprod_{f,q} Y$; that is $C_{\mathfrak{X}} = C_Z \prod_{f^*,q^*} C_Y$. Recall that the functor $\widetilde{q}^*$ maps each object $(L, M; \phi)$ of the category $C_{\mathfrak{X}}$ to the object L of C_Z and each morphism (ξ, γ) to ξ.

Let $L \xrightarrow{\mathfrak{e}} L'$ be an arrow of $\mathfrak{E}_Z$. Then $f^*(\mathfrak{e})$ is a morphism of $\mathfrak{E}_X$. Since $X \xrightarrow{q} Y$ is a morphism of $\mathfrak{I}^{\mathfrak{es}}$, there exists a morphism $M \xrightarrow{\mathfrak{t}} M'$ of $\mathfrak{E}_Y$ and a commutative diagram

$$\begin{array}{ccc} f^*(L) & \xrightarrow{f^*(\mathfrak{e})} & f^*(L') \\ \psi \downarrow \wr & & \wr \downarrow \psi' \\ q^*(M) & \xrightarrow{q^*(\mathfrak{t})} & q^*(M') \end{array}$$

whose vertical arrows are isomorphisms. By the definition of the right exact category $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$, this means that $(\mathfrak{e}, \mathfrak{t})$ is a morphism $(L, M; \psi) \longrightarrow (L', M'; \psi')$ of $C_{\mathfrak{X}}$ which belongs to $\mathfrak{E}_{\mathfrak{X}}$. The localization functor $\widetilde{q}^*$ maps it to $\mathfrak{e}$. Thus, $\mathfrak{E}_Z = \widetilde{q}^*(\mathfrak{E}_{\mathfrak{X}})$, hence $\widetilde{q} \in \mathfrak{E}_{\mathfrak{X}}$. This shows that $\mathfrak{I}^{\mathfrak{es}}$ is stable under cobase change.

It remains to verify that for every morphism $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ of $\mathfrak{I}^{\mathfrak{es}}$ the square

$$\begin{array}{ccc} (C_{\mathfrak{Y}}, \mathfrak{E}_{\mathfrak{Y}}) & \xrightarrow{p_1^*} & (C_Y, \mathfrak{E}_Y) \\ p_2^* \downarrow & & \downarrow q^* \\ (C_Y, \mathfrak{E}_Y) & \xrightarrow{q^*} & (C_X, \mathfrak{E}_X) \end{array} \qquad (3)$$

is cocartesian. Here $C_{\mathfrak{Y}} = C_Y \prod_{q^*,q^*} C_Y$.

Consider a quasi-commutative diagram

$$\begin{array}{ccc} (C_{\mathfrak{Y}}, \mathfrak{E}_{\mathfrak{Y}}) & \xrightarrow{p_1^*} & (C_Y, \mathfrak{E}_Y) \\ p_2^* \downarrow & & \downarrow v^* \\ (C_Y, \mathfrak{E}_Y) & \xrightarrow{u^*} & (C_W, \mathfrak{E}_W) \end{array} \qquad (4)$$

of 'exact' functors. Since the square

$$\begin{array}{ccc} C_{\mathfrak{Y}} & \xrightarrow{p_1^*} & C_Y \\ p_2^* \downarrow & & \downarrow q^* \\ C_Y & \xrightarrow{q^*} & C_X \end{array}$$

is cocartesian, there exists a unique up to isomorphism functor $C_X \xrightarrow{w^*} C_W$ such that $v^* \simeq w^* q^* \simeq u^*$. The claim is that w^* is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_W, \mathfrak{E}_W)$.

Since $q \in \mathfrak{I}^{\mathfrak{es}}$, every morphism of $\mathfrak{E}_X$ is isomorphic to a morphism of $q^*(\mathfrak{E}_Y)$ and v^* maps $\mathfrak{E}_Y$ to $\mathfrak{E}_W$. Therefore w^* maps $\mathfrak{E}_X$ to $\mathfrak{E}_W$. The fact that q^* and $v^* \simeq w^*q^*$ are 'exact' functors implies that the functor w^* is 'exact'. ∎

5.3.2. Proposition. *The left exact category* $(\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{I}^{\mathfrak{es}})$ *has enough injective objects.*

Proof. (a) The canonical embedding of $|Cat|^o$ into $\mathfrak{Esp}_{\mathfrak{r}}$ which provides each 'space' X with the coarsest right exact structure is an exact fully faithful functor

$$(|Cat|^o, \mathfrak{I}^{\mathfrak{s}}) \xrightarrow{\mathfrak{J}_*} (\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{I}^{\mathfrak{es}})$$

which is right adjoint to the functor

$$(\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{I}^{\mathfrak{es}}) \xrightarrow{\mathfrak{J}^*} (|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$$

forgetting right exact structures. Since the functor $\mathfrak{J}^*$ maps inflations to inflations, by (the dual version of) III.1.3.1, its right adjoint $\mathfrak{J}_*$ maps injective objects to injective objects.

(b) *Another elementary injective.* We denote by $(\mathfrak{S}, \mathfrak{E}_{\mathfrak{S}})$ the right exact 'space' defined as follows. Objects of the category $C_{\mathfrak{S}}$ are $\mathfrak{v}_n$, $n \geq 0$, where $\mathfrak{v}_0$ a final object; morphisms are generated by identical morphisms and arrows

$$\mathfrak{v}_{n+1} \xrightarrow{\mathfrak{s}_n^i} \mathfrak{v}_n, \quad 0 \leq i \leq n,\ n \geq 0, \quad \text{and} \quad \mathfrak{v}_n \xrightarrow{\delta_n^i} \mathfrak{v}_{n+1}, \quad 0 \leq i \leq n-1,\ n \geq 1,$$

subject to the relations:

$$\begin{aligned}
&\mathfrak{s}_n^j \circ \mathfrak{s}_{n+1}^i = \mathfrak{s}_n^i \circ \mathfrak{s}_{n+1}^{j+1}, \quad \text{if} \quad i \leq j, \\
&\delta_{n+1}^j \circ \delta_n^i = \delta_{n+1}^i \circ \delta_n^{j-1}, \quad \text{if} \quad i < j \\
&\mathfrak{s}_n^j \circ \delta_n^i = \begin{cases} \delta_n^i \circ \mathfrak{s}_{n-1}^{j-1} & \text{if} \quad i < j, \\ id_{\mathfrak{v}_n} & \text{if} \quad j = i \text{ or } j = i+1, \\ \delta_n^{i-1} \circ \mathfrak{s}_{n-1}^j, & \text{if} \quad i > j+1. \end{cases} \quad \Bigg| \quad n \geq 1
\end{aligned} \tag{5}$$

The class of deflations $\mathfrak{E}_{\mathfrak{S}}$ is generated by identical morphisms and morphisms $\mathfrak{s}_n^i$.

The claim is that the right exact 'space' $(\mathfrak{S}, \mathfrak{E}_{\mathfrak{S}})$ is an injective object of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{I}^{\mathfrak{es}})$.

In fact, let $(\mathfrak{S}, \mathfrak{E}_{\mathfrak{S}}) \xrightarrow{\mathfrak{f}} (X, \mathfrak{E}_X)$ be an inflation (that is a morphism from $\mathfrak{I}^{\mathfrak{es}}$. Then its inverse image functor $C_X \xrightarrow{\mathfrak{f}^*} C_{\mathfrak{S}}$ is 'exact' and (since all isomorphisms of the category $C_{\mathfrak{S}}$ are identical) it maps $\mathfrak{E}_X$ surjectively onto the set $\mathfrak{E}_{\mathfrak{S}}$ of deflations of the 'space' $(\mathfrak{S}, \mathfrak{E}_{\mathfrak{S}})$. In particular, $\mathfrak{s}_0 = \mathfrak{f}^*(\mathfrak{e})$ for some deflation $M \xrightarrow{\mathfrak{e}} L$ of $(X, \mathfrak{E}_X)$. There is an 'exact' functor

$$(C_{\mathfrak{S}}, \mathfrak{E}_{\mathfrak{S}}) \xrightarrow{\gamma^*} (C_X, \mathfrak{E}_X)$$

which is determined uniquely up to isomorphism by the equality $\gamma^*(\mathfrak{s}_0) = \mathfrak{e}$.

Indeed, the 'exactness' of γ^* implies that the kernel pair $(\mathfrak{s}_0^0, \mathfrak{s}_0^1)$ of the deflation $\mathfrak{s}_0$ should go to the kernel pair of the deflation $\mathfrak{e}$. The morphism $[0] \xrightarrow{d_0^0} [1]$ is mapped to the diagonal morphism $M \longrightarrow M \times_L M$; etc.. We use here the fact that all deflations and their splittings obtained from one deflation via kernel pairs and pull-backs along already obtained deflations, can be organized into the data like (5).

(c) Every right exact 'space' has an inflation to a product of a set of copies of the injective object described in (b) above and an injective object coming from $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$ (products of sets of copies of the object $\mathfrak{x}_1$). One can construct such covers economically by starting covering with deflations: picking a deflation $\mathfrak{e} \in \mathfrak{E}_X$, mapping $\mathfrak{s}_0^0$ to $\mathfrak{e}$ which induces an 'exact' functor $(C_{\mathfrak{S}}, \mathfrak{E}_{\mathfrak{S}})$ to $(C_X, \mathfrak{E}_X)$ covering a whole tower of deflations and their splittings. Then pick another deflation, etc.. This way, one covers eventually (using a transfinite induction) essentially all deflations. What remains is ordinary arrows (that is *non-deflations*), which are covered by a coproduct of copies of $C_{\mathfrak{x}_1}$. Altogether is an inverse image functor of an inflation to an injective object of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{I}^{\mathfrak{es}})$. ∎

5.4. The left exact category of right exact Karoubian 'spaces'. Let $\mathfrak{KEsp}_{\mathfrak{r}}$ denote the full subcategory of $\mathfrak{Esp}_{\mathfrak{r}}$ generated by right exact 'spaces' represented by right exact Karoubian svelte categories (see I.3.4.1). We call the objects of the subcategory $\mathfrak{KEsp}_{\mathfrak{r}}$ *Karoubian 'spaces'.*

5.4.1. Proposition. *The inclusion functor* $\mathfrak{KEsp}_{\mathfrak{r}} \xrightarrow{\mathfrak{K}^*} \mathfrak{Esp}_{\mathfrak{r}}$ *has a canonical right adjoint,* $\mathfrak{Esp}_{\mathfrak{r}} \xrightarrow{\mathfrak{K}_*} \mathfrak{KEsp}_{\mathfrak{r}}$.

Proof. The functor $\mathfrak{Esp}_{\mathfrak{r}} \xrightarrow{\mathfrak{K}_*} \mathfrak{KEsp}_{\mathfrak{r}}$. assigns to each right exact 'space' $(X, \mathfrak{E}_X)$ the 'space' $(X_K, \mathfrak{E}_{X_K})$ represented by the Karoubian envelope, $(C_{X_K}, \mathfrak{E}_{X_K}))$ of the right exact category $(C_X, \mathfrak{E}_X)$ (see I.3.4.1 and I.3.4.2). The canonical full embedding

$$(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{k}_X^*} (C_{X_K}, \mathfrak{E}_{X_K})$$

of the right exact category C_X into its Karoubian envelope is the inverse image functor of (the value at $(X, \mathfrak{E}_X)$ of) an adjunction morphism

$$\mathfrak{K}^*\mathfrak{K}_*(X) = (X_K, \mathfrak{E}_{X_K}) \xrightarrow{\mathfrak{k}_X^*} (X, \mathfrak{E}_X).$$

The adjunction morphism $Id_{\mathfrak{KEsp}_{\mathfrak{r}}} \longrightarrow \mathfrak{K}_*\mathfrak{K}^*$ is a natural isomorphism. ∎

5.4.2. The canonical left exact structure on the category $\mathfrak{KEsp}_\mathfrak{r}$. The left exact structure $\mathfrak{J}^{\mathfrak{es}}$ on the category of 'spaces' $\mathfrak{Esp}_\mathfrak{r}$ induces a left exact structure on $\mathfrak{KEsp}_\mathfrak{r}$, which we denote by $\mathfrak{J}^{\mathfrak{Ke}}$.

5.4.3. Proposition. *The left exact category* $(\mathfrak{KEsp}_\mathfrak{r}, \mathfrak{J}^{\mathfrak{Ke}})$ *has enough injective objects.*

Proof. The argument follows that of 2.5.1. Still, because of the importance of this fact and in order to fix the notations, we outline it below.

(i) The inclusion functor $\mathfrak{KEsp}_\mathfrak{r} \xrightarrow{\mathfrak{K}^*} \mathfrak{Esp}_\mathfrak{r}$ preserves small colimits and (by definition of the left exact structure $\mathfrak{J}^{\mathfrak{Ke}}$) maps $\mathfrak{J}^{\mathfrak{Ke}}$ to $\mathfrak{J}^{\mathfrak{es}}$. In particular, $\mathfrak{K}^*$ is an 'exact' functor from the left exact category $(\mathfrak{KEsp}_\mathfrak{r}, \mathfrak{J}^{\mathfrak{Ke}})$ to the left exact category $(\mathfrak{Esp}_\mathfrak{r}, \mathfrak{J}^{\mathfrak{es}})$.

(ii) The fact that $\mathfrak{K}^*$ maps inflations to inflations implies that its right adjoint, $\mathfrak{K}_*$ maps injective objects to injective objects. The claim is that there are enough injective objects obtained this way; that is, for any Karoubian right exact 'space' $(X, \mathfrak{E}_X)$, there exists an inflation into $\mathfrak{K}_*(Y, \mathfrak{E}_Y)$ for some injective object $(Y, \mathfrak{E}_Y)$ of the left exact category $(\mathfrak{Esp}_\mathfrak{r}, \mathfrak{J}^{\mathfrak{es}})$.

(iii) Notice that if $(X, \mathfrak{E}_X)$ is a Karoubian right exact 'space' and a morphism

$$\mathfrak{K}^*(X, \mathfrak{E}_X) \xrightarrow{\ \mathfrak{f}\ } (Y, \mathfrak{E}_Y)$$

is an inflation (that is it belongs to $\mathfrak{J}^{\mathfrak{es}}$), then the adjoint morphism

$$(X, \mathfrak{E}_X) \xrightarrow{\ \widehat{\mathfrak{f}}\ } \mathfrak{K}_*(Y, \mathfrak{E}_Y)$$

(which is isomorphic to $\mathfrak{K}_*(\mathfrak{f})$) is an inflation.

In fact, $\widehat{\mathfrak{f}}$ is a morphism from the right exact 'space' $(X, \mathfrak{E}_X)$ to the Karoubian envelope $(Y_K, \mathfrak{E}_Y)$ of the right exact 'space' $(Y, \mathfrak{E}_Y)$; and the inflation $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (Y, \mathfrak{E}_Y)$ is the composition of $(X, \mathfrak{E}_X) \xrightarrow{\widehat{\mathfrak{f}}} (Y_K, \mathfrak{E}_{Y_K})$ and the canonical morphism $(Y_K, \mathfrak{E}_{Y_K}) \xrightarrow{\mathfrak{k}_Y} (Y, \mathfrak{E}_Y)$ (corresponding to the full embedding C_Y into its Karoubian envelope C_{Y_K}). Therefore, by 2.4.3, the morphism $(X, \mathfrak{E}_X) \xrightarrow{\widehat{\mathfrak{f}}} (Y_K, \mathfrak{E}_{Y_K})$ belongs to $\mathfrak{J}^{\mathfrak{es}}$. Both $(X, \mathfrak{E}_X)$ and $(Y_K, \mathfrak{E}_{Y_K})$ are Karoubian 'spaces'; so that $\widehat{\mathfrak{f}} \in \mathfrak{J}^{\mathfrak{Ke}}$.

(iv) Since the left exact category $(\mathfrak{Esp}_\mathfrak{r}, \mathfrak{J}^{\mathfrak{es}})$ has enough injective objects, for every right exact Karoubian 'space' $(X, \mathfrak{E}_X)$, we have an inflation (actually, a canonical inflation) $\mathfrak{K}^*(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ to an injective object $(\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ of the left exact category $(\mathfrak{Esp}_\mathfrak{r}, \mathfrak{J}^{\mathfrak{es}})$. By (iii) above, the adjoint morphism $(X, \mathfrak{E}_X) \xrightarrow{\widehat{\mathfrak{f}}} \mathfrak{K}_*(\mathfrak{X}, \mathfrak{E}_\mathfrak{X}) = (\mathfrak{X}_K, \mathfrak{E}_{\mathfrak{X}_K})$ is an inflation; and by (ii), $\mathfrak{K}_*(\mathfrak{X}, \mathfrak{E}_\mathfrak{X}) = (\mathfrak{X}_K, \mathfrak{E}_{\mathfrak{X}_K})$ is an injective object of the left exact category $(\mathfrak{KEsp}_\mathfrak{r}, \mathfrak{J}^{\mathfrak{Ke}})$ of Karoubian right exact 'spaces'. ■

5.5. Some other left exact structures on the category of right exact 'spaces'. Let $\mathfrak{L}_{\mathfrak{es}}$ be the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ of right exact 'spaces' such that q^* is a localization functor and each arrow of $\mathfrak{E}_X$ is isomorphic to $q^*(\mathfrak{e})$ for some $\mathfrak{e} \in \mathfrak{E}_Y$.

If Σ_{q^*} is a left or right multiplicative system, then the morphism q belongs to the class $\mathfrak{I}^{\mathfrak{es}}$; so that, in this case, $\mathfrak{E}_X$ is the smallest right exact structure containing $q^*(\mathfrak{E}_Y)$.

5.5.1. Proposition. *The class $\mathfrak{L}_{\mathfrak{es}}$ is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces'.*

Proof. The class $\mathfrak{L}_{\mathfrak{es}}$ contains, obviously, all isomorphisms, and it is easy to see that $\mathfrak{L}_{\mathfrak{es}}$ is closed under composition. It remains to show that the class $\mathfrak{L}_{\mathfrak{es}}$ is stable under cobase change and its arrows are cocovers of a subcanonical copretopology.

Let $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{L}_{\mathfrak{es}}$ and $(X, \mathfrak{E}_X) \xrightarrow{f} (Z, \mathfrak{E}_Z)$ an arbitrary morphism. The claim is that the canonical morphism $Z \xrightarrow{\widetilde{q}} Z \coprod_{f,q} Y$ belongs to $\mathfrak{L}_{\mathfrak{es}}$.

Consider the corresponding cartesian (in pseudo-categorical sense) square of right exact categories:

$$\begin{array}{ccc} (C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}) & \xrightarrow{p^*} & (C_Y, \mathfrak{E}_Y) \\ \widetilde{q}^* \downarrow & & \downarrow q^* \\ (C_Z, \mathfrak{E}_Z) & \xrightarrow{f^*} & (C_X, \mathfrak{E}_X) \end{array} \qquad (2)$$

where $\mathfrak{X} = Z \coprod_{f,q} Y$; that is $C_{\mathfrak{X}} = C_Z \prod_{f^*,q^*} C_Y$. Recall that the functor $\widetilde{q}^*$ maps each object $(L, M; \phi)$ of the category $C_{\mathfrak{X}}$ to the object L of C_Z and each morphism (ξ, γ) to ξ. By 1.5.1(a), $\widetilde{q}^*$ is a localization functor (because q^* is a localization functor).

Let $L \xrightarrow{\mathfrak{e}} L'$ be an arrow of $\mathfrak{E}_Z$. Then $f^*(\mathfrak{e})$ is a morphism of $\mathfrak{E}_X$. Since $X \xrightarrow{q} Y$ is a morphism of $\mathfrak{L}_{\mathfrak{es}}$, there exists a morphism $M \xrightarrow{\mathfrak{t}} M'$ of $\mathfrak{E}_Y$ and a commutative diagram

$$\begin{array}{ccc} f^*(L) & \xrightarrow{f^*(\mathfrak{e})} & f^*(L') \\ \psi \downarrow \wr & & \wr \downarrow \psi' \\ q^*(M) & \xrightarrow{q^*(\mathfrak{t})} & q^*(M') \end{array}$$

whose vertical arrows are isomorphisms. By the definition of the right exact category $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$, this means that $(\mathfrak{e}, \mathfrak{t})$ is a morphism $(L, M; \psi) \longrightarrow (L', M'; \psi')$ of $C_{\mathfrak{X}}$ which belongs to $\mathfrak{E}_{\mathfrak{X}}$. The localization functor $\widetilde{q}^*$ maps it to $\mathfrak{e}$. Thus, $\mathfrak{E}_Z = \widetilde{q}^*(\mathfrak{E}_{\mathfrak{X}})$, hence $\widetilde{q} \in \mathfrak{E}_{\mathfrak{X}}$. This shows that $\mathfrak{L}_{\mathfrak{es}}$ is stable under cobase change.

It remains to verify that for every morphism $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ of $\mathfrak{L}_{\mathfrak{es}}$ the square

$$\begin{array}{ccc} (C_{\mathfrak{Y}}, \mathfrak{E}_{\mathfrak{Y}}) & \xrightarrow{p_1^*} & (C_Y, \mathfrak{E}_Y) \\ {\scriptstyle p_2^*} \downarrow & & \downarrow {\scriptstyle q^*} \\ (C_Y, \mathfrak{E}_Y) & \xrightarrow{q^*} & (C_X, \mathfrak{E}_X) \end{array} \qquad (3)$$

is cocartesian. Here $C_{\mathfrak{Y}} = C_Y \prod\limits_{q^*, q^*} C_Y$.

Consider a quasi-commutative diagram

$$\begin{array}{ccc} (C_{\mathfrak{Y}}, \mathfrak{E}_{\mathfrak{Y}}) & \xrightarrow{p_1^*} & (C_Y, \mathfrak{E}_Y) \\ {\scriptstyle p_2^*} \downarrow & & \downarrow {\scriptstyle v^*} \\ (C_Y, \mathfrak{E}_Y) & \xrightarrow{u^*} & (C_W, \mathfrak{E}_W) \end{array} \qquad (4)$$

of 'exact' functors. Since, by 1.5.3, that the square

$$\begin{array}{ccc} C_{\mathfrak{Y}} & \xrightarrow{p_1^*} & C_Y \\ {\scriptstyle p_2^*} \downarrow & & \downarrow {\scriptstyle q^*} \\ C_Y & \xrightarrow{q^*} & C_X \end{array}$$

is cocartesian, there exists a unique up to isomorphism functor $C_X \xrightarrow{w^*} C_W$ such that $v^* \simeq w^* q^* \simeq u^*$. The claim is that w^* is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_W, \mathfrak{E}_W)$. Since $q \in \mathfrak{L}_{\mathfrak{es}}$, every morphism of $\mathfrak{E}_X$ is isomorphic to a morphism of $q^*(\mathfrak{E}_Y)$ and v^* maps $\mathfrak{E}_Y$ to $\mathfrak{E}_W$. Therefore w^* maps $\mathfrak{E}_X$ to $\mathfrak{E}_W$. The fact that q^* and $v^* \simeq w^* q^*$ are 'exact' functors implies that the functor w^* is 'exact'. ∎

5.5.2. Corollary. *Each of the classes of morphisms of 'spaces'* $\mathfrak{L}_\ell$, $\mathfrak{L}_{\mathfrak{r}}$, $\mathfrak{L}_{\mathfrak{e}}$, $\mathfrak{L}^{\mathfrak{c}}$, *and* $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{e}}$ *(cf. 3.1) induces a structure of a left exact category on the category* $\mathfrak{Esp}_{\mathfrak{r}}$ *of right exact 'spaces'.*

Proof. The class $\mathfrak{L}_\ell$ induces the class $\mathfrak{L}^{\mathfrak{es}}_\ell$ of morphisms of the category $\mathfrak{Esp}_{\mathfrak{r}}$ formed by all arrows $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ from $\mathfrak{L}_{\mathfrak{es}}$ such that the morphism of 'spaces' $X \xrightarrow{q} Y$ belongs to $\mathfrak{L}_\ell$. Similarly, we define the classes $\mathfrak{L}^{\mathfrak{es}}_\ell$, $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{r}}$, $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{es}}$, and $\mathfrak{L}^{\mathfrak{e},\mathfrak{c}}_{\mathfrak{es}}$. ∎

5.5.3. The left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$. For a right exact 'space' $(X, \mathfrak{E}_X)$, let $Sq(X, \mathfrak{E}_X)$ denote the class of all cartesian squares in the category C_X with at least two parallel arrows from $\mathfrak{E}_X$.

The class $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ consists of all morphisms $(X,\mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y,\mathfrak{E}_Y)$ of right exact 'spaces' such that its inverse image functor, $\mathfrak{q}^*$, is equivalent to a localization functor and each square of $Sq(X,\mathfrak{E}_X)$ is isomorphic to some square of $\mathfrak{q}^*(Sq(Y,\mathfrak{E}_Y))$.

5.5.4. Proposition. *The class $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' which is coarser than $\mathfrak{L}_{\mathfrak{es}}$ and finer than $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{r}}$.*

Proof. The argument is left to the reader. ∎

5.6. Right exact 'spaces' over a 'space'. The category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' has initial objects and no final object. Final objects appear if we fix a right exact 'space' $\mathcal{S} = (S,\mathfrak{E}_{\mathcal{S}})$ and consider the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{S}$ instead of $\mathfrak{Esp}_{\mathfrak{r}}$. The category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{S}$ has a natural final object and cokernels of all morphisms. It also inherits left exact structures from $\mathfrak{Esp}_{\mathfrak{r}}$, in particular those defined above (see 5.5.2). Therefore, our theory of derived functors (satellites) can be applied to functors from $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{S}$.

6. Left exact category of right exact k-'spaces'.

For a commutative unital ring k, we denote by $\mathfrak{Esp}^{\mathfrak{r}}_k$ the category whose objects are right exact 'spaces' $(X,\mathfrak{E}_X)$ such that C_X is a k-linear additive category and morphisms are morphisms of right exact 'spaces' whose inverse image functors are k-linear. Let $\mathfrak{Esp}^{\mathfrak{r}}_k \xrightarrow{\mathfrak{F}^{\mathfrak{r}}_k} \mathfrak{Esp}_{\mathfrak{r}}$ be the natural functor forgetting k-linear structure. Notice that the functor $\mathfrak{F}^{\mathfrak{r}}_k$ preserves and reflects all small colimits; in particular, it preserves and reflects arbitrary push-forwards.

We denote by $\mathfrak{I}^{\mathfrak{es}}_k$ the preimage of the left exact structure $\mathfrak{I}^{\mathfrak{es}}$; that is $\mathfrak{I}^{\mathfrak{es}}_k$ consists of all morphisms of right exact k-'spaces' $(X,\mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (Y,\mathfrak{E}_Y)$ such that every arrow of the category C_X is isomorphic to an arrow $\mathfrak{f}^*(\xi)$ for some $\xi \in HomC_Y$ and every arrow of $\mathfrak{E}_X$ is isomorphic to the image of an arrow of $\mathfrak{E}_Y$. Since the functor $\mathfrak{F}^{\mathfrak{r}}_k$ preserves and reflects push-forwards, it follows that $\mathfrak{I}^{\mathfrak{es}}_k$ is a left exact structure on the category of right exact k-'spaces' and $\mathfrak{F}^{\mathfrak{r}}_k$ is an 'exact' functor from $(\mathfrak{Esp}^{\mathfrak{r}}_k,\mathfrak{I}^{\mathfrak{es}}_k)$ to $(\mathfrak{Esp}_{\mathfrak{r}},\mathfrak{I}^{\mathfrak{es}})$.

6.1. Proposition. *The left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_k,\mathfrak{I}^{\mathfrak{es}}_k)$ has enough injective objects.*

Proof. The argument that follows is a k-linear version of the proof of 5.3.2.

(a) *Injective objects from $(|Cat_k|^o,\mathfrak{I}^{\mathfrak{s}})$.* The canonical embedding of $|Cat_k|^o$ into $\mathfrak{Esp}^{\mathfrak{r}}_k$ which provides each 'space' X with the coarsest right exact structure is an exact fully faithful functor

$$(|Cat_k|^o,\mathfrak{I}^{\mathfrak{s}}) \xrightarrow{\mathfrak{J}_*} (\mathfrak{Esp}^{\mathfrak{r}}_k,\mathfrak{I}^{\mathfrak{es}}_k)$$

which is right adjoint to the functor

$$(\mathfrak{Esp}^{\mathfrak{r}}_k,\mathfrak{I}^{\mathfrak{es}}_k) \xrightarrow{\mathfrak{J}^*} (|Cat_k|^o,\mathfrak{I}^{\mathfrak{s}})$$

forgetting right exact structures. Since the functor $\mathfrak{J}^*$ maps inflations to inflations, by (the dual version of) III.1.3.1, its right adjoint $\mathfrak{J}_*$ maps injective objects to injective objects.

(b) *The k-linear version of* $(\mathfrak{S}, \mathfrak{E}_{\mathfrak{S}})$. We denote by $(\mathfrak{S}_k, \mathfrak{E}_{\mathfrak{S}_k})$ the smallest right exact k-'space' having the same right exact structure: $\mathfrak{E}_{\mathfrak{S}_k} = \mathfrak{E}_{\mathfrak{S}}$. Explicitly, the category $C_{\mathfrak{S}_k}$ has the same objects, $\{\mathfrak{v}_n \mid n \geq 0\}$, as the category $C_{\mathfrak{S}}$. Morphisms between objects form free k-modules described as follows:

$$\begin{aligned} C_{\mathfrak{S}_k}(\mathfrak{v}_{n+1}, \mathfrak{v}_n) &= \bigoplus_{0 \leq i \leq n} k \cdot \mathfrak{s}_n^i, \quad n \geq 0, \quad \text{and} \\ C_{\mathfrak{S}_k}(\mathfrak{v}_n, \mathfrak{v}_{n+1}) &= \bigoplus_{0 \leq i \leq n-1} k \cdot \delta_n^i, \quad n \geq 1. \end{aligned} \tag{1}$$

The composition rules are imposed by the relations for $\mathfrak{s}_n^i$ and δ_m^j (see 5.3.2 (5)).

An inverse image functor of any inflation $(\mathfrak{S}_k, \mathfrak{E}_{\mathfrak{S}_k}) \xrightarrow{\mathfrak{f}} (X, \mathfrak{E}_X)$ is determined by the map of deflations, which is surjective. Therefore, the splitting constructed in the (part (b) of the) argument of 5.3.2 uniquely extends to a splitting of the functor $\mathfrak{f}^*$.

(c) The construction of an inflation of a right exact k-space $(X, \mathfrak{E}_X)$ into an injective object is an adaptation of the procedure sketched in the part (c) of the argument of 5.3.2.

Let $(X, \mathfrak{E}_X)$ be a right exact 'space'. Replacing (in case of need) $(X, \mathfrak{E}_X)$ by an isomorphic right exact 'space' with a small underlying category, we assume that the category C_X is small. As in 5.3.2, we start the construction from deflations.

(c1) For every $\mathfrak{e} \in \mathfrak{E}_X$, we map the deflation $\mathfrak{s}_0^0 \in \mathfrak{E}_{\mathfrak{S}_k}$ to $\mathfrak{e}$, which induces an 'exact' functor

$$(C_{\mathfrak{S}_k}, \mathfrak{E}_{\mathfrak{S}_k}) \xrightarrow{\mathfrak{f}_{\mathfrak{e}}^*} (C_X, \mathfrak{E}_X).$$

The functors $\mathfrak{f}_{\mathfrak{e}}^*$, $\mathfrak{e} \in \mathfrak{E}_X$, determine the functor

$$\coprod_{\mathfrak{e} \in \mathfrak{E}_X} (C_{\mathfrak{S}_k}, \mathfrak{E}_{\mathfrak{S}_k}) \xrightarrow{\mathfrak{f}_{\mathfrak{E}_X}^*} (C_X, \mathfrak{E}_X). \tag{2*}$$

which is an inverse image functor of a morphism

$$(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}_{\mathfrak{E}_X}} \prod_{\mathfrak{e} \in \mathfrak{E}_X} (\mathfrak{S}_k, \mathfrak{E}_{\mathfrak{S}_k}) \tag{2}$$

of right exact 'spaces'. Notice that functor $\mathfrak{f}_{\mathfrak{E}_X}^*$ 'covers' all deflations of $(X, \mathfrak{E}_X)$.

(c2) What remains 'uncovered' by the functor $\mathfrak{f}_{\mathfrak{E}_X}^*$ is the class of arrows

$$HomC_X - \bigcup_{\mathfrak{e} \in \mathfrak{E}_X} \mathfrak{f}_{\mathfrak{e}}^*(HomC_{\mathfrak{S}_k})$$

which we denote by $\mathfrak{Ord}(X, \mathfrak{E}_X)$ (meaning *ordinary arrows*).

For each arrow $\alpha \in \mathfrak{Ord}(X, \mathfrak{E}_X)$, consider the functor

$$C_{\mathfrak{x}_k} \xrightarrow{\gamma_\alpha^*} C_X$$

(from the part (b) of the argument 4.4.4), which maps a to L, b to M and each element $\lambda \in k = C_{\mathfrak{x}_k}(a, b)$ to the morphism $\lambda \cdot \alpha$. The functors $\{\gamma_\alpha^* \mid \alpha \in \mathfrak{Ord}(X, \mathfrak{E}_X)\}$ define an 'exact' (by a trivial reason) functor

$$\coprod_{\alpha \in \mathfrak{Ord}(X, \mathfrak{E}_X)} C_{\mathfrak{x}_k} \xrightarrow{\gamma_X^*} (C_X, \mathfrak{E}_X) \tag{3*}$$

which is an inverse image functor of a morphism

$$(X, \mathfrak{E}_X) \xrightarrow{\gamma_X} \prod_{\alpha \in \mathfrak{Ord}(X, \mathfrak{E}_X)} \mathfrak{x}_k. \tag{3}$$

(c3) The morphisms (2) and (3) determine a morphism

$$(X, \mathfrak{E}_X) \xrightarrow{\gamma_X} \Big(\prod_{\alpha \in \mathfrak{Ord}(X, \mathfrak{E}_X)} \mathfrak{x}_k \Big) \prod \Big(\prod_{\mathfrak{e} \in \mathfrak{E}_X} (\mathfrak{S}_k, \mathfrak{E}_{\mathfrak{S}_k}) \Big) \tag{4}$$

which is an inflation by construction. Since any product of injective objects is an injective, the morphism (4) is an inflation into an injective object, whence the assertion. ∎

6.2. Note. The construction of the inflation of a right exact 'space' into an injective object described in the argument of 6.1 is canonical, but, of course, not economical. One can decrease the injective object by taking representatives of isomorphism classes of deflations and 'ordinary' arrows. The morphism of the right exact k-'space' $(X, \mathfrak{E}_X)$ into an injective object obtained this way will be still an inflation.

6.3. Additivization. Let $\mathfrak{Esp}_k^{\mathfrak{r},\mathfrak{a}}$ denote the full subcategory of the category $\mathfrak{Esp}_k^{\mathfrak{r}}$ of right exact k-'spaces' generated by the 'spaces' represented by additive k-linear categories.

6.3.1. Proposition. *The inclusion functor* $\mathfrak{Esp}_k^{\mathfrak{r},\mathfrak{a}} \xrightarrow{\mathfrak{J}^*} \mathfrak{Esp}_k^{\mathfrak{r}}$ *has a right adjoint.*

Proof. To any right exact k-'space' $(X, \mathfrak{E}_X)$, the functor $\mathfrak{J}_*$ assigns the smallest additive subcategory of the category of sheaves of k-modules on the presite $(X, \mathfrak{E}_X)$ endowed with the right exact structure generated by the image of $\mathfrak{E}_X$. ∎

6.3.2. The canonical left exact structure. The left exact structure $\mathfrak{I}_k^{\mathfrak{es}}$ on $\mathfrak{Esp}_k^{\mathfrak{r}}$ induces a left exact structure on the category $\mathfrak{Esp}_k^{\mathfrak{r},\mathfrak{a}}$, which we denote by $\widetilde{\mathfrak{I}}_k^{\mathfrak{es}}$.

6.3.3. Proposition. *The left exact category* $(\mathfrak{Esp}_k^{\mathfrak{r},\mathfrak{a}}, \widetilde{\mathfrak{I}}_k^{\mathfrak{es}})$ *has enough injective objects.*

Proof. The argument follows the same idea as the argument of 5.4.3 (or 2.5.1). Namely, the existence of a right adjoint to the inclusion functor $\mathfrak{Esp}_k^{\mathfrak{r},\mathfrak{a}} \xrightarrow{\mathfrak{J}^*} \mathfrak{Esp}_k^{\mathfrak{r}}$ together with the fact that the inclusion functor maps inflations to inflations implies that its right adjoint, $\mathfrak{J}_*$, maps injective objects to injective objects and there are enough of injective objects obtained this way. Details are left to the reader. ■

6.4. Left exact category of Karoubian right exact k-'spaces'. Let $\mathfrak{KEsp}_k^{\mathfrak{r}}$ denote the full subcategory of the category $\mathfrak{Esp}_k^{\mathfrak{r}}$ generated by Karoubian right exact k-spaces. The canonical left exact structure $\mathfrak{I}_k^{\mathfrak{es}}$ induces a left exact structure on the category $\mathfrak{KEsp}_k^{\mathfrak{r}}$, which we denote by $\mathfrak{I}_k^{\mathfrak{Ke}}$.

6.4.1. Proposition. *The left exact category* $(\mathfrak{KEsp}_k^{\mathfrak{r}}, \mathfrak{I}_k^{\mathfrak{Ke}})$ *has enough injective objects.*

Proof. The argument follows the pattern of the proof of 5.4.3. ■

6.4.2. Remark. Notice that any *pointed* Karoubian right exact category $(C_X, \mathfrak{E}_X)$ has finite products. This follows from the fact that the morphism of any object of C_X to a zero object splits, hence it is a deflation; and deflations are stable under base change.

In particular, any Karoubian k-linear right exact category is additive.

6.5. Left exact structures formed by localizations. Each of the left exact structures $\mathfrak{L}_{\mathfrak{es}}$, $\mathfrak{L}_\ell^{\mathfrak{es}}$, $\mathfrak{L}_{\mathfrak{r}}^{\mathfrak{es}}$, $\mathfrak{L}_{\mathfrak{es}}^{\mathfrak{c}}$, and $\mathfrak{L}_{\mathfrak{es}}^{\mathfrak{c};\mathfrak{c}}$ we introduced on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of 'spaces' (see 5.5.) induces a left exact structure on the category $\mathfrak{Esp}_k^{\mathfrak{r}}$ of right exact k-'spaces'. We denote them by respectively $\mathfrak{L}_{\mathfrak{es}}(k)$, $\mathfrak{L}_\ell^{\mathfrak{es}}(k)$, $\mathfrak{L}_{\mathfrak{r}}^{\mathfrak{es}}(k)$, $\mathfrak{L}_{\mathfrak{es}}^{\mathfrak{c}}(k)$, and $\mathfrak{L}_{\mathfrak{es}}^{\mathfrak{c};\mathfrak{c}}(k)$.

6.6. Left and right canonical structures. Let $\mathfrak{I}_\ell^{\mathfrak{es}}(k)$ (resp. by $\mathfrak{I}_{\mathfrak{r}}^{\mathfrak{es}}(k)$) denote the class of all morphisms of $\mathfrak{I}_k^{\mathfrak{es}}$ such that $\Sigma_{f^*} \stackrel{\text{def}}{=} \{s \in HomC_Y \mid f^*(s) \text{ is invertible}\}$ is a left (resp. right) multiplicative system. We denote by $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{es}}(k)$ the intersection of the classes $\mathfrak{I}_\ell^{\mathfrak{es}}(k)$ and $\mathfrak{I}_{\mathfrak{r}}^{\mathfrak{es}}(k)$.

6.6.1. Proposition. *(a) The classes of morphisms* $\mathfrak{I}_\ell^{\mathfrak{es}}(k)$ *and* $\mathfrak{I}_{\mathfrak{r}}^{\mathfrak{es}}(k)$ *(hence their intersection* $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{es}}(k)$*) are left exact structures on the category* $\mathfrak{Esp}_k^{\mathfrak{r}}$ *of right exact k-'spaces'.*

(b) The left exact category $(\mathfrak{Esp}_k^{\mathfrak{r}}, \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{es}}(k))$ *has enough injective objects.*

Proof. (a) The assertion follows from 3.3.1(a) (or its argument).

(b) One can observe that the canonical inflations to injective objects belong to the subclass $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{es}}(k)$. ■

7. Left exact category of exact k-'spaces'.

We denote by $\mathfrak{Esp}_k^{\mathfrak{e}}$ the full subcategory of the category $\mathfrak{Esp}_k^{\mathfrak{r}}$ generated by the exact 'spaces', that is right exact 'spaces' $(X, \mathfrak{E}_X)$ for which the category $(C_X, \mathfrak{E}_X)$ is exact.

7.1. Proposition. *The inclusion functor* $\mathfrak{Esp}^{\mathfrak{e}}_k \xrightarrow{\mathfrak{J}^*} \mathfrak{Esp}^{\mathfrak{r}}_k$ *has a right adjoint.*

Proof. By I.7.5, for any svelte k-linear right exact category $(C_X, \mathfrak{E}_X)$, there exists an exact category $(C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}})$ and a fully faithful k-linear 'exact' functor

$$(C_X, \mathfrak{E}_X) \xrightarrow{\gamma^*_X} (C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}}) \qquad (1)$$

which is universal; that is any 'exact' k-linear functor from $(C_X, \mathcal{E}_X)$ to an exact k-linear category factorizes uniquely through γ^*_X. This means that the map which assigns to every right exact k-'space' $(X, \mathfrak{E}_X)$ the exact k-'space' $(X_\mathfrak{e}, \mathfrak{E}_{X_\mathfrak{e}})$ extends to a functor $\mathfrak{Esp}^{\mathfrak{r}}_k \xrightarrow{\mathfrak{J}_*} \mathfrak{Esp}^{\mathfrak{e}}_k$ which is right adjoint to the inclusion functor $\mathfrak{J}^*$, and (1) is an inverse image functor of the adjunction morphism

$$\mathfrak{J}^* \circ \mathfrak{J}_*(X, \mathfrak{E}_X) = (X_\mathfrak{e}, \mathfrak{E}_{X_\mathfrak{e}}) \xrightarrow{\gamma_X} (X, \mathfrak{E}_X).$$

The other adjunction morphism is identical. ■

7.2. The canonical left exact structure. The left exact structure $\mathfrak{I}^{\mathfrak{es}}_k$ on $\mathfrak{Esp}^{\mathfrak{r}}_k$ induces a left exact structure on the category $\mathfrak{Esp}^{\mathfrak{e}}_k$, which we denote by $\mathfrak{I}^{\mathfrak{e}}_k$. Since the inclusion functor $\mathfrak{J}^*$ preserves colimits, in particular push-forwards, and, by construction, maps inflations to inflations, it is an 'exact' functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{e}}_k, \mathfrak{I}^{\mathfrak{e}}_k)$ to the left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_k, \mathfrak{I}^{\mathfrak{es}}_k)$.

7.3. Proposition. *The left exact category* $(\mathfrak{Esp}^{\mathfrak{e}}_k, \mathfrak{I}^{\mathfrak{e}}_k)$ *has enough injective objects.*

Proof. Since the functor $\mathfrak{J}^*$ maps inflations to inflations, its right adjoint $\mathfrak{J}_*$ maps injective objects to injective objects. There are enough injective objects are of the form $\mathfrak{J}_*(\mathfrak{X}, \mathfrak{E}_\mathfrak{X}) = (\mathfrak{X}_\mathfrak{e}, \mathfrak{E}_{\mathfrak{X}_\mathfrak{e}})$, where $(\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ runs through injective objects of the left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_k, \mathfrak{I}^{\mathfrak{es}}_k)$, and the canonical inflations of an exact k-'space' $(X, \mathfrak{E}_X)$ into an injective object is the image by the functor $\mathfrak{J}_*$ of the canonical inflation of $\mathfrak{J}^*(X, \mathfrak{E}_X)$ to an injective object of the left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_k, \mathfrak{I}^{\mathfrak{es}}_k)$. The argument is similar to that of 5.4.3. ■

8. Diagrams.

We fix a diagram scheme $\mathfrak{D}$. The category of diagrams $\mathfrak{D} \longrightarrow C_X$ will be denoted by $C^{\mathfrak{D}}_X$ and the corresponding 'space' by $X^{\mathfrak{D}}$; it is defined by $C_{X^{\mathfrak{D}}} = C^{\mathfrak{D}}_X$.

The map $X \longmapsto X^{\mathfrak{D}}$ extends naturally to an endofunctor $\mathcal{P}^{\mathfrak{D}}$ of the category of 'spaces' $|Cat|^o$: a morphism $X \xrightarrow{f} Y$ with an inverse image functor f^* is mapped to the morphism $X^{\mathfrak{D}} \xrightarrow{f_{\mathfrak{D}}} Y^{\mathfrak{D}}$ with an inverse image functor $C^{\mathfrak{D}}_Y \xrightarrow{f^*_{\mathfrak{D}}} C^{\mathfrak{D}}_X$.

8.1. Proposition. *The functor* $\mathcal{P}^{\mathfrak{D}}$ *preserves colimits.*

Proof. For any family $\{X_i \mid i \in J\}$ of 'spaces', set $X_J = \coprod_{i \in J} X_j$. Then

$$C_{X_J^{\mathfrak{D}}} = C_{X_J}^{\mathfrak{D}} = \big(\prod_{i \in J} C_{X_i}\big)^{\mathfrak{D}} = \prod_{i \in J} C_{X_i}^{\mathfrak{D}} = \prod_{i \in J} C_{X_i^{\mathfrak{D}}} = C_{\coprod_{i \in J} X_i^{\mathfrak{D}}}.$$

For any pair of arrows $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, their cokernel, $\mathcal{C}(f,g)$, is represented by the kernel of their adjoint functors

$$C_{\mathcal{C}(f,g)} = Ker(f^*, g^*) \xrightarrow{\mathfrak{c}^*} C_Y \overset{f^*}{\underset{g^*}{\rightrightarrows}} C_X.$$

It follows from the description of $Ker(f^*, g^*)$ that $Ker(f^*, g^*)^{\mathfrak{D}}$ is naturally isomorphic to $Ker(f^*_{\mathfrak{D}}, g^*_{\mathfrak{D}})$, which means that $\mathcal{C}(f,g)^{\mathfrak{D}} \simeq \mathcal{C}(f_{\mathfrak{D}}, g_{\mathfrak{D}})$. ∎

8.1.1. Note. A more conceptual proof of 8.1 is based on observation that the functor $\mathcal{P}^{\mathfrak{D}}$ has a right adjoint, $\mathcal{P}_{\mathfrak{D}}$, which assigns to every 'space' X the 'space' $\mathcal{P}_{\mathfrak{D}}(X)$ represented by the category $\mathcal{P}\mathfrak{a}(\mathfrak{D}) \times C_X$ and to every morphism of 'spaces' $X \xrightarrow{f} Y$ with an inverse image functor $C_Y \xrightarrow{f^*} C_X$ the morphism $\mathcal{P}_{\mathfrak{D}}(f)$ having inverse image functor $Id \times f^*$. Here $\mathcal{P}\mathfrak{a}(\mathfrak{D})$ denotes the category of paths of the diagram $\mathfrak{D}$.

8.2. The left exact structure $\mathfrak{I}^{\mathfrak{D}}$. Let $\mathfrak{I}^{\mathfrak{D}}$ denote the class of all morphisms $X \xrightarrow{\mathfrak{f}} Y$ of 'spaces' such that $X^{\mathfrak{D}} \xrightarrow{\mathfrak{f}_{\mathfrak{D}}} Y^{\mathfrak{D}}$ belongs to $\mathfrak{I}^{\mathfrak{s}}$.

8.2.1. Proposition. *(a) $\mathfrak{I}^{\mathfrak{D}}$ is a left exact structure on $|Cat|^o$.*

(b) The functor $\mathcal{P}^{\mathfrak{D}}$ is an 'exact' functor from the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{D}})$ to the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$.

(c) The left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{D}})$ has enough injective objects.

Proof. (a) By definition $\mathfrak{I}^{\mathfrak{D}} = \mathcal{P}^{\mathfrak{D}^{-1}}(\mathfrak{I}^{\mathfrak{s}})$. This implies immediately that $\mathfrak{I}^{\mathfrak{D}}$ contains all isomorphisms and is closed under composition. By 8.1, the functor $\mathcal{P}^{\mathfrak{D}}$ preserves colimits; in particular, it preserves push-forwards. Therefore, since $\mathfrak{I}^{\mathfrak{s}}$ is stable under push-forwards, its preimage, $\mathfrak{I}^{\mathfrak{D}}$, has the same property.

(b) By definition of the left exact structure $\mathfrak{I}^{\mathfrak{D}}$, the functor $\mathcal{P}^{\mathfrak{D}}$ maps $\mathfrak{I}^{\mathfrak{D}}$ ot $\mathfrak{I}^{\mathfrak{s}}$. Since, by 8.1 (or 8.1.1), the functor $\mathcal{P}^{\mathfrak{D}}$ preserves colimits, in particular, it preserves push-forwards, $\mathcal{P}^{\mathfrak{D}}$ is an 'exact' functor from the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{D}})$ to the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$.

(c) The fact that $\mathcal{P}^{\mathfrak{D}}$ maps inflations to inflations implies that its right adjoint, $\mathcal{P}_{\mathfrak{D}}$ (cf. 8.1.1) maps injective objects to injective objects. The claim is that there are enough injective objects of the form $\mathcal{P}_{\mathfrak{D}}(\mathfrak{X})$, where $\mathfrak{X}$ runs through injective objects of the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$.

In fact, since the left exact category $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$ has enough injective objects, for any 'space' X, there is a morphism $\mathcal{P}^{\mathfrak{D}}(X) \xrightarrow{\mathfrak{f}} \mathfrak{X}$ from the class $\mathfrak{I}^{\mathfrak{s}}$ with $\mathfrak{X}$ an injective of $(|Cat|^o, \mathfrak{I}^{\mathfrak{s}})$. By adjunction, $\mathfrak{f}$ determines a morphism $X \xrightarrow{\widehat{\mathfrak{f}}} \mathcal{P}_{\mathfrak{D}}(\mathfrak{X})$ which is the composition of the adjunction morphism $X \xrightarrow{\eta_X} \mathcal{P}_{\mathfrak{D}}\mathcal{P}^{\mathfrak{D}}(X)$ and $\mathcal{P}_{\mathfrak{D}}(\mathcal{P}^{\mathfrak{D}}(X) \xrightarrow{\mathfrak{f}} \mathfrak{X})$. Applying $\mathcal{P}^{\mathfrak{D}}$ to this composition, we obtain a commutative square

$$\begin{array}{ccc}
\mathcal{P}^{\mathfrak{D}}(X) & \xrightarrow{\mathcal{P}^{\mathfrak{D}}(\eta(X)} & \mathcal{P}^{\mathfrak{D}}\mathcal{P}_{\mathfrak{D}}\mathcal{P}^{\mathfrak{D}}(X) \\
\mathfrak{f} \downarrow & & \downarrow \mathcal{P}^{\mathfrak{D}}\mathcal{P}_{\mathfrak{D}}(\mathfrak{f}) \\
\mathfrak{X} & \xleftarrow{\epsilon(\mathfrak{X})} & \mathcal{P}^{\mathfrak{D}}\mathcal{P}_{\mathfrak{D}}(\mathfrak{X})
\end{array}$$

Since $\mathcal{P}^{\mathfrak{D}}(X) \xrightarrow{\mathfrak{f}} \mathfrak{X}$ belongs to $\mathfrak{I}^{\mathfrak{s}}$, it follows from this commutative square and the observation 2.4.3 that $\mathcal{P}^{\mathfrak{D}}(\widehat{\mathfrak{f}}) \in \mathfrak{I}^{\mathfrak{s}}$. But, this means precisely that $\widehat{\mathfrak{f}} \in \mathfrak{I}^{\mathfrak{D}}$. ■

8.2.2. Remark. The left exact structure $\mathfrak{I}^{\mathfrak{D}}$ is coarser than $\mathfrak{I}^{\mathfrak{s}}$.
In fact, for every $X \xrightarrow{\mathfrak{f}} Y$, we have a commutative diagram

$$\begin{array}{ccc}
C_Y & \xrightarrow{\mathfrak{f}^*} & C_X \\
\mathfrak{j}_{Y*} \downarrow & & \downarrow \mathfrak{j}_{X*} \\
C_Y^{\mathfrak{D}} & \xrightarrow{\mathfrak{f}^*_{\mathfrak{D}}} & C_X^{\mathfrak{D}}
\end{array}$$

where the vertical arrows are canonical full embeddings identifying every object of the category with the constant diagram with values in this object (and its identical arrow).

The fact that $\mathfrak{f}_{\mathfrak{D}} \in \mathfrak{I}^{\mathfrak{s}}$ means that every morphism of $C_X^{\mathfrak{D}}$ is isomorphic to $\mathfrak{f}^*_{\mathfrak{D}}$ of some morphism of $C_Y^{\mathfrak{D}}$. In particular, this holds for morphisms between constant functors. But, if the morphism $\mathfrak{j}_{X*}(a \to b)$ is isomorphic to a morphism $\mathfrak{f}^*(\mathcal{D}_1 \xrightarrow{\lambda} \mathcal{D}_2)$ for some diagrams $\mathfrak{D} \longrightarrow C_Y$, then there are morphisms between constant diagrams having this property: it suffices to choose object $z \in Ob\mathfrak{D}$ and take the morphism in $\mathfrak{j}_{Y*}(\mathcal{D}_1(z) \xrightarrow{\lambda(z)} \mathcal{D}(z))$ of constant functors.

8.3. Diagrams in right exact categories.

8.3.1. The standard right exact structure on the 'space' of diagrams. Let $(X, \mathfrak{E}_X)$ be a right exact 'space'. There is an obvious right exact structure $\mathfrak{E}_{X^{\mathfrak{D}}}$ on the 'space' $X^{\mathfrak{D}}$: a morphism of diagrams $\mathcal{D}_1 \xrightarrow{\mathfrak{t}} \mathcal{D}_2$ belongs to $\mathfrak{E}_{X^{\mathfrak{D}}}$ if $\mathcal{D}_1(a) \xrightarrow{\mathfrak{t}(a)} \mathcal{D}_2(a)$ is a deflation for all $a \in Ob\mathfrak{D}$.

8.3.2. The cartesian right exact structure. Fix a right exact 'space' $(X, \mathfrak{E}_X)$. We denote by $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$ the class of all morphisms $\mathcal{D}_1 \xrightarrow{\mathfrak{t}} \mathcal{D}_2$ from $\mathfrak{E}_{X^{\mathfrak{D}}}$ such that for every arrow $a \xrightarrow{\gamma} b$ of $\mathfrak{D}$, the square

$$\begin{array}{ccc} \mathcal{D}_1(a) & \xrightarrow{\mathcal{D}_1(\gamma)} & \mathcal{D}_1(b) \\ {\scriptstyle \mathfrak{t}(a)}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}(b)} \\ \mathcal{D}_2(a) & \xrightarrow{\mathcal{D}_2(\gamma)} & \mathcal{D}_2(b) \end{array}$$

is cartesian.

8.3.3. Proposition. *The class of morphisms* $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$ *is a right exact structure on the category* $C_X^{\mathfrak{D}} = C_{X^{\mathfrak{D}}}$.

Proof. The class $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$ contains all isomorphisms and (since the composition of cartesian squares is a cartesian square) it is closed under compositions. It remains to show that $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$ is stable under base change. In fact, consider a cartesian square

$$\begin{array}{ccc} \mathcal{D}_4 & \xrightarrow{\widetilde{\xi}} & \mathcal{D}_1 \\ {\scriptstyle \mathfrak{t}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}} \\ \mathcal{D}_3 & \xrightarrow{\xi} & \mathcal{D}_2 \end{array}$$

where the morphism $\mathcal{D}_1 \xrightarrow{\mathfrak{t}} \mathcal{D}_2$ belongs to $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$ and $\mathcal{D}_3 \xrightarrow{\xi} \mathcal{D}_2$ is an arbitrary diagram morphism. The claim is that $\mathfrak{t}' \in \mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$; that is for every arrow $a \xrightarrow{\gamma} b$ in $\mathfrak{D}$, the square

$$\begin{array}{ccc} \mathcal{D}_4(a) & \xrightarrow{\mathcal{D}_4(\gamma)} & \mathcal{D}_4(b) \\ {\scriptstyle \mathfrak{t}'(a)}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}'(b)} \\ \mathcal{D}_3(a) & \xrightarrow{\mathcal{D}_3(\gamma)} & \mathcal{D}_3(b) \end{array} \qquad (1)$$

is cartesian. In fact, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{D}_4(a) & \xrightarrow{\mathcal{D}_4(\gamma)} & \mathcal{D}_4(b) & \xrightarrow{\widetilde{\xi}(b)} & \mathcal{D}_1(b) \\ {\scriptstyle \mathfrak{t}'(a)}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}(b)} & cart & \downarrow{\scriptstyle \mathfrak{t}'(b)} \\ \mathcal{D}_3(a) & \xrightarrow{\mathcal{D}_3(\gamma)} & \mathcal{D}_3(b) & \xrightarrow{\xi(b)} & \mathcal{D}_2(b) \end{array} \qquad (2)$$

with a right square cartesian and a commutative diagram

$$\begin{array}{ccccc} \mathcal{D}_4(a) & \xrightarrow{\widetilde{\xi}(a)} & \mathcal{D}_1(a) & \xrightarrow{\mathcal{D}_1(\gamma)} & \mathcal{D}_1(b) \\ {\scriptstyle \mathfrak{t}'(a)}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}(a)} & cart & \downarrow{\scriptstyle \mathfrak{t}(b)} \\ \mathcal{D}_3(a) & \xrightarrow{\xi(a)} & \mathcal{D}_2(a) & \xrightarrow{\mathcal{D}_2(\gamma)} & \mathcal{D}_2(b) \end{array} \qquad (3)$$

whose both squares are cartesian. The latter implies that the composition of the two squares (3),

$$\begin{array}{ccc} \mathcal{D}_4(a) & \xrightarrow{\mathcal{D}_1(\gamma)\circ\widetilde{\xi}(a)} & \mathcal{D}_1(b) \\ {\scriptstyle \mathfrak{t}'(a)}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}(b)} \\ \mathcal{D}_3(a) & \xrightarrow{\mathcal{D}_2(\gamma)\circ\xi(a)} & \mathcal{D}_2(b) \end{array}$$

is cartesian. But, this composition coincides with the composition of the squares (2). Since the composition of the squares (2) is cartesian and the right square of (2) is cartesian, its left square, (1), is cartesian. ■

8.3.4. Pointed diagrams and the cartesian right exact structure. We call a diagram scheme $\mathfrak{D}$ *pointed* if it has a final object, $\mathfrak{pt}$. In this case, for any 'space' X, the standard full embedding

$$C_X \xrightarrow{\gamma^{\mathfrak{D}}_{X*}} C_{X^{\mathfrak{D}}} = C^{\mathfrak{D}}_X$$

has a canonical left adjoint,

$$C_{X^{\mathfrak{D}}} \xrightarrow{\gamma^{\mathfrak{D}*}_X} C_X, \quad (\mathfrak{D} \xrightarrow{\mathcal{F}} C_X) \longmapsto \mathrm{colim}\mathcal{F} = \mathcal{F}(\mathfrak{pt}). \tag{4}$$

8.3.4.1. Proposition. *(a) For every right exact 'space' $(X, \mathfrak{E}_X)$, the morphism of 'spaces' $X \xrightarrow{\gamma^{\mathfrak{D}}_X} X^{\mathfrak{D}}$ is a morphism*

$$(X, \mathfrak{E}_X) \xrightarrow{\gamma^{\mathfrak{D}}_X} (X^{\mathfrak{D}}, \mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}})$$

which belongs to $\mathfrak{I}^{\mathfrak{es}}$.

(b) The inverse image functor $\gamma^{\mathfrak{D}*}_X$ establishes an equivalence between the category of deflations of every object $\mathcal{F}$ of the right exact category $(C_{X^{\mathfrak{D}}}, \mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}})$ and the category of deflations of the object $\gamma^{\mathfrak{D}*}_X(\mathcal{F}) = \mathcal{F}(\mathfrak{pt})$ of the right exact category $(C_X, \mathfrak{E}_X)$.

Proof. (a) The functor $\gamma^{\mathfrak{D}*}_X$ maps deflations to deflations, because if $\mathcal{G} \xrightarrow{\mathfrak{s}} \mathcal{F}$ is a deflation of the diagrams (– a morphism of $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$), then $\mathcal{G}(a) \xrightarrow{\mathfrak{s}(a)} \mathcal{F}(a)$ for every $a \in Ob\mathfrak{D}$; in particular, $\mathcal{G}(\mathfrak{pt}) \xrightarrow{\mathfrak{s}(\mathfrak{pt})} \mathcal{F}(\mathfrak{pt})$ is a deflation. Similarly, $\mathcal{F} \longmapsto \mathcal{F}(\mathfrak{pt})$ preserves pull-backs of deflations, because pull-backs of diagrams are taken object-wise.

(b) Let $L \xrightarrow{\mathfrak{t}} \gamma^{\mathfrak{D}*}_X(\mathcal{F}) = \mathcal{F}(\mathfrak{pt})$ be a morphism of $\mathfrak{E}_X$. Then for every $a \in Ob\mathfrak{D}$, there exists a cartesian square

$$\begin{array}{ccc} \mathcal{L}(a) & \longrightarrow & L \\ {\scriptstyle \widetilde{\mathfrak{t}}(a)}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}} \\ \mathcal{F}(a) & \longrightarrow & \mathcal{F}(\mathfrak{pt}) \end{array}$$

It follows from the universal property of cartesian squares that the choice of squares for each $a \in Ob\mathfrak{D}$ (which is unique up to isomorphism) uniquely determines a diagram $\mathfrak{D} \xrightarrow{\mathcal{L}} C_X$ such that $\mathcal{L}(\mathfrak{pt}) = L$ and a cartesian deflation $\mathcal{L} \xrightarrow{\widetilde{\mathfrak{t}}} \mathcal{F}$ such that $\widetilde{\mathfrak{t}}(\mathfrak{pt}) = \mathfrak{t}$.

It follows from the uniqueness (up to isomorphism) of this construction that every morphism

$$\begin{array}{ccc} M & \xrightarrow{\xi} & L \\ \mathfrak{s} \searrow & & \swarrow \mathfrak{t} \\ & \mathcal{F}(\mathfrak{pt}) & \end{array}$$

of deflations of the object $\mathcal{F}(\mathfrak{pt})$ uniquely extends to a morphism

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\widetilde{\xi}} & \mathcal{L} \\ \widetilde{\mathfrak{s}} \searrow & & \swarrow \widetilde{\mathfrak{t}} \\ & \mathcal{F} & \end{array}$$

of the corresponding cartesian deflations of the diagram $\mathcal{F}$. ■

8.3.4.2. Proposition. *Let $\mathfrak{D}$ be a pointed diagram scheme with the final object $\mathfrak{pt}$. Then projective objects of the right exact category $(C_{X^{\mathfrak{D}}}, \mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}})$ are all diagrams $\mathfrak{D} \xrightarrow{\mathcal{F}} C_X$ such that $\mathcal{F}(\mathfrak{pt})$ is a projective object of $(C_X, \mathfrak{E}_X)$.*

Proof. Let $\mathcal{G} \xrightarrow{\lambda} \mathcal{F}$ be a morphism of $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$. In particular, $\mathcal{G}(\mathfrak{pt}) \xrightarrow{\lambda(\mathfrak{pt})} \mathcal{F}(\mathfrak{pt})$ is a deflation. If $\mathcal{F}(\mathfrak{pt})$ is projective, there exists a splitting, $\mathcal{F}(\mathfrak{pt}) \xrightarrow{\psi_{\mathfrak{pt}}} \mathcal{G}(\mathfrak{pt})$ of $\lambda(\mathfrak{pt})$; that is $\lambda(\mathfrak{pt}) \circ \psi_{\mathfrak{pt}} = id_{\mathcal{F}(\mathfrak{pt})}$. Thus, for every $a \in Ob\mathfrak{D}$, we have a commutative square

$$\begin{array}{ccc} \mathcal{F}(a) & \xrightarrow{\psi_{\mathfrak{tt}}\mathcal{F}(\mathfrak{p}_a)} & \mathcal{G}(\mathfrak{pt}) \\ id \downarrow & & \downarrow \lambda(\mathfrak{pt}) \\ \mathcal{F}(a) & \xrightarrow{\mathcal{F}(\mathfrak{p}_a)} & \mathcal{F}(\mathfrak{pt}) \end{array}$$

where $\mathfrak{p}_a$ is the unique arrow $a \to \mathfrak{pt}$. Since, by definition of a deflation, the square

$$\begin{array}{ccc} \mathcal{G}(a) & \xrightarrow{\mathcal{G}(\mathfrak{p}_a)} & \mathcal{G}(\mathfrak{pt}) \\ \lambda(a) \downarrow & cart & \downarrow \lambda(\mathfrak{pt}) \\ \mathcal{F}(a) & \xrightarrow{\mathcal{F}(\mathfrak{p}_a)} & \mathcal{F}(\mathfrak{pt}) \end{array}$$

is cartesian, there exists a unique morphism $\mathcal{F}(a) \xrightarrow{\psi(a)} \mathcal{G}(a)$ such that

$$\lambda(a) \circ \psi(a) = id_{\mathcal{F}(a)} \quad \text{and} \quad \mathcal{G}(\mathfrak{p}_a) \circ \psi(a) = \psi_{\mathfrak{pt}} \circ \mathcal{F}(\mathfrak{p}_a).$$

This constructs a morphism $\psi = (\psi(a) \mid a \in Ob\mathfrak{D})$ from $\mathcal{F}$ to $\mathcal{G}$ which splits the deflation $\mathcal{G} \xrightarrow{\lambda} \mathcal{F}$. This shows that $\mathcal{F}$ is a projective, if $\mathcal{F}(\mathfrak{pt})$ is a projective.

It follows from this argument (or from 8.3.4.1) that, conversely, if $\mathcal{F}$ is a projective object of $(C_{X^{\mathfrak{D}}}, \mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}})$, then $\mathcal{F}(\mathfrak{pt})$ is a projective object of $(C_X, \mathfrak{E}_X)$. ■

8.3.4.3. Note. It is easy to see that a projective $\mathcal{F}$ of the right exact category $(C_{X^{\mathfrak{D}}}, \mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}})$ is pointed iff the projective object $\mathcal{F}(\mathfrak{pt})$ of $(C_X, \mathfrak{E}_X)$ is pointed.

8.3.5. An application.

8.3.5.1. The class $\Sigma_{G,\mathfrak{E}_X}$. Fix a right exact category $(C_X, \mathfrak{E}_X)$ and a functor $C_X \xrightarrow{G} C_{\mathcal{Z}}$. As usual, we denote by Σ_G the class of all arrows s of the category C_X which the functor G maps to isomorphisms. We denote by $\Sigma_{G,\mathfrak{E}_X}$ the class of all arrows s such that their pull-backs along all deflations (in particular, s itself) belong to Σ_G.

It is easy to see that $\Sigma_{G,\mathfrak{E}_X}$ contains all isomorphisms, closed under compositions and, of course, stable under pull-backs along arbitrary deflations.

8.3.5.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be right exact category with enough projective objects and G a functor from C_X to a complete category. Then $S^{\bullet}_{-}(G)(s)$ is an isomorphism for every $s \in \Sigma_{G,\mathfrak{E}_X}$.*

Proof. (a) Consider the category $C_{X^{\rightarrow}} = C_X^{\rightarrow}$ of arrows of the category C_X endowed with the cartesian right exact structure $\mathfrak{E}^{\mathfrak{c}}_{X^{\rightarrow}}$. Let $C_{X^{\rightarrow}_G}$ denote the full subcategory of the category $C_{X^{\rightarrow}}$ whose objects are arrows of $\Sigma_{G,\mathfrak{E}_X}$. Since the class $\Sigma_{G,\mathfrak{E}_X}$ is stable under pull-backs along deflations, it follows that $C_{X^{\rightarrow}}$ is an exact subcategory of the right exact category $(C_{X^{\rightarrow}}, \mathfrak{E}^{\mathfrak{c}}_{X^{\rightarrow}})$. Let $C_{X^{\rightarrow}} \underset{F_{\mathfrak{t}}}{\overset{F_{\mathfrak{s}}}{\rightrightarrows}} C_X$ be source and target functors and $C_{X^{\rightarrow}_G} \underset{F^G_{\mathfrak{t}}}{\overset{F^G_{\mathfrak{s}}}{\rightrightarrows}} C_X$ their restrictions to the subcategory $C_{X^{\rightarrow}_G}$.

(b) The functors $F_{\mathfrak{s}}$ and $F_{\mathfrak{t}}$ are 'exact' functors from $(C_{X^{\rightarrow}}, \mathfrak{E}^{\mathfrak{c}}_{X^{\rightarrow}})$ to $(C_X, \mathfrak{E}_X)$, which implies that $F^G_{\mathfrak{s}}$ and $F^G_{\mathfrak{t}}$ are 'exact' functors from $(C_{X^{\rightarrow}_G}, \mathfrak{E}^{\mathfrak{c}}_{X^{\rightarrow}_G})$ to $(C_X, \mathfrak{E}_X)$, where $\mathfrak{E}^{\mathfrak{c}}_{X^{\rightarrow}_G}$ is the induced right exact structure. Let $F^G_{\mathfrak{s}} \xrightarrow{\rho} F^G_{\mathfrak{t}}$ denote the restriction of the natural morphism $F_{\mathfrak{s}} \longrightarrow F_{\mathfrak{t}}$. It follows that $G \circ F^G_{\mathfrak{s}} \xrightarrow{G(\rho)} G \circ F^G_{\mathfrak{t}}$ is a functor isomorphism.

(c) Notice that the right exact category $(C_{X^{\rightarrow}_G}, \mathfrak{E}^{\mathfrak{c}}_{X^{\rightarrow}_G})$ has enough projective objects.

In fact, let $M \xrightarrow{\mathfrak{u}} L$ be an arrow of $\Sigma_{G,\mathfrak{E}_X}$ (regarded as an object of the category $C_{X^{\rightarrow}_G}$). Since, by hypothesis, the right exact category $(C_X, \mathfrak{E}_X)$ has enough projective objects, there exists a deflation $\mathcal{P} \xrightarrow{\mathfrak{e}} L$ with $\mathcal{P}$ projective. By 8.3.4.2, the upper horizontal arrow in the cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{P}} & \xrightarrow{\ \widetilde{\mathfrak{u}}\ } & \mathcal{P} \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{u}} \\ M & \xrightarrow{\ \mathfrak{u}\ } & L \end{array}$$

is a projective object of $(C_{X^{\to}}, \mathfrak{E}^{\mathfrak{c}}_{X^{\to}})$ and the square itself is a deflation – morphism of $\mathfrak{E}^{\mathfrak{c}}_{X^{\to}}$. Since, $\Sigma_{G,\mathfrak{E}_X}$ is stable under pull-backs along deflations, the $(\widetilde{\mathcal{P}} \xrightarrow{\widetilde{\mathfrak{u}}} \mathcal{P}$ is an object of the subcategory $C_{X^{\to}_G}$, hence it is a projective of $(C_{X^{\to}_G}, \mathfrak{E}^{\mathfrak{c}}_{X^{\to}_G})$.

(d) It remains to apply (the dual version of) III.3.6.7, which gives an isomorphism

$$S^{\bullet}_{+}(G) \circ F^G_{\mathfrak{s}} \xrightarrow{\ S^{\bullet}_{+}(G)(\rho)\ } S^{\bullet}_{+}(G) \circ F^G_{\mathfrak{t}}.$$

This proves the assertion. ∎

8.4. Two power functors.

8.4.1. Proposition. *The maps*

$$(X, \mathfrak{E}_X) \longmapsto (X^{\mathfrak{D}}, \mathfrak{E}_{X^{\mathfrak{D}}}) \quad and \quad (X, \mathfrak{E}_X) \longmapsto (X^{\mathfrak{D}}, \mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}})$$

extend naturally to endofunctors, respectively

$$\mathfrak{Esp}_r \xrightarrow{\ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{r}}\ } \mathfrak{Esp}_r \quad and \quad \mathfrak{Esp}_r \xrightarrow{\ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}}\ } \mathfrak{Esp}_r$$

of the category of right exact 'spaces'.

Proof. (a) By definition, morphisms $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ have 'exact' inverse image functor $(C_Y, \mathfrak{E}_Y) \xrightarrow{f^*} (C_X, \mathfrak{E}_X)$; that is f^* maps deflations to deflations and preserves pull-backs of deflations. Since pull-backs of the diagrams are taken object-wise, the functor

$$C^{\mathfrak{D}}_Y \xrightarrow{\ f^*\ } C^{\mathfrak{D}}_X, \quad \mathcal{F} \longmapsto f^* \circ \mathcal{F},$$

has the same property with respect to the right exact structures $\mathfrak{E}_{Y^{\mathfrak{D}}}$ and $\mathfrak{E}_{X^{\mathfrak{D}}}$ on the categories respectively $C^{\mathfrak{D}}_Y$ and $C^{\mathfrak{D}}_X$.

By the similar reasons, because the *cartesian* right exact structure is defined in terms of pull-backs of deflations (along arrows of the diagrams), the functor $f^*_{\mathfrak{D}}$ maps the cartesian right exact structure $\mathfrak{E}^{\mathfrak{c}}_{Y^{\mathfrak{D}}}$ to the cartesian right exact structure $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$. ∎

8.4.2. Proposition. *The endofunctor*

$$\mathfrak{Esp}_r \xrightarrow{\ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}}\ } \mathfrak{Esp}_r$$

has a right adjoint.

Proof. For a right exact 'space' $(X, \mathfrak{E}_X)$, let $\mathcal{P}^{\mathfrak{c}}_{\mathfrak{D}}(X, \mathfrak{E}_X)$ denotes the right exact 'space' $(X_{\mathfrak{D}}, \mathfrak{E}_{X_{\mathfrak{D}}})$, where $X_{\mathfrak{D}}$ is a 'space' represented by the category $\mathcal{P}\mathfrak{a}(\mathfrak{D}) \times C_X$ and $\mathfrak{E}_{X_{\mathfrak{D}}} = Iso(\mathcal{P}\mathfrak{a}(\mathfrak{D})) \times \mathfrak{E}_X$. The map $(X, \mathfrak{E}_X) \longmapsto (X_{\mathfrak{D}}, \mathfrak{E}_{X_{\mathfrak{D}}})$ extends naturally to a functor

$$\mathfrak{Esp}_{\mathfrak{r}} \xrightarrow{\mathcal{P}^{\mathfrak{c}}_{\mathfrak{D}}} \mathfrak{Esp}_{\mathfrak{r}}.$$

The functor $\mathcal{P}^{\mathfrak{c}}_{\mathfrak{D}}$ is a right adjoint to the functor $\mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}}$.

The adjunction arrow

$$\mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}} \circ \mathcal{P}^{\mathfrak{c}}_{\mathfrak{D}}(X) \xrightarrow{\epsilon_X} X$$

has inverse image functor

$$C_X \xrightarrow{\epsilon^*_X} (\mathcal{P}\mathfrak{a}(\mathfrak{D}) \times C_X)^{\mathfrak{D}} \simeq Funct(\mathcal{P}\mathfrak{a}(\mathfrak{D}), \mathcal{P}\mathfrak{a}(\mathfrak{D}) \times C_X)$$

which maps every object M of C_X to the functor $Id_{\mathcal{P}\mathfrak{a}(\mathfrak{D})} \times M$ and acts accordingly on morphisms. Notice that every arrow $(a, M) \xrightarrow{(\xi, \mathfrak{t})} (b, L)$ of the category $\mathcal{P}\mathfrak{a}(\mathfrak{D}) \times C_X$, the square

$$\begin{array}{ccc} a \times M & \xrightarrow{\xi \times M} & b \times M \\ {\scriptstyle a \times \mathfrak{t}} \downarrow & & \downarrow {\scriptstyle b \times \mathfrak{t}} \\ a \times L & \xrightarrow{\xi \times L} & b \times L \end{array}$$

is cartesian. In particular, the functor ϵ^*_X maps morphisms of $\mathfrak{E}_X$ to morphisms of $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$ and preserves pull-backs of deflations.

The second adjunction arrow

$$(X, \mathfrak{E}_X) \xrightarrow{\eta_X} \mathcal{P}^{\mathfrak{c}}_{\mathfrak{D}} \circ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}}(X, \mathfrak{E}_X)$$

has as an inverse image functor the evaluation functor

$$\mathcal{P}\mathfrak{a}(\mathfrak{D}) \times Funct(\mathcal{P}\mathfrak{a}(\mathfrak{D}), C_X) \xrightarrow{\eta^*_X} C_X, \quad (a, \mathcal{F}) \longmapsto \mathcal{F}(a).$$

It follows from the definition of deflations of $\mathcal{P}\mathfrak{a}(\mathfrak{D}) \times Funct(\mathcal{P}\mathfrak{a}(\mathfrak{D}), C_X)$ that the functor η^*_X maps deflations (which are of the form $(\mathfrak{u}, \mathfrak{t}) \in Iso(\mathcal{P}\mathfrak{a}(\mathfrak{D}) \times \mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}})$ to deflations and preserves pull-backs of deflations.

Indeed, every cartesian square in $\mathcal{P}\mathfrak{a}(\mathfrak{D}) \times Funct(\mathcal{P}\mathfrak{a}(\mathfrak{D}), C_X)$ whose vertical arrows are deflations is isomorphic to the cartesian square of the form

$$\begin{array}{ccc} b \times \mathcal{G}_1 & \xrightarrow{(\xi, \phi')} & (a, \mathcal{F}_1) \\ {\scriptstyle (b, \widetilde{\mathfrak{t}})} \downarrow & cart & \downarrow {\scriptstyle (a, \mathfrak{t})} \\ b \times \mathcal{G}_2 & \xrightarrow{(\xi, \phi)} & (a, \mathcal{F}_2) \end{array} \qquad (1)$$

The functor η_X^* maps the square (1) to the square which is the composition of two squares:

$$\begin{array}{ccccc} \mathcal{G}_1(b) & \xrightarrow{\phi'(b)} & \mathcal{F}_1(b) & \xrightarrow{\mathcal{F}_1(\xi)} & \mathcal{F}_1(a) \\ {\scriptstyle \widetilde{\mathfrak{t}}(b)}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}(b)} & & \downarrow{\scriptstyle \mathfrak{t}(a)} \\ \mathcal{G}_2(b) & \xrightarrow{\phi(b)} & \mathcal{F}_2(b) & \xrightarrow{\mathcal{F}_2(\xi)} & \mathcal{F}_2(a) \end{array} \qquad (2)$$

The left square of (2) is cartesian, because the square

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{\phi'} & \mathcal{F}_1 \\ {\scriptstyle \widetilde{\mathfrak{t}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}} \\ \mathcal{G}_2 & \xrightarrow{\phi} & \mathcal{F}_2 \end{array}$$

is cartesian and limits in the category of functors are taken object-wise. The right square of (2) is cartesian, because $\mathcal{F}_1 \xrightarrow{\mathfrak{t}} \mathcal{F}_2$ is a *cartesian* deflation, that is it belongs to the class $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{D}}}$ (cf. 8.3.2). Since the composition of cartesian squares is a cartesian square, this shows that the functor η_X^* maps the cartesian square (1) to a cartesian square. So that η_X^* preserves pull-backs of deflations. ■

8.4.3. Corollary. *The endofunctor*

$$\mathfrak{Esp}_r \xrightarrow{\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}} \mathfrak{Esp}_r$$

preserves small colimits. In particular it preserves push-forwards.

8.5. The left exact structure. We denote by $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$ the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (Y, \mathfrak{E}_Y)$ of right exact 'spaces' such that $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(\mathfrak{f})$ belongs to the left exact structure $\mathfrak{I}^{\mathfrak{es}}$.

8.5.1. Observation. The class of morphisms $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$ is contained in the left exact structure $\mathfrak{I}^{\mathfrak{es}}$. The argument is an adaptation of 8.2.2.

8.5.2. Proposition. *(a) The class of morphisms $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$ is a left exact structure on the category $\mathfrak{Esp}_r$ of right exact 'spaces'.*

(b) The functor $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_r, \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}})$ to the left exact category $(\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{es}})$.

Proof. (a) Evidently, the class of morphisms $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$ contains all isomorphisms and is closed under composition. By 8.4.3, the functor $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}$ preserves (small colimits, in particular it preserves) push-forwards. This implies that the class of morphisms $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$ is stable under push-forwards; that is it forms a copretopology. Since, by 8.5.1, the copretopology $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$ is coarser than the left exact structure $\mathfrak{I}^{\mathfrak{es}}$, it is a left exact structure.

(b) By definition of the left exact structure $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$, the functor $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}$ maps $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$ to $\mathfrak{I}^{\mathfrak{es}}$. By 8.4.3, it preserves push-forwards; in particular, it preserves push-forwards of morphisms of $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$, hence the assertion. ■

8.5.3. Proposition. *The left exact category* $(\mathfrak{Esp}_r, \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}})$ *has enough injective objects.*

Proof. The argument below follows the pattern of that of 8.2.1(c).

By 8.5.2(b), $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_r, \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}})$ to the left exact category $(\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{es}})$; in particular, it maps inflations to inflations.

The latter implies that its right adjoint, $\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}$ maps injective objects to injective objects. The claim is that there are enough injective objects of the form $\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$, where $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ runs through injective objects of the left exact category $(\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{es}})$.

In fact, since the left exact category $(\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{es}})$ has enough injective objects, for any right exact 'space' $(X, \mathfrak{E}_X)$, there is a morphism $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ from the class $\mathfrak{I}^{\mathfrak{es}}$ with $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ an injective of $(\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{es}})$. By adjunction, $\mathfrak{f}$ determines a morphism

$$(X, \mathfrak{E}_X) \xrightarrow{\widehat{\mathfrak{f}}} \mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}),$$

which is the composition of the adjunction morphism

$$(X, \mathfrak{E}_X) \xrightarrow{\eta_X} \mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(X, \mathfrak{E}_X) \quad \text{and} \quad \mathcal{P}_{\mathfrak{D}}(\mathcal{P}^{\mathfrak{D}}(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})).$$

Applying $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}$ to this composition, we obtain a commutative square

$$\begin{array}{ccc} \mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(X, \mathfrak{E}_X) & \xrightarrow{\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(\eta_X} & \mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(X, \mathfrak{E}_X) \\ \mathfrak{f}\downarrow & & \downarrow \mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}(\mathfrak{f}) \\ (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}) & \xleftarrow{\epsilon_{\mathfrak{X}}} & \mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}) \end{array}$$

Since, by hypothesis, the morphism $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ belongs to the class $\mathfrak{I}^{\mathfrak{es}}$, it follows from this commutative square and the observation 2.4.3 that $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(\widehat{\mathfrak{f}}) \in \mathfrak{I}^{\mathfrak{es}}$. But, this means precisely that $\widehat{\mathfrak{f}} \in \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$. ■

8.5.4. Remark. The argument of 5.3.2 provides, for every right exact 'space' $(X, \mathfrak{E}_X)$ a canonical inflations (that is a morphism from $\mathfrak{I}^{\mathfrak{es}}$ of $(X, \mathfrak{E}_X)$ into an injective object $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ of the left exact category $(\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{es}})$. This canonical injective object is the product of $|\mathfrak{E}_X|$ copies (where $|\mathfrak{E}_X|$ is the set of isomorphism classes of deflations) of the elementary injective $(\mathfrak{S}, \mathfrak{E}_{\mathfrak{S}})$ and a number of copies of the elementary injective $\mathfrak{x}_1$ (see the argument of 5.3.2). These canonical (up to isomorphism) construction, for every right exact 'space' an inflation (that is a morphism from $\mathfrak{I}^{\mathfrak{es}}$) to an injective object of $(\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{es}})$ induces a canonical construction of a morphism from $\mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}}$ to an injective of the left exact category

$(\mathfrak{Esp}_r, \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}})$. Namely, given a canonical morphism $\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ into an injective object, we take (following the argument of 8.5.3) the composition of the adjunction morphism

$$(X, \mathfrak{E}_X) \xrightarrow{\eta_X} \mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}} \mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(X, \mathfrak{E}_X)$$

and the morphism

$$\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}} \mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}}(X, \mathfrak{E}_X) \xrightarrow{\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}(\mathfrak{f})} \mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}).$$

Notice that the functor $cP_{\mathfrak{D}}^{\mathfrak{c}}$, as every functor having a left adjoint, preserves small limits, in particular, it preserves small products. Therefore, the injective object $\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ is the product of $|\mathfrak{E}_X|$ copies of the (image of the) *elementary* injective $\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}(\mathfrak{S}, \mathfrak{E}_{\mathfrak{S}})$ and a the product of a set of copies of the elementary injective $\mathcal{P}_{\mathfrak{D}}^{\mathfrak{c}}(\mathfrak{x}_1)$ (we do not indicate deflations here, because they are all isomorphisms).

8.6. Functorial dependence on scheme of diagrams. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are diagram schemes and $\mathfrak{D}_1 \xrightarrow{\gamma} \mathfrak{D}_2$ is a surjective morphism, then the functor

$$(\mathfrak{Esp}_r, \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}_1}) \xrightarrow{\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}_1}} (\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{es}})$$

uniquely factors through the functor

$$(\mathfrak{Esp}_r, \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}_2}) \xrightarrow{\mathcal{P}_{\mathfrak{c}}^{\mathfrak{D}_2}} (\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{es}}).$$

In other words, the surjective morphism $\mathfrak{D}_1 \xrightarrow{\gamma} \mathfrak{D}_2$ induces an 'exact' functor

$$(\mathfrak{Esp}_r, \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}_1}) \xrightarrow{\gamma^*} (\mathfrak{Esp}_r, \mathfrak{I}_{\mathfrak{c}}^{\mathfrak{D}_2}).$$

8.7. Diagrams with finite support. Fix a diagram scheme $\mathfrak{D}$. We say that a diagram $\mathfrak{D} \xrightarrow{\mathcal{F}} C_X$ has a *finite support* if $\mathcal{F}(Hom\mathfrak{D})$ has only a finite number of non-identical arrows. For every category C_X, we denote by $C_{X_{\mathfrak{f}}^{\mathfrak{D}}}$ the full subcategory of the category $C_{X^{\mathfrak{D}}}$ whose objects are diagrams with a finite support. The embedding

$$C_{X_{\mathfrak{f}}^{\mathfrak{D}}} \xrightarrow{\gamma_{\mathfrak{f}}^{\mathfrak{D}}(X)^*} C_{X^{\mathfrak{D}}}$$

is interpreted as an inverse image functor of a morphism

$$X^{\mathfrak{D}} \xrightarrow{\gamma_{\mathfrak{f}}^{\mathfrak{D}}(X)} X_{\mathfrak{f}}^{\mathfrak{D}}.$$

The correspondence $X \longmapsto \gamma_{\mathfrak{f}}^{\mathfrak{D}}(X)$ is functorial; i.e. it defines a functor morphism

$$\mathcal{P}^{\mathfrak{D}} \xrightarrow{\gamma_{\mathfrak{f}}^{\mathfrak{D}}} \mathcal{P}_{\mathfrak{f}}^{\mathfrak{D}}.$$

The functor $\mathcal{P}_{\mathfrak{f}}^{\mathfrak{D}}$ induces an endofunctor $\mathcal{P}_{\mathfrak{c},\mathfrak{f}}^{\mathfrak{D}}$ of the category $\mathfrak{Esp}_{\mathfrak{r}}$.

9. Functors.

Fix a small category $\mathfrak{A}$. We denote the category of functors $\mathfrak{A} \longrightarrow C_X$ by $C_{X(\mathfrak{A})}$. If $(X, \mathfrak{E}_X)$ is a right exact category, we denote by $\mathfrak{E}_{X(\mathfrak{A})}$ the *standard* right exact structure on $C_{X(\mathfrak{A})}$ (– a functor morphism $\mathcal{F} \xrightarrow{\mathfrak{t}} \mathcal{G}$ is a deflation iff $\mathcal{F}(a) \xrightarrow{\mathfrak{t}(a)} \mathcal{G}(a)$ is a deflation for every $a \in Ob\mathfrak{A}$) and by $\mathfrak{E}_{X(\mathfrak{A})}^{\mathfrak{c}}$ the *cartesian* right exact structure, which is induced by the cartesian structure on the category of diagrams.

9.1. Proposition. *(a) The maps*

$$(X, \mathfrak{E}_X) \longmapsto (X(\mathfrak{A}), \mathfrak{E}_{X(\mathfrak{A})}) \quad \textit{and} \quad (X, \mathfrak{E}_X) \longmapsto (X(\mathfrak{A}), \mathfrak{E}_{X(\mathfrak{A})}^{\mathfrak{c}})$$

extend naturally to endofunctors, respectively

$$\mathfrak{Esp}_r \xrightarrow{\mathfrak{F}_{\mathfrak{r}}^{\mathfrak{A}}} \mathfrak{Esp}_r \quad \textit{and} \quad \mathfrak{Esp}_r \xrightarrow{\mathfrak{F}_{\mathfrak{c}}^{\mathfrak{A}}} \mathfrak{Esp}_r$$

of the category of right exact 'spaces'.

(b) The endofunctor $\mathfrak{Esp}_r \xrightarrow{\mathfrak{F}_{\mathfrak{c}}^{\mathfrak{A}}} \mathfrak{Esp}_r$ *has a right adjoint.*

Proof. (a) The assertion follows from 8.4.1.

(b) Let $\mathfrak{F}_{\mathfrak{A}}^{\mathfrak{c}}$ denote the functor which assigns to every right exact 'space' $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ the right exact 'space' $(\mathfrak{X}_{\mathfrak{A}}, \mathfrak{E}_{\mathfrak{X}_{\mathfrak{A}}})$, where $C_{\mathfrak{X}_{\mathfrak{A}}} = \mathfrak{A} \times C_X$ and $\mathfrak{E}_{\mathfrak{X}_{\mathfrak{A}}} = Iso(\mathfrak{A}) \times \mathfrak{E}_{\mathfrak{X}}$. If $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}) \xrightarrow{\mathfrak{f}} (\mathfrak{Y}, \mathfrak{E}_{\mathfrak{Y}})$ is a morphism of right exact 'spaces' with an inverse image functor $\mathfrak{f}^*$, then the functor $Id_{\mathfrak{A}} \times \mathfrak{f}^*$ is an inverse image functor of the morphism $\mathfrak{F}_{\mathfrak{A}}^{\mathfrak{c}}(\mathfrak{f})$.

The adjunction arrow

$$\mathfrak{F}_{\mathfrak{c}}^{\mathfrak{A}} \circ \mathfrak{F}_{\mathfrak{A}}^{\mathfrak{c}}(X) \xrightarrow{\epsilon_X} X$$

has a canonical inverse image functor

$$C_X \xrightarrow{\epsilon_X^*} Funct(\mathfrak{A}, \mathfrak{A} \times C_X)$$

which maps every object M of C_X to the functor $a \longmapsto a \times M$ and acts accordingly on morphisms. For every arrow $(a, M) \xrightarrow{(\xi, \mathfrak{t})} (b, L)$ of the category $\mathfrak{A} \times C_X$, the square

$$\begin{array}{ccc} a \times M & \xrightarrow{\xi \times M} & b \times M \\ {\scriptstyle a \times \mathfrak{t}} \downarrow & & \downarrow {\scriptstyle b \times \mathfrak{t}} \\ a \times L & \xrightarrow{\xi \times L} & b \times L \end{array}$$

is cartesian. Therefore, the functor ϵ_X^* maps morphisms of $\mathfrak{E}_X$ to morphisms of $\mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{A}}}$ and preserves pull-backs of deflations.

The second adjunction arrow

$$(X, \mathfrak{E}_X) \xrightarrow{\eta_X} \mathfrak{F}^{\mathfrak{c}}_{\mathfrak{A}} \circ \mathfrak{F}^{\mathfrak{A}}_{\mathfrak{c}}(X, \mathfrak{E}_X)$$

has as an inverse image functor the evaluation functor

$$\mathfrak{A} \times Funct(\mathfrak{A}, C_X) \xrightarrow{\eta_X^*} C_X, \quad (a, \mathcal{F}) \longmapsto \mathcal{F}(a).$$

It follows from the definition of deflations of $\mathfrak{A} \times Funct(\mathfrak{A}, C_X)$ that the functor η_X^* maps deflations (which are of the form $(\mathfrak{u}, \mathfrak{t}) \in Iso(\mathfrak{A} \times \mathfrak{E}^{\mathfrak{c}}_{X^{\mathfrak{A}}})$ to deflations and preserves pull-backs of deflations (see the corresponding part of the argument of 8.4.2). ∎

9.2. The left exact structure $\mathfrak{I}^{\mathfrak{F}_{\mathfrak{A}}}_{\mathfrak{c}}$. We denote by $\mathfrak{I}^{\mathfrak{F}_{\mathfrak{A}}}_{\mathfrak{c}}$ the class of all morphisms $\mathfrak{f}$ of right exact 'spaces' such that $\mathfrak{F}^{\mathfrak{A}}_{\mathfrak{c}}(\mathfrak{f}) \in \mathfrak{I}^{\mathfrak{es}}$.

It follows from (the argument of) 8.5.2 that $\mathfrak{F}^{\mathfrak{A}}_{\mathfrak{c}}(\mathfrak{f}) \in \mathfrak{I}^{\mathfrak{es}}$ is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' which is coarser than $\mathfrak{I}^{\mathfrak{es}}$.

9.3. Proposition. *The left exact category $(\mathfrak{Esp}_r, \mathfrak{I}^{\mathfrak{F}_{\mathfrak{A}}}_{\mathfrak{c}})$ has enough injective objects.*

Proof. The argument is similar to that of 8.5.3. ∎

9.4. Proposition. *Suppose that $\mathfrak{A}$ is a category with a final object, $\mathfrak{pt}$. For any 'space' X, let $\mathfrak{p}_X^{\mathfrak{A}*}$ denote the functor*

$$C_{X(\mathfrak{A})} = Funct(\mathfrak{A}, C_X) \longrightarrow C_X, \quad \mathcal{F} \longmapsto \mathcal{F}(\mathfrak{pt}).$$

(a) For every right exact 'space' $(X, \mathfrak{E}_X)$, the morphism of 'spaces' $X \xrightarrow{\mathfrak{p}_X^{\mathfrak{A}}} X(\mathfrak{A})$ is a morphism

$$(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{p}_X^{\mathfrak{A}}} (X(\mathfrak{A}), \mathfrak{E}^{\mathfrak{c}}_{X(\mathfrak{A})})$$

which belongs to $\mathfrak{I}^{\mathfrak{es}}$.

(b) The inverse image functor $\mathfrak{p}_X^{\mathfrak{A}*}$ establishes an equivalence between the category of deflations of every object $\mathcal{F}$ of the right exact category $(C_{X(\mathfrak{A})}, \mathfrak{E}^{\mathfrak{c}}_{X(\mathfrak{A})})$ and the category of deflations of the object $\mathfrak{p}_X^{\mathfrak{A}*}(\mathcal{F}) = \mathcal{F}(\mathfrak{pt})$ of the right exact category $(C_X, \mathfrak{E}_X)$.

(c) Projective objects (resp. pointed projective objects) of the right exact category $(X(\mathfrak{A}), \mathfrak{E}^{\mathfrak{c}}_{X(\mathfrak{A})})$ are all functors $\mathfrak{A} \xrightarrow{\mathcal{F}} C_X$ such that $\mathcal{F}(\mathfrak{pt})$ is a projective (resp. pointed projective) of the right exact category $(C_X, \mathfrak{E}_X)$.

Proof. (a) and (b) follows from (the argument) of 8.4.3.1 and (c) is a consequence of (the argument of) 8.3.4.2. ∎

10. 'Spaces' and right exact 'spaces'.

10.1. The functor $\mathfrak{J}^!$. The natural fully faithful functor

$$|Cat|^o \xrightarrow{\mathfrak{J}_*} \mathfrak{Esp}_{\mathfrak{r}}, \quad X \longmapsto (X, Iso(C_X)),$$

is a right adjoint to the forgetful functor

$$\mathfrak{Esp}_{\mathfrak{r}} \xrightarrow{\mathfrak{J}^*} |Cat|^o, \quad (X, \mathfrak{E}_X) \longmapsto X.$$

Notice that the functor $\mathfrak{J}_*$ has also a right adjoint functor,

$$\mathfrak{Esp}_{\mathfrak{r}} \xrightarrow{\mathfrak{J}^!} |Cat|^o, \quad (X, \mathfrak{E}_X) \longmapsto \mathfrak{E}_X^{-1} X,$$

which maps every right exact 'space' $(X, \mathfrak{E}_X)$ to its localization $\mathfrak{E}_X^{-1} X$ at the class of deflations. Since inverse image functor of every morphism of the category $\mathfrak{Esp}_{\mathfrak{r}}$ maps deflations to deflations, the map $(X, \mathfrak{E}_X) \longmapsto \mathfrak{E}_X^{-1} X$ extends to morphisms; hence it defines a functor which we denote by $\mathfrak{J}^!$. The inverse image functor of the adjunction arrow

$$\mathfrak{J}_*\mathfrak{J}^!(X, \mathfrak{E}_X) \xrightarrow{\epsilon^!_X} (X, \mathfrak{E}_X)$$

is (given by) the localization functor

$$(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}^{-1}_{\mathfrak{E}_X}} (\mathfrak{E}_X^{-1} C_X, Iso(\mathfrak{E}_X^{-1} C_X)).$$

The other adjunction arrow, $X \xrightarrow{\eta_X} \mathfrak{J}^!\mathfrak{J}_*(X)$, is an (identical) isomorphism.

Thus, the forgetful functor $\mathfrak{J}^*$ is a localization (because its right adjoint is fully faithful) and it can be regarded as an inverse image functor of an affine morphism.

10.1.1. Note. Thanks to the fact that the adjunction morphism $\mathfrak{J}^*\mathfrak{J}_* \xrightarrow{\epsilon_{\mathfrak{J}}} Id_{|Cat|^o}$ is an isomorphism, we have a canonical morphism $\mathfrak{J}^! \xrightarrow{\wp} \mathfrak{J}^*$ which is the composition of the isomorphism $\mathfrak{J}^! \xrightarrow{\mathfrak{J}^!\epsilon_{\mathfrak{J}}^{-1}} \mathfrak{J}^*\mathfrak{J}_*\mathfrak{J}^!$ and the adjunction arrow $\mathfrak{J}^*\mathfrak{J}_*\mathfrak{J}^! \xrightarrow{\mathfrak{J}^*\epsilon^!} \mathfrak{J}^*$.

10.2. The functor $\mathfrak{J}^!$ and correspondences. The category $C_{\mathfrak{E}_X^{-1}X} \stackrel{\mathrm{def}}{=} \mathfrak{E}_X^{-1} C_X$ representing the 'space' $\mathfrak{E}_X^{-1} X$ is described as follows. It has the same objects as the category C_X. Morphisms from an object M to an object L are equivalence classes of the pairs of arrows $M \xleftarrow{\mathfrak{t}} \mathfrak{M} \xrightarrow{\xi} L$, where $\mathfrak{t} \in \mathfrak{E}_X$, with respect to the following relation: $M \xleftarrow{\mathfrak{t}} \mathfrak{M} \xrightarrow{\xi} L$ is equivalent to $M \xleftarrow{\widetilde{\mathfrak{t}}} \widetilde{\mathfrak{M}} \xrightarrow{\widetilde{\xi}} L$, if there exists a deflation $L \xrightarrow{\mathfrak{s}} L'$

such that the compositions of $\mathfrak{s} \circ \xi$ and $\mathfrak{s} \circ \widetilde{\xi}$ with appropriate projections in the cartesian square

$$\begin{array}{ccccccc} & & \mathcal{M} & \xrightarrow{\mathfrak{t}_2} & \mathfrak{M} & \xrightarrow{\xi} & L \\ & & {\scriptstyle \mathfrak{t}_1}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}} & & \\ L & \xleftarrow{\widetilde{\xi}} & \widetilde{\mathfrak{M}} & \xrightarrow{\widetilde{\mathfrak{t}}} & M & & \end{array}$$

coincide, that is $\mathfrak{s} \circ \xi \circ \mathfrak{t}_2 = \mathfrak{s} \circ \widetilde{\xi} \circ \mathfrak{t}_1$. One can see that this condition does not depend on the choice of a cartesian square. The composition of morphisms $[M_1 \xleftarrow{\mathfrak{t}_1} \mathfrak{M}_1 \xrightarrow{\xi_1} M_2]$ and $[M_2 \xleftarrow{\mathfrak{t}_2} \mathfrak{M}_2 \xrightarrow{\xi_2} M_3]$ is defined via the diagram

$$\begin{array}{ccccccc} & & \mathfrak{M}_3 & \xrightarrow{\widetilde{\xi}_1} & \mathfrak{M}_2 & \xrightarrow{\xi_2} & M_3 \\ & & {\scriptstyle \mathfrak{t}'_2}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}_2} & & \\ M_1 & \xleftarrow{\mathfrak{t}_1} & \mathfrak{M}_1 & \xrightarrow{\xi_1} & M_2 & & \end{array}$$

with a cartesian square: the composition is the equivalence class

$$[M_1 \xleftarrow{\mathfrak{t}_1 \circ \mathfrak{t}'_2} \mathfrak{M}_3 \xrightarrow{\xi_2 \circ \widetilde{\xi}_1} M_3].$$

There is a canonical functor

$$C_X \xrightarrow{\mathfrak{q}^*_{\mathfrak{E}_X}} C_{\mathfrak{E}_X^{-1}X} \tag{1}$$

which is identical on objects and maps every morphism $M \xrightarrow{\gamma} L$ to the equivalence class of $M \xleftarrow{id_M} M \xrightarrow{\gamma} L$. It follows from the definition of the morphisms of the category $C_{\mathfrak{E}_X^{-1}X}$ that two arrows, $M \overset{\alpha}{\underset{\beta}{\rightrightarrows}} L$ have the same image in $C_{\mathfrak{E}_X^{-1}X}$ iff $\mathfrak{s} \circ \alpha = \mathfrak{s} \circ \beta$ for some deflations $\mathfrak{s}$. It follows from the definition of the composition and the equivalence classes, that the functor $\mathfrak{q}^*_{\mathfrak{E}_X}$ maps every deflation (– a morphism of $\mathfrak{E}_X$) to an invertible morphism; explicitly, if $M \xrightarrow{\mathfrak{t}} L$ is a deflation, then

$$\mathfrak{q}^*_{\mathfrak{E}_X}(\mathfrak{t})^{-1} = [L \xleftarrow{\mathfrak{t}} M \xrightarrow{id_M} M].$$

Thus, the canonical functor (1) uniquely factors through the localization functor

$$C_X \xrightarrow{\mathfrak{Q}^*_{\mathfrak{E}_X}} \mathfrak{E}_X^{-1} C_X.$$

That is $\mathfrak{q}^*_{\mathfrak{E}_X} = \mathfrak{p}^*_{\mathfrak{E}_X} \circ \mathfrak{Q}^*_{\mathfrak{E}_X}$ for a unique functor $\mathfrak{E}_X^{-1} C_X \xrightarrow{\mathfrak{p}^*_{\mathfrak{E}_X}} C_{\mathfrak{E}_X^{-1}X}$.

On the other hand, it follows from the description of the (standard) quotient category $\mathfrak{E}_X^{-1}C_X$ (cf. [GZ, I.1]) that there exists a natural functor $C_{\mathfrak{E}_X^{-1}X} \xrightarrow{\lambda_X^*} \mathfrak{E}_X^{-1}C_X$ which is quasi-inverse to the canonical functor $\mathfrak{E}_X^{-1}C_X \xrightarrow{\mathfrak{p}^*_{\mathfrak{E}_X}} C_{\mathfrak{E}_X^{-1}X}$.

10.3. The functors $\mathfrak{J}^*$, $\mathfrak{J}_*$, $\mathfrak{J}^!$ and the power functors. Fix a diagram scheme $\mathfrak{D}$. The functors $\mathfrak{J}_*$ and $\mathfrak{J}^*$ commute with the power functors, that is

$$\mathfrak{J}_* \circ \mathcal{P}^{\mathfrak{D}} = \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}} \circ \mathfrak{J}_* \quad \text{and} \quad \mathfrak{J}^* \circ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}} = \mathcal{P}^{\mathfrak{D}} \circ \mathfrak{J}^*.$$

The first equality implies an isomorphism $\mathcal{P}_{\mathfrak{D}} \circ \mathfrak{J}^! \simeq \mathfrak{J}^! \circ \mathcal{P}^{\mathfrak{c}}_{\mathfrak{D}}$ which, in turn, gives an isomorphism

$$\mathcal{P}^{\mathfrak{D}} \circ \mathcal{P}_{\mathfrak{D}} \circ \mathfrak{J}^! \circ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}} \simeq \mathcal{P}^{\mathfrak{D}} \circ \mathfrak{J}^! \circ \mathcal{P}^{\mathfrak{c}}_{\mathfrak{D}} \circ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}}.$$

Applying to the latter isomorphism the adjunction morphisms

$$Id_{\mathfrak{Esp}_{\mathfrak{r}}} \longrightarrow \mathcal{P}^{\mathfrak{c}}_{\mathfrak{D}} \circ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}} \quad \text{and} \quad \mathcal{P}^{\mathfrak{D}} \circ \mathcal{P}_{\mathfrak{D}} \longrightarrow Id_{|Cat|^o}$$

(from the right to the left), we obtain a canonical morphism

$$\mathcal{P}^{\mathfrak{D}} \circ \mathfrak{J}^! \xrightarrow{\lambda_{\mathfrak{D}}} \mathfrak{J}^! \circ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}}.$$

This morphism is compatible with the morphism $\mathfrak{J}^! \xrightarrow{\wp} \mathfrak{J}^*$ (cf. 10.1.1); that is we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{P}^{\mathfrak{D}} \circ \mathfrak{J}^! & \xrightarrow{\mathcal{P}^{\mathfrak{D}}\wp} & \mathcal{P}^{\mathfrak{D}} \circ \mathfrak{J}^* \\
{\scriptstyle \lambda_{\mathfrak{D}}}\downarrow & & \downarrow{\scriptstyle id} \\
\mathfrak{J}^! \circ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}} & \xrightarrow{\wp\mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}}} & \mathfrak{J}^* \circ \mathcal{P}^{\mathfrak{D}}_{\mathfrak{c}}
\end{array}$$

11. Complements.

11.1. The path 'space' of a right exact 'space'. Fix a right exact svelte category $(C_X, \mathfrak{E}_X)$. Let $C_{\mathfrak{X}}$ be the quotient of the category $C_{\mathfrak{Pa}(X)}$ of paths of the category C_X by the relations $\mathfrak{s} \circ \widetilde{\mathfrak{f}} = \mathfrak{f} \circ \mathfrak{t}$, where

$$\begin{array}{ccc}
\widetilde{N} & \xrightarrow{\widetilde{\mathfrak{f}}} & M \\
{\scriptstyle \mathfrak{t}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{s}} \\
N & \xrightarrow{\mathfrak{f}} & L
\end{array}$$

runs through cartesian squares in C_X whose vertical arrows belong to $\mathfrak{E}_X$. In particular, $ObC_{\mathfrak{X}} = ObC_X$. We denote by $\mathfrak{E}_{\mathfrak{X}}$ the image in $C_{\mathfrak{X}}$ of all paths of morphisms of $\mathfrak{E}_X$ and by $\mathfrak{Pa}(X, \mathfrak{E}_X)$ the pair $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$.

11.1.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category and $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}) = \mathfrak{Pa}(X, \mathfrak{E}_X)$ (see above).*

(a) The class of morphisms $\mathfrak{E}_{\mathfrak{X}}$ is a right exact structure on the category $C_{\mathfrak{X}}$.

(b) The canonical functor $C_{\mathfrak{Pa}(X)} \xrightarrow{\varepsilon_X^*} C_X$ (identical on objects and mapping paths of arrows to their composition) factors uniquely through a functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{p}_X^*} C_X$ which is an inverse image functor of a morphism $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{p}_X} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ that belongs to $\mathfrak{L}_{\mathfrak{sq}}^{\mathfrak{es}}$.

Proof. (a) It follows (from the fact that the composition of cartesian squares is a cartesian square) that $\mathfrak{E}_{\mathfrak{X}}$ is a right exact structure on $C_{\mathfrak{X}}$.

(b) The functor $C_{\mathfrak{Pa}(X)} \xrightarrow{\varepsilon_X^*} C_X$ is (equivalent to) a localization functor which factors uniquely through $C_{\mathfrak{X}} \xrightarrow{\mathfrak{p}_X^*} C_X$. Therefore, $\mathfrak{p}_X^*$ is (equivalent to) a localization functor. It follows from definitions that $\mathfrak{p}_X^*$ maps cartesian squares with deflations among their arrows to cartesian squares of the same type. Moreover, all cartesian squares with this property are obtained this way. Therefore, the morphism

$$(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{p}_X} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$$

of right exact 'spaces' belongs to the class $\mathfrak{L}_{\mathfrak{sq}}^{\mathfrak{es}}$. ■

11.2. The left exact structure $\mathfrak{L}_{\mathfrak{r}}^{\mathfrak{E}}$. Fix a right exact category $(C_X, \mathfrak{E}_X)$. We say that a class Σ of deflations is $\mathfrak{E}_X$*-saturated* if it is the intersection of a saturated system of arrows of C_X and $\mathfrak{E}_X$.

11.2.1. Lemma. *Let Σ be an $\mathfrak{E}_X$-saturated class of deflations. Then Σ is a right multiplicative system iff it is stable under base change.*

Proof. Let Σ be an $\mathfrak{E}_X$-saturated system of deflations. In particular, it contains all isomorphisms of C_X and is closed under compositions.

If Σ is stable under base change, it is a right multiplicative system.

Conversely, if Σ is a right multiplicative system, then, by [GZ, I.3.1], the localization functor $C_X \xrightarrow{\mathfrak{q}^*} \Sigma^{-1}C_X$ is right exact. In particular, it maps all cartesian squares of C_X to cartesian squares of $\Sigma^{-1}C_X$. Since $\mathfrak{E}_X$ is stable under base change, every diagram $M \xrightarrow{\mathfrak{s}} L \xrightarrow{f} N$ with $\mathfrak{s} \in \mathfrak{E}_X$ can be completed to a cartesian square

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\widetilde{f}} & M \\ \mathfrak{t} \downarrow & & \downarrow \mathfrak{s} \\ N & \xrightarrow{f} & L \end{array} \qquad (1)$$

and $\mathfrak{t} \in \mathfrak{E}_X$. If $\mathfrak{s} \in \Sigma$, then the localization $\mathfrak{q}^*$ maps (1) to a cartesian square whose right vertical arrow, $\mathfrak{q}^*(\mathfrak{s})$, is an isomorphism. Therefore its left vertical arrow is an isomorphism. Since Σ is $\mathfrak{E}_X$-saturated, this implies that $\mathfrak{t} \in \Sigma$. ■

We denote by $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$ the preorder (under the inclusion) of all $\mathfrak{E}_X$-saturated right multiplicative systems Σ of $\mathfrak{E}_X$ having the following property:

(#) If the right horizontal arrows in the commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{M}} & \overset{\mathfrak{p}'_1}{\underset{\mathfrak{p}'_2}{\rightrightarrows}} & \mathcal{M} & \xrightarrow{\tilde{\mathfrak{e}}} & \mathcal{L} \\
{\scriptstyle \tilde{\mathfrak{t}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{s}} & & \downarrow{\scriptstyle \mathfrak{s}'} \\
M \times_L M & \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & L
\end{array}$$

are deflations, the pairs of arrows are kernel pairs of these deflations and two left vertical arrows belong to Σ, then the remaining vertical arrow belongs to Σ.

11.2.2. Proposition. *(a) For any morphism $(Y, \mathfrak{E}_Y) \xrightarrow{\mathfrak{q}} (X, \mathfrak{E}_X)$ of the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces', the intersection $\Sigma_{\mathfrak{q}^*} \bigcap \mathfrak{E}_X = \{\mathfrak{t} \in \mathfrak{E}_X \mid \mathfrak{q}^*(\mathfrak{t})$ is invertible$\}$ belongs to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$.*

(b) For any $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$, the localization functor $C_X \xrightarrow{\mathfrak{q}^*} \Sigma^{-1}C_X = C_{\mathfrak{X}}$ is an inverse image functor of a morphism $(\mathfrak{X}, \mathfrak{E}^{\mathfrak{st}}_{\mathfrak{X}}) \xrightarrow{\mathfrak{q}} (X, \mathfrak{E}_X)$ of $\mathfrak{Esp}_{\mathfrak{r}}$. As usual, $\mathfrak{E}^{\mathfrak{st}}_{\mathfrak{X}}$ denote the finest right exact structure on $C_{\mathfrak{X}}$.

Proof. (a) By definition of morphisms of the category $\mathfrak{Esp}_{\mathfrak{r}}$, its inverse image functor maps pull-backs of deflations to pull-backs of deflations. Therefore the intersection $\Sigma_{\mathfrak{q}^*} \bigcap \mathfrak{E}_X = \{\mathfrak{t} \in \mathfrak{E}_X \mid \mathfrak{q}^*(\mathfrak{t})$ is invertible $\}$ is (by definition) saturated and stable under base change. The property (#) follows from the 'exactness' of the localization functor $\mathfrak{q}^*$.

(b) Let $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$. Since Σ is a right multiplicative system, the localization functor $C_X \xrightarrow{\mathfrak{q}^*} \Sigma^{-1}C_X = C_{\mathfrak{X}}$ is left exact. In particular, it maps all cartesian squares to cartesian squares. It remains to show that it maps deflations to strict epimorphisms.

Let $M \xrightarrow{\mathfrak{e}} L$ be a morphism of $\mathfrak{E}_X$ and $M \times_L M \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} M$ its kernel pair. Let $\mathfrak{q}^*(M) \xrightarrow{\xi'} \mathfrak{q}^*(N)$ be a morphism which equalizes the pair $\mathfrak{q}^*\big(M \times_L M \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} M\big)$. Since Σ is a right multiplicative system, the morphism ξ' is the composition $\mathfrak{q}^*(\xi)\mathfrak{q}^*(\mathfrak{s})^{-1}$ for some morphisms $M \xleftarrow{\mathfrak{s}} \mathcal{M} \xrightarrow{\xi} N$, where $\mathfrak{s} \in \Sigma$. Thus we have a diagram

$$\begin{array}{ccccc}
\mathcal{M}_1 & \xrightarrow{\mathfrak{u}_1} & M \times_L M & \xleftarrow{\mathfrak{u}_2} & \mathcal{M}_2 \\
{\scriptstyle \mathfrak{t}_1}\downarrow & cart & {\scriptstyle \mathfrak{p}_1}\downarrow\downarrow{\scriptstyle \mathfrak{p}_2} & cart & \downarrow{\scriptstyle \mathfrak{t}_2} \\
\mathcal{M} & \xrightarrow{\mathfrak{s}} & M & \xleftarrow{\mathfrak{s}} & \mathcal{M}
\end{array}$$

whose both squares are cartesian, all arrows are deflations, and all horizontal arrows belong to Σ. Therefore, there exists a cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{v}'_1} & \mathcal{M}_2 \\ {\scriptstyle \mathfrak{v}_2}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{u}_2} \\ \mathcal{M}_1 & \xrightarrow{\mathfrak{u}_1} & M \times_L M \end{array}$$

whose all arrows belong to Σ. Altogether leads to a commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \overset{\mathfrak{p}'_1}{\underset{\mathfrak{p}'_2}{\rightrightarrows}} & \mathcal{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \mathcal{L} \\ {\scriptstyle \widetilde{\mathfrak{t}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{s}} & & \downarrow{\scriptstyle \mathfrak{s}'} \\ M \times_L M & \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & L \end{array}$$

whose rows are exact diagrams and two (left) vertical arrows belong to Σ. Therefore, the remaining vertical arrow belongs to Σ. The localization functor $\mathfrak{q}^*$ maps the compositions $\xi \circ \mathfrak{p}'_1$ and $\xi \circ \mathfrak{p}'_2$ to the same arrow. This means precisely that there exists a morphism $\lambda \in \Sigma$ such that $\xi \circ \mathfrak{p}'_1 \circ \lambda = \xi \circ \mathfrak{p}'_2 \circ \lambda$ (cf. [GZ, I.2.2]). Since all morphisms of Σ are epimorphisms, the latter equality implies that the morphism ξ equalizes the pair $\widetilde{\mathcal{M}} \overset{\mathfrak{p}'_1}{\underset{\mathfrak{p}'_2}{\rightrightarrows}} \mathcal{M}$. Therefore, it factors uniquely through the morphism $\mathcal{M} \xrightarrow{\widetilde{\mathfrak{e}}} \mathcal{L}$; i.e. $\xi = \widetilde{\xi} \circ \widetilde{\mathfrak{e}}$. The pair of arrows $L \xleftarrow{\mathfrak{s}'} \mathcal{L} \xrightarrow{\widetilde{\xi}} N$ determines a unique morphism $\mathfrak{q}^*(L) \longrightarrow \mathfrak{q}^*(N)$ whose composition with $q^*(\mathfrak{e})$ equals to ξ'. ■

We denote by $\mathfrak{L}^{\mathfrak{E}}_{\mathfrak{r}}$ the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ whose inverse image functor is equivalent to the localization functor at a system which belongs to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$.

11.2.3. Proposition. *The class of morphisms* $\mathfrak{L}^{\mathfrak{E}}_{\mathfrak{r}}$ *is a left exact structure on the category* $\mathfrak{Esp}_{\mathfrak{r}}$ *of right exact 'spaces'.*

Proof. The class of morphisms $\mathfrak{L}^{\mathfrak{E}}_{\mathfrak{r}}$ contains all isomorphisms and is closed under compositions and cobase change. ■

Chapter V

K-Theory of Right Exact 'Spaces'.

In Section 1, we assign to each right exact 'space' $(X, \mathfrak{E}_X)$ an abelian group $K_0(X, \mathfrak{E}_X)$. This assignment is a contravariant functor from the category of right exact 'spaces' and their morphisms (having 'exact' inverse image functors) to the category $\mathbb{Z}-mod$ of abelian groups. We observe that the map $(X, \mathfrak{E}_X) \longmapsto K_0(X, \mathfrak{E}_X)$ is functorial with respect to *weakly 'exact'* morphisms between right exact 'spaces' represented by categories with initial objects, i.e. morphisms whose inverse image functors preserve conflations.

In Section 2, we show that the functor K_0 is right 'exact' with respect to the left exact structure $\mathfrak{I}_{\mathfrak{c}}^{\rightarrow}$ – one of the *canonical* left exact structures introduced in Chapter IV.

Notice that the category of right exact 'spaces' does not have final objects; so that we cannot apply the formalism of cohomological functors developed in Chapters II and III. An obvious way to acquire final objects is to consider the category of right exact 'spaces' over a 'space'. We do this in Section 3, introducing the *relative* K_0-functors and their derived functors with respect to a left exact structure on the category of right exact 'spaces' over a right exact 'space'. It follows from the results of Section 2 that the relative K_0 functors are right 'exact' with respect to the left exact structures induced by $\mathfrak{I}_{\mathfrak{c}}^{\rightarrow}$.

In Section 4, we specialize results and constructions of Section 3 to the 'spaces' over a 'point', which is the subcategory $\mathfrak{Esp}_{\mathfrak{r}}^*$ of right exact spaces with initial objects and 'exact' morphisms whose inverse image functors map initial objects to initial objects. Notice that this category is pointed having a canonical zero object x – the 'point', which is represented by the right exact 'space' with one morphism. The 'space' x is interpreted as the affine scheme associated with the "field" $\mathbb{F}_1$. So that $\mathfrak{Esp}_{\mathfrak{r}}^*$ can be regarded as the category of right exact 'spaces' over $\mathbb{F}_1$. It is endowed by the left exact structure $\mathfrak{I}_{\mathfrak{c}*}^{\rightarrow}$ induced by $\mathfrak{I}_{\mathfrak{c}}^{\rightarrow}$.

It is important to realize that the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{I}_{\mathfrak{c}*}^{\rightarrow})$ is not the ultimate domain for a higher K-theory. On the contrary, it serves as a device for producing higher K-theories on other left exact categories. Namely, every functor from a left exact category $(C_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ (having final objects) to $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{I}_{\mathfrak{c}*}^{\rightarrow})$ which preserves conflations gives rise to an 'exact' higher K-theory on the left exact category $(C_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$. In Section 5, we apply this consideration to the natural functor from the left exact category of 'spaces' represented by svelte abelian categories endowed with the left exact structure formed by exact localizations, obtaining this way a universal K-theory of ('spaces' represented by) svelte abelian categories. In Section 6, we construct the universal K-theory of k-linear exact categories via the obvious functor from a left exact category $(\mathfrak{Esp}_k^{\mathfrak{e}}, \mathfrak{I}_k^{\mathfrak{e}})$ of 'spaces' represented by k-linear exact categories to $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{I}_{\mathfrak{c}*}^{\rightarrow})$.

The purpose of the following sections is creation the standard tools of higher K-theory which generalize the corresponding facts of Quillen's K-theory. Notice that the most difficult general theorem of Quilllen's K-theory – the long exact sequence of an exact localization of an abelian category, is obtained in our approach almost for free and for all right exact categories. Curiously, *devissage*, which is easily established in Quillen's theory, requires preparation which takes a considerable part of Chapter VI. Of course, our devissage holds in a much more general setting. The remaining techniques – reduction by resolution and characteristic filtrations and sequences, appear here: Section 7 is dedicated to "reduction by resolution", Section 8 treats characteristic 'exact' filtrations and sequences. Section 9 contains an analog of Quillen's Q-construction for right exact categories with initial objects.

1. The functor K_0.

1.1. The group $\mathbb{Z}_0|C_X|$. For a svelte category C_X, we denote by $|C_X|$ the set of isomorphism classes of objects of C_X, by $\mathbb{Z}|C_X|$ the free abelian group generated by $|C_X|$, and by $\mathbb{Z}_0(C_X)$ the subgroup of $\mathbb{Z}|C_X|$ generated by differences $[M]-[N]$ for all arrows $M \longrightarrow N$ of the category C_X. Here $[M]$ denotes the isomorphism class of an object M.

1.2. Proposition. *(a) The maps $X \longmapsto \mathbb{Z}|C_X|$ and $X \longmapsto \mathbb{Z}_0(C_X)$ extend naturally to presheaves of $\mathbb{Z}$-modules on the category of 'spaces' $|Cat|^o$ (i.e. to functors from $(|Cat|^o)^{op}$ to $\mathbb{Z}-mod$).*

(b) If the category C_X has an initial (resp. final) object x, then $\mathbb{Z}_0(C_X)$ is the subgroup of $\mathbb{Z}|C_X|$ generated by differences $[M]-[x]$, where $[M]$ runs through the set $|C_X|$ of isomorphism classes of objects of C_X.

Proof. The argument is left to the reader. ∎

1.3. Remarks. (a) Evidently, there are natural isomorphisms $\mathbb{Z}|C_X| \simeq \mathbb{Z}|C_X^{op}|$ and $\mathbb{Z}_0(C_X) \simeq \mathbb{Z}_0(C_X^{op})$.

(b) Let $\mathbb{Z}_0(C_X)$ be regarded as a groupoid with one object, $\bullet$. Then the map which assigns to every object of C_X the object $\bullet$ and to any morphism $M \longrightarrow N$ of C_X the difference $[M]-[N]$ is a functor from C_X to the groupoid $\mathbb{Z}_0(C_X)$.

1.4. The group K_0 of a right exact 'space'. Let $(X, \mathfrak{E}_X)$ be a right exact 'space'. We denote by $K_0(X, \mathfrak{E}_X)$ the quotient of the group $\mathbb{Z}_0|C_X|$ by the subgroup generated by the expressions $[M']-[M]+[L]-[L']$ for all cartesian squares

$$\begin{array}{ccc} M' & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ L' & \xrightarrow{f} & L \end{array}$$

whose vertical arrows are deflations.

We call $K_0(X, \mathfrak{E}_X)$ the *group* K_0 of the right exact 'space' $(X, \mathfrak{E}_X)$.

1.4.1. Example: the group K_0 of a 'space'. Any 'space' X is identified with the *trivial* right exact 'space' $(X, Iso(C_X))$. We set $K_0(X) = K_0(X, Iso(C_X))$. That is $K_0(X)$ coincides with the group $\mathbb{Z}_0(C_X)$.

1.5. Proposition. *(a) The map $(X, \mathfrak{E}_X) \longmapsto K_0(X, \mathfrak{E}_X)$ extends to a contravariant functor, K_0, from the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces' (cf. 6.8) to the category $\mathbb{Z}-mod$ of abelian groups.*

(b) Let $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{Esp}_\mathfrak{r}$ having the following property:

(†) *if M' and L' are non-isomorphic objects of C_X which can be connected by non-oriented sequence of arrows (i.e. they belong to one connected component of the associated groupoid), then there exist objects M and L of C_Y which have the same property and such that $f^*(M) \simeq M'$, $f^*(L) \simeq L'$.*
Then

$$K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(f)} K_0(X, \mathfrak{E}_X)$$

is a group epimorphism.

Proof. (a) Let $(X, \mathfrak{E}_X)$ and $(Y, \mathfrak{E}_Y)$ be right exact 'spaces' and $(C_Y, \mathfrak{E}_Y) \xrightarrow{f^*} (C_X, \mathfrak{E}_X)$ an 'exact' functor. Then f^* induces a morphism

$$K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(f)} K_0(X, \mathfrak{E}_X)$$

uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{Z}_0(C_Y) & \xrightarrow{\mathbb{Z}_0(f^*)} & \mathbb{Z}_0(C_X) \\ p_Y \downarrow & & \downarrow p_X \\ K_0(Y, \mathfrak{E}_Y) & \xrightarrow{K_0(f)} & K_0(X, \mathfrak{E}_X) \end{array} \qquad (1)$$

of $\mathbb{Z}$-modules. Here $\mathbb{Z}_0(f^*)$ denotes the morphism of abelian groups induced by the functor f^*. The vertical arrows, p_Y and p_X, are natural epimorphisms.

(b) Suppose that $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ is a morphism of $\mathfrak{Esp}_\mathfrak{r}$ having the property (†). Then $\mathbb{Z}_0(C_Y) \xrightarrow{\mathbb{Z}_0(f^*)} \mathbb{Z}_0(C_X)$ is a group epimorphism. Thus, $K_0(f) \circ p_Y = p_X \circ \mathbb{Z}_0(f^*)$ is an epimorphism, which implies that $K_0(f)$ is an epimorphism. ■

1.5.1. Corollary. *Let $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{Esp}_\mathfrak{r}$ whose inverse image functor, f^*, induces a surjective map $|C_Y| \longrightarrow |C_X|$ of isomorphism classes of objects. If the groupoid associated with the category C_Y is connected, then $K_0(f)$ is a surjective map. In particular, $K_0(f)$ is surjective if the category C_Y has initial or final objects.*

Proof. The assertion follows from 1.5(b). ∎

1.5.2. Corollary. *For any 'exact' localization* $(X,\mathfrak{E}_X)\xrightarrow{\mathfrak{q}}(Y,\mathfrak{E}_Y)$ *(i.e.* $\mathfrak{q}^*$ *is equivalent to a localization functor), the map* $K_0(\mathfrak{q})$ *is an epimorphism.*

Proof. If $\mathfrak{q}^*$ is equivalent to a localization functor, then each object of C_X is isomorphic to an object of $\mathfrak{q}^*(C_Y)$ and any morphism $\mathfrak{q}^*(M)\longrightarrow\mathfrak{q}^*(L)$ is the composition of the form $\mathfrak{q}^*(\mathfrak{s}_n)^{-1}\circ\mathfrak{q}^*(f_n)\circ\ \cdots\ \circ\mathfrak{q}^*(\mathfrak{s}_1)^{-1}\circ\mathfrak{q}^*(f_1)$ for some chain of arrows

$$M\xrightarrow{f_1}\widetilde{M}_1\xleftarrow{\mathfrak{s}_1}M_1\xrightarrow{f_1}\ \ldots\ \xrightarrow{f_n}\widetilde{M}_n\xleftarrow{\mathfrak{s}_n}M_n=L.$$

So that the condition (†) of 1.5(b) holds. ∎

1.5.3. Corollary. *For every morphism* $(X,\mathfrak{E}_X)\xrightarrow{\mathfrak{f}}(Y,\mathfrak{E}_Y)$ *such that* $X\xrightarrow{\mathfrak{f}}Y$ *belongs to the class* $\mathfrak{I}^{\mathfrak{s}}$ *the map* $K_0(\mathfrak{f})$ *is an epimorphism. In particular,* $K_0(\mathfrak{f})$ *is an epimorphism for every* $\mathfrak{f}\in\mathfrak{I}^{\mathfrak{es}}$*.*

Proof. Since $\mathfrak{f}\in\mathfrak{I}^{\mathfrak{s}}$, its inverse image functor is essentially surjective on arrows. This implies that the map $\mathbb{Z}_0(C_Y)\xrightarrow{\mathbb{Z}_0(\mathfrak{f})}\mathbb{Z}_0(C_X)$ is surjective. Therefore, it follows from the diagram (1) that $K_0(Y,\mathfrak{E}_Y)\xrightarrow{K_0(\mathfrak{f})}K_0(X,\mathfrak{E}_X)$ is surjective. ∎

1.6. Proposition. *Let* $(X,\mathfrak{E}_X)$ *be a right exact 'space' such that the category* C_X *has initial objects. Then the group* $K_0(X,\mathfrak{E}_X)$ *is the quotient of the free abelian group* $\mathbb{Z}|C_X|$ *generated by the isomorphism classes of objects of* C_X *by the subgroup generated by* $[M]-[L]-[N]$ *for all conflations* $N\longrightarrow M\longrightarrow L$ *and the isomorphism class of initial objects of* C_X*.*

Proof. (a) The expressions $[M]-[L]-[N]$, where $N\xrightarrow{\mathfrak{k}}M\xrightarrow{\mathfrak{e}}L$ runs through conflations of $(C_X,\mathfrak{E}_X)$, are among the relations because each of them corresponds to a cartesian square

$$\begin{array}{ccc} N & \longrightarrow & x \\ {\scriptstyle\mathfrak{k}}\downarrow & cart & \downarrow \\ M & \xrightarrow{\mathfrak{e}} & L \end{array}$$

where x is an initial object of C_X.

(b) On the other hand, let

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{L} \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\mathfrak{e}} & L \end{array} \qquad (1)$$

be a cartesian square whose horizontal arrows are deflations. Therefore we have a commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{\widetilde{\mathfrak{k}}} & \widetilde{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{L} \\ {\scriptstyle id}\downarrow & & {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ N & \xrightarrow{\mathfrak{k}} & M & \xrightarrow{\mathfrak{e}} & L \end{array}$$

whose rows are conflations. The rows give relations $[\widetilde{M}] - [\widetilde{L}] - [N]$ and $[M] - [L] - [N]$. Their difference, $[\widetilde{M}] - [M] + [L] - [\widetilde{L}]$, is the relation corresponding to the cartesian square (1). Hence the assertion. ■

1.6.1. A curious observation. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and $M \xrightarrow{\mathfrak{t}} L$ a deflation. Let $Ker_2(\mathfrak{t})$ denote the *kernel pair* of the morphism $\mathfrak{t}$. It follows from the cartesian square

$$\begin{array}{ccc} Ker_2(\mathfrak{t}) & \xrightarrow{p_1} & M \\ {\scriptstyle p_2}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}} \\ M & \xrightarrow{\mathfrak{t}} & L \end{array}$$

that, in the group $K_0(X, \mathfrak{E}_X)$, we have:

$$[Ker_2(\mathfrak{t})] - [M] = [M] - [L]. \tag{2}$$

On the other hand, if the category C_X has initial objects, then $[M]-[L] = [Ker(\mathfrak{t})]$ (see 1.6). So that, in this case, the equality (2) can be rewritten as $[Ker_2(\mathfrak{t})]-[M] = [Ker(\mathfrak{t})]$, or

$$[Ker_2(\mathfrak{t})] = [M] + [Ker(\mathfrak{t})]. \tag{3}$$

If the category C_X is additive, then $Ker_2(\mathfrak{t})$ is naturally isomorphic to the coproduct of M and $Ker(\mathfrak{t})$ (cf. I.4.3.2(b)), which, of course, implies the equality (3).

Curiously, the equality (3) holds without additivity hypothesis, and, if the difference $[M] - [L]$ between the isomorphisms classes of the source and the target of a deflation $\mathfrak{t}$ is interpreted as the isomorphism class of its kernel, the formula (3) acquires sense for an arbitrary right exact 'space' $(X, \mathfrak{E}_X)$.

1.7. The categories $\mathfrak{Esp}_{\mathfrak{r}}^{w}$ and $\mathfrak{Esp}_{\mathfrak{r}}^{*}$. Let $\mathfrak{Esp}_{\mathfrak{r}}^{w}$ denote the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ such that C_X has initial objects; and morphisms $(X, \mathfrak{E}_X) \longrightarrow (Y, \mathfrak{E}_Y)$ are given by morphisms of 'spaces' $X \xrightarrow{f} Y$ whose inverse image functors preserve conflations. In particular, they map initial objects to initial objects.

We denote by $\mathfrak{Esp}_{\mathfrak{r}}^{*}$ the subcategory of $\mathfrak{Esp}_{\mathfrak{r}}$ whose objects are right exact 'spaces' $(C_X, \mathfrak{E}_X)$ such that the category C_X has initial objects and morphisms are defined by the requirement that their inverse image functor maps initial objects to initial objects.

It follows that $\mathfrak{Esp}_{\mathfrak{r}}^*$ is a subcategory of the category $\mathfrak{Esp}_{\mathfrak{r}}^w$. The k-linear versions of these categories coincide.

1.8. Proposition. *(a) The map* $(X, \mathfrak{E}_X) \longmapsto K_0(X, \mathfrak{E}_X)$ *extends to a contravariant functor,* K_0^w*, from the category* $\mathfrak{Esp}_{\mathfrak{r}}^w$ *to the category* $\mathbb{Z} - mod$ *of abelian groups.*

(b) Let $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ *be a morphism of* $\mathfrak{Esp}_{\mathfrak{r}}^w$ *such that* f^* *induces a surjective map* $|C_Y| \longrightarrow |C_X|$ *of the isomorphism classes of objects. Then*

$$K_0^w(Y, \mathfrak{E}_Y) \xrightarrow{K_0^w(f)} K_0^w(X, \mathfrak{E}_X)$$

is a group epimorphism. In particular, the functor K_0 *maps 'exact' localizations to epimorphisms.*

Proof. The assertions follow from 1.6. ■

2. Exactness properties of the functor K_0.

One of the inconveniences of the functor K_0 is that it assigns to a right exact 'space' an object of different nature – a group. This break of continuity creates difficulties in studying its properties. Below, we correct the problem replacing K_0 by its intermediate categorical version – a groupoid $\widetilde{\mathcal{K}}_0$. A smooth transition to the definition and study of $\widetilde{\mathcal{K}}_0$ involves some other functors on 'spaces' and right exact 'spaces' which are important on their own right.

2.1. 'Spaces' and associated groupoids. The functor $\mathfrak{Gr}^*$. Let $|\mathfrak{Gr}|^o$ denote the full subcategory of $|Cat|^o$ generated by 'spaces' represented by groupoids. We denote by $\mathfrak{Gr}^*(X)$ the 'space' represented by the groupoid obtained by inverting all arrows of the category C_X. The map $X \longmapsto \mathfrak{Gr}^*(X)$ extends to a functor

$$|Cat|^o \xrightarrow{\mathfrak{Gr}^*} |\mathfrak{Gr}|^o. \qquad (1)$$

The functor $\mathfrak{Gr}^*$ is left adjoint to the natural full embedding

$$|\mathfrak{Gr}|^o \xrightarrow{\mathfrak{Gr}_*} |Cat|^o. \qquad (2)$$

So that $\mathfrak{Gr}_*$ is a continuous localization functor. Actually, it is not only continuous, but affine, because the functor (2) has a right adjoint

$$|Cat|^o \xrightarrow{\mathfrak{Gr}^!} |\mathfrak{Gr}|^o. \qquad (3)$$

which maps each 'space' X to the 'space' $\mathfrak{Gr}^!(X)$ represented by the subcategory of C_X with the same objects whose arrows are all isomorphisms of the category C_X.

2.1.1. Left exact structures. The canonical left exact structure $\mathfrak{I}^{\mathfrak{s}}$ on the category of 'spaces' $|Cat|^o$ induces a left exact structure, $\mathfrak{I}^{\mathfrak{s}}_{\mathfrak{Gr}}$, on the category $|\mathfrak{Gr}|^o$. It follows that $\mathfrak{Gr}^*$ and $\mathfrak{Gr}_*$ define 'exact' functors between the corresponding left exact categories:

$$(|Cat|^o, \mathfrak{I}^{\mathfrak{s}}) \xrightarrow{\mathfrak{Gr}^*} (|\mathfrak{Gr}|^o, \mathfrak{I}^{\mathfrak{s}}_{\mathfrak{Gr}}) \xrightarrow{\mathfrak{Gr}_*} (|Cat|^o, \mathfrak{I}^{\mathfrak{s}}). \tag{4}$$

In fact, both functors map inflations to inflations, and both preserved small colimits (as all functors having a right adjoint). In particular, they preserve push-forwards and map push-forwards of inflations to push-forwards of inflations.

2.2. Preorders. The functor $\mathfrak{S}\mathfrak{k}^*$. Let $|\mathfrak{Ord}|^o$ denote the full subcategory of $|Cat|^o$ generated by 'spaces' represented by preorders. For an arbitrary 'space' X, we denote by $\mathfrak{S}\mathfrak{k}^*(X)$ the 'space' represented by the preorder whose objects are isomorphism classes of objects of the category C_X and an arrow from a class $|M|$ to a class $|L|$ exists iff there are arrows from M to L. The map $X \longmapsto \mathfrak{S}\mathfrak{k}^*(X)$ extends naturally to a functor

$$|Cat|^o \xrightarrow{\mathfrak{S}\mathfrak{k}^*} |\mathfrak{Ord}|^o. \tag{1}$$

The functor $\mathfrak{S}\mathfrak{k}^*$ is left adjoint to the full embedding

$$|\mathfrak{Ord}|^o \xrightarrow{\mathfrak{S}\mathfrak{k}_*} |Cat|^o.$$

2.2.1. Left exact structures on 'spaces' represented by preorders. We endow the category $|\mathfrak{Ord}|^o$ with a left exact structure, $\mathfrak{I}^{\mathfrak{s}}_{\mathfrak{or}}$ induced by $\mathfrak{I}^{\mathfrak{s}}$: a morphism $X \xrightarrow{\mathfrak{f}} Y$ of $|\mathfrak{Ord}|^o$ belongs to $\mathfrak{I}^{\mathfrak{s}}_{\mathfrak{or}}$ iff its inverse image functor is a surjective morphism of preorders.

It follows that both functor $\mathfrak{S}\mathfrak{k}^*$ and $\mathfrak{S}\mathfrak{k}_*$ map inflations to inflations and the functor $\mathfrak{S}\mathfrak{k}^*$ defines an 'exact' functor

$$(|Cat|^o, \mathfrak{I}^{\mathfrak{s}}) \xrightarrow{\mathfrak{Gr}^*} (|\mathfrak{Ord}|^o, \mathfrak{I}^{\mathfrak{s}}_{\mathfrak{or}}).$$

2.2.2. Preorder-groupoids. The functor $\mathfrak{Gs}^*$. Let $|\mathfrak{Gsk}|^o$ denote the full subcategory of the category $|\mathfrak{Ord}|^o$ formed by 'spaces' represented by preorders whose all arrows are invertible. We call such 'spaces' *preorder-groupoids*.

The composition $\mathfrak{Gr}^* \circ \mathfrak{S}\mathfrak{k}_* \circ \mathfrak{S}\mathfrak{k}^*$ induces a functor from $|Cat|^o$ to $|\mathfrak{Gsk}|^o$. We denote this functor by $\mathfrak{Gs}^*$. The functor $\mathfrak{Gs}^*$ is left adjoint to the full embedding

$$|\mathfrak{Gsk}|^o \xrightarrow{\mathfrak{Gs}_*} |Cat|^o.$$

2.3

2.3. The 'spaces' of deflations. Fix a right exact 'space' $(X, \mathfrak{E}_X)$. Let $C_{X^{\mathfrak{c}}_{\mathfrak{E}}}$ denote the subcategory of the category $C_{X^2} = C_X^2$ of arrows whose objects are deflations of $(X, \mathfrak{E}_X)$ (that is arrows of $\mathfrak{E}_X$) and morphisms from a deflation $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{L}$ to a deflation $\mathcal{M}' \xrightarrow{\mathfrak{t}'} \mathcal{L}'$ is a cartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & \mathcal{M}' \\ \mathfrak{t} \downarrow & cart & \downarrow \mathfrak{t}' \\ \mathcal{L} & \xrightarrow{g} & \mathcal{L}' \end{array}$$

The "source" and "target" functors $C_{X^2} \overset{\mathfrak{s}}{\underset{\mathfrak{t}}{\rightrightarrows}} C_X$ induce a pair of functors

$$C_{X^{\mathfrak{c}}_{\mathfrak{E}}} \overset{\mathfrak{p}^*_{\mathfrak{s}}}{\underset{\mathfrak{p}^!_{\mathfrak{t}}}{\rightrightarrows}} C_X$$

whose cokernel we denote by $C_{X/\mathfrak{E}}$.

Notice that the functor $\mathfrak{p}^*_{\mathfrak{s}}$ is left adjoint and the functor $\mathfrak{p}^!_{\mathfrak{t}}$ is right adjoint to the fully faithful functor $C_X \xrightarrow{\mathfrak{j}^{\mathfrak{c}}_X} C_{X^{\mathfrak{c}}_{\mathfrak{E}}}$ which assigns to every object of C_X its identical morphism and acts correspondingly on arrows.

2.3.1. Functorialities. The map $(X, \mathfrak{E}_X) \longmapsto X^{\mathfrak{c}}_{\mathfrak{E}}$ extends to a functor

$$\mathfrak{Esp}_{\mathfrak{r}} \xrightarrow{\mathfrak{F}_{\mathfrak{c}}} |Cat|^o$$

because inverse image of any morphism of right exact 'spaces' preserves deflations and pull-backs of deflations.

2.3.2. Proposition. *The functor* $\mathfrak{Esp}_{\mathfrak{r}} \xrightarrow{\mathfrak{F}_{\mathfrak{c}}} |Cat|^o$ *is an 'exact' functor from the left exact category* $(\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{I}^{\rightarrow}_{\mathfrak{c}})$ *to the left exact category* $(|Cat|^o, \mathfrak{I}^{\rightarrow})$.

Proof. We lift the functor $\mathfrak{F}_{\mathfrak{c}}$ to a quotient $\widetilde{\mathfrak{F}}_{\mathfrak{c}}$ of the functor $\mathcal{P}^{\rightarrow}_{\mathfrak{c}}$. ∎

2.4. The functor $\widetilde{\mathcal{K}}_0$.

2.4.1. The groupoid $\widetilde{\mathcal{K}}_0$ of a right exact 'space'. Combining with functors introduced in 2.2, we obtain a commutative diagram

$$\begin{array}{ccccc}
C_{X^{\mathfrak{c}}_{\mathfrak{E}}} & \overset{\mathfrak{p}_1^*}{\underset{\mathfrak{p}_2^*}{\rightrightarrows}} & C_X & \xrightarrow{\pi^*_{X/\mathfrak{E}}} & C_{X/\mathfrak{E}} \\
\downarrow & & \downarrow & & \downarrow \\
C_{\mathfrak{S}\mathfrak{k}^*(X^{\mathfrak{c}}_{\mathfrak{E}})} & \overset{\mathfrak{p}_1^*}{\underset{\mathfrak{p}_2^*}{\rightrightarrows}} & C_{\mathfrak{S}\mathfrak{k}^*(X)} & \xrightarrow{\pi^*_{X/\mathfrak{E}}} & C_{\mathfrak{S}\mathfrak{k}^*(X/\mathfrak{E})} \\
\downarrow & & \downarrow & & \downarrow \\
C_{\mathfrak{G}\mathfrak{s}^*(X^{\mathfrak{c}}_{\mathfrak{E}})} & \overset{\mathfrak{p}_1^*}{\underset{\mathfrak{p}_2^*}{\rightrightarrows}} & C_{\mathfrak{G}\mathfrak{s}^*(X)} & \xrightarrow{\pi^*_{X/\mathfrak{E}}} & C_{\mathfrak{G}\mathfrak{s}^*(X/\mathfrak{E})}
\end{array} \qquad (1)$$

with exact rows.

We denote the preorder-groupoid $C_{\mathfrak{G}\mathfrak{s}^*(X/\mathfrak{E})}$ in the right lower corner of the diagram (1) by $\widetilde{\mathcal{K}}_0(X, \mathfrak{E}_X)$ and call it the $\widetilde{\mathcal{K}}_0$ *groupoid* of the right exact 'space' $(X, \mathfrak{E}_X)$.

2.4.2. 'Exactness' properties of $\widetilde{\mathcal{K}}_0$. Notice that the whole diagram (1) consists of canonical functors and (therefore) is functorial in $(X, \mathfrak{E}_X)$.

2.4.2.1. Proposition. *The functor $\widetilde{\mathcal{K}}_0$ is right 'exact' with respect to the left exact structure $\mathfrak{J}^{\to}_{\mathfrak{c}}$.*

Proof. The fact is that the whole diagram (1) is a right 'exact' functor in $(X, \mathfrak{E}_X)$ with respect to the left exact structure $\mathfrak{J}^{\to}_{\mathfrak{c}}$. ∎

2.4.2.2. Proposition. *The functor K_0 is right 'exact' with respect to the left exact structure $\mathfrak{J}^{\to}_{\mathfrak{c}}$.*

Proof. We have a contravariant functor $\mathbb{Z}_0$ from the category of 'spaces' to the category of commutative groupoids with one object, which assigns to each 'space' X the subgroupoid of $\mathbb{Z}|C_X|$ generated by differences $[M]-[L]$ for every morphism $L \longrightarrow M$. There is a natural functor $C_X \longrightarrow \mathbb{Z}_0(X)$ which assigns to every morphism $L \longrightarrow M$ the difference $[M]-[L]$.

The functor $\mathbb{Z}_0$ respects coproducts, that is $\mathbb{Z}_0(\coprod_{i\in J} X_i) \simeq \prod_{i\in J} \mathbb{Z}_0(X_i)$. In general, $\mathbb{Z}_0$ does not preserve cokernels of pairs of arrows. Fortunately, its restriction to preorders (or preorder-groupoids) does. Notice that $\mathbb{Z}_0 \circ \widetilde{\mathcal{K}}_0 = K_0$. Applying $\mathbb{Z}_0$ to the last row of the diagram (1), we obtain an exact sequence of groups (identified with groupoids with one object) and group morphisms

$$\mathbb{Z}_0(\mathfrak{G}\mathfrak{s}^*(X^{\mathfrak{c}}_{\mathfrak{E}})) \overset{\mathbb{Z}_0(\mathfrak{p}_1)}{\underset{\mathbb{Z}_0(\mathfrak{p}_2)}{\rightrightarrows}} \mathbb{Z}_0(\mathfrak{G}\mathfrak{s}^*(X)) \xrightarrow{\mathbb{Z}_0(\pi_{X/\mathfrak{E}})} \mathbb{Z}_0\widetilde{\mathcal{K}}_0(X, \mathfrak{E}_X) = K_0(X, \mathfrak{E}_X). \qquad (2)$$

This shows that K_0 is a right 'exact' functor from $(\mathfrak{Esp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^{\to})^{op}$ to $\mathbb{Z}-mod$. ∎

3. The relative functors K_0 and their higher images.

3.1. Universal relative K-functors. Fix a right exact 'space' $\mathcal{Y} = (Y, \mathfrak{E}_\mathcal{Y})$. The functor $(\mathfrak{Esp}_\mathfrak{r})^{op} \xrightarrow{K_0} \mathbb{Z}-mod$ induces a functor

$$(\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y})^{op} \xrightarrow{K_0^\mathcal{Y}} \mathbb{Z}-mod$$

defined by

$$K_0^\mathcal{Y}(\mathcal{X}, \xi) = K_0^\mathcal{Y}(\mathcal{X}, \mathcal{X} \xrightarrow{\xi} \mathcal{Y}) = Cok(K_0(\mathcal{Y}) \xrightarrow{K_0(\xi)} K_0(\mathcal{X}))$$

and acting correspondingly on morphisms.

The domain of the functor $K_0^\mathcal{Y}$, the category $\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}$, has a final object, cokernels of morphisms, and natural left exact structures induced by left exact structures on $\mathfrak{Esp}_\mathfrak{r}$. Fix a left exact structure $\mathfrak{I}$ on $\mathfrak{Esp}_\mathfrak{r}$ (say, one of those defined in IV.8.3.2) and denote by $\mathfrak{I}_\mathcal{Y}$) the left exact structure on $\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}$ induced by $\mathfrak{I}$. Notice that, since the category $\mathbb{Z}-mod$ is complete (and cocomplete), there is a well defined satellite endofunctor

$$\mathcal{H}om((\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y})^{op}, \mathbb{Z}-mod) \xrightarrow{\mathcal{S}_{\mathfrak{I}_\mathcal{Y}}} \mathcal{H}om((\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y})^{op}, \mathbb{Z}-mod), \quad F \longmapsto \mathcal{S}_{\mathfrak{I}_\mathcal{Y}} F.$$

So that, for every functor F from $(\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y})^{op}$ to $\mathbb{Z}-mod$, there is a unique up to isomorphism universal ∂^*-functor $(\mathcal{S}^i_{\mathfrak{I}_\mathcal{Y}} F, \mathfrak{d}_i \mid i \geq 0)$.

In particular, there is a universal contravariant ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}} = (K_i^{\mathcal{Y},\mathfrak{I}}, \mathfrak{d}_i \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}, \mathfrak{I}_\mathcal{Y})$ of right exact 'spaces' over $\mathcal{Y}$ to the category $\mathbb{Z}-mod$ of abelian groups; that is $K_i^{\mathcal{Y},\mathfrak{I}} = \mathcal{S}^i_{\mathfrak{I}_\mathcal{Y}} K_0^{\mathcal{Y},\mathfrak{I}}$ for all $i \geq 0$.

We call the groups $K_i^{\mathcal{Y},\mathfrak{I}}(\mathcal{X}, \xi)$ *universal* K-groups of the right exact 'space' $(\mathcal{X}, \xi)$ over the right exact 'space' $\mathcal{Y} = (C_Y, \mathfrak{E}_Y)$ with respect to the left exact structure $\mathfrak{I}$.

3.2. The principal left exact structure. Let $\mathfrak{I}_{\mathfrak{c}/\mathcal{Y}}^{\to}$ denote the left exact structure on $\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}$ induced by the left exact structure $\mathfrak{I}_\mathfrak{c}^{\to}$ on $\mathfrak{Esp}_\mathfrak{r}$.

3.3. Proposition. *Let $\mathcal{Y} = (Y, \mathfrak{E}_Y)$ be a right exact 'space', and let $\mathfrak{I}$ be a left exact structure on the category $\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}$ which is coarser than $\mathfrak{I}_{\mathfrak{c}/\mathcal{Y}}^{\to}$ (cf. 3.2). Then the universal ∂^*-functor $K_\bullet^\mathcal{Y} = (K_i^\mathcal{Y}, \mathfrak{d}_i \mid i \geq 0)$ from the left exact category $(\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}, \mathfrak{I}_\mathcal{Y})$ to the category $\mathbb{Z}-mod$ of abelian groups is 'exact'; i.e. for any conflation*

$$(X, \xi) \xrightarrow{\mathfrak{q}} (X', \xi') \xrightarrow{\mathfrak{c}_\mathfrak{q}} (X'', \xi''),$$

the associated long sequence

$$\ldots \xrightarrow{K_1^{\mathcal{Y}}(\mathfrak{q})} K_1^{\mathcal{Y}}(X,\xi) \xrightarrow{\mathfrak{d}_0} K_0^{\mathcal{Y}}(X'',\xi'') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{e}_{\mathfrak{q}})} K_0^{\mathcal{Y}}(X',\xi') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{q})} K_0^{\mathcal{Y}}(X,\xi) \longrightarrow 0$$

is exact.

Proof. (a) Since the left exact structure $\mathfrak{I}_{\mathcal{Y}}$ is coarser than $\mathfrak{I}^{\rightarrow}_{\mathfrak{c}/\mathcal{Y}}$, the functor $K_0^{\mathcal{Y}}$ is right 'exact' on $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$. This fact can be read from the commutative diagram

$$\begin{array}{ccccccccc}
 & & Im(\xi_{\mathfrak{Z}}) & \longleftarrow & K_0(\mathcal{Y}) & \longrightarrow & Im(\xi_{\mathcal{X}}) \\
 & & \downarrow & & & & \downarrow \\
K_0(\mathfrak{Z}) & \longrightarrow & \widetilde{\mathcal{K}}_0(\mathfrak{Z}) & \xrightarrow{\mathfrak{j}} & K_0(\mathcal{Z}) \prod K_0(\mathfrak{X}) & \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} & K_0(\mathcal{X}) \\
\downarrow & & \downarrow & cart & \downarrow & cart & \downarrow \\
K_0^{\mathcal{Y}}(\mathfrak{Z},\xi_{\mathfrak{Z}}) & \longrightarrow & Ker(\mathfrak{p}_1^{\mathcal{Y}},\mathfrak{p}_2^{\mathcal{Y}}) & \xrightarrow{\mathfrak{j}^{\mathcal{Y}}} & K_0^{\mathcal{Y}}(\mathcal{Z},\xi_{\mathcal{Z}}) \prod K_0^{\mathcal{Y}}(\mathfrak{X},\xi_{\mathfrak{X}}) & \overset{\mathfrak{p}_1^{\mathcal{Y}}}{\underset{\mathfrak{p}_2^{\mathcal{Y}}}{\rightrightarrows}} & K_0^{\mathcal{Y}}(\mathcal{X},\xi_{\mathcal{X}})
\end{array} \qquad (1)$$

corresponding to a cocartesian square

$$\begin{array}{ccc}
(\mathcal{X},\xi_X) & \longrightarrow & (\mathcal{Z},\xi_Z) \\
\downarrow & cocart & \downarrow \\
(\mathfrak{X},\xi_{\mathfrak{X}}) & \longrightarrow & (\mathfrak{Z},\xi_{\mathfrak{X}})
\end{array}$$

in $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ whose vertical arrows are inflations.

All lower vertical arrows of the diagram (1) are surjective. Since the (composition) of two cartesian squares of (1) is a cartesian square and $Im(\xi_{\mathfrak{X}})$ and $Im(\xi_{\mathcal{X}})$ are kernels of the corresponding vertical arrows of the cartesian square, they are naturally isomorphic. The morphism $K_0(\mathfrak{Z}) \longrightarrow \widetilde{\mathcal{K}}_0(\mathfrak{Z})$ factors through $Ker(\mathfrak{p}_1 \circ \mathfrak{j}, \mathfrak{p}_2 \circ \mathfrak{j})$ and, by hypothesis, this factorization,

$$K_0(\mathfrak{Z}) \longrightarrow Ker(\mathfrak{p}_1 \circ \mathfrak{j}, \mathfrak{p}_2 \circ \mathfrak{j}),$$

is an epimorphism. The latter implies that the map

$$K_0(\mathfrak{Z}) \coprod Im(\xi_{\mathfrak{Z}}) \longrightarrow \widetilde{\mathcal{K}}_0(\mathfrak{Z}) \qquad (2)$$

determined by the morphisms $K_0(\mathfrak{Z}) \longrightarrow \widetilde{\mathcal{K}}_0(\mathfrak{Z}) \longleftarrow Im(\xi_{\mathfrak{Z}})$ (see the diagram (1)) is surjective. The surjectivity of the map (2) together with the surjectivity of the morphism $K_0(\mathfrak{Z}) \longrightarrow \mathcal{K}_0^{\mathcal{Y}}(\mathfrak{Z},\xi_{\mathfrak{Z}})$ imply the surjectivity of

$$K_0^{\mathcal{Y}}(\mathfrak{Z},\xi_{\mathfrak{Z}}) \longrightarrow Ker(\mathfrak{p}_1^{\mathcal{Y}},\mathfrak{p}_2^{\mathcal{Y}})$$

which proves the right 'exactness' of the functor $K_0^{\mathcal{Y}}$.

(b) In particular, for any conflation

$$(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi') \xrightarrow{\mathfrak{c}_{\mathfrak{q}}} (X'',\xi'')$$

of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^*/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$, the sequence

$$K_0^{\mathcal{Y}}(X'',\xi'') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{c}_{\mathfrak{q}})} K_0^{\mathcal{Y}}(X',\xi') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{q})} K_0^{\mathcal{Y}}(X,\xi) \longrightarrow 0$$

of $\mathbb{Z}$-modules is exact. Therefore, by II.6.3, the universal ∂^*-functor $K_\bullet^{\mathcal{Y}} = (K_i^{\mathcal{Y}}, \mathfrak{d}_i |\ i \geq 0)$ from $(\mathfrak{Esp}_{\mathfrak{r}}^*/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$ to $\mathbb{Z} - mod$ is 'exact'. ∎

It is convenient to have the following generalization of the previous assertion.

3.4. Proposition. *Let $\mathcal{Y} = (Y, \mathfrak{E}_Y)$ be a right exact 'space', $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ be a left exact category, and $\mathfrak{F}$ a functor $C_{\mathfrak{S}} \longrightarrow \mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ which maps conflations of $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ to conflations of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{I}^{\rightarrow}_{\mathfrak{c}/\mathcal{Y}})$. Then there exists a (unique up to isomorphism) universal ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}} = (K_i^{\mathfrak{S},\mathfrak{F}}, \mathfrak{d}_i \mid i \geq 0)$ from the right exact category $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})^{op}$ to $\mathbb{Z} - mod$ whose zero component, $K_0^{\mathfrak{S},\mathfrak{F}}$, is the composition of the functor $\mathcal{C}_{\mathfrak{S}}^{op} \xrightarrow{\mathfrak{F}^{op}} \mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}^{op}$ and the functor $K_0^{\mathcal{Y}}$.*

The ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ is 'exact'.

Proof. The existence of the ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ follows, by 3.3.2, from the completeness (– existence of limits of small diagrams) of the category $\mathbb{Z} - mod$ of abelian groups. The main thrust of the proposition is in the 'exactness' of $K_\bullet^{\mathfrak{S},\mathfrak{F}}$.

By hypothesis, the functor $\mathfrak{F}$ maps conflations to conflations. Therefore, it follows from 3.1 that for any conflation $\mathfrak{X} \longrightarrow \mathfrak{X}' \longrightarrow \mathfrak{X}''$ of the left exact category $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$, the sequence of abelian groups $K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{X}'') \longrightarrow K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{X}') \longrightarrow K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{X}) \longrightarrow 0$ is exact. By II.7.1, this implies the 'exactness' of the ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}}$. ∎

4. The higher K-theory of 'spaces' over the 'point'.

4.0. The pointed category of right exact 'spaces'. Let $|Cat_*|^o$ denote the subcategory of the category $|Cat|^o$ of 'spaces' whose objects are 'spaces' represented by categories with initial objects and morphisms are those morphisms of 'spaces' whose inverse image functor maps initial objects to initial objects. The category $|Cat_*|^o$ is pointed: it has a canonical zero (that is both initial and final) object, x, which is represented by the category with one (identical) morphism. Thus, the final objects of the category $|Cat|^o$ of all 'spaces' are zero objects of the subcategory $|Cat_*|^o$.

Each morphism $X \xrightarrow{\mathfrak{f}} Y$ of the category $|Cat_*|^o$ has a cokernel, $Y \xrightarrow{\mathfrak{c}_{\mathfrak{f}}} \mathcal{C}(\mathfrak{f})$, where the category $C_{\mathcal{C}(\mathfrak{f})}$ representing the 'space' $\mathcal{C}(\mathfrak{f})$ is the kernel $Ker(\mathfrak{f}^*)$ of the functor $\mathfrak{f}^*$.

By definition, $Ker(\mathfrak{f}^*)$ is the full subcategory of the category C_Y generated by all objects of C_Y which the functor $\mathfrak{f}^*$ maps to initial objects. The inverse image functor $\mathfrak{c}_\mathfrak{f}^*$ of the canonical morphism $\mathfrak{c}_\mathfrak{f}$ is the natural embedding $Ker(\mathfrak{f}^*) \longrightarrow C_Y$.

The category $\mathfrak{Esp}_\mathfrak{r}^*$ formed by right exact 'spaces' with initial objects and morphisms whose inverse image functor is 'exact' and maps initial objects to initial objects (cf. 1.7), is pointed and the forgetful functor

$$\mathfrak{Esp}_\mathfrak{r}^* \xrightarrow{\mathfrak{J}^*} |Cat_*|^o, \quad (X, \mathfrak{E}_X) \longmapsto X,$$

is a left adjoint to the canonical full embedding $|Cat_*|^o \xrightarrow{\mathfrak{J}_*} \mathfrak{Esp}_\mathfrak{r}^*$ which assigns to every 'space' X the right exact category $(X, Iso(C_X))$. Both functors, $\mathfrak{J}^*$ and $\mathfrak{J}_*$, map zero objects to zero objects. Similarly, the canonical right adjoint $\mathfrak{J}^!$ to the functor $\mathfrak{J}_*$ also maps the category $\mathfrak{Esp}_\mathfrak{r}^*$ to the category $|Cat_*|^o$, because localization functors map initial objects to initial objects and $\mathfrak{J}^!(X, \mathfrak{E}_X) = \mathfrak{E}_X^{-1}X$.

Let x be a zero object of the category $\mathfrak{Esp}_\mathfrak{r}^*$. Then $\mathfrak{Esp}_\mathfrak{r}^*/x$ is naturally isomorphic to $\mathfrak{Esp}_\mathfrak{r}^*$ and $K_0^x = K_0$.

4.1. The left exact structure $\mathfrak{I}_{\mathfrak{c}*}^{\rightarrow}$. We denote by $\mathfrak{I}_{\mathfrak{c}*}^{\rightarrow}$ the canonical left exact structure $\mathfrak{I}_{\mathfrak{c}/_x}^{\rightarrow}$, where x is an initial object. It does not depend on the choice of the zero object x.

4.2. Proposition. *Let $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ be a left exact category, and let $\mathfrak{F}$ be a functor $C_\mathfrak{S} \longrightarrow \mathfrak{Esp}_\mathfrak{r}^*$ which maps conflations of $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ to conflations of the left exact category $(\mathfrak{Esp}_\mathfrak{r}^*, \mathfrak{I}_{\mathfrak{c}*}^{\rightarrow})$. Let $\mathcal{G}$ be a functor from $(\mathfrak{Esp}_\mathfrak{r}^*)^{op}$ to a category C_Z with limits of 'small' filtered systems and initial objects. Then*

(a) There exists a universal ∂^*-functor $\mathcal{G}_\bullet^{\mathfrak{S},\mathfrak{F}} = (\mathcal{G}_i^{\mathfrak{S},\mathfrak{F}}, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$ from $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})^{op}$ to C_Z whose zero component, $\mathcal{G}_0^{\mathfrak{S},\mathfrak{F}}$, is the composition of the functor

$$\mathcal{C}_\mathfrak{S}^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_\mathfrak{r}^*)^{op}$$

and the functor $\mathcal{G}$.

(b) If $(C_Z, \mathfrak{E}_Z)$ is a right exact category and the functor $\mathcal{G}$ is left 'exact', then the ∂^*-functor $\mathcal{G}_\bullet^{\mathfrak{S},\mathfrak{F}}$ is 'exact'. In particular, the ∂^*-functor $\mathcal{G}_\bullet = (\mathcal{G}_i, \mathfrak{d}_i \mid i \geq 0)$ from $(\mathfrak{Esp}_\mathfrak{r}^*, \mathfrak{I}_{\mathfrak{c}*}^{\rightarrow})$ to $(C_Z, \mathfrak{E}_Z)$ is 'exact'.

Proof. The assertion is a special case of II.3.4. ■

4.2.1. Corollary. *Let $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ be a left exact category, and $C_\mathfrak{S} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_\mathfrak{r}^*$ a functor which maps conflations of $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ to conflations of the left exact category $(\mathfrak{Esp}_\mathfrak{r}^*, \mathfrak{I}_{\mathfrak{c}*}^{\rightarrow})$. Then there exists a universal ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}} = (K_i^{\mathfrak{S},\mathfrak{F}}, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$ from $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})^{op}$ to $\mathbb{Z} - mod$ whose zero component, $K_0^{\mathfrak{S},\mathfrak{F}}$, is the composition of the functor*

$$\mathcal{C}_\mathfrak{S}^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_\mathfrak{r}^*)^{op}$$

and the functor K_0.

The ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ is 'exact'. In particular, the ∂^*-functor $K_\bullet = (K_i, \mathfrak{d}_i|\ i \geq 0)$ from the left exact category $(\mathfrak{Esp}_\mathfrak{r}^*, \mathfrak{I}_{\mathfrak{c}*}^{\to})$ to the abelian category $\mathbb{Z}-mod$ is 'exact'.

4.3. The class of morphisms $\mathfrak{I}_{\mathfrak{c}*}^{\to\circledast}$. We denote by $\mathfrak{I}_{\mathfrak{c}*}^{\to\circledast}$ the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (Y, \mathfrak{E}_Y)$ of $\mathfrak{I}_{\mathfrak{c}*}^{\to}$ such that $Cok(\mathfrak{f})$ is a zero object, or, equivalently, $Ker(\mathfrak{f}^*)$ is a trivial category.

4.4. Proposition. *The class $\mathfrak{I}_{\mathfrak{c}*}^{\to\circledast}$ is a left exact structure on the category $\mathfrak{Esp}_\mathfrak{r}^*$ of right exact 'spaces' with initial objects.*

Proof. The assertion is a special (dual) case of the following fact (see II.4.4.1): given a right exact 'space' $(X, \mathfrak{E}_X)$ such that the category C_X has initial objects, the class $\mathfrak{E}_X^{\circledast}$ of all morphisms of $\mathfrak{E}_X$ having a trivial kernel is a right exact structure on the category C_X. ■

4.5. Proposition. *Let $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ be a left exact category, $\mathfrak{F}$ a functor $C_\mathfrak{S} \longrightarrow \mathfrak{Esp}_\mathfrak{r}^*$ which maps conflations of $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ to conflations of the left exact category $(\mathfrak{Esp}_\mathfrak{r}^*, \mathfrak{I}_{\mathfrak{c}*}^{\to\circledast})$, and $K_\bullet^{\mathfrak{S},\mathfrak{F}} = (K_i^{\mathfrak{S},\mathfrak{F}}, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$ a universal ∂^*-functor from $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})^{op}$ to $\mathbb{Z}-mod$ whose zero component, $K_0^{\mathfrak{S},\mathfrak{F}}$, is the composition of $\mathcal{C}_\mathfrak{S}^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_\mathfrak{r}^*)^{op}$ and K_0 (cf. 4.2.1).*

If $\mathcal{X} \xrightarrow{\mathfrak{q}} \mathcal{Y}$ is a morphism of $\mathfrak{I}_\mathfrak{S}$ with trivial cokernel, then the morphisms

$$K_i^{\mathfrak{S},\mathfrak{F}}(\mathcal{Y}) \xrightarrow{K_i^{\mathfrak{S},\mathfrak{F}}} K_i^{\mathfrak{S},\mathfrak{F}}(\mathcal{X})$$

are isomorphisms for all $i \geq 0$.

Proof. Let $\mathfrak{I}_\mathfrak{S}^{\circledast}$ denote the class of all morphisms of $\mathfrak{I}_\mathfrak{S}$ having a trivial cokernel. By (the dual version of) 3.3.7.1, the class $\mathfrak{I}_\mathfrak{S}^{\circledast}$ is a left exact structure on the category $C_\mathfrak{S}$.

Since the functor $C_\mathfrak{S} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_\mathfrak{r}^*$ maps conflations to conflations, it maps final objects of the category $C_\mathfrak{S}$ to zero objects of $\mathfrak{Esp}_\mathfrak{r}^*$. In particular, $\mathfrak{F}$ maps morphisms of $\mathfrak{I}_\mathfrak{S}^{\circledast}$ to morphisms of $\mathfrak{I}_{\mathfrak{c}*}^{\to\circledast}$. By 4.2.1, the ∂^*-functor is 'exact', so that for any conflation

$$\mathcal{X} \xrightarrow{\mathfrak{q}} \mathcal{X}' \xrightarrow{\mathfrak{c}_\mathfrak{q}} \mathcal{X}'',$$

the sequence

$$K_0^{\mathfrak{S},\mathfrak{F}}(\mathcal{X}'') \xrightarrow{K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{c}_\mathfrak{q})} K_0^{\mathfrak{S},\mathfrak{F}}(\mathcal{X}') \xrightarrow{K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{q})} K_0^{\mathfrak{S},\mathfrak{F}}(\mathcal{X}) \longrightarrow 0$$

is exact. If $\mathfrak{q} \in \mathfrak{I}_\mathfrak{S}^{\circledast}$, then $K_0^{\mathfrak{S},\mathfrak{F}}(\mathcal{X}'') = K_0(\mathfrak{F}(\mathcal{X}'')) = 0$. So that in this case the morphism $K_0^{\mathfrak{S},\mathfrak{F}}(\mathcal{X}') \xrightarrow{K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{q})} K_0^{\mathfrak{S},\mathfrak{F}}(\mathcal{X})$ is an isomorphism. The assertion follows now from 5.3.7.2. ■

4.6. Corollary. *For every morphism* $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (X', \mathfrak{E}_{X'})$ *of* $\mathfrak{I}_{\mathfrak{c}*}^{\to\circledast}$ *the corresponding map*

$$K_i(X', \mathfrak{E}_{X'}) \xrightarrow{K_i(\mathfrak{q})} K_i(X, \mathfrak{E}_X)$$

is an isomorphism for all $i \geq 0$.

5. Universal K-theory of abelian categories.

Let $\mathfrak{Esp}_k^{\mathfrak{a}}$ denote the category whose objects are 'spaces' X represented by k-linear abelian categories and morphisms $X \xrightarrow{f} Y$ are represented by k-linear exact functors.

There is a natural functor

$$\mathfrak{Esp}_k^{\mathfrak{a}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^* \tag{1}$$

which assigns to every object X of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ the right exact (actually, exact) 'space' $(X, \mathfrak{E}_X^{st})$, where $\mathfrak{E}_X^{st}$ is the *canonical* (i.e. the finest) right exact structure on the category C_X, and maps each morphism $X \xrightarrow{f} Y$ to the morphism $(X, \mathfrak{E}_X^{st}) \xrightarrow{f} (Y, \mathfrak{E}_Y^{st})$ of right exact 'spaces'. One can see that the functor $\mathfrak{F}$ maps the zero object of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ (represented by the zero category) to a zero object of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$.

5.1. Proposition. *Let* C_X *and* C_Y *be* k*-linear abelian categories endowed with the canonical exact structure. Any exact localization functor* $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ *is an inverse image functor of a morphism of* $\mathfrak{I}_{\mathfrak{c}*}^{\to}$.

Proof. We need to show that every cartesian square

$$\begin{array}{ccc}
\mathfrak{q}^*(\mathcal{M}) & \xrightarrow{\mathfrak{p}_2} & \mathfrak{q}^*(M) \\
\mathfrak{p}_1 \downarrow & cart & \downarrow \widetilde{\mathfrak{t}} \\
\mathfrak{q}^*(\mathcal{L}) & \xrightarrow{\widetilde{h}} & \mathfrak{q}^*(L)
\end{array} \tag{2}$$

in C_X whose vertical arrows are deflations (that is epimorphisms) is isomorphic to the image of a cartesian square of the same type.

In fact, each morphism $\mathfrak{q}^*(M) \xrightarrow{\widetilde{h}} \mathfrak{q}^*(N)$ is of the form $\mathfrak{q}^*(h)\mathfrak{q}^*(s)^{-1}$ for some morphisms $M' \xrightarrow{h} N$ and $M' \xrightarrow{s} M$ such that $\mathfrak{q}^*(s)$ is invertible. The morphism h is a (unique) composition $\mathfrak{j} \circ \mathfrak{e}$, where $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}$ is an epimorphism. Since the functor $\mathfrak{q}^*$ is exact, $\mathfrak{q}^*(\mathfrak{j})$ is a monomorphism and $\mathfrak{q}^*(\mathfrak{e})$ is an epimorphism. Therefore, $\widetilde{h}$ is an epimorphism iff $\mathfrak{q}^*(\mathfrak{j})$ is an isomorphism. Thus, we include the cartesian square (2) into a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{q}^*(\mathcal{M}) & \xrightarrow{\mathfrak{p}_2} & \mathfrak{q}^*(M) & \xleftarrow{\mathfrak{q}^*(\mathfrak{s})} & \mathfrak{q}^*(M') \\
{\scriptstyle \mathfrak{p}_1}\downarrow & cart & \downarrow{\scriptstyle \widetilde{\mathfrak{t}}} & & \downarrow{\scriptstyle \mathfrak{q}^*(\mathfrak{t})} \\
\mathfrak{q}^*(\mathcal{L}) & \xrightarrow{\widetilde{h}} & \mathfrak{q}^*(L) & \xleftarrow{\mathfrak{q}^*(\mathfrak{j})} & \mathfrak{q}^*(L') \\
{\scriptstyle id}\downarrow & & \uparrow{\scriptstyle \mathfrak{q}^*(h)} & cart & \uparrow{\scriptstyle \mathfrak{q}^*(h')} \\
\mathfrak{q}^*(\mathcal{L}) & \xleftarrow{\mathfrak{q}^*(\mathfrak{u})} & \mathfrak{q}^*(L') & \xleftarrow{\mathfrak{q}^*(\widetilde{\mathfrak{j}})} & \mathfrak{q}^*(L'')
\end{array}$$

whose lower right square is the image of a cartesian square

$$\begin{array}{ccc}
L & \xleftarrow{\mathfrak{j}} & L' \\
{\scriptstyle h}\uparrow & cart & \uparrow{\scriptstyle h'} \\
L' & \xleftarrow{\widetilde{\mathfrak{j}}} & L''
\end{array}$$

and the left horizontal arrows and $\mathfrak{q}^*(\mathfrak{u})$ are isomorphisms.

This shows that the cartesian square (2) is isomorphic to the image of a pull-back

$$\begin{array}{ccc}
\mathfrak{L} & \xrightarrow{\widetilde{h'}} & M' \\
{\scriptstyle \mathfrak{t}_1}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}} \\
L'' & \xrightarrow{h'} & L'
\end{array}$$

of the deflation $M' \xrightarrow{\mathfrak{t}} L'$ along the morphism $L'' \xrightarrow{h'} L'$. ■

5.2. A left exact structure on the category $\mathfrak{Esp}^{\mathfrak{a}}_k$.

5.2.1. Proposition. *The category $\mathfrak{Esp}^{\mathfrak{a}}_k$ of 'spaces' represented by abelian k-linear categories has arbitrary colimits, and the functor*

$$\mathfrak{Esp}^{\mathfrak{a}}_k \xrightarrow{\mathfrak{F}} \mathfrak{Esp}^*_{\mathfrak{r}}$$

preserves colimits.

Proof. The product of any set of abelian categories is an abelian category. And for any pair of exact functors

$$C_X \overset{g_1^*}{\underset{g_2^*}{\rightrightarrows}} C_Y$$

from one abelian category to another, the kernel $Ker(g_1^*, g_2^*)$ is an abelian category and the natural functor $Ker(g_1^*, g_2^*) \xrightarrow{\mathfrak{c}^*_{g_1,g_2}} C_X$ is exact. Hence the assertion. ∎

5.2.2. Corollary. *Let $\mathfrak{I}_k^{\mathfrak{a}}$ denote the preimage in the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ of the left exact structure $\mathfrak{I}_{\mathfrak{c}*}^{\to}$. The functor $\mathfrak{Esp}_k^{\mathfrak{a}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^*$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_k^{\mathfrak{a}}, \mathfrak{I}_k^{\mathfrak{a}})$ to the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{I}_{\mathfrak{c}*}^{\to})$.*

Proof. This follows from the fact that the functor

$$\mathfrak{Esp}_k^{\mathfrak{a}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^*$$

preserves colimits (in particular, it preserves push-forwards) and that, by definition of $\mathfrak{I}_k^{\mathfrak{a}}$, the functor $\mathfrak{F}$ maps morphisms of $\mathfrak{I}_k^{\mathfrak{a}}$ to inflations – morphisms of $\mathfrak{I}_{\mathfrak{c}*}^{\to}$. ∎

5.2.3. Proposition. *(a) The class $\mathfrak{L}^{\mathfrak{a}}$ of all morphisms $X \xrightarrow{q} Y$ of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ such that $C_Y \xrightarrow{q^*} C_X$ is a localization functor is a left exact structure on $\mathfrak{Esp}_k^{\mathfrak{a}}$ which is coarser than $\mathfrak{I}_k^{\mathfrak{a}}$.*

(b) Every morphism of $\mathfrak{I}_k^{\mathfrak{a}}$ is a canonical composition of a morphism of $\mathfrak{L}^{\mathfrak{a}}$ and a conservative morphism of $\mathfrak{I}_k^{\mathfrak{a}}$.

(c) The class ${}^{\mathfrak{c}}\mathfrak{I}_k^{\mathfrak{a}}$ of conservative morphisms from $\mathfrak{I}_k^{\mathfrak{a}}$ coincides with the class of morphisms of inflations with trivial cokernel.

Proof. (a) The assertion follows from 5.1.

(b) Every morphism $X \xrightarrow{\mathfrak{f}} Y$ of $|Cat|^o$ is the composition $\mathfrak{q}_{\mathfrak{f}} \circ \mathfrak{f}_{\mathfrak{c}}$ of a localization $\mathfrak{q}_{\mathfrak{f}}$ at $\Sigma_{\mathfrak{f}}$ and a conservative morphism. If $\mathfrak{f}$ is a morphism of $\mathfrak{Esp}_k^{\mathfrak{a}}$, its inverse image functor is an exact k-linear functor, hence $\mathfrak{q}_{\mathfrak{f}}^*$ is an exact k-linear localization functor and $\mathfrak{f}_{\mathfrak{c}}^*$ is an exact k-linear conservative functor.

(c) The latter means precisely that $\mathfrak{f}_{\mathfrak{c}}^*$ is an exact k-linear functor with a trivial kernel. Hence the assertion. ∎

5.3. The Grothendieck functor. The composition $K_0^{\mathfrak{a}}$ of the functor

$$(\mathfrak{Esp}_k^{\mathfrak{a}})^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^*)^{op}$$

and the functor $(\mathfrak{Esp}_{\mathfrak{r}}^*)^{op} \xrightarrow{K_0^*} \mathbb{Z} - mod$ assigns to each object X of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ the abelian group $K_0^*(X, \mathfrak{E}_X^{st})$ which coincides with the Grothendieck group of the abelian category C_X. We call $K_0^{\mathfrak{a}}$ the *Grothendieck functor.*

5.4. Proposition. *There exists a universal ∂^*-functor $K_\bullet^{\mathfrak{a}} = (K_i^{\mathfrak{a}}, \mathfrak{d}_i^{\mathfrak{a}} \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}_k^{\mathfrak{a}}, \mathfrak{I}_k^{\mathfrak{a}})^{op}$ to the category $\mathbb{Z} - mod$ whose zero component is the*

Grothendieck functor K_0. The universal ∂^*-functor $K^{\mathfrak{a}}_\bullet$ is 'exact'; that is for any inflation $X \xrightarrow{\mathfrak{f}} X'$, the canonical long sequence

$$\ldots \xrightarrow{K^{\mathfrak{a}}_1(\mathfrak{f})} K^{\mathfrak{a}}_1(X) \xrightarrow{\mathfrak{d}^{\mathfrak{a}}_0(\mathfrak{f})} K^{\mathfrak{a}}_0(X'') \xrightarrow{K^{\mathfrak{a}}_0(\mathfrak{c}_\mathfrak{f})} K^{\mathfrak{a}}_0(X') \xrightarrow{K^{\mathfrak{a}}_0(\mathfrak{f})} K^{\mathfrak{a}}_0(X) \longrightarrow 0 \qquad (3)$$

is exact.

Proof. By 5.2.1(b), the functor $\mathfrak{Esp}^{\mathfrak{a}}_k \xrightarrow{\mathfrak{F}} \mathfrak{Esp}^*_{\mathfrak{r}}$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{I}^{\mathfrak{a}}_k)$ to the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{I}^{\to}_{\mathfrak{c}*})$ which maps the zero object of the category $\mathfrak{Esp}^{\mathfrak{a}}_k$ (– the 'space' represented by the zero category) to a zero object of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$. Therefore, $\mathfrak{F}$ maps conflations to conflations.

The assertion follows now from 4.2.1 applied to the functor $\mathfrak{F}$. ■

5.4.1. Note. Let $X \xrightarrow{\mathfrak{f}} X'$ be an inflation and

$$X \xrightarrow{\mathfrak{f}_\mathfrak{c}} \widetilde{\mathcal{X}} = \Sigma_{\mathfrak{f}}^{-1} X' \xrightarrow{\mathfrak{q}_\mathfrak{f}} X'$$

its canonical decomposition into an exact localization and a conservative inflation. It follows from 5.2.3(a) that $\widetilde{\mathcal{X}} = \Sigma_{\mathfrak{f}}^{-1} X' \xrightarrow{\mathfrak{q}_\mathfrak{f}} X'$ is an inflation and from 5.2.3(c) that $X \xrightarrow{\mathfrak{f}_\mathfrak{c}} \widetilde{\mathcal{X}}$ is an inflation with trivial cokernel. The functor $K^{\mathfrak{a}}_0$ maps inflations with trivial cokernel to isomorphisms. Therefore, by II.4.4.2, the higher K-functors, $K^{\mathfrak{a}}_i$, do the same. Thus, we have the long exact sequence (3) corresponding to the inflation $X \xrightarrow{\mathfrak{f}} X'$, another long exact sequence corresponding to the exact localization $\widetilde{\mathcal{X}} = \Sigma_{\mathfrak{f}}^{-1} X' \xrightarrow{\mathfrak{q}_\mathfrak{f}} X'$ and isomorphism between them

$$\begin{array}{ccccccccccc}
\ldots & \xrightarrow{K^{\mathfrak{a}}_1(\mathfrak{f})} & K^{\mathfrak{a}}_1(X) & \xrightarrow{\mathfrak{d}^{\mathfrak{a}}_0(\mathfrak{f})} & K^{\mathfrak{a}}_0(X'') & \xrightarrow{K^{\mathfrak{a}}_0(\mathfrak{c}_\mathfrak{f})} & K^{\mathfrak{a}}_0(X') & \xrightarrow{K^{\mathfrak{a}}_0(\mathfrak{f})} & K^{\mathfrak{a}}_0(\widetilde{\mathcal{X}}) & \longrightarrow & 0 \\
 & K^{\mathfrak{a}}_1(\mathfrak{f}_\mathfrak{c}) \downarrow\wr & & & id\downarrow\wr & & id\downarrow\wr & & \wr\downarrow K^{\mathfrak{a}}_0(\mathfrak{f}_\mathfrak{c}) & & \\
\ldots & \xrightarrow{K^{\mathfrak{a}}_1(\mathfrak{f})} & K^{\mathfrak{a}}_1(X) & \xrightarrow{\mathfrak{d}^{\mathfrak{a}}_0(\mathfrak{f})} & K^{\mathfrak{a}}_0(X'') & \xrightarrow{K^{\mathfrak{a}}_0(\mathfrak{c}_\mathfrak{f})} & K^{\mathfrak{a}}_0(X') & \xrightarrow{K^{\mathfrak{a}}_0(\mathfrak{f})} & K^{\mathfrak{a}}_0(X) & \longrightarrow & 0
\end{array}$$

given by the identical morphisms and isomorphisms $K^{\mathfrak{a}}_i(\mathfrak{f}_\mathfrak{c})$.

5.5. The universal ∂^*-functor $K^{\mathfrak{a}}_\bullet$ and the Quillen's K-theory. For a 'space' X represented by a svelte k-linear abelian category C_X, we denote by $K^{\mathfrak{Q}}_i(X)$ the i-th Quillen's K-group of the category C_X. For each $i \geq 0$, the map $X \longmapsto K^{\mathfrak{Q}}_i(X)$ extends naturally to a functor

$$(\mathfrak{Esp}^{\mathfrak{a}}_k)^{op} \xrightarrow{K^{\mathfrak{Q}}_i} \mathbb{Z} - mod$$

It follows from the Quillen's localization theorem [Q, 5.5] that, for any exact localization $X \xrightarrow{\mathfrak{q}} X'$ and each $i \geq 0$, there exists a *connecting morphism*

$$K^{\mathfrak{Q}}_{i+1}(X) \xrightarrow{\mathfrak{d}^{\mathfrak{Q}}_i(\mathfrak{q})} K^{\mathfrak{Q}}_0(X''),$$

where $C_{X''} = Ker(\mathfrak{q}^*)$, such that the sequence

$$\dots \xrightarrow{K_1^{\mathfrak{Q}}(\mathfrak{q})} K_1^{\mathfrak{Q}}(X) \xrightarrow{\mathfrak{d}_0^{\mathfrak{Q}}(\mathfrak{q})} K_0^{\mathfrak{Q}}(X'') \xrightarrow{K_0^{\mathfrak{Q}}(\mathfrak{e}_{\mathfrak{q}})} K_0^{\mathfrak{Q}}(X') \xrightarrow{K_0^{\mathfrak{Q}}(\mathfrak{q})} K_0^{\mathfrak{Q}}(X) \longrightarrow 0 \qquad (4)$$

is exact. It follows (from the proof of the Quillen's localization theorem) that the connecting morphisms $\mathfrak{d}_i^{\mathfrak{Q}}(\mathfrak{q})$, $i \geq 0$, depend functorially on the localization morphism $\mathfrak{q}$.

In other words, $K_\bullet^{\mathfrak{Q}} = (K_i^{\mathfrak{Q}}, \mathfrak{d}_i^{\mathfrak{Q}} \mid i \geq 0)$ is an 'exact' ∂^*-functor from the right exact category $(\mathfrak{Esp}_k^{\mathfrak{a}}, \mathfrak{L}_k^{\mathfrak{a}})^{op}$ to the category $\mathbb{Z} - mod$ of abelian groups.

Naturally, we call the ∂^*-functor $K_\bullet^{\mathfrak{Q}}$ the *Quillen's K-functor.*

5.5.1. The canonical morphism. Since $K_\bullet^{\mathfrak{a}} = (K_i^{\mathfrak{a}}, \mathfrak{d}_i^{\mathfrak{a}} \mid i \geq 0)$ is a universal ∂^*-functor from $(\mathfrak{Esp}_k^{\mathfrak{a}}, \mathfrak{L}^{\mathfrak{a}})^{op}$ to $\mathbb{Z} - mod$, the identical isomorphism $K_0^{\mathfrak{Q}} \longrightarrow K_0^{\mathfrak{a}}$ extends uniquely to a ∂^*-functor morphism

$$K_\bullet^{\mathfrak{Q}} \xrightarrow{\varphi_\bullet^{\mathfrak{Q}}} K_\bullet^{\mathfrak{a}}. \qquad (5)$$

5.6. Remark. There is a canonical functorial morphism of the universal determinant group $K_1^{det}(X)$ (introduced by Bass [Ba, p. 389]) to the Quillen's $K_1^{\mathfrak{Q}}(X)$. If X is *affine*, i.e. C_X is the category of modules over a ring, this morphism is an isomorphism. It is known [Ger, 5.2] that if C_X is the category of coherent sheaves on the complete non-singular curve of genus 1 over $\mathbb{C}$, then $K_1^{det}(X) \longrightarrow K_1^{\mathfrak{Q}}(X)$ is not a monomorphism. In particular, the composition $K_1^{det}(X) \longrightarrow K_1^{\mathfrak{a}}(X)$ of the morphism $K_1^{det}(X) \longrightarrow K_1^{\mathfrak{Q}}(X)$ and the canonical morphism $K_1^{\mathfrak{Q}}(X) \xrightarrow{\varphi_1^{\mathfrak{Q}}(X)} K_1^{\mathfrak{a}}(X)$ is not a monomorphism.

6. Universal K-theory of k-linear right exact categories.

Let $\mathfrak{Esp}_k^{\mathfrak{r}}$ denote the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$, where the 'space' X is represented by a k-linear svelte additive category and morphisms $(X, \mathfrak{E}_X) \longrightarrow (Y, \mathfrak{E}_Y)$ are given by morphisms of 'spaces' $X \xrightarrow{f} Y$ whose inverse image functors are k-linear 'exact' functors. By 1.4, the 'exactness' of a morphism f means precisely that its inverse image functor, f^*, maps conflations to conflations.

There is a natural functor

$$\mathfrak{Esp}_k^{\mathfrak{r}} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}} \mathfrak{Esp}_{\mathfrak{r}}^* \qquad (1)$$

which maps objects and morphisms of the category $\mathfrak{Esp}_k^{\mathfrak{r}}$ to the corresponding objects and morphisms of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$.

6.1. Proposition. *The functor* $\mathfrak{Esp}_k^{\mathfrak{r}} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}} \mathfrak{Esp}_{\mathfrak{r}}^*$ *preserves colimits and maps the zero object of the category* $\mathfrak{Esp}_k^{\mathfrak{r}}$ *to the zero object of the category* $\mathfrak{Esp}_{\mathfrak{r}}^*$.

Proof. The product of any set of right exact k-linear categories (taken in $\mathfrak{Esp}^*_\mathfrak{r}$) is a right exact k-linear category. For any pair of 'exact' functors

$$(C_X, \mathfrak{E}_X) \overset{g_1^*}{\underset{g_2^*}{\rightrightarrows}} (C_Y, \mathfrak{E}_Y)$$

from one k-linear right exact category to another, the kernel $Ker(g_1^*, g_2^*)$ is a k-linear right exact category and the natural functor $Ker(g_1^*, g_2^*) \xrightarrow{\mathfrak{c}^*_{g_1,g_2}} (C_X, \mathfrak{E}_X)$ is 'exact'. ∎

6.2. Corollary. *The class of morphisms* $\mathfrak{I}^\mathfrak{r}_k = \mathfrak{F}_\mathfrak{r}^{-1}(\mathfrak{I}^{\to}_{\mathfrak{c}*})$ *is a left exact structure on the category* $\mathfrak{Esp}^\mathfrak{r}_k$ *and* $\mathfrak{F}_r$ *is an 'exact' functor from the left exact category* $(\mathfrak{Esp}^\mathfrak{r}_k, \mathfrak{I}^\mathfrak{r}_k)$ *to the left exact category* $(\mathfrak{Esp}^*_\mathfrak{r}, \mathfrak{I}^{\to}_{\mathfrak{c}\circledast})$.

Proof. Since the functor $\mathfrak{F}_\mathfrak{r}$ preserves cocartesian squares, the preimage $\mathfrak{F}_\mathfrak{r}^{-1}(\tau)$ of any copretopology τ on $\mathfrak{Esp}^*_\mathfrak{r}$ is a copretopology on the category $\mathfrak{Esp}^\mathfrak{r}_k$. In particular, $\mathfrak{I}^\mathfrak{r}_k = \mathfrak{F}_\mathfrak{r}^{-1}(\mathfrak{I}^{\to}_{\mathfrak{c}*})$ is the class of cocovers of a copretopology. The copretopology $\mathfrak{I}^\mathfrak{r}_k$ is subcanonical, i.e. $\mathfrak{I}^\mathfrak{r}_k$ is a left exact structure on the category $\mathfrak{Esp}^\mathfrak{r}_k$.

The copretopology $\mathfrak{I}^\mathfrak{r}_k$ is subcanonical iff for any morphism $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ the cocartesian square

$$\begin{array}{ccc}
(X, \mathfrak{E}_X) & \xrightarrow{\ \mathfrak{q}\ } & (Y, \mathfrak{E}_Y) \\
{\scriptstyle \mathfrak{q}} \downarrow & & \downarrow {\scriptstyle \mathfrak{p}_1} \\
(Y, \mathfrak{E}_Y) & \xrightarrow{\ \mathfrak{p}_2\ } & (\mathfrak{X}, \mathfrak{E}_\mathfrak{X})
\end{array} \qquad (2)$$

is cartesian, or, equivalently, the diagram

$$(X, \mathfrak{E}_X) \xrightarrow{\ \mathfrak{q}\ } (Y, \mathfrak{E}_Y) \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} (\mathfrak{X}, \mathfrak{E}_\mathfrak{X}) \qquad (3)$$

is exact. The claim is that, indeed, the diagram (3) is exact.

In fact, let $(Z, \mathfrak{E}_Z) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ be a morphism which equalizes the pair of arrows $(Y, \mathfrak{E}_Y) \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} (\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$. Since the functor $\mathfrak{F}_\mathfrak{r}$ transforms (2) into a cartesian square, there exists a unique morphism $\mathfrak{F}_\mathfrak{r}(Z, \mathfrak{E}_Z) \xrightarrow{h} \mathfrak{F}_\mathfrak{r}(X, \mathfrak{E}_X)$ such that $\mathfrak{F}_\mathfrak{r}(\mathfrak{q}) \circ h = \mathfrak{F}_\mathfrak{r}(f)$. It follows that the inverse image h^* of h is a k-linear functor $C_X \longrightarrow C_Z$. Therefore h is the image of (a uniquely determined) morphism $(Z, \mathfrak{E}_Z) \longrightarrow (X, \mathfrak{E}_X)$, hence the morphism f factors uniquely through $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$. ∎

6.2.1. Note. Let $\mathfrak{L}^{\to}_{\mathfrak{c}*}$ denote the class of all localizations (that is all morphisms whose inverse image is a localization) which belong to $\mathfrak{I}^{\to}_{\mathfrak{c}*}$; And let $\mathfrak{L}^\mathfrak{r}_k = \mathfrak{F}_\mathfrak{r}^{-1}(\mathfrak{L}^{\to}_{\mathfrak{c}*})$. Then $\mathfrak{L}^\mathfrak{r}_k$ is a left exact structure on the category $\mathfrak{Esp}^\mathfrak{r}_k$ of k-linear right exact 'spaces'.

This follows from the fact $\mathfrak{L}_{\mathfrak{c}*}^{\rightarrow}$ is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}^*$.

6.3. The functor $K_0^{\mathfrak{r}}$. We denote by $K_0^{\mathfrak{r}}$ the composition of the functor

$$(\mathfrak{Esp}_k^{\mathfrak{r}})^{op} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^*)^{op}$$

and the functor $(\mathfrak{Esp}_{\mathfrak{r}}^*)^{op} \xrightarrow{K_0^*} \mathbb{Z} - mod.$

6.4. Proposition. *There exists a universal ∂^*-functor $K_\bullet^{\mathfrak{r}} = (K_i^{\mathfrak{r}}, \mathfrak{d}_i^{\mathfrak{r}} \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}_k^{\mathfrak{r}}, \mathfrak{I}_k^{\mathfrak{r}})^{op}$ to the category $\mathbb{Z} - mod$ whose zero component is the functor $K_0^{\mathfrak{r}}$. The universal ∂^*-functor $K_\bullet^{\mathfrak{r}}$ is 'exact'; that is for any exact localization $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (X', \mathfrak{E}_{X'})$ which belongs to $\mathfrak{L}^{\mathfrak{r}}$, the canonical long sequence*

$$\begin{array}{ccccccc}
K_1^{\mathfrak{r}}(X.\mathfrak{E}_X) & \xleftarrow{K_1^{\mathfrak{r}}(\mathfrak{q})} & K_1^{\mathfrak{r}}(X', \mathfrak{E}_{X'}) & \xleftarrow{K_1^{\mathfrak{r}}(\mathfrak{c}_{\mathfrak{q}})} & K_1^{\mathfrak{r}}(X'', \mathfrak{E}_{X''}) & \xleftarrow{\mathfrak{d}_1^{\mathfrak{r}}(\mathfrak{q})} & \dots \\
\mathfrak{d}_0^{\mathfrak{r}}(\mathfrak{q}) \big\downarrow & & & & & & \\
K_0^{\mathfrak{r}}(X'', \mathfrak{E}_{X''}) & \xrightarrow{K_0^{\mathfrak{r}}(\mathfrak{c}_{\mathfrak{q}})} & K_0^{\mathfrak{r}}(X', \mathfrak{E}_{X'}) & \xrightarrow{K_0^{\mathfrak{r}}(\mathfrak{q})} & K_0^{\mathfrak{r}}(X, \mathfrak{E}_X) & \longrightarrow & 0
\end{array} \qquad (4)$$

is exact.

Proof. The functor $\mathfrak{Esp}_k^{\mathfrak{r}} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}} \mathfrak{Esp}_{\mathfrak{r}}^*$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_k^{\mathfrak{r}}, \mathfrak{I}_k^{\mathfrak{r}})$ to the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{I}_{\mathfrak{c}*}^{\rightarrow})$ which maps the zero object of the category $\mathfrak{Esp}_k^{\mathfrak{r}}$ (– the 'space' represented by the zero category) to a zero object of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$. Therefore, $\mathfrak{F}_{\mathfrak{r}}$ maps conflations to conflations. It remains to apply 4.2.1. ∎

6.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact svelte k-linear additive category, $(C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$ the associated exact k-linear category, and $(C_X, \mathfrak{E}_X) \xrightarrow{\gamma_X^*} (C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$ the canonical fully faithful 'exact' universal functor (see I.7.5) regarded as an inverse image functor of a morphism $(X_{\mathfrak{e}}, \mathfrak{E}_{X_{\mathfrak{e}}}) \xrightarrow{\gamma_X} (X, \mathfrak{E}_X)$.*

The map $K_0(X, \mathfrak{E}_X) \xrightarrow{K_0(\gamma_X)} K_0(X_{\mathfrak{e}}, \mathfrak{E}_{X_{\mathfrak{e}}})$ is a group epimorphism.

Proof. The assertion follows from the description of the exact category $(C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$ (see the argument of I.7.5). Details are left to the reader. ∎

6.6. The category of exact k-'spaces' and Grothendieck-Quillen functor. Let $\mathfrak{Esp}_k^{\mathfrak{e}}$ denote the full subcategory of the category $\mathfrak{Esp}_k^{\mathfrak{r}}$ whose objects are pairs $(X, \mathfrak{E}_X)$ such that $(C_X, \mathfrak{E}_X)$ is an exact k-linear category.

It follows from I.7.5 that the inclusion functor, $\mathfrak{Esp}_k^{\mathfrak{e}} \xrightarrow{\mathfrak{J}^*} \mathfrak{Esp}_k^{\mathfrak{r}}$ has a right adjoint, $\mathfrak{J}_*$ which assigns to each right exact k-space $(X, \mathfrak{E}_X)$ the associated exact k-space $(X_{\mathfrak{e}}, \mathfrak{E}_{X_{\mathfrak{e}}})$.

The adjunction arrow $\mathfrak{J}^*\mathfrak{J}_* \longrightarrow Id_{\mathfrak{Esp}_k^{\mathfrak{r}}}$ assigns to each object $(X, \mathfrak{E}_X)$ of $\mathfrak{Esp}_k^{\mathfrak{r}}$ the morphism $(X_{\mathfrak{e}}, \mathfrak{E}_{X_{\mathfrak{e}}}) \xrightarrow{\gamma_X} (X, \mathfrak{E}_X)$ (see 6.5). The adjunction morphism $Id_{\mathfrak{Esp}_k^{\mathfrak{e}}} \longrightarrow \mathfrak{J}_*\mathfrak{J}^*$ is the identity morphism.

Thus, $\mathfrak{Esp}_k^{\mathfrak{r}} \xrightarrow{\mathfrak{J}_*} \mathfrak{Esp}_k^{\mathfrak{e}}$ is a localization functor. According to 6.5, the functor $(\mathfrak{Esp}_k^{\mathfrak{r}})^{op} \xrightarrow{K_0^{\mathfrak{r}}} \mathbb{Z} - mod$ factors through the localization functor

$$(\mathfrak{Esp}_k^{\mathfrak{r}})^{op} \xrightarrow{\mathfrak{J}_*^{op}} (\mathfrak{Esp}_k^{\mathfrak{e}})^{op}.$$

That is the functor $K_0^{\mathfrak{r}}$ is isomorphic to the composition $K_0^{\mathfrak{e}} \circ \mathfrak{J}_*^{op}$, where $K_0^{\mathfrak{e}}$ denote the restriction of $K_0^{\mathfrak{r}}$ to the subcategory $(\mathfrak{Esp}_k^{\mathfrak{e}})^{op}$, i.e. the composition $K_0^{\mathfrak{r}} \circ \mathfrak{J}^{*^{op}}$.

For each exact k-space $(X, \mathfrak{E}_X)$, the group $K_0^{\mathfrak{e}}(X, \mathfrak{E}_X)$ coincides with the Grothendieck group K_0 of the exact category $(C_X, \mathfrak{E}_X)$ as it was defined by Quillen [Q].

6.7. Proposition. *The restriction $\mathfrak{I}_k^{\mathfrak{e}}$ of the left exact structure $\mathfrak{I}_k^{\mathfrak{r}}$ on $\mathfrak{Esp}_k^{\mathfrak{r}}$ to the subcategory $\mathfrak{Esp}_k^{\mathfrak{e}}$ is a left exact structure on $\mathfrak{Esp}_k^{\mathfrak{e}}$.*

Proof. The inclusion functor $\mathfrak{Esp}_k^{\mathfrak{e}} \xrightarrow{\mathfrak{J}^*} \mathfrak{Esp}_k^{\mathfrak{r}}$ preserves all colimits; in particular, it preserves cocartesian squares. The latter implies that $\mathfrak{I}_k^{\mathfrak{e}} = \mathfrak{J}^{*^{-1}}(\mathfrak{I}_k^{\mathfrak{r}})$ is a left exact structure on the category $\mathfrak{Esp}_k^{\mathfrak{r}}$. ∎

6.7.1. The universal K-functor on exact k-'spaces'. In particular, we have a universal ∂^*-functor $K_\bullet^{\mathfrak{e}} = (K_i^{\mathfrak{e}}, \mathfrak{d}_i^{\mathfrak{e}} \mid i \geq 0)$ from $(\mathfrak{Esp}_k^{\mathfrak{e}}, \mathfrak{I}_k^{\mathfrak{e}})^{op}$ to $\mathbb{Z} - mod$ which is exact.

6.8. Remarks on K-theory of k-linear exact categories. The category $\mathfrak{Esp}_k^{\mathfrak{e}}$ of exact k-spaces has an automorphism $\mathfrak{D}$ which assigns to each 'space' $(X, \mathfrak{E}_X)$ the dual 'space' $(X, \mathfrak{E}_X)^o$ represented by the opposite exact category $(C_X, \mathfrak{E}_X)^{op}$.

6.8.1. Proposition. *Let F be a contravariant functor from the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ of exact k-'spaces' to a category $C_{\mathcal{Z}}$ with filtered limits. If for each 'space' $(X, \mathfrak{E}_X)$, there is an isomorphism $F(X, \mathfrak{E}_X) \simeq F((X, \mathfrak{E}_X)^o)$ functorial in $(X, \mathfrak{E}_X)$, then the universal ∂^*-functor $S_\bullet^{\mathfrak{L}^{\mathfrak{e}}} F$ is isomorphic to its composition with the duality automorphism $\mathfrak{D}$ of the category $\mathfrak{Esp}_k^{\mathfrak{e}}$.*

Proof. The argument is left to the reader. ∎

6.8.2. Corollary. *There is a natural isomorphism of universal ∂^*-functors*

$$K_\bullet^{\mathfrak{e}} \simeq K_\bullet^{\mathfrak{e}} \circ \mathfrak{D}.$$

Proof. In fact, $K_0(X, \mathfrak{E}_X)$ is naturally isomorphic to $K_0((X, \mathfrak{E}_X)^o)$, because the (identical) isomorphism $ObC_X \xrightarrow{\sim} Ob(C_X^{op})$ implies a canonical isomorphism $\mathbb{Z}|C_X| \simeq \mathbb{Z}|C_X^{op}|$, relations defining K_0 correspond to conflations, and the dualization functor $\mathfrak{D}$ induces an isomorphism between the corresponding categories of conflations. ■

7. Reduction by resolution.

7.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and C_Y its fully exact subcategory such that*

(a) If $M' \longrightarrow M \longrightarrow M''$ is a conflation with $M \in ObC_Y$, then $M' \in ObC_Y$.

(b) For any $M'' \in ObC_X$, there exists a deflation $M \longrightarrow M''$ with $M \in ObC_Y$.

Then the morphism $K_\bullet(Y, \mathfrak{E}_Y) \longrightarrow K_\bullet(X, \mathfrak{E}_X)$ is an isomorphism.

Proof. (i) Suppose that F_0 is a contravariant functor from $\mathfrak{Esp}_\mathfrak{r}$ to a category with filtered limits such that the conditions (a) and (b) imply that $F_0(Y, \mathfrak{E}_Y) \xrightarrow{F(\mathfrak{j})} F_0(X, \mathfrak{E}_X)$ is an isomorphism. Then the morphisms

$$S^n_- F_0(Y, \mathfrak{E}_Y) \longrightarrow S^n_- F_0(X, \mathfrak{E}_X)$$

are isomorphisms for all $n \geq 0$.

1) If $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ is a morphism of $\mathfrak{I}_\mathfrak{c}^{\rightarrow}$ and

$$\begin{array}{ccc} (\mathfrak{X}, \mathfrak{E}_\mathfrak{X}) & \xrightarrow{\mathfrak{j}_1} & (\mathfrak{Y}, \mathfrak{E}_\mathfrak{Y}) \\ \mathfrak{f} \uparrow & cocart & \uparrow \widetilde{\mathfrak{f}} \\ (X, \mathfrak{E}_X) & \xrightarrow{\mathfrak{j}} & (Y, \mathfrak{E}_Y) \end{array} \qquad (2)$$

a cocartesian square, then the functor

$$(C_\mathfrak{Y}, \mathfrak{E}_\mathfrak{Y}) \xrightarrow{\mathfrak{j}_1^*} (\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$$

inherits the conditions (a) and (b).

In fact, consider the pseudo-cartesian square

$$\begin{array}{ccc} C_\mathfrak{Y} & \xrightarrow{\mathfrak{j}_1^*} & C_\mathfrak{X} \\ \widetilde{\mathfrak{f}}^* \downarrow & cart & \downarrow \mathfrak{f}^* \\ C_Y & \xrightarrow{\mathfrak{j}^*} & C_X \end{array}$$

of inverse image functors corresponding to the diagram (2). The functor

$$C_\mathfrak{Y} \xrightarrow{\mathfrak{j}_1^*} C_\mathfrak{X}, \quad (L, M; \mathfrak{j}^*(L) \xrightarrow{\sim} \mathfrak{f}^*(M)) \longmapsto M,$$

is fully faithful without any conditions on $\mathfrak{f}$.

Suppose $X \xrightarrow{\mathfrak{f}} \mathfrak{X}$ is a canonical inflation to an injective object (cf. IV.5.3). Then any morphism of $\mathfrak{E}_X$ is the image of some morphism of $\mathfrak{E}_{\mathfrak{X}}$ (see the argument of IV.5.3.2). Let M be an object of $C_{\mathfrak{X}}$. By the condition (b), there is a deflation $\mathfrak{j}^*(L) \xrightarrow{\mathfrak{t}} \mathfrak{f}^*(M)$. This deflation is the image of an arrow $\mathfrak{M} \xrightarrow{\mathfrak{t}'} M$ of $\mathfrak{E}_{\mathfrak{X}}$. So that $\mathfrak{j}_1$ satisfies the condition (b).

The condition (a) is also inherited by $\mathfrak{j}_1^*$, because the functor $\mathfrak{f}^*$ preserves kernels and, by hypothesis, $\mathfrak{f}^*(Ker(\mathfrak{t}') \simeq Ker(\mathfrak{t}) \simeq \mathfrak{j}^*(L_0)$ for some object L_0 of C_Y, which shows that the kernel of $\mathfrak{j}_1^*(L, \mathfrak{M}; id) \xrightarrow{\mathfrak{t}'} M$ is isomorphic to the image of an object of $C_{\mathfrak{Y}}$.

2) By IV.8.5.3, the left exact category $(\mathfrak{Esp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^{\rightarrow})$ has enough injectives. The assertion (i) follows now from IV.8.3.5.2.

It remains to verify that the functor K_0 satisfy the conditions (i).

(ii) It follows from the conditions (a) and (b) that the map

$$K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(\mathfrak{j})} K_0(X, \mathfrak{E}_X)$$

is surjective. In fact, by condition (b), for every object M of the category C_X, there exists a deflation $\mathfrak{M} \xrightarrow{\mathfrak{t}} M$ such that $\mathfrak{M} \in ObC_Y$. By condition (a), the kernel of this deflation belongs to the subcategory C_Y. Therefore, the element $[M]$ of the group $K_0(X, \mathfrak{E}_X)$ is equal to $[\mathfrak{M}] - [Ker(\mathfrak{t})]$, which is the image of an element of the group $K_0(Y, \mathfrak{E}_Y)$.

(iii) The map $K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(\mathfrak{j})} K_0(X, \mathfrak{E}_X)$ is injective.

Notice that the kernel of the map $K_0(\mathfrak{j})$ consists of combinations (with coefficients in $\mathbb{Z}$) of the expressions $[M] - [L] - [N]$, where $N \longrightarrow M \longrightarrow L$ runs through conflations of $(C_X, \mathfrak{E}_X)$. The claim is that each of these combinations is equal to zero.

In fact, let $N \xrightarrow{\mathfrak{i}_\mathfrak{t}} M \xrightarrow{\mathfrak{t}} L$ be conflation in C_X. Thanks to the condition (b), it can be inserted into a commutative diagram

$$\begin{array}{ccccc}
\mathcal{M} & \xrightarrow{\mathfrak{u}} & M' & \xrightarrow{\widetilde{\mathfrak{t}}} & \mathcal{L} \\
 & & {\scriptstyle \mathfrak{s}_1}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{s}} \\
 & & M & \xrightarrow{\mathfrak{t}} & L
\end{array}$$

where all arrows are deflations, the square is cartesian, and $\mathcal{L}$, $\mathcal{M}$ are objects of the subcategory C_Y. Therefore, we obtain a commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{N}} & \longrightarrow & \widetilde{\mathcal{M}} & \longrightarrow & \widetilde{\mathcal{L}} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathfrak{i}_\mathfrak{s}} \\
\mathcal{N} & \longrightarrow & \mathcal{M} & \xrightarrow{\widetilde{\mathfrak{t}}\mathfrak{u}} & \mathcal{L} \\
\downarrow & & {\scriptstyle \mathfrak{s}_1\mathfrak{u}}\downarrow & & \downarrow{\scriptstyle \mathfrak{s}} \\
N & \xrightarrow{\mathfrak{i}_\mathfrak{t}} & M & \xrightarrow{\mathfrak{t}} & L
\end{array} \qquad (3)$$

whose rows and columns are conflations. Therefore,

$$[M] - [L] - [N] = ([\mathcal{M}] - [\mathcal{L}] - [\mathcal{N}]) - ([\widetilde{\mathcal{M}}] - [\widetilde{\mathcal{L}}] - [\widetilde{\mathcal{N}}]),$$

because $[\mathcal{L}] = [\widetilde{\mathcal{L}}] + [L]$, $[\mathcal{M}] = [\widetilde{\mathcal{M}}] + [M]$ and $[\mathcal{N}] = [\widetilde{\mathcal{N}}] + [N]$.

It follows from the condition (a) (applied to the columns of the diagram (2)) that two upper rows of (2) are conflations in $(C_Y, \mathfrak{E}_Y)$. ∎

7.1.1. Note. The condition (a) in 7.1 can be replaced by the condition

(a2) If $M \xrightarrow{\mathfrak{t}} N$ is a deflation in C_X such that M belongs to the subcategory C_Y, then $M \times_N M$ is an object of C_Y.

In the additive case, the condition (a2) is equivalent to the condition (a), because then $M \times_N M$ is isomorphic to $M \coprod Ker(\mathfrak{t})$. It follows from the observation 1.6.1 that, in a sense, the condition (a2) is equivalent to (a) *with respect to* K_0 regardless of additivity (or any other properties) of the category C_X.

7.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_\mathfrak{Z}, \mathfrak{E}_\mathfrak{Z})$ be right exact categories with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ an 'exact' ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_\mathfrak{Z}, \mathfrak{E}_\mathfrak{Z})$. Let C_Y be the full subcategory of C_X generated by T-acyclic objects (that is objects V such that $T_i(V)$ is an initial object of $C_\mathfrak{Z}$ for $i \geq 1$). Assume that for every $M \in ObC_X$, there is a deflation $P \longrightarrow M$ with $P \in ObC_Y$, and that $T_n(M)$ is an initial object of $C_\mathfrak{Z}$ for n sufficiently large. Then the natural map $K_\bullet(Y, \mathfrak{E}_Y) \longrightarrow K_\bullet(X, \mathfrak{E}_X)$ is an isomorphism.*

Proof. Let C_{Y_n} denote the full subcategory of the category C_X generated by all objects M such that $T_i(M)$ is an initial object of $C_\mathfrak{Z}$ for $i \geq n$.

(i) All the subcategories C_{Y_n} are fully exact.

Indeed, if $N \longrightarrow M \longrightarrow L$ is a conflation in $(C_X, \mathfrak{E}_X)$ such that N and L are objects of the subcategory C_{Y_n}, then, thanks to the 'exactness' of the ∂^*-functor T, we have an exact sequence

$$\dots \longrightarrow T_{m+1}(L) \longrightarrow T_m(N) \longrightarrow T_m(M) \longrightarrow T_m(L) \longrightarrow \dots$$

If $m \geq n$, then the objects $T_m(N)$ and $T_m(L)$ are initial. Since the kernel of a morphism of an object $\mathcal{M}$ to an initial object is isomorphic to $\mathcal{M}$, it follows that $T_m(M)$ is an initial object.

(ii) Let $N \longrightarrow M \longrightarrow L$ be a conflation in $(C_X, \mathfrak{E}_X)$ such that $M \in ObC_{Y_n}$ and $L \in ObC_{Y_{n+1}}$. Then N is an object of C_{Y_n}.

In fact, we have an 'exact' sequence

which yields the 'exact' sequence $z \longrightarrow T_m(N) \longrightarrow z$ for all $m \geq n$, where z is an initial object of the category $C_\mathfrak{Z}$. Therefore, $T_m(N)$ is an initial object for $m \geq n$.

(iii) This shows that the subcategory C_{Y_n} of the right exact category $(C_{Y_{n+1}}, \mathfrak{E}_{Y_{n+1}})$ satisfies the condition (a) of 7.1. The condition (b) of 7.1 holds, because $C_Y = C_{Y_1} \subseteq C_{Y_n}$

and, by hypothesis, for every $M \in ObC_X$, there exists a deflation $P \longrightarrow M$ with $P \in ObC_Y$. Applying 7.1, we obtain that the natural map $K_\bullet(Y_n, \mathfrak{E}_{Y_n}) \longrightarrow K_\bullet(Y_{n+1}, \mathfrak{E}_{Y_{n+1}})$ is an isomorphism for all $n \geq 1$. Since, by hypothesis, $C_X = \bigcup_{n\geq 1} C_{Y_n}$, the isomorphisms $K_\bullet(Y_n, \mathfrak{E}_{Y_n}) \xrightarrow{\sim} K_\bullet(Y_{n+1}, \mathfrak{E}_{Y_{n+1}})$ imply that the natural map $K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(X, \mathfrak{E}_X)$ is an isomorphism. ■

8. Characteristic 'exact' filtrations and sequences.

8.1. The right exact 'spaces' $(X_n, \mathfrak{E}_{X_n})$. For a right exact exact 'space' $(X, \mathfrak{E}_X)$, let C_{X_n} be the category whose objects are sequences $M_n \longrightarrow M_{n-1} \longrightarrow \dots \longrightarrow M_0$ of n morphisms of $\mathfrak{E}_X$, $n \geq 1$, and morphisms between sequences are commutative diagrams

$$\begin{array}{ccccccc} M_n & \longrightarrow & M_{n-1} & \longrightarrow & \dots & \longrightarrow & M_0 \\ {\scriptstyle f_n}\downarrow & & {\scriptstyle f_{n-1}}\downarrow & & \dots & & \downarrow{\scriptstyle f_0} \\ M'_n & \longrightarrow & M'_{n-1} & \longrightarrow & \dots & \longrightarrow & M'_0 \end{array}$$

Notice that if x is an initial object of the category C_X, then $x \longrightarrow \dots \longrightarrow x$ is an initial object of C_{X_n}.

We denote by $\mathfrak{E}_{X_n}$ the class of all morphisms (f_i) of the category C_{X_n} such that $f_i \in \mathfrak{E}_X$ for all $0 \leq i \leq n$.

8.1.1. Proposition. *(a) The pair $(C_{X_n}, \mathfrak{E}_{X_n})$ is a right exact category.*

(b) The map which assigns to each right exact 'space' $(X, \mathfrak{E}_X)$ the right exact 'space' $(X_n, \mathfrak{E}_{X_n})$ extends naturally to an 'exact' endofunctor of the left exact category $(\mathfrak{Esp}_r, \mathfrak{I}_{\mathfrak{c}}^{\rightarrow})$ of right 'exact' 'spaces' which induces an 'exact' endofunctor $\mathcal{P}_n$ of its exact subcategory $(\mathfrak{Esp}_r^*, \mathfrak{I}_{\mathfrak{c}*}^{\rightarrow})$.

Proof. The argument is left to the reader. ■

8.2. Proposition. (Additivity of 'characteristic' filtrations) *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects and $f_n^* \xrightarrow{\mathfrak{t}_n} f_{n-1}^* \xrightarrow{\mathfrak{t}_{n-1}} \dots \xrightarrow{\mathfrak{t}_1} f_0^*$ a sequence of deflations of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ such that the functors $\mathfrak{k}_i^* = Ker(\mathfrak{t}_i^*)$ are 'exact' for all $1 \leq i \leq n$. Then $K_\bullet(f_n) = K_\bullet(f_0) + \sum_{1\leq i\leq n} K_\bullet(\mathfrak{k}_i)$.*

Proof. (a) For $1 \leq i \leq n$, let $\mathfrak{p}_{Y,i}^*$ denote the functor $C_{Y_n} \longrightarrow C_Y$ which assigns to every object $\mathcal{M} = (M_n \xrightarrow{\gamma_n} \dots \xrightarrow{\gamma_1} M_0)$ of C_{Y_n} the object M_i and to every morphism $f = (f_m)$ the morphism f_i. The assignment to any object $\mathcal{M} = (M_n \xrightarrow{\gamma_n} \dots \xrightarrow{\gamma_1} M_0)$ of C_{Y_n} of C_{Y_n} the deflation $M_i \xrightarrow{\gamma_i} M_{i-1}$ is a functor morphism $\mathfrak{p}_{Y,i}^* \xrightarrow{\mathfrak{t}_i^Y} \mathfrak{p}_{Y,i-1}^*$. Let $\mathfrak{k}_{Y,i}^*$ denote the kernel of $\mathfrak{t}_i^Y$, i.e. the functor $C_{Y_n} \longrightarrow C_Y$ that assigns to an object $\mathcal{M} = (M_n \xrightarrow{\gamma_n} \dots \xrightarrow{\gamma_1} M_0)$

the kernel of $M_i \xrightarrow{\gamma_i} M_{i-1}$. Thus, we obtain a diagram

$$\begin{array}{ccccccccc} \mathfrak{k}^*_{Y,n} & & \mathfrak{k}^*_{Y,n-1} & & \dots & & \mathfrak{k}^*_{Y,1} & & \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \\ \mathfrak{p}^*_{Y,n} & \xrightarrow{\mathfrak{t}^Y_n} & \mathfrak{p}^*_{Y,n-1} & \longrightarrow & \dots & \longrightarrow & \mathfrak{p}^*_{Y,1} & \xrightarrow{\mathfrak{t}^Y_1} & \mathfrak{p}^*_{Y,0} \end{array} \qquad (1)$$

of functors from C_{Y_n} to C_Y whose horizontal arrows are deflations.

The functors $\mathfrak{p}^*_{Y,i-1}$ and $\mathfrak{k}^*_{Y,i}$ map initial objects to initial objects and pull-backs of deflations to pull-backs of deflations; i.e. they are inverse image functors of morphisms of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$. These morphisms depend functorially on the right exact 'space' $(Y, \mathfrak{E}_Y)$, that is they form functor morphisms

$$\mathcal{P}_n \xrightarrow{\mathfrak{p}_i} Id_{\mathfrak{Esp}^*_{\mathfrak{r}}}, \quad 0 \leq i \leq n, \quad \text{and} \quad \mathcal{P}_n \xrightarrow{\mathfrak{k}_i} Id_{\mathfrak{Esp}^*_{\mathfrak{r}}}, \quad 1 \leq i \leq n.$$

These morphisms induce morphisms

$$K_\bullet \xleftarrow{K_\bullet(\mathfrak{k}_i)} K_\bullet \circ \mathcal{P}_n \xrightarrow{K_\bullet(\mathfrak{p}_i)} K_\bullet$$

of ∂^*-functors. The claim is that the morphism $K_\bullet(\mathfrak{p}_n)$ coincides with the morphism $K_\bullet(\mathfrak{p}_0) + \sum_{1 \leq i \leq n} K_\bullet(\mathfrak{k}_i)$.

In fact, the zero components of these morphisms coincide. Since $K_\bullet$ is a universal ∂^*-functor, this implies that the entire morphisms coincide with each other.

(b) The argument above proves, in a functorial way, the assertion 8.2 for the special case of the sequence of deflations $\mathfrak{p}^*_{Y,n} \xrightarrow{\mathfrak{t}^Y_n} \mathfrak{p}^*_{Y,n-1} \longrightarrow \dots \longrightarrow \mathfrak{p}^*_{Y,1} \xrightarrow{\mathfrak{t}^Y_1} \mathfrak{p}^*_{Y,0}$. of 'exact' functors from C_{Y_n} to C_Y. That is

$$K_\bullet(\mathfrak{p}_{Y,n}) = K_\bullet(\mathfrak{p}_{Y,0}) + \sum_{1 \leq i \leq n} K_\bullet(\mathfrak{k}_{Y,i}). \qquad (2)$$

Consider now the general case.

A sequence of deflations $f^*_n \xrightarrow{\mathfrak{t}_n} f^*_{n-1} \xrightarrow{\mathfrak{t}_{n-1}} \dots \xrightarrow{\mathfrak{t}_1} f^*_0$ of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ defines an 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{\widetilde{\mathfrak{f}}^*_n} (C_{Y_n}, \mathfrak{E}_{Y_n})$. The kernels $\mathfrak{k}^*_i = Ker(\mathfrak{t}_i)$ map initial objects to initial objects. The fact that they are 'exact' (which is equivalent to the condition that they map deflations to deflations) means that they are inverse image functors of morphisms of $\mathfrak{Esp}^*_{\mathfrak{r}}$, hence the morphisms $K_\bullet(\mathfrak{k}_i)$ are well defined. Therefore, the morphism $K_\bullet(f_0) + \sum_{1\le i\le n} K_\bullet(\mathfrak{k}_i)$ from $K_\bullet(X,\mathfrak{E}_X)$ to $K_\bullet(Y,\mathfrak{E}_Y)$ is well defined. One can see that

$$K_\bullet(f_n) = K_\bullet(\mathfrak{p}_{Y,n}) \circ K_\bullet(\widetilde{\mathfrak{f}}_n) \quad \text{and}$$
$$K_\bullet(f_0) + \sum_{1\le i\le n} K_\bullet(\mathfrak{k}_i) = \big(K_\bullet(\mathfrak{p}_{Y,0}) + \sum_{1\le i\le n} K_\bullet(\mathfrak{k}_{Y,i})\big) \circ K_\bullet(\widetilde{\mathfrak{f}}_n)$$

So that the assertion follows from the equality (2). ■

8.3. Corollary. *Let $(C_X,\mathfrak{E}_X)$ and $(C_Y,\mathfrak{E}_Y)$ be right exact categories with initial objects and $g^* \longrightarrow f^* \longrightarrow h^*$ a conflation of 'exact' functors from $(C_X,\mathfrak{E}_X)$ to $(C_Y,\mathfrak{E}_Y)$. Then $K_\bullet(f) = K_\bullet(g) + K_\bullet(h)$.*

8.4. Corollary. (Additivity for 'characteristic' 'exact' sequences) *Let*

$$\mathfrak{f}_n^* \longrightarrow \mathfrak{f}_{n-1}^* \longrightarrow \dots \longrightarrow \mathfrak{f}_1^* \longrightarrow \mathfrak{f}_0^*$$

be an 'exact' sequence of 'exact' functors from $(C_X,\mathfrak{E}_X)$ to $(C_Y,\mathfrak{E}_Y)$ which map initial objects to initial objects. Suppose that $\mathfrak{f}_1^* \longrightarrow \mathfrak{f}_0^*$ is a deflation and $\mathfrak{f}_n^* \longrightarrow \mathfrak{f}_{n-1}^*$ is the kernel of $\mathfrak{f}_{n-1}^* \longrightarrow \mathfrak{f}_{n-2}^*$. Then the morphism $\sum_{0\le i\le n}(-1)^i K_\bullet(\mathfrak{f}_i)$ from $K_\bullet(X,\mathfrak{E}_X)$ to $K_\bullet(Y,\mathfrak{E}_Y)$ is equal to zero.

Proof. The assertion follows from 8.3 by induction.

A more conceptual proof goes along the lines of the argument of 8.2. Namely, we assign to each right exact category $(C_Y,\mathfrak{E}_Y)$ the right exact category $(C_{Y_n^\mathfrak{c}},\mathfrak{E}_{Y_n^\mathfrak{c}})$ whose objects are 'exact' sequences $\mathcal{L} = (L_n \longrightarrow L_{n-1} \longrightarrow \dots \longrightarrow L_1 \longrightarrow L_0)$, where $L_1 \longrightarrow L_0$ is a deflation and $L_n \longrightarrow L_{n-1}$ is the kernel of $L_{n-1} \longrightarrow L_{n-2}$. This assignment defines an endofunctor $\mathfrak{P}_n^\mathfrak{c}$ of the category $\mathfrak{Esp}_\mathfrak{r}^*$ of right exact 'spaces' with initial objects, and maps $\mathcal{L} \longmapsto L_i$ determine morphisms $\mathfrak{P}_n^\mathfrak{c} \longrightarrow Id_{\mathfrak{Esp}_\mathfrak{r}^*}$. The rest of the argument is left to the reader. ■

9. Complements.

9.1. Another description of the functor K_0. Fix a right exact category $(C_X,\mathfrak{E}_X)$. Let $C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ denote the category having the same objects as C_X and with morphisms defined as follows. For any pair M, L of objects, consider all diagrams (if any) of the form $M \xleftarrow{\mathfrak{e}} \widetilde{M} \xrightarrow{f} L$, where $\mathfrak{e}$ is a deflation and f an arbitrary morphism of C_X. We consider isomorphisms between such diagrams of the form (id_M,ϕ,id_L) and define morphisms from M to L as isomorphism classes of these diagrams. The composition of the morphisms $N \xleftarrow{\mathfrak{t}} \widetilde{N} \xrightarrow{g} M$ and $M \xleftarrow{\mathfrak{e}} \widetilde{M} \xrightarrow{f} L$ is the morphism represented by the pair $(\mathfrak{t}\circ\widetilde{\mathfrak{e}}, f\circ g')$

in the diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{N}} & \stackrel{g'}{\longrightarrow} & \widetilde{M} & \stackrel{f}{\longrightarrow} & L \\
{\scriptstyle \widetilde{\mathfrak{e}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} & & \\
\widetilde{N} & \stackrel{g}{\longrightarrow} & M & & \\
{\scriptstyle \mathfrak{t}}\downarrow & & & & \\
M & & & &
\end{array}$$

with cartesian square. If the category C_X is svelte (i.e. it represents a 'space'), then $C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ is a well defined svelte category.

There is a canonical functor $C_X \xrightarrow{\mathfrak{r}_X^*} C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ which is identical on objects and maps each morphism $M \xrightarrow{f} L$ to the morphism represented by the diagram $M \xleftarrow{id} M \xrightarrow{f} L$.

Let $C_{|\mathfrak{E}_X|}$ denote the subcategory of C_X formed by all deflations. The map which assigns to every morphism $M \xrightarrow{\mathfrak{e}} N$ of $\mathfrak{E}_X$ the morphism of $C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ represented by the diagram $N \xleftarrow{\mathfrak{e}} M \xrightarrow{id} M$ is a functor $C^{op}_{|\mathfrak{E}_X|} \xrightarrow{\mathfrak{l}_X^*} C_{\mathfrak{L}(X,\mathfrak{E}_X)}$.

Let $\mathcal{G}(X)$ denote the group $\mathbb{Z}_0|C_X|$ which is identified with the corresponding groupoid with one object. Let $\mathfrak{p}_X$ denote the map $HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} \longrightarrow \mathcal{G}(X)$ which assigns to a morphism $[N \xleftarrow{\mathfrak{e}} M \xrightarrow{f} L]$ represented by the diagram $N \xleftarrow{\mathfrak{e}} M \xrightarrow{f} L$ the element $[M]-[N]$ of the group $\mathcal{G}(X)$. We have a (non-commutative) diagram

$$\begin{array}{ccc}
Hom^2C_{\mathfrak{L}(X,\mathfrak{E}_X)} & \xrightarrow{\mathfrak{p}_X \times \mathfrak{p}_X} & \mathcal{G}(X) \times \mathcal{G}(X) \\
{\scriptstyle \mathfrak{c}}\downarrow & & \downarrow{\scriptstyle +} \\
HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} & \xrightarrow{\mathfrak{p}_X} & \mathcal{G}(X)
\end{array} \qquad (1)$$

where Hom^2C_Z stands for the class of composable morphisms of the category C_Z and the vertical arrows are compositions. Taking the compositions in the diagram (1), we obtain a pair of arrows

$$Hom^2C_{\mathfrak{L}(X,\mathfrak{E}_X)} \underset{\mathfrak{v}_X}{\overset{\mathfrak{u}_X}{\rightrightarrows}} \mathcal{G}(X). \qquad (2)$$

Consider the *cokernel of (2) in the category of groupoids*, which is, by definition, the universal groupoid morphism equalizing the pair of maps (2).

9.1.1. Proposition. *The cokernel of the pair (2) in the category of groupoids is (isomorphic to) the group $K_0(X,\mathfrak{E}_X)$ defined in Section 1.*

Proof. The fact follows from the definitions. ∎

9.1.2. Note. The map $HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} \xrightarrow{\mathfrak{p}_X} \mathcal{G}(X)$ is the composition of the map $HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} \xrightarrow{\pi_X} \mathfrak{E}_X$ and the map $\mathfrak{E}_X \xrightarrow{\lambda_X} \mathcal{G}(X)$ which assigns to each deflation $M \longrightarrow L$

the element $[M] - [L]$ of $\mathcal{G}(X)$. One can see that $\pi_X \circ \mathfrak{l}_X^*$ is the identical map, and the map λ_X is a functor $C^{op}_{|\mathfrak{E}_X|} \longrightarrow \mathcal{G}(X)$.

9.1.3. Functorialities. Any 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}^*} (C_Y, \mathfrak{E}_Y)$ between right exact categories induces a functor $C_{\mathfrak{L}(X,\mathfrak{E}_X)} \xrightarrow{\mathfrak{L}(\mathfrak{f})^*} C_{\mathfrak{L}(Y,\mathfrak{E}_Y)}$ such that the diagram

$$\begin{array}{ccccccccc} K_0(X) & \longleftarrow & \mathcal{G}(X) & \xleftarrow{\lambda_X} & C^{op}_{|\mathfrak{E}_X|} & \xrightarrow{\mathfrak{l}_X^*} & C_{\mathfrak{L}(X,\mathfrak{E}_X)} & \xleftarrow{\mathfrak{r}_X^*} & C_X \\ {\scriptstyle K_0(\mathfrak{f})}\big\downarrow & & {\scriptstyle \mathcal{G}(\mathfrak{f})}\big\downarrow & & \big\downarrow & & {\scriptstyle \mathfrak{L}(\mathfrak{f})^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}^*} \\ K_0(Y) & \longleftarrow & \mathcal{G}(Y) & \xleftarrow{\lambda_Y} & C^{op}_{|\mathfrak{E}_Y|} & \xrightarrow{\mathfrak{l}_Y^*} & C_{\mathfrak{L}(Y,\mathfrak{E}_Y)} & \xleftarrow{\mathfrak{r}_Y^*} & C_Y \end{array} \qquad (3)$$

commutes, as well as the diagram

$$\begin{array}{ccc} HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} & \xrightarrow{\pi_X} & \mathfrak{E}_X \\ {\scriptstyle \mathfrak{f}^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{E}(\mathfrak{f})^*} \\ HomC_{\mathfrak{L}(Y,\mathfrak{E}_Y)} & \xrightarrow{\pi_Y} & \mathfrak{E}_Y \end{array} \qquad (4)$$

9.2. The Q-construction for right exact categories with initial objects. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We denote by $\mathfrak{I}_X$ the class of all inflations of $(C_X, \mathfrak{E}_X)$ (i.e. morphisms which are kernels of deflations) and by $\mathfrak{I}_X^\infty$ the smallest subcategory of C_X containing $\mathfrak{I}_X$.

We denote by $C_{\mathcal{Q}(X,\mathfrak{E}_X)}$ the subcategory of the category $C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ formed by all morphisms $M \xleftarrow{\mathfrak{e}} \widetilde{M} \xrightarrow{\mathfrak{j}} L$, where ($\mathfrak{e}$ is a deflation and) $\mathfrak{j} \in \mathfrak{I}_X^\infty$.

9.2.1. Note. If $(C_X, \mathfrak{E}_X)$ is an exact k-linear category, then $\mathfrak{I}_X^\infty = \mathfrak{I}_X$ and the category $C_{\mathcal{Q}(X,\mathfrak{E}_X)}$ coincides with the Quillen's category $\mathcal{Q}C_X$ associated with the exact category $(C_X, \mathfrak{E}_X)$ (see [Q, p. 102]).

Let

$$Hom^2 C_{\mathcal{Q}(X,\mathfrak{E}_X)} \underset{\mathfrak{b}_X}{\overset{\mathfrak{a}_X}{\rightrightarrows}} \mathcal{G}(X). \qquad (1)$$

be the composition of the pair of maps 9.1(2) with the embedding

$$Hom^2 C_{\mathcal{Q}(X,\mathfrak{E}_X)} \longrightarrow Hom^2 C_{\mathfrak{L}(X,\mathfrak{E}_X)};$$

and let $Cok_{\mathfrak{G}}(\mathfrak{a}_X, \mathfrak{b}_X)$ denote the cokernel of the pair of maps $(\mathfrak{a}_X, \mathfrak{b}_X)$ in the category of groupoids.

9.2.2. Proposition. *The unique map* $Cok_{\mathfrak{G}}(\mathfrak{a}_X, \mathfrak{b}_X) \longrightarrow K_0(X, \mathfrak{E}_X)$ *which makes commute the diagram*

$$\begin{array}{ccccc}
Hom^2 C_{\mathfrak{Q}(X,\mathfrak{E}_X)} & \overset{\mathfrak{a}_X}{\underset{\mathfrak{b}_X}{\rightrightarrows}} & \mathcal{G}(X) & \longrightarrow & Cok_{\mathfrak{G}}(\mathfrak{a}_X, \mathfrak{b}_X) \\
\downarrow & & id \downarrow & & \downarrow \\
Hom^2 C_{\mathfrak{L}(X,\mathfrak{E}_X)} & \overset{\mathfrak{u}_X}{\underset{\mathfrak{v}_X}{\rightrightarrows}} & \mathcal{G}(X) & \longrightarrow & K_0(X, \mathfrak{E}_X)
\end{array}$$

is a group(oid) isomorphism.

Proof. The assertion is a consequence of 1.6. ■

9.3. The category $C_{\mathfrak{E}\backslash X}$**.** Fix a right exact 'space' $(X, \mathfrak{E}_X)$. For two objects, M and L, consider the class of all diagrams of the form $M \overset{\mathfrak{t}}{\longleftarrow} M_1 \overset{\xi}{\longrightarrow} L$, where $\mathfrak{t} \in \mathfrak{E}_X$. We say that the diagram $M \overset{\mathfrak{t}}{\longleftarrow} M_1 \overset{\xi}{\longrightarrow} L$, is *equivalent* to a diagram $M \overset{\mathfrak{t}'}{\longleftarrow} M_1' \overset{\xi'}{\longrightarrow} L$, if the compositions of ξ and ξ' with appropriate projections in the cartesian square

$$\begin{array}{ccccc}
 & & \mathfrak{M} & \xrightarrow{\mathfrak{t}_2} & M_1 & \xrightarrow{\xi} & L \\
 & & \mathfrak{t}_1 \downarrow & cart & \downarrow \mathfrak{t} & & \\
L & \xleftarrow{\xi'} & M_1' & \xrightarrow{\mathfrak{t}'} & M & &
\end{array}$$

coincide, that is $\xi \circ \mathfrak{t}_2 = \xi' \circ \mathfrak{t}_1$. Since cartesian squares with arrows $\mathfrak{t}, \mathfrak{t}'$) are all isomorphic to each other, this condition does not depend on the choice of a cartesian square. It follows (from the fact that the square built of cartesian squares is cartesian) that the relation defined this way is, indeed, an equivalence relation.

The equivalence classes of these diagrams with fixed objects M and L form a set, which we denote by $C_{\mathfrak{E}\backslash X}(M, L)$. The elements of the set $C_{\mathfrak{E}\backslash X}(M, L)$ are interpreted as morphisms from M to L. The composition of morphisms $[M_1 \overset{\mathfrak{t}_1}{\longleftarrow} \mathfrak{M}_1 \overset{\xi_1}{\longrightarrow} M_2]$ and $[M_2 \overset{\mathfrak{t}_2}{\longleftarrow} \mathfrak{M}_2 \overset{\xi_2}{\longrightarrow} M_3]$ is defined via the diagram

$$\begin{array}{ccccccc}
 & & \mathfrak{M}_3 & \xrightarrow{\widetilde{\xi_1}} & \mathfrak{M}_2 & \xrightarrow{\xi_2} & M_3 \\
 & & \mathfrak{t}_2' \downarrow & cart & \downarrow \mathfrak{t}_2 & & \\
M_1 & \xleftarrow{\mathfrak{t}_1} & \mathfrak{M}_1 & \xrightarrow{\xi_1} & M_2 & &
\end{array}$$

with a cartesian square: the composition is the equivalence class

$$[M_1 \xleftarrow{\mathfrak{t}_1 \circ \mathfrak{t}_2'} \mathfrak{M}_3 \xrightarrow{\xi_2 \circ \widetilde{\xi_1}} M_3].$$

Altogether defines a category $C_{\mathfrak{E}\backslash X}$ which has the same class of objects as the category C_X; and its morphisms and their compositions are defined above.

9.3.1. A canonical embedding. There is a canonical functor

$$C_X \xrightarrow{\mathfrak{q}^*_{\mathfrak{E}\backslash X}} C_{\mathfrak{E}\backslash X} \qquad (1)$$

which is identical on objects and maps every morphism $M \xrightarrow{\gamma} L$ to the equivalence class of $M \xleftarrow{id_M} M \xrightarrow{\gamma} L$. It follows from the definition of the morphisms of the category $C_{\mathfrak{E}\backslash X}$ that the functor (1) is faithful: two arrows of the category C_X with the same source and target, $M \underset{\beta}{\overset{\alpha}{\rightrightarrows}} L$, have the same image in $C_{\mathfrak{E}\backslash X}$ iff they coincide.

9.3.2. Proposition. *The class* $\widetilde{\mathfrak{E}}_{\mathfrak{E}\backslash X} = \{\mathfrak{q}^*_{\mathfrak{E}\backslash X}(\mathfrak{s}) \mid \mathfrak{s} \in \mathfrak{E}_X\}$ *is a subcanonical pretopology on the category* $C_{\mathfrak{E}\backslash X}$. *Every morphism of* $\widetilde{\mathfrak{E}}_{\mathfrak{E}\backslash X}$ *has a canonical splitting.*

Proof. (a) It follows from the definition of morphisms of $C_{\mathfrak{E}\backslash X}$ (– equivalence classes) and the composition of morphisms that, for every morphism $M \xrightarrow{\mathfrak{s}} L$ of $\mathfrak{E}_X$, the equivalence class $[L \xleftarrow{\mathfrak{s}} M \xrightarrow{\mathfrak{s}} L]$ is the identity morphism: $[L \xleftarrow{id_L} L \xrightarrow{id_L} L] = \mathfrak{q}^*_{\mathfrak{E}\backslash X}(id_L)$.

On the other hand, $[L \xleftarrow{\mathfrak{s}} M \xrightarrow{\mathfrak{s}} L]$ is the composition of $[L \xleftarrow{\mathfrak{s}} M \xrightarrow{id_M} M]$ and $[M \xleftarrow{id_M} M \xrightarrow{\mathfrak{s}} L] = \mathfrak{q}^*_{\mathfrak{E}\backslash X}(\mathfrak{s})$. So that the morphism $[L \xleftarrow{\mathfrak{s}} M \xrightarrow{id_M} M]$ is a left inverse of $\mathfrak{q}^*_{\mathfrak{E}\backslash X}(\mathfrak{s})$; in other words, the epimorphism $\mathfrak{q}^*_{\mathfrak{E}\backslash X}(\mathfrak{s})$ splits. The composition in the opposite order is, by definition,

$$\mathfrak{q}^*_{\mathfrak{E}\backslash X}(\mathfrak{s}) \circ [L \xleftarrow{\mathfrak{s}} M \xrightarrow{id_M} M] = [M \xleftarrow{\mathfrak{p}_1} M \times_L M \xrightarrow{\mathfrak{p}_2} M].$$

So that the projector $\mathfrak{q}^*_{\mathfrak{E}\backslash X}(M) \xrightarrow{\mathfrak{p}_{\mathfrak{s}}} \mathfrak{q}^*_{\mathfrak{E}\backslash X}(M)$ corresponding by the canonical splitting of $\mathfrak{q}^*_{\mathfrak{E}\backslash X}(\mathfrak{s})$ is given by the kernel pair of the deflation $M \xrightarrow{\mathfrak{s}} L$.

(b) Let $M \xrightarrow{\mathfrak{s}} L$ be a deflation and $L_1 \xleftarrow{\mathfrak{t}} \mathcal{L} \xrightarrow{\xi} L$ a diagram representing a morphism of the category $C_{\mathfrak{E}\backslash X}$ (that is $\mathfrak{t} \in \mathfrak{E}_X$). Consider the diagram

$$\begin{array}{ccccc}
\mathfrak{M} & \xleftarrow{id_{\mathfrak{M}}} & \mathfrak{M} & \xrightarrow{\xi'} & M \\
{\scriptstyle \mathfrak{t}\widetilde{\mathfrak{s}}}\downarrow & & {\scriptstyle \widetilde{\mathfrak{s}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{s}} \\
L_1 & \xleftarrow{\mathfrak{t}} & \mathcal{L} & \xrightarrow{\xi} & L
\end{array} \qquad (2)$$

with cartesian square. The observation is that the that both squares in the diagram (2) are cartesian.

In fact, let $[\mathcal{N} \xleftarrow{\mathfrak{u}_1} N_1 \xrightarrow{\zeta_1} M]$ and $[\mathcal{N} \xleftarrow{\mathfrak{u}_2} N_2 \xrightarrow{\zeta_1} L_1]$ be morphisms of the category $C_{\mathfrak{E}\backslash X}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{N} & \xrightarrow{[\mathfrak{u}_1,\zeta_1]} & M \\
{\scriptstyle [\mathfrak{u}_2,\zeta_2]}\downarrow & & \downarrow{\scriptstyle [id_M,\mathfrak{s}]} \\
L_1 & \xrightarrow{[\mathfrak{t},\xi]} & L
\end{array} \qquad (3)$$

commutes. The compositions of morphisms of the diagram (3) are encoded in the diagram

$$\begin{array}{ccccc}
\mathcal{N} & \xrightarrow{id_{\mathcal{N}}} & \mathcal{N} & \xleftarrow{\mathfrak{u}_1} & N_1 \\
{\scriptstyle \mathfrak{u}_2}\uparrow & & & & \downarrow{\scriptstyle \zeta_1} \\
N_2 & \xrightarrow{\zeta_2} & L_1 & & M \\
{\scriptstyle \mathfrak{t}_1}\uparrow & cart & \uparrow{\scriptstyle \mathfrak{t}} & & \downarrow{\scriptstyle \mathfrak{s}} \\
\mathfrak{N} & \xrightarrow{\zeta_2'} & \mathcal{L} & \xrightarrow{\xi} & L
\end{array} \qquad (4)$$

in the category C_X with a cartesian inner square. The diagram (4) gives rise to the diagram

$$\begin{array}{ccccccc}
N_1 & \xrightarrow{\mathfrak{u}_1} & \mathcal{N} & \xleftarrow{\mathfrak{u}_1} & N_1 & \xrightarrow{\zeta_1} & M \\
{\scriptstyle \mathfrak{u}}\uparrow & cart & {\scriptstyle \mathfrak{u}_2\mathfrak{t}_1}\uparrow & & & & \downarrow{\scriptstyle \mathfrak{s}} \\
\widetilde{\mathfrak{N}} & \xrightarrow{\widetilde{\mathfrak{u}}_1} & \mathfrak{N} & \xrightarrow{\zeta_2'} & \mathcal{L} & \xrightarrow{\xi} & L
\end{array}$$

with cartesian left square with all arrows deflations. The commutativity of the square (3) means, by definition, the commutativity of the diagram

$$\begin{array}{ccccc}
\widetilde{\mathfrak{N}} & \xrightarrow{\mathfrak{u}} & N_1 & \xrightarrow{\zeta_1} & M \\
{\scriptstyle \widetilde{\mathfrak{u}}_1}\downarrow & & & & \downarrow{\scriptstyle \mathfrak{s}} \\
\mathfrak{N} & \xrightarrow{\zeta_2'} & \mathcal{L} & \xrightarrow{\xi} & L
\end{array}$$

But, then, thanks to the cartesian square in (1), we have a commutative diagram

$$\begin{array}{ccccc}
 & & \widetilde{\mathfrak{N}} & \xrightarrow{\mathfrak{u}} & N_1 \\
 & & {\scriptstyle \mathfrak{v}}\downarrow & & \downarrow{\scriptstyle \zeta_1} \\
\widetilde{\mathfrak{N}} & \xrightarrow{\mathfrak{v}} & \mathfrak{M} & \xrightarrow{\xi'} & M \\
{\scriptstyle \widetilde{\mathfrak{u}}_1}\downarrow & & {\scriptstyle \widetilde{\mathfrak{s}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{s}} \\
\mathfrak{N} & \xleftarrow{\zeta_2'} & \mathcal{L} & \xrightarrow{\xi} & L
\end{array} \qquad (5)$$

with a uniquely determined morphism $\widetilde{\mathfrak{N}} \xrightarrow{\mathfrak{v}} \mathfrak{M}$.

Thus, we obtain a morphism $[\mathcal{N} \xleftarrow{\mathfrak{u}_1\mathfrak{u}} \widetilde{\mathfrak{N}} \xrightarrow{\mathfrak{v}} \mathfrak{M}]$ from $\mathcal{N}$ to $\mathfrak{M}$ such that

$$\mathfrak{q}^*(\mathfrak{t} \circ \widetilde{\mathfrak{s}}) \circ [\mathfrak{u}_1 \circ \mathfrak{u}, \mathfrak{v}] = [\mathfrak{u}_2, \zeta_2] \quad \text{and} \quad [id_{\mathfrak{M}}, \xi'] \circ [\mathfrak{u}_1 \circ \mathfrak{u}, \mathfrak{v}] = [\mathfrak{u}_1, \zeta_1]$$

(see (1) and (2)). The uniqueness of such morphism follows from the construction. ■

9.3.3. A right exact structure on $C_{\mathfrak{E}\backslash X}$. It follows from 9.3.2 that the class $\mathfrak{E}_{\mathfrak{E}\backslash X}$ of morphisms of the category $C_{\mathfrak{E}\backslash X}$ generated by $\widetilde{\mathfrak{E}}_{\mathfrak{E}\backslash X} = \{\mathfrak{q}^*_{\mathfrak{E}\backslash X}(\mathfrak{s}) \mid \mathfrak{s} \in \mathfrak{E}_X\}$ and all isomorphisms is a right exact structure on $C_{\mathfrak{E}\backslash X}$.

It follows that the functor

$$C_X \xrightarrow{\mathfrak{q}^*_{\mathfrak{E}\backslash X}} C_{\mathfrak{E}\backslash X}$$

is an inverse is an 'exact' functor $(C_X, \mathfrak{E}_X) \longrightarrow (C_{\mathfrak{E}\backslash X}, \mathfrak{E}_{\mathfrak{E}\backslash X})$.

9.3.4. The universal meaning of the right exact 'space' $(\mathfrak{E}\backslash X, \mathfrak{E}_{\mathfrak{E}\backslash X})$. Consider the category $\mathfrak{Sp}_{\mathfrak{r}}$ whose objects are pairs $(X, \mathfrak{P}_X)$, where X is a 'space' and $\mathfrak{P}_X$ is a class of split idempotents of the category C_X such that the associated class $\mathfrak{E}^{\mathfrak{p}}_{\mathfrak{X}}$ of split epimorphisms (which consists of cokernels of the pairs $M \overset{\mathfrak{p}}{\underset{id_M}{\rightrightarrows}} M$, where $\mathfrak{p} = \mathfrak{p}^2$) is a right exact structure on C_X. Morphisms from $(X, \mathfrak{P}_X)$ to $(Y, \mathfrak{P}_Y)$ are the corresponding morphisms $(X, \mathfrak{E}^{\mathfrak{p}}_X) \longrightarrow (X, \mathfrak{E}^{\mathfrak{p}}_Y)$ of right exact 'spaces'. The composition of morphisms is standard. So that the map

$$\begin{array}{ccc} (X, \mathfrak{P}_X) & & (X, \mathfrak{E}^{\mathfrak{p}}_X) \\ \mathfrak{f} \big\downarrow & \longmapsto & \big\downarrow \mathfrak{f} \\ (Y, \mathfrak{P}_Y) & & (Y, \mathfrak{E}^{\mathfrak{p}}_Y) \end{array}$$

is a functor $\mathfrak{Sp}_{\mathfrak{r}} \longrightarrow \mathfrak{Esp}_{\mathfrak{r}}$.

9.3.4.1. Proposition. *The functor*

$$\mathfrak{Sp}_{\mathfrak{r}} \xrightarrow{\mathfrak{Q}^*} \mathfrak{Esp}_{\mathfrak{r}}, \quad (X, \mathfrak{E}^{\mathfrak{p}}_X) \longrightarrow (X, \mathfrak{E}^{\mathfrak{p}}_Y),$$

is left adjoint to the functor $\mathfrak{Esp}_{\mathfrak{r}} \xrightarrow{\mathfrak{Q}_*} \mathfrak{Sp}_{\mathfrak{r}}$ *which maps every right exact 'space'* $(X, \mathfrak{E}_X)$ *to* $(X, \mathfrak{P}\mathfrak{E}_X)$, *where* $\mathfrak{P}\mathfrak{E}_X$ *consists of all idempotents* $[M \xleftarrow{\mathfrak{p}_1} M \times_{\mathfrak{s},\mathfrak{s}} M \xrightarrow{\mathfrak{p}_2} M]$ *– the equivalence classes of the kernel pair of a deflation* $M \xrightarrow{\mathfrak{s}} L$, *where* $\mathfrak{s}$ *runs through the class* $\mathfrak{E}_X$ *of all deflations.*

Proof. The canonical functor $C_X \xrightarrow{\mathfrak{q}^*_{\mathfrak{E}\backslash X}} C_{\mathfrak{E}\backslash X}$ is the inverse image functor of the adjunction morphism

$$\mathfrak{Q}^*\mathfrak{Q}_*(X, \mathfrak{E}_X) = (\mathfrak{E}\backslash X, \mathfrak{E}_{\mathfrak{E}\backslash X}) \xrightarrow{\mathfrak{q}_{\mathfrak{E}\backslash X}} (X, \mathfrak{E}_X).$$

The other adjunction morphism,

$$(X, \mathfrak{P}_X) \longrightarrow \mathfrak{Q}_* \mathfrak{Q}^*(X, \mathfrak{P}_X)$$

is an isomorphism. ■

9.3.5. Relation with the localization at the class of deflations. It follows from the description of the localization of the category C_X at the class $\mathfrak{E}_X$ given in IV.9.2 that there is a natural surjective functor

$$C_{\mathfrak{E}\backslash X} \xrightarrow{\pi^*_{\mathfrak{E}\backslash x}} \mathfrak{E}_X^{-1} C_X$$

which is identical on objects and map morphisms to their equivalence classes with respect to the relation: two morphisms $M \rightrightarrows L$ are equivalent if they are equalized by a morphism $\mathfrak{q}^*_{\mathfrak{E}\backslash X}(\mathfrak{s})$ for some deflation $L \xrightarrow{\mathfrak{s}} L'$ (here we abuse the fact that the categories C_X and $C_{\mathfrak{E}\backslash X}$ have the same objects). So that our right exact category $(C_{\mathfrak{E}\backslash X}, \mathfrak{E}_{\mathfrak{E}\backslash X})$ is an intermediate step of the localization at the class of deflations $\mathfrak{E}_X$: first the 'exact' functor

$$(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}^*_{\mathfrak{E}\backslash X}} (C_{\mathfrak{E}\backslash X}, \mathfrak{E}_{\mathfrak{E}\backslash X})$$

inverts deflations on the right (that is they obtain a right inverse); then the exact functor

$$(C_{\mathfrak{E}\backslash X}, \mathfrak{E}_{\mathfrak{E}\backslash X}) \xrightarrow{\pi^*_{\mathfrak{E}\backslash x}} (\mathfrak{E}_X^{-1} C_X, Iso(\mathfrak{E}_X^{-1} C_X)$$

finishes the localization: the localization functor $\mathfrak{q}^*_{\mathfrak{E}_X}$ at $\mathfrak{E}_X$ is the composition of the functors $\mathfrak{q}^*_{\mathfrak{E}\backslash X}$ and $\pi^*_{\mathfrak{E}\backslash x}$.

9.4. Complements to "reduction by resolution". The assertions of this section are of less general nature than those of Section 7: they require certain conditions which hold by trivial reasons in exact categories and by less trivial reasons in a wide class of non-additive categories (including the categories of algebras over operads and far beyond).

9.4.1. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *be a right exact category with initial objects; and let*

$$\begin{array}{ccccc}
Ker(f') & \xrightarrow{\beta_1'} & Ker(f) & \xrightarrow{\alpha_1'} & Ker(f'') \\
{\scriptstyle \mathfrak{k}'}\downarrow & & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_1) & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
Ker(\alpha_2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2''
\end{array} \qquad (3)$$

be a commutative diagram (determined by its lower right square) such that $Ker(\mathfrak{k}'')$ and $Ker(\beta_2)$ are trivial. Then

(a) The upper row of (3) is 'exact', and the morphism β_1' is the kernel of α_1'.

(b) Suppose, in addition, that the arrows f', α_1 and α_2 in (3) are deflations and $(C_X, \mathfrak{E}_X)$ has the following property:

(#) If $M \xrightarrow{\mathfrak{e}} N$ is a deflation and $M \xrightarrow{p} M$ an idempotent morphism (i.e. $p^2 = p$) which has a kernel and such that the composition $\mathfrak{e} \circ p$ is a trivial morphism, then the composition of the canonical morphism $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ and $M \xrightarrow{\mathfrak{e}} N$ is a deflation.

Then the upper row of (3) is a conflation.

Proof. (a) It follows from C1.5.1 that the upper row of (3) is 'exact'. It follows from the argument of C1.5.1 that the morphism $Ker(f') \xrightarrow{\beta_1'} Ker(f)$ is the kernel morphism of $Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$.

(b) The following argument is an appropriate modification of the proof the 'snake' lemma C1.5.2.

(b1) We have a commutative diagram

$$\begin{array}{ccccccc}
 & & \widetilde{A}_1 & \xrightarrow{id} & Ker(f''\alpha_1) & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
 & & {\scriptstyle id}\downarrow & & {\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_1) & \xrightarrow{\psi'} & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
{\scriptstyle id}\downarrow & & {\scriptstyle h}\downarrow & cart & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
Ker(\alpha_1) & \xrightarrow{f'} & Ker(\alpha_2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2''
\end{array} \tag{4}$$

with cartesian squares as indicated. It follows (from the left lower cartesian square of (4)) that $Ker(h)$ is naturally isomorphic to $Ker(f)$.

(b2) Since the upper right square of (4) is cartesian, we have a commutative diagram

$$\begin{array}{ccccc}
Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & Ker(f''\alpha_1) = \widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
{\scriptstyle id}\downarrow & & {\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_1) & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
Ker(\alpha_2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2''
\end{array} \tag{5}$$

(b3) Since $Ker(\alpha_1) \xrightarrow{f'} Ker(\alpha_2)$ is a deflation, there exists a cartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\gamma} & \widetilde{A}_1 \\ {\scriptstyle \mathfrak{p}}\big\downarrow & cart & \big\downarrow{\scriptstyle h} \\ Ker(\alpha_1) & \xrightarrow{f'} & Ker(\alpha_2) \end{array} \qquad (6)$$

whose upper horizontal arrow, γ, is also a deflation.

The commutative diagram (5) shows, among other things, that the arrow f' factors through h (see the diagram (4)), there exists a splitting, $Ker(\alpha_1) \xrightarrow{\mathfrak{s}} \mathcal{M}$, of the morphism $\mathfrak{p}$. Set $p = \mathfrak{s} \circ \mathfrak{p}$. The morphism $\mathcal{M} \xrightarrow{p} \mathcal{M}$ is an idempotent which has the same kernel as $\mathfrak{p}$, because $\mathfrak{s}$ is a monomorphism.

(b4) Let $\mathcal{M} \xrightarrow{\varphi} Ker(f'')$ denote the composition of the deflations $\mathcal{M} \xrightarrow{\gamma} \widetilde{A}_1$ and $\widetilde{A}_1 \xrightarrow{\widetilde{\alpha}_1} Ker(f'')$. The composition $\varphi \circ p$ is trivial.

In fact, $\varphi \circ p = \widetilde{\alpha}_1 \circ \gamma \circ \mathfrak{s} \circ \mathfrak{p}$, and, by the origin of the morphism $\mathfrak{s}$, the composition $\gamma \circ \mathfrak{s}$ coincides with $\mathfrak{k}(\widetilde{\alpha}_1)$; so that $\varphi \circ p = (\widetilde{\alpha}_1 \circ \mathfrak{k}(\widetilde{\alpha}_1)) \circ \mathfrak{p}$ which shows the triviality of $\varphi \circ p$.

(b5) Suppose that the condition (#) holds. Then the triviality of $\varphi \circ p$ implies that the composition φ with the canonical morphism $Ker(p) \xrightarrow{\mathfrak{k}(p)} \mathcal{M}$ is a deflation. It follows from the commutative diagram

$$\begin{array}{ccccc} Ker(p) & \xrightarrow{id} & Ker(h) & \xrightarrow{\sim} & Ker(f) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \alpha'_1} \\ \mathcal{M} & \xrightarrow{\gamma} & \widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\ {\scriptstyle \mathfrak{p}}\big\downarrow & cart & \big\downarrow{\scriptstyle h} & & \\ Ker(\alpha_1) & \xrightarrow{f'} & Ker(\alpha_2) & & \end{array} \qquad (7)$$

that the composition of $Ker(p) \xrightarrow{\mathfrak{k}(p)} \mathcal{M}$ with $\mathcal{M} \xrightarrow{\varphi} Ker(f'')$ equals to the composition of $Ker(f) \xrightarrow{\alpha'_1} Ker(f'')$ with an isomorphism $Ker(p) \xrightarrow{\sim} Ker(f)$. Therefore, the morphism $Ker(f) \xrightarrow{\alpha'_1} Ker(f'')$ is a deflation. Together with (a) above, this means that the upper row of the diagram (3) is a conflation. ■

9.4.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects having the property (#) of 9.4.1. Let C_Y be a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ which has the following properties:*

(a) If $N \longrightarrow M \longrightarrow L$ is a conflation in $(C_X, \mathfrak{E}_X)$ and L, M are objects of C_Y, then N belongs to C_Y too.

(b) For any deflation $M \longrightarrow \mathcal{L}$ with $\mathcal{L} \in ObC_Y$, there exist a deflation $\mathcal{M} \longrightarrow \mathcal{L}$ with $\mathcal{M} \in ObC_Y$ and a morphism $\mathcal{M} \longrightarrow M$ such that the diagram

$$\begin{array}{ccc} & \mathcal{M} & \\ \swarrow & & \searrow \\ M & \longrightarrow & \mathcal{L} \end{array}$$

commutes.

(c) If P, $\mathcal{M}$ are objects of C_Y and $P \longrightarrow x$ is a morphism to initial object, then $P \coprod \mathcal{M}$ exists (in C_X) and the sequence $P \longrightarrow P \coprod \mathcal{M} \longrightarrow \mathcal{M}$ (where the left arrow is the canonical coprojection and the right arrow corresponds to the $\mathcal{M} \xrightarrow{id} \mathcal{M}$ and the composition of $P \longrightarrow x \longrightarrow \mathcal{M}$) is a conflation.

Let C_{Y_n} be a full subcategory of C_X generated by all objects L having a C_Y-resolution of the length $\leq n$. And set $C_{Y_\infty} = \bigcup_{n \geq 0} C_{Y_n}$. Then C_{Y_n} is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$ for all $n \leq \infty$ and the natural morphisms

$$K_\bullet(Y, \mathfrak{E}_Y) \xrightarrow{\sim} K_\bullet(Y_1, \mathfrak{E}_{Y_1}) \xrightarrow{\sim} \ \dots \ \xrightarrow{\sim} K_\bullet(Y_n, \mathfrak{E}_{Y_n}) \xrightarrow{\sim} \ K_\bullet(Y_\infty, \mathfrak{E}_{Y_\infty})$$

are isomorphisms for all $n \geq 0$.

Proof. Let $N \longrightarrow M \longrightarrow L$ be a conflation in $(C_X, \mathfrak{E}_X)$. Then for any integer $n \geq 0$, we have

(i) If $L \in ObC_{Y_{n+1}}$ and $M \in ObC_{Y_n}$, then $N \in ObC_{Y_n}$.

(ii) If N and L are objects of $C_{Y_{n+1}}$, then M is an object of $C_{Y_{n+1}}$.

(iii) If M and L are objects of $C_{Y_{n+1}}$, then N is an object of $C_{Y_{n+1}}$.

It suffices to prove the assertion for $n = 0$.

(i) Since $L \in ObC_{Y_1}$, there exists a conflation $P' \longrightarrow P \longrightarrow L$, where P and P' are objects of C_Y. Thus, we have a commutative diagram

$$\begin{array}{ccccc} x & \longrightarrow & P' & \xrightarrow{id} & P' \\ \downarrow & & \downarrow & & \downarrow \\ N & \longrightarrow & \widetilde{P} & \longrightarrow & P \\ {\scriptstyle id}\downarrow & & \downarrow & cart & \downarrow \\ N & \longrightarrow & M & \longrightarrow & L \end{array} \tag{8}$$

whose rows and columns are conflations. Here x is an initial object of the category C_X. Since M and P' belong to C_Y and C_Y is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$, in particular, it is closed under extensions, the object $\widetilde{P}$ belongs to C_Y. Since $\widetilde{P}$ and P are objects of C_Y, it follows from the condition (a) that $N \in ObC_Y$.

(ii) Since $L \in ObC_{Y_1}$, there exists a deflation $P \longrightarrow L$ with $P \in ObC_Y$. Applying (b) to the deflation $\widetilde{P} \longrightarrow P$ in (3), we obtain a deflation $\mathcal{M} \longrightarrow P$ such that $\mathcal{M} \in ObC_Y$ and the composition $\mathcal{M} \longrightarrow L$ factors through the deflation $M \longrightarrow L$ (see (8)). Since $N \in ObC_{Y_1}$, there exists a conflation $\widetilde{\mathcal{P}}' \longrightarrow \mathcal{P} \longrightarrow N$ where $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ are objects of C_Y. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{P}}' & \longrightarrow & \widetilde{\mathcal{M}} & \longrightarrow & \widetilde{\mathcal{P}}'' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P} & \longrightarrow & \mathcal{P} \coprod \mathcal{M} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow & & \downarrow \\ N & \longrightarrow & M & \longrightarrow & L \end{array} \tag{9}$$

whose two lower rows and the left and the right columns are conflations. By 9.4.1(b), the upper row of (9) is a conflation too. Applying (i) to the right column of (9), we obtain that $\widetilde{\mathcal{P}}'' \in ObC_Y$. This implies that $\widetilde{\mathcal{M}} \in ObC_Y$, whence $M \in ObC_{Y_1}$.

(iii) Since $M \in ObC_{Y_1}$, there is a commutative diagram

$$\begin{array}{ccccc} P' & \xrightarrow{id} & P' & \xrightarrow{\lambda} & x \\ \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & \widetilde{P} & \longrightarrow & L \\ \downarrow & cart & \downarrow & & \downarrow id \\ N & \longrightarrow & M & \longrightarrow & L \end{array} \tag{10}$$

whose rows and columns are conflations. Here x is an initial object of C_X and λ is a unique morphism $P' \longrightarrow x$ determined by the fact that $P' \longrightarrow K$ is the kernel of $K \longrightarrow N$. Since $L \in ObC_{Y_1}$, applying (i) to the middle row, we obtain that $K \in ObC_Y$. So, $N \in ObC_Y$. ■

9.4.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects having the property (#) of 9.4.1. Let C_Y be a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ satisfying the conditions (a) and (c) of 9.4.2. Let $M' \longrightarrow M \longrightarrow M''$ be a conflation in $(C_X, \mathfrak{E}_X)$, and let $\mathcal{P}' \longrightarrow M'$, $\mathcal{P}'' \longrightarrow M''$ be C_Y-resolutions of the length $n \geq 1$. Suppose that resolution $\mathcal{P}'' \longrightarrow M'$ is projective. Then there exists a C_Y-resolution $\mathcal{P} \longrightarrow M$ of the length n such that $\mathcal{P}_i = \mathcal{P}'_i \coprod \mathcal{P}''_i$ for all $i \geq 1$ and the splitting 'exact' sequence $\mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}''$ is an 'exact' sequence of complexes.*

Proof. We have the diagram

$$\begin{array}{ccccc} \mathcal{P}'_0 & & & & \mathcal{P}''_0 \\ \downarrow & & & & \downarrow \\ M' & \longrightarrow & M & \longrightarrow & M'' \end{array}$$

whose row is a conflation and vertical arrows are deflations. Since, by hypothesis, $\mathcal{P}_0''$ is a projective object of $(C_X, \mathfrak{E}_X)$ and $M \longrightarrow M''$ is a deflation, the right vertical arrow, $\mathcal{P}_0'' \longrightarrow M''$, factors through $M \longrightarrow M''$. Therefore (like in the argument 9.4.2(ii)), we obtain a commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{e}') & \longrightarrow & Ker(\mathfrak{e}) & \longrightarrow & Ker(\mathfrak{e}'') \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_0' & \longrightarrow & \mathcal{P}_0' \coprod \mathcal{P}_0'' & \longrightarrow & \mathcal{P}_0'' \\ {\scriptstyle \mathfrak{e}'} \downarrow & & {\scriptstyle \mathfrak{e}} \downarrow & & \downarrow {\scriptstyle \mathfrak{e}''} \\ M' & \longrightarrow & M & \longrightarrow & M'' \end{array}$$

By 9.4.1(b), the upper row of this diagram is a conflation, which allows to repeat the step with the diagram

$$\begin{array}{ccccc} \mathcal{P}_1' & & & & \mathcal{P}_1'' \\ \downarrow & & & & \downarrow \\ Ker(\mathfrak{e}') & \longrightarrow & Ker(\mathfrak{e}) & \longrightarrow & Ker(\mathfrak{e}'') \end{array}$$

whose vertical arrows are deflations; etc.. ∎

Chapter VI

Relative 'Spaces'. Devissage.

In the first three sections, we introduce the Gabriel multiplication of subcategories of a right exact category with initial objects, upper and lower infinitesimal neighborhoods of a subcategory and revisit fully exact subcategories observing that the upper infinitesimal neighborhood of a subcategory $\mathcal{B}$ is the smallest fully exact subcategory containing $\mathcal{B}$. In Section 4, we define cofiltrations of an object as sequences of deflations and prove a useful generalization of Zassenhouse's lemma. In Section 5, we introduce semitopologizing, topologizing and thick subcategories of a right exact category and establish some of their properties. Section 6 we introduce the (left exact) category of relative right exact 'spaces' and obtain devissage for higher images of a functor G provided the devissage holds for G on a certain class of relative right exact 'spaces'. In particular, we establish devissage for the universal K-functor.

1. The Gabriel multiplication in right exact categories.

Fix a right exact category $(C_X, \mathfrak{E}_X)$ with initial objects. Let $\mathbb{T}$ and $\mathbb{S}$ be subcategories of the category C_X. The *Gabriel product* $\mathbb{S}\bullet\mathbb{T}$ is the full subcategory of C_X whose objects M fit into conflations $L \xrightarrow{g} M \xrightarrow{h} N$ such that $L \in Ob\mathbb{S}$ and $N \in Ob\mathbb{T}$.

1.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be a right exact category with initial objects. For any subcategories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ of the category C_X, there is the inclusion*

$$\mathcal{A}\bullet(\mathcal{B}\bullet\mathcal{D}) \subseteq (\mathcal{A}\bullet\mathcal{B})\bullet\mathcal{D}.$$

Proof. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ be subcategories of C_X. Let M be an object of $\mathcal{A}\bullet(\mathcal{B}\bullet\mathcal{D})$; i.e. there is a conflation $L \longrightarrow M \longrightarrow N$ such that $L \in Ob\mathcal{A}$ and $N \in Ob\mathcal{B}\bullet\mathcal{D}$. The latter means that there is a conflation $N_1 \longrightarrow N \longrightarrow N_2$ with $N_1 \in Ob\mathcal{B}$ and $N_2 \in Ob\mathcal{D}$. Thus, we have a commutative diagram

$$\begin{array}{ccccc}
L & \longrightarrow & M_1 & \longrightarrow & N_1 \\
{\scriptstyle id}\downarrow & & \downarrow & cart & \downarrow \\
L & \longrightarrow & M & \longrightarrow & N \\
 & & \downarrow & & \downarrow \\
 & & N_2 & \xrightarrow{id} & N_2
\end{array}$$

whose two upper right square is cartesian, and two upper rows and two right columns are conflations. So, we have a conflation $M_1 \longrightarrow M \longrightarrow N_2$ with $N_2 \in Ob\mathcal{D}$ and $M_1 \in Ob\mathcal{A}{\bullet}\mathcal{B}$, hence M is an object of the subcategory $(\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{D}$. ■

1.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be an exact category. Then the Gabriel multiplication is associative.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ be subcategories of C_X. By 1.1, we have the inclusion $\mathcal{A} \bullet (\mathcal{B} \bullet \mathcal{D}) \subseteq (\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{D}$. The opposite inclusion holds by duality, because $(\mathcal{A} \bullet \mathcal{B})^{op} = \mathcal{B}^{op} \bullet \mathcal{A}^{op}$. Here we use the fact that the category opposite to an exact category is exact. ■

2. The infinitesimal neighborhoods of a subcategory.

Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We denote by $\mathbb{O}_X$ the full subcategory of C_X generated by all initial objects of C_X. For any subcategory $\mathcal{B}$ of C_X, we define subcategories $\mathcal{B}^{(n)}$ and $\mathcal{B}_{(n)}$, $0 \le n \le \infty$, by setting $\mathcal{B}^{(0)} = \mathbb{O}_X = \mathcal{B}_{(0)}$, $\mathcal{B}^{(1)} = \mathcal{B} = \mathcal{B}_{(1)}$, and

$$\mathcal{B}^{(n)} = \mathcal{B}^{(n-1)} \bullet \mathcal{B} \quad \text{for} \quad 2 \le n < \infty; \quad \text{and} \quad \mathcal{B}^{(\infty)} = \bigcup_{n \ge 1} \mathcal{B}^{(n)};$$

$$\mathcal{B}_{(n)} = \mathcal{B} \bullet \mathcal{B}_{(n-1)} \quad \text{for} \quad 2 \le n < \infty; \quad \text{and} \quad \mathcal{B}_{(\infty)} = \bigcup_{n \ge 1} \mathcal{B}_{(n)}$$

It follows that $\mathcal{B}^{(n)} = \mathcal{B}_{(n)}$ for $n \le 2$ and, by 1.1, $\mathcal{B}_{(n)} \subseteq \mathcal{B}^{(n)}$ for $3 \le n \le \infty$.

We call the subcategory $\mathcal{B}^{(n+1)}$ the *upper n^{th} infinitesimal neighborhood* of $\mathcal{B}$ and the subcategory $\mathcal{B}_{(n+1)}$ the *lower n^{th} infinitesimal neighborhood* of $\mathcal{B}$. It follows that $\mathcal{B}^{(n+1)}$ is the strictly full subcategory of C_X generated by all $M \in ObC_X$ such that there exists a sequence of arrows

$$M_0 \xrightarrow{\ \mathfrak{j}_1\ } M_1 \xrightarrow{\ \mathfrak{j}_2\ } \dots \xrightarrow{\ \mathfrak{j}_n\ } M_n = M$$

with the property: $M_0 \in Ob\mathcal{B}$, and for each $n \ge i \ge 1$, there exists a deflation $M_i \xrightarrow{\mathfrak{e}_i} N_i$ with $N_i \in Ob\mathcal{B}$ such that $M_{i-1} \xrightarrow{\mathfrak{j}_i} M_i \xrightarrow{\mathfrak{e}_i} N_i$ is a conflation.

Similarly, $\mathcal{B}_{(n+1)}$ is a strictly full subcategory of C_X generated by all $M \in ObC_X$ such that there exists a sequence of deflations

$$M = M_n \xrightarrow{\ \mathfrak{e}_n\ } \dots \xrightarrow{\ \mathfrak{e}_2\ } M_1 \xrightarrow{\ \mathfrak{e}_1\ } M_0$$

such that M_0 and $Ker(\mathfrak{e}_i)$ are objects of $\mathcal{B}$ for $1 \le i \le n$.

2.1. Note. Let x be an initial object of the category C_X and (x) the subcategory of C_X which consists of id_x. It follows that $\mathcal{A} \subseteq (x) \bullet \mathcal{A}$ for any subcategory $\mathcal{A}$ of C_X. In particular, $\mathcal{B}_{(n)} \subseteq \mathcal{B}_{(n+1)}$ for all $n \ge 0$, if $\mathcal{B}$ contains an initial object of the category C_X.

3. Fully exact subcategories of a right exact category.

Fix a right exact category $(C_X, \mathcal{E}_X)$. Notice that a subcategory $\mathcal{A}$ of C_X is a *fully exact* subcategory of $(C_X, \mathcal{E}_X)$ iff $\mathcal{A} \bullet \mathcal{A} = \mathcal{A}$.

3.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be a right exact category with initial objects. For any subcategory $\mathcal{B}$ of C_X, the subcategory $\mathcal{B}^{(\infty)}$ is the smallest fully exact subcategory of $(C_X, \mathcal{E}_X)$ containing $\mathcal{B}$.*

Proof. (a) If $\mathcal{A}$ be a fully exact subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, i.e. $\mathcal{A} = \mathcal{A} \bullet \mathcal{A}$. then $\mathcal{B}^{(\infty)} \subseteq \mathcal{A}$, for any subcategory $\mathcal{B}$ of $\mathcal{A}$.

(b) For any subcategory $\mathcal{B}$ of the category C_X, there is the inclusion

$$\mathcal{B}^{(n)} \bullet \mathcal{B}^{(m)} \subseteq \mathcal{B}^{(m+n)} \tag{1}$$

for any pair n, m of nonnegative integers.

In fact, $\mathcal{B}^n \bullet \mathcal{B} = \mathcal{B}_{n+1}$ by definition of $\mathcal{B}_{n+1}$. Assuming that the inclusion holds for n and m, we obtain:

$$\mathcal{B}^{(n)} \bullet \mathcal{B}^{(m+1)} = \mathcal{B}^{(n)} \bullet (\mathcal{B}^{(m)} \bullet \mathcal{B}) \subseteq (\mathcal{B}^{(n)} \bullet \mathcal{B}^{(m)}) \bullet \mathcal{B} \subseteq \mathcal{B}^{(n+m)} \bullet \mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}^{(n+m+1)}.$$

Here the first inclusion follows from 1.1 and the second one holds by induction hypothesis.

(c) It follows from the inclusions (1) that $\mathcal{B}^{(\infty)} = \mathcal{B}^{(\infty)} \bullet \mathcal{B}^{(\infty)}$, that is $\mathcal{B}^{(\infty)}$ is a fully exact subcategory of $(C_X, \mathcal{E}_X)$ containing $\mathcal{B}$. By (a) above, it is the smallest fully exact subcategory containing $\mathcal{B}$. ∎

3.2. Note. Another consequence of (1) is that if $\mathcal{B}$ is a subcategory containing an initial object of the category C_X, then

$$\mathcal{B}^{(n)} \subseteq (x) \bullet \mathcal{B}^{(n)} \subseteq \mathcal{B} \bullet \mathcal{B}^{(n)} \subseteq \mathcal{B}^{(n+1)}$$

(compare with 2.1).

4. Cofiltrations. Zassenhouse's Lemma.

4.1. Cofiltrations. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with initial objects. A *cofiltration of the length* n+1 of an object M is a sequence of deflations

$$M = M_n \xrightarrow{\mathfrak{e}_n} \dots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0. \tag{1}$$

The cofiltration (1) is said to be *equivalent* to a cofiltration

$$M = \widetilde{M}_m \xrightarrow{\widetilde{\mathfrak{e}}_n} \dots \xrightarrow{\widetilde{\mathfrak{e}}_2} \widetilde{M}_1 \xrightarrow{\widetilde{\mathfrak{e}}_1} \widetilde{M}_0$$

if $m = n$ and there exists a permutation σ of $\{0, \dots, n\}$ such that $Ker(\mathfrak{e}_i) \simeq Ker(\widetilde{\mathfrak{e}}_{\sigma(i)})$ for $1 \le i \le n$ and $M_0 \simeq \widetilde{M_0}$. Evidently, this is, indeed, an equivalence relation.

The following assertion is a version (and a generalization) of Zassenhouse's Lemma.

4.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ have the following property:*
(‡) *for any pair of deflations $M_1 \xleftarrow{\mathfrak{t}_1} M \xrightarrow{\mathfrak{t}_2} M_2$, there is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_1} & M_1 \\ {\scriptstyle \mathfrak{t}_2}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{p}_2} \\ M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array}$$

of deflations such that the unique morphism $M \longrightarrow M_1 \times_{M_3} M_2$ is a deflation.

Then any two cofiltrations of an object M have equivalent refinements.

Proof. Let

$$M = M_n \xrightarrow{\mathfrak{e}_n} \dots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0 \quad \text{and}$$
$$M = \widetilde{M}_m \xrightarrow{\widetilde{\mathfrak{e}}_n} \dots \xrightarrow{\widetilde{\mathfrak{e}}_2} \widetilde{M}_1 \xrightarrow{\widetilde{\mathfrak{e}}_1} \widetilde{M}_0$$

be cofiltrations. If $n = 0$, then the second cofiltration is a refinement of the first one.

(a) Suppose that $n = 1 = m$; that is we have a pair of deflations $\widetilde{M}_1 \xleftarrow{\widetilde{\mathfrak{e}}_1} M \xrightarrow{\mathfrak{e}_1} M_1$. Thanks to the property (‡), there exists a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{e}_1} & M_1 \\ {\scriptstyle \widetilde{\mathfrak{e}}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{p}_1} \\ \widetilde{M}_1 & \xrightarrow{\mathfrak{p}'_1} & N \end{array}$$

whose all arrows are deflations, and the unique arrow $M \xrightarrow{\mathfrak{e}_3} M_2 = M_1 \times_N \widetilde{M}_1$ is a deflation too. Since the right lower square in the commutative diagram

$$\begin{array}{ccccc} & & Ker(\widetilde{\mathfrak{e}}_2) & \xrightarrow{\sim} & Ker(\mathfrak{p}_1) \\ & & {\scriptstyle \widetilde{\mathfrak{k}}_2}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{k}_1} \\ Ker(\mathfrak{e}_2) & \xrightarrow{\mathfrak{k}_2} & M_2 & \xrightarrow{\mathfrak{e}_2} & M_1 \\ \wr\big\downarrow & & {\scriptstyle \widetilde{\mathfrak{e}}_2}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{p}_1} \\ Ker(\mathfrak{p}'_1) & \xrightarrow{\mathfrak{k}'_1} & \widetilde{M}_1 & \xrightarrow{\mathfrak{p}'_1} & N \end{array}$$

is cartesian, its upper horizontal and left vertical arrows are isomorphisms. This shows that the cofiltrations

$$M \xrightarrow{\mathfrak{e}_3} M_2 \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{p}_1} N \quad \text{and}$$
$$M \xrightarrow{\mathfrak{e}_3} M_2 \xrightarrow{\widetilde{\mathfrak{e}}_2} \widetilde{M}_1 \xrightarrow{\mathfrak{p}'_1} N$$

are equivalent to each other.

(b) Let $n > 1$ and $m = 1$. Then, applying (a) to the deflations $\widetilde{M}_0 \xleftarrow{\widetilde{\mathfrak{e}}_1} M \xrightarrow{\mathfrak{e}_n} M_{n-1}$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} M & \xrightarrow{\mathfrak{e}'} & M' & \xrightarrow{\mathfrak{e}_n} & M_{n-1} & \xrightarrow{\mathfrak{e}_n} & M_{n-2} & \longrightarrow \dots \longrightarrow & M_0 \\ & & {\scriptstyle \widetilde{\mathfrak{e}}_1}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{p}_1} & & & & \\ & & \widetilde{M}_1 & \xrightarrow{\mathfrak{p}'_1} & N & & & & \end{array}$$

which provides an induction argument.

(c) Finally, (b) provides the main induction step in the general case. Details are left to the reader. ■

5. Semitopologizing, topologizing, and thick subcategories of a right exact category.

Fix a right exact category $(C_X, \mathcal{E}_X)$ with initial objects.

5.1. Definitions. (a) We call a full subcategory $\mathcal{T}$ of the category C_X *semitopologizing* if the following condition holds:

(a1) If $M \xrightarrow{\mathfrak{e}} L$ is a deflation which belongs to $\mathcal{T}$, then $Ker(\mathfrak{e})$ is an object of $\mathcal{T}$.

(a2) If $N \longrightarrow M$ and $M \longrightarrow L$ are deflations whose composition belongs to $\mathcal{T}$ (that is N and L are objects of $\mathcal{T}$), then both of them belong to $\mathcal{T}$, i.e. M is an object of $\mathcal{T}$.

(b) We call a semitopologizing subcategory $\mathcal{T}$ of the category C_X *topologizing* if it is a right exact subcategory of $(C_X, \mathfrak{E}_X)$, that is if

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{f'} & M \\ {\scriptstyle \widetilde{\mathfrak{e}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ N & \xrightarrow{f} & L \end{array}$$

is a cartesian square in C_X and the objects M, L, and N belong to the subcategory $\mathcal{T}$, then $\widetilde{N}$ is an object of $\mathcal{T}$.

(c) We call a subcategory $\mathcal{T}$ of C_X a *thick* subcategory of $(C_X, \mathcal{E}_X)$ if it is topologizing and fully exact, i.e. $\mathcal{T} \bullet \mathcal{T} = \mathcal{T}$.

5.1.1. Note. It follows from (a2) that every semitopologizing subcategory of $(C_X, \mathfrak{E}_X)$ is strictly full. Applying (a1) to isomorphisms, we obtain that every semitopologizing subcategory of $(C_X, \mathfrak{E}_X)$ contains all initial objects of the category C_X.

5.2. Proposition. *(a) Let $(C_X, \mathfrak{E}_X)$ be a right exact pointed category such that all morphisms to zero objects are deflations (say, $(C_X, \mathfrak{E}_X)$ is Karoubian). Then*

(i) A full subcategory $\mathcal{T}$ of C_X is semitopologizing iff for every deflation $M \xrightarrow{\mathfrak{t}} L$ with $M \in Ob\mathcal{T}$, the object L and $Ker(\mathfrak{e})$ belong to $\mathcal{T}$.

(ii) Any topologizing subcategory of $(C_X, \mathfrak{E}_X)$ is closed under finite products.

(b) If C_X is an abelian category and $\mathfrak{E}_X$ is the canonical exact structure on C_X, then topologizing subcategories of $(C_X, \mathfrak{E}_X)$ are topologizing subcategories of the abelian category C_X in the sense of Gabriel [Gab].

Proof. (a) Let $\mathcal{T}$ be a semitopologizing subcategory of $(C_X, \mathfrak{E}_X)$.

(i) By observation 5.1.1, it is a strictly full subcategory of C_X containing all zero objects of C_X. Let $M \xrightarrow{\mathfrak{t}} L$ be a deflation with $M \in Ob\mathcal{T}$. By hypothesis, unique morphisms of L and M to a zero object $x_\bullet$ of C_X are deflations and $x_\bullet$ is an object of $\mathcal{T}$. Since the unique morphism $M \longrightarrow x_\bullet$ is a deflation and it is the composition of the deflation $M \xrightarrow{\mathfrak{t}} L$ and $L \longrightarrow x_\bullet$, it follows from the definition of a semitopologizing subcategory that $L \in Ob\mathcal{T}$. Therefore, $Ker(\mathfrak{t})$ is an object of $\mathcal{T}$.

(ii) Let M, N be objects of $\mathcal{T}$ and $x_\bullet$ a zero object of C_X. By hypothesis, the unique morphisms $M \longrightarrow x_\bullet$ and $N \longrightarrow x_\bullet$ are deflations. Therefore, the cartesian square

$$\begin{array}{ccc} M\prod N & \xrightarrow{p_M} & M \\ {\scriptstyle p_N}\big\downarrow & & \big\downarrow \\ N & \longrightarrow & x_\bullet \end{array}$$

is contained in $\mathcal{T}$.

(b) If $(C_X, \mathfrak{E}_X)$ is an abelian category with the canonical structure, then it follows from (a) above that any topologizing subcategory $\mathcal{T}$ of $(C_X, \mathfrak{E}_X)$ is closed under finite (co)products, and if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence with $M \in Ob\mathcal{T}$, then M' and M'' are objects of $\mathcal{T}$. This means that $\mathcal{T}$ is a topologizing subcategory of the abelian category C_X in the sense of Gabriel. On the other hand any topologizing subcategory of C_X in the sense of Gabriel is closed under any finite limits and colimits (taken in C_X), in particular, it is closed under arbitrary pull-backs. ∎

5.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a k-linear additive right exact category such that all morphisms to zero objects are deflations.*

(a) Any topologizing subcategory of $(C_X, \mathfrak{E}_X)$ is closed under finite products.

(b) If $(C_X, \mathfrak{E}_X)$ is an exact category, then any topologizing subcategory of $(C_X, \mathfrak{E}_X)$ is an exact (sub)category.

Proof. (a) This follows from 5.2(a).

(b) Fix a topologizing subcategory $\mathcal{T}$ of an exact k-linear category $(C_X, \mathfrak{E}_X)$. Let $M \xrightarrow{\mathfrak{j}} M' \xrightarrow{\mathfrak{e}} M''$ be a conflation in $\mathcal{T}$ and $M \xrightarrow{f} L$ an arbitrary morphism of $\mathcal{T}$. Since $(C_X, \mathfrak{E}_X)$ is an exact category, there is cocartesian square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{j}} & M' \\ f \downarrow & cocart & \downarrow f' \\ N & \xrightarrow{\widetilde{\mathfrak{j}}} & N' \end{array} \tag{1}$$

whose horizontal arrows are inflations. Notice that the pair of morphisms

$$M \xrightarrow{(f,\mathfrak{j})} N \times M' = N \oplus M' \xrightarrow{\widetilde{\mathfrak{j}}+f'} N' \tag{2}$$

is a conflation. In fact, the Gabriel-Quillen embedding is 'exact', hence it sends the cocartesian square (1) to a cocartesian square of the abelian category of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$. And for abelian categories the fact is easy to check. Since the Gabriel-Quillen embedding reflects conflations, it follows that (2) is a conflation.

By (a) above, $N \oplus M' \in Ob\mathcal{T}$, because N and M' are objects of $\mathcal{T}$. Therefore, the object N' belongs to $\mathcal{T}$. ■

5.4. Proposition. *Let $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (Y, \mathfrak{E}_Y)$ be a morphism of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$. If $\mathcal{T}$ is a semitopologizing (resp. topologizing, resp. thick) subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, then ${\mathfrak{f}^*}^{-1}(\mathcal{T})$ is a semitopologizing (resp. topologizing, resp. thick) subcategory of $(C_Y, \mathfrak{E}_Y)$.*

Proof. By the definition of morphisms of $\mathfrak{Esp}_{\mathfrak{r}}^*$, the inverse image functor $\mathfrak{f}^*$ is an 'exact' (that is preserving pull-backs of deflations) functor from $(C_Y, \mathfrak{E}_Y)$ to $(C_X, \mathfrak{E}_X)$ which maps initial objects to initial objects. The assertion follows from definitions. ■

5.5. Proposition. *Let*

$$\begin{array}{ccc} (Z, \mathfrak{E}_Z) & \xrightarrow{g} & (Y, \mathfrak{E}_Y) \\ f \downarrow & cocart & \downarrow p_1 \\ (X, \mathfrak{E}_X) & \xrightarrow{p_2} & (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}) \end{array}$$

be a cocartesian square in the category $\mathfrak{Esp}_{\mathfrak{r}}^*$, and let C_{X_0}, C_{Y_0} be semitopologizing subcategories of resp. $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$. Then

$$C_{\mathfrak{X}_0} = C_{X_0} \prod_{f_0^*, g_0^*} C_{Y_0}$$

is a semitopologizing subcategory of $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$.

If the subcategories C_{X_0} *and* C_{Y_0} *are topologizing (resp. thick), then* $C_{\mathfrak{X}_0}$ *is a topologizing (resp. thick) subcategory of the category* $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$.

Proof. (a) By IV.5.2, $\mathfrak{E}_{\mathfrak{X}}$ consists of all morphisms

$$(M, L; \phi) \xrightarrow{(\xi,\gamma)} (M', L'; \phi')$$

of the category $C_{\mathfrak{X}}$ such that $\xi \in \mathfrak{E}_X$ and $\gamma \in \mathfrak{E}_Y$. And $Ker(\xi, \gamma) = (Ker(\xi), Ker(\gamma); \phi'')$, where ϕ'' is a uniquely determined (once $Ker(\xi)$ and $Ker(\gamma)$ are fixed) isomorphism. Therefore, if

$$(M, L; \phi) \xrightarrow{\xi,\gamma} (M', L'; \phi') \xrightarrow{\xi',\gamma'} (M", L"; \phi")$$

are deflations, the objects $(M, L; \phi)$ and $(M", L"; \phi")$ belong to the subcategory $C_{\mathfrak{X}_0}$, and both categories C_{X_0} and C_{Y_0} are semitopological, then $(M', L'; \phi')$ and $Ker(\xi, \gamma)$ are objects of $C_{\mathfrak{X}_0}$, which shows that $C_{\mathfrak{X}_0}$ is a semitopological subcategory of the category $C_{\mathfrak{X}}$.

(b) Suppose now that C_{X_0} and C_{Y_0} are topologizing subcategories of respectively $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$. By definition of morphisms of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$, the inverse image functors $C_X \xrightarrow{f^*} C_Z$ and $C_Y \xrightarrow{g^*} C_Z$ are 'exact'; i.e. they preserve pull-backs of deflations. This implies that for any deflation $(M, L; \phi) \xrightarrow{(\xi,\gamma)} (M', L'; \phi')$ and an arbitrary morphism $(M'', L''; \phi'') \xrightarrow{(\alpha,\beta)} (M', L'; \phi')$ of the category $C_{\mathfrak{X}}$, there exists a cartesian square

$$\begin{array}{ccc}
(\widetilde{M}, \widetilde{L}; \widetilde{\phi}) & \xrightarrow{(p_2,p_2')} & (M, L; \phi) \\
{\scriptstyle (p_1,p_1')}\downarrow & & \downarrow{\scriptstyle (\xi,\gamma)} \\
(M'', L''; \phi'') & \xrightarrow{(\alpha,\beta)} & (M', L'; \phi')
\end{array}$$

determined uniquely up to isomorphism by the fact that the squares

$$\begin{array}{ccc}
\widetilde{L} & \xrightarrow{p_2} & L \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \xi} \\
L'' & \xrightarrow{\xi} & L'
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\widetilde{M} & \xrightarrow{p_2'} & M \\
{\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle \gamma} \\
M'' & \xrightarrow{\gamma} & M'
\end{array}$$

are both cartesian. Therefore, if L and L'' are objects of the topologizing subcategory C_{Y_0}, then $\widetilde{L} \in ObC_{Y_0}$. Similarly, $\widetilde{M} \in ObC_{X_0}$ if M and M'' are objects of C_{X_0}. This shows that $C_{\mathfrak{X}_0}$ is a topologizing subcategory of $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$.

(c) If the subcategories C_{X_0} and C_{Y_0} are fully exact, then the subcategory $C_{\mathfrak{X}_0}$ is fully exact. So that if C_{X_0} and C_{Y_0} *thick* (that is fully exact and topologizing), this and (b) above imply that $C_{\mathfrak{X}_0}$ is a fully exact subcategory. ∎

6. An application to K-functors: devissage.

6.1. Proposition. (Devissage for K_0.) *Let $(X, \mathfrak{E}_X)$ be a right exact 'space' with the following property (which appeared in 4.2):*

($\ddagger$) *for any pair of deflations $M_1 \overset{\mathfrak{t}_1}{\longleftarrow} M \overset{\mathfrak{t}_2}{\longrightarrow} M_2$, there is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_1} & M_1 \\ {\scriptstyle \mathfrak{t}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_2} \\ M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array}$$

of deflations such that the unique morphism $M \longrightarrow M_1 \times_{M_3} M_2$ is a deflation.

Then for every topologizing subcategory C_Y of the right exact category $(C_X, \mathfrak{E}_X)$, the natural morphisms

$$K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(Y_\infty, \mathfrak{E}_{Y_\infty}) \longrightarrow K_0(Y^\infty, \mathfrak{E}_{Y^\infty}) \qquad (1)$$

are isomorphisms.

Proof. Let M be an object of C_{Y_∞}, and let

$$M = M_n \xrightarrow{\mathfrak{e}_n} \dots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0 \qquad (2)$$

be its C_Y-cofiltration. That is the arrows of (2) are deflations and the objects M_0 and $Ker(\mathfrak{e}_i)$, $1 \le i \le n$, belong to the subcategory C_Y. Since the subcategory C_Y is (semi)topological, any refinement of a C_Y-cofiltration is a C_Y-cofiltration.

(a1) The map

$$[M] \longmapsto [M_0]_{C_Y} + \sum_{1 \le i \le n} [Ker(\mathfrak{e}_i)]_{C_Y} \qquad (3)$$

applied to a refinement of the cofiltration (2) gives the same result. Here $[N]_{C_Y}$ denotes the image of the object N in $K_0(Y)$.

In fact, for any sequence of deflations $\mathcal{M}_m \xrightarrow{\mathfrak{t}_m} \dots \xrightarrow{\mathfrak{t}_2} \mathcal{M}_1 \xrightarrow{\mathfrak{t}_1} \mathcal{M}_0$, we have a commutative diagram

$$\begin{array}{ccccccccccc} \mathfrak{K}_m & \xrightarrow{\widetilde{\mathfrak{t}}_m} & \mathfrak{K}_{m-1} & \xrightarrow{\widetilde{\mathfrak{t}}_{m-1}} & \dots & \xrightarrow{\widetilde{\mathfrak{t}}_3} & \mathfrak{K}_2 & \xrightarrow{\widetilde{\mathfrak{t}}_2} & \mathfrak{K}_1 & \xrightarrow{\widetilde{\mathfrak{t}}_1} & x \\ \downarrow & cart & \downarrow & cart & \dots & cart & \downarrow & cart & \downarrow & cart & \downarrow \\ \mathcal{M}_m & \xrightarrow{\mathfrak{t}_m} & \mathcal{M}_{m-1} & \xrightarrow{\mathfrak{t}_{m-1}} & \dots & \xrightarrow{\mathfrak{t}_3} & \mathcal{M}_2 & \xrightarrow{\mathfrak{t}_2} & \mathcal{M}_1 & \xrightarrow{\mathfrak{t}_1} & \mathcal{M}_0 \end{array} \qquad (4)$$

formed by cartesian squares. Here x is an initial object of the category C_X. Since the 'composition' of cartesian squares is a cartesian square, it follows that

$$\mathfrak{K}_1 = Ker(\mathfrak{t}_1),\ \mathfrak{K}_2 = Ker(\mathfrak{t}_1\mathfrak{t}_2),\ \dots,\ \mathfrak{K}_m = Ker(\mathfrak{t}_1\mathfrak{t}_2\dots\mathfrak{t}_m).$$

Since each square

$$\begin{array}{ccc} \mathfrak{K}_\ell & \xrightarrow{\widetilde{\mathfrak{t}}_\ell} & \mathfrak{K}_{\ell-1} \\ \downarrow & cart & \downarrow \\ \mathcal{M}_\ell & \xrightarrow{\mathfrak{t}_\ell} & \mathcal{M}_{\ell-1} \end{array}$$

of the diagram (4) is cartesian, all morphisms $\widetilde{\mathfrak{t}}_\ell$ are deflations and $Ker(\widetilde{\mathfrak{t}}_\ell) \simeq Ker(\mathfrak{t}_\ell)$ for all $1 \le \ell \le m$. Therefore,

$$[Ker(\mathfrak{t}_1\mathfrak{t}_2 \ldots \mathfrak{t}_m)] = [\mathfrak{K}_m] = \sum_{1\le i\le n} [Ker(\widetilde{\mathfrak{t}}_i)] = \sum_{1\le i\le n} [Ker(\mathfrak{t}_i)]. \tag{5}$$

This shows that the right hand side of (3) remains the same when each of the deflations $\mathfrak{e}_i$ is further decomposed into a sequence of deflations.

(a2) By 4.2, any two finite cofiltrations of an object of C_X have equivalent refinements. Together with (a1) above, this implies that the map (3) does not depend on the choice of C_Y-cofiltrations of objects. Thus, (3) defines a map, $\widetilde{\psi}$, from the set $|C_{Y_\infty}|$ of isomorphism classes of objects of the category C_{Y_∞} to the group $K_0^*(Y, \mathfrak{E}_Y)$.

(a3) For any conflation $M' \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} M''$ in $(C_{Y_\infty}, \mathfrak{E}_{Y_\infty})$, we have

$$\widetilde{\psi}([M]) = \widetilde{\psi}([M']) + \widetilde{\psi}([M'']).$$

Indeed, let $M \longrightarrow \ \ldots \ \longrightarrow M_0$ be some C_Y-cofiltration of M. By 4.2, this cofiltration and the cofiltration $M \xrightarrow{\mathfrak{e}} M''$ have equivalent refinements which are, forcibly, C_Y-cofiltrations. Consider the obtained this way refinement

$$M = M_n \xrightarrow{\mathfrak{e}_n} \ldots \xrightarrow{\mathfrak{e}_m} M_{m-1} = M'' \xrightarrow{\mathfrak{e}_m} \ldots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0$$

and the associated commutative diagram

$$\begin{array}{ccccccccccc} \mathfrak{K}_n & \xrightarrow{\widetilde{\mathfrak{e}}_n} & \mathfrak{K}_{n-1} & \xrightarrow{\widetilde{\mathfrak{e}}_{n-1}} & \ldots & \xrightarrow{\widetilde{\mathfrak{e}}_m} & x & & & & \\ \downarrow & cart & \downarrow & cart & \ldots & cart & \downarrow & & & & \\ M & \xrightarrow{\mathfrak{e}_n} & \mathcal{M}_{n-1} & \xrightarrow{\mathfrak{e}_{n-1}} & \ldots & \xrightarrow{\mathfrak{e}_m} & M'' & \xrightarrow{\mathfrak{e}_{m-1}} & \ldots & \xrightarrow{\mathfrak{e}_1} & \mathcal{M}_0 \end{array} \tag{6}$$

built of cartesian squares. Here x is an initial object of the category C_X. Since $M \xrightarrow{\mathfrak{e}} M''$ equals to the composition $\mathfrak{e}_m \circ \ldots \circ \mathfrak{e}_n$, it follows from the argument of (a1) that $\mathfrak{K}_n \simeq M' = Ker(\mathfrak{e})$ and the upper row of (6) is a C_Y-filtration of the object M'. The latter implies that

$$\widetilde{\psi}([M']) = \sum_{m\le i\le n} [Ker(\widetilde{\mathfrak{e}}_i)]_{C_Y} = \sum_{m\le i\le n} [Ker(\mathfrak{e}_i)]_{C_Y}$$

(see (5) above). From the lower row, we obtain

$$\widetilde{\psi}([M'']) = [M_0]_{C_Y} + \sum_{m<i\leq n} [Ker(\mathfrak{e}_i)]_{C_Y} \quad \text{and}$$
$$\widetilde{\psi}([M]) = [M_0]_{C_Y} + \sum_{1\leq i\leq n} [Ker(\mathfrak{e}_i)]_{C_Y}.$$

Therefore, $\widetilde{\psi}([M]) = \widetilde{\psi}([M']) + \widetilde{\psi}([M''])$.

(a4) The map $|C_X| \xrightarrow{\widetilde{\psi}} K_0(Y, \mathfrak{E}_Y)$ extends uniquely to a $\mathbb{Z}$-module morphism

$$\mathbb{Z}|C_{Y_\infty}| \xrightarrow{\mathbb{Z}\widetilde{\psi}} K_0(Y, \mathfrak{E}_Y). \tag{7}$$

It follows from (a3) that the morphism (7) factors through a (uniquely determined) $\mathbb{Z}$-module morphism

$$K_0(Y_\infty, \mathfrak{E}_{Y_\infty}) \xrightarrow{\psi_0} K_0(Y, \mathfrak{E}_Y).$$

The claim is that the morphism ψ_0 is invertible and its inverse is

$$K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(\mathfrak{j})} K_0(Y_\infty, \mathfrak{E}_{Y_\infty}).$$

It is immediate that $\psi_0 \circ K_0(\mathfrak{j}) = id_{K_0(Y,\mathfrak{E}_Y)}$.

The equality $K_0(\mathfrak{j}) \circ \psi_0 = id_{K_0(Y_\infty,\mathfrak{E}_{Y_\infty})}$ is also easy to see: if M is an object of C_{Y_∞} endowed with a C_Y-cofiltration $M = M_n \xrightarrow{\mathfrak{e}_n} \dots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0$, then

$$K_0(\mathfrak{j}) \circ \psi_0([M]) = K_0(\mathfrak{j})([M_0]_{C_Y} + \sum_{1\leq i\leq n} [Ker(\mathfrak{e}_i)]_{C_Y}) = [M_0] + \sum_{1\leq i\leq n} [Ker(\mathfrak{e}_i)] = [M].$$

This proves that the map

$$K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(Y_\infty, \mathfrak{E}_{Y_\infty}) \tag{1.1}$$

is an isomorphism.

(b) Consider the natural maps

$$K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(Y_{(2)}, \mathfrak{E}_{Y_{(2)}}) \longrightarrow K_0(Y_\infty, \mathfrak{E}_{Y_\infty}),$$

where $C_{Y_{(2)}} = C_Y \bullet C_Y$. It follows from the argument above that the map

$$K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(Y_{(2)}, \mathfrak{E}_{Y_{(2)}}) \tag{1.2}$$

is an epimorphism. It is also a monomorphism, because the composition of two maps (1.2) is the isomorphism (1.1). Therefore, the map (1.2) is an isomorphism.

Since there is an inclusion $C_Y \subseteq C_Y \bullet C_Y$ for any topologizing subcategory C_Y of $(C_X, \mathfrak{E}_X)$ (see 2.1), the subcategory C_{Y^∞} is the union of the subcategories C_{Y^n} defined by

$$C_{Y^1} = C_Y, \;\; C_{Y^{n+1}} = C_{Y^n} \bullet C_{Y^n}.$$

Therefore, we have a sequence of isomorphisms

$$K_0(Y, \mathfrak{E}_Y) \xrightarrow{\sim} K_0(Y^2, \mathfrak{E}_{Y^2}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} K_0(Y^n, \mathfrak{E}_{Y^n}) \xrightarrow{\sim} K_0(Y^{n+1}, \mathfrak{E}_{Y^{n+1}}) \xrightarrow{\sim} \cdots$$

which implies that the map

$$K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(Y^\infty, \mathfrak{E}_{Y^\infty})$$

is an isomorphism. ■

6.2. The left exact category of relative 'spaces'. Consider the category $\mathfrak{RE}\mathfrak{sp}_\mathfrak{r}$ whose objects – *relative 'spaces'*, are pairs $(\mathcal{X}, Y)$, where $\mathcal{X} = (X, \mathfrak{E}_X)$ is a right exact 'space' and C_Y is a topologizing subcategory of the right exact category $(C_X, \mathfrak{E}_X)$ with the induced right exact structure. A morphism from $(\mathcal{X}, Y)$ to $(\mathcal{X}', Y')$ is a morphism of right exact 'spaces' $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{X}'$ whose inverse image functor maps $C_{Y'}$ to C_Y. The composition of morphisms is defined in an obvious way.

We call a morphism $(\mathcal{X}, Y) \xrightarrow{\mathfrak{f}} (\mathfrak{X}, \mathfrak{Y})$ an *inflation* if the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathfrak{X}$ belongs to $\mathfrak{I}_\mathfrak{c}^\to$ and $C_\mathfrak{Y} = \mathfrak{f}^{*^{-1}}(C_Y)$. We denote by $\mathfrak{I}_\mathfrak{r}^\to$ the class of all inflations of relative 'spaces'.

6.2.1. Proposition. *(a) The class $\mathfrak{I}_\mathfrak{r}^\to$ is a left exact structure on the category $\mathfrak{RE}\mathfrak{sp}_\mathfrak{r}$ of relative right exact 'spaces'.*

(b) The left exact category $(\mathfrak{RE}\mathfrak{sp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{r}^\to)$ has enough injective objects.

Proof. (a) This follows from the definition of the class $\mathfrak{I}_\mathfrak{r}^\to$ and the fact that $\mathfrak{I}_\mathfrak{c}^\to$ is a left exact structure on the category $\mathfrak{E}\mathfrak{sp}_\mathfrak{r}$ of right exact 'spaces'.

(b) Injective objects of the left exact category $(\mathfrak{RE}\mathfrak{sp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{r}^\to)$ are relative right exact 'spaces' $(\mathcal{X}, Y)$ such that $\mathcal{X}$ is an injective object of the left exact category $(\mathfrak{E}\mathfrak{sp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^\to)$ of the right exact 'spaces'.

In fact, if $(\mathcal{X}, Y) \xrightarrow{\mathfrak{f}} (\mathcal{X}', Y')$ is an inflation, then, by definition, $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{X}'$ is a morphism of $\mathfrak{I}_\mathfrak{c}^\to$. Since $\mathcal{X}$ is an injective object of the left exact category $(\mathfrak{E}\mathfrak{sp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^\to)$, the inflation $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{X}'$ splits; that is there is a morphism $\mathcal{X}' \xrightarrow{\gamma} \mathcal{X}$ such that $\gamma \circ \mathfrak{f} = id_\mathcal{X}$. In particular, its inverse image functor, γ^*, maps the topologizing subcagegory C_Y to the topologizing subcategory $\mathfrak{f}^{*^{-1}}(C_Y) = C_{Y'}$; that is γ is a morphism from $(\mathcal{X}', Y')$ to $(\mathcal{X}, Y)$.

Let $(\mathcal{X}, Y)$ be an arbitrary object of $\mathfrak{RE}\mathfrak{sp}_\mathfrak{r}$. Since $(\mathfrak{E}\mathfrak{sp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^\to)$ has enough injective objects, there exists a morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathfrak{X}$ from $\mathfrak{I}_\mathfrak{c}^\to$ with $\mathfrak{X}$ an injective object of $(\mathfrak{E}\mathfrak{sp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^\to)$.

The morphism $\mathfrak{f}$ can be regarded as an inflation from $(\mathcal{X}, Y)$ to the injective object $(\mathfrak{X}, \mathfrak{Y})$, where $C_{\mathfrak{Y}} = \mathfrak{f}^{*^{-1}}(C_Y)$. ■

6.2.2. Three 'exact' functors on the category of relative right exact 'spaces'. The maps

$$(\mathcal{X}, Y) \longmapsto Y, \quad (\mathcal{X}, Y) \longmapsto Y_\infty, \quad (\mathcal{X}, Y) \longmapsto Y^\infty$$

are naturally extended to 'exact' functors respectively $\mathfrak{F}_0$, $\mathfrak{F}_\infty$, $\mathfrak{F}^\infty$ from the left exact category $(\mathfrak{R}\mathfrak{Esp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{r}^\to)$ to the left exact category $(\mathfrak{Esp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^\to)$ of right exact 'spaces'.

The natural morphisms $Y^\infty \longrightarrow Y_\infty \longrightarrow Y$ are functorial in $(\mathcal{X}, Y)$; i.e. they define morphisms of functors

$$\mathfrak{F}^\infty \xrightarrow{\lambda_\infty} \mathfrak{F}_\infty \xrightarrow{\lambda_0} \mathfrak{F}_0.$$

Since the functors $\mathfrak{F}_0$, $\mathfrak{F}_\infty$, $\mathfrak{F}^\infty$ are 'exact' and the left exact category $(\mathfrak{R}\mathfrak{Esp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{r}^\to)$ has enough injective objects, it follows from III.3.6.6 that, for any contravariant functor G from the category $(\mathfrak{Esp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^\to)$ of right exact 'spaces' to a complete category, we have a (quasi-)commutative diagram

$$\begin{array}{ccccc} S_+^\bullet(G \circ \mathfrak{F}_0) & \xrightarrow{S_+^\bullet(G(\lambda_0))} & S_+^\bullet(G \circ \mathfrak{F}_\infty) & \xrightarrow{S_+^\bullet(G(\lambda_\infty))} & S_+^\bullet(G \circ \mathfrak{F}^\infty) \\ \wr\downarrow & & \wr\downarrow & & \downarrow\wr \\ S_+^\bullet(G) \circ \mathfrak{F}_0 & \xrightarrow{S_+^\bullet(G)(\lambda_0)} & S_+^\bullet(G) \circ \mathfrak{F}_\infty & \xrightarrow{S_+^\bullet(G)(\lambda_\infty)} & S_+^\bullet(G) \circ \mathfrak{F}^\infty \end{array} \quad (8)$$

of functor morphisms whose vertical arrows are isomorphisms.

6.3. Proposition. *Let G be a left 'exact' contravariant functor from the category $(\mathfrak{Esp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^\to)$ of right exact 'spaces' to a complete category such that, for every right exact 'space' $(X, \mathfrak{E}_X)$ with the property (‡) of 6.1 and any topologizing subcategory C_Y of $(C_X, \mathfrak{E}_X)$, the natural morphisms*

$$G(Y, \mathfrak{E}_Y) \xrightarrow{\lambda_0(\mathcal{X}, Y)} G(Y_\infty, \mathfrak{E}_{Y_\infty}) \xrightarrow{\lambda_\infty(\mathcal{X}, Y)} G(Y^\infty, \mathfrak{E}_{Y^\infty})$$

are isomorphisms. Then, for every relative right exact 'space' $(\mathcal{X}, Y)$, the natural morphisms

$$S_+^m(G)(Y, \mathfrak{E}_Y) \longrightarrow S_+^m(G)(Y_\infty, \mathfrak{E}_{Y_\infty}) \longrightarrow S_+^m(G)(Y^\infty, \mathfrak{E}_{Y^\infty}) \quad (9)$$

are isomorphisms for $m \geq 1$.

(b) Suppose $(X, \mathfrak{E}_X)$ is a right exact 'space' satisfying the property (‡) of 6.1. Then, for every topologizing subcategory C_Y of the right exact category $(C_X, \mathfrak{E}_X)$, the morphisms (1) are isomorphisms for all $m \geq 0$.

Proof. (i) It follows from the fact that the left exact category $(\mathfrak{REsp}_{\mathfrak{r}}, \mathfrak{I}_{\mathfrak{r}}^{\rightarrow})$ has enough injective objects (see 6.2.1(b)), the 'exactness' of the functors $\mathfrak{F}_0,\ \mathfrak{F}_\infty,\ \mathfrak{F}^\infty$ (cf. 6.2.2) and III.4.1.4 that the morphisms (9) are epimorphisms for all $m \geq 0$.

(ii) Let $(C_X, \mathfrak{E}_X)$ be a right exact category satisfying the property (‡) of 6.1 and $(C_X, \mathfrak{E}_X) \xrightarrow{\mathcal{F}} (C_Z, \mathfrak{E}_Z)$ an 'exact' functor. Then the right exact subcategory $Ker(\mathcal{F})$ of $(C_X, \mathfrak{E}_X)$ satisfies the property (‡) of 6.1.

In fact, let $M_1 \xleftarrow{\mathfrak{t}_1} M \xrightarrow{\mathfrak{t}_2} M_2$ be a pair of deflations in $Ker(\mathcal{F})$. Since $(C_X, \mathfrak{E}_X)$ satisfies the property (†), there is a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_1} & M_1 \\ {\scriptstyle \mathfrak{t}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_2} \\ M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array}$$

of deflations in $(C_X, \mathfrak{E}_X)$ such that the unique morphism $M \longrightarrow M_1 \times_{M_3} M_2$ is a deflation. In other words, there is a diagram

$$\begin{array}{ccccc} M & \xrightarrow{\xi} & \mathfrak{M} & \xrightarrow{\mathfrak{u}_1} & M_1 \\ & & {\scriptstyle \mathfrak{u}_2}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{p}_2} \\ & & M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array} \qquad (10)$$

in C_X whose all arrows are deflations, the square is cartesian, and $\mathfrak{u}_i \circ \xi = \mathfrak{t}_i,\ i = 1, 2$.

The claim is that the object M_3 belongs to $Ker(\mathcal{F})$.

(ii') Since the functor $\mathcal{F}$ is exact, it maps the diagram (10) to the diagram of the same kind, and the objects $\mathcal{F}(M),\ \mathcal{F}(M_1),\ \mathcal{F}(M_2)$ are initial. In particular, $\mathcal{F}(\mathfrak{u}_1) \circ \mathcal{F}(\xi)$ is an isomorphism. The latter implies that $\mathcal{F}(\xi)$ is a strict monomorphism. Since $F(\xi)$ is a deflation (in particular, an epimorphism), it is an isomorphism. Thus, $\mathcal{F}(\mathfrak{M})$ is an initial object and, therefore, both projections, $\mathcal{F}(\mathfrak{u}_i),\ i = 1, 2$, are isomorphisms.

(ii") Notice that if

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{p_1} & L_1 \\ {\scriptstyle p_2}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{s}_2} \\ L_2 & \xrightarrow{\mathfrak{s}_1} & L_3 \end{array}$$

is a cartesian square whose projections are isomorphisms, then the other two arrows are monomorphisms.

Indeed, replacing the square by an isomorphic square, we can assume that the projections $p_1,\ p_2$ are identical morphisms, hence $\mathfrak{s}_1 = \mathfrak{s}_2$. If a pair of arrows $N \overset{\zeta_1}{\underset{\zeta_2}{\rightrightarrows}} L_1$ equalizes the morphism $L_1 \xrightarrow{\mathfrak{s}_1} L_3$, then it is a composition of the unique morphism $N \longrightarrow \mathfrak{L}$ and the pair of projections (p_1, p_2). So that $\zeta_1 = \zeta_2$.

(ii"') It follows from (ii') and (ii") that the morphisms $\mathcal{F}(\mathfrak{p}_1)$ and $\mathcal{F}(\mathfrak{p}_2)$ are monomorphisms. Since $\mathcal{F}(\mathfrak{p}_1)$ and $\mathcal{F}(\mathfrak{p}_2)$ are deflations (hence strict epimorphisms), this implies that they are isomorphisms.

(iii) Every relative right exact 'space' $(\mathcal{X}, Y)$ has a canonical resolution

$$\begin{array}{c}(\mathcal{X}, Y) \stackrel{\mathrm{def}}{=} \widetilde{\mathfrak{X}}_0 \longrightarrow \mathfrak{X}_0 \longrightarrow \widetilde{\mathfrak{X}}_1 \longrightarrow \mathfrak{X}_1 \longrightarrow \cdots \\ \cdots \longrightarrow \widetilde{\mathfrak{X}}_n \longrightarrow \mathfrak{X}_n \longrightarrow \widetilde{\mathfrak{X}}_{n+1} \longrightarrow \cdots\end{array} \tag{11}$$

where

$$(\widetilde{\mathcal{X}}_n, \widetilde{Y}_n) = \mathfrak{X}_n \longrightarrow \mathfrak{X}_n \longrightarrow \widetilde{\mathfrak{X}}_{n+1}$$

is a conflation and $\mathfrak{X}_n = (\mathcal{X}_n, Y_n)$ is an injective object and for every $n \geq 0$.

It follows from the definition of inflations that the resolution (11) is uniquely determined by (X, Y) and a(ny) sequence

$$\mathcal{X} \stackrel{\mathrm{def}}{=} \widetilde{\mathcal{X}}_0 \longrightarrow \mathcal{X}_0 \longrightarrow \widetilde{\mathcal{X}}_1 \longrightarrow \mathcal{X}_1 \longrightarrow \cdots \longrightarrow \widetilde{\mathcal{X}}_n \longrightarrow \mathcal{X}_n \longrightarrow \widetilde{\mathcal{X}}_{n+1} \longrightarrow \cdots \tag{11'}$$

of morphisms of right exact 'spaces' such that $\mathcal{X}_n$ is an injective object and

$$\widetilde{\mathcal{X}}_n \longrightarrow \mathcal{X}_n \longrightarrow \widetilde{\mathcal{X}}_{n+1}$$

is a conflation for every $n \geq 0$.

(iv) Every (canonical) injective object of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{I}_{\mathfrak{c}}^{\rightarrow})$ has the property (‡). In particular, all injective objects $\mathcal{X}_n$, $n \geq 0$, in the sequence (11') have the property (‡). Since the right exact 'spaces' $\widetilde{\mathcal{X}}_n$ for $n \geq 1$ are cokernels of 'exact' morphisms (that is their representing categories are kernels of 'exact' functors), it follows from (ii) above that $\widetilde{\mathcal{X}}_n$ also have the property (‡).

(v) For every $n \geq 0$, we have a commutative diagram

$$\begin{array}{ccccccccc}
G \circ \mathfrak{F}_0(\widetilde{\mathfrak{X}}_n) & \longrightarrow & G \circ \mathfrak{F}_0(\mathfrak{X}_n) & \longrightarrow & G \circ \mathfrak{F}_0(\widetilde{\mathfrak{X}}_{n+1}) & \longrightarrow & S_+(G \circ \mathfrak{F}_0)(\widetilde{\mathfrak{X}}_n) & \longrightarrow & z \\
{\scriptstyle G\lambda_0 \widetilde{\mathfrak{X}}_n)} \downarrow & & {\scriptstyle G\lambda_0(\mathfrak{X}_n)} \downarrow & & {\scriptstyle G\lambda_0(\widetilde{\mathfrak{X}}_{n+1})} \downarrow & & \downarrow {\scriptstyle S_+ G\lambda_0(\mathfrak{X}_n)} & & \\
G \circ \mathfrak{F}_\infty(\widetilde{\mathfrak{X}}_n) & \longrightarrow & G \circ \mathfrak{F}_\infty(\mathfrak{X}_n) & \longrightarrow & G \circ \mathfrak{F}_\infty(\widetilde{\mathfrak{X}}_{n+1}) & \longrightarrow & S_+(G \circ \mathfrak{F}_\infty)(\widetilde{\mathfrak{X}}_n) & \longrightarrow & z \\
{\scriptstyle G\lambda_\infty \widetilde{\mathfrak{X}}_n)} \downarrow & & {\scriptstyle G\lambda_\infty(\mathfrak{X}_n)} \downarrow & & {\scriptstyle G\lambda_\infty(\widetilde{\mathfrak{X}}_{n+1})} \downarrow & & \downarrow {\scriptstyle S_+ G\lambda_\infty(\mathfrak{X}_n)} & & \\
G \circ \mathfrak{F}^\infty(\widetilde{\mathfrak{X}}_n) & \longrightarrow & G \circ \mathfrak{F}^\infty(\mathfrak{X}_n) & \longrightarrow & G \circ \mathfrak{F}^\infty(\widetilde{\mathfrak{X}}_{n+1}) & \longrightarrow & S_+(G \circ \mathfrak{F}^\infty)(\widetilde{\mathfrak{X}}_n) & \longrightarrow & z
\end{array} \tag{12}$$

with 'exact' rows (see III.4.1 (3)) and, for $n \geq 1$, the commutative diagrams

$$\begin{array}{ccccc}
S_+^{n+1}(G)\circ\mathfrak{F}_0(\widetilde{\mathfrak{X}}_0) & \xrightarrow{S_+^{n+1}(G)(\lambda_0)} & S_+^{n+1}(G)\circ\mathfrak{F}_\infty(\widetilde{\mathfrak{X}}_0) & \xrightarrow{S_+^{n+1}(G)(\lambda_\infty)} & S_+^{n+1}(G)\circ\mathfrak{F}^\infty(\widetilde{\mathfrak{X}}_0) \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr \\
S_+^{n+1}(G\circ\mathfrak{F}_0)(\widetilde{\mathfrak{X}}_0) & \xrightarrow{S_+^{n+1}(G(\lambda_0))} & S_+^{n+1}(G\circ\mathfrak{F}_\infty)(\widetilde{\mathfrak{X}}_0) & \xrightarrow{S_+^{n+1}(G(\lambda_\infty))} & S_+^{n+1}(G\circ\mathfrak{F}^\infty)(\widetilde{\mathfrak{X}}_0) \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr \\
S_+(G\circ\mathfrak{F}_0)(\widetilde{\mathfrak{X}}_n) & \xrightarrow{S_+(G(\lambda_0))} & S_+(G\circ\mathfrak{F}_\infty)(\widetilde{\mathfrak{X}}_n) & \xrightarrow{S_+(G(\lambda_\infty))} & S_+(G\circ\mathfrak{F}^\infty)(\widetilde{\mathfrak{X}}_n) \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr \\
S_+(G)\circ\mathfrak{F}_0(\widetilde{\mathfrak{X}}_n) & \xrightarrow{S_+(G)(\lambda_0)} & S_+(G)\circ\mathfrak{F}_\infty(\widetilde{\mathfrak{X}}_n) & \xrightarrow{S_+(G)(\lambda_\infty)} & S_+(G)\circ\mathfrak{F}^\infty(\widetilde{\mathfrak{X}}_n)
\end{array} \qquad (12')$$

whose vertical arrows are isomorphisms (cf. III.4.1 (5) and the diagram (8) in 6.2.2).

Suppose that $G\lambda_i(\mathfrak{X})$ (where $i = 0$ or/and ∞) is an isomorphism for every $\mathfrak{X} = (\mathcal{X}, Y)$ such that the right exact 'space' $\mathcal{X}$ has the property (‡). Then it follows from (iv) that the arrows $G\lambda_i(\mathfrak{X}_n)$ are isomorphisms for $n \geq 0$ and the arrows $G\lambda_i(\widetilde{\mathfrak{X}}_n)$ are isomorphisms for $n \geq 1$. Together with the 'exactness' of rows of the diagram (12), this implies that $S_+G\lambda_i(\widetilde{\mathfrak{X}}_n)$ is an isomorphism for $n \geq 0$.

It follows now from the diagram (12') that $S_+^nG(\lambda_i)$ is an isomorphism for any $n \geq 1$. Thus, if $G\lambda_0(\mathfrak{X})$ and $G\lambda_\infty(\mathfrak{X})$ are isomorphisms for every $\mathfrak{X} = (\mathcal{X}, Y)$ such that the right exact 'space' $\mathcal{X}$ has the property (‡), then the natural morphisms

$$S_+^n(G)\circ\mathfrak{F}_0 \xrightarrow{S_+^n(G)(\lambda_0)} S_+^n(G)\circ\mathfrak{F}_\infty \xrightarrow{S_+^n(G)(\lambda_\infty)} S_+^n(G)\circ\mathfrak{F}^\infty$$

are isomorphisms for every $n \geq 1$. ∎

6.3.1. Corollary. *(a) For every relative right exact 'space' $(\mathcal{X}, Y)$, the natural morphisms*

$$K_m(Y, \mathfrak{E}_Y) \longrightarrow K_m(Y_\infty, \mathfrak{E}_{Y_\infty}) \longrightarrow K_m(Y^\infty, \mathfrak{E}_{Y^\infty}) \qquad (9)$$

are isomorphisms if $m \geq 1$ and epimorphisms for $m = 0$.

(b) Suppose $(X, \mathfrak{E}_X)$ is a right exact 'space' satisfying the property (‡) of 6.1. Then, for every topologizing subcategory C_Y of the right exact category $(C_X, \mathfrak{E}_X)$, the morphisms (1) are isomorphisms for all $m \geq 0$.

Proof. (a) Applying 6.3 (and 6.1) to the K-functor, we obtain that the canonical morphisms

$$K_n\circ\mathfrak{F}_0 \xrightarrow{K^n(\lambda_0)} K_n\circ\mathfrak{F}_\infty \xrightarrow{K_n(\lambda_\infty)} K_n\circ\mathfrak{F}^\infty$$

are isomorphisms for every $n \geq 1$.

(b) The assertion is the combination of (a) above and 6.1. ∎

Complementary Facts

In Section C1 (which complements Section 4 of Chapter I), we look at some examples, which acquire importance somewhere in the text. In Section C2, we pay tribute to standard techniques of homological algebra by expanding the most popular facts on diagram chasing to right exact categories. They appear here mainly as a curiosity and are used only once in the main body of the manuscript. Section C3 is dedicated to localizations of exact and (co)suspended categories. In particular, t-structures of (co)suspended categories appear on the scene. Again, a work by Keller and Vossieck, [KV1], suggested the notions. Section C4 is dedicated to cohomological functors on suspended categories and can be regarded as a natural next step after the works [KeV] and [Ke1]. It is heavily relied on Appendix K. We consider cohomological functors on suspended categories with values in exact categories and prove the existence of a universal cohomological functor. The construction of the universal functor gives, among other consequences, an equivalence between the bicategory of Karoubian suspended svelte categories with triangle functors as 1-morphisms and the bicategory of exact svelte $\mathbb{Z}_+$-categories with enough injective objects whose 1-morphisms are 'exact' functors. We show that if the suspended category is triangulated, then the universal cohomological functor takes values in an abelian category, and our construction recovers the abelianization of triangulated categories by Verdier [Ve2]. It is also observed that the *triangulation* of suspended categories induces an abelianization of the corresponding exact $\mathbb{Z}_+$-categories. We conclude with a discussion of homological dimension and resolutions of suspended categories and exact categories with enough injective objects. These resolutions suggest that the 'right' objects to consider from the very beginning are exact (resp. abelian) and (co)suspended (resp. triangulated) $\mathbb{Z}_+^n$-categories. All the previously discussed facts (including the content of Appendix K) extend easily to this setting. In Section C5, we define the *weak costable* category of a right exact category as the localization of the right exact category at a certain class of arrows related with its projective objects. If the right exact category in question is exact, then its costable category is isomorphic to the costable category in the conventional sense (reminded in Appendix K). If a right exact category has enough *pointable* projective objects (in which case all its projective objects are pointable), then its weak costable category is naturally equivalent to the *costable* category of this right exact category defined in Chapter III. We study right exact categories of modules over monads and associated stable and costable categories. The general constructions acquire here a concrete shape. We introduce the notion of a *Frobenius* monad. The category of modules over a Frobenius monad is a Frobenius category, hence its stable category is triangulated. We consider the case of modules over an augmented monad which includes as special cases most of standard homological algebra based on complexes and their homotopy and derived categories.

C1. Complements on kernels and cokernels.

C1.1. Kernels of morphisms of 'spaces'. The category $|Cat|^o$ of 'spaces' has an initial object x represented by the category with one object and one (identical) morphism. By [KR, 2.2], the category $|Cat|^o$ has small limits (and colimits). In particular, any morphism of $|Cat|^o$ has a kernel. The kernel of a morphism $X \xrightarrow{f} Y$ of $|Cat|^o$ can be explicitly described as follows.

Let $C_Y \xrightarrow{f^*} C_X$ be an inverse image functor of f. For any two objects L, M of the category C_X, we denote by $\mathfrak{I}_f(L, M)$ the set of all arrows $L \longrightarrow M$ which factor through an object of the subcategory $f^*(C_Y)$. The class $\mathfrak{I}_f$ of arrows of C_X obtained this way is a two-sided ideal; i.e. it is closed under compositions on both sides with arbitrary arrows of C_X. We denote by C_{X_f} the quotient of the category C_X by the ideal $\mathfrak{I}_f$; that is $ObC_{X_f} = ObC_X$, $C_{X_f}(L, M) = C_X(L, M)/\mathfrak{I}_f(L, M)$ for all objects L, M, and the composition is induced by the composition in C_X. Each object M of the image of the subcategory $f^*(C_Y)$ in C_{X_f} has the property that $C_{X_f}(L, M)$ and $C_{X_f}(M, L)$ consist of at most one arrow. This allows to define a category $C_{K(f)}$ by replacing the image of $f^*(C_Y)$ by one object z and one morphism, id_z. (i.e. $ObC_{K(f)} = ObC_X/f^*(C_Y)$). If objects L and M are not equal to z, then we set $C_{K(f)}(L, M) = C_{X_f}(L, M)$. The set $C_{K(f)}(L, z)$ (resp. $C_{K(f)}(z, M)$) consists of one element iff there exists a morphism from L to an object of $f^*(C_Y)$ (resp. from an object of $f^*(C_Y)$ to M); otherwise, it is empty.

We denote by $\mathfrak{k}(f)^*$ the natural projection $C_X \longrightarrow C_{K(f)}$. Thus, we have a commutative square of functors

$$\begin{array}{ccc} C_{K(f)} & \xleftarrow{\mathfrak{c}(f)^*} & C_X \\ \pi_z^* \uparrow & & \uparrow f^* \\ C_x & \longleftarrow & C_Y \end{array}$$

where π_z^* maps the unique object of C_x to z. This square corresponds to a cartesian square

$$\begin{array}{ccc} K(f) & \xrightarrow{\mathfrak{c}(f)} & X \\ \pi_z \downarrow & & \downarrow f \\ x & \longrightarrow & Y \end{array}$$

of morphisms of 'spaces'; i.e. the morphism $K(f) \xrightarrow{\mathfrak{c}(f)} X$ is the kernel of $X \xrightarrow{f} Y$.

Similarly to *Sets*, the category $|Cat|^o$ has a unique final object represented by the empty category. Since there are no functors from non-empty categories to the empty category, the cokernel of any morphism of $|Cat|^o$ is the unique morphism to the final object.

C1.2. Kernels and cokernels of morphisms of relative objects. Fix an object V of a category C_X and consider the category C_X/V.

C1.2.1. Cokernels. This category has a final object, (V, id_V), so we can discuss cokernels of its morphisms. Notice that the forgetful functor $C_X/V \longrightarrow C_X$ is exact, in particular, it preserves push-forwards. Therefore, the cokernel of a morphism $(M,g) \xrightarrow{f} (N,h)$ exists iff a push-forward $N\coprod_M V = N\coprod_{f,g} V$ exists and is equal to $(N\coprod_M V, h')$, where $N\coprod_M V \xrightarrow{h'} V$ is determined by $N \xrightarrow{h} V$.

C1.2.2. Kernels of morphisms of relative objects. If the category C_X has an initial object x, then $(x, x \to V)$ is an initial object of the category C_X/V. The forgetful functor $C_X/V \longrightarrow C_X$ preserves pull-backs; in particular, it preserves kernels of morphisms. So that the kernel of a morphism $(M,g) \xrightarrow{f} (N,h)$ exists iff the kernel $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ of $M \xrightarrow{f} N$ exists; and it is equal to $(Ker(f), g\circ\mathfrak{k}(f)) \xrightarrow{\mathfrak{k}(f)} (M,g)$.

C1.3. Application: cokernels of morphisms of relative 'spaces'. Fix a 'space' $\mathcal{S}$ and consider the category $|Cat|^o/\mathcal{S}$ of 'spaces' over $\mathcal{S}$. According to C1.2.1, the cokernel of a morphism $(X,g) \xrightarrow{f} (Y,h)$ of 'spaces' over $\mathcal{S}$ is the pair $(Cok(f), \widetilde{h})$, where $C_{Cok(f)}$ is the pull-back (in the pseudo-categorical sense) of the pair of inverse image functors $C_{\mathcal{S}} \xrightarrow{g^*} C_X \xleftarrow{h^*} C_Y$. That is objects of the category $C_{Cok(f)}$ are triples $(M,N;\phi)$, where $M \in ObC_{\mathcal{S}}$, $N \in ObC_Y$ and ϕ an isomorphism $g^*(M) \xrightarrow{\sim} f^*(N)$. Morphisms from $(M,N;\phi)$ to $(M',N';\phi')$ are given by a pair of arrows $M \xrightarrow{\mathfrak{u}} M'$, $N \xrightarrow{\mathfrak{v}} N'$ such that the square

$$\begin{array}{ccc} g^*(M) & \xrightarrow{g^*(\mathfrak{u})} & g^*(M') \\ \phi \downarrow\wr & & \wr\downarrow \phi' \\ f^*(N) & \xrightarrow{f^*(\mathfrak{v})} & f^*(N') \end{array}$$

commutes. The functor $C_{\mathcal{S}} \xrightarrow{\widetilde{h}^*} C_{Cok(f)}$ which assigns to every object L of the category $C_{\mathcal{S}}$ the object $(g^*(L), h^*(L); \psi(L))$, where ψ is an isomorphism $g^* \xrightarrow{\sim} f^*h^*$, is an inverse image functor of the morphism $\widetilde{h}$.

C1.4. Categories with initial objects and associated pointed categories. Let C_X be a category with an initial object, x. Then the category $C_{X_x} = C_X/x$ is a pointed category with a zero object (x, id_x).

C1.4.1. Example: augmented monads. Let $C_{\mathfrak{X}}$ be the category $\mathfrak{Mon}(X)$ of monads on the category C_X. The category $C_{\mathfrak{X}}$ has a canonical initial object x which is the identical monad (Id_{C_X}, id). The category $C_{\mathfrak{X}_x}$ coincides with the category $\mathfrak{Mon}^+(X)$ of *augmented* monads. Its objects are pairs $(\mathcal{F}, \epsilon)$, where $\mathcal{F} = (F, \mu)$ is a monad on C_X and ϵ is a monad morphism $\mathcal{F} \longrightarrow (Id_{C_X}, id)$ called an *augmentation morphism.* One can see that a functor morphism $F \xrightarrow{\epsilon} Id_{C_X}$ is an augmentation morphism iff $(M, \epsilon(M))$ is an $\mathcal{F}$-module morphism for every $M \in ObC_X$. In other words, there is a bijective correspondence between augmentation morphisms and sections $C_X \longrightarrow \mathcal{F} - mod$ of the forgetful functor $\mathcal{F} - mod \xrightarrow{f_*} C_X$.

C1.5. Pointed category of 'spaces'. Consider first a simpler case – the category Cat^{op}. It has an initial object, x, which is represented by the category with one object and one (identical) morphism. The associated pointed category Cat^{op}/x is equivalent to the category whose objects are pairs $(X, \mathcal{O}_X)$, where X is a 'space' and $\mathcal{O}_X$ an object of the category C_X representing X. Morphisms from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ are pairs (f^*, ϕ), where f^* is a functor $C_Y \longrightarrow C_X$ and ϕ is an isomorphism $f^*(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$. The composition of $(X, \mathcal{O}_X) \xrightarrow{(f^*,\phi)} (Y, \mathcal{O}_Y) \xrightarrow{(g^*,\psi)} (Z, \mathcal{O}_Z)$ is given by $(g^*, \psi) \circ (f^*, \phi) = (f^* \circ g^*, \phi \circ f^*(\psi))$.

The pointed category $|Cat|^o/x$ associated with the category of 'spaces' $|Cat|^o$ admits a similar realization after fixing a pseudo-functor

$$|Cat|^o \longrightarrow Cat^{op}, \quad X \longmapsto C_X, \ f \longmapsto f^*; \ (gf)^* \xrightarrow{\mathfrak{c}_{f,g}} f^*g^*,$$

– a section of the natural projection $Cat^{op} \longrightarrow |Cat|^o$. Namely, it is equivalent to a category $|Cat|^o_x$ whose objects are (as above) pairs $(X, \mathcal{O}_X)$, where $\mathcal{O}_X \in ObC_X$, morphisms from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ are pairs (f, ϕ), where f is a morphism of 'spaces' $X \longrightarrow Y$ and ϕ is an isomorphism $f^*(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$. The composition of $(X, \mathcal{O}_X) \xrightarrow{(f,\phi)} (Y, \mathcal{O}_Y) \xrightarrow{(g,\psi)} (Z, \mathcal{O}_Z)$ is the morphism $(g \circ f, \phi \circ f^*(\psi) \circ \mathfrak{c}_{f,g})$.

C1.5.1. Cokernels of morphisms. One can deduce from the description of cokernels in C1.3 in terms of the realization of the category $|Cat|^o_x$ given above, that the cokernel of a morphism $(X, \mathcal{O}_X) \xrightarrow{(f,\phi)} (Y, \mathcal{O}_Y)$ is isomorphic to $(Y, \mathcal{O}_Y) \xrightarrow{(\mathfrak{c}(f),\psi)} (\mathfrak{C}(f), \mathcal{O}_{\mathfrak{C}(f)})$, where $C_{\mathfrak{C}(f)}$ is a subcategory of C_Y whose objects are $M \in ObC_Y$ such that $f^*(M) \simeq \mathcal{O}_Y$ and morphisms are all arrows between these objects which f^* transforms into isomorphisms. The 'structure' object $\mathcal{O}_{\mathfrak{C}(f)}$ coincides with $\mathcal{O}_Y$; the inverse image functor of $\mathfrak{c}(f)$ is the inclusion functor $C_{\mathfrak{C}(f)} \longrightarrow C_Y$; and the isomorphism ψ is identical.

C1.6. The category of k-'spaces'. We call 'spaces' represented by k-linear additive categories *k-spaces*. We denote by $|Cat_k|^o$ the category whose objects are k-'spaces' and morphisms $X \longrightarrow Y$ are isomorphism classes of k-linear functors $C_Y \longrightarrow C_X$. The category $|Cat_k|^o$ is pointed: its zero object is represented by the zero category. It is easy to see that every morphism $X \xrightarrow{f} Y$ has a canonical cokernel $Y \xrightarrow{\mathfrak{c}} Cok(f)$, where $C_{Cok(f)}$ is the subcategory $Ker(f^*)$ of C_Y (– the full subcategory generated by all objects L such that $f^*(L) = 0$) and $\mathfrak{c}^*$ is the inclusion functor $Ker(f^*) \longrightarrow C_Y$.

The kernel $Ker(f) \xrightarrow{\mathfrak{k}(f)} X$ of f admits a simple description which is a linear version of the one in C1.1. Namely, $C_{Ker(f)}$ is the quotient of the category C_X by the ideal $\mathfrak{I}_f$

formed by all morphisms of C_X which factor through objects of $f^*(C_Y)$. The inverse image of $\mathfrak{k}(f)^*$ is the canonical projection $C_X \longrightarrow C_X/\mathfrak{I}_f$.

C1.6.1. k-'Spaces' over $\mathbf{Sp}(k)$. Consider now the full subcategory $|Cat_k|^o_{\mathbf{Sp}(k)}$ of the category of k-'spaces' over the affine scheme $\mathbf{Sp}(k)$ whose objects are pairs (X, f) where $X \xrightarrow{f} \mathbf{Sp}(k)$ is continuous (i.e. f^* has a right adjoint, f_*). This category admits a realization in the style of C1.5. Namely, it is equivalent to the category whose objects are pairs $(X, \mathcal{O}_X)$, where X is a k-'space' and $\mathcal{O}_X$ is an object of the category C_X such that there exist infinite coproducts of copies of $\mathcal{O}_X$ and cokernels of morphisms between these coproducts. Morphisms from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ are pairs (f, ϕ), where f^* is a k-linear functor $C_Y \longrightarrow C_X$ and ϕ an isomorphism $f^*(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$. The composition is defined as in C1.5 (see [KR, 4.5]). By C1.2.2, kernels of morphisms (as well as other limits) are inherited from $|Cat_k|^o$. That is the kernel of a morphism $(X, \mathcal{O}_X) \xrightarrow{(f,\phi)} (Y, \mathcal{O}_Y)$ is the morphism $(Ker(f), \mathcal{O}_{Ker(f)}) \xrightarrow{(\mathfrak{k}(f), id)} (X, \mathcal{O}_X)$, where $C_{Ker(f)} = C_X/\mathfrak{I}_f$, $\mathfrak{k}(f)^*$ is the canonical projection $C_X \longrightarrow C_X/\mathfrak{I}_f$, and $\mathcal{O}_{Ker(f)}$ is the image of $\mathcal{O}_X$.

The cokernel $(Y, \mathcal{O}_Y) \xrightarrow{(\mathfrak{c}(f),\psi)} (\mathfrak{C}_f, \mathcal{O}_{\mathfrak{C}_f})$ of (f, ϕ) is described following C1.3. Objects of the category $C_{\mathfrak{C}_f}$ are triples $(M, N; \phi)$, where $M \in ObC_Y$, $N \in Obk-mod$, and α is an isomorphism $f^*(M) \longrightarrow \gamma^*(N)$. Here γ^* is a functor $k-mod \longrightarrow C_X$ which maps k to $\mathcal{O}_X$ and preserves colimits (which determines γ^* uniquely up to isomorphism). Morphisms are defined as in C1.3. The structure object $\mathcal{O}_{\mathfrak{C}_f}$ is $(\mathcal{O}_Y, k, \phi)$. The inverse image functor $\mathfrak{c}(f)^*$ of $\mathfrak{c}(f)$ is the projection $(M, N; \alpha) \longmapsto M$.

C1.7. The (bi)categories $Cat_\star$ and $Cat_{\mathfrak{pt}}$. Let $Cat_\star$ denote the category whose objects are pairs (C_X, x), where C_X is a category and x its initial object; morphisms $(C_X, x) \longrightarrow (C_Y, y)$ are pairs (F, ϕ), where F is a functor $C_X \longrightarrow C_Y$ and ϕ a morphism $F(x) \longrightarrow y$. The composition of two morphisms, $(C_X, x) \xrightarrow{(F,\phi)} (C_Y, y) \xrightarrow{(G,\gamma)} (C_Z, z)$, is given by $(G, \gamma) \circ (F, \phi) = (G \circ F, \gamma \circ G(\phi))$.

Every morphism $(C_X, x) \xrightarrow{(F,\phi)} (C_Y, y)$ defines a functor $C_{X_x} \xrightarrow{F_\phi} C_{Y_y}$ between the corresponding pointed categories; and the map $(F, \phi) \longmapsto F_\phi$ respects compositions and maps identical morphisms to identical functors; i.e. the correspondence

$$(C_X, x) \longmapsto C_{X_x},\ (F, \phi) \longmapsto F_\phi$$

is a functor, $\mathfrak{J}_*$, from the category $Cat_\star$ onto the full subcategory $Cat_{\mathfrak{pt}}$ of Cat whose objects are pointed categories. The functor $\mathfrak{J}_*$ is a right adjoint to the functor $Cat_{\mathfrak{pt}} \xrightarrow{\mathfrak{J}^*} Cat_\star$ which assigns to each pointed category C_X an object (C_X, x) of the category $Cat_\star$ and to every functor $C_X \xrightarrow{F} C_Y$ between pointed categories the morphism $C_X, x) \xrightarrow{(F,\phi)} (C_Y, y)$ in which the arrow $F(x) \xrightarrow{\phi} y$ is uniquely defined. The adjunction arrow $Id_{Cat_{\mathfrak{pt}}} \xrightarrow{\eta} \mathfrak{J}_*\mathfrak{J}^*$

assigns to each pointed category C_X the natural isomorphism $C_X \xrightarrow{\sim} C_{X_x}$ (where x is the zero object of C_X involved in the definition of $\mathfrak{J}^*$). The other adjunction arrow, $\mathfrak{J}^*\mathfrak{J}_* \xrightarrow{\epsilon} Id_{Cat_\star}$, assigns to each object (C_Y, y) of $Cat_\star$ the forgetful functor $C_{Y_y} \longrightarrow C_Y$. Notice by passing that the image of this forgetful functor is the full subcategory of C_Y generated by all objects having a morphism to an initial object.

C1.8. Induced right exact structures. A pretopology τ on C_X induces a pretopology τ_V on the category C_X/V for any $V \in ObC_X$; hence τ induces a pretopology τ_x on C_{X_x}. In particular, a structure $\mathfrak{E}_X$ of a right exact category on C_X induces a structure $\mathfrak{E}_{X_x}$ of a right exact category on C_{X_x}. If $(C_X, \mathfrak{E}_X)$ has enough projective objects, then $(C_{X_x}, \mathfrak{E}_{X_x})$ has enough projective objects. Finally, if the class $\mathfrak{E}_X^{spl}$ of split epimorphisms of C_X is stable under base change, then the class $\mathfrak{E}_{X_x}^{spl}$ of split epimorphisms of C_{X_x} has this property.

C1.9. Monads on categories with an initial object and monads on corresponding pointed categories.

C1.9.1. Definition. Fix an object (C_X, x) of the category $Cat_\star$. A *monad on* (C_X, x) is a pair $(\mathcal{F}, \phi)$, where $\mathcal{F} = (F, \mu)$ is a monad on C_X and $F(x) \xrightarrow{\phi} x$ is an $\mathcal{F}$-module structure on the initial object x.

We denote by $\mathfrak{Mon}(C_X, x)$ the category whose objects are monads on (C_X, x); morphisms from $(\mathcal{F}, \phi)$ to $(\mathcal{F}', \phi')$ are monad morphisms $\mathcal{F} \xrightarrow{g} \mathcal{F}'$ such that $\phi = \phi' \circ g(x)$.

C1.9.2. Lemma. *Every monad $(\mathcal{F}, \phi)$ on (C_X, x) defines a monad $\mathcal{F}_\phi = (F_\phi, \mu_\phi)$ on the corresponding pointed category C_{X_x}. The map $(\mathcal{F}, \phi) \longmapsto \mathcal{F}_\phi$ extends to an isomorphism between the category $\mathfrak{Mon}(C_X, x)$ of monads on C_{X_x} and the category $\mathfrak{Mon}(C_{X_x})$ of monads on C_{X_x}.*

Proof is left to the reader. ■

C1.9.3. A remark on augmented monads. Every augmented monad $(\mathcal{F}, \epsilon)$ on the category C_X (see C1.4.1) defines a monad $(\mathcal{F}, \epsilon(x))$ on (C_X, x), hence a monad on the associated pointed category C_{X_x}. The map $(\mathcal{F}, \epsilon) \longmapsto (\mathcal{F}, \epsilon(x))$ is functorial; so that we have functors

$$\mathfrak{Mon}^+(C_X) \longrightarrow \mathfrak{Mon}(C_X, x) \xrightarrow{\sim} \mathfrak{Mon}(C_{X_x}).$$

On the other hand, it is easy to see that there is a natural isomorphism between the category $\mathfrak{Mon}^+(C_X)$ of augmented monads on C_X and the category $\mathfrak{Mon}^+(C_{X_x})$ of augmented monads on the pointed category C_{X_x}.

C2. Diagram chasing.

C2.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and kernels of morphisms; and let*

$$\begin{array}{ccccccc} & & & & & & x \\ & & & & & & \downarrow \\ & & Ker(f') & \xrightarrow{\beta_1'} & Ker(f) & \xrightarrow{\alpha_1'} & Ker(f'') \\ & & {\scriptstyle \mathfrak{k}'}\downarrow & & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\ & & A_1' & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\ & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ x & \longrightarrow & A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \end{array} \qquad (1)$$

be a commutative diagram whose two lower rows and the right column are 'exact'. Then its upper row, $Ker(f') \xrightarrow{\beta_1'} Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$, is 'exact'.

Proof. Let x be an initial object of C_X; and let $Ker(\beta_2) = x$. The diagram (1) gives rise to the commutative diagram

$$\begin{array}{ccccccc} Ker(f') & \xrightarrow{\mathfrak{e}_1'} & Ker(\widetilde{f}) & \xrightarrow{\mathfrak{j}_1'} & Ker(f) & \xrightarrow{\alpha_1'} & Ker(f'') \\ {\scriptstyle \mathfrak{k}'}\downarrow & cart & {\scriptstyle \widetilde{\mathfrak{k}}}\downarrow & cart & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\ A_1' & \xrightarrow{\mathfrak{e}_1} & Ker(\alpha_1) & \xrightarrow{\mathfrak{j}_1} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\ {\scriptstyle f'}\downarrow & & {\scriptstyle \widetilde{f}}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ A_2' & \xrightarrow{id} & Ker(\alpha_2) = A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \end{array} \qquad (1.1)$$

where $\mathfrak{e}_1$ is a deflation, $\mathfrak{j}_1 \circ \mathfrak{e}_1 = \beta_1$, $\mathfrak{j}_1' \circ \mathfrak{e}_1' = \beta_1'$, and $A_2' \xrightarrow{\beta_2} A_2$ is the kernel of α_2. The claim is that the morphism $Ker(f) \xrightarrow{\mathfrak{e}_1'} Ker(\alpha_1')$ is a deflation.

By 2.3.4.1 (or 2.3.4.3), the upper left square of (1.1) is cartesian, because the left lower horizontal arrow is identical. Since $A_1' \xrightarrow{\mathfrak{e}_1} Ker(\alpha_1)$ is a deflation, this implies that $Ker(f') \xrightarrow{\mathfrak{e}_1'} Ker(\widetilde{f})$ is a deflation.

Since $Ker(\beta_2)$ is trivial, it follows from 2.3.4.3 (applied to the middle section of the diagram (1.1)) that the upper middle square of (1.1) is cartesian.

Notice that $Ker(\widetilde{f}) \xrightarrow{\mathfrak{j}_1'} Ker(f)$ is the kernel of $Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$.

In fact, by hypothesis, the kernel of $Ker(f'') \xrightarrow{\mathfrak{k}''} A''_1$ is trivial. Therefore, by 3.3.4.3, the right square of the commutative diagram

$$\begin{array}{ccccc}
Ker(\alpha'_1) & \xrightarrow{\mathfrak{j}'_1} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') \\
{\scriptstyle \widetilde{\mathfrak{k}}'}\downarrow & cart & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_1) & \xrightarrow{\mathfrak{j}_1} & A_1 & \xrightarrow{\alpha_1} & A''_1
\end{array}$$

is cartesian (whenever $Ker(\alpha'_1)$ exists). Therefore, by the universality of cartesian squares, there is a natural isomorphism $Ker(\widetilde{f}) \xrightarrow{\sim} Ker(\alpha'_1)$. ■

The following assertion is a non-additive version of the 'snake lemma'. Its proof is not reduced to the element-wise diagram chasing, like the argument of the classical 'snake lemma'. Therefore, it requires more elaboration than its abelian prototype.

C2.2. Proposition ('snake lemma'). *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x; and let*

$$\begin{array}{ccccccc}
 & & & & & & x \\
 & & & & & & \downarrow \\
 & & Ker(f') & \xrightarrow{\beta'_1} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') \\
 & & {\scriptstyle \mathfrak{k}'}\downarrow & & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\
 & & A'_1 & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A''_1 \\
 & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
x & \longrightarrow & A'_2 & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A''_2 \\
 & & {\scriptstyle \mathfrak{e}'}\downarrow & & {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\
 & & A'_3 & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A''_3
\end{array} \qquad (2)$$

be a commutative diagram whose vertical columns and middle rows are 'exact', the arrows α_1, $\mathfrak{e}'$, $\mathfrak{e}$, $\mathfrak{e}''$ are deflations, and the kernel of $Ker(f'') \xrightarrow{\mathfrak{k}''} A''_1$ is trivial.

(a) Suppose that each deflation of $(C_X, \mathfrak{E}_X)$ is isomorphic to its coimage and the unique arrow $x \longrightarrow A''_3$ is a monomorphism. Then there exists a natural morphism $Ker(f'') \xrightarrow{\mathfrak{d}} A'_3$ such that the sequence

$$\begin{array}{ccccccccc}
Ker(f') & \xrightarrow{\beta'_1} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') & & & & \\
 & & & & \downarrow{\scriptstyle \mathfrak{d}} & & & & \\
 & & & & A'_3 & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A''_3
\end{array} \qquad (3)$$

is a complex. Moreover, its subsequences $Ker(f') \xrightarrow{\beta_1'} Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$ *and* $Ker(f'') \xrightarrow{\mathfrak{d}} A_3' \xrightarrow{\beta_2} A_3$ *are 'exact'.*

(b) Suppose, in addition, that

(b1) $\mathfrak{E}_X$ *is saturated in the following sense: if* $\lambda \circ \mathfrak{s}$ *is a deflation and* $\mathfrak{s}$ *is a deflation, then* λ *is a deflation;*

(b2) the following condition holds:

(#) If $M \xrightarrow{\mathfrak{e}} N$ *is a deflation and* $M \xrightarrow{p} M$ *an idempotent morphism (i.e.* $p^2 = p$*) which has a kernel and such that the composition* $\mathfrak{e} \circ p$ *is a trivial morphism, then the composition of the canonical morphism* $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ *and* $M \xrightarrow{\mathfrak{e}} N$ *is a deflation.*

Then the entire sequence (3) is 'exact'.

Proof. (i) Since α_1 is a deflation, there exists a cartesian square

$$\begin{array}{ccc}
\widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
{\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\
A_1 & \xrightarrow{\alpha_1} & A_1''
\end{array}$$

where $\widetilde{\alpha}_1$ is a deflation too. It follows from 2.3.4.1 that $\widetilde{A}_1 = Ker(\alpha_2 f) = Ker(f''\alpha_1)$. This is seen from the commutative diagram

$$\begin{array}{ccccc}
\widetilde{A}_1 & \xrightarrow{id} & Ker(f''\alpha_1) & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
{\scriptstyle id}\downarrow & & {\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
{\scriptstyle h}\downarrow & cart & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \\
{\scriptstyle \mathfrak{e}'}\downarrow & & {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\
A_3' & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A_3''
\end{array} \qquad (4)$$

with cartesian squares as indicated.

(ii) By 2.3.3, we have a commutative diagram

$$\begin{array}{ccccc}
Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & \widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
{\scriptstyle \wr}\downarrow & & {\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_1) & \xrightarrow{\mathfrak{k}(\alpha_1)} & A_1 & \xrightarrow{\alpha_1} & A_1''
\end{array}$$

whose (rows are conflations and the) left vertical arrow is an isomorphism. Thus, we obtain a commutative diagram

$$
\begin{array}{ccccc}
Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & \widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
\mathfrak{e}_1 \Big\uparrow & & \widetilde{\mathfrak{k}}'' \Big\downarrow & cart & \Big\downarrow \mathfrak{k}'' \\
A_1' & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
f' \Big\downarrow & & f \Big\downarrow & & \Big\downarrow f'' \\
A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \\
\mathfrak{e}' \Big\downarrow & & \mathfrak{e} \Big\downarrow & & \Big\downarrow \mathfrak{e}'' \\
A_3' & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A_3''
\end{array}
\qquad (5)
$$

Since the second row of the diagram (2) is 'exact', the morphism $\mathfrak{e}_1$ is a deflation.

(iii) Combining the diagram (4) with (the left upper corner of) (5), we obtain a commutative diagram

$$
\begin{array}{ccccccc}
A_1' & \xrightarrow{\mathfrak{e}_1} & Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & Ker(f''\alpha_1) & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
 & & \mathfrak{k}(\widetilde{\alpha}_1) \Big\downarrow & & \widetilde{\mathfrak{k}}'' \Big\downarrow & cart & \Big\downarrow \mathfrak{k}'' \\
 & & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
 & & h \Big\downarrow & cart & f \Big\downarrow & & \Big\downarrow f'' \\
 & & A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \\
 & & \mathfrak{e}' \Big\downarrow & & \mathfrak{e} \Big\downarrow & & \Big\downarrow \mathfrak{e}'' \\
 & & A_3' & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A_3''
\end{array}
\qquad (6)
$$

where $\widetilde{\mathfrak{k}}''\circ\mathfrak{k}(\widetilde{\alpha}_1)\circ\mathfrak{e}_1 = \beta_1$. Therefore, $\beta_2\circ(h\circ\mathfrak{k}(\widetilde{\alpha}_1)\circ\mathfrak{e}_1) = f\circ(\widetilde{\mathfrak{k}}''\circ\mathfrak{k}(\widetilde{\alpha}_1)\circ\mathfrak{e}_1) = f\circ\beta_1 = \beta_2\circ f'$. Since the left middle square of (6) is cartesian, this implies that $h \circ \mathfrak{k}(\widetilde{\alpha}_1) \circ \mathfrak{e}_1 = f'$.

Therefore, $\mathfrak{e}' \circ h \circ \mathfrak{k}(\widetilde{\alpha}_1) \circ \mathfrak{e}_1 = \mathfrak{e}' \circ f'$ is a trivial morphism.

(iv) Notice that, by 2.1.2, the kernel morphism $Ker(\mathfrak{e}' \circ h \circ \mathfrak{k}(\widetilde{\alpha}_1)) \longrightarrow Ker(\widetilde{\alpha}_1)$ is a monomorphism, because A_3' has a morphism to x, hence $x \longrightarrow A_3'$ is a (split) monomorphism. Since $\mathfrak{e}_1$ is a deflation, in particular a strict epimorphism, it follows from 2.3.4.4 that the composition $(\mathfrak{e}'\circ h)\circ\mathfrak{k}(\widetilde{\alpha}_1)$ is trivial. By hypothesis, $\widetilde{\alpha}_1$ (being a deflation) is isomorphic to the coimage morphism, i.e. $Ker(f'')$ is naturally isomorphic to $Coim(\widetilde{\alpha}_1)$. Therefore, the morphism $\mathfrak{e}' \circ h$ factors through $\widetilde{\alpha}_1$, i.e. $\mathfrak{e}' \circ h = \mathfrak{d} \circ \widetilde{\alpha}_1$. Since $\widetilde{\alpha}_1$ is a deflation, in particular an epimorphism, the latter equality determines the morphism $Ker(f'') \xrightarrow{\mathfrak{d}} A_3'$ uniquely.

(v) By C2.1, the sequence $Ker(f') \xrightarrow{\beta_1'} Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$ is 'exact'.

(vi) The composition of $Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$ and $Ker(f'') \xrightarrow{\mathfrak{d}} A_3'$ is trivial. In fact, the diagram (6) induces a commutative diagram

$$\begin{array}{ccccccccc}
A_1' & & Ker(h) & \xrightarrow{\sim} & Ker(f) & \xrightarrow{\alpha_1'} & Ker(f'') \\
{\scriptstyle \mathfrak{e}_1}\downarrow & & {\scriptstyle \mathfrak{k}(h)}\downarrow & & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
& & {\scriptstyle h}\downarrow & cart & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
& & A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \\
& & {\scriptstyle \mathfrak{e}'}\downarrow & & {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\
& & A_3' & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A_3''
\end{array} \qquad (7)$$

where the isomorphism $Ker(h) \xrightarrow{\sim} Ker(f)$ is due the fact that the left middle square of the diagram (7) is cartesian. We can and will assume that this isomorphism is identical. The morphism $Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$ is the composition of $Ker(f) \xrightarrow{\mathfrak{k}(h)} Ker(\alpha_2 f)$ and $Ker(\alpha_2 f) \xrightarrow{\widetilde{\alpha}_1} Ker(f'')$. Therefore, $\mathfrak{d} \circ \alpha_1' = \mathfrak{d} \circ \widetilde{\alpha}_1 \circ \mathfrak{k}(h) = \mathfrak{e}' \circ h \circ \mathfrak{k}(h)$, which shows that the composition $\mathfrak{d} \circ \alpha_1'$ is trivial, because already the composition $h \circ \mathfrak{k}(h)$ is trivial.

(vii) The argument above can be summarized in the commutative diagram

$$\begin{array}{ccccc}
A_1' & & Ker(f) & \xrightarrow{\gamma} & Ker(\mathfrak{d}) \\
{\scriptstyle \mathfrak{e}_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}(h)} & & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{d})} \\
Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\alpha}_1} & K(f'') \\
{\scriptstyle \mathfrak{e}_2}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathfrak{d}} \\
Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e}')} & A_2' & \xrightarrow{\mathfrak{e}'} & A_3'
\end{array} \qquad (8)$$

where $Ker(\widetilde{\alpha}_1) \xrightarrow{\mathfrak{e}_2} Ker(\mathfrak{e}')$ is a deflation. Taking into consideration the cartesian square

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\widetilde{\gamma}} & Ker(\mathfrak{d}) \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{d})} \\
Ker(\alpha_2 f) & \xrightarrow{\widetilde{\alpha}_1} & K(f'')
\end{array} \qquad (9)$$

we extend (8) to the commutative diagram

$$
\begin{array}{ccccccccc}
 & & Ker(f) & \xrightarrow{id} & Ker(h) & \xrightarrow{\gamma} & Ker(\mathfrak{d}) \\
 & & {\scriptstyle id}\big\downarrow & & {\scriptstyle \mathfrak{k}(\widetilde{h})}\big\downarrow & & \big\downarrow{\scriptstyle id} \\
A'_1 & & Ker(\widetilde{h}) & \xrightarrow{\mathfrak{k}(\widetilde{h})} & \mathcal{M} & \xrightarrow{\widetilde{\gamma}} & Ker(\mathfrak{d}) \\
{\scriptstyle \mathfrak{e}_1}\big\downarrow & & {\scriptstyle \mathfrak{k}(\widetilde{h})}\big\downarrow & & {\scriptstyle \mu}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{k}(\mathfrak{d})} \\
Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}_1} & \mathcal{M} & \xrightarrow{\mu} & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\alpha}_1} & K(f'') \\
{\scriptstyle \mathfrak{e}_2}\big\downarrow & & {\scriptstyle \widetilde{h}}\big\downarrow & cart & {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{d}} \\
Ker(\mathfrak{e}') & \xrightarrow{id} & Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e}')} & A'_2 & \xrightarrow{\mathfrak{e}'} & A'_3
\end{array}
\qquad (10)
$$

where $\mu \circ \mathfrak{k}(\widetilde{h}) = \mathfrak{k}(h)$, and $\mu \circ \mathfrak{k}_1 = \mathfrak{k}(\widetilde{\alpha}_1)$.

Since the square (9) is cartesian and $\widetilde{\alpha}_1$ is a deflation, its pull-back, $\widetilde{\gamma}$, is a deflation too. Notice that the commutativity of the left lower square and the fact that $\mathfrak{e}_2$ is a strict epimorphism imply that $\widetilde{h}$ is a strict epimorphism.

Consider the cartesian square

$$
\begin{array}{ccc}
\widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{e}'_3} & \mathcal{M} \\
{\scriptstyle \mathfrak{p}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{h}} \\
A'_1 & \xrightarrow{\mathfrak{e}_3} & Ker(\mathfrak{e}')
\end{array}
\qquad (11)
$$

where $\mathfrak{e}_3 = \mathfrak{e}_2 \circ \mathfrak{e}_1$. Since $\mathfrak{e}_3$ is a deflation, the arrow $\widetilde{\mathcal{M}} \xrightarrow{\mathfrak{e}'_3} \mathcal{M}$ is a deflation. Since $\widetilde{h} \circ \mathfrak{k}_1 \circ \mathfrak{e}_1 = \mathfrak{e}_3$, the projection $\mathfrak{p}$ has a splitting, $A'_1 \xrightarrow{\mathfrak{s}} \widetilde{\mathcal{M}}$; i.e. $\mathfrak{p} \circ \mathfrak{s} = id$. Set $p = \mathfrak{s} \circ \mathfrak{p}$ and $\varphi = \widetilde{\gamma} \circ \mathfrak{e}'_3$. It follows that $\widetilde{\mathcal{M}} \xrightarrow{p} \widetilde{\mathcal{M}}$ is an idempotent (– a projector), φ is a deflation, the composition $\varphi \circ p = \widetilde{\gamma} \circ (\mathfrak{e}'_3 \circ \mathfrak{s}) \circ \mathfrak{p} = \widetilde{\gamma} \circ (\mathfrak{k}_1 \circ \mathfrak{e}_1) \circ \mathfrak{p}$ is trivial, because $\mathfrak{k}(\mathfrak{d}) \circ \widetilde{\gamma} \circ \mathfrak{k}_1 = \widetilde{\alpha}_1 \circ \mathfrak{k}(\widetilde{\alpha}_1)$ is trivial and $Ker(\mathfrak{d}) \xrightarrow{\mathfrak{k}(\mathfrak{d})} K(f'')$ is a monomorphism. The latter follows from the fact that A'_3 has a morphism to x, hence the unique arrow $x \longrightarrow A'_3$ is a (split) monomorphism.

Since the square (11) is cartesian, it follows from 2.3.4.1 that $Ker(\mathfrak{p})$ is naturally isomorphic to $Ker(\widetilde{h}) = Ker(f)$. And, by 2.3.4.3, $Ker(p)$ is naturally isomorphic to $Ker(\mathfrak{p})$, because $p = \mathfrak{s} \circ \mathfrak{p}$ and $\mathfrak{s}$ is a monomorphism.

Thus, $Ker(p)$ is naturally isomorphic to $Ker(f)$.

(viii) Suppose that the condition (#) of the proposition holds. Then the composition of $Ker(p) \longrightarrow \widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}} \xrightarrow{\varphi} Ker(\mathfrak{d})$ is a deflation; hence the composition of the morphisms $Ker(f) \xrightarrow{\mathfrak{k}(\widetilde{h})} \mathcal{M}$ and $\mathcal{M} \xrightarrow{\widetilde{\gamma}} Ker(\mathfrak{d})$ (i.e. the morphism $Ker(f) \xrightarrow{\gamma} Ker(\mathfrak{d})$ in the diagram (10)) is a deflation.

(ix) The claim is that $\mathfrak{d}$ is the composition of the morphism $Ker(\beta_3) \xrightarrow{\mathfrak{k}(\beta_3)} A_3'$ and a deflation $Ker(f'') \xrightarrow{\mathfrak{d}'} Ker(\beta_3)$. Since $\widetilde{\alpha}_1$ is a deflation, it suffices to prove a similar assertion for $\mathfrak{d} \circ \widetilde{\alpha}_1 = \mathfrak{e}' \circ h$.

We have a commutative diagram

$$\begin{array}{ccccccc}
 & & \mathfrak{B} & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
 & & {\scriptstyle \mathfrak{t}_h}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{t}_f} & & \big\downarrow{\scriptstyle \mathfrak{t}''} \\
 & & \mathcal{B} & \xrightarrow{\widetilde{\beta}_3} & Ker(\mathfrak{e}) & \xrightarrow{\lambda} & Ker(\mathfrak{e}'') \\
 & & {\scriptstyle \psi}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{k}(\mathfrak{e})} & & \big\downarrow{\scriptstyle \mathfrak{k}(\mathfrak{e}'')} \\
\mathcal{B} & \xrightarrow{\psi} & A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \\
{\scriptstyle \widetilde{\mathfrak{e}}'}\big\downarrow & cart & {\scriptstyle \mathfrak{e}'}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{e}} & & \big\downarrow{\scriptstyle \mathfrak{e}''} \\
Ker(\beta_3) & \xrightarrow{\mathfrak{k}(\beta_3)} & A_3' & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A_3''
\end{array} \qquad (12)$$

where $f = \mathfrak{k}(\mathfrak{e}) \circ \mathfrak{t}_f$, $f'' = \mathfrak{k}(\mathfrak{e}'') \circ \mathfrak{t}''$ and the remaining new arrows are determined by the commutativity of the diagram (12) and by being a part of a cartesian square. By hypothesis, the columns of the diagram (2) are 'exact'; in particular, the morphism $\mathfrak{t}_f$ is a deflation. Therefore, the morphism $\mathfrak{B} \xrightarrow{\mathfrak{t}_h} \mathcal{B}$ is a deflation. Being the composition of two cartesian diagrams, the diagram

$$\begin{array}{ccc}
\mathfrak{B} & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 \\
{\scriptstyle \psi \circ \mathfrak{t}_h}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{k}(\mathfrak{e}) \circ \mathfrak{t}_f = f} \\
A_2' & \xrightarrow{\beta_2} & A_2
\end{array}$$

is cartesian, as well as the diagram

$$\begin{array}{ccc}
Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 \\
{\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
A_2' & \xrightarrow{\beta_2} & A_2
\end{array}$$

Therefore, they are isomorphic to each other. So, we can and will assume that $\mathfrak{B} = Ker(\alpha_2 f)$ and $h = \psi \circ \mathfrak{t}_h$. It follows from (the left part of) the diagram (12) that

$$\mathfrak{e}' \circ h = \mathfrak{e}' \circ \psi \circ \mathfrak{t}_h = \mathfrak{k}(\beta_3) \circ (\widetilde{\mathfrak{e}}' \circ \mathfrak{t}_h),$$

that is $\mathfrak{e}\circ h$ is the composition of $Ker(\beta_3) \xrightarrow{\mathfrak{k}(\beta_3)} A'_3$ and the deflation $\widetilde{\mathfrak{e}}'\circ\mathfrak{t}_h$.

(x) The composition $\alpha_3\circ\beta_3$ is trivial by 2.3.4.4, because the composition $\alpha_3\circ\beta_3\circ\mathfrak{e}' = \mathfrak{e}''\circ\alpha_2\circ\beta_2$ is trivial, $x \longrightarrow A''_3$ is a monomorphism (by hypothesis), and $\mathfrak{e}'$ is a deflation, hence a strict epimorphism. The claim is that, if $(C_X, \mathfrak{E}_X)$ has the property (#), then the morphism $A'_3 \xrightarrow{\beta_3} A_3$ is the composition of the kernel morphism $Ker(\alpha_3) \xrightarrow{\mathfrak{k}_3} A''_3$ and a deflation $A'_3 \xrightarrow{\mathfrak{t}_3} Ker(\alpha_3)$.

Since in the upper right square of the diagram (12), the arrows α_1, $\mathfrak{t}''$, and $\mathfrak{t}_f$ are deflations, the forth arrow, $Ker(\mathfrak{e}) \xrightarrow{\lambda} Ker(\mathfrak{e}'')$, is a deflation too (due to the saturatedness condition (b1)). Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & Ker(\mathfrak{e}) & \xrightarrow{\lambda} & Ker(\mathfrak{e}'') \\
 & & & & {\scriptstyle\mathfrak{v}}\big\downarrow & & \big\downarrow{\scriptstyle id} \\
 & & A'_2 & \xrightarrow{\mathfrak{t}'_3} & \mathfrak{D} & \xrightarrow{\mathfrak{p}_2} & Ker(\mathfrak{e}'') \\
 & & {\scriptstyle\mathfrak{t}'_3}\big\downarrow & & {\scriptstyle\beta'_2}\big\downarrow & cart & \big\downarrow{\scriptstyle\mathfrak{k}(\mathfrak{e}'')} \\
A'_2 & \xrightarrow{\mathfrak{t}'_3} & \mathfrak{D} & \xrightarrow{\beta'_2} & A_2 & \xrightarrow{\alpha_2} & A''_2 \\
{\scriptstyle\mathfrak{e}'}\big\downarrow & & {\scriptstyle\mathfrak{u}}\big\downarrow & cart & {\scriptstyle\mathfrak{e}}\big\downarrow & & \big\downarrow{\scriptstyle\mathfrak{e}''} \\
A'_3 & \xrightarrow{\mathfrak{t}_3} & Ker(\alpha_3) & \xrightarrow{\mathfrak{k}(\alpha_3)} & A_3 & \xrightarrow{\alpha_3} & A''_3
\end{array} \qquad (13)$$

where $\beta'_2\circ\mathfrak{v} = \mathfrak{k}(\mathfrak{e})$, $\beta'_2\circ\mathfrak{t}'_2 = \beta_2$.

The upper left corner of the commutative diagram (13) gives rise to the commutative diagram

$$\begin{array}{ccccc}
\widetilde{A}'_2 & \xrightarrow{\widetilde{\mathfrak{t}}'_3} & \widetilde{\mathfrak{D}} & \xrightarrow{\widetilde{\mathfrak{p}}_2} & Ker(\mathfrak{e}) \\
{\scriptstyle\lambda'}\big\downarrow & cart & {\scriptstyle\widetilde{\lambda}}\big\downarrow & cart & \big\downarrow{\scriptstyle\lambda} \\
A'_2 & \xrightarrow{\mathfrak{t}'_3} & \mathfrak{D} & \xrightarrow{\mathfrak{p}_2} & Ker(\mathfrak{e}'')
\end{array} \qquad (14)$$

whose both squares are cartesian. Since λ is a deflation, all vertical arrows of (14) are deflations, as well as the arrows $\mathfrak{p}_2$ and $\widetilde{\mathfrak{p}}_2$. The morphism $Ker(\mathfrak{e}) \xrightarrow{\mathfrak{v}} \mathfrak{D}$ determines a splitting $Ker(\mathfrak{e}) \xrightarrow{\mathfrak{s}_2} \widetilde{\mathfrak{D}}$ of the projection $\widetilde{\mathfrak{p}}_2$. Let p_2 denote the composition $\mathfrak{s}_2\circ\mathfrak{p}_2$. It follows that p_2 is an idempotent $\widetilde{\mathfrak{D}} \longrightarrow \widetilde{\mathfrak{D}}$ and the composition

$$\mathfrak{k}(\alpha_3)\circ(\mathfrak{u}\circ\widetilde{\lambda})\circ p_2 = \mathfrak{e}\circ\beta'_2\circ(\widetilde{\lambda}\circ\mathfrak{s}_2)\circ\widetilde{\mathfrak{p}}_2 = \mathfrak{e}\circ\beta'_2\circ\mathfrak{v}\circ\widetilde{\mathfrak{p}}_2 = (\mathfrak{e}\circ\mathfrak{k}(\mathfrak{e}))\circ\widetilde{\mathfrak{p}}_2$$

is trivial. Therefore, $(\mathfrak{u}\circ\widetilde{\lambda})\circ p_2$ is trivial. The kernel of the idempotent p_2 is isomorphic to the kernel of $\widetilde{\mathfrak{p}}_2$. Since the right square of (14) is cartesian, there is a natural isomorphism

$Ker(\mathfrak{p}_2) \simeq Ker(\widetilde{\mathfrak{p}}_2)$. It follows from the right cartesian square of (13) that there is a natural isomorphism $Ker(\mathfrak{p}_2) \simeq Ker(\alpha_2) = A'_2$.

If the right exact category $(C_X, \mathfrak{E}_X)$ has the property (#), then the above implies that the morphism $A'_2 \xrightarrow{\mathfrak{u}\circ\mathfrak{t}'_3} Ker(\alpha_3)$ is a deflation. Since $\mathfrak{u} \circ \mathfrak{t}'_3 = \mathfrak{t}_3 \circ \mathfrak{e}'$ and $\mathfrak{e}'$ is a deflation, the morphism $A'_3 \xrightarrow{\mathfrak{t}_3} Ker(\alpha_3)$ is a deflation. ■

C2.3. Remarks about conditions of the 'snake lemma'. Fix a right exact category $(C_X, \mathfrak{E}_X)$. The main condition of the 'snake lemma' C2.2, the one which guarantees the existence of the *connecting morphism* $\mathfrak{d}$, is that each deflation $M \xrightarrow{\mathfrak{e}} N$ is isomorphic to its coimage morphism $M \xrightarrow{\mathfrak{c}(\mathfrak{e})} Coim(\mathfrak{e}) = M/Ker(\mathfrak{e})$.

If the category C_X is additive, then every strict epimorphism which has a kernel, in particular, every deflation, is isomorphic to its coimage morphism.

The latter property holds in many non-additive categories, for instance in the category Alg_k of unital associative k-algebras (see 2.3.5.3).

Similarly, the property

(#) If $M \xrightarrow{\mathfrak{e}} N$ is a deflation and $M \xrightarrow{p} M$ an idempotent morphism (i.e. $p^2 = p$) which has a kernel and such that the composition $\mathfrak{e} \circ p$ is a trivial morphism, then the composition of the canonical morphism $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ and the deflation $M \xrightarrow{\mathfrak{e}} N$ is a deflation.

which ensures 'exactness' of the 'snake' sequence (3) holds in any additive category.

In fact, if the category C_X is additive, then the existence of the kernel of p means precisely that the idempotent $q = id_M - p$ is splittable; i.e. $M \xrightarrow{q} M$ is the composition of $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ and a (strict) epimorphism $M \xrightarrow{\mathfrak{t}} Ker(p)$ such that $\mathfrak{t} \circ \mathfrak{k}(p) = id$. The condition $\mathfrak{e} \circ p$ is trivial (that is $\mathfrak{e} \circ p = 0$) is equivalent to the equalities $\mathfrak{e} = \mathfrak{e} \circ q = (\mathfrak{e} \circ \mathfrak{k}(p)) \circ \mathfrak{t}$ which imply (under saturatedness condition, cf. C2.2(b1)) that $\mathfrak{e} \circ \mathfrak{k}(p)$ is a deflation.

C2.3.1. Example. The property (#) holds in the category Alg_k. In fact, let $A \xrightarrow{\varphi} B$ be a strict algebra epimorphism, and $A \xrightarrow{p} A$ an idempotent endomorphism such that the composition $\varphi \circ p$ is a trivial morphism; that it equals to the composition of an augmentation morphism $A \xrightarrow{\pi} k$ and the k-algebra structure $k \xrightarrow{i_B} B$. In particular, $A = k \oplus A_+$, where $A_+ = K(\pi)$ is the kernel of the augmentation π in the usual sense.

On the other hand, $Ker(p) = k \oplus K(p)$, and, since $p \circ p = p$ and the ideal $K(p) \stackrel{\mathrm{def}}{=} \{y \in A \mid p(y) = 0\}$ coincides $\{x - p(x) \mid x \in A\}$. Similarly, $Ker(\varphi \circ p) = k \oplus K(\varphi \circ p)$, and it follows that $K(\varphi \circ p) = A_+$.

Every element x of A is uniquely represented as $\lambda \cdot 1_A + x_+$, where 1_A is the unit element of the algebra A and $x_+ \in A_+$. Therefore, $x - p(x) = x_+ - p(x_+)$ and

$$\varphi(\mu \cdot 1_A + (x - p(x))) = \mu \cdot 1_B + \varphi(x_+ - p(x_+)) = \mu \cdot 1_B + \varphi(x_+) = \varphi(\mu \cdot 1_A + x_+).$$

Since $\mu \in k$ and $x_+ \in A_+$ are arbitrary and φ is a strict epimorphism (that is a surjective map), this shows that $\varphi \circ \mathfrak{k}(p)$ is a strict epimorphism.

C3. Localizations of exact categories and (co)quasi-suspended categories. t-Structures.

C3.1. Remarks on localizations. Let $C_X \xrightarrow{u^*} C_Z$ be a functor. Suppose that the category C_Z is *cocomplete*, i.e. it has colimits of arbitrary small diagrams (equivalently, it has infinite coproducts and cokernels of pairs of arrows). By [GZ, II.1], the functor u^* equals to the composition of the Yoneda embedding $C_X \xrightarrow{h_X} \widehat{C}_X$ of the category C_X into the category $\widehat{C}_X$ of presheaves of sets on C_X and a continuous (that is having a right adjoint) functor $\widehat{C}_X \xrightarrow{\widetilde{u}^*} C_Z$. Since every presheaf of sets on a category is a colimit of a canonical diagram of representable presheaves and the functor $\widetilde{u}^*$ preserves colimits, it is determined uniquely up to isomorphism.

In particular, every functor $C_X \xrightarrow{q^*} C_Y$ gives rise to a commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{q^*} & C_Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ \widehat{C}_X & \xrightarrow{\widehat{q}^*} & \widehat{C}_Y \end{array} \tag{1}$$

with a continuous functor $\widehat{q}^*$ determined by the commutativity of (1) uniquely up to isomorphism.

C3.1.1. Lemma. *(a) The functor*

$$\widehat{C}_Y \xrightarrow{\widehat{q_*}} \widehat{C}_X, \quad F \longmapsto F \circ q^*, \tag{2}$$

is a canonical right adjoint to $\widehat{q}^*$.

(b) If the functor q^* has a right adjoint q_*, then the diagram

$$\begin{array}{ccc} C_X & \xleftarrow{q_*} & C_Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ \widehat{C}_X & \xleftarrow{\widehat{q_*}} & \widehat{C}_Y \end{array} \tag{1_*}$$

quasi-commutes.

Proof. (a) Recall that the functor $\widehat{q}^*$ is determined uniquely up to isomorphism by the equality $\widehat{q}^*(h_Y(L)) = h_Y(q^*(L))$ for all $L \in ObC_X$. For every $L \in ObC_X$ and $F \in Ob\widehat{C}_Y$, we have

$$\widehat{C}_X(h_X(L), F \circ q^*) \simeq F(q^*(L)) \simeq \widehat{C}_Y(h_Y(q^*(L)), F) \simeq \widehat{C}_Y(\widehat{q}^*(h_Y(L)), F).$$

Since all isomorphisms here are functorial, it follows that the functor (2) is a right adjoint to $\widehat{q}^*$.

(b) For any $L \in ObC_Y$,

$$\widehat{q}_*(h_Y(L)) = h_Y(L) \circ q^* = C_Y(q^*(-), L) \simeq C_X(-, q_*(L)) = h_X(q_*(L)),$$

hence the assertion. ∎

C3.1.1.1. Corollary. *For every functor* $C_X \xrightarrow{q^*} C_Y$, *the functor* $\widehat{q}_*$ *has a right adjoint,* $\widehat{q}^!$. *In particular,* $\widehat{q}_*$ *is exact.*

Proof. The fact follows from C3.1.1(a). ∎

C3.1.2. Proposition. *If* $C_X \xrightarrow{q^*} C_Y$ *is a localization, then the continuous functor* $\widehat{q}^*$ *in (1) is a localization too.*

Proof. The functor $\widehat{C}_X \xrightarrow{\widehat{q}^*} \widehat{C}_Y$ is decomposed into a localization $\widehat{C}_X \xrightarrow{\widehat{q}^*_{\mathfrak{f}}} C_Z$ at $\Sigma_{\widehat{q}^*} = \{s \in Hom\widehat{C}_X \mid \widehat{q}^*(s) \text{ is invertible}\}$ and a conservative functor $C_Z \xrightarrow{\widehat{q}^*_c} \widehat{C}_Y$. Since q^* is a localization and the composition $\widehat{q}^*_{\mathfrak{f}} \circ h_X$ makes invertible all arrows of $\Sigma_{q^*} = \{s \in Hom\widehat{C}_X \mid q^*(s) \text{ is invertible}\}$, there exists a unique functor $C_Y \xrightarrow{\Psi} C_Z$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{q^*} & C_Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle \Psi} \\ \widehat{C}_X & \xrightarrow{\widehat{q}^*_{\mathfrak{f}}} & C_Z \end{array} \qquad (3)$$

commutes. The localization $\widehat{q}^*_{\mathfrak{f}}$ is continuous, i.e. it has a right adjoint which is, forcibly, a fully faithful functor. Therefore, by [GZ, I.1.4], the category C_Z has limits and colimits of arbitrary (small) diagrams. Therefore, the functor $C_Y \xrightarrow{\Psi} C_Z$ is the composition of the Yoneda imbedding $C_Y \xrightarrow{h_Y} \widehat{C}_Y$ and a continuous functor $\widehat{C}_Y \xrightarrow{\Psi'} C_Z$; the latter is defined uniquely up to isomorphism. Thus, we have the equalities

$$\begin{aligned} \widehat{q}^*_{\mathfrak{f}} \circ h_X = \Psi \circ q^* = \Psi' \circ h_Y \circ q^* = \Psi' \circ \widehat{q}^* \circ h_X = (\Psi' \circ \widehat{q}^*_c) \circ \widehat{q}^*_{\mathfrak{f}} \circ h_X \\ (\widehat{q}^*_c \circ \Psi') \circ h_Y \circ q^* = \widehat{q}^*_c \circ \Psi \circ q^* = \widehat{q}^*_c \circ \widehat{q}^*_{\mathfrak{f}} \circ h_X = \widehat{q}^* \circ h_X \simeq h_Y \circ q^* \end{aligned} \qquad (4)$$

The equality $\widehat{q}^*_{\mathfrak{f}} \circ h_X = (\Psi' \circ \widehat{q}^*_c) \circ \widehat{q}^*_{\mathfrak{f}} \circ h_X$ implies, thanks to the continuity of the functors $\Psi' \circ \widehat{q}^*_c$ and $\widehat{q}^*_{\mathfrak{f}}$ and the universal properties of the localization $\widehat{q}^*_{\mathfrak{f}}$, that the composition $\Psi' \circ \widehat{q}^*_c$ is isomorphic to the identity functor.

Similarly, thanks to the universal properties of the localization q^*, the isomorphism $(\widehat{q}^*_c \circ \Psi') \circ h_Y \circ q^* \simeq h_Y \circ q^*$ implies that $(\widehat{q}^*_c \circ \Psi') \circ h_Y \simeq h_Y$. Since the functor $\widehat{q}^*_c \circ \Psi'$

is continuous and every presheaf of sets on C_Y is a colimit of a (canonical) diagram of representable presheaves, it follows from the latter isomorphism that the composition $\widehat{q}_c^* \circ \Psi'$ is isomorphic to the identical functor. All together shows that $\widehat{q}_c^*$ and Ψ' are mutually quasi-inverse category equivalences. ■

C3.1.3. Note. Suppose that C_X and C_Y are k-linear categories and $C_X \xrightarrow{q^*} C_Y$ a k-linear functor. If the category C_Y is cocomplete, then it follows from the assertion [GZ, II.1] mentioned above that there exists a unique up to isomorphism continuous functor $\mathcal{M}_k(X) \xrightarrow{\widehat{q}^*} C_Y$ such that $q^* = \widehat{q}^* \circ h_X$. Here, as above, $\mathcal{M}_k(X)$ is the category of k-presheaves on the category C_X. This establishes an equivalence between the category $\mathcal{H}om_k(C_X, C_Y)$ of k-linear functors $C_X \longrightarrow C_Y$ and the category $\mathcal{H}om_k^c(C_X, C_Y)$ of continuous k-linear functors $\mathcal{M}_k(X) \longrightarrow C_Y$.

If a k-linear functor $C_X \xrightarrow{q^*} C_Y$ is equivalent to a localization functor (i.e. it is the composition of the localization functor at $\Sigma_{q^*} \stackrel{\mathrm{def}}{=} \{s \in HomC_X \mid q^*(s) \text{ is invertible}\}$ and a category equivalence $\Sigma_{q^*}^{-1} C_X \longrightarrow C_Y$), then the argument of C3.1.1 with the categories of presheaves of sets replaced by the categories of k-presheaves shows that the natural extension $\mathcal{M}_k(X) \xrightarrow{\widehat{q}^*} \mathcal{M}_k(Y)$ is equivalent to a continuous localization.

C3.2. Right weakly 'exact' functors and 'exact' localizations. Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact categories. A *right weakly 'exact'* functor $(C_X, \mathcal{E}_X) \longrightarrow (C_Y, \mathcal{E}_Y)$ is a functor $C_X \xrightarrow{\varphi^*} C_Y$ such that for every conflation $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$, there is a commutative diagram

$$\begin{array}{ccccc} \varphi^*(L) & \xrightarrow{\varphi^*(\mathfrak{j})} & \varphi^*(M) & \xrightarrow{\varphi^*(\mathfrak{e})} & \varphi^*(N) \\ \mathfrak{e}' \searrow & & \nearrow \mathfrak{j}' & & \\ & L_1 & & & \end{array}$$

in which $\mathfrak{e}'$ is a deflation and $L_1 \xrightarrow{\mathfrak{j}'} \varphi^*(M) \xrightarrow{\varphi^*(\mathfrak{e})} \varphi^*(N)$ is a conflation.

Recall that the Gabriel-Quillen embedding $C_X \xrightarrow{\mathfrak{j}_X^*} C_{X_\mathfrak{E}}$ is the composition of the Yoneda embedding $C_X \xrightarrow{h_X} \mathcal{M}_k(X)$ and the sheafification functor $\mathcal{M}_k(X) \xrightarrow{\mathfrak{q}_X^*} C_{X_\mathfrak{E}}$.

C3.2.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories and $(C_X, \mathcal{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathcal{E}_Y)$ a right 'exact' k-linear functor.*

(a) There is a unique up to isomorphism continuous k-linear functor $C_{X_\mathfrak{E}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_\mathfrak{E}}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ \mathfrak{j}_X^* \downarrow & & \downarrow \mathfrak{j}_Y^* \\ C_{X_\mathfrak{E}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_\mathfrak{E}} \end{array}$$

commutes. Here the vertical arrows are the Gabriel-Quillen embeddings.

(b) If the functor $C_X \xrightarrow{\varphi^*} C_Y$ *is a localization, then the functor* $C_{X_\mathfrak{E}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_\mathfrak{E}}$ *is a localization.*

(c) Suppose that the following condition holds: for every $L \in ObC_X$ *and every deflation* $N \xrightarrow{\mathfrak{e}} \varphi^*(L)$*, there exist a deflation* $M \xrightarrow{\mathfrak{t}} L$ *and a commutative diagram*

$$\begin{array}{ccc} \varphi^*(M) & \xrightarrow{g} & N \\ \varphi^*(\mathfrak{t}) \searrow & & \swarrow \mathfrak{e} \\ & \varphi^*(L) & \end{array}$$

Then a right adjoint $C_{Y_\mathfrak{E}} \xrightarrow{\widetilde{\varphi}_*} C_{X_\mathfrak{E}}$ *to the functor* $\widetilde{\varphi}^*$ *has a right adjoint,* $\widetilde{\varphi}^!$*. In particular, the functor* $\widetilde{\varphi}_*$ *is exact.*

Proof. (a) Objects of the category $C_{X_\mathfrak{E}}$ – k-sheaves on the pretopology $(C_X, \mathfrak{E}_X)$, are naturally identified with right 'exact' k-linear functors from C_X to the abelian category $\mathcal{M}_k(X)^{op}$. Therefore, since the functor $C_X \xrightarrow{\varphi^*} C_Y$ is right 'exact', the composition with it maps $C_{Y_\mathfrak{E}}$ to $C_{X_\mathfrak{E}}$. By C3.1.1, we can (and will) assume that the functor

$$\mathcal{M}_k(Y) \xrightarrow{\widehat{\varphi}_*} \mathcal{M}_k(X)$$

is given by $F \longmapsto F \circ \varphi^*$. Thus, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_k(X) & \xleftarrow{\widehat{\varphi}_*} & \mathcal{M}_k(Y) \\ \mathfrak{q}_{X*} \uparrow & & \uparrow \mathfrak{q}_{Y*} \\ C_{X_\mathfrak{E}} & \xleftarrow{\widetilde{\varphi}_*} & C_{Y_\mathfrak{E}} \end{array}$$

whose vertical arrows are inclusion functors. This diagram yields, by adjunction, a quasi-commutative diagram

$$\begin{array}{ccc} \mathcal{M}_k(X) & \xrightarrow{\widehat{\varphi}^*} & \mathcal{M}_k(Y) \\ \mathfrak{q}_X^* \downarrow & & \downarrow \mathfrak{q}_Y^* \\ C_{X_\mathfrak{E}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_\mathfrak{E}} \end{array} \qquad (1)$$

where the vertical arrows are sheafification functors. The sheafification functors are exact localizations. An isomorphism $\mathfrak{q}_Y^* \widehat{\varphi}^* \simeq \widetilde{\varphi}^* \mathfrak{q}_X^*$ implies that $\widetilde{\varphi}^* \simeq \mathfrak{q}_Y^* \widehat{\varphi}^* \mathfrak{q}_{X*}$, because the adjunction arrow $Id_{C_{X_\mathfrak{E}}} \longrightarrow \mathfrak{q}_X^* \mathfrak{q}_{X*}$ is an isomorphism. Together with the isomorphism $\mathfrak{q}_Y^* \widehat{\varphi}^* \simeq \widetilde{\varphi}^* \mathfrak{q}_X^*$, this implies that the canonical morphism $\mathfrak{q}_Y^* \widehat{\varphi}^* \longrightarrow \mathfrak{q}_Y^* \widehat{\varphi}^* \mathfrak{q}_{X*} \mathfrak{q}_X^*$ is an isomorphism. The claim is that the functor $\widetilde{\varphi}_* \stackrel{\mathrm{def}}{=} \mathfrak{q}_X^* \widehat{\varphi}_* \mathfrak{q}_{Y*}$ is a right adjoint to $\widetilde{\varphi}^*$. In fact, the composition of morphisms

$$Id_{C_{Y_{\mathfrak{E}}}} \xrightarrow{\sim} \mathfrak{q}_Y^* \mathfrak{q}_{Y*} \longrightarrow \mathfrak{q}_Y^* \widehat{\varphi}_* \widehat{\varphi}^* \mathfrak{q}_{Y*} \longrightarrow \mathfrak{q}_Y^* \widehat{\varphi}_* \mathfrak{q}_{X*} \mathfrak{q}_X^* \widehat{\varphi}^* \mathfrak{q}_{Y*} \xrightarrow{\sim} \widetilde{\varphi}_* \widetilde{\varphi}^*$$

and

$$\widetilde{\varphi}^* \widetilde{\varphi}_* \xrightarrow{\sim} \mathfrak{q}_X^* \widehat{\varphi}^* \mathfrak{q}_{Y*} \mathfrak{q}_Y^* \widehat{\varphi}_* \mathfrak{q}_{X*} \xrightarrow{\sim} \mathfrak{q}_X^* \widehat{\varphi}^* \widehat{\varphi}_* \mathfrak{q}_{X*} \longrightarrow \mathfrak{q}_X^* \mathfrak{q}_{X*} \xrightarrow{\sim} Id_{C_{X_{\mathfrak{E}}}}$$

are adjunction arrows.

(b) By C3.1.1 (and C3.1.2), the continuous functor $\mathcal{M}_k(X) \xrightarrow{\widehat{\varphi}^*} \mathcal{M}_k(Y)$ is a localization. Thus, the three arrows of the quasi-commutative diagram (1) are localizations, hence the forth one, $\widetilde{\varphi}^*$, is a localization.

(c) The condition (c) means that for every $L \in ObC_Y$ and every presheaf F of k-modules on C_X, the value of the associated sheaf $\mathfrak{q}_X(F)$ at $\varphi^*(L)$ can be computed using only deflations (– covers) of the form $\varphi^*(M \xrightarrow{\mathfrak{t}} L)$, where $M \xrightarrow{\mathfrak{t}} L$ is a deflation. This implies that the diagram

$$\begin{array}{ccc} \mathcal{M}_k(X) & \xleftarrow{\widehat{\varphi}_*} & \mathcal{M}_k(Y) \\ {\scriptstyle \mathfrak{q}_X^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{q}_Y^*} \\ C_{X_{\mathfrak{E}}} & \xleftarrow{\widetilde{\varphi}_*} & C_{Y_{\mathfrak{E}}} \end{array} \qquad (1_*)$$

quasi-commutes. Therefore, by the argument similar to (a) above, the functor $\mathfrak{q}_Y^* \widehat{\varphi}^! \mathfrak{q}_{X*}$ is a right adjoint to $\widetilde{\varphi}_*$. ■

C3.3. Example. Suppose that C_X is a k-linear category with the smallest exact structure (given by split conflations). Then any k-linear functor (in particular, any right or left 'exact' k-linear functor) $C_X \xrightarrow{\varphi^*} C_Y$ is 'exact'. The category $C_{X_{\mathfrak{E}}}$ coincides with the category $\mathcal{M}_k(X)$ of k-presheaves on C_X, and the functor

$$C_{X_{\mathfrak{E}}} = \mathcal{M}_k(X) \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{E}}}$$

is isomorphic to the composition of the functor $\mathcal{M}_k(X) \xrightarrow{\widehat{\varphi}^*} \mathcal{M}_k(Y)$ and the sheafification functor $\mathcal{M}_k(Y) \xrightarrow{\mathfrak{q}_Y^*} C_{Y_{\mathfrak{E}}}$. Therefore, a right adjoint $\widetilde{\varphi}_*$ to $\widetilde{\varphi}^*$ is isomorphic to the composition $\widehat{\varphi}_* \mathfrak{q}_{Y*}$, which is not, usually, an exact functor.

C3.3.1. Example. Let $(C_X, \mathcal{E}_X)$ be an exact k-linear category. Suppose that C_Y is an additive k-linear category endowed with the smallest exact structure, $\mathcal{E}_Y^{spl}$. Then a functor $C_X \xrightarrow{\varphi^*} C_Y$ is right 'exact' functor from $(C_X, \mathcal{E}_X)$ to $(C_X, \mathcal{E}_Y^{spl})$ iff it maps every deflation of the exact category $(C_X, \mathcal{E}_X)$ to a split epimorphism (i.e. a coretraction). Notice that the condition (c) of C3.2.1 holds because every deflation in C_Y splits. Therefore, by C3.2.1(c), the functor $\widetilde{\varphi}_*$ has a right adjoint, $\widetilde{\varphi}^!$.

If the exact structure on $(C_X, \mathcal{E}_X)$ is also the smallest one (i.e. $\mathcal{E}_X = \mathcal{E}_X^{spl}$), then $C_{X_\mathfrak{e}} = \mathcal{M}_k(X)$ and $C_{Y_\mathfrak{e}} = \mathcal{M}_k(Y)$; i.e. in this case $\widetilde{\varphi}^* = \widehat{\varphi}^*$ and, therefore, a right adjoint to the functor $\widetilde{\varphi}_*$ coincides with $\widehat{\varphi}^!$.

C3.4. Remark. Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact categories. If $C_Y \xrightarrow{\varphi^*} C_X$ is an arbitrary functor, one can still define functors

$$C_{Y_\mathfrak{e}} \xrightarrow{\widetilde{\varphi}^*} C_{X_\mathfrak{e}} \xrightarrow{\widetilde{\varphi}_*} C_{Y_\mathfrak{e}} \xrightarrow{\widetilde{\varphi}^!} C_{X_\mathfrak{e}}$$

by the formulas

$$\widetilde{\varphi}^* = \mathfrak{q}_X^* \widehat{\varphi}^* \mathfrak{q}_{Y*}, \ \ \widetilde{\varphi}_* = \mathfrak{q}_Y^* \widehat{\varphi}_* \mathfrak{q}_{X*}, \ \ \widetilde{\varphi}^! = \mathfrak{q}_X^* \widehat{\varphi}^! \mathfrak{q}_{Y*}. \qquad (2)$$

C3.5. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories and $(C_X, \mathcal{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathcal{E}_Y)$ a right 'exact' k-linear functor. Suppose that φ^* is a localization functor. Then φ^* is 'exact' iff the class of arrows $\Sigma_{\varphi^*} = \{s \in HomC_X \mid \varphi^*(s)$ is an isomorphism$\}$ satisfies the following condition:*

(#) If the rows of a commutative diagram

$$\begin{array}{ccccc} L & \longrightarrow & M & \longrightarrow & N \\ \downarrow & & \downarrow & & \downarrow \\ L' & \longrightarrow & M' & \longrightarrow & N' \end{array} \qquad (2)$$

are conflations and any two of its vertical arrows belong to Σ_{φ^*}, then the remaining arrow belongs to Σ_{φ^*}.

Proof. (i) Consider first the case when φ^* is the identical functor. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \end{array} \qquad (3)$$

be a commutative diagram in C_Y such that $L \longrightarrow M \longrightarrow N$ and $L' \longrightarrow M' \longrightarrow N'$ are conflations. If two of the three vertical arrows are isomorphisms, then the third arrow is an isomorphism as well.

In fact, the Gabriel-Quillen embedding transforms the diagram (2) into a commutative diagram with exact rows. If two of the vertical arrows of such diagram are isomorphisms, then the third one is an isomorphism. The Gabriel-Quillen embedding is a fully faithful functor, in particular, it is conservative. Therefore, all vertical arrows in the original diagram are isomorphisms.

(ii) Suppose that the functor $C_X \xrightarrow{\varphi^*} C_Y$ is 'exact'; i.e. it maps conflations to conflations. In particular, φ^* maps a diagram (2) with two arrows from Σ_{φ^*} to a diagram whose rows are conflations and two vertical arrows are isomorphisms. By (i) above, the third arrow is an isomorphism too; i.e. all vertical arrows of the diagram (2) belong to Σ_{φ^*}.

(iii) Suppose now that $C_X \xrightarrow{\varphi^*} C_Y$ is a localization functor which is right 'exact' and satisfies the condition (#). The claim is that the functor φ^* is 'exact'.

Let $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ be a conflation in C_X. The functor φ^* being right 'exact' means that there is a commutative diagram

$$\begin{array}{ccccc} \varphi^*(L) & \xrightarrow{\varphi^*(\mathfrak{j})} & \varphi^*(M) & \xrightarrow{\varphi^*(\mathfrak{e})} & \varphi^*(N) \\ \mathfrak{e}' \searrow & & \nearrow \mathfrak{j}' & & \\ & \widetilde{L} & & & \end{array} \qquad (4)$$

such that $\widetilde{L} \xrightarrow{\mathfrak{j}'} \varphi^*(M) \xrightarrow{\varphi^*(\mathfrak{e})} \varphi^*(N)$ is a conflation in C_Y and $\mathfrak{e}' \in \mathfrak{E}_Y$. Since φ^* is a localization, we can and will assume that $\widetilde{L} = \varphi^*(L')$ for some $L' \in ObC_X$. Let $\mathfrak{j}'$ be the composition of arrows $\varphi^*(L') \xrightarrow{\varphi^*(\mathfrak{j}'')} \varphi^*(M_1)$ and $\varphi^*(M_1) \xrightarrow{\varphi^{*-1}(s)} \varphi^*(M)$ for some $s \in \Sigma_{\varphi^*}$.

Consider the cocartesian square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{e}} & N \\ s \downarrow & & \downarrow s' \\ M_1 & \xrightarrow{\mathfrak{e}_1} & N_1 \end{array} \qquad (4)$$

By hypothesis, $\mathfrak{e}_1$ is a deflation and φ^* maps (4) to a cocartesian square. The square (4) is embedded into a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & N \\ s'' \downarrow & & s \downarrow & & \downarrow s' \\ L_1 & \xrightarrow{\mathfrak{j}_1} & M_1 & \xrightarrow{\mathfrak{e}_1} & N_1 \end{array} \qquad (5)$$

whose rows are deflations. Since the vertical arrows s, s' in (5) belong to Σ_{φ^*}, the remaining vertical arrow, s'', belongs to Σ_{φ^*}.

The equality $\varphi^*(\mathfrak{e}_1 \circ \mathfrak{j}'') = 0$ means that $\mathfrak{e}_1 \circ \mathfrak{j}'' \circ \mathfrak{t} = 0$ for some $\mathfrak{t} \in \Sigma_{\varphi^*}$. Therefore we have a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & N \\ s'' \downarrow & & \downarrow s & & \downarrow s' \\ L_1 & \xrightarrow{\mathfrak{j}_1} & M_1 & \xrightarrow{\mathfrak{e}_1} & N_1 \\ g \uparrow & & \uparrow \mathfrak{j}'' & & \\ L'' & \xrightarrow{\mathfrak{t}} & L' & & \end{array} \qquad (6)$$

with a uniquely defined $L'' \xrightarrow{g} L_1$. Thus, we have a commutative diagram

$$\begin{array}{ccc} \varphi^*(L) & \xrightarrow{\varphi^*(s'')} & \varphi^*(L_1) \\ \mathfrak{e}_2 \searrow & & \nearrow \varphi^*(g) \\ & \varphi^*(L'') & \end{array} \tag{7}$$

where the arrow $\mathfrak{e}_2$ is the composition of the deflation $\varphi^*(L) \xrightarrow{\mathfrak{e}'} \varphi^*(L)$ and the isomorphism $\varphi^*(L') \xrightarrow{\varphi^*(s'')^{-1}} \varphi^*(L'')$. Since $\varphi^*(L) \xrightarrow{\varphi^*(s'')} \varphi^*(L_1)$ is an isomorphism, it follows from the commutativity of (7) that the arrow $\mathfrak{e}_2$ is a retraction; in particular it is a strict monomorphism. On the other hand, $\mathfrak{e}_2$ is a deflation, hence an epimorphism. Therefore, $\mathfrak{e}_2$ is an isomorphism, which implies that the deflation $\mathfrak{e}'$ in the diagram (4) is an isomorphism. Therefore, φ^* maps the deflation $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ to a deflation. ■

C3.5.1. Corollary. *Let $(C_{\mathfrak{X}}, \mathcal{E}_X)$ and $(C_{\mathfrak{Y}}, \mathcal{E}_Y)$ be exact categories and $C_X \xrightarrow{\varphi^*} C_Y$ a left 'exact' functor. Suppose that φ^* is a localization functor. Then the functor φ^* is 'exact' iff the class of arrows $\Sigma_{\varphi^*} = \{s \in HomC_X \mid \varphi^*(s)$ is an isomorphism$\}$ satisfies the condition (#) of C3.5.*

Proof. The assertion is dual to that of C3.5. ■

C3.6. Proposition. *Let $(C_{\mathfrak{X}}, \mathcal{E}_X)$ and $(C_{\mathfrak{Y}}, \mathcal{E}_Y)$ be exact categories, $C_{\mathfrak{X}} \xrightarrow{\phi^*} C_{\mathfrak{Y}}$ an 'exact' functor, and*

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\phi^*} & C_{\mathfrak{Y}} \\ \phi^*_s \searrow & & \nearrow \phi^*_c \\ & \Sigma^{-1}_{\phi^*} C_{\mathfrak{X}} & \end{array}$$

its canonical decomposition into a localization and a conservative functor. The functors ϕ^*_s and ϕ^*_c are 'exact'.

Proof. We call a pair of arrows $L \longrightarrow M \longrightarrow N$ in $\Sigma^{-1}_{\phi^*} C_{\mathfrak{X}}$ a *conflation* if it is isomorphic to the image of a conflation of $C_{\mathfrak{X}}$. We leave to the reader verifying that this defines a structure of an exact category on the quotient category $\Sigma^{-1}_{\phi^*} C_{\mathfrak{X}}$. It follows that the functors ϕ^*_s and ϕ^*_c are 'exact'. ■

C3.7. Proposition. *Let $(C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}})$ be an exact svelte category, S a family of arrows of $C_{\mathfrak{X}}$; and let $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -))$ be the pseudo-functor which assigns to every exact category $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$ the category of 'exact' functors from $(C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}})$ to $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$ mapping every arrow of S to an isomorphism. The pseudo-functor $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -)$ is representable.*

Proof. Let $\mathfrak{F}_S$ be the family of all 'exact' functors which map S to isomorphisms, and let $\bar{S}$ denote the family of all arrows which are transformed into isomorphisms by all functors from $\mathfrak{F}_S$. Since the category $C_{\mathfrak{X}}$ is svelte, there exists a subset Ω of $\mathfrak{F}_S$ such that the family of all arrows of $C_{\mathfrak{X}}$ made invertible by functors of Ω coincides with $\bar{S}$.

The product of any set of exact categories is an exact category. In particular, the product $C_{\mathfrak{X}_\Omega}$ of targets of functors of Ω is an exact category and the canonical functor $C_{\mathfrak{X}} \xrightarrow{F_\Omega} C_{\mathfrak{X}_\Omega}$ is an 'exact' functor. By C3.6, the functor F_Ω factors through an 'exact' localization $C_{\mathfrak{X}} \xrightarrow{F_S} C_{\bar{S}^{-1}\mathfrak{X}}$. The 'exact' functor F_S is the universal arrow representing the pseudo-functor $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -)$. ■

We need versions of the above facts for exact categories with actions.

C3.7.1. Proposition. *Let $(C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}})$ be an exact $\mathbb{Z}_+$-category, S a family of arrows of $C_{\mathfrak{X}}$; and let $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -))$ be the pseudo-functor which assigns to every exact $\mathbb{Z}_+$-category $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$ the category of 'exact' $\mathbb{Z}_+$-functors from $(C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}})$ to $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$ mapping every arrow of S to an isomorphism. The pseudo-functor $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -)$ is representable.*

C3.8. Multiplicative systems in quasi-(co)suspended categories. Fix a quasi-cosuspended category $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$. We call a class Σ of arrows of $C_{\mathfrak{X}}$ a *multiplicative system* of the quasi-cosuspended category $\mathfrak{T}_- C_{\mathfrak{X}}$ if it is $\theta_{\mathfrak{X}}$-invariant, closed under composition, contains all isomorphisms, and satisfies the following condition:

(L1) for every pair of triangles

$$\theta_X(L) \xrightarrow{\mathfrak{d}} N \xrightarrow{g} M \xrightarrow{f} L \quad \text{and} \quad \theta_X(L') \xrightarrow{\mathfrak{d}'} N' \xrightarrow{g'} M' \xrightarrow{f'} L'$$

and a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ t \downarrow & & \downarrow s \\ M' & \xrightarrow{f'} & L' \end{array}$$

where s and t are elements of Σ, there exists a morphism $N \xrightarrow{u} N'$ in Σ such that (u, t, s) is a morphism of triangles, i.e. the diagram

$$\begin{array}{ccccccc} \theta_X(L) & \xrightarrow{\mathfrak{d}} & N & \xrightarrow{g} & M & \xrightarrow{f} & L \\ \theta_{\mathfrak{X}}(s) \downarrow & & u \downarrow & & t \downarrow & & \downarrow s \\ \theta_X(L') & \xrightarrow{\mathfrak{d}'} & N' & \xrightarrow{g'} & M' & \xrightarrow{f'} & L' \end{array}$$

commutes.

We denote by $\mathcal{SM}_-(\mathfrak{X})$ the preorder (with respect to the inclusion) of all multiplicative systems and by $\mathcal{SM}^{\mathfrak{s}}_-(\mathfrak{X})$ the preorder of saturated multiplicative systems of the quasi-cosuspended category $\mathfrak{T}_- C_{\mathfrak{X}}$.

Recall that a multiplicative system Σ in $C_{\mathfrak{X}}$ is *saturated* iff the following condition holds: if α, β, γ are arrows of $C_{\mathfrak{X}}$ such that the compositions $\alpha\beta$ and $\beta\gamma$ belong to Σ, then $\beta \in \Sigma$ (equivalently, all three arrows belong to Σ).

C3.8.1. Proposition. *(a) Let $\mathfrak{T}\_C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ and $\mathfrak{T}\_C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, Tr_{\mathfrak{Y}})$ be quasi-cosuspended categories and $\mathfrak{T}\_C_{\mathfrak{X}} \xrightarrow{F} \mathfrak{T}\_C_{\mathfrak{Y}}$ a triangle functor. The family of arrows $\Sigma_F = \{s \in HomC_{\mathfrak{X}} \mid F(s)$ is invertible$\}$ is a saturated multiplicative system in $\mathfrak{T}\_C_{\mathfrak{X}}$.*

(b) Let $\mathfrak{T}\_C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ be a quasi-cosuspended category, $(C_Z, \mathcal{E}_Z)$ an exact category and $\mathfrak{T}\_C_{\mathfrak{X}} \xrightarrow{H} (C_Z, \mathcal{E}_Z)$ a homological functor. Then

$$\Sigma_{H,\theta_{\mathfrak{X}}} = \{s \in HomC_{\mathfrak{X}} \mid H\theta_{\mathfrak{X}}^n(s) \text{ is invertible for all } n \geq 0\}$$

is a saturated multiplicative system in $\mathfrak{T}\_C_{\mathfrak{X}}$.

Proof. (a) For any functor F, the family Σ_F is closed under composition and contains all isomorphisms. The $\theta_{\mathfrak{X}}$-invariance of Σ_F and the property (L1) follow from the axioms of quasi-cosuspended categories.

(b) The system $\Sigma_{H,\theta_{\mathfrak{X}}}$ is closed under composition, contains all isomorphisms, and is $\theta_{\mathfrak{X}}$-invariant by construction. It remains to verify the property (L1). Let

$$\theta_X(L) \xrightarrow{\mathfrak{d}} N \xrightarrow{g} M \xrightarrow{f} L \quad \text{and} \quad \theta_X(L') \xrightarrow{\mathfrak{d}'} N' \xrightarrow{g'} M' \xrightarrow{f'} L'$$

be a pair of triangles and

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ t \downarrow & & \downarrow s \\ M' & \xrightarrow{f'} & L' \end{array}$$

a commutative diagram with s and t elements of $\Sigma_{H,\theta_{\mathfrak{X}}}$. By the property (S3) of quasi-cosuspended categories, there exists a morphism $N \xrightarrow{u} N'$ in Σ such that (u, t, s) is a morphism of triangles, i.e. the diagram

$$\begin{array}{ccccccc} \theta_X(L) & \xrightarrow{\mathfrak{d}} & N & \xrightarrow{g} & M & \xrightarrow{f} & L \\ \theta_{\mathfrak{X}}(s) \downarrow & & u \downarrow & & t \downarrow & & \downarrow s \\ \theta_X(L') & \xrightarrow{\mathfrak{d}'} & N' & \xrightarrow{g'} & M' & \xrightarrow{f'} & L' \end{array} \qquad (1)$$

commutes. Let $\mathcal{H}$ denote the composition of the homological functor $C_{\mathfrak{X}} \xrightarrow{H} C_Z$ with the Gabriel-Quillen embedding $C_Z \longrightarrow C_{Z_{\mathfrak{E}}}$, we obtain for every nonnegative integer n a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{H}\theta_X^{n+1}(L) & \xrightarrow{\mathcal{H}\theta_X^n(\mathfrak{d})} & \mathcal{H}\theta_X^n(N) & \xrightarrow{\mathcal{H}\theta_X^n(g)} & \mathcal{H}\theta_X^n(M) & \xrightarrow{\mathcal{H}\theta_X^n(f)} & \mathcal{H}\theta_X^n(L) \\
\mathcal{H}\theta_{\mathfrak{X}}^{n+1}(s) \downarrow \wr & & \mathcal{H}\theta_X^n(u) \downarrow & & \mathcal{H}\theta_X^n(t) \downarrow \wr & & \wr \downarrow \mathcal{H}\theta_X^n(s) \\
\mathcal{H}\theta_X^{n+1}(L') & \xrightarrow{\mathcal{H}\theta_X^n(\mathfrak{d}')} & \mathcal{H}\theta_X^n(N') & \xrightarrow{\mathcal{H}\theta_X^n(g')} & \mathcal{H}\theta_X^n(M') & \xrightarrow{\mathcal{H}\theta_X^n(f')} & \mathcal{H}\theta_X^n(L')
\end{array} \qquad (2)$$

in the abelian category $C_{Z_{\mathfrak{E}}}$ whose rows are exact sequences and three of the for vertical arrows are isomorphisms. Therefore the fourth vertical arrow, $\mathcal{H}\theta_X^n(u)$ is an isomorphism for all $n \geq 0$; i.e. u belongs to $\Sigma_{H,\theta_{\mathfrak{X}}}$. ∎

C3.8.2. Proposition. *(a) Let* $\mathfrak{T}\_C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ *and* $\mathfrak{T}\_C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, Tr_{\mathfrak{Y}})$ *be quasi-cosuspended categories. Every triangle functor* $\mathfrak{T}\_C_{\mathfrak{X}} \xrightarrow{F} \mathfrak{T}\_C_{\mathfrak{Y}}$ *is uniquely represented as the composition of a triangle localization* $\mathfrak{T}\_C_{\mathfrak{X}} \xrightarrow{F_{\mathfrak{s}}} \mathfrak{T}\_C_{\mathfrak{X}_{\mathfrak{s}}}$ *and a conservative triangle functor* $\mathfrak{T}\_C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{F_{\mathfrak{c}}} \mathfrak{T}\_C_{\mathfrak{Y}}$.

(b) Let $\mathfrak{T}\_C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ *be a quasi-cosuspended category and* $(C_Z, \mathcal{E}_Z)$ *an exact category. Every homological functor* $\mathfrak{T}\_C_{\mathfrak{X}} \xrightarrow{H} (C_Z, \mathcal{E}_Z)$ *is uniquely represented as the composition of a triangle localization* $\mathfrak{T}\_C_{\mathfrak{X}} \xrightarrow{H_{\mathfrak{s}}} \mathfrak{T}\_C_{\mathfrak{X}_{\mathfrak{s}}}$ *and a conservative homological functor* $\mathfrak{T}\_C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{H_{\mathfrak{c}}} (C_Z, \mathcal{E}_Z)$.

Proof. Let Σ denote the multiplicative system Σ_F of C3.8.1(a), or $\Sigma_{H,\theta_{\mathfrak{X}}}$ of C3.8.1(b). Then the quotient category $\Sigma^{-1}C_{\mathfrak{X}}$ is an additive k-linear category having a unique structure $(\widetilde{\theta}, Tr_{\Sigma^{-1}\mathfrak{X}})$ of a quasi-cosuspended category such that the localization functor

$$C_{\mathfrak{X}} \xrightarrow{q_{\Sigma}^*} \Sigma^{-1}C_{\mathfrak{X}} = C_{\Sigma^{-1}\mathfrak{X}}$$

is a strict triangle functor. Here *strict* means that the quasi-cosuspension functor $\widetilde{\theta} = \theta_{\Sigma^{-1}\mathfrak{X}}$ is uniquely determined by the equality $\widetilde{\theta} \circ q_{\Sigma}^* = q_{\Sigma}^* \circ \theta_{\mathfrak{X}}$, and $Tr_{\Sigma^{-1}\mathfrak{X}}^-$ is the class of all sequences $\widetilde{\theta}(L) \longrightarrow N \longrightarrow M \longrightarrow L$ in $C_{\Sigma^{-1}\mathfrak{X}}$ which are isomorphic to the images of triangles of $Tr_{\mathfrak{X}}$ by the localization functor q_{Σ}^*. Details are left to the reader. ∎

C3.8.3. Proposition. *Let* $\mathfrak{T}\_C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}}^-)$ *be a svelte quasi-cosuspended category,* S *a family of arrows of the category* $C_{\mathfrak{X}}$*, and* $Tr_S(\mathfrak{T}\_C_{\mathfrak{X}}, -)$ *the pseudo-functor which assigns to every quasi-cosuspended category* $\mathfrak{T}\_C_{\mathfrak{Y}}$ *the category of all triangular functors* F *from* $\mathfrak{T}\_C_{\mathfrak{X}}$ *to* $\mathfrak{T}\_C_{\mathfrak{Y}}$ *transforming all arrows of* S *into isomorphisms. The pseudo-functor* $Tr_S(\mathfrak{T}\_C_{\mathfrak{X}}, -)$ *is representable.*

Proof. Let $\mathfrak{F}_S$ be the family of all triangular functors which map S to isomorphisms, and let $\bar{S}$ denote the family of arrows which are transformed into isomorphisms by all functors from $\mathfrak{F}_S$. Since the category $C_{\mathfrak{X}}$ is svelte, there exists a subset Ω of $\mathfrak{F}_S$ such that the family of all arrows of $C_{\mathfrak{X}}$ made invertible by functors of Ω coincides with $\bar{S}$.

The product of any set of quasi-cosuspended categories is a quasi-cosuspended category. In particular, the product $C_{\mathfrak{X}_\Omega}$ of targets of functors of Ω is a quasi-cosuspended category and the canonical functor $C_{\mathfrak{X}} \xrightarrow{F_\Omega} C_{\mathfrak{X}_\Omega}$ is a triangle functor. By C3.8.2, the functor F_Ω factors through a triangle localization $\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{F_S} \mathfrak{T}_-C_{\bar{S}^{-1}\mathfrak{X}}$. The triangle functor F_S is the universal arrow representing the pseudo-functor $Tr_S(\mathfrak{T}_-C_{\mathfrak{X}}, -)$. ■

C3.9. Triangle subcategories. A full subcategory $\mathcal{B}$ of the category $C_{\mathfrak{X}}$ is called a *triangle subcategory* of $\mathfrak{T}_-C_{\mathfrak{X}}$ if it is $\theta_{\mathfrak{X}}$-stable and has the following property: any morphism $M \xrightarrow{f} L$ of $\mathcal{B}$ is embedded into a triangle

$$\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{f} L$$

such that $N \in Ob\mathcal{B}$.

A full triangle subcategory $\mathcal{B}$ of $\mathfrak{T}_-C_{\mathfrak{X}}$ is called a *thick triangle subcategory* if it is closed under extensions, i.e. if $\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{f} L$ is a triangle with L and N objects of $\mathcal{B}$, then M belongs to $\mathcal{B}$.

C3.9.1. Saturated triangle subcategories. A full triangle subcategory $\mathcal{B}$ of a quasi-cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}}$ is called *saturated* if it coincides with its Karoubian envelope in $\mathfrak{T}_-C_{\mathfrak{X}}$; i.e. any retract of an object of $\mathcal{B}$ is an object of $\mathcal{B}$.

Evidently, every thick triangle subcategory of $\mathfrak{T}_-C_{\mathfrak{X}}$ is saturated.

It is known that the converse is true if $\mathfrak{T}_-C_{\mathfrak{X}}$ is a triangulated category: a full triangle subcategory of a triangulated category is thick iff it is saturated.

C3.10. Triangle subcategories and multiplicative systems. Let $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ be a quasi-cosuspended k-linear category; and let $\mathcal{B}$ be its triangle subcategory. Let $\Sigma(\mathcal{B})$ denote the family of all arrows $N \xrightarrow{t} M$ of the category $C_{\mathfrak{X}}$ such that there exists a triangle $\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{t} M \xrightarrow{f} L$ with $L \in Ob\mathcal{B}$. Set

$$\Sigma_\infty(\mathcal{B}) = \{s \in HomC_{\mathfrak{X}} \mid \theta^n(s) \in \Sigma(\mathcal{B}) \text{ for some } n \geq 0\}.$$

C3.10.1. Proposition. *Let $\mathcal{B}$ be a full triangle subcategory of a quasi-cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$. Then the class $\Sigma_\infty(\mathcal{B})$ is a multiplicative system. It is saturated iff the subcategory $\mathcal{B}$ is saturated.*

Proof. It follows from the definitions of $\Sigma(\mathcal{B})$ and $\Sigma_\infty(B)$ that both systems are $\theta_{\mathfrak{X}}$-stable and contain all isomorphisms. ■

For a full triangle subcategory $\mathcal{B}$ of the quasi-cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}}$, we set $C_{\mathfrak{X}}/\mathcal{B} = \Sigma(\mathcal{B})^{-1}C_{\mathfrak{X}}$.

C3.10.2. Proposition. *Let $\mathfrak{T}_-C_{\mathfrak{X}}$ and $\mathfrak{T}_-C_{\mathfrak{Y}}$ be quasi-cosuspended categories, and let $\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{F} \mathfrak{T}_-C_{\mathfrak{Y}}$ be a triangle functor. Then*

(a) $Ker(F)$ is a thick triangle subcategory of $\mathfrak{T}_-C_{\mathfrak{X}}$;

(b) $\theta_{\mathfrak{X}}(\Sigma_F) \subseteq \Sigma(Ker(F)) \subseteq \Sigma_F$. In particular, $\Sigma_F = \Sigma(Ker(F))$ if the quasi-cosuspension $\theta_{\mathfrak{X}}$ is a conservative functor.

Proof. (a) If $\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{f} L$ is a triangle in $C_{\mathfrak{X}}$ with L and N objects of $Ker(F)$, then the functor F maps it to the triangle $0 \longrightarrow 0 \longrightarrow F(M) \longrightarrow 0$, hence $F(M) = 0$.

(b) Let $N \xrightarrow{t} M$ be a morphism of $\Sigma(F)$; i.e. there exists a triangle

$$\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{t} M \xrightarrow{f} L$$

with $L \in ObKer(F)$. The functor F maps it to the triangle

$$0 \longrightarrow F(N) \xrightarrow{F(t)} F(M) \longrightarrow 0$$

which means, precisely, that $F(t)$ is an isomorphism. This shows that $\Sigma(Ker(F)) \subseteq \Sigma_F$.

Conversely, let $M \xrightarrow{s} L$ be a morphism of Σ_F and $\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{s} L$ a triangle. The functor F maps it to the triangle

$$\ldots \longrightarrow F\theta_{\mathfrak{X}}(M) \xrightarrow{\sim} F\theta_{\mathfrak{X}}(L) \xrightarrow{F(h)} F(N) \xrightarrow{F(g)} F(M) \xrightarrow{\sim} F(L).$$

Therefore, $F\theta_{\mathfrak{X}}^n(N) = 0$ for all $n \geq 0$. This shows that $\theta_{\mathfrak{X}}^n(s) \in \Sigma(Ker(F))$ for $n \geq 1$. ∎

C3.11. Coaisles and t-structures in a quasi-cosuspended category.

C3.11.1. Coaisles in a quasi-cosuspended category. Let $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ be a quasi-cosuspended category. Its thick triangle subcategory $\mathcal{U}$ is called a *coaisle* if the inclusion functor $\mathcal{U} \xrightarrow{\mathfrak{j}_*} C_{\mathfrak{X}}$ has a left adjoint, $\mathfrak{j}^*$.

C3.11.2. Proposition [KeV1]. *Let $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ be a triangulated k-linear category (i.e. the quasi-cosuspension $\theta_{\mathfrak{X}}$ is an auto-equivalence). Then a strictly full subcategory $\mathcal{U}$ of $C_{\mathfrak{X}}$ is a coaisle iff it is $\theta_{\mathfrak{X}}$-stable and for each $M \in ObC_{\mathfrak{X}}$, there is a triangle*

$$\theta_{\mathfrak{X}}(M^{\mathcal{U}}) \longrightarrow M_{\perp\mathcal{U}} \longrightarrow M \longrightarrow M^{\mathcal{U}}, \qquad (1)$$

where $M^{\mathcal{U}}$ is an object of $\mathcal{U}$ and $M_{\perp\mathcal{U}}$ is an object of ${}^{\perp}\mathcal{U}$. The triangle (1) is unique up to isomorphism.

Proof. Suppose $\mathcal{U}$ is a coaisle in $\mathfrak{T}_-C_{\mathfrak{X}}$, i.e. it is $\theta_{\mathfrak{X}}$-stable and the inclusion functor $\mathcal{U} \xrightarrow{\mathfrak{j}_*} C_{\mathfrak{X}}$ has a left adjoint, $\mathfrak{j}^*$. Fix an adjunction morphism $Id_{C_{\mathfrak{X}}} \xrightarrow{\eta} \mathfrak{j}_*\mathfrak{j}^*$. Then we have, for any $M \in ObC_{\mathfrak{X}}$, a triangle

$$\theta_{\mathfrak{X}}\mathfrak{j}_*\mathfrak{j}^*(M) = \theta_{\mathfrak{X}}(M^{\mathcal{U}}) \xrightarrow{\partial(M)} M_{\perp\mathcal{U}} = \mathfrak{K}(M) \xrightarrow{\mathfrak{k}(M)} M \xrightarrow{\eta(M)} \mathfrak{j}_*\mathfrak{j}^*(M) = M^{\mathcal{U}} \qquad (2)$$

Since, by hypothesis, j^* is a triangle functor, its application to the triangle (2) produces a triangle in the quasi-cosuspended category $\mathfrak{T}_-\mathcal{U}$. Since $\mathfrak{j}^*\eta$ is an isomorphism, $\mathfrak{j}^*(\mathfrak{K}(M)) = 0$, i.e. $\mathfrak{K}(M) = M_{\perp\mathcal{U}}$ belongs to the kernel of the localization functor $\mathfrak{j}^*$. It is easy to see that $Ker(\mathfrak{j}^*)$ coincides with ${}^\perp\mathcal{U}$.

Conversely, suppose that for every $M \in ObC_{\mathfrak{X}}$, there exists a triangle (1) with $M^{\mathcal{U}} \in Ob\mathcal{U}$ and $M_{\perp\mathcal{U}} \in Ob{}^\perp\mathcal{U}$. ■

C3.11.2. Cores of t-structures. The core of a t-structure $\mathcal{U} \xrightarrow{\mathfrak{j}_*} C_{\mathfrak{X}}$ is the subcategory $\mathcal{U} \cap {}^\perp\theta_{\mathfrak{X}}(\mathcal{U})$.

C4. Universal cohomological and homological functors.

See preliminaries on exact categories in Section 7 of Chapter I and on (co)suspended categories in Appendix K. Categories (suspended, cosuspended, exact) and functors of this section are k-linear for a fixed commutative unital ring k.

C4.1. k-Presheaves on a k-linear $\mathbb{Z}_+$-category. Fix a k-linear $\mathbb{Z}_+$-category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$. Let $\mathcal{M}_k(\mathfrak{X}) \xrightarrow{\Theta^*_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X})$ denote the continuous (i.e. having a right adjoint) extension of the functor $C_{\mathfrak{X}} \xrightarrow{\theta_{\mathfrak{X}}} C_{\mathfrak{X}}$. This extension is determined uniquely up to isomorphism by the quasi-commutativity of the diagram

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\theta_{\mathfrak{X}}} & C_{\mathfrak{X}} \\ {\scriptstyle h_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle h_{\mathfrak{X}}} \\ \mathcal{M}_k(\mathfrak{X}) & \xrightarrow{\Theta^*_{\mathfrak{X}}} & \mathcal{M}_k(\mathfrak{X}) \end{array}$$

where $h_{\mathfrak{X}}$ is the Yoneda embedding.

Let Θ_* be a right adjoint to $\Theta^*_{\mathfrak{X}}$. Notice that the projective objects of the category $\mathcal{M}_k(\mathfrak{X})$ are direct summands of coproducts of representable presheaves. Since $\Theta^*_{\mathfrak{X}}$ maps representable presheaves to representable objects and preserves arbitrary coproducts, it maps projective objects of $\mathcal{M}_k(\mathfrak{X})$ to projective objects. Therefore, thanks to the fact that the category $\mathcal{M}_k(\mathfrak{X})$ has enough projective objects, the functor Θ_* is exact.

C4.1.1. Note. Whenever it is convenient, we shall identify a k-linear $\mathbb{Z}_+$-category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$ with the equivalent to it full subcategory of the $\mathbb{Z}_+$-category $(\mathcal{M}_k(\mathfrak{X}), \Theta^*_{\mathfrak{X}})$ generated by representable presheaves.

C4.2. Cohomological and homological functors. Let $\mathfrak{T}_+C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^+_{\mathfrak{X}})$ be a suspended category and $(C_Z, \mathcal{E}_Z)$ an exact category. A functor $C_{\mathfrak{X}} \xrightarrow{\Phi} C_Z$ is called a *cohomological* functor on $\mathfrak{T}_+C_{\mathfrak{X}}$ with values in $(C_Z, \mathcal{E}_Z)$ (and we write $\mathfrak{T}_+C_{\mathfrak{X}} \xrightarrow{\Phi} (C_Z, \mathcal{E}_Z)$), if for any triangle $L \longrightarrow M \longrightarrow N \longrightarrow \theta_{\mathfrak{X}}(L)$, the sequence

$$\Phi(L) \longrightarrow \Phi(M) \longrightarrow \Phi(N) \longrightarrow \Phi(\theta_{\mathfrak{X}}(L)) \longrightarrow \Phi(\theta_{\mathfrak{X}}(M)) \longrightarrow \dots \qquad (1)$$

is 'exact' and for any morphism $L \xrightarrow{f} M$ of $C_{\mathfrak{X}}$, there exists a kernel of $\Phi(f)$ and the canonical monomorphism $Ker(\Phi(f)) \longrightarrow \Phi(L)$ is an inflation.

Dually, if $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ is a cosuspended category, then a functor $C_{\mathfrak{X}} \xrightarrow{\Psi} C_Z$ is called a *homological* functor $\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{\Psi} (C_Z, \mathcal{E}_Z)$ if the dual functor $C^{op}_{\mathfrak{X}} \xrightarrow{\Psi^{op}} C^{op}_Z$ is cohomological. In other words, for any triangle $\theta_{\mathfrak{X}}(N) \longrightarrow L \longrightarrow M \longrightarrow N$, the sequence

$$\ldots \longrightarrow \Psi(\theta_{\mathfrak{X}}(M)) \longrightarrow \Psi(\theta_{\mathfrak{X}}(N)) \longrightarrow \Psi(L) \longrightarrow \Psi(M) \longrightarrow \Psi(N)$$

is 'exact' and for any morphism $L \xrightarrow{f} M$ of $C_{\mathfrak{X}}$, there exists a cokernel of $\Psi(f)$ and the canonical epimorphism $\Psi(M) \longrightarrow Cok(\Psi(f))$ is a deflation in $(C_Z, \mathcal{E}_Z)$.

C4.2.1. Example. Let $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ be a k-linear cosuspended category. Then for every $W \in ObC_{\mathfrak{X}}$, the sequence

$$\ldots \longrightarrow C_{\mathfrak{X}}(W, \theta_{\mathfrak{X}}(M)) \longrightarrow C_{\mathfrak{X}}(W, \theta_{\mathfrak{X}}(L)) \longrightarrow C_{\mathfrak{X}}(W, N) \longrightarrow C_{\mathfrak{X}}(W, M) \longrightarrow C_{\mathfrak{X}}(W, L) \quad (2)$$

is exact. This means precisely that the Yoneda embedding

$$C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X}), \quad M \longmapsto C_{\mathfrak{X}}(-, M),$$

is a homological functor.

Let $\mathfrak{T}_+C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^+_{\mathfrak{X}})$ be a suspended category. For every object V of $C_{\mathfrak{X}}$ and every triangle $L \longrightarrow M \longrightarrow N \longrightarrow \theta_{\mathfrak{X}}(L)$, the sequence

$$C_{\mathfrak{X}}(L, V) \longleftarrow C_{\mathfrak{X}}(M, V) \longleftarrow C_{\mathfrak{X}}(N, V) \longleftarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(L), V) \longleftarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(M), V) \longleftarrow \ldots \quad (3)$$

is exact. In other words, the functor $h^o_{\mathfrak{X}}$ dual to the Yoneda embedding

$$C^{op}_{\mathfrak{X}} \longrightarrow \mathcal{M}_k(\mathfrak{X}^o), \quad M \longmapsto C_{\mathfrak{X}}(M, -),$$

is a cohomological functor.

C4.3. Universal homological functors.

C4.3.1. The category $C_{X_{\mathfrak{a}}}$. For any k-linear category C_X, let $C_{X_{\mathfrak{a}}}$ denote the full subcategory of the category $\mathcal{M}_k(X)$ of k-presheaves on C_X whose objects are k-presheaves having a left resolution formed by representable presheaves.

C4.3.2. Proposition. *(a) The subcategory $C_{X_{\mathfrak{a}}}$ is closed under extensions; i.e. $C_{X_{\mathfrak{a}}}$ is an exact subcategory of the abelian category $\mathcal{M}_k(X)$. In particular, $C_{X_{\mathfrak{a}}}$ is an additive k-linear category.*

(b) Suppose that the category C_X is Karoubian. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence in $\mathcal{M}_k(X)$. If two of the objects M', M, M'' belong to the subcategory $C_{X_\mathfrak{a}}$, then the third object belongs to $C_{X_\mathfrak{a}}$.

(b1) More generally, if C_X is Karoubian and

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \dots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow 0$$

is an exact sequence in $\mathcal{M}_k(X)$ with at least $n-1$ objects from the subcategory $C_{X_\mathfrak{a}}$, then the remaining object belongs to $C_{X_\mathfrak{a}}$.

Proof. (a) Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence in $\mathcal{M}_k(X)$. Let $P' \longrightarrow M'$ and $P'' \longrightarrow M''$ be projective resolutions. Then, by [Ba, I.6.7], there exists a differential on the graded object $P = P' \oplus P''$ such that the splitting exact sequence $0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$ is an exact sequence of complexes which are resolutions of the exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$. If the complexes P' and P'' are formed by representable presheaves, then P is a complex of representable presheaves, hence M is an object of the subcategory $C_{X_\mathfrak{a}}$.

(b) The assertion (b) follows from [Ba, I.6.8] and (b1) is a special case of [Ba, I.6.9]. ■

C4.3.3. Lemma. *If $\mathfrak{T}_-C_\mathfrak{X} = (C_\mathfrak{X}, \theta_\mathfrak{X}, \mathfrak{Tr}^-_\mathfrak{X})$ is a cosuspended category, then objects of $C_{\mathfrak{X}_\mathfrak{a}}$ are all objects M of $\mathcal{M}_k(X)$ such that there exists an exact sequence*

$$M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0,$$

where M_0 and M_1 are representable presheaves.

Proof. In fact, let $M_1 \xrightarrow{f} M_0 \xrightarrow{e} M \longrightarrow 0$ be such an exact sequence. Since M_0 and M_1 are representable, there exists a triangle $\Theta^*_\mathfrak{X}(M_0) \xrightarrow{\mathfrak{d}} M_2 \xrightarrow{g} M_1 \xrightarrow{f} M_0$ which gives rise to a resolution

$$\dots \longrightarrow \Theta^*_\mathfrak{X}(M_1) \xrightarrow{\Theta^*_\mathfrak{X}(f)} \Theta^*_\mathfrak{X}(M_0) \xrightarrow{\mathfrak{d}} M_2 \xrightarrow{g} M_1 \xrightarrow{f} M_0 \xrightarrow{e} M$$

of the object M. ■

C4.3.4. Proposition. *Let $\mathfrak{T}_-C_\mathfrak{X} = (C_\mathfrak{X}, \theta_\mathfrak{X}, \mathfrak{Tr}^-_\mathfrak{X})$ be a cosuspended category. Then the corestriction $C_\mathfrak{X} \xrightarrow{\mathfrak{H}_\mathfrak{X}} C_{\mathfrak{X}_\mathfrak{a}}$ of the Yoneda embedding $C_\mathfrak{X} \xrightarrow{h_\mathfrak{X}} \mathcal{M}_k(X)$ to the subcategory $C_{\mathfrak{X}_\mathfrak{a}}$ is a universal homological functor in the following sense: for any exact category $(C_Z, \mathcal{E}_Z)$ and a homological functor $\mathfrak{T}_-C_\mathfrak{X} \xrightarrow{\mathcal{H}} (C_Z, \mathcal{E}_Z)$, there exists a unique up to isomorphism 'exact' functor $(C_{\mathfrak{X}_\mathfrak{a}}, \mathcal{E}_{\mathfrak{X}_\mathfrak{a}}) \xrightarrow{\mathcal{H}_\mathfrak{a}} (C_Z, \mathcal{E}_Z)$ such that $\mathcal{H} \simeq \mathcal{H}_\mathfrak{a} \circ \mathfrak{H}_\mathfrak{X}$.*

The category $C_{\mathfrak{X}_{\mathfrak{a}}}$ has a unique up to isomorphism $\mathbb{Z}_+$-category structure $C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\theta_{\mathfrak{X}_{\mathfrak{a}}}} C_{\mathfrak{X}_{\mathfrak{a}}}$ such that the functor $\mathfrak{H}_{\mathfrak{X}}$ is a $\mathbb{Z}_+$-functor $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}) \longrightarrow (C_{\mathfrak{X}_{\mathfrak{a}}}, \theta_{\mathfrak{X}_{\mathfrak{a}}})$.

Proof. (a) Fix an exact category $(C_Z, \mathcal{E}_Z)$ with the class of deflations $\mathfrak{E}_Z$. Let $\mathfrak{q}_Z^{\mathfrak{J}}$ denote the Gabriel-Quillen embedding $C_Z \longrightarrow C_{Z_{\mathfrak{E}}}$. Since $C_{Z_{\mathfrak{E}}}$ is a Grothendieck category, in particular it is cocomplete (i.e. closed under colimits), any functor $C_{\mathfrak{X}} \xrightarrow{\mathcal{H}} C_Z$ gives a rise to a quasi-commutative diagram

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\mathcal{H}} & C_Z \\ {\scriptstyle h_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{q}_Z^{\mathfrak{J}}} \\ \mathcal{M}_k(\mathfrak{X}) & \xrightarrow{\mathcal{H}^*} & C_{Z_{\mathfrak{E}}} \end{array} \tag{1}$$

in which the functor $\mathcal{H}^*$ has a right adjoint, $\mathcal{H}_*$. Since the functor $\mathcal{H}^*$ preserves colimits of small diagrams (thanks to the existence of a right adjoint) and every object of the category $\mathcal{M}_k(\mathfrak{X})$ is a colimit of a small diagram of representable presheaves, $\mathcal{H}^*$ is determined uniquely up to isomorphism by the quasi-commutativity of the diagram (1).

If $C_{\mathfrak{X}} \xrightarrow{\mathcal{H}} C_Z$ is a homological functor $\mathfrak{T}_- C_{\mathfrak{X}} \longrightarrow (C_Z, \mathcal{E}_Z)$, then the composition of $\mathcal{H}$ and $C_Z \xrightarrow{\mathfrak{q}_Z^{\mathfrak{J}}} C_{Z_{\mathfrak{E}}}$ is a homological functor, because the functor $\mathfrak{q}_Z^{\mathfrak{J}}$ is 'exact' and homological functors are stable under the composition with 'exact' functors.

(b) The diagram (1) induces the quasi-commutative diagram

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\mathcal{H}} & C_Z \\ {\scriptstyle \mathfrak{H}_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{q}_Z^{\mathfrak{J}}} \\ C_{\mathfrak{X}_{\mathfrak{a}}} & \xrightarrow{\mathcal{H}_{\mathfrak{a}}^*} & C_{Z_{\mathfrak{E}}} \end{array} \tag{2}$$

The claim is that the functor $\mathcal{H}_{\mathfrak{a}}^*$ (– the restriction of the functor $\mathcal{H}^*$ to $C_{\mathfrak{X}_{\mathfrak{a}}}$) is 'exact'.

In fact, let $M' \longrightarrow M \longrightarrow M''$ be a conflation in $C_{\mathfrak{X}_a}$. Since the functor $\mathcal{H}^*$ is right exact, the sequence $\mathcal{H}^*(M') \longrightarrow \mathcal{H}^*(M) \longrightarrow \mathcal{H}^*(M'') \longrightarrow 0$ is exact. It remains to show that $\mathcal{H}^*(M') \longrightarrow \mathcal{H}^*(M)$ is a monomorphism.

Let $P_1' \xrightarrow{f'} P_0' \xrightarrow{e'} M' \longrightarrow 0$ and $P_1'' \xrightarrow{f''} P_0'' \xrightarrow{e''} M'' \longrightarrow 0$ be exact sequences in $C_{\mathfrak{X}_{\mathfrak{a}}}$ such that the objects P_i', P_i'', $i = 0, 1$, are representable. The morphisms $P_1' \xrightarrow{f'} P_0'$ and $P_1'' \xrightarrow{f''} P_0''$ can be inserted into triangles resp. $\Theta_{\mathfrak{X}}^*(P_0') \xrightarrow{\mathfrak{d}'} P_2' \xrightarrow{g'} P_1' \xrightarrow{f'} P_0'$ and $\Theta_{\mathfrak{X}}^*(P_0'') \xrightarrow{\mathfrak{d}''} P_2'' \xrightarrow{g''} P_1'' \xrightarrow{f''} P_0''$ which give rise to the complexes

$$\mathcal{P}' = \big(\ldots \longrightarrow \Theta_{\mathfrak{X}}^*(P_1') \xrightarrow{\Theta_{\mathfrak{X}}^*(f')} \Theta_{\mathfrak{X}}^*(P_0') \xrightarrow{\mathfrak{d}'} P_2' \xrightarrow{g'} P_1' \xrightarrow{f'} P_0' \big)$$

and

$$\mathcal{P}'' = \big(\dots \longrightarrow \Theta^*_{\mathfrak{X}}(P''_1) \xrightarrow{\Theta^*_{\mathfrak{X}}(f'')} \Theta^*_{\mathfrak{X}}(P''_0) \xrightarrow{\mathfrak{d}''} P''_2 \xrightarrow{g''} P''_1 \xrightarrow{f''} P''_0\big)$$

By (the argument of) C4.3.2(a), there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}' & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{P}'' & \longrightarrow & 0 \\ & & {\scriptstyle e'}\downarrow & & {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e''} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array} \qquad (3)$$

in which $\mathcal{P}' \xrightarrow{e'} M'$, $\mathcal{P}'' \xrightarrow{e''} M''$, and $\mathcal{P} \xrightarrow{e} M$ are projective resolutions and

$$0 \longrightarrow \mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}'' \longrightarrow 0$$

is an exact sequence of projective complexes. Since $\mathcal{H}^* \circ h_{\mathfrak{X}}$ is a cohomological functor, the complexes $\mathcal{H}^*(\mathcal{P}')$ and $\mathcal{H}^*(\mathcal{P}'')$ are exact. Together with the exactness of the sequence

$$0 \longrightarrow \mathcal{H}^*(\mathcal{P}') \longrightarrow \mathcal{H}^*(\mathcal{P}) \longrightarrow \mathcal{H}^*(\mathcal{P}'') \longrightarrow 0$$

this implies the exactness of the complex $\mathcal{H}^*(\mathcal{P})$. Now it follows from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}^*(\mathcal{P}') & \longrightarrow & \mathcal{H}^*(\mathcal{P}) & \longrightarrow & \mathcal{H}^*(\mathcal{P}'') & \longrightarrow & 0 \\ & & {\scriptstyle \mathcal{H}^*(e')}\downarrow & & {\scriptstyle \mathcal{H}^*(e)}\downarrow & & \downarrow{\scriptstyle \mathcal{H}^*(e'')} & & \\ 0 & \longrightarrow & \mathcal{H}^*(M') & \longrightarrow & \mathcal{H}^*(M) & \longrightarrow & \mathcal{H}^*(M'') & \longrightarrow & 0 \end{array}$$

that $\mathcal{H}^*(M') \longrightarrow \mathcal{H}^*(M)$ is a monomorphism; hence the sequence

$$0 \longrightarrow \mathcal{H}^*(M') \longrightarrow \mathcal{H}^*(M) \longrightarrow \mathcal{H}^*(M'') \longrightarrow 0$$

is exact.

(c) There is a unique up to isomorphism functor $C_{\mathfrak{X}_\mathfrak{a}} \xrightarrow{\mathcal{H}_\mathfrak{a}} C_Z$ such that $\mathcal{H}^*_\mathfrak{a} \simeq \mathfrak{q}^{\mathfrak{I}}_Z \circ \mathcal{H}_\mathfrak{a}$. The functor $\mathcal{H}_\mathfrak{a}$ is an 'exact' functor $(C_{\mathfrak{X}_\mathfrak{a}}, \mathcal{E}_{\mathfrak{X}_\mathfrak{a}}) \longrightarrow (C_Z, \mathcal{E}_Z)$.

Let M be an object of $C_{\mathfrak{X}_\mathfrak{a}}$, and let $P_1 \xrightarrow{f} P_0 \xrightarrow{e} M \longrightarrow 0$ be an exact sequence with representable objects P_0 and P_1. Since $\mathcal{H}$ is a homological functor, there exists a cokernel of the morphism $\mathcal{H}(f)$. We set $\mathcal{H}_\mathfrak{a}(M) = Cok(\mathcal{H}(f))$. Since the functor $\mathcal{H}^*$ is right exact, it maps $P_1 \xrightarrow{f} P_0 \xrightarrow{e} M \longrightarrow 0$ to an exact sequence. Therefore, because the Gabriel-Quillen embedding $(C_Z, \mathcal{E}_Z) \xrightarrow{\mathfrak{q}^{\mathfrak{I}}_Z} (C_{Z_\mathfrak{e}}, \mathcal{E}_{Z_\mathfrak{e}})$ is an 'exact' functor, we have an isomorphism $\mathfrak{q}^{\mathfrak{I}}_Z(\mathcal{H}_\mathfrak{a}(M)) \simeq \mathcal{H}^*(M)$. Since the functor $\mathfrak{q}^{\mathfrak{I}}_Z$ is fully faithful, it follows that the object $\mathcal{H}_\mathfrak{a}(M)$ is defined uniquely up to isomorphism. By a standard argument, once the objects $\mathcal{H}_{\mathfrak{a}}(M)$ and $\mathcal{H}_{\mathfrak{a}}(N)$ are fixed, any morphism $M \xrightarrow{g} N$ determines uniquely a morphism $\mathcal{H}_{\mathfrak{a}}(M) \longrightarrow \mathcal{H}_{\mathfrak{a}}(N)$.

The 'exactness' of $\mathcal{H}_{\mathfrak{a}}$ follows from the isomorphism $\mathcal{H}_{\mathfrak{a}}^* \simeq \mathfrak{q}_Z^{\mathfrak{J}} \circ \mathcal{H}_{\mathfrak{a}}$, because the functor $\mathcal{H}_{\mathfrak{a}}^*$ is 'exact' (by (b) above) and the functor $\mathfrak{q}_Z^{\mathfrak{J}}$ reflects 'exactness': if $L' \longrightarrow L \longrightarrow L''$ is a sequence in C_Z such that the sequence $0 \longrightarrow \mathfrak{q}_Z^{\mathfrak{J}}(L') \longrightarrow \mathfrak{q}_Z^{\mathfrak{J}}(L) \longrightarrow \mathfrak{q}_Z^{\mathfrak{J}}(L'') \longrightarrow 0$ is exact, then $L' \longrightarrow L \longrightarrow L''$ is a conflation.

(d) The isomorphism $\mathcal{H}_{\mathfrak{a}}^* \simeq \mathfrak{q}_Z^{\mathfrak{J}} \circ \mathcal{H}_{\mathfrak{a}}$ implies that $\mathfrak{q}_Z^{\mathfrak{J}} \circ (\mathcal{H}_{\mathfrak{a}} \circ \mathfrak{H}_{\mathfrak{X}}) \simeq \mathcal{H}_{\mathfrak{a}}^* \circ \mathfrak{H}_{\mathfrak{X}} \simeq \mathfrak{q}_Z^{\mathfrak{J}} \circ \mathcal{H}$. Since the functor $\mathfrak{q}_Z^{\mathfrak{J}}$ is fully faithful, it follows that $\mathcal{H} \simeq \mathcal{H}_{\mathfrak{a}} \circ \mathfrak{H}_{\mathfrak{X}}$. It follows from the definition of the exact category $C_{\mathfrak{X}_{\mathfrak{a}}}$ and the exactness of the functor $\mathcal{H}_{\mathfrak{a}}$ that it is determined by the isomorphism $\mathcal{H} \simeq \mathcal{H}_{\mathfrak{a}} \circ \mathfrak{H}_{\mathfrak{X}}$ uniquely up to isomorphism.

(e) The extension $\mathcal{M}_k(\mathfrak{X}) \xrightarrow{\Theta_{\mathfrak{X}}^*} \mathcal{M}_k(\mathfrak{X})$ of the functor $C_{\mathfrak{X}} \xrightarrow{\theta_{\mathfrak{X}}^*} C_{\mathfrak{X}}$ maps representable presheaves to representable presheaves and has a right adjoint functor. In particular, $\Theta_{\mathfrak{X}}^*$ is a right exact functor, and it maps an exact sequence $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ in $\mathcal{M}_k(\mathfrak{X})$ with representable presheaves P_1 and P_0 to an exact sequence of the same type. By C4.3.3, this implies that the subcategory $C_{\mathfrak{X}_{\mathfrak{a}}}$ is $\Theta_{\mathfrak{X}}^*$-stable. Therefore, $\Theta_{\mathfrak{X}}^*$ induces a functor $C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\theta_{\mathfrak{X}_{\mathfrak{a}}}} C_{\mathfrak{X}_{\mathfrak{a}}}$ such that the diagram

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\theta_{\mathfrak{X}}} & C_{\mathfrak{X}} \\ {\scriptstyle \mathfrak{H}_{\mathfrak{X}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{H}_{\mathfrak{X}}} \\ C_{\mathfrak{X}_{\mathfrak{a}}} & \xrightarrow{\theta_{\mathfrak{X}_{\mathfrak{a}}}} & C_{\mathfrak{X}_{\mathfrak{a}}} \end{array}$$

quasi-commutes, i.e. $\mathfrak{H}_{\mathfrak{X}}$ is a $\mathbb{Z}_+$-functor $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}) \longrightarrow (C_{\mathfrak{X}_{\mathfrak{a}}}, \theta_{\mathfrak{X}_{\mathfrak{a}}})$. ∎

C4.3.5. Remarks. (a) The universal property described in C4.3.4 determines the exact category $(C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}})$ and the functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}}$ uniquely up to equivalence.

(b) It follows from the definition of the category $C_{\mathfrak{X}_{\mathfrak{a}}}$ that its projective objects are retracts of representable presheaves. In particular, if the category $C_{\mathfrak{X}}$ is Karoubian, then every projective object of the exact category $C_{\mathfrak{X}_{\mathfrak{a}}}$ is isomorphic to an object of the form $\mathfrak{H}_X(M)$ for some $M \in ObC_{\mathfrak{X}}$. In other words, the canonical embedding $C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}}$ induces an equivalence between $C_{\mathfrak{X}}$ and the full subcategory of the category $C_{\mathfrak{X}_{\mathfrak{a}}}$ generated by all projective objects of $C_{\mathfrak{X}_{\mathfrak{a}}}$.

The following proposition is a *cosuspended* version of Theorem 2.2.1 in [Ve2].

C4.3.6. Proposition. *The map which assigns to each cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}}^-)$ the exact category $C_{\mathfrak{X}_{\mathfrak{a}}}$ is functorial in the following sense: to every triangle functor $\widetilde{\Phi} = (\Phi, \phi)$ from a cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}}$ to a cosuspended category $\mathfrak{T}_-C_{\mathfrak{Y}}$, there corresponds an 'exact' $\mathbb{Z}_+$-functor $(C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}}) \xrightarrow{\Phi_{\mathfrak{a}}} (C_{\mathfrak{Y}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{Y}_{\mathfrak{a}}})$ which maps projective objects to projective objects. The functor $\Phi_{\mathfrak{a}}$ is determined uniquely up to isomorphism* *by the quasi-commutativity of the diagram*

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\ \Phi\ } & C_{\mathfrak{Y}} \\ {\scriptstyle \mathfrak{H}_{\mathfrak{X}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{H}_{\mathfrak{Y}}} \\ C_{\mathfrak{X}_{\mathfrak{a}}} & \xrightarrow{\ \Phi_{\mathfrak{a}}\ } & C_{\mathfrak{Y}_{\mathfrak{a}}} \end{array} \qquad (1)$$

Proof. (a) Since $\widetilde{\Phi} = (\Phi, \phi)$ is a triangle functor and $\mathfrak{H}_{\mathfrak{Y}}$ is a homological functor, the composition, $\mathfrak{H}_{\mathfrak{Y}} \circ \Phi$ is a homological functor. By the universal property of the homological functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}}$ (see C4.3.4), there exists a unique (up to isomorphism) exact functor $C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\widetilde{\Phi}_{\mathfrak{a}}} C_{\mathfrak{Y}_{\mathfrak{a}}}$ such that the diagram (1) quasi-commutes. The quasi-commutativity of the diagram (1) implies that $\widetilde{\Phi}_{\mathfrak{a}}$ maps representable presheaves to representable presheaves. Since projective objects of the categories $C_{\mathfrak{X}_{\mathfrak{a}}}$ and $C_{\mathfrak{Y}_{\mathfrak{a}}}$ are all possible retracts (direct summands) of representable presheaves, it follows that $\widetilde{\Phi}_{\mathfrak{a}}$ maps projective objects to projective objects.

The isomorphism $\Phi \circ \theta_{\mathfrak{X}} \xrightarrow{\phi} \theta_{\mathfrak{Y}} \circ \Phi$ induces an isomorphism $\Phi_{\mathfrak{a}} \circ \Theta^{\mathfrak{a}}_{\mathfrak{X}} \xrightarrow{\phi_{\mathfrak{a}}} \Theta^{\mathfrak{a}}_{\mathfrak{Y}} \circ \Phi_{\mathfrak{a}}$, where $\Theta^{\mathfrak{a}}_{\mathfrak{X}}$ is the endofunctor $C_{\mathfrak{X}_{\mathfrak{a}}} \longrightarrow C_{\mathfrak{X}_{\mathfrak{a}}}$ induced by $\Theta^*_{\mathfrak{X}}$. So that the pair $(\Phi_{\mathfrak{a}}, \phi_{\mathfrak{a}})$ is a $\mathbb{Z}_+$-functor $(C_{\mathfrak{X}_{\mathfrak{a}}}, \Theta^{\mathfrak{a}}_{\mathfrak{X}}) \longrightarrow (C_{\mathfrak{X}_{\mathfrak{a}}}, \Theta^{\mathfrak{a}}_{\mathfrak{X}})$ and the diagram (1) is a diagram of $\mathbb{Z}_+$-functors. ∎

Let $\mathfrak{T}_- C_{\mathfrak{X}}$ be a cosuspended category and $(C_Z, \mathcal{E}_Z)$ an exact category. We denote by $\mathfrak{Ex}((C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}}), (C_Z, \mathcal{E}_Z))$ the category whose objects are 'exact' functors from $(C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}})$ to $(C_Z, \mathcal{E}_Z)$ and morphisms are morphisms of functors. Let $\mathcal{H}om(C_{\mathfrak{X}}, C_Z)$ denote the category whose objects are functors from $C_{\mathfrak{X}}$ to C_Z and morphisms are morphisms of functors.

C4.3.7. Proposition. *The composition with the functor* $C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}}$ *defines a fully faithful functor*

$$\mathfrak{Ex}((C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}}), (C_Z, \mathcal{E}_Z)) \longrightarrow \mathcal{H}om(C_{\mathfrak{X}}, C_Z)$$

which induces an equivalence of the category $\mathfrak{Ex}((C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}}), (C_Z, \mathcal{E}_Z))$ *with the full subcategory of* $\mathcal{H}om(C_{\mathfrak{X}}, C_Z)$ *generated by homological functors.*

Proof. The assertion is a corollary of (actually, it is equivalent to) C4.3.4. ∎

C4.3.8. Triangle functors. Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ and $\mathfrak{T}_- C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, \mathfrak{Tr}^-_{\mathfrak{Y}})$ be cosuspended categories, and let $\widetilde{\Phi} = (\Phi, \phi)$ be a triangle functor $\mathfrak{T}_- C_{\mathfrak{X}} \longrightarrow \mathfrak{T}_- C_{\mathfrak{Y}}$. Then we have a quasi-commutative diagram of $\mathbb{Z}_+$-categories and $\mathbb{Z}_+$-functors

$$\begin{array}{ccccc} (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}) & \xrightarrow{\ \mathfrak{H}_{\mathfrak{X}}\ } & (C_{\mathfrak{X}_{\mathfrak{a}}}, \theta_{\mathfrak{X}_{\mathfrak{a}}}) & \xrightarrow{\ \mathfrak{Q}_{\mathfrak{X}_{\mathfrak{a}}}\ } & (C_{\mathfrak{X}^{\mathfrak{e}}_{\mathfrak{a}}}, \Theta_{\mathfrak{X}^{\mathfrak{e}}_{\mathfrak{a}}}) \\ {\scriptstyle \Phi}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{\mathfrak{a}}} & & \big\downarrow{\scriptstyle \Phi^*_{\mathfrak{E}}} \\ (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}) & \xrightarrow{\ \mathfrak{H}_{\mathfrak{Y}}\ } & (C_{\mathfrak{Y}_{\mathfrak{a}}}, \theta_{\mathfrak{Y}_{\mathfrak{a}}}) & \xrightarrow{\ \mathfrak{Q}_{\mathfrak{Y}_{\mathfrak{a}}}\ } & (C_{\mathfrak{Y}^{\mathfrak{e}}_{\mathfrak{a}}}, \Theta_{\mathfrak{Y}^{\mathfrak{e}}_{\mathfrak{a}}}) \end{array} \qquad (1)$$

in which $\mathfrak{Q}_{\mathfrak{X}_\mathfrak{a}}$ and $\mathfrak{Q}_{\mathfrak{Y}_\mathfrak{a}}$ are Gabriel-Quillen embeddings, the functor $\Phi_\mathfrak{a}$ is exact, and the functor $\Phi^*_{\mathfrak{E}}$ has a right adjoint, $\Phi_{\mathfrak{E}*}$, which is an exact functor.

C4.4. The category $C_{\mathfrak{X}_\mathfrak{m}}$ and abelianization of triangulated categories. Fix a k-linear cosuspended category $\mathfrak{T}_-C_\mathfrak{X} = (C_\mathfrak{X}, \theta_\mathfrak{X}, \mathfrak{Tr}^-_\mathfrak{X})$. We denote by $C_{\mathfrak{X}_\mathfrak{m}}$ the strictly full subcategory of the category $\mathcal{M}_k(\mathfrak{X})$ of k-presheaves on C_X whose objects are (isomorphic to) images of morphisms between representable presheaves. In other words, an object of $\mathcal{M}_k(\mathfrak{X})$ belongs to $C_{\mathfrak{X}_\mathfrak{m}}$ iff it is a subobject and a quotient object of some representable presheaves. An immediate consequence of this description is that the category $C_{\mathfrak{X}_\mathfrak{m}}$ is Karoubian. It is easy to show that the subcategory $C_{\mathfrak{X}_\mathfrak{m}}$ is closed under finite coproducts in $\mathcal{M}_k(\mathfrak{X})$; i.e. $C_{\mathfrak{X}_\mathfrak{m}}$ is an additive subcategory of $\mathcal{M}_k(\mathfrak{X})$.

Notice that $C_{\mathfrak{X}_\mathfrak{m}}$ is a subcategory of $C_{\mathfrak{X}_\mathfrak{a}}$. In fact, by the definition of the subcategory $C_{\mathfrak{X}_\mathfrak{m}}$, for every its object M, there exist an epimorphism $M_0 \xrightarrow{\mathfrak{e}} M$ and a monomorphism $M \xrightarrow{\mathfrak{j}} L_0$, where M_0 and L_0 are representable presheaves. There is a triangle

$$\Theta^*_\mathfrak{X}(L_0) \xrightarrow{\mathfrak{d}} M_1 \xrightarrow{g} M_0 \xrightarrow{\mathfrak{j}\circ\mathfrak{e}} L_0.$$

Since this triangle is an exact sequence, we have an exact sequence

$$M_1 \xrightarrow{g} M_0 \xrightarrow{\mathfrak{e}} M \longrightarrow 0$$

with M_0, M_1 representable presheaves. By C4.3.3, M is an object of $C_{\mathfrak{X}_\mathfrak{a}}$.

It follows that an object of $C_{\mathfrak{X}_\mathfrak{a}}$ belongs to the subcategory $C_{\mathfrak{X}_\mathfrak{m}}$ iff it is a subobject of a representable presheaf.

C4.4.1. Proposition. *(a) The subcategory $C_{\mathfrak{X}_\mathfrak{m}}$ is $\Theta^*_\mathfrak{X}$-stable.*

(b) For every morphism α of $C_{\mathfrak{X}_\mathfrak{m}}$, the kernel and cokernel of $\Theta^*_\mathfrak{X}(\alpha)$ belong to the subcategory $C_{\mathfrak{X}_\mathfrak{m}}$.

Proof. (i) Let $K \xrightarrow{\alpha} K'$ be morphism of $C_{\mathfrak{X}_\mathfrak{m}}$; i.e. there exist $M \xrightarrow{f} L$ and $M' \xrightarrow{f'} L'$ such that $K = Im(f)$, $K' = Im(f')$, and presheaves M and M' are representable. Let $\Theta^*_\mathfrak{X}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{f} L$ and $\Theta^*_\mathfrak{X}(L') \xrightarrow{h'} N' \xrightarrow{g'} M' \xrightarrow{f'} L'$ be triangles. Then there is a commutative diagram

$$\begin{array}{ccccccccccc}
\Theta^*_\mathfrak{X}(M) & \xrightarrow{\Theta^*_\mathfrak{X}(f)} & \Theta^*_\mathfrak{X}(L) & \xrightarrow{h} & N & \xrightarrow{g} & M & \xrightarrow{\mathfrak{e}} & K & \xrightarrow{\mathfrak{j}} & L \\
\Theta^*_\mathfrak{X}(\xi_1) \downarrow & & \widetilde{\xi}_0 \downarrow & & \downarrow \xi_2 & & \downarrow \xi_1 & & \downarrow \alpha & & \\
\Theta^*_\mathfrak{X}(M') & \xrightarrow{\Theta^*_\mathfrak{X}(f')} & \Theta^*_\mathfrak{X}(L') & \xrightarrow{h'} & N' & \xrightarrow{g'} & M' & \xrightarrow{\mathfrak{e}'} & K' & \xrightarrow{\mathfrak{j}'} & L'
\end{array} \tag{7}$$

constructed as follows. The arrow $M \xrightarrow{\xi_1} M'$ is due to the fact that M is a projective object of $\mathcal{M}_k(\mathfrak{X})$ and $M' \xrightarrow{\mathfrak{e}'} K'$ is an epimorphism. Similarly, the morphism $N \xrightarrow{\xi_2} N'$

exists because the sequence $N' \xrightarrow{g'} M' \xrightarrow{\mathfrak{e}'} K'$ is exact and the object N is projective. By the property (SP2), the sequences

$$\Theta^*_{\mathfrak{X}}(M) \xrightarrow{-\Theta^*_{\mathfrak{X}}(f)} \Theta^*_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M$$

and

$$\Theta^*_{\mathfrak{X}}(M') \xrightarrow{-\Theta^*_{\mathfrak{X}}(f')} \Theta^*_{\mathfrak{X}}(L') \xrightarrow{h'} N' \xrightarrow{g'} M'$$

are triangles. By (SP3), there exists a morphism $\Theta^*_{\mathfrak{X}}(L) \xrightarrow{\widetilde{\xi}_0} \Theta^*_{\mathfrak{X}}(L')$ such that the diagram

$$\begin{array}{ccccccc}
\Theta^*_{\mathfrak{X}}(M) & \xrightarrow{-\Theta^*_{\mathfrak{X}}(f)} & \Theta^*_{\mathfrak{X}}(L) & \xrightarrow{h} & N & \xrightarrow{g} & M \\
\Theta^*_{\mathfrak{X}}(\xi_1) \downarrow & & \widetilde{\xi}_0 \downarrow & & \downarrow \xi_2 & & \downarrow \xi_1 \\
\Theta^*_{\mathfrak{X}}(M') & \xrightarrow{-\Theta^*_{\mathfrak{X}}(f')} & \Theta^*_{\mathfrak{X}}(L') & \xrightarrow{h'} & N' & \xrightarrow{g'} & M'
\end{array}$$

commutes. Therefore, the diagram (7) commutes.

(ii) Since the functor $\Theta^*_{\mathfrak{X}}$ is right exact, the arrows $\Theta^*_{\mathfrak{X}}(\mathfrak{e})$ and $\Theta^*_{\mathfrak{X}}(\mathfrak{e}')$ in the commutative diagram

$$\begin{array}{ccccccccc}
\Theta^*_{\mathfrak{X}}(N) & \xrightarrow{\Theta^*_{\mathfrak{X}}(g)} & \Theta^*_{\mathfrak{X}}(M) & \xrightarrow{\Theta^*_{\mathfrak{X}}(\mathfrak{e})} & \Theta^*_{\mathfrak{X}}(K) & \xrightarrow{\Theta^*_{\mathfrak{X}}(\mathfrak{j})} & \Theta^*_{\mathfrak{X}}(L) & \xrightarrow{h} & N \\
\Theta^*_{\mathfrak{X}}(\xi_2) \downarrow & & \downarrow \Theta^*_{\mathfrak{X}}(\xi_1) & & \Theta^*_{\mathfrak{X}}(\alpha) \downarrow & & \widetilde{\xi}_0 \downarrow & & \downarrow \xi_2 \\
\Theta^*_{\mathfrak{X}}(N') & \xrightarrow{\Theta^*_{\mathfrak{X}}(g')} & \Theta^*_{\mathfrak{X}}(M') & \xrightarrow{\Theta^*_{\mathfrak{X}}(\mathfrak{e}')} & \Theta^*_{\mathfrak{X}}(K') & \xrightarrow{\Theta^*_{\mathfrak{X}}(\mathfrak{j}')} & \Theta^*_{\mathfrak{X}}(L') & \xrightarrow{h'} & N'
\end{array} \qquad (8)$$

are epimorphisms. It follows from the exactness of the rows in (7) that the arrows $\Theta^*_{\mathfrak{X}}(\mathfrak{j})$ and $\Theta^*_{\mathfrak{X}}(\mathfrak{j}')$ are monomorphisms.

An argument similar to that of [Ve2, 3.2.5] applied to the commutative diagram (7) shows that the kernel and cokernel of the morphism $\Theta^*_{\mathfrak{X}}(\alpha)$ belong to the subcategory $C_{\mathfrak{X}_\mathfrak{m}}$. Since α is an arbitrary morphism of $C_{\mathfrak{X}_\mathfrak{m}}$, it follows, in particular, that the subcategory $C_{\mathfrak{X}_\mathfrak{m}}$ is $\Theta^*_{\mathfrak{X}}$-stable; i.e. it has a natural structure of a $\mathbb{Z}_+$-category and the Yoneda embedding induces a $\mathbb{Z}_+$-functor $(C_{\mathfrak{X}}, \theta) \longrightarrow (C_{\mathfrak{X}_\mathfrak{m}}, \Theta^*_{\mathfrak{X}})$. ■

C4.4.2. Note. Since the Yoneda functor $C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X})$ takes values in $C_{\mathfrak{X}_\mathfrak{m}}$, the $\mathbb{Z}_+$-category $(C_{\mathfrak{X}_\mathfrak{m}}, \Theta_{\mathfrak{X}_\mathfrak{m}})$ has enough projective objects. It follows that the 'translation' functor $C_{\mathfrak{X}_\mathfrak{m}} \xrightarrow{\Theta_{\mathfrak{X}_\mathfrak{m}}} C_{\mathfrak{X}_\mathfrak{m}}$ induced by $\Theta^*_{\mathfrak{X}}$ maps projective objects to projective objects.

C4.4.3. Proposition. *Suppose that the cosuspension functor $C_{\mathfrak{X}} \xrightarrow{\theta_{\mathfrak{X}}} C_{\mathfrak{X}}$ is a category equivalence, i.e. $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ is a triangulated category. Then $C_{\mathfrak{X}_\mathfrak{m}}$ is an abelian category which coincides with $C_{\mathfrak{X}_\mathfrak{a}}$.*

Proof. If the suspension functor $C_{\mathfrak{X}} \xrightarrow{\theta_{\mathfrak{X}}} C_{\mathfrak{X}}$ is a category equivalence, then its extension $\Theta^*_{\mathfrak{X}}$ is a category equivalence. In this case, it follows from C4.4.1(ii) that the subcategory $C_{\mathfrak{X}_{\mathfrak{m}}}$ contains kernels and cokernels of all its morphisms, hence $C_{\mathfrak{X}_{\mathfrak{m}}}$ is an abelian subcategory of $\mathcal{M}_k(\mathfrak{X})$. Since every object of the category $C_{\mathfrak{X}_{\mathfrak{a}}}$ is the cokernel of a morphism between representable objects, it follows that $C_{\mathfrak{X}_{\mathfrak{a}}} \subseteq C_{\mathfrak{X}_{\mathfrak{m}}}$. Therefore $C_{\mathfrak{X}_{\mathfrak{a}}} = C_{\mathfrak{X}_{\mathfrak{m}}}$. ∎

C4.4.4. Note. Proposition C4.4.3 together with 3.2.4 and 5.2.6 recover, in particular, the 'abelianization' theory for triangulated categories [Ve2, II.3].

C4.5. Triangulation and abelianization of cosuspended categories.

C4.5.1. Inverting endofunctors. A $\mathbb{Z}$-category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$ is called *strict* if the endofunctor $\theta_{\mathfrak{X}}$ is an auto-morphism of the category $C_{\mathfrak{X}}$.

There is a standard construction which assigns to each $\mathbb{Z}_+$-category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$ a strict $\mathbb{Z}$-category $(C_{\mathfrak{X}_{\mathfrak{s}}}, \theta_{\mathfrak{X}_{\mathfrak{s}}})$. Objects of the category $C_{\mathfrak{X}_{\mathfrak{s}}}$ are pairs (n, M), where $n \in \mathbb{Z}$ and $M \in ObC_{\mathfrak{X}}$. Morphisms are defined by

$$C_{\mathfrak{X}_{\mathfrak{s}}}((s,M),(t,N)) \stackrel{\text{def}}{=} \underset{n > s,t}{\operatorname{colim}} C_{\mathfrak{X}}(\theta^{n-s}_{\mathfrak{X}}(M), \theta^{n-t}_{\mathfrak{X}}(N)). \qquad (1)$$

The composition is determined by the compositions

$$C_{\mathfrak{X}}(\theta^{n-r}_{\mathfrak{X}}(L), \theta^{n-s}_{\mathfrak{X}}(M)) \times C_{\mathfrak{X}}(\theta^{n-s}_{\mathfrak{X}}(M), \theta^{n-t}_{\mathfrak{X}}(N)) \longrightarrow C_{\mathfrak{X}}(\theta^{n-r}_{\mathfrak{X}}(L), \theta^{n-t}_{\mathfrak{X}}(N)).$$

The functor $\theta_{X_{\mathfrak{s}}}$ is defined on objects by $\theta_{X_{\mathfrak{s}}}(s,M) = (s-1,M)$. It follows from (1) above that there is a natural isomorphism

$$C_{\mathfrak{X}_{\mathfrak{s}}}((s,M),(t,N)) \xrightarrow{\sim} C_{\mathfrak{X}_{\mathfrak{s}}}(\theta_{X_{\mathfrak{s}}}(s,M), \theta_{X_{\mathfrak{s}}}(t,N)) = C_{\mathfrak{X}_{\mathfrak{s}}}((s-1,M),(t-1,N)),$$

which is the action of $\theta_{X_{\mathfrak{s}}}$ on morphisms.

There is a functor $C_{\mathfrak{X}} \xrightarrow{\Phi_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{s}}}$ which maps an object M of $C_{\mathfrak{X}}$ to the object $(0, M)$ and a morphism $M \longrightarrow N$ to its image in

$$C_{\mathfrak{X}_{\mathfrak{s}}}((0,M),(0,N)) \stackrel{\text{def}}{=} \underset{n \geq 1}{\operatorname{colim}}\, C_{\mathfrak{X}}(\theta^n_{\mathfrak{X}}(M), \theta^n_{\mathfrak{X}}(N)).$$

The morphism

$$\theta_{\mathfrak{X}_{\mathfrak{s}}} \circ \Phi_{\mathfrak{X}}(M) = (-1, M) \xrightarrow{\varphi_{\mathfrak{X}}(M)} \Phi_{\mathfrak{X}} \circ \theta_{\mathfrak{X}}(M) = (0, \theta_{\mathfrak{X}}(M))$$

is the image of the identical morphism $\theta_{\mathfrak{X}}(M) \longrightarrow \theta_{\mathfrak{X}}(M)$.

Let $\mathbb{Z}_+ - Cat_k$ denote the category of svelte k-linear $\mathbb{Z}_+$-categories, and let $\mathbb{Z} - \mathfrak{Cat}_k$ denote its full subcategory generated by k-linear strict $\mathbb{Z}$-categories.

C4.5.1.1. Proposition. *The map which assigns to a $\mathbb{Z}_+$-category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$ the strict $\mathbb{Z}$-category $(C_{\mathfrak{X}_\mathfrak{s}}, \theta_{\mathfrak{X}_\mathfrak{s}})$ extends to a functor $\mathbb{Z}_+ - Cat_k \xrightarrow{\mathfrak{J}^*} \mathbb{Z} - \mathfrak{Cat}_k$ which is a left adjoint to the inclusion functor $\mathbb{Z} - \mathfrak{Cat}_k \xrightarrow{\mathfrak{J}_*} \mathbb{Z}_+ - Cat_k$.*

Proof. The morphisms $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}) \xrightarrow{(\Phi_{\mathfrak{X}}, \varphi_{\mathfrak{X}})} (C_{\mathfrak{X}_\mathfrak{s}}, \theta_{\mathfrak{X}_\mathfrak{s}})$ defined above form an adjunction morphism from identical functor on $\mathbb{Z} - Cat_k$ to the composition $\mathfrak{J}_*\mathfrak{J}^*$. The second adjunction morphism is a natural isomorphism. ■

C4.5.2. Cosuspended categories and strict triangulated categories. The construction of C4.5.1 extends to a functor from the category of cosuspended categories to the category of *strict triangulated* categories. Recall that a triangulated category $\mathfrak{T}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$ is *strict* if $\theta_{\mathfrak{X}}$ is an auto-morphism of the category $C_{\mathfrak{X}}$.

C4.5.2.1. Proposition [KeV]. *To any cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$, there corresponds a strict triangulated category $\mathfrak{T}_-C_{\mathfrak{X}_\mathfrak{s}}$ and a triangle functor*

$$\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{(\Phi_{\mathfrak{X}}, \varphi_{\mathfrak{X}})} \mathfrak{T}_-C_{\mathfrak{X}_\mathfrak{s}}$$

such that for every triangulated category $\mathfrak{T}C_{\mathfrak{Y}}$, the functor

$$\widetilde{\mathfrak{Tr}}^-_k(\mathfrak{T}_-C_{\mathfrak{X}_\mathfrak{s}}, \mathfrak{T}C_{\mathfrak{Y}}) \longrightarrow \widetilde{\mathfrak{Tr}}^-_k(\mathfrak{T}_-C_{\mathfrak{X}}, \mathfrak{T}C_{\mathfrak{Y}}) \qquad (1)$$

of composition with $(\Phi_{\mathfrak{X}}, \varphi_{\mathfrak{X}})$ is an equivalence of categories.

(a) If $\mathfrak{T}C_{\mathfrak{Y}}$ is a strict triangulated category, then (1) is an isomorphism of categories.

(b) If $\mathfrak{T}_-C_{\mathfrak{X}}$ is a triangulated category, then $(\Phi_{\mathfrak{X}}, \varphi_{\mathfrak{X}})$ is a triangle equivalence.

Proof. By C4.5.1, objects of the category $\mathfrak{T}_-C_{\mathfrak{X}_\mathfrak{s}}$ are pairs (n, M), where $n \in \mathbb{Z}$ and $M \in ObC_{\mathfrak{X}}$. The triangles are sequences

$$\theta_{X_\mathfrak{s}}(r, L) = (r-1, L) \longrightarrow (t, N) \longrightarrow (s, M) \longrightarrow (r, L)$$

associated to sequences

$$\theta_{\mathfrak{X}}\theta^{n-r}_{\mathfrak{X}}(L) \xrightarrow{w} \theta^{n-t}_{\mathfrak{X}}(N) \xrightarrow{v} \theta^{n-s}_{\mathfrak{X}}(M) \xrightarrow{u} \theta^{n-r}_{\mathfrak{X}}(L)$$

such that $((-1)^n w, v, u)$ is a triangle. ■

Let $\mathfrak{T}_-Cat_k$ (resp. $\mathfrak{Tr}\mathfrak{Cat}_k$) denote the category whose objects are svelte cosuspended (resp. svelte triangulated strict) k-linear categories and morphisms are triangle functors.

C4.5.3. Proposition. *The map which assigns to each cosuspended category the corresponding strict triangulated category extends to a functor* $\mathfrak{T}_-Cat_k \xrightarrow{{}^{\mathfrak{t}}\mathfrak{J}^*} \mathfrak{TrCat}_k$ *which is a left adjoint to the inclusion functor.*

Proof. See C4.5.1.1. ■

C4.5.4. Proposition. *Let* $\mathfrak{T}_-C_{\mathfrak{X}}$ *be a cosuspended k-linear category. The functor* $\mathbb{Z}_+ - Cat_k \xrightarrow{\mathfrak{J}^*} \mathbb{Z} - \mathfrak{Cat}_k$ *maps the natural embedding* $C_{\mathfrak{X}_{\mathfrak{m}}} \longrightarrow C_{\mathfrak{X}_{\mathfrak{a}}}$ *of* $\mathbb{Z}_+$*-categories to an equivalence between abelian strict* $\mathbb{Z}$*-categories.*

Proof. It follows from the construction of the functor $\mathfrak{J}^*$ that it is compatible with the 'triangularization' functor $\mathfrak{T}_-Cat_k \xrightarrow{{}^{\mathfrak{t}}\mathfrak{J}^*} \mathfrak{TrCat}_k$ of C4.5.3. The constructions of the categories $C_{\mathfrak{X}_{\mathfrak{m}}}$ and $C_{\mathfrak{X}_{\mathfrak{a}}}$ are also compatible with the functors triangularization functor and the functor $\mathfrak{J}^*$. By C4.4.3, the categories $C_{\mathfrak{X}_{\mathfrak{m}}}$ and $C_{\mathfrak{X}_{\mathfrak{a}}}$ coincide if $\mathfrak{T}_-C_{\mathfrak{X}}$ is a triangulated category, hence the assertion. ■

C4.6. Complements.

C4.6.0. Exact categories and exact categories with enough projective objects. Let $(C_X, \mathfrak{E}_X)$ be an exact category and $C_{X_{\mathfrak{P}}}$ its full subcategory generated by all objects M of C_X such that there exists a deflation $P \longrightarrow M$, where P is a projective object of $(C_X, \mathfrak{E}_X)$. It follows from (the argument of) C4.3.2 that the subcategory $C_{X_{\mathfrak{P}}}$ is *fully exact* (i.e. it is closed under extensions). In particular, it is an exact subcategory of $(C_X, \mathfrak{E}_X)$. By construction, this exact subcategory, $(C_{X_{\mathfrak{P}}}, \mathfrak{E}_{X_{\mathfrak{P}}})$, has enough projective objects.

Let $\mathfrak{Cat}_{\mathfrak{ex}}$ denote the bicategory of exact categories (whose 1-morphisms are 'exact' functors) and $\mathfrak{Cat}_{\mathfrak{ex}}^{\mathfrak{P}}$ its full subcategory generated by exact categories with enough projective objects. The map which assigns to every exact category $(C_X, \mathfrak{E}_X)$ its fully exact subcategory $(C_{X_{\mathfrak{P}}}, \mathfrak{E}_{X_{\mathfrak{P}}})$ extends to a 2-functor from $\mathfrak{Cat}_{\mathfrak{ex}}$ to $\mathfrak{Cat}_{\mathfrak{ex}}^{\mathfrak{P}}$ which is left adjoint to the inclusion functor $\mathfrak{Cat}_{\mathfrak{ex}}^{\mathfrak{P}} \longrightarrow \mathfrak{Cat}_{\mathfrak{ex}}$ (in the 2-categorical sense).

C4.6.1. Costable categories in terms of complexes. Let $(C_X, \mathfrak{E}_X)$ be an exact category. Consider the full subcategory $C_{\mathcal{P}_0X}$ of the homotopy category $\mathcal{H}(C_X)$ whose objects are acyclic complexes $\mathcal{P} = (\dots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow M \longrightarrow 0)$ such that objects P_i, $i \geq 0$, are projective.

The category $C_{\mathcal{P}_0X}$ has a natural $\mathbb{Z}_+$-action given by the 'translation' functor θ_- which assigns to every object $\mathcal{P} = (\dots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow M \longrightarrow 0)$ the object $\theta_-(\mathcal{P}) = (\dots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} Cok(d_0) \longrightarrow 0)$.

C4.6.1.1. Lemma. *Let* $(C_X, \mathcal{E}_X)$ *be an exact category with enough projective objects. Then the costable category* $C_{\mathfrak{S}_-X}$ *of* C_X *is* $\mathbb{Z}_+$*-equivalent to the category* $C_{\mathcal{P}_0X}$*.*

Proof. The equivalence is given by the functor $C_{\mathcal{P}_0 X} \longrightarrow C_{\mathfrak{S}_- X}$ which assigns to every object $\mathcal{P} = (\ldots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow M \longrightarrow 0)$ of $C_{\mathcal{P}_0 X}$ the (image in $C_{\mathfrak{S}_- X}$ of the) cokernel of $P_1 \xrightarrow{d_0} P_0$. The quasi-inverse functor assigns to each object M of $C_{\mathfrak{S}_- X}$ (the image in $C_{\mathcal{P}_0 X}$ of) its projective resolution $(\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0)$.

It follows from the definitions that both functors are compatible with the $\mathbb{Z}_+$-actions on the respective categories. ∎

C4.6.2. Homological dimension.

C4.6.2.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be an exact category with enough projective objects, $\mathfrak{T}_- C_{\mathfrak{S}_- X} = (C_{\mathfrak{S}_- X}, \theta, \mathfrak{Tr}_{\mathfrak{S}_- X})$ its costable cosuspended category, and $C_X \xrightarrow{\mathfrak{P}_X} C_{\mathfrak{S}_- X}$ the canonical projection.*

(a) The following condition on an object M of C_X are equivalent:

(a1) $hd(M) \leq n$;

(a2) $\theta^n(\mathfrak{P}_X(M)) = 0$.

(b) An object M of C_X is projective iff its image in the costable category is zero.

Proof. Consider first the case $n = 0$. Then the condition (a1) means that the object M is projective. The condition (a2) reads: the image of M in the costable category is zero. The implication $(a) \Rightarrow (b)$ follows from the definition of the costable category.

On the other hand, the image of M in the costable category is zero iff the image of the identical morphism id_M is zero. The latter means that id_M factors through a projective object, i.e. M is a retract of a projective object, hence it is projective.

Suppose now that $n \geq 1$. Let $(\ldots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow M)$ be a projective resolution of the object M. By the definition of the cosuspension θ, there is an isomorphism $\theta(\mathfrak{P}_X(M)) \simeq \mathfrak{P}_X(im(d_0))$. Therefore, $\theta^n(\mathfrak{P}_X(M)) \simeq \mathfrak{P}_X(im(d_{n-1})$. The homological dimension of M is less or equal to n iff $im(d_{n-1})$ is a projective object, or, equivalently, $\mathfrak{P}_X(im(d_{n-1}) = 0$. ∎

C4.6.2.1.1. Corollary. *Let $(C_X, \mathcal{E}_X)$ be an exact category with enough projective objects. The following conditions are equivalent:*

(a) $hd(C_X, \mathcal{E}_X) \leq n$;

(b) $\theta^n = 0$.

In particular, $hd(C_X, \mathcal{E}_X) = 0$ iff the costable category of $(C_X, \mathcal{E}_X)$ is trivial.

C4.6.2.2. Homological dimension of objects of a cosuspended category. Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}}^-)$ be a cosuspended category. We say that an object M of $C_{\mathfrak{X}}$ has homological dimension n if $\theta^n(M) = 0$ and $\theta^{n-1}(M) \neq 0$. In particular, an object of $C_{\mathfrak{X}}$ is of homological dimension zero iff it is zero.

C4.6.2.3. Proposition. *Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}}^-)$ be a cosuspended category.*

(a) The full subcategory $C_{\mathfrak{X}_{h\omega}}$ of the category $C_{\mathfrak{X}}$ generated by the objects of finite homological dimension is a thick cosuspended subcategory of $\mathfrak{T}_- C_{\mathfrak{X}}$.

(b) The subcategory $C_{\mathfrak{X}_{h\omega}}$ is contained in the kernel of the canonical "triangularization" functor $\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{(\Phi_{\mathfrak{X}},\varphi_{\mathfrak{X}})} \mathfrak{T}_-C_{\mathfrak{X}(\mathbb{Z})}$ *(see C4.5.2.1.)*

Proof. (a) Recall that a full cosuspended subcategory $\mathcal{B}$ of $\mathfrak{T}_-C_{\mathfrak{X}}$ is called a *thick cosuspended subcategory* if it is closed under extensions, i.e. if $\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{f} L$ is a triangle and L and N are objects of $\mathcal{B}$, then M is an object of $\mathcal{B}$ too.

By K8.4(b), for every triangle $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$, the sequence of representable functors

$$\ldots \longrightarrow C_{\mathfrak{X}}(-,\theta_{\mathfrak{X}}(L)) \xrightarrow{C_{\mathfrak{X}}(-,w)} C_{\mathfrak{X}}(-,N) \xrightarrow{C_{\mathfrak{X}}(-,v)} C_{\mathfrak{X}}(-,M) \xrightarrow{C_{\mathfrak{X}}(-,u)} C_{\mathfrak{X}}(-,L)$$

is exact. In particular, there is an exact sequence of representable functors

$$\ldots \longrightarrow C_{\mathfrak{X}}(-,\theta^n_{\mathfrak{X}}(N)) \longrightarrow C_{\mathfrak{X}}(-,\theta^n_{\mathfrak{X}}(M)) \longrightarrow C_{\mathfrak{X}}(-,\theta^n_{\mathfrak{X}}(L)) \longrightarrow \ldots \quad (1)$$

for every positive integer n. If the objects L and N have finite homological dimension, i.e. $\theta^n_{\mathfrak{X}}(L))$ and $\theta^n_{\mathfrak{X}}(N))$ are zero objects for some n, then it follows from the exactness of the sequence (1) that $\theta^n_{\mathfrak{X}}(M)) = 0$.

(b) Triangulated categories are precisely cosuspended categories whose cosuspension functor is an auto-equivalence. Therefore, every nonzero object of a triangulated category has an infinite homological dimension. ∎

C4.6.2.4. Homological dimension of a cosuspended category. Homological dimension of the cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}}$ is, by definition, the supremum of homological dimensions of its objects. In particular, $hd(C_{\mathfrak{X}}) \leq n$ for some finite n iff $\theta^n_{\mathfrak{X}} = 0$.

C4.6.3. The stable and costable categories of an arbitrary exact category. Let $(C_X, \mathcal{E}_X)$ be an exact category with the class of deflations (resp. inflations) $\mathfrak{E}_X$ (resp. $\mathfrak{M}_X$). Let $C_X \xrightarrow{\mathfrak{q}^{\mathfrak{I}}_X} C_{X_{\mathfrak{E}}}$ be the Gabriel-Quillen embedding. Since $C_{X_{\mathfrak{E}}}$ is a Grothendieck category, it has enough injective objects. In particular, $C_{X_{\mathfrak{E}}}$ has the stable suspended category $(C_{\mathfrak{S}_+X_{\mathfrak{E}}}, \Theta_{X_{\mathfrak{E}}}, \mathfrak{Tr}_{\mathfrak{S}_+X_{\mathfrak{E}}})$ with infinite coproducts and products.

The composition of the Gabriel-Quillen embedding and the projection $C_{X_{\mathfrak{E}}} \longrightarrow C_{\mathfrak{S}_+X_{\mathfrak{E}}}$ gives a functor $C_X \longrightarrow C_{\mathfrak{S}_+X_{\mathfrak{E}}}$. We call the *stable category* of the exact category C_X the triple $(C_{\mathfrak{T}_+X}, \Theta_X, \mathfrak{Tr}_{\mathfrak{T}_+X})$, where $C_{\mathfrak{T}_+X}$ is the smallest $\Theta_{X_{\mathfrak{E}}}$-invariant full subcategory of $C_{\mathfrak{S}_+X_{\mathfrak{E}}}$ containing the image of C_X, Θ_X is the endofunctor of $C_{\mathfrak{T}_+X}$ induced by $\Theta_{X_{\mathfrak{E}}}$, and $\mathfrak{Tr}_{\mathfrak{T}_+X}$ is the class of all triangles from $\mathfrak{Tr}_{\mathfrak{S}_+X_{\mathfrak{E}}}$ which belong to the subcategory $C_{\mathfrak{T}_+X}$.

One can see that $(C_{\mathfrak{T}_+X}, \Theta_X, \mathfrak{Tr}_{\mathfrak{T}_+X})$ is a full suspended subcategory of the suspended category $(C_{\mathfrak{T}_+X_{\mathfrak{E}}}, \Theta_{X_{\mathfrak{E}}})$. If the exact category C_X has enough injective objects, then the suspended category $(C_{\mathfrak{T}_+X}, \Theta_X)$ is equivalent to the stable category of C_X defined earlier.

The costable category $(C_{\mathfrak{T}_-X}, \theta_X, \mathfrak{Tr}_{\mathfrak{T}_-X})$ of the exact category C_X is defined dually.

C4.6.4. Canonical resolutions.

C4.6.4.1. The resolution of a cosuspended category. Let $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}})$ be a cosuspended category. The universal homological functor is the full embedding the $\mathbb{Z}_+$-categories $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}) \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} (C_{\mathfrak{X}_{\mathfrak{a}}}, \Theta_{\mathfrak{X}_{\mathfrak{a}}})$ which realizes $C_{\mathfrak{X}}$ as a subcategory of the full subcategory of $C_{\mathfrak{X}_{\mathfrak{a}}}$ generated by projective objects of $(C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}})$. Since the exact category $C_{\mathfrak{X}_{\mathfrak{a}}}$ has enough projective objects, its costable category $C_{\mathfrak{S}_-\mathfrak{X}_{\mathfrak{a}}}$ is (the underlying category of) a cosuspended category with the cosuspension functor θ_2. Since the functor $\Theta_{\mathfrak{X}_{\mathfrak{a}}}$ maps projective objects to projective objects, it induces an endofunctor θ_1 on the costable category $C_{\mathfrak{S}_-\mathfrak{X}_{\mathfrak{a}}}$. It follows from the exactness of the functor $\Theta_{\mathfrak{X}_{\mathfrak{a}}}$ that $\theta_1 \circ \theta_2 \simeq \theta_2 \circ \theta_1$; i.e. $C_{\mathfrak{T}\mathfrak{X}_{\mathfrak{a}}}$ is a cosuspended $\mathbb{Z}_+$-category. In particular, it is a $\mathbb{Z}_+ \times \mathbb{Z}_+$-category. The canonical universal homological functor embeds the cosuspended $\mathbb{Z}_+$-category $C_{\mathfrak{S}_-\mathfrak{X}_{\mathfrak{a}}}$ into an exact $\mathbb{Z}_+ \times \mathbb{Z}_+$-category $C_{(\mathfrak{S}_-\mathfrak{X}_{\mathfrak{a}})_{\mathfrak{a}}}$, etc.. As a result of this procedure, we obtain a sequence of categories and functors

$$\begin{gathered} C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a}}}} C_{\mathfrak{X}_1} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_1}} C_{\mathfrak{X}_{\mathfrak{a},1}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},1}}} \dots \\ \dots \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n-1}}} C_{\mathfrak{X}_n} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_n}} C_{\mathfrak{X}_{\mathfrak{a},n}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n}}} C_{\mathfrak{X}_{n+1}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_{n+1}}} \dots \end{gathered} \tag{1}$$

where $\mathfrak{X}_{\mathfrak{a},n} = (\mathfrak{X}_n)_{\mathfrak{a}}$, $\mathfrak{X}_{n+1} = \mathfrak{S}_-\mathfrak{X}_{\mathfrak{a},n}$ for $n \geq 0$ and $\mathfrak{X}_0 = \mathfrak{X}$.

It follows that $\mathfrak{X}_n$ is represented by a cosuspended $\mathbb{Z}_+^n$-category (hence a $\mathbb{Z}_+^{n+1}$-category), $\mathfrak{X}_{\mathfrak{a},n}$ is represented by an exact $\mathbb{Z}_+^{n+1}$-category; and the universal homological functor $\mathfrak{H}_{\mathfrak{X}_n}$ and the canonical projections $\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n}}$ are $\mathbb{Z}_+^n$-functors. All exact categories $(C_{\mathfrak{X}_{\mathfrak{a},n}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a},n}})$ have enough projective objects.

For every exact category $(C_X, \mathcal{E}_X)$ with enough projective objects, let Φ_X denote the composition of the projection $C_X \xrightarrow{\mathfrak{P}_X} C_{\mathfrak{S}_-X}$ to the costable category and the universal homological functor $C_{\mathfrak{S}_-X} \xrightarrow{\mathfrak{H}_{\mathfrak{S}_-X}} C_{\mathfrak{S}_-X_{\mathfrak{a}}}$.

Set $\Phi_n = \mathfrak{H}_{\mathfrak{X}_n} \circ \mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n-1}}$. Then we have a sequence of functors

$$\begin{gathered} C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a}}}} C_{\mathfrak{X}_{\mathfrak{a},1}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},1}}} C_{\mathfrak{X}_{\mathfrak{a},2}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},2}}} \dots \\ \dots \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n-2}}} C_{\mathfrak{X}_{\mathfrak{a},n-1}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n-1}}} C_{\mathfrak{X}_{\mathfrak{a},n}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n}}} C_{\mathfrak{X}_{\mathfrak{a},n+1}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n+1}}} \dots \end{gathered} \tag{2}$$

in which the composition of any two consecutive arrows equals to zero. The kernel of the functor $C_{\mathfrak{X}_{\mathfrak{a},n}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n}}} C_{\mathfrak{X}_{\mathfrak{a},n+1}}$ coincides with the full subcategory of the category $C_{\mathfrak{X}_{\mathfrak{a},n}}$ generated by all its projective objects. It coincides with the Karoubian envelope in $C_{\mathfrak{X}_{\mathfrak{a},n}}$ of the image of the functor $\Phi_{\mathfrak{X}_{\mathfrak{a},n-1}}$.

C4.6.4.2. The resolution of an exact category with enough projective objects. Let $(C_X, \mathcal{E}_X)$ be an exact category with enough projective objects. Let $C_{\mathcal{P}X}$ denote the full subcategory of the category C_X generated by all projective objects of $(C_X, \mathcal{E}_X)$. Then we have a sequence

$$\begin{array}{c} C_{\mathcal{P}X} \xrightarrow{\mathfrak{K}_X} C_X \xrightarrow{\mathfrak{P}_X} C_{\mathfrak{X}_0} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_0}} C_{\mathfrak{X}_{\mathfrak{a},0}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},0}}} C_{\mathfrak{X}_1} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_1}} C_{\mathfrak{X}_{\mathfrak{a},1}} \dots \\ \dots \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n-1}}} C_{\mathfrak{X}_n} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_n}} C_{\mathfrak{X}_{\mathfrak{a},n}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n}}} C_{\mathfrak{X}_{n+1}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_{n+1}}} \dots \end{array} \tag{3}$$

where $\mathfrak{X}_0 = \mathfrak{S}_- X$, i.e. $C_{\mathfrak{X}_0}$ is the costable category of the exact category $(C_X, \mathcal{E}_X)$, and the rest is defined as in (1) above. Again, one can ignore the intermediate cosuspended categories and obtain a complex of exact categories

$$\begin{array}{c} C_{\mathcal{P}X} \xrightarrow{\mathfrak{K}_X} C_X \xrightarrow{\Phi_X} C_{\mathfrak{X}_{\mathfrak{a},0}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},0}}} C_{\mathfrak{X}_{\mathfrak{a},1}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},1}}} \dots \\ \dots \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n-1}}} C_{\mathfrak{X}_{\mathfrak{a},n}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n}}} C_{\mathfrak{X}_{\mathfrak{a},n+1}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n+1}}} C_{\mathfrak{X}_{\mathfrak{a},n+2}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n+2}}} \dots \end{array} \tag{4}$$

C4.6.4.3. Note. If $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}})$ is a triangulated category (i.e. $\theta_{\mathfrak{X}}$ is an auto-equivalence), then all the cosuspended $\mathbb{Z}^n_+$-categories $\mathfrak{T}_- C_{\mathfrak{X}_n} = (C_{\mathfrak{X}_n}, \theta_{\mathfrak{X}_n}, \mathfrak{Tr}_{\mathfrak{X}_n})$ constructed above are triangulated $\mathbb{Z}^n$-categories and all exact $\mathbb{Z}^n_+$-categories $C_{\mathfrak{X}_{\mathfrak{a},n}}$ are abelian $\mathbb{Z}^n$-categories.

C5. The weak costable category of a right exact category.

C5.1. Definition. Let $(C_X, \mathfrak{E}_X)$ be a right exact category such that the category C_X has an initial object, x. We denote by $\mathfrak{Pr}(X, \mathfrak{E}_X)$ the full subcategory of C_X whose objects are projective objects. Let $\widetilde{\mathcal{S}}_X$ denote the class of all arrows t_1 in the commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e}')} & P & \xrightarrow{\mathfrak{e}} & M \\ t_1 \downarrow & & \downarrow t_0 & & \downarrow id_M \\ Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e})} & V & \xrightarrow{\mathfrak{e}} & M \end{array}$$

where $\mathfrak{e}$, $\mathfrak{e}'$ are deflations, t_0 (hence t_1) are split epimorphisms, and P (hence V) is an object of $\mathfrak{Pr}(X, \mathfrak{E}_X)$. Let $\mathcal{S}_X$ be the smallest saturated system containing $\widetilde{\mathcal{S}}_X$ and all deflations $P \longrightarrow P'$ with P and P' in $\mathfrak{Pr}(X, \mathfrak{E}_X)$. We call the quotient category $\mathcal{S}_X^{-1} C_X$ the *weak costable* category of the right exact category $(C_X, \mathfrak{E}_X)$ and denote it by $C_{\mathcal{S}_- X}$.

C5.1.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and enough projective objects. For any object N of the costable category, let $\theta^w_X(N)$ denote the image in $C_{\mathcal{S}_- X}$ of $Ker(\mathfrak{e})$, where $P \xrightarrow{\mathfrak{e}} N$ is a deflation with P projective (we identify* *objects of $C_{\mathcal{S}_-X}$ with objects of C_X). The object $\theta^w_X(N)$ is determined uniquely up to isomorphism. The map $N \longmapsto \theta^w_X(N)$ extends to a functor $C_{\mathcal{S}_-X} \longrightarrow C_{\mathcal{S}_-X}$.*

Proof. Let $P' \xrightarrow{\mathfrak{e}'} N \xleftarrow{\mathfrak{e}''} P''$ be deflations with P' and P'' projective objects. Since $(C_X, \mathfrak{E}_X)$ has enough projective objects, there exists (by the argument C5.3.1(a)) a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{t'_0} & P' \\ {\scriptstyle t''_0}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}'} \\ P'' & \xrightarrow{\mathfrak{e}''} & N \end{array}$$

whose arrows are deflations and the object P is projective. Therefore, we have a commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}'} & P' & \xrightarrow{\mathfrak{e}''} & N \\ {\scriptstyle t'_1}\uparrow & & \uparrow{\scriptstyle t'_0} & & \uparrow{\scriptstyle id_N} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}} & P & \xrightarrow{\mathfrak{e}''} & N \\ {\scriptstyle t''_1}\downarrow & & \downarrow{\scriptstyle t''_0} & & \downarrow{\scriptstyle id_N} \\ Ker(\mathfrak{e}'') & \xrightarrow{\mathfrak{k}''} & P'' & \xrightarrow{\mathfrak{e}''} & N \end{array}$$

Since t'_0 and t''_0 are deflations to projective objects, they are split epimorphisms. Therefore, t'_1 and t''_1 are split epimorphisms, i.e. they belong to $\widetilde{\mathcal{S}}_X$ (cf. 2.5).

Consider a diagram $N \xrightarrow{f} L \xleftarrow{\mathfrak{e}'} M$, where $\mathfrak{e}'$ is a deflation. Then we have a commutative diagram

$$\begin{array}{ccccc} Ker(\sigma) & \xrightarrow{\mathfrak{k}(\sigma)} & P & \xrightarrow{\sigma} & N \\ {\scriptstyle \mathfrak{t}_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}_0} & & \downarrow{\scriptstyle id_N} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}(\mathfrak{e})} & \mathfrak{N} & \xrightarrow{\mathfrak{e}} & N \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e}')} & M & \xrightarrow{\mathfrak{e}'} & L \end{array} \qquad (1)$$

in which the right lower square is cartesian, the morphism f_1 is uniquely determined by the choice of f_0 (hence both f_0 and f_1 are determined by f uniquely up to isomorphism), $\mathfrak{t}_0$ is a deflation, and $\mathfrak{t}_1$ is (a deflation) uniquely determined by $\mathfrak{t}_0$. Applying the localization $C_X \xrightarrow{\mathfrak{q}^*_{\mathcal{S}_X}} C_{\mathcal{S}_-X}$, we obtain morphisms

$$\theta^w_X(N) \xrightarrow{\sim} \mathfrak{q}^*_{\mathcal{S}_X}(Ker(\sigma)) \xrightarrow{\mathfrak{q}^*_{\mathcal{S}_X}(\mathfrak{t}_1)} \mathfrak{q}^*_{\mathcal{S}_X}(Ker(\mathfrak{e})) \xrightarrow{\mathfrak{q}^*_{\mathcal{S}_X}(f_1)} \mathfrak{q}^*_{\mathcal{S}_X}(Ker(\mathfrak{e}')). \qquad (2)$$

The only choice in this construction is that of the deflation $P \xrightarrow{\mathfrak{t}_0} \mathfrak{N}$. If $P' \xrightarrow{\mathfrak{s}_0} \mathfrak{N}$ is another choice, then there exists a commutative square

$$\begin{array}{ccc} P'' & \xrightarrow{\mathfrak{s}''_0} & P \\ {\scriptstyle \mathfrak{t}''_0}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}_0} \\ P' & \xrightarrow{\mathfrak{s}_0} & \mathfrak{N} \end{array}$$

whose arrows are deflations and the object P'' is a projective. Therefore, $\mathfrak{t}''_0$ and $\mathfrak{s}''_0$ are split deflations, and we have a commutative diagram

$$\begin{array}{ccccc} Ker(\sigma') & \xrightarrow{\mathfrak{k}(\sigma')} & P' & \xrightarrow{\sigma'} & N \\ {\scriptstyle \mathfrak{s}''_1}\uparrow & & \uparrow{\scriptstyle \mathfrak{s}''_0} & & \uparrow{\scriptstyle id_N} \\ Ker(\sigma'') & \xrightarrow{\mathfrak{k}(\sigma'')} & P'' & \xrightarrow{\sigma''} & N \\ {\scriptstyle \mathfrak{t}''_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}''_0} & & \downarrow{\scriptstyle id_N} \\ Ker(\sigma) & \xrightarrow{\mathfrak{k}(\sigma)} & P & \xrightarrow{\sigma} & N \end{array} \qquad (3)$$

whose vertical arrows belong to $\mathcal{S}_X$, i.e. their images in the costable category are isomorphisms. This implies that the composition $\theta^w_X(N) \longrightarrow \mathfrak{q}^*_{\mathcal{S}_X}(Ker(\mathfrak{e}'))$ of morphisms of (2) does not depend on the choice of the deflation $P \xrightarrow{\mathfrak{t}_0} \mathfrak{N}$. Taking M projective, we obtain a morphism $\theta^w_X(N) \xrightarrow{\theta_X(f)} \theta^w_X(L)$ which is uniquely defined once the choice of objects $\theta^w_X(N)$ and $\theta^w_X(L)$ is fixed. ■

C5.2. The weak cosuspension functor. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with enough projective objects and initial objects. Let $C_{\mathcal{S}\_X}$ its cosuspended category. The functor $C_{\mathcal{S}\_X} \xrightarrow{\theta^w_X} C_{\mathcal{S}\_X}$ defined in C5.1.1 is called the *weak cosuspension* functor.

Notice that the weak costable category $C_{\mathcal{S}\_X}$ of $(C_X, \mathfrak{E}_X)$ has initial objects. If the category C_X is pointed, then $C_{\mathcal{S}\_X}$ is pointed and the image in $C_{\mathcal{S}\_X}$ of each projective object of $(C_X, \mathfrak{E}_X)$ is a zero object.

C5.2.1. Note. It follows from C6.7.1 that if the category C_X is additive, then the weak costable category $C_{\mathcal{S}\_X}$ with the weak cosuspension functor θ^w_X is equivalent to the costable category $C_{\mathfrak{T}\_X}$ with the cosuspension functor θ_X.

C5.3. Right exact categories of modules over monads and their weak costable categories. Suppose that C_X is a category with initial objects and such that the class $\mathfrak{E}^{spl}_X$ of split epimorphisms of C_X is stable under base change; so that $(C_X, \mathfrak{E}^{spl}_X)$ is a right exact category. Let $\mathcal{F} = (F, \mu)$ be a monad on C_X. Set $C_{\mathfrak{X}} = \mathcal{F} - mod$. Let

$C_{\mathfrak{X}} \xrightarrow{f_*} C_X$ be the forgetful functor, f^* its canonical left adjoint, and ε the standard adjunction morphism $f^*f_* \longrightarrow Id_{C_{\mathfrak{X}}}$. We denote by $\mathfrak{E}_{\mathfrak{X}}$ the right exact structure on $C_{\mathfrak{X}}$ induced by $\mathfrak{E}_X^{spl}$ via the forgetful functor f_*. By 5.5, $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ is a right exact category with enough projective objects: all modules of the form $(F(L), \mu(L))$, $L \in ObC_X$, are projective objects of $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$, and every module $\mathcal{M} = (M, \xi)$ has a canonical deflation $f^*f_*(\mathcal{M}) \xrightarrow{\varepsilon(\mathcal{M})} \mathcal{M}$.

We denote by $\Omega_{\mathcal{F}}$ the kernel of the adjunction morphism $f^*f_* \xrightarrow{\varepsilon} Id_{C_{\mathfrak{X}}}$ and call it the functor of *Kähler differentials.*

C5.3.1. Standard triangles. Let $\mathcal{M} = (M, \xi_{\mathcal{M}})$ and $\mathcal{L} = (L, \xi_{\mathfrak{L}})$ be $\mathcal{F}$-modules and $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ a deflation (i.e. the epimorphism $M \xrightarrow{f_*(\mathfrak{e})} L$ splits). Then we have a commutative diagram

$$\begin{array}{ccccc} \Omega_{\mathcal{F}}(\mathcal{L}) & \xrightarrow{\mathfrak{k}_{\mathcal{F}}(\mathcal{L})} & f^*f_*(\mathcal{L}) & \xrightarrow{\varepsilon(\mathcal{L})} & \mathcal{L} \\ \partial \downarrow & & \downarrow \mathfrak{t}_0 & & \downarrow id_{\mathcal{L}} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array} \qquad (4)$$

which contains (and defines) the *standard triangle*

$$\Omega_{\mathcal{F}}(\mathcal{L}) \xrightarrow{\partial} Ker(\mathfrak{e}) \xrightarrow{\mathfrak{k}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \qquad (5)$$

corresponding to the deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$.

The image of (5) in the weak costable category $C_{\mathcal{S}_-X}$ is a standard triangle of $C_{\mathcal{S}_-X}$.

C5.4. Example: right exact categories of unital algebras. Let C_X be the category Alg_k of associative unital k-algebras. The category C_X has an initial object – the k-algebra k, and the associated pointed category C_{X_k} is the category of augmented k-algebras.

C5.4.1. The functor of Kähler differentials. Kähler differentials appear when we have a pair of adjoint functors $C_X \xrightarrow{f_*} C_Y \xrightarrow{f^*} C_X$. Presently, the role of the category C_Y is played by the category of k-modules. The forgetful functor $Alg_k \xrightarrow{f_*} k - mod$ has a canonical left adjoint f^* which assigns to every k-module M the tensor algebra $T_k(M) = \bigoplus_{n \geq 0} M^{\otimes n}$. Therefore, the class of all split k-module epimorphisms induces via f_* a structure $\mathfrak{E}_X$ of a right exact category on $C_X = Alg_k$. In this case, the tensor algebra $f^*(M) = T_k(M)$ is a projective object of $(C_X, \mathfrak{E}_X)$ for every k-module M; and for every k-algebra A, the adjunction morphism

$$f^*f_*(A) = T_k(f_*(A)) \xrightarrow{\varepsilon(A)} A,$$

(determined by the k-algebra structure and the multiplication $f_*(A)\otimes_k f_*(A) \longrightarrow f_*(A)$ in A) is a canonical projective deflation. By definition, the functor Ω_k of *Kähler differentials* assigns to each k-algebra A the kernel of the adjunction morphism $\varepsilon(A)$, which coincides with the augmented k-algebra $k \oplus \Omega_k^+(A)$, where $\Omega_k^+(A)$ is the kernel $K(\varepsilon(A))$ of the algebra morphism $\varepsilon(A)$ in the usual sense (i.e. in the category of non-unital algebras).

C5.4.2. The functor of non-additive Kähler differentials. The category Alg_k has small products and kernels of pairs of arrows $A \rightrightarrows B$, hence it has limits of arbitrary small diagrams. As any functor having a left adjoint, the forgetful functor $Alg_k \xrightarrow{f_*} k{-}mod$ preserves limits. In particular, f_* preserves pull-backs and, therefore, kernel pairs of algebra morphisms. Therefore, each k-algebra morphism $A \xrightarrow{\varphi} B$ has a canonical kernel pair $A \times_B A \underset{p_2}{\overset{p_1}{\rightrightarrows}} A$. Using the fact that $A \times_B A$ is computed as the pull-back of k-modules, we can represent $A \times_B A$ as the k-module $f_*(A) \oplus K(f_*(\varphi))$ with the multiplication induced by the isomorphism

$$f_*(A) \oplus Ker(f_*(\varphi)) \xrightarrow{\ \sim\ } f_*(A) \times_{f_*(B)} f_*(A), \quad x \oplus y \longmapsto (x, x+y).$$

That is the multiplication is given by the formula $(a \oplus b)(a' \oplus b') = aa' \oplus (ab' + ba' + bb')$. We denote this algebra by $A\#K(\varphi)$.

Applying this to the adjunction arrow $f^*f_* \xrightarrow{\varepsilon} Id_{C_X}$, we obtain a canonical isomorphism between the functor $\widetilde{\Omega}_k$ of non-additive Kähler differentials and $f^*f_*\#\Omega_k^+$, where $\Omega_k^+(A)$ is the kernel of the algebra morphism $T_k(f_*(A)) \xrightarrow{\varepsilon(A)} A$ in the category of non-unital k-algebras (cf. C5.4.1). Thus, for every k-algebra A, we have a commutative diagram similar to the one in the additive case:

$$\begin{array}{ccccccc}
k \oplus \Omega_k^+(A) & \xrightarrow{\sim} & \Omega_k(A) & \underset{0_k}{\overset{\mathfrak{k}(\varepsilon)}{\rightrightarrows}} & T_k(f_*(A)) & \xrightarrow{\varepsilon} & A \\
\widetilde{\mathfrak{j}}_k \big\downarrow & & \mathfrak{j}_k \big\downarrow & & \big\downarrow id & & \big\downarrow id \\
T_k(f_*(A))\#\Omega_k^+(A) & \xrightarrow{\sim} & \widetilde{\Omega}_k(A) & \underset{\lambda_2}{\overset{\lambda_1}{\rightrightarrows}} & T_k(f_*(A)) & \xrightarrow{\varepsilon} & A
\end{array} \qquad (6)$$

Here $0_k = 0_k(A)$ is the 'zero' morphism – the composition of the augmentation morphism $\Omega_k(A) \longrightarrow k$ and the k-algebra structure $k \longrightarrow T_k(f_*(A))$.

The morphism $\widetilde{\mathfrak{j}}_k$ (hence $\mathfrak{j}_k$) becomes an isomorphism in the costable category.

C5.4.3. Another canonical right exact structure. Let $\mathfrak{E}_X^{\mathfrak{s}}$ denote the class of all strict epimorphisms of k-algebras. The class $\mathfrak{E}_X^{\mathfrak{s}}$ is stable under base change, i.e. $(C_X, \mathfrak{E}_X)$ is a right exact category. For every projective k-module V, the tensor algebra $T_k(V)$ is a projective object of $(C_X, \mathfrak{E}_X^{\mathfrak{s}})$, because the forgetful functor $Alg_k \xrightarrow{f_*} k - mod$

is exact (hence it maps strict epimorphisms to epimorphisms of k-modules). By 5.3.1, its left adjoint f^* maps projective objects of $k-mod$ to projective objects of $(C_X, \mathfrak{E}^{\mathfrak{s}}_X)$. That is for every projective k-module V the tensor algebra $T_k(V)$ of V is a projective object of $(C_X, \mathfrak{E}^{\mathfrak{s}}_X)$. Since the adjunction arrow $f^*f_* \xrightarrow{\varepsilon} Id_{C_X}$ is a strict epimorphism and $k-mod$ has enough projective objects, the right exact category $(C_X, \mathfrak{E}^{\mathfrak{s}}_X)$ has enough projective objects: for any k-algebra A, there exists a strict k-algebra epimorphism $T_k(V) \xrightarrow{\mathfrak{e}} A$ for some projective k-module V. By 2.2.1, the kernel $Ker(\mathfrak{e})$ coincides with the augmented k-algebra $k \oplus K(\mathfrak{e})$, where $K(\mathfrak{e})$ is the kernel of $\mathfrak{e}$ in the usual sense – a two-sided ideal in $T_k(V)$.

C5.4.4. Remarks. (a) The forgetful functor $Alg_k \xrightarrow{f_*} k-mod$ is conservative and preserves cokernels of pairs of arrows. Therefore, by Beck's Theorem, there is a canonical equivalence (in this case, an isomorphism) between the category Alg_k and the category $\mathcal{F}-mod$ of modules over the monad $\mathcal{F} = (f_*f^*, \mu) = (T_k(-), \mu)$ associated with the pair of adjoint functors f_*, f^* and the adjunction morphism $f^*f_* \xrightarrow{\varepsilon} Id_{Alg_k}$.

(b) Consider the category $Aff_k = Alg_k^{op}$ of affine (noncommutative) k-schemes. Right exact structures on Alg_k define left exact structures on Aff_k and vice versa. Inflations in Aff_k corresponding strict epimorphisms of algebras are precisely closed immersions of (noncommutative) affine schemes.

(c) The example C5.4 is generalized to algebras in an additive monoidal category.

C5.5. The left exact category of comodules over a comonad and its weak stable category. Fix a comonad $\mathcal{G} = (G, \delta)$ on a category C_X with final objects and split monomorphisms stable under cobase change; i.e. $(C_X, \mathfrak{I}^{spl}_X)$ is a left exact category.

C5.5.1. The suspension functor. Let $\mathcal{G}_+$ denote the functor $C_{\mathfrak{Y}} \longrightarrow C_{\mathfrak{Y}}$ which assigns to every $\mathcal{G}$-comodule $\mathcal{M} = (M, \nu)$ the cokernel of the adjunction morphism

$$\mathcal{M} \xrightarrow{\ \nu\ } g_*g^*(\mathcal{M}) = (G(M), \delta(M)).$$

The functor $\mathcal{G}_+$ is a canonical *suspension* functor on the category $(\mathfrak{Y}\backslash\mathcal{G}) - comod$ which induces a suspension functor on the stable category $\mathcal{S}_+C_{\mathfrak{Y}}$ of the exact category $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$.

C5.5.2. Lemma. *A morphism $\mathcal{M} \xrightarrow{\phi} \mathcal{M}'$ of $C_{\mathfrak{Y}}$ becomes a trivial morphism in the stable category $\mathcal{T}_+C_{\mathfrak{Y}}$ iff it factors through an adjunction arrow (3); i.e. there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \phi\ } & \mathcal{M}' \\ \nu \searrow & & \nearrow h \\ & g_*g^*(\mathcal{M}) & \end{array}$$

for some morphism $g_*g^*(\mathcal{M}) = (G(M), \delta(M)) \xrightarrow{\ h\ } \mathcal{M}'$.

Proof. By definition of the stable category, the image of an arrow $\mathcal{M} \xrightarrow{\phi} \mathcal{M}'$ of $C_{\mathfrak{Y}}$ in the stable category $\mathcal{T}_+ C_{\mathfrak{Y}}$ is trivial iff it factors through an $\mathcal{E}_{\mathfrak{Y}}$-injective object $\mathcal{N}$. By 5.5.3, the adjunction arrow $\mathcal{N} \longrightarrow g_* g^*(\mathcal{N})$ splits. Therefore, the arrow $\mathcal{M} \xrightarrow{\phi} \mathcal{M}'$ becomes trivial in the stable category iff it factors through a morphism $\mathcal{M} \longrightarrow g_*(N)$ for an object N of C_X. Every such arrow factors through the adjunction morphism $\mathcal{M} \longrightarrow g_* g^*(\mathcal{M})$; hence the assertion. ■

C5.5.3. Standard triangles. For any conflation $\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N}$ in $C_{\mathfrak{Y}} = \mathcal{G} - $ *comod*, the standard triangle

$$\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N} \xrightarrow{\mathfrak{d}} \mathcal{G}_+(\mathcal{L})$$

is defined via a commutative diagram

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{N} \\ {\scriptstyle id_{\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathfrak{d}} \\ \mathcal{L} & \xrightarrow{\eta_\gamma(\mathcal{L})} & \mathcal{G}(\mathcal{L}) & \xrightarrow{\lambda_\gamma(\mathcal{L})} & \mathcal{G}_+(\mathcal{L}) \end{array} \qquad (1)$$

where $\mathcal{G} = g_* g^*$ and $G \xrightarrow{\lambda_g} \mathcal{G}_+$ is the canonical deflation. The morphism γ in (1) exists by the $\mathcal{E}_{\mathfrak{Y}}$-injectivity of $G(\mathcal{L})$. The morphism $\mathcal{N} \xrightarrow{\mathfrak{d}} \mathcal{G}_+(\mathcal{L})$ is uniquely determined by the choice of γ (because $\mathfrak{e}$ is an epimorphism). The image of $\mathfrak{d}$ in the stable category $\mathcal{T}_+ C_{\mathfrak{Y}}$ does not depend on the choice of γ.

C5.6. Frobenious morphisms of 'spaces', Frobenious monads. Let $Y \xrightarrow{f} X$ be a continuous morphism of 'spaces' with an inverse image functor f^* and a direct image functor f_*. We say that f is a *Frobenious* morphism if there exists an auto-equivalence Ψ of the category C_X such that the composition $f^* \circ \Psi$ is a right adjoint to f_*.

It is clear that every isomorphism is a Frobenious morphism and the composition of Frobenious morphisms is a Frobenious morphism.

It follows that every Frobenious morphism $Y \xrightarrow{f} X$ with a conservative direct image functor is affine. Therefore, the category C_Y can be identified with the category $\mathcal{F} - mod$ of modules over the monad $\mathcal{F} = (F, \mu)$ on a category C_X associated with the pair of adjoint functors f^*, f_*. Conversely, we call a monad $\mathcal{F}$ on the category C_X a *Frobenious* monad if the forgetful functor $\mathcal{F} - mod \xrightarrow{f_*} C_X$ is a direct image functor of a Frobenious morphism; i.e. there exists an equivalence $C_X \xrightarrow{\Psi} C_X$ such that the functor

$$C_X \xrightarrow{f^* \circ \Psi} \mathcal{F} - mod, \quad V \longmapsto (F(\Psi(V)), \mu(\Psi(V)),$$

is a right adjoint to the forgetful functor f_*. In particular, the monad $\mathcal{F}$ is continuous.

C5.6.1. Proposition. *Let $\mathcal{F}$ be a Frobenious monad on a category C_X such that $(C_X, \mathfrak{E}_X^{spl})$ is a right exact category. Then the right exact category $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$, where $C_{\mathfrak{X}}$ is the category $\mathcal{F} - mod$ of $\mathcal{F}$-modules and $\mathfrak{E}_{\mathfrak{X}}$ is a right exact structure induced by $\mathfrak{E}_X^{spl}$, is a Frobenious category.*

Proof. Let f_* denote the forgetful functor $\mathcal{F} - mod \longrightarrow C_X$ and f^* its canonical left adjoint. Let Ψ be a functor $C_X \longrightarrow C_X$ such that the composition $f^! = f^* \circ \Psi$ is a right adjoint to f_*. Then every injective object of the category $\mathcal{F} - mod$ is a retract of an object of the form $f^*(\Psi(V))$ for some $V \in ObC_X$. On the other hand, every projective object of $\mathcal{F} - mod$ is a retract of an object of the form $f^*(L)$ for some $L \in ObC_X$. Therefore, every injective $\mathcal{F}$-module is projective. If the functor Ψ is an auto-equivalence, then $f^* \simeq f^! \circ \Psi^*$, where Ψ^* is a quasi-inverse to Ψ. That the functor $f^! \circ \Psi$ is a left adjoint to f_*. By duality, it follows from the argument above that every projective object of $\mathcal{F} - mod$ is injective. ■

C5.7. The costable category associated with an augmented monad. Let $\mathcal{F} = (F, \mu)$ be an augmented monad on a k-linear additive category C_X; i.e. $F = Id_{C_X} \oplus F_+$. The category $\mathcal{F} - mod$ of $\mathcal{F}$-modules is isomorphic to the category $\mathcal{F}_+ - mod_1$ of $\mathcal{F}_+$-actions. Recall that the objects of $\mathcal{F}_+ - mod_1$ are pairs (M, ξ), where $M \in ObC_X$ and ξ is a morphism $F_+(M) \longrightarrow M$ satisfying associativity condition with respect to multiplication $F_+^2 \xrightarrow{\mu_+} F_+$, i.e. $\xi \circ \mu_+(M) = \xi \circ F_+(\xi)$. Morphisms are defined naturally.

Notice that the monad $\mathcal{F}$ is continuous (i.e. the functor F has a right adjoint) iff the functor F_+ has a right adjoint.

It follows that $\Omega_{\mathcal{F}} \longrightarrow f^* f_*$ factors through the subfunctor $\mathcal{F}_+$ of $f^* f_*$ corresponding to the subsemimonad (F_+, μ^+) of $\mathcal{F}$. The full subcategory $\mathcal{T}_{\mathcal{F}_+}$ of $\mathcal{F} - mod$ generated by all $\mathcal{F}$-modules $\mathcal{M}$ such that $\Omega_{\mathcal{F}}(\mathcal{M}) \longrightarrow \mathcal{F}_+(\mathcal{M})$ is an isomorphism (i.e. the action of F_+ on $\mathcal{M}$ is zero) is isomorphic to the category C_X.

C5.7.1. Infinitesimal neighborhoods. Let $\mathcal{T}_{\mathcal{F}_+}^{(n)}$ denote the *n-th infinitesimal neighborhood* of $\mathcal{T}_{\mathcal{F}_+}$, $n \geq 1$. It is the full subcategory of $\mathcal{F} - mod$ generated by modules $\mathcal{M} = (M, \xi)$ such that the n-th iteration $F_+^n(M) \xrightarrow{\xi_n^+} M$ of the action of F_+ on M is zero. In particular, $\mathcal{T}_{\mathcal{F}_+}^{(1)} = \mathcal{T}_{\mathcal{F}_+}$.

Since ξ_n^+ is an $\mathcal{F}$-module morphism for any $n \geq 1$, an $\mathcal{F}$-module $\mathcal{M} = (M, \xi)$ is an object of $\mathcal{T}_{\mathcal{F}_+}^{(n)}$ iff $F_+^{\cdot n} \hookrightarrow \Omega_{\mathcal{F}}$, where $F_+^{\cdot n}$ is the image of the iterated multiplication $F_+^n \xrightarrow{\mu_n^+} F_+$. One can see that $F_+^{\cdot n}$ is a two-sided ideal in the monad $\mathcal{F}$. If the quotient functor $F/F_+^{\cdot n}$ is well defined (which is the case if cokernels of morphisms exist in C_X), then there is a unique monad structure μ_n on the quotient $F/F_+^{\cdot n}$ such that the quotient morphism $F \longrightarrow F/F_+^{\cdot n}$ is a monad morphism from $\mathcal{F}$ to $\mathcal{F}/F_+^{\cdot n} = (F/F_+^{\cdot n}, \mu_n)$ and the category $\mathcal{T}_{\mathcal{F}_+}^{(n)}$ is equivalent to the category $\mathcal{F}/F_+^{\cdot n}$-modules. Clearly, $\mathcal{F}/F_+^{\cdot n}$ is an augmented monad: $F/F_+^{\cdot n} \simeq Id_{C_X} \oplus F_+/F_+^{\cdot n}$.

It follows from the preceding discussion that $F_+^{\cdot(n-1)}/F_+^{\cdot n} \hookrightarrow \Omega_{\mathcal{F}/F_+^{\cdot n}} \hookrightarrow F_+/F_+^{\cdot n}$. In particular, $\Omega_{\mathcal{F}/F_+^{\cdot 2}} = \mathcal{F}_+/F_+^{\cdot 2}$.

C5.7.2. Free actions. Let C_X be a k-linear category with the exact structure $\mathcal{E}^{spl}$; and let $\mathfrak{L}$ be a k-linear endofunctor on C_X. Consider the category $\mathfrak{L} - \mathfrak{act}$ whose objects are pairs (M, ξ), where $M \in ObC_X$ and ξ is a morphism $\mathfrak{L}(M) \longrightarrow M$. Morphisms between actions are defined in a standard way. We endow $\mathfrak{L} - \mathfrak{act}$ with the exact structure induced by the forgetful functor $\mathfrak{L} - \mathfrak{act} \xrightarrow{f_*} C_X$. If C_X has countable coproducts and the functor $\mathfrak{L}$ preserves countable coproducts, then the category $\mathfrak{L} - \mathfrak{act}$ is isomorphic to $\mathbb{T}(\mathfrak{L}) - mod$, where $\mathbb{T}(\mathfrak{L}) = (T(\mathfrak{L}), \mu)$ is a free monad generated by the endofunctor $\mathfrak{L}$; i.e. $T(\mathfrak{L}) = \oplus_{n \geq 0} \mathfrak{L}^n$ and μ is the multiplication defined by the identical morphisms $\mathfrak{L}^n \circ \mathfrak{L}^m \longrightarrow \mathfrak{L}^{n+m}$, $n, m \geq 0$.

The category C_X is isomorphic to the full subcategory $\mathcal{T}_{\mathfrak{L}}$ of $\mathfrak{L} - \mathfrak{act}$ generated by zero actions. The n-th infinitesimal neighborhood of $\mathcal{T}_{\mathfrak{L}}$ is the full subcategory $\mathcal{T}_{\mathfrak{L}}^{(n)}$ of $\mathfrak{L} - \mathfrak{act}$ generated by all actions (M, ξ) such that the n-th iteration $\mathfrak{L}^n(M) \xrightarrow{\xi_n} M$ of the action ξ is zero. The category $\mathcal{T}_{\mathfrak{L}}^{(n+1)}$ is equivalent to the category $\mathbb{T}_{\mathfrak{L},n} - mod$ of modules over the monad $\mathbb{T}_{\mathfrak{L},n} = (T_{\mathfrak{L},n}, \mu_n)$, where $T_{\mathfrak{L},n} = \bigoplus_{0 \leq m \leq n} \mathfrak{L}^m$ and the multiplication defined by morphisms $\mathfrak{L}^k \circ \mathfrak{L}^m \longrightarrow \mathfrak{L}^{k+m}$, $0 \leq k, m \leq n$, which are identical if $k + m < n$ and zero otherwise.

It follows from C5.7.1 that $\mathfrak{L}^n \hookrightarrow \Omega_{\mathbb{T}_{\mathfrak{L},n}} \hookrightarrow T^+_{\mathfrak{L},n} \stackrel{\text{def}}{=} \bigoplus_{1 \leq m \leq n} \mathfrak{L}^m$.

In particular, $\Omega_{\mathbb{T}_{\mathfrak{L},2}} = \mathfrak{L}$. Here $\mathfrak{L}$ denotes the functor $\mathfrak{L} - \mathfrak{act} \longrightarrow \mathfrak{L} - \mathfrak{act}$ which assigns to an object (M, ξ) the object $(\mathfrak{L}(M), \mathfrak{L}(\xi))$ and acts on morphisms accordingly.

C5.7.2.1. Projective objects and injective objects of an infinitesimal neighborhood. Projective objects of the category $\mathcal{T}_{\mathfrak{L}}^{(n+1)} = \mathbb{T}_{\mathfrak{L},n} - mod$ are retracts of relatively free objects. The latter are $\mathbb{T}_{\mathfrak{L},n}$-modules of the form $\mathbb{T}_{\mathfrak{L},n}(V)$, $V \in ObC_X$.

Suppose that $\mathfrak{L}$ has a right adjoint functor, $\mathfrak{L}_*$. Then the functor $T_{\mathfrak{L},n} = \bigoplus_{0 \leq m \leq n} \mathfrak{L}^m$ has a right adjoint equal to $T^!_{\mathfrak{L},n} = \bigoplus_{0 \leq m \leq n} \mathfrak{L}^m_*$; that is $\mathbb{T}_{\mathfrak{L},n}$ is a continuous monad. Therefore, by G1.4, the injective objects of $\mathbb{T}_{\mathfrak{L},n} - mod$ are retracts of $\mathbb{T}_{\mathfrak{L},n}$-modules of the form $\mathbb{T}^!_{\mathfrak{L},n}(V) = (T^!_{\mathfrak{L},n}(V), \gamma_n(V))$, $V \in ObC_X$.

C5.7.2.2. Proposition. *Suppose that $\mathfrak{L}$ is an auto-equivalence of the category C_X. Then $\mathcal{T}_{\mathfrak{L}}^{(n+1)} = \mathbb{T}_{\mathfrak{L},n} - mod$ is a Frobenius category.*

Proof. It suffices to show that $\mathbb{T}_{\mathfrak{L},n}$ is a Frobenious monad. An adjunction arrow $\mathfrak{L} \circ \mathfrak{L}_* \longrightarrow Id_{C_X}$ induces a canonical morphism from $\mathbb{T}_{\mathfrak{L},n}(\mathfrak{L}^n_*(V))$ to the injective object $\mathbb{T}^!_{\mathfrak{L},n}(V)$. If $\mathfrak{L}$ is an auto-equivalence, then this canonical morphism is an isomorphism. ■

C5.7.3. Example. Let C_X be the product of $\mathbb{Z}$ copies of a k-linear category C_Y; i.e. objects of C_X are sequences $M = (M_i|\ i \in \mathbb{Z})$ of objects of C_Y. Let $\mathfrak{L}$ be the translation functor: $\mathfrak{L}(M)_i = M_{i-1}$. Objects of the category $\mathfrak{L} - \mathfrak{act}$ of $\mathfrak{L}$-actions are arbitrary sequences of arrows $(\ldots \xrightarrow{d_{n+1}} M_{n+1} \xrightarrow{d_n} M_n \xrightarrow{d_{n-1}} \ldots)$. Objects of the subcategory $\mathcal{T}_{\mathfrak{L}}^{(n)}$ are sequences such that the composition of any n consecutive arrows is zero. In particular, $\mathcal{T}_{\mathfrak{L}}^{(2)}$ coincides with the category of complexes on C_Y and its subcategory $\mathcal{T}_{\mathfrak{L}} = \mathcal{T}_{\mathfrak{L}}^{(1)}$ is the category of complexes with zero differential. By C5.7.2.2, $\mathcal{T}_{\mathfrak{L}}^{(n)}$ is a Frobenious category for every n. Therefore, its costable category is triangulated. Notice that the costable category of $\mathcal{T}_{\mathfrak{L}}^{(2)}$ coincides with the homotopy category of unbounded complexes.

Chapter VII

A Sketch of a More General Theory

The purpose here is to extend basic notions and constructions of homological algebra to arbitrary right and left exact categories. This means that we do not require the existence of initial or final objects in our categories, including the categories, in which homological functors take their values.

We start, in Section 1, with a natural definition of kernels of arrows of an arbitrary category and show that the main properties of kernels summarized in Chapter I survive the generalization. In order to acquire flexibility, we introduce in Section 2 the notion of a *virtual* kernel and argue that the existence of morphisms with non-trivial virtual kernels imposes very precise choice of categories: they should be *virtually semi-complete*, which means, by definition, that each connected component has *pointed objects*. We make an application of properties of kernels extending the notion of a fully exact subcategory to arbitrary right (or left) exact categories. In Section 3, we introduce ∂^*-functors from a right exact category to an arbitrary category. We define *universal* ∂^*-functors (otherwise called *right derived* functors of their zero component) in a standard way (that is by a universal property) and constructively prove their existence (i.e. write a formula) in the case when the target category has pull-backs and limits of filtered diagrams. By duality, we obtain ∂-functors from a left exact category to an arbitrary category and *universal* ∂-functors, otherwise called *left derived* functors. In Section 4, following the pattern of Chapter II, we establish the existence of the universal left derived functor from a given virtually semi-complete left exact category to *semi-complete* categories. Following the scenario of Section 4 of Chapter III, we define, in Section 5, the *stable* category of the category of presheaves of sets associated with a virtually semi-complete left exact category. In Section 6, we define prestable and stable categories of a left (or right) exact category. Section 7 gives a brief account on 'exactness' properties of derived functors and the fact that, under certain condition on the target right exact category, 'exact' ∂^*-functors are universal. In Section 8, we discuss shortly homology of 'spaces' with coefficients in arbitrary right exact category. In Section 9, we apply our machinery to define the "absolute" higher K-theory of arbitrary right exact 'spaces', which gives rise to absolute K-theories of arbitrary left exact categories over the left exact category of right exact 'spaces'.

1. Kernels of arrows.

1.0. Definition. Let $\mathcal{M} \xrightarrow{f} \mathcal{L}$ be a morphism of a svelte category C_X and $C_X/\mathcal{L}^+$ the disjoint union of the category $C_X/\mathcal{L}$ and the "point" – the category with one morphism. We denote by $\mathfrak{D}_f$ the functor $C_X/\mathcal{L}^+ \longrightarrow C_X/\mathcal{L}$, which is identical on $C_X/\mathcal{L}$ and maps the "point" to the object $(\mathcal{M}, \mathcal{M} \xrightarrow{f} \mathcal{L})$. We denote the limit of the functor $C_X/\mathcal{L}^+ \xrightarrow{\mathfrak{D}_f} C_X/\mathcal{L}$, (if any) by $(Ker(f), \xi_f)$ and the canonical morphism $(Ker(f), \xi_f) \longrightarrow (\mathcal{M}, f)$ by $\mathfrak{k}(f)$. We call the object $(Ker(f), Ker(f) \xrightarrow{\mathfrak{k}(f)} \mathcal{M})$ of the category $C_X/\mathcal{M}$ the *kernel* of the morphism $\mathcal{M} \xrightarrow{f} \mathcal{L}$.

1.1. Kernels of arrows in categories with pull-backs. Let C_X be a svelte category with pull-backs. For every morphism $\mathcal{M} \xrightarrow{f} \mathcal{L}$ of C_X, let

$$C_X/\mathcal{L} \xrightarrow{\mathfrak{P}_f} C_X/\mathcal{M}$$

be a functor which assigns to each object $(\mathcal{N}, \mathcal{N} \xrightarrow{\xi} \mathcal{L})$ of the category $C_X/\mathcal{L}$ the object $(\mathcal{M}_{\xi,f}, \mathcal{M}_{\xi,f} \xrightarrow{f_\xi} \mathcal{M})$, where $\mathcal{M}_{\xi,f} \xrightarrow{\xi'_f} \mathcal{M}$ is the pull-back of the morphism $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$ along $\mathcal{M} \xrightarrow{f} \mathcal{L}$. The action of $\mathfrak{P}_f$ on morphisms is natural. The kernel $(Ker(f), Ker(f) \xrightarrow{\mathfrak{k}(f)} \mathcal{M})$ is the limit of the functor $\mathfrak{P}_f$. It follows that if the category C_X has (pull-backs and) limits of filtered diagrams, then every morphism of C_X has a kernel.

1.2. Note. It follows from 1.1 that, if C_X is a category with initial objects, then the notion of the kernel of a morphism coincides with the one introduced in I.4.1.

1.3. Proposition. *Let*

$$\begin{array}{ccc} M_{g,f} & \xrightarrow{f_g} & N_g \\ g'_f \downarrow & cart & \downarrow g \\ M & \xrightarrow{f} & N \end{array} \qquad (1)$$

be a cartesian square. Then $Ker(f)$ *exists iff* $Ker(f_g)$ *exists, and they are naturally isomorphic to each other.*

Proof. It follows from the observation 1.2(a) that it suffices to establish the fact for a category C_X with pull-backs and limits of filtered diagrams. In this case, every morphism of C_X has a kernel. It follows from the definition of the kernel that the commutative diagram (1) yields a canonical morphism $Ker(f) \xrightarrow{\phi} Ker(f_g)$. On the other hand, for any morphism $\mathcal{L} \xrightarrow{\xi} N$, there is a diagram

$$\begin{array}{ccccc} M_{g,f} & \xrightarrow{f_g} & N_g & \xleftarrow{\xi_g} & \mathcal{L}_\xi \\ g'_f \downarrow & cart & \downarrow g & cart & \downarrow g_\xi \\ M & \xrightarrow{f} & N & \xleftarrow{\xi} & \mathcal{L} \end{array} \qquad (2)$$

formed by cartesian squares, which gives rise to a morphism $Ker(f_g) \xrightarrow{\psi} Ker(f)$. It follows from the universal property of limits that the morphism ψ is inverse to ϕ. ■

1.3.1. Proposition. *Let* $M \xrightarrow{f} N$ *be a morphism of a category* C_X, *which has a kernel pair,* $Ker_2(f) = M \times_N M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M$. *Then the morphism* $M \xrightarrow{f} N$ *has a kernel iff the projection* $Ker_2(f) \xrightarrow{p_1} M$ *has a kernel; and there is a natural isomorphism* $Ker(f) \xrightarrow{\sim} Ker(p_1)$.

Proof. The fact follows from 1.3. ■

1.4. Proposition. *Let* $L \xrightarrow{f} M$ *and* $M \xrightarrow{g} N$ *be morphisms of a category* C_X. *Suppose that there exists the kernel* $(Ker(g), Ker(g) \xrightarrow{\mathfrak{k}(g)} M)$ *of the morphism* $M \xrightarrow{g} N$. *Then the kernel of the composition* $L \xrightarrow{g\circ f} N$ *(if any) is the pull-back of the canonical morphism* $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ *along* $L \xrightarrow{f} M$.

Proof. Consider a cone $\mathfrak{L} \xrightarrow{\widetilde{\lambda}} \mathfrak{D}_{g\circ f}$, where $C_X/N^+ \xrightarrow{\mathfrak{D}_{g\circ f}} C_X/N$ is the functor associated with the morphism $L \xrightarrow{g\circ f} L$ (see 1.0). This cone can be written as commutative diagrams

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\widetilde{\lambda}(\xi)} & N_\xi \\ {\scriptstyle \widetilde{\lambda}^0_{g\circ f}}\downarrow & & \downarrow{\scriptstyle \xi} \\ L & \xrightarrow{g\circ f} & N \end{array} \qquad (1)$$

where (N_ξ, ξ) runs through the objects of the category C_X/N.

Since there exists the kernel $(Ker(g), \mathfrak{k}(g))$ of the morphism $M \xrightarrow{g} N$, the diagram (1) uniquely decomposes into the diagram

$$\begin{array}{ccccc} \mathfrak{L} & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{\lambda_\xi} & N_\xi \\ {\scriptstyle \widetilde{\lambda}^0_{g\circ f}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}(g)} & & \downarrow{\scriptstyle \xi} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

with commutative squares. This shows that the kernel of the composition $L \xrightarrow{g\circ f} N$ is the pull-back of $Ker(g) \xrightarrow{\mathfrak{k}(g)} N$ along $L \xrightarrow{f} M$. In particular, the existence of this pull-back is equivalent to the existence of the kernel of $g \circ f$. ■

1.4.1. Corollary. *Let* $L \xrightarrow{f} M$ *and* $M \xrightarrow{g} N$ *be morphisms such that there exist kernels of* g *and* $g \circ f$. *Then* $Ker(f)$ *exists iff there exists the kernel of the canonical morphism* $Ker(g \circ f) \xrightarrow{\widetilde{f}} Ker(g)$ *and both kernels are isomorphic to each other.*

Proof. By 1.4, the square

$$\begin{array}{ccc} Ker(gf) & \xrightarrow{\ \widetilde{f}\ } & Ker(g) \\ \mathfrak{k}(gf)\Big\downarrow & cart & \Big\downarrow \mathfrak{k}(g) \\ L & \xrightarrow{\ f\ } & M \end{array}$$

is cartesian. Therefore, by 1.3, the $Ker(f)$ is naturally isomorphic to $Ker(\widetilde{f})$. ∎

1.5. The coimage of a morphism. Let $M \xrightarrow{f} N$ be an arrow which has a kernel. It follows from the definition of the kernel, that there is a uniquely defined commutative square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\ \mathfrak{k}(f)\ } & M \\ 0_f\Big\downarrow & & \Big\downarrow f \\ M & \xrightarrow{\ f\ } & N \end{array}$$

or, what is the same, a pair of arrows $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M$, which is equalized by the morphism $M \xrightarrow{f} N$. If the cokernel of this pair of arrows exists, it will be called the *coimage of* f and denoted by $Coim(f)$, or. loosely, $M/Ker(f)$.

Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. Then f is the composition of the canonical strict epimorphism $M \xrightarrow{p_f} Coim(f)$ and a uniquely defined morphism $Coim(f) \xrightarrow{j_f} N$.

1.5.1. Note. If the category C_X has an initial object, then the notion of the coimage of a morphism coincides with the one introduced in I.4.5.

1.6. Trivial morphisms, trivial objects, trivial kernels.

1.6.1. Definitions. (a) We call a morphism $M \xrightarrow{f} N$ *trivial*, if $Ker(f)$ exists and the canonical morphism $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a split epimorphism.

(b) We call an object M *trivial*, if the identical morphism $M \xrightarrow{id_M} M$ is trivial.

(c) We say that a morphism $M \xrightarrow{f} N$ of the category C_X has *trivial* kernel, if $(Ker(f), Ker(f) \xrightarrow{\mathfrak{k}(f)} M)$ is an initial object of the category C_X/M.

1.6.2. Remarks. (i) If the category C_X has an initial object x, then a morphism $M \xrightarrow{f} N$ of C_X is trivial iff it factors through the unique morphism $x \longrightarrow N$.

(ii) This implies, in particular, that if the category C_X has initial objects, then an object of C_X is trivial iff it is initial.

(iii) If $M \xrightarrow{f} N$ is a morphism with a trivial kernel, then

(a) the canonical morphism $M \xrightarrow{\mathfrak{p}_f} Coim(f)$ is an isomorphism;

(b) the object $Ker(f)$ is trivial.

Conversely, if the object $Ker(f)$ is trivial, then $M \xrightarrow{f} N$ is a morphism with a trivial kernel. So that the definitions 1.6.1 do not create any ambiguity: a morphism with a trivial kernel is a morphism whose kernel is a trivial object.

1.6.3. Proposition. *(a) For every morphism $L \xrightarrow{\mathfrak{f}} M$ having the kernel, the composition of $Ker(\mathfrak{f}) \xrightarrow{\mathfrak{k}(\mathfrak{f})} L$ and $L \xrightarrow{\mathfrak{f}} M$ is a trivial morphism, as well as the morphism $Ker(\mathfrak{f}) \xrightarrow{0_{\mathfrak{f}}} L$.*

(b) Composition of two morphisms is trivial if one of the morphisms is trivial.

(c) If $L \xrightarrow{f} M$ is a trivial morphism, then it equalizes any pair of arrows from M to any object.

(d) If there is a trivial morphism $M \xrightarrow{g} N$, then the converse is true. More precisely, a morphism $L \xrightarrow{f} M$ is trivial, if it equalizes any pair of arrows from M to M.

Proof. (a1) It follows from 1.4 that the kernel of the composition of $Ker(\mathfrak{f}) \xrightarrow{\mathfrak{k}(\mathfrak{f})} L$ and $L \xrightarrow{\mathfrak{f}} M$ is naturally isomorphic to the kernel pair of the morphism $Ker(\mathfrak{f}) \xrightarrow{\mathfrak{k}(\mathfrak{f})} L$, which is illustrated by the diagram

$$\begin{array}{ccccc} Ker(\mathfrak{f}\circ\mathfrak{k}(\mathfrak{f})) & \longrightarrow & Ker(\mathfrak{f}) & & \\ \downarrow & cart & \downarrow & & \\ Ker(\mathfrak{f}) & \xrightarrow{\mathfrak{k}(\mathfrak{f})} & L & \xrightarrow{\mathfrak{f}} & M \end{array}$$

with a cartesian square. So that the canonical projections $Ker(\mathfrak{f}\circ\mathfrak{k}(\mathfrak{f})) \longrightarrow Ker(\mathfrak{f})$ are split epimorphisms.

(a2) The triviality of $Ker(\mathfrak{f}) \xrightarrow{0_{\mathfrak{f}}} L$.

For every $L_\xi \xrightarrow{\xi} L$, consider the cartesian square

$$\begin{array}{ccc} \mathfrak{M}_\xi & \longrightarrow & L_\xi \\ {\scriptstyle \mathfrak{p}_\xi}\downarrow & cart & \downarrow{\scriptstyle \xi} \\ Ker(\mathfrak{f}) & \xrightarrow{0_{\mathfrak{f}}} & L \end{array}$$

By definition of $Ker(\mathfrak{f})$, there exists a canonical morphism $Ker(\mathfrak{f}) \xrightarrow{\mathfrak{k}_\xi} L_\xi$ such that $0_{\mathfrak{f}} = \xi \circ \mathfrak{k}_\xi$, which depends functorially on ξ. This morphism determines a splitting

$Ker(\mathfrak{f}) \xrightarrow{\mathfrak{j}_\xi} \mathfrak{M}_\xi$ of the projection $\mathfrak{M}_\xi \xrightarrow{\mathfrak{p}_\xi} Ker(\mathfrak{f})$ also functorial in ξ. These splittings determine the morphism $Ker(\mathfrak{f}) \xrightarrow{\mathfrak{j}} \lim_\xi \mathfrak{M}_\xi$ which splits the canonical morphism $\lim_\xi \mathfrak{M}_\xi \xrightarrow{\mathfrak{p}} Ker(\mathfrak{f})$. By 1.1, $\lim_\xi \mathfrak{M}_\xi \simeq Ker(0_\mathfrak{f})$.

(b) Let $L \xrightarrow{g} M$ and $M \xrightarrow{f} N$ be morphisms. Then, by 1.4, we have a diagram

$$\begin{array}{ccccccc} Ker(\widetilde{g}) & \xrightarrow{\mathfrak{k}(\widetilde{g})} & Ker(f \circ g) & \xrightarrow{\widetilde{g}} & Ker(f) & & \\ \phi \downarrow \wr & & \mathfrak{k}(f \circ g) \downarrow & cart & \downarrow \mathfrak{k}(f) & & \\ Ker(g) & \xrightarrow{\mathfrak{k}(g)} & L & \xrightarrow{g} & M & \xrightarrow{f} & N \end{array} \qquad (1)$$

whose right square is cartesian and, by 1.3, left vertical arrow is an isomorphism.

(b1) If the morphism $M \xrightarrow{f} N$ is trivial, i.e. the right vertical arrow of the diagram (1) is a split epimorphism, then its pull-back – the morphism

$$Ker(f \circ g) \xrightarrow{\mathfrak{k}(f \circ g)} L$$

is a split monomorphism, which means that $L \xrightarrow{f \circ g} N$ is a trivial morphism.

(b2) Suppose the morphism $L \xrightarrow{g} M$ is trivial; i.e. that is a splitting $L \xrightarrow{\mathfrak{j}} Ker(g)$ of the morphism $Ker(g) \xrightarrow{\mathfrak{k}(g)} L$. Then the composition $L \xrightarrow{\phi^{-1}\mathfrak{j}} Ker(f \circ g)$ is a splitting of the morphism $Ker(f \circ g) \xrightarrow{\mathfrak{k}(f \circ g)} L$.

(c1) For any morphism $L \xrightarrow{\mathfrak{f}} M$, the canonical morphism $Ker(f) \xrightarrow{0_\mathfrak{f}} L$ equalizes any pair of arrows from L to another object.

(c) Let $L \xrightarrow{f} M$ be a trivial morphism; i.e. there is a morphism $L \xrightarrow{\iota} Ker(f)$ whose composition with the canonical morphism $Ker(f) \xrightarrow{\mathfrak{k}(f)} L$ is the identical morphism $L \longrightarrow L$. Therefore, $f = f \circ (\mathfrak{k}(f) \circ \iota) = f \circ 0_f \circ \iota$. Since the morphism $Ker(f) \xrightarrow{0_f} L$ equalizes any pair of arrows from L to the same target, same holds for the morphism $L \xrightarrow{f} M$.

(d) Let there exist a trivial morphism $M \xrightarrow{g} N$; that is the canonical morphism $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ is a strict epimorphism split by some $M \xrightarrow{\iota_g} Ker(g)$. By hypothesis, the morphism $L \xrightarrow{f} M$ equalizes the pair of arrows $(id_M, 0_g \circ \iota_g)$; that is $f = 0_g \circ (\iota_g \circ f)$. Since, by (a), the morphism 0_g is trivial, it follows from (b) that the morphism $L \xrightarrow{f} M$ is trivial. ∎

1.6.3.1. Corollary. *A morphism $M \xrightarrow{f} N$ is trivial iff it factors through any arrow $L \xrightarrow{\xi} N$. In particular, an object M is trivial iff any morphism to M is a split epimorphism.*

Proof. (i) In fact, it follows from the definition of the kernel of an arrow that there exists a morphism $Ker(f) \xrightarrow{f(\xi)} L$ such that the diagram

$$\begin{array}{ccc} Ker(f) & \xrightarrow{f(\xi)} & L \\ {\scriptstyle \mathfrak{k}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \xi} \\ M & \xrightarrow{f} & N \end{array}$$

commutes. So that if $M \xrightarrow{f} N$ is trivial, i.e. there exists a splitting $Ker(f) \xrightarrow{\mathfrak{j}} Ker(f)$ of the morphism $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$, then $f = f \circ \mathfrak{k}(f) \circ \mathfrak{j} = \xi \circ (f(\xi) \circ \mathfrak{j})$.

(ii) Conversely, suppose that $M \xrightarrow{f} N$ factors through any morphism to N. Take any morphism $\widehat{L} \xrightarrow{\zeta} Ker(\widehat{f})$ and denote by $L \xrightarrow{\xi} N$ the morphism such that $\widehat{\xi} = \widehat{f} \circ \mathfrak{k}(\widehat{f}) \circ \zeta$. By hypothesis, $M \xrightarrow{f} N$ factors through $L \xrightarrow{\xi} N$, which implies that $\widehat{M} \xrightarrow{\widehat{f}} \widehat{N}$ factors through $\widehat{f} \circ \mathfrak{k}(\widehat{f})$ and, by 1.6.3(a), the latter is a trivial morphism. Therefore, by 1.6.3(b), the morphism $\widehat{M} \xrightarrow{\widehat{f}} \widehat{N}$ is trivial, which is equivalent to the triviality of $M \xrightarrow{f} N$. ∎

1.6.4. Proposition. *The following conditions on a pair of arrows* $L \xrightarrow{f} M \xrightarrow{g} N$ *are equivalent:*

(i) the composition $L \xrightarrow{g \circ f} N$ *is a trivial morphism;*

(ii) the morphism $L \xrightarrow{f} M$ *factors through* $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$.

Proof. Consider the diagram

$$\begin{array}{ccccc} Ker(g \circ f) & \xrightarrow{\widetilde{g}} & Ker(g) & & \\ {\scriptstyle \mathfrak{k}(g \circ f)}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{k}(g)} & & \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array} \qquad (1)$$

whose square is cartesian by 1.4. If the composition $g \circ f$ is trivial, then the left vertical arrow has a splitting $L \xrightarrow{\sigma} Ker(g \circ f)$. So that $f = f \circ (\mathfrak{k}(g \circ f) \circ \sigma) = \mathfrak{k}(g) \circ (\widetilde{g} \circ \sigma)$.

Conversely, if $f = \mathfrak{k}(g) \circ \gamma$ for some morphism γ, then $g \circ f = (g \circ \mathfrak{k}(g)) \circ \gamma$; and, by 1.6.3(a), the composition $g \circ \mathfrak{k}(g)$ is a trivial morphism. Therefore, by 1.6.3(b), the morphism $g \circ f$ is trivial. ∎

1.6.5. Proposition. *(a) The kernel of a monomorphism is trivial.*

(b) Let $M \xrightarrow{g} N$ *be a morphism with a trivial kernel. Then a morphism* $L \xrightarrow{f} M$ *has a kernel iff the composition* $g \circ f$ *has a kernel, and these two kernels are naturally isomorphic one to another.*

(c) Let

$$\begin{array}{ccc} L & \xrightarrow{\ f\ } & M \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \widetilde{M} & \xrightarrow{\ \phi\ } & N \end{array}$$

be a commutative square such that the kernels of the arrows f and ϕ exist and the kernel of g is trivial. Then the kernel of the composition $\phi \circ \gamma$ is isomorphic to the kernel of the morphism f, and the left square of the commutative diagram

$$\begin{array}{ccccccc} Ker(f) & \xrightarrow{\ \sim\ } & Ker(\phi\gamma) & \xrightarrow{\ \mathfrak{k}(f)\ } & L & \xrightarrow{\ f\ } & M \\ & & {\scriptstyle\widetilde{\gamma}}\big\downarrow & cart & {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ & & Ker(\phi) & \xrightarrow{\ \mathfrak{k}(\phi)\ } & \widetilde{M} & \xrightarrow{\ \phi\ } & N \end{array}$$

is cartesian.

Proof. (a) It follows from the definition of the kernel that, if $\mathcal{M} \xrightarrow{f} \mathcal{L}$ is a monomorphism, then $Ker(f)$ is the limit of the forgetful functor $C_X/M \longrightarrow C_X$, which means precisely that $(Ker(f), \mathfrak{k}(f))$ is the initial object of the category C_X/M.

(b) It follows from 1.4 (see also 1.4.1) that the square in the diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\ \widetilde{f}\ } & Ker(g) & & \\ {\scriptstyle \mathfrak{k}(gf)}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{k}(g)} & & \\ L & \xrightarrow{\ f\ } & M & \xrightarrow{\ g\ } & N \end{array}$$

is cartesian. By hypothesis, the kernel of of the morphism $M \xrightarrow{g} N$ is trivial, that is $(Ker(g), \mathfrak{k}(g))$ is an initial object of the category C_X/M. This implies that $Ker(g \circ f)$ is isomorphic to $Ker(f)$ (see 1.1).

(c) Since the kernel of $M \xrightarrow{g} N$ is trivial, it follows from (a) that $Ker(f)$ is naturally isomorphic to the kernel $g \circ f = \phi \circ \gamma$. ■

2. Virtual kernels and virtually (semi-)complete categories.

Let C_X be a svelte category and $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ a morphism of C_X. By definition, the kernel of the morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ (if any) is the pair $(Ker(\mathfrak{f}), Ker(\mathfrak{f}) \xrightarrow{\mathfrak{k}(\mathfrak{f})} \mathcal{M})$, such that the object $(Ker(\mathfrak{f}), \mathfrak{f} \circ \mathfrak{k}(\mathfrak{f}))$ of the category $C_X/\mathcal{L}$ is the limit of the diagram $C_X/\mathcal{L}^+ \xrightarrow{\ \mathfrak{D}_\mathfrak{f}\ } C_X/\mathcal{L}$ defined in 1.0.

2.1. Virtual kernels. We define the *virtual kernel* of a morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ of the category C_X as the canonical morphism of the limit $\lim(h_{X/\mathcal{L}} \circ \mathfrak{D}_{\mathfrak{f}})$ of the composition of the diagram $C_X/\mathcal{L}^+ \xrightarrow{\mathfrak{D}_{\mathfrak{f}}} C_X/\mathcal{L}$ with the Yoneda embedding

$$C_X/\mathcal{L} \xrightarrow{h_{X/\mathcal{L}}} C_X^\wedge/\widehat{\mathcal{L}} = (C_X/\mathcal{L})^\wedge$$

to the object $(\widehat{\mathcal{M}}, \widehat{\mathcal{M}} \xrightarrow{\widehat{\mathfrak{f}}} \widehat{\mathcal{L}})$ of the category $C_X^\wedge/\widehat{\mathcal{L}}$. Thus, the virtual kernel can be viewed as a pair $(Ker_{\mathfrak{v}}(\mathfrak{f}), Ker_{\mathfrak{v}}(\mathfrak{f}) \xrightarrow{\mathfrak{k}_{\mathfrak{v}}(\mathfrak{f})} \widehat{\mathcal{M}})$.

2.1.1. The virtual kernels and the existence of kernels. Since the category $C_X^\wedge$ of presheaves of sets has all limits, the virtual kernel of any morphism exists. The kernel of the morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ exists iff the presheaf of sets $Ker_{\mathfrak{v}}(\mathfrak{f})$ is representable.

2.2. The (non-)triviality of virtual kernels. Let $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$ be an arbitrary morphism and

$$\begin{array}{ccc} \mathfrak{M}_{\xi,\mathfrak{f}} & \xrightarrow{\mathfrak{f}_\xi} & \widehat{\mathcal{L}}_\xi \\ \xi'_{\mathfrak{f}} \downarrow & cart & \downarrow \widehat{\xi} \\ \widehat{\mathcal{M}} & \xrightarrow{\widehat{\mathfrak{f}}} & \widehat{\mathcal{L}} \end{array}$$

a cartesian square in the category of presheaves of sets $C_X^\wedge$.

(a) If the presheaf $\mathfrak{M}_{\xi,\mathfrak{f}}$ is the trivial presheaf $\bar{\emptyset}$ (– the initial object of the category $C_X^\wedge$), then the virtual kernel $Ker_{\mathfrak{v}}(\mathfrak{f})$ of the morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ is trivial too.

(b) Suppose that the pull-back of the morphism $\widehat{\mathcal{M}} \xrightarrow{\widehat{\mathfrak{f}}} \widehat{\mathcal{L}}$ along any representable morphism $\widehat{\mathcal{L}}_\xi \xrightarrow{\widehat{\xi}} \widehat{\mathcal{L}}$ is non-trivial (that is $\mathfrak{M}_{\xi,\mathfrak{f}}$ is a non-trivial presheaf).

This means that, for every morphism $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$, there exists a commutative diagram

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\mathfrak{t}} & \mathcal{L}_\xi \\ \xi' \downarrow & & \downarrow \xi \\ \mathcal{M} & \xrightarrow{\mathfrak{f}} & \mathcal{L} \end{array}$$

If this condition holds, then the virtual kernel $Ker_{\mathfrak{v}}(\mathfrak{f})$ is determined by the cartesian square

$$\begin{array}{ccc} Ker_{\mathfrak{v}}(\mathfrak{f}) & \xrightarrow{\mathfrak{f}_\xi} & \lim h_{X/\mathcal{L}} \\ \mathfrak{k}_{\mathfrak{v}}(\mathfrak{f}) \downarrow & cart & \downarrow \mathfrak{j}_{\mathcal{L}} \\ \widehat{\mathcal{M}} & \xrightarrow{\widehat{\mathfrak{f}}} & \widehat{\mathcal{L}} \end{array} \qquad (1)$$

(b1) If $\lim h_{X/\mathcal{L}}$ is the trivial presheaf, then, of course, $Ker_{\mathfrak{v}}(\mathfrak{f})$ is trivial too, because the only morphism to $\overline{\emptyset}$ is the identical isomorphism.

(b2) If $\lim h_{X/\mathcal{L}}$ is a non-trivial, then $Ker_{\mathfrak{v}}(\mathfrak{f})$ is non-trivial too.

This follows from the fact that the canonical morphism $\lim h_{X/\mathcal{L}} \xrightarrow{\mathfrak{j}_{\mathcal{L}}} \mathcal{L}$ factors through $\widehat{\mathcal{M}} \xrightarrow{\widehat{\mathfrak{f}}} \widehat{\mathcal{L}}$. Therefore, by the universal property of cartesian squares, the projection $Ker_{\mathfrak{v}}(\mathfrak{f}) \xrightarrow{\mathfrak{f}_{\mathfrak{E}}} \lim h_{X/\mathcal{L}}$ splits. In particular, there is a morphism from a non-trivial presheaf $\lim h_{X/\mathcal{L}}$ to $Ker_{\mathfrak{v}}(\mathfrak{f})$, which implies that $Ker_{\mathfrak{v}}(\mathfrak{f})$ is non-trivial too.

2.2.1. Thus, the non-triviality of the virtual kernel of a morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ depends on non-triviality of the presheaf of sets $\lim h_{X/\mathcal{L}}$. In particular, it depends only on the target of the morphism $\mathfrak{f}$ – the object $\mathcal{L}$.

2.3. Virtually complete categories.

2.3.0. Virtually initial objects. For any svelte category C_X, we call the limit of the Yoneda embedding $C_X \xrightarrow{h_X} C_X^\wedge$ the *virtually initial object* of the category C_X.

2.3.0.1. Lemma. *A svelte category C_X has an initial object iff the presheaf* $\lim h_X$ *is representable.*

Proof. If the category C_X has an initial object x, then $\lim h_X \simeq \widehat{x} = C_X(-,x)$. Conversely, if the presheaf $\lim h_X$ is representable by an object $\mathfrak{x}$, then this object is an initial object of the category C_X. ■

2.3.0.2. The category $C_{X^{\circledast}}$. We denote by $C_{X^{\circledast}}$ the category $\lim h_X \backslash C_X^\wedge$ and by $C_X \xrightarrow{h_X^{\circledast}} C_{X^{\circledast}}$ the functor induced by the Yoneda embedding $C_X \xrightarrow{h_X} C_X^\wedge$.

(a) If the virtually initial object $\lim h_X$ is trivial, i.e. $\lim h_X = \emptyset_X$, then $C_{X^{\circledast}}$ coincides with the category $C_X^\wedge$ and, therefore, the functor $C_X \xrightarrow{h_X^{\circledast}} C_{X^{\circledast}}$ is the usual Yoneda embedding.

(b) It follows from (the argument of) 2.3.0.1 that if the category C_X has an initial object $\mathfrak{x}$, then $C_{X^{\circledast}} = \widehat{\mathfrak{x}} \backslash C_X^\wedge$. So that, in this case, the functor $C_X \xrightarrow{h_X^{\circledast}} C_{X^{\circledast}}$ coincides with the "reduced" Yoneda embedding introduced in I.2.0.2(b).

2.3.1. Definition. We call a svelte category C_X *virtually complete*, if its virtually initial object – the limit of the Yoneda embedding $C_X \xrightarrow{h_X} C_X^\wedge$, is a non-trivial presheaf.

2.3.1.1. It follows from 2.3.0.1 that every category with initial objects is virtually complete.

2.3.2. Proposition. *The following conditions on a svelte category C_X are equivalent:*

(a) The category C_X is virtually complete.

(b) The category C_X has pointed objects (that is objects $\mathcal{M}$ for which there exists a cone $\mathcal{M} \longrightarrow Id_{C_X}$).

Proof. (a)⇒(b). Non-triviality of $\lim h_X$ means that $\lim h_X(\mathcal{M}) \neq \emptyset$ for some object $\mathcal{M}$ of the category C_X, or, equivalently, there exists a morphism $\widehat{\mathcal{M}} \longrightarrow \lim h_X$. The composition of this morphism with the universal cone $\lim h_X \longrightarrow h_X$ is the image of a cone $\mathcal{M} \longrightarrow Id_{C_X}$.

(b)⇒(a). Each cone $\mathcal{M} \longrightarrow Id_{C_X}$ determines a cone $\widehat{\mathcal{M}} \longrightarrow h_X$. The colimit of all such cones is a universal cone $\lim h_X \longrightarrow h_X$. This shows, in particular, that the presheaf $\lim h_X$ is non-trivial. ∎

2.3.2.1. Corollary. *Every virtually complete category C_X is connected and quasi-filtered.*

Proof. (a) A virtually complete category C_X is connected, because, by 2.3.2, there exists a cone $\mathcal{M} \longrightarrow Id_{C_X}$.

(b) Recall that a category C_X is called *quasi-filtered* if any pair of arrows

$$L \longrightarrow M \longleftarrow L'$$

of C_X can be complemented to a commutative square

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & L' \\ \downarrow & & \downarrow \\ L & \longrightarrow & M \end{array}$$

(say, C_X has pull-backs, or each connected component of C_X has initial objects) and, for any pair of arrows $\mathcal{M} \rightrightarrows \mathcal{N}$, there is an equalizer $\mathcal{L} \longrightarrow \mathcal{M}$.

By I.4.6.2.3, a category C_X is virtually complete iff there exists a cone $\mathcal{V} \xrightarrow{\zeta} Id_{C_X}$ for some object $\mathcal{V}$ of C_X. In particular, we have a commutative square

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\zeta(L')} & L' \\ {\scriptstyle \zeta(L)}\downarrow & & \downarrow{\scriptstyle \beta} \\ L & \xrightarrow{\alpha} & M \end{array}$$

for any pair of arrows $L \xrightarrow{\alpha} M \xleftarrow{\beta} L'$. Also, the morphism $\mathcal{V} \xrightarrow{\zeta(\mathcal{M})} \mathcal{M}$ equalizes any pair of arrows $\mathcal{M} \rightrightarrows \mathcal{N}$. ∎

2.3.3. Proposition. *Let C_X be a svelte connected category. The following conditions are equivalent:*

(a) The category C_X is quasi-filtered and the category $C_X/\mathcal{L}$ is virtually complete for some object $\mathcal{L}$ of C_X.

(b) The category $C_X/\mathfrak{L}$ is virtually complete for any $\mathfrak{L} \in ObC_X$.
(c) The category C_X is virtually complete.

Proof. (c)⇒(b). By 2.3.2, a category C_X is virtually complete iff there exists a cone $\mathcal{M} \xrightarrow{\gamma} Id_{C_X}$ for some object $\mathcal{M}$ of the category C_X. This cone induces, for any $\mathfrak{L} \in ObC_X$, a cone $(\mathcal{M}, \mathcal{M} \xrightarrow{\gamma(\mathfrak{L})} \mathfrak{L}) \xrightarrow{\gamma_{\mathfrak{L}}} Id_{C_X/\mathfrak{L}}$, where $\gamma_{\mathfrak{L}}(\mathcal{N}, \mathcal{N} \xrightarrow{\xi} \mathfrak{L}) = \gamma(\mathcal{N})$.

(b)⇒(a). We need to show that the category C_X is quasi-filtered; that is any pair of arrows

$$L \longrightarrow M \longleftarrow L'$$

of C_X can be complemented to a commutative square.

In fact, any pair of arrows $L \longrightarrow M \longleftarrow L'$ can be viewed as a pair of morphisms of the category C_X/M to the final object (M, id_M). By condition (b), the category C_X/M is virtually complete. Therefore, by 2.3.2.1, it is quasi-filtered. So that there exists a commutative square

$$\begin{array}{ccc} \mathfrak{L} & \longrightarrow & L' \\ \big\downarrow & & \big\downarrow \\ L & \longrightarrow & M \end{array}$$

which shows that the category C_X is quasi-filtered.

(a)⇒(c). Suppose that the category $C_X/\mathcal{L}$ is virtually complete; that is, by 2.3.2, there exists a cone $(\mathfrak{v}, \mathfrak{v} \xrightarrow{\xi_{\mathcal{L}}} \mathcal{L}) \xrightarrow{\xi} Id_{C_X/\mathcal{L}}$. The claim is that, if the category C_X is quasi-filtered, then this cone determines a cone $\mathfrak{v} \xrightarrow{\xi} Id_{C_X}$.

Let $\mathcal{L} \xrightarrow{\gamma} \mathfrak{L}$ be a morphism. We define $\mathfrak{v} \xrightarrow{\xi_{\mathfrak{L}}} \mathfrak{L}$ as the composition $\gamma \circ \xi_{\mathcal{L}}$.

Notice that, thanks to the fact that, the category C_X is quasi-filtered, the composition $\xi_{\mathfrak{L}} = \gamma \circ \xi_{\mathcal{L}}$ does not depend on the choice of the morphism $\mathcal{L} \xrightarrow{\gamma} \mathfrak{L}$.

Indeed, if $\mathcal{L} \xrightarrow{\gamma_1} \mathfrak{L}$ is another morphism, then, since, by hypothesis, the category C_X is filtered, there is a commutative square

$$\begin{array}{ccc} \widetilde{\mathfrak{M}} & \xrightarrow{\lambda_1} & \mathcal{L} \\ {\scriptstyle \lambda_2}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} \\ \mathcal{L} & \xrightarrow{\gamma_1} & \mathfrak{L} \end{array}$$

and a morphism $\mathfrak{M} \xrightarrow{\beta} \widetilde{\mathfrak{M}}$ equalizing the pair of arrows $\widetilde{\mathfrak{M}} \overset{\lambda_1}{\underset{\lambda_2}{\rightrightarrows}} \mathcal{L}$. So that we have a commutative square

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\lambda} & \mathcal{L} \\ {\scriptstyle \lambda}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} \\ \mathcal{L} & \xrightarrow{\gamma_1} & \mathfrak{L} \end{array}$$

where $\lambda = \lambda_1 \circ \beta$. The cone $(\mathfrak{v}, \mathfrak{v} \xrightarrow{\xi_{\mathcal{L}}} \mathcal{L}) \xrightarrow{\ \ \xi\ \ } Id_{C_X/\mathcal{L}}$ yields a morphism

$$(\mathfrak{v}, \xi_{\mathcal{L}}) \xrightarrow{\ \xi(\mathfrak{M},\lambda)\ } (\mathfrak{M}, \lambda).$$

In other words, $\lambda \circ \xi(\mathfrak{M}, \lambda) = \xi_{\mathcal{L}}$. So that

$$\gamma_1 \circ \xi_{\mathcal{L}} = \gamma_1 \circ \lambda \circ \xi(\mathfrak{M}, \lambda) = \gamma \circ \lambda \circ \xi(\mathfrak{M}, \lambda) = \gamma_1 \circ \xi_{\mathcal{L}}.$$

Consider now a pair of arrows, $\mathcal{L} \xrightarrow{\gamma} \mathfrak{L} \xleftarrow{\mathfrak{t}} \mathfrak{L}_{\mathfrak{t}}$. Since the category C_X is quasi-filtered, there is a commutative square

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\ \ \beta\ \ } & \mathfrak{L}_{\mathfrak{t}} \\ {\scriptstyle \mathfrak{u}}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} \\ \mathcal{L} & \xrightarrow{\ \ \gamma\ \ } & \mathfrak{L} \end{array} \qquad (1)$$

We define a morphism $\mathfrak{v} \xrightarrow{\xi_{\mathfrak{L}_{\mathfrak{t}}}} \mathfrak{L}_{\mathfrak{t}}$ by $\xi_{\mathfrak{L}_{\mathfrak{t}}} = \beta \circ \xi_{\mathfrak{M}}$.

The morphism $\xi_{\mathfrak{L}_{\mathfrak{t}}}$ does not depend on the choice of the square (1).

In fact, let

$$\begin{array}{ccc} \mathfrak{N} & \xrightarrow{\ \ \beta_1\ \ } & \mathfrak{L}_{\mathfrak{t}} \\ {\scriptstyle \mathfrak{u}_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} \\ \mathcal{L} & \xrightarrow{\ \ \gamma\ \ } & \mathfrak{L} \end{array}$$

be another commutative square. Since the the category C_X is quasi-filtered, there is a commutative square

$$\begin{array}{ccc} \mathfrak{V} & \xrightarrow{\ \ \alpha\ \ } & \mathfrak{M} \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathfrak{N} & \xrightarrow{\ \ \beta_1\ \ } & \mathfrak{L}_{\mathfrak{t}} \end{array}$$

Therefore, $\beta_1 \circ \xi_{\mathfrak{N}} = \beta_1\alpha_1 \circ \xi_{\mathfrak{V}} = \beta\alpha \circ \xi_{\mathfrak{V}} = \beta \circ \xi_{\mathfrak{M}}$. ∎

2.3.4. Proposition. *(a) Let $\mathcal{L}$ be an object of a svelte category C_X. The following conditions are equivalent:*

(i) The virtual kernel of some morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ is non-trivial.

(ii) The virtual kernel of any morphism $\mathcal{M} \longrightarrow \mathcal{L}$ is non-trivial.

(iii) The category $C_X/\mathcal{L}$ is virtually complete.

(b) If the category C_X is connected and quasi-filtered, then the conditions above are equivalent to each of the following conditions:

(iv) The virtual kernel of any morphism of the category C_X is non-trivial.

(v) The category C_X is virtually complete.

Proof. (a) The implication (i)⇒(ii)⇒(iii) follow from the observations 2.2.

The implication (iii)⇒(ii) follows from the fact that the limit of the Yoneda embedding $C_X/\mathcal{L} \xrightarrow{h_{X/\mathcal{L}}} (C_X/\mathcal{L})^\wedge = C_X^\wedge/\widehat{\mathcal{L}}$ is either trivial (and then the virtual kernel of any morphism to $\mathcal{L}$ is the trivial presheaf), or it is non-trivial. In the latter case, by definition, the category $C_X/\mathcal{L}$ is virtually complete.

(b) The equivalence of the conditions (i), (ii), (iii) to each of the conditions (iv) and (v) follows from 2.3.3. ∎

2.3.5. Proposition. *Let C_X be a svelte virtually complete category with colimits. The following conditions are equivalent:*

(a) The category C_X is complete and cocomplete; that is it has limits and final objects, as well as initial objects.

(b) The category C_X is virtually complete.

Proof. The implication (a)⇒(b) holds, because every category with initial objects is virtually complete.

(b)⇒(a). If C_X is a virtually complete category. Then there are cones of the form $\mathcal{M} \longrightarrow Id_{C_X}$. For any diagram $\mathcal{D} \xrightarrow{\mathfrak{D}} C_X$, the cone $\mathcal{M} \longrightarrow Id_{C_X}$ induces a cone $\mathcal{M} \longrightarrow \mathfrak{D}$. We denote by $C_X/\mathfrak{D}$ the category of such cones and consider the forgetful functor $C_X/\mathfrak{D} \xrightarrow{\mathfrak{f}_{\mathfrak{D}}} C_X$. One can see that $\mathrm{colim}(\mathfrak{f}_{\mathfrak{D}}) \simeq \lim \mathfrak{D}$. ∎

2.4. Semi-complete and virtually semi-comlete categories.

2.4.0. The "reduced" category of presheaves of sets and "reduced" Yoneda embedding. Let $C_X = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X_\mathfrak{i}}$ be the decomposition of the category C_X into the disjoint union of its connected components. We call $\coprod_{\mathfrak{i}\in\pi_0(X)} C_{X_\mathfrak{i}^\circledast}$ the *"reduced" category of presheaves of sets* on the category C_X and denote it by $C_{X^{\mathfrak{r}\mathfrak{e}}}$.

2.4.0.1. We denote by $C_X \xrightarrow{h_X^{\mathfrak{r}\mathfrak{e}}} C_{X^{\mathfrak{r}\mathfrak{e}}}$ the coproduct of the canonical full embeddings $C_{X_\mathfrak{i}} \xrightarrow{h_{X_\mathfrak{i}}^\circledast} C_{X_\mathfrak{i}^\circledast}$, $\mathfrak{i} \in \pi_0(X)$. and call the fully faithful functor $C_X \xrightarrow{h_X^{\mathfrak{r}\mathfrak{e}}} C_{X^{\mathfrak{r}\mathfrak{e}}}$ the *reduced Yoneda embedding.*

2.4.1. Semi-complete categories. We call a category $C_{\mathfrak{X}}$ *semi-complete* if each of its connected components is a svelte category with limits of small diagrams.

In particular, each connected component $C_{\mathfrak{X}_\mathfrak{i}}$ of $C_{\mathfrak{X}}$ has initial objects, which are limits of a category equivalence $C_{\mathfrak{Y}_\mathfrak{i}} \longrightarrow C_{\mathfrak{X}_\mathfrak{i}}$ with $C_{\mathfrak{Y}_\mathfrak{i}}$ a small category.

2.4.2. Virtually semi-complete categories. We call a category C_X *virtually semi-complete* if each of its connected components is virtually complete.

2.4.3. Note. Every category whose all connected components have initial objects are virtually semi-complete (because every category with initial objects is virtually complete). In particular, all semi-complete categories are virtually semi-complete.

2.4.4. A semi-complete category associated with a virtually semi-complete category. Let $C_X = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X_\mathfrak{i}}$ be the decomposition of the category C_X into the disjoint union of its connected components. Notice that the category C_X is virtually semi-complete iff the "reduced" category $C_{X^{\mathfrak{r}\mathfrak{e}}} = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X_\mathfrak{i}^\circledast}$ of presheaves of sets on C_X is semi-complete. In this case, we refer to $C_{X^{\mathfrak{r}\mathfrak{e}}} = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X_\mathfrak{i}^\circledast}$ as the *semi-complete category associated with a virtually semi-complete category* C_X.

2.4.5. Proposition. *Every virtually semi-complete category is quasi-filtered.*

Proof. A category is quasi-filtered iff all its connected components are quasi-filtered. So that the assertion follows from 2.3.2.1. ∎

2.5. Proposition. *The following conditions on a svelte category C_X are equivalent:*
(a) Every morphism of the category C_X has a non-trivial virtual kernel.
(b) For any object $\mathcal{L}$ of the category C_X, the category $C_X/\mathcal{L}$ is virtually complete.
(c) The category C_X is virtually semi-complete.

Proof. The assertion follows from 2.3.3 and 2.3.4. ∎

2.6. Kernels in virtually semi-complete categories. Let C_X be a virtually semi-complete category. If a morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ belongs to the connected component $C_{X_\mathfrak{i}}$ of the category C_X, then its virtual kernel, $Ker_\mathfrak{v}(\mathfrak{f})$, is determined by a cartesian square

$$\begin{array}{ccc} Ker_\mathfrak{v}(\mathfrak{f}) & \longrightarrow & \mathfrak{x}_\mathfrak{i} \\ {\scriptstyle \mathfrak{k}_\mathfrak{v}(\mathfrak{f})}\downarrow & cart & \downarrow \\ \widehat{\mathcal{M}} & \xrightarrow{\widehat{\mathfrak{f}}} & \widehat{\mathcal{L}} \end{array} \qquad (2)$$

where $\mathfrak{x}_\mathfrak{i} = \lim h_{X_\mathfrak{i}}$ – the virtually initial object of the connected component $C_{X_\mathfrak{i}}$.

The kernel of the morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ (if any) is an object of the subcategory $C_{X_\mathfrak{i}}$ which represents the presheaf $Ker_\mathfrak{v}(\mathfrak{f})$ and $Ker(\mathfrak{f}) \xrightarrow{\mathfrak{k}(\mathfrak{f})} \mathcal{M}$ is a morphism, which represents the left vertical arrow $Ker_\mathfrak{v}(\mathfrak{f}) \xrightarrow{\mathfrak{k}_\mathfrak{v}(\mathfrak{f})} \widehat{\mathcal{M}}$ of the cartesian square (2).

2.7. Remarks. (a) The observation 2.6 shows that, if a connected component $C_{X_\mathfrak{i}}$ is virtually complete, then our general notion of kernel is reduced to the case of kernels in category $C_{X_i^\circledast}$ with initial objects and all limits (in particular, pull-backs), which enables us to use facts and techniques of Chapters I and II.

(b) On the other hand, if the component $C_{X_\mathfrak{i}}$ is not virtually complete, then the kernel of the image in $C^\wedge_{X_\mathfrak{i}}$ of any morphism from $C_{X_\mathfrak{i}}$ is the (unique morphism from) the trivial presheaf. In particular, there are no representable kernels of arrows.

Therefore, in the constructions related with kernels of arrows of a category C_X, we shall assume, normally, that C_X is virtually semi-complete.

2.8. Virtually trivial morphisms and objects. Virtually trivial kernels.

2.8.1. Virtually trivial morphisms. We call a morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ of a category C_X *virtually trivial* if the canonical morphism $Ker_\mathfrak{v}(\mathfrak{f}) \xrightarrow{\mathfrak{k}_\mathfrak{v}(\mathfrak{f})} \widehat{\mathcal{M}}$ is a split epimorphism.

This implies, by 2.4.1, that the connected component, $C_{X_\mathfrak{i}}$, of the object $\mathcal{L}$ is virtually complete; i.e. the category $C_{X_\mathfrak{i}^\circledast}$ has a non-trivial initial object $\mathfrak{x}_\mathfrak{i}$.

So that the morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ is virtually trivial iff its image $\widehat{\mathcal{M}} \xrightarrow{\widehat{\mathfrak{f}}} \widehat{\mathcal{L}}$ in $C^\wedge_{X_\mathfrak{i}}$ factors through a virtually initial object $\mathfrak{x}_\mathfrak{i}$ of the category $C_{X_\mathfrak{i}}$.

2.8.1.1. Note. It follows that *trivial* morphisms defined in 1.6.1 are virtually trivial. Moreover, the only difference between these two notions is the existence of a kernel: a virtually trivial morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ is trivial iff $Ker(\mathfrak{f})$ exists, or, what is the same, the virtual kernel $Ker_\mathfrak{v}(\mathfrak{f})$ of the morphism $\mathfrak{f}$ is representable.

2.8.2. Virtually trivial objects. By definition, an object $\mathcal{M}$ of C_X is *virtually trivial* if the identical morphism $\mathcal{M} \xrightarrow{id_\mathcal{M}} \mathcal{M}$ is virtually trivial. It follows from 2.8.1 that this can happen only if $\widehat{\mathcal{M}}$ is an initial object of the category $C_{X_\mathfrak{i}^\circledast}$ associated with the connected component $C_{X_\mathfrak{i}}$ of the object $\mathcal{M}$; that is $\widehat{\mathcal{M}} \simeq \mathfrak{x}_\mathfrak{i} = \lim h_{C_{X_\mathfrak{i}}}$. But, $\widehat{\mathcal{M}}$ is an initial object of $C_{X_\mathfrak{i}^\circledast}$ iff $\mathcal{M}$ is an initial object of the category $C_{X_\mathfrak{i}}$.

Thus, virtually trivial objects are, precisely, initial objects of connected components of the category C_X. Since *trivial* objects (defined in 1.6.1) are virtually trivial and initial objects of connected components of the category C_X are trivial, these two notions – *virtually trivial* and *trivial*, coincide.

2.8.3. Virtually trivial kernels. We say that a morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$ has a *virtually trivial* kernel if $Ker_\mathfrak{v}(\mathfrak{f})$ is a trivial object of the category C^*_X. By 2.8.2, this means that $Ker_\mathfrak{v}(\mathfrak{f})$ is (representable by) an initial object of the connected component $C_{X_\mathfrak{i}}$ of the object $\mathcal{M}$.

2.8.4. Proposition. *Suppose that there is a virtually trivial morphism* $\mathcal{M} \xrightarrow{g} \mathcal{N}$. *Then, for any morphism* $\mathcal{L} \xrightarrow{\mathfrak{f}} \mathcal{M}$, *the canonical morphism* $Ker_\mathfrak{v}(\mathfrak{f}) \xrightarrow{\mathfrak{k}_\mathfrak{v}(\mathfrak{f})} \widehat{\mathcal{L}}$ *is* *a strict monomorphism. In particular, if $Ker(\mathfrak{f})$ exists (that is the presheaf $Ker_{\mathfrak{v}}(\mathfrak{f})$ is representable), then $Ker(\mathfrak{f}) \xrightarrow{\mathfrak{k}(\mathfrak{f})} \mathcal{M}$ is a strict monomorphism.*

Proof. Morphisms $\mathcal{L} \xrightarrow{\mathfrak{f}} \mathcal{M}$ and $\mathcal{M} \xrightarrow{g} \mathcal{N}$ belong to a connected component $C_{X_{\mathfrak{i}}}$ of the category C_X. Let $\mathfrak{x}_{\mathfrak{i}}$ be an initial object of the category $C_{X_{\mathfrak{i}}^{\circledast}}$ of presheaves of sets on the category $C_{X_{\mathfrak{i}}}$. The triviality of the morphism $\mathcal{M} \xrightarrow{g} \mathcal{N}$ means that it factors through the virtually initial object $\mathfrak{x}_{\mathfrak{i}}$ of the category $C_{X_{\mathfrak{i}}}$. In particular, there exists a morphism $\widehat{\mathcal{M}} \longrightarrow \mathfrak{x}_{\mathfrak{i}}$, which implies that the unique morphism $\mathfrak{x}_{\mathfrak{i}} \longrightarrow \widehat{\mathcal{M}}$ is a split monomorphism; therefore, it is a strict monomorphism. The canonical morphism $Ker_{\mathfrak{v}}(\mathfrak{f}) \xrightarrow{\mathfrak{k}_{\mathfrak{v}}(\mathfrak{f})} \widehat{\mathcal{L}}$, being a pull-back of a strict monomorphism, is itself a strict monomorphism. ■

2.8.4.1. Corollary. *Suppose that there is a virtually trivial morphism $\mathcal{M} \xrightarrow{g} \mathcal{N}$. Then, a morphism $\mathcal{L} \xrightarrow{\mathfrak{f}} \mathcal{M}$ is virtually trivial iff $\mathcal{M} \xrightarrow{id_{\mathcal{L}}} \mathcal{L}$ is a kernel of $\mathcal{L} \xrightarrow{\mathfrak{f}} \mathcal{M}$.*

Proof. By 2.8.4, the canonical morphism $Ker_{\mathfrak{v}}(\mathfrak{f}) \xrightarrow{\mathfrak{k}_{\mathfrak{v}}(\mathfrak{f})} \widehat{\mathcal{L}}$ is a strict monomorphism. If the morphism $\mathcal{L} \xrightarrow{\mathfrak{f}} \mathcal{M}$ is virtually trivial, then, by definition, $Ker_{\mathfrak{v}}(\mathfrak{f}) \xrightarrow{\mathfrak{k}_{\mathfrak{v}}(\mathfrak{f})} \widehat{\mathcal{L}}$ is a split epimorphism. Therefore, it is an isomorphism. ■

2.10. Virtually semi-complete categories versus semi-complete categories.

2.10.1. Proposition. *Let C_X be a virtually semi-complete category and C_Y a semi-complete category. The functor of the composition with the "reduced" Yoneda embedding*

$$\mathcal{H}om(C_{X^{\mathfrak{rc}}}, C_Y) \xrightarrow{\circ h_X^{\mathfrak{rc}}} \mathcal{H}om(C_X, C_Y), \quad G \longmapsto G \circ h_X^{\mathfrak{rc}}, \tag{1}$$

establishes an equivalence between the category $\mathcal{H}om^{\mathfrak{vc}}(C_X, C_Y)$ of functors from C_X to C_Y which map virtually trivial morphisms to virtually trivial morphisms and the full subcategory $\mathfrak{Hom}(C_{X^{\mathfrak{rc}}}, C_Y)$ of the category $\mathcal{H}om(C_{X^{\mathfrak{rc}}}, C_Y)$ generated by all continuous functors $C_{X^{\mathfrak{rc}}} \longrightarrow C_Y$.

Proof. The argument is an adaptation of the proof of I.2.0.2. ■

2.10.2. The category of semi-complete categories. We denote by $Cat_{\mathfrak{sc}}$ the subcategory of the category Cat whose objects are semi-complete categories and morphisms are continuous functors.

2.10.3. The category of virtually semi-complete categories. We denote by $Cat_{\mathfrak{sc}}^{\mathfrak{v}}$ the subcategory of the category Cat whose objects are semi-complete categories and morphisms functors which map virtually trivial morphisms to virtually trivial morphisms.

2.10.4. Proposition. *The inclusion functor $Cat_{\mathfrak{sc}} \longrightarrow Cat_{\mathfrak{sc}}^{\mathfrak{v}}$ has a left adjoint.*

Proof. The assertion follows from 2.10.1. ■

2.11. Digression: fully exact subcategories of a right exact category.

2.11.1. Conflations. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. A *conflation* is the pair of arrows $Ker(\mathfrak{e}) \xrightarrow{\mathfrak{k}(\mathfrak{e})} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$, where $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ is a deflation.

2.11.1.1. Remark. The existence of a conflation $Ker(\mathfrak{e}) \xrightarrow{\mathfrak{k}(\mathfrak{e})} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ implies that the connected component of the object $\mathcal{M}$ is virtually complete (see 2.7(b)). So that the notion of conflation (and everything based on this notion, in particular, the content of this section) makes sense only for *virtually semi-complete* right exact categories: connected components which are not virtually complete do not participate.

2.11.2 Definition. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. We call a strictly full subcategory $\mathcal{B}$ of the category C_X a *fully exact* subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, if $\mathcal{B}$ is *closed under extensions* in the following sense: if $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ is a deflation such that $Ker(\mathfrak{e})$ exists and belongs to $\mathcal{B}$ and the object $\mathcal{L}$ belongs to $\mathcal{B}$, then $\mathcal{M}$ is an object of the subcategory $\mathcal{B}$ too.

2.11.3. Induced right exact structure. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and $\mathcal{B}$ a full subcategory of the category C_X. Let $\mathfrak{E}'_{\mathcal{B},X}$ denote the class of all deflations $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ such that $Ker(\mathfrak{e})$ (exists and) belongs to $\mathcal{B}$ as well as the objects $\mathcal{M}$ and $\mathcal{L}$. We denote by $\mathfrak{E}_{\mathcal{B},X}$ the union of $\mathfrak{E}'_{\mathcal{B},X}$ and the class of all isomorphisms of the category $\mathcal{B}$.

2.11.3.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category and $\mathcal{B}$ its fully exact subcategory. Then the class $\mathfrak{E}_{X,\mathcal{B}}$ is a structure of a right exact category on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \longrightarrow C_X$ is an 'exact' functor $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}}) \longrightarrow (C_X, \mathfrak{E}_X)$.*

Proof. It is clear that $Iso(C_X) \circ \mathfrak{E}'_{\mathcal{B},X} \circ Iso(C_X) = \mathfrak{E}'_{\mathcal{B},X}$. It remains to show that the class $\mathfrak{E}'_{\mathcal{B},X}$ is stable under arbitrary pull-backs and the composition: $\mathfrak{E}'_{\mathcal{B},X} \circ \mathfrak{E}'_{\mathcal{B},X} \subseteq \mathfrak{E}'_{\mathcal{B},X}$.

(a) The first assertion follows from 1.3. In fact, let

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\widetilde{\mathfrak{t}}} & \mathfrak{L} \\ \downarrow & cart & \downarrow \\ \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{L} \end{array}$$

be a cartesian square whose right vertical arrow belongs to $\mathcal{B}$ and the lower horizontal arrow is a deflation from $\mathfrak{E}'_{\mathcal{B},X}$. The latter means that $Ker(\mathfrak{t})$ exists and is an object of the subcategory $\mathcal{B}$. By 1.3, $Ker(\mathfrak{t})$ is naturally isomorphic to $Ker(\widetilde{\mathfrak{t}})$. Since $\mathcal{B}$ is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$, it follows that the object $\mathfrak{M}$ belongs to $\mathcal{B}$. Therefore, the deflation $\mathfrak{M} \xrightarrow{\widetilde{\mathfrak{t}}} \mathfrak{L}$ belongs to the class $\mathfrak{E}'_{\mathcal{B},X}$.

(b) Let $\mathcal{N} \xrightarrow{\mathfrak{s}} \mathcal{M}$ and $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{L}$ be morphisms of $\mathfrak{E}'_{\mathcal{B},X}$. In particular, $Ker(\mathfrak{s})$ exists and belongs to the subcategory $\mathcal{B}$. By 1.4, we have a cartesian square

$$\begin{array}{ccc} Ker(\mathfrak{t}\circ\mathfrak{s}) & \xrightarrow{\mathfrak{s}'} & Ker(\mathfrak{t}) \\ {\scriptstyle \mathfrak{k}(\mathfrak{ts})}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{t})} \\ \mathcal{N} & \xrightarrow{\mathfrak{s}} & \mathcal{M} \end{array}$$

whose existence follows from the invariance of deflations under pull-backs. By 1.3, the existence of $Ker(\mathfrak{s})$ implies that $Ker(\mathfrak{s}')$ exists and is isomorphic to $Ker(\mathfrak{s})$. Since $Ker(\mathfrak{s})$ and $Ker(\mathfrak{t})$ are objects of the subcategory $\mathcal{B}$ and $\mathcal{B}$ is a fully exact subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, it follows that $Ker(\mathfrak{t}\circ\mathfrak{s})$ is an object of $\mathcal{B}$. ■

3. Derived functors.

3.1. ∂^*-Functors. Fix a svelte right exact category $(C_X, \mathfrak{E}_X)$.
A *∂^*-functor* from $(C_X, \mathcal{E}_X)$ to a category C_Y is a sequence of functors

$$C_X \xrightarrow{T_i} C_Y, \quad i \geq 0,$$

together with an assignment to every cartesian square

$$\begin{array}{ccc} \mathcal{M}_{\xi,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\xi} & \mathcal{L}_\xi \\ {\scriptstyle \xi_\mathfrak{e}}\downarrow & cart & \downarrow{\scriptstyle \xi} \\ \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array} \tag{1}$$

whose horizontal arrows are deflations, and every $i \geq 0$ a morphism

$$T_{i+1}(\mathcal{L}) \xrightarrow{\mathfrak{d}_i(\mathfrak{e},\xi)} T_i(\mathcal{M}_{\xi,\mathfrak{e}}), \tag{2_i}$$

functorially depending on the cartesian square (or, what is the same, on the pair of arrows $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xleftarrow{\xi} \mathcal{L}_\xi$), whose composition with the morphism $T_i(\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\xi_\mathfrak{e}} \mathcal{M})$ is virtually trivial. In other words, the image of the morphism (2) in the category C_Y^* factors through the virtual kernel of the morphism $T_i(\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\xi_\mathfrak{e}} \mathcal{M})$.

3.1.0. Note about target categories of ∂^*-functors. Let $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ be a ∂^*-functor from a right exact category $(C_X, \mathcal{E}_X)$ to a category C_Y. The fact that the image of the morphism

$$T_1(\mathcal{L}) \xrightarrow{\mathfrak{d}_0(\mathfrak{e},\xi)} T_0(\mathcal{M}_{\xi,\mathfrak{e}}), \tag{2_0}$$

in the category C_Y^* factors through the virtual kernel of the morphism $T_0(\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\xi_\mathfrak{e}} \mathcal{M})$ implies, by 2.4.1, that the functor T_0 (hence all functors $T_i,\ i \geq 0$) takes values *only* in the virtually complete connected components of the category C_Y.

3.1.0.1. So that we can (and will) assume, without loss of generality, that ∂^*-functors take values in virtually semi-complete categories.

3.1.1. The 'long sequence'. Thus, we have a long "sequence"

$$\xrightarrow{\mathfrak{d}_{i+1}(\mathfrak{e},\xi)} T_{i+1}\begin{pmatrix} \mathcal{M}_{\xi,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\xi} & \mathcal{L}_\xi \\ \xi_\mathfrak{e}\downarrow & cart & \downarrow \xi \\ \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{pmatrix} \xrightarrow{\mathfrak{d}_i(\mathfrak{e},\xi)} T_i\begin{pmatrix} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \xi_\mathfrak{e}\uparrow & cart & \uparrow \xi \\ \mathcal{M}_{\xi,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\xi} & \mathcal{L}_\xi \end{pmatrix} \xrightarrow{\mathfrak{d}_{i-1}(\mathfrak{e},\xi)} \tag{3}$$

functorially depending on the cartesian square, or, what is the same, on the pair of arrows $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xleftarrow{\xi} \mathcal{L}_\xi$ whose left arrow is a deflation.

3.1.2. Morphisms of ∂^*-functors. Let $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ and $T' = (T'_i, \mathfrak{d}'_i|\ i \geq 0)$ be ∂^*-functors from a right exact category $(C_X, \mathcal{E}_X)$ to a category C_Y. A morphism from T to T' is a family $f = (T_i \xrightarrow{f_i} T'_i \mid i \geq 0)$ of functor morphisms such that, for any cartesian square (1) whose horizontal arrows are deflations, and every $i \geq 0$, the diagram

$$\begin{array}{ccc} T_{i+1}(\mathcal{L}) & \xrightarrow{\mathfrak{d}_i(\mathfrak{e},\xi)} & T_i(\mathcal{M}_{\xi,\mathfrak{e}}) \\ f_{i+1}(\mathcal{L}) \downarrow & & \downarrow f_i(\mathcal{M}_{\xi,\mathfrak{e}}) \\ T'_{i+1}(\mathcal{L}) & \xrightarrow{\mathfrak{d}'_i(\mathfrak{e},\xi)} & T'_i(\mathcal{M}_{\xi,\mathfrak{e}}) \end{array} \tag{4}$$

commutes. The composition of morphisms is naturally defined. Thus, we have the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$ of ∂^*-functors from $(C_X, \mathcal{E}_X)$ to C_Y.

3.1.3. Contravariant functoriality. Let $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X}) \xrightarrow{\Phi} (C_X, \mathfrak{E}_X)$ be an 'exact' functor (that is Φ preserves deflations and pull-backs of deflations). For any ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ from the right exact category $(C_X, \mathfrak{E}_X)$ to a category C_Y, the composition

$$T \circ \Phi = (T_i \circ \Phi, \mathfrak{d}_i\Phi|\ i \geq 0)$$

is a ∂^*-functor from $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ to C_Y. The map $(T, \Phi) \longmapsto T \circ \Phi$ extends to a functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \times \mathcal{E}x_*((C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X}), (C_X, \mathfrak{E}_X)) \longrightarrow \mathcal{H}om^*((C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X}), C_Y), \tag{5}$$

where $\mathcal{E}x((C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X}), (C_X, \mathfrak{E}_X))$ denotes the full subcategory of $\mathcal{H}om(C_\mathfrak{X}, C_X)$ whose objects are 'exact' functors $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X}) \longrightarrow (C_X, \mathfrak{E}_X)$.

3.1.4. Covariant functoriality. Let C_Y and C_Z be virtually semi-complete categories and $C_Y \xrightarrow{F} C_Z$ a functor which maps virtually trivial morphisms to virtually trivial morphisms. For any ∂^*-functor $T = (T_i, \mathfrak{d}_i | \ i \geq 0)$ from a right exact category $(C_X, \mathfrak{E}_X)$ to the category C_Y, the composition

$$F \circ T = (F \circ T_i, F\mathfrak{d}_i \mid i \geq 0)$$

is a ∂^*-functor from $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ to C_Z.

3.1.4.1. Note. Let $C_Y = \coprod_{\mathfrak{i} \in \pi_0(X)} C_{Y_\mathfrak{i}}$ and $C_Z = \coprod_{\gamma \in \pi_0(Z)} C_{Z_\gamma}$ be the decompositions of the categories C_Y and C_Z into the disjoint union of connected components.

Any functor $C_Y \xrightarrow{F} C_Z$ maps connected components of the category C_Y to connected components of the category C_Z. In other words, the functor F induces a map

$$\pi_0(Y) \xrightarrow{\pi_0(F)} \pi_0(Z)$$

and, for every $\mathfrak{i} \in \pi_0(Y)$, a functor $C_{Y_\mathfrak{i}} \xrightarrow{F_\mathfrak{i}} C_{Z_{\pi_0(F)(\mathfrak{i})}}$.

It follows from 2.8.1 that the functor $C_Y \xrightarrow{F} C_Z$ maps virtually trivial morphisms to virtually trivial morphisms iff, for every $\mathfrak{i} \in \pi_0(Y)$, the associated functor

$$C_{Y_\mathfrak{i}^{\circledast}} \xrightarrow{F_\mathfrak{i}^*} C_{Z^{\circledast}_{\pi_0(F)(\mathfrak{i})}}$$

maps initial objects of the category $C_{Y_\mathfrak{i}^{\circledast}}$ to initial objects of the category $C_{Y^{\circledast}_{\pi_0(F)(\mathfrak{i})}}$.

3.2. Universal ∂^*-functors. A ∂^*-functor $T = (T_i, \mathfrak{d}_i | \ i \geq 0)$ from a right exact category $(C_X, \mathcal{E}_X)$ to a virtually semi-complete category C_Y is called *universal* if, for every ∂^*-functor $T' = (T'_i, \mathfrak{d}'_i | \ i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y and every functor morphism $T'_0 \xrightarrow{g} T_0$, there exists a unique morphism $f = (T'_i \xrightarrow{f_i} T_i \mid i \geq 0)$ from T' to T such that $f_0 = g$.

3.2.1. Interpretation. Consider the functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \tag{1}$$

which assigns to every ∂^*-functor (resp. every morphism of ∂^*-functors) its zero component. For any functor $C_X \xrightarrow{F} C_Y$, we have a presheaf of sets $\mathcal{H}om(\Psi^*(-), F)$ on the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$. Suppose that this presheaf is representable by an object (i.e. a ∂^*-functor) $\Psi_*(F)$. Then $\Psi_*(F)$ is a universal ∂^*-functor.

Conversely, if $T = (T_i, \mathfrak{d}_i | \ i \geq 0)$ is a universal ∂^*-functor, then $T \simeq \Psi_*(T_0)$.

3.2.2. Right derived functors. If $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ is a universal ∂^*-functor, then, for every $i \geq 1$, the functor T_i will be called the *i-th right derived* functor of the functor T_0. It follows from the universality of T that, for all $i \geq 0$, the (i+1)-th derived functor of T_0 is the first right derived functor of the i-th derived functor T_i.

3.3. The construction of the first right derived functor. Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category and C_Y a virtually semi-complete category with pull-backs.

Fix a functor $C_X \xrightarrow{F} C_Y$.

3.3.1. An intermediate diagram and its limit. For any object $\mathcal{L}$ of the category C_X, let $\mathfrak{D}\mathcal{S}\_F(\mathcal{L})$ denote the diagram

$$\begin{array}{ccc} \mathcal{V}_{\alpha,F(\xi_{\mathfrak{e}})} & \xrightarrow{\mathfrak{d}^{\alpha}_{\xi,\mathfrak{e}}} & \\ \big\downarrow & cart & F \left(\begin{array}{ccc} \mathcal{M}_{\xi,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\xi} & \mathcal{L}_\xi \\ \xi_{\mathfrak{e}}\big\downarrow & cart & \big\downarrow \xi \\ \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array}\right) \\ \mathcal{V}_\alpha & \xrightarrow{\alpha} & \end{array} \qquad (1)$$

where $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ runs through deflations of the object $\mathcal{L}$ and $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$ and $\mathcal{V}_\alpha \xrightarrow{\alpha} F(\mathcal{M})$ through (the set of representatives of isomorphism classes of) arbitrary arrows.

The diagram $\mathfrak{D}\mathcal{S}\_F(\mathcal{L})$ is filtered. So that if the category C_Y has limits of filtered diagrams, then the limit of the diagram $\mathfrak{D}\mathcal{S}\_F(\mathcal{L})$ exists for any object $\mathcal{L}$ of the category C_X. We denote this limit by $\mathcal{S}\_F(\mathcal{L})$.

3.3.2. The connecting morphism. It follows from the definition of $\mathcal{S}\_F(\mathcal{L})$ that, for every cartesian square

$$\begin{array}{ccc} \mathcal{M}_{\xi,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\xi} & \mathcal{L}_\xi \\ \xi_{\mathfrak{e}}\big\downarrow & cart & \big\downarrow \xi \\ \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array}$$

whose horizontal arrows are deflations, there is a canonical diagram

$$\begin{array}{ccccc} \mathcal{S}\_F(\mathcal{L}) & \xrightarrow{\mathfrak{d}_{\xi,\mathfrak{e}}} & F(\mathcal{M}_{\xi,\mathfrak{e}}) & \xrightarrow{F(\mathfrak{e}_\xi)} & F(\mathcal{L}_\xi) \\ & & F(\xi_{\mathfrak{e}})\big\downarrow & & \big\downarrow F(\xi) \\ & & F(\mathcal{M}) & \xrightarrow{F(\mathfrak{e})} & F(\mathcal{L}) \end{array} \qquad (2)$$

which depends functorially on the morphisms $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xleftarrow{\xi} \mathcal{L}_\xi$.

3.3.3. Limits and "triangles". Let C_Y be a virtually semi-complete category with limits of filtered diagrams. By 2.4.5, C_Y is quasi-filtered and, therefore, has kernels of arrows. It follows from the definition of $\mathcal{S}_-F(\mathcal{L})$ that it is the limit of $Ker(F(\xi_{\mathfrak{e}}))$, where $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ runs through deflations of the object $\mathcal{L}$ and $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$ through arbitrary arrows.

Thanks to the existence of limits of filtered diagrams, this limit can be split in two consecutive limits:

$$\mathcal{S}_-F(\mathcal{L}) = \lim_{\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}} \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} Ker(F(\xi'_{\mathfrak{e}}))$$

In particular, for each deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$, we have a canonical morphism

$$\mathcal{S}_-F(\mathcal{L}) \xrightarrow{\mathfrak{d}_{\mathfrak{e}}} \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} Ker(F(\xi'_{\mathfrak{e}})),$$

which is a part of "triangle"

$$\mathcal{S}_-F(\mathcal{L}) \xrightarrow{\mathfrak{d}_{\mathfrak{e}}} \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} Ker(F(\xi'_{\mathfrak{e}})) \xrightarrow{\bar{\mathfrak{k}}(\mathfrak{e})} F(\mathcal{M}) \xrightarrow{F(\mathfrak{e})} F(\mathcal{L}). \tag{3}$$

If all deflations of the object $\mathcal{L}$ have kernels (which is the case when the category $C_X/\mathcal{L}$ has initial objects), then the limit $\lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} Ker(F(\xi'_{\mathfrak{e}}))$ is isomorphic to the morphism $F(Ker(\mathfrak{e}) \xrightarrow{\mathfrak{k}(\mathfrak{e})} \mathcal{M})$. So that $\mathcal{S}_-F(\mathcal{L})$ is isomorphic to the limit of kernels of the morphisms $F(Ker(\mathfrak{e}) \xrightarrow{\mathfrak{k}(\mathfrak{e})} \mathcal{M})$, where $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ runs through the category of deflations of the object $\mathcal{L}$. In this case, the diagram 3.3.2(2) can be replaced by a more familiar form of a triangle:

$$\mathcal{S}_-F(\mathcal{L}) \xrightarrow{\mathfrak{d}(\mathfrak{e})} F(Ker(\mathfrak{e})) \xrightarrow{F(\mathfrak{k}(\mathfrak{e}))} F(\mathcal{M}) \xrightarrow{F(\mathfrak{e})} F(\mathcal{L}). \tag{4}$$

In particular, if the categories C_X and C_Y have initial objects, we recover the right derived functor introduced in II.3.2.

3.3.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category, C_Y a virtually semi-complete category with limits of filtered diagrams, and $C_X \xrightarrow{F} C_Y$ a functor. Suppose that the category C_X is quasi-filtered (say, C_X has pull-backs, or C_X is virtually semi-complete; see 2.4.5).*

Then the map $\mathcal{L} \longmapsto \mathcal{S}_-F(\mathcal{L})$ extends to a functor $C_X \xrightarrow{\mathcal{S}_-F} C_Y$.

Proof. Replacing the svelte category C_X by an equivalent small category, we assume that C_X is a small category. Replacing the category C_Y by the associated semi-complete category $C_{Y^{\mathfrak{r}\mathfrak{e}}} = \coprod_{\mathfrak{i}\in\pi_0(Y)} C_{Y_\mathfrak{i}^{\circledast}}$ (see 2.4.4), we assume that the category C_Y has pull-backs. Since the canonical embedding

$$C_Y = \coprod_{\mathfrak{i}\in\pi_0(Y)} C_{Y_\mathfrak{i}} \xrightarrow{\ h_Y^{\mathfrak{r}\mathfrak{e}}\ } C_{Y^{\mathfrak{r}\mathfrak{e}}} = \coprod_{\mathfrak{i}\in\pi_0(Y)} C_{Y_\mathfrak{i}^{\circledast}}$$

preserves limits and, for any $\mathcal{L} \in ObC_X$, the presheaf of sets $\mathcal{S}_-(h_Y^{\mathfrak{r}\mathfrak{e}}\circ F)(\mathcal{L})$ is representable by $\mathcal{S}_-F(\mathcal{L})$ (see 3.3.3), we can do this reduction without loss of generality.

(a) For any morphism $\mathfrak{L} \xrightarrow{\mathfrak{f}} \mathcal{L}$ of the category C_X, consider the commutative diagrams

$$\begin{array}{ccccc}
\mathcal{S}_-F(\mathcal{L}) & \longrightarrow & \mathcal{V}_{\alpha,F(\mathfrak{f}'_\mathfrak{e}\gamma'_{\mathfrak{e}_\mathfrak{f}})} & \xrightarrow{\mathfrak{d}^\alpha_{\gamma,\mathfrak{e}_\xi}} \\
 & & \big\downarrow & cart \\
 & & \mathcal{V}_{\alpha,F(\xi'_\mathfrak{e})} & \xrightarrow{\mathfrak{d}^\alpha_{\xi,\mathfrak{e}}} \\
 & & \big\downarrow & cart \\
 & & \mathcal{V}_\alpha & \xrightarrow{\alpha} \\
 & & \big\uparrow & cart \\
 & & \mathcal{V}_{\alpha,F(\mathfrak{f}'_\mathfrak{e})} & \xrightarrow{\mathfrak{d}^\alpha_{\mathfrak{f}}} \\
 & & \big\uparrow & cart \\
\mathcal{S}_-F(\mathfrak{L}) & \longrightarrow & \mathcal{V}_{\alpha,F(\mathfrak{f}'_\mathfrak{e}\gamma'_{\mathfrak{e}_\mathfrak{f}})} & \xrightarrow{\mathfrak{d}^\alpha_{\gamma,\mathfrak{e}_\xi}}
\end{array}
\quad F\left(\begin{array}{ccccc}
\mathcal{M}_{\mathfrak{f}\gamma,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}\gamma}} & \mathfrak{L}_\gamma & & \\
\mathfrak{f}'_{\gamma,\mathfrak{e}_\xi}\big\downarrow & cart & \big\downarrow \mathfrak{f}_\gamma & & \\
\mathcal{M}_{\xi,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\xi} & \mathcal{L}_\xi & \xleftarrow{\mathfrak{f}_\gamma} & \mathfrak{L}_\gamma \\
\xi'_\mathfrak{e}\big\downarrow & cart & \big\downarrow \xi & & \big\downarrow \gamma \\
\mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xleftarrow{\mathfrak{f}} & \mathfrak{L} \\
\mathfrak{f}'_\mathfrak{e}\big\uparrow & cart & \big\uparrow \mathfrak{f} & & \\
\mathcal{M}_{\mathfrak{f},\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\mathfrak{f}} & \mathfrak{L} & & \\
\gamma'_{\mathfrak{e}_\mathfrak{f}}\big\uparrow & cart & \big\uparrow \gamma & & \\
\mathcal{M}_{\mathfrak{f}\gamma,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}\gamma}} & \mathfrak{L}_\gamma & &
\end{array}\right) \qquad (1)$$

built around the diagram 3.3.1(1) and the morphism $\mathfrak{L} \xrightarrow{\mathfrak{f}} \mathcal{L}$. Here $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ runs through deflations of the object $\mathcal{L}$, $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$ and $\mathcal{V}_\alpha \xrightarrow{\alpha} F(\mathcal{M})$ through arbitrary arrows, and $(\gamma, \mathfrak{f}_\gamma)$ runs through the set of all pairs of arrows $\mathcal{L}_\xi \xleftarrow{\mathfrak{f}_\gamma} \mathfrak{L}_\gamma \xrightarrow{\gamma} \mathfrak{L}$ satisfying $\mathfrak{f}\circ\gamma = \xi\circ\mathfrak{f}_\gamma$. By hypothesis, this set is non-empty.

Observations. (a1) If the morphism $\mathfrak{L} \xrightarrow{\mathfrak{f}} \mathcal{L}$ has arbitrary pull-backs, then the pairs of arrows $\mathcal{L}_\xi \xleftarrow{\mathfrak{f}_\gamma} \mathfrak{L}_\gamma \xrightarrow{\gamma} \mathfrak{L}$ satisfying $\mathfrak{f}\circ\gamma = \xi\circ\mathfrak{f}_\gamma$ are in natural bijective correspondence with all morphisms $\mathfrak{L}_\gamma \xrightarrow{\lambda} \mathfrak{L}_{\xi,\mathfrak{f}}$ determined by the choice of a cartesian square

$$\begin{array}{ccc}
\mathfrak{L}_{\xi,\mathfrak{f}} & \xrightarrow{\mathfrak{f}_\xi} & \mathcal{L}_\xi \\
\xi'_\mathfrak{f}\big\downarrow & cart & \big\downarrow \xi \\
L & \xrightarrow{\mathfrak{f}} & \mathcal{L}
\end{array}$$

(a2) If there are no pull-backs of some morphisms to $\mathcal{L}$ along $\mathfrak{L} \xrightarrow{\mathfrak{f}} \mathcal{L}$, then we still can fix a commutative square

$$\begin{array}{ccc} \mathfrak{L}_0 & \xrightarrow{\mathfrak{f}_0} & \mathcal{L}_\xi \\ {\scriptstyle \xi_0}\downarrow & & \downarrow{\scriptstyle \xi} \\ L & \xrightarrow{\mathfrak{f}} & \mathcal{L} \end{array}$$

(whose existence is guaranteed by hypothesis) and consider only pairs $(\gamma, \mathfrak{f}_\gamma)$ of the form $(\xi_0 \circ \lambda, \mathfrak{f}_0 \circ \lambda)$ for an arbitrary $\mathfrak{L}_\gamma \xrightarrow{\lambda} \mathfrak{L}_{\cdot 0}$, because the limits we are taking do not depend on the choice of this commutative square.

(a3) It follows from (a1) and (a2) that we can replace the diagram (1) by the diagram

$$\begin{array}{ccccc} & & \mathcal{V}_\alpha & \xrightarrow{\alpha} & \\ & & \uparrow & cart & \\ & & \mathcal{V}_{\alpha, F(\mathfrak{f}'_\mathfrak{e})} & \xrightarrow{\mathfrak{d}^\alpha_\mathfrak{f}} & F\left(\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ {\scriptstyle \mathfrak{f}'_\mathfrak{e}}\uparrow & cart & \uparrow{\scriptstyle \mathfrak{f}} \\ \mathcal{M}_{\mathfrak{f},\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\mathfrak{f}} & \mathfrak{L} \\ {\scriptstyle \gamma'_{\mathfrak{e}_\mathfrak{f}}}\uparrow & cart & \uparrow{\scriptstyle \gamma} \\ \mathcal{M}_{\mathfrak{f}\gamma,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}\gamma}} & \mathfrak{L}_\gamma \end{array}\right) \\ & & \uparrow & cart & \\ \mathcal{S}\_F(\mathcal{L}) & \longrightarrow & \mathcal{V}_{\alpha, F(\mathfrak{f}'_\mathfrak{e}\gamma'_{\mathfrak{e}_\mathfrak{f}})} & \xrightarrow{\mathfrak{d}^\alpha_{\gamma,\mathfrak{e}_\xi}} & \\ & & \uparrow & & \\ & & \mathcal{S}\_F(\mathfrak{L}) & & \end{array} \qquad (2)$$

where $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ runs through deflations of the object $\mathcal{L}$ and $\mathfrak{L}_\gamma \xrightarrow{\gamma} \mathfrak{L}$ through arbitrary morphisms.

(b) The diagram (2) incorporates two cones:

$$\mathcal{S}\_F(\mathcal{L}) \longrightarrow \text{DIAGRAM} \quad \text{and} \quad \mathcal{S}\_F(\mathfrak{L}) \longrightarrow \text{DIAGRAM}, \qquad (3)$$

where DIAGRAM is

$$\begin{array}{ccc} \mathcal{V}_{\alpha, F(\mathfrak{f}'_\mathfrak{e}\gamma'_{\mathfrak{e}_\mathfrak{f}})} & \xrightarrow{\mathfrak{d}^\alpha_{\mathfrak{f}\gamma,\mathfrak{e}}} & \\ \downarrow & cart & F\left(\begin{array}{ccc} \mathcal{M}_{\mathfrak{f}\gamma,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}\gamma}} & \mathfrak{L}_\gamma \\ {\scriptstyle \mathfrak{f}'_\mathfrak{e}\gamma'_{\mathfrak{e}_\mathfrak{f}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}\gamma} \\ \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array}\right) \\ \mathcal{V}_\alpha & \xrightarrow{\alpha} & \end{array} \qquad (4)$$

It follows from this construction and the definition of $\mathcal{S}\_F(\mathcal{L})$ that the first cone, $\mathcal{S}\_F(\mathcal{L}) \longrightarrow$ DIAGRAM, is universal. Therefore, there exists a unique morphism

$$\mathcal{S}\_F(\mathfrak{L}) \xrightarrow{\mathcal{S}\_F(\mathfrak{f})} \mathcal{S}\_F(\mathcal{L})$$

such that

$$\begin{array}{ccc} \mathcal{S}\_F(\mathfrak{L}) & \xrightarrow{\mathcal{S}\_F(\mathfrak{f})} & \mathcal{S}\_F(\mathcal{L}) \\ & \searrow \quad \swarrow & \\ & \text{DIAGRAM} & \end{array}$$

commutes. ∎

3.3.5. Remark. Consider the class of all arrows $\mathcal{M} \xrightarrow{f} \mathcal{L}$ of a category C_X such that, for any arrow $\mathcal{L}_\zeta \xrightarrow{\zeta} \mathcal{L}$, there exists a commutative square

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \mathcal{L}_\zeta \\ {\scriptstyle \zeta'}\downarrow & & \downarrow {\scriptstyle \zeta} \\ \mathcal{M} & \xrightarrow{f} & \mathcal{L} \end{array}$$

This class of arrows contains all isomorphisms of C_X and is closed under composition. Therefore, it defines a subcategory $C_{X_\mathfrak{f}}$ of the category C_X, which is, by construction, quasi-filtered. Evidently, $ObC_{X_\mathfrak{f}} = ObC_X$, and it is easy to see that $HomC_{X_\mathfrak{f}}$ is closed under pull-backs along morphisms of C_X in the sense that if

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \mathfrak{L} \\ \downarrow & cart & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{L} \end{array}$$

is a cartesian square in C_X whose right vertical arrow belongs to $HomC_{X_\mathfrak{f}}$, then its left vertical arrow belongs to $HomC_{X_\mathfrak{f}}$ as well.

Evidently, $HomC_{X_\mathfrak{f}}$ contains all morphisms stable under arbitrary pull-backs. In particular, if $\mathfrak{E}_X$ is a right exact structure on C_X, then $\mathfrak{E}_X \subseteq HomC_{X_\mathfrak{f}}$; hence the pair $(C_{X_\mathfrak{f}}, \mathfrak{E}_X)$ is a right exact subcategory of $(C_X, \mathfrak{E}_X)$. Suppose that a functor $C_X \xrightarrow{F} C_Y$ is such that $\mathcal{S}\_F(L)$ exists for all $L \in ObC_X$. It follows from the argument of 3.3.4 that the map $L \longmapsto \mathcal{S}\_F(L)$ extends to a functor from the category $C_{X_\mathfrak{f}}$ to the category C_Y.

3.4. Some observations and details. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and $C_X \xrightarrow{F} C_Y$ a functor. We assume that the category C_X is quasi-filtered and the category C_Y is virtually semi-complete and has pull-backs and limits of filtered diagrams. In particular, the category C_Y has kernels of all arrows, and $\mathcal{S}\_F(\mathcal{L})$ exists for all functors $C_X \xrightarrow{F} C_Y$ and all $\mathcal{L} \in ObC_X$. By 3.3.4, these conditions imply that $\mathcal{S}\_F$ is a well defined functor from C_X to C_Y.

3.4.1. Functoriality of $\mathcal{S}\_F$ in terms of kernels. Since the category C_Y has kernels, the cone

$$\mathcal{S}\_F(\mathcal{L}) \longrightarrow \text{DIAGRAM}$$

in the argument of 3.3.4 factors through the limit of the DIAGRAM, with respect to the morphisms $\mathcal{V}_\alpha \xrightarrow{\alpha} F(\mathcal{M})$; that is $\mathcal{S}_{-}F(\mathcal{L}) \longrightarrow$ DIAGRAM can be replaced by the diagram

$$\begin{array}{ccccc} Ker(F(\mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_\mathfrak{f}})) & \longrightarrow & \mathcal{V}_{\alpha,F(\mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_\mathfrak{f}})} & \xrightarrow{\mathfrak{d}^\alpha_{\mathfrak{f}\gamma,\mathfrak{e}}} & \\ \uparrow & & \downarrow & cart & F\left(\begin{array}{ccc} \mathcal{M}_{\mathfrak{f}\gamma,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}\gamma}} & \mathfrak{L}_\gamma \\ \mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_\mathfrak{f}}\downarrow & cart & \downarrow \mathfrak{f}\gamma \\ \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array}\right) \\ \mathcal{S}_{-}F(\mathcal{L}) & & \mathcal{V}_\alpha & \xrightarrow{\alpha} & \end{array} \qquad (1)$$

where $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ runs through deflations of the object $\mathcal{L}$ and $\mathfrak{L}_\gamma \xrightarrow{\xi} \mathfrak{L}$ and $\mathcal{V}_\alpha \xrightarrow{\alpha} F(\mathcal{M})$ are arbitrary arrows.

Similarly, the cone $\mathcal{S}_{-}F(\mathfrak{L}) \longrightarrow$ DIAGRAM can be replaced by the cone

$$\begin{array}{ccccc} & & \mathcal{V}_\alpha & \xrightarrow{\alpha} & \\ & & \uparrow & cart & \\ \mathcal{S}_{-}F(\mathfrak{L}) & & \mathcal{V}_{\alpha,F(\mathfrak{f})} & \xrightarrow{\mathfrak{d}^\alpha_{\mathfrak{f}}} & F\left(\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \mathfrak{f}'_{\mathfrak{e}}\uparrow & cart & \uparrow \mathfrak{f} \\ \mathcal{M}_{\mathfrak{f},\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\mathfrak{f}} & \mathfrak{L} \\ \gamma'_{\mathfrak{e}_\mathfrak{f}}\uparrow & cart & \uparrow \gamma \\ \mathcal{M}_{\mathfrak{f}\gamma,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}\gamma}} & \mathfrak{L}_\gamma \end{array}\right) \\ \downarrow & & \uparrow & cart & \\ Ker(F(\gamma'_{\mathfrak{e}_\mathfrak{f}})) & \longrightarrow & \mathcal{V}_{\alpha,F(\mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_\mathfrak{f}})} & \xrightarrow{\mathfrak{d}^\alpha_{\gamma,\mathfrak{e}_\xi}} & \end{array} \qquad (2)$$

which is determined uniquely up to isomorphism by the same varying parameters as the diagram (1): deflations $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ of the object $\mathcal{L}$ and arbitrary morphisms $\mathfrak{L}_\gamma \xrightarrow{\gamma} \mathfrak{L}$ and $\mathcal{V}_\alpha \xrightarrow{\alpha} F(\mathcal{M})$.

Thus, we have a commutative diagram

$$\begin{array}{ccccccc} & & Ker(F(\mathfrak{f}'_{\mathfrak{e}})) & \xrightarrow{\mathfrak{k}(F(\mathfrak{f}'_{\mathfrak{e}}))} & & & \\ & & \zeta_{\mathfrak{e},\gamma}\uparrow & cart & F\left(\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \mathfrak{f}'_{\mathfrak{e}}\uparrow & cart & \uparrow \mathfrak{f} \\ \mathcal{M}_{\mathfrak{f},\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\mathfrak{f}} & \mathfrak{L} \\ \gamma'_{\mathfrak{e}_\mathfrak{f}}\uparrow & cart & \uparrow \gamma \\ \mathcal{M}_{\mathfrak{f}\gamma,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}\gamma}} & \mathfrak{L}_\gamma \end{array}\right) \\ \mathcal{S}_{-}F(\mathcal{L}) & \longrightarrow & Ker(F(\mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_\mathfrak{f}})) & \xrightarrow{\mathfrak{k}(F(\mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_\mathfrak{f}}))} & & & \\ \mathcal{S}_{-}F(\mathfrak{f})\uparrow & & \mathfrak{k}(\zeta_{\mathfrak{e},\gamma})\uparrow & & \uparrow \mathfrak{k}(F(\gamma'_{\mathfrak{e}_\mathfrak{f}})) & & \\ \mathcal{S}_{-}F(\mathfrak{L}) & \longrightarrow & Ker(F(\gamma'_{\mathfrak{e}_\mathfrak{f}})) & \xrightarrow{id} & Ker(F(\gamma'_{\mathfrak{e}_\mathfrak{f}})) & & \end{array} \qquad (3)$$

depending on deflations $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ of the object $\mathcal{L}$ and arbitrary morphisms $\mathfrak{L}_\gamma \xrightarrow{\gamma} \mathfrak{L}$ and $\mathcal{V}_\alpha \xrightarrow{\alpha} F(\mathcal{M})$. with cartesian squares and their images as indicated.

Notice that (3) can be viewed as a cone with the vertex $\mathcal{S}\_F(\mathfrak{L})$.

As it was already mentioned in the argument of 3.3.4, the cone

$$\mathcal{S}\_F(\mathcal{L}) \xrightarrow{\lambda^F_{\mathfrak{f},\mathfrak{e},\gamma}(\mathfrak{L})} \begin{array}{c} Ker(F(\mathfrak{f}'_{\mathfrak{e}})) \\ \zeta_{\mathfrak{e},\gamma}\uparrow \\ Ker(F(\mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_{\mathfrak{f}}})) \end{array} \begin{array}{c} \xrightarrow{\mathfrak{k}(F(\mathfrak{f}'_{\mathfrak{e}}))} \\ cart \\ \xrightarrow{\mathfrak{k}(F(\mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_{\mathfrak{f}}}))} \end{array} F\left(\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \mathfrak{f}'_{\mathfrak{e}}\uparrow & cart & \uparrow\mathfrak{f} \\ \mathcal{M}_{\mathfrak{f},\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}}} & \mathfrak{L} \\ \gamma'_{\mathfrak{e}_{\mathfrak{f}}}\uparrow & cart & \uparrow\gamma \\ \mathcal{M}_{\mathfrak{f}\gamma,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}\gamma}} & \mathfrak{L}_{\gamma} \end{array}\right) \qquad (4)$$

is universal. Altogether implies that the morphism $\mathcal{S}\_F(\mathfrak{L}) \xrightarrow{\mathcal{S}\_F(\mathfrak{f})} \mathcal{S}\_F(\mathcal{L})$ is an arrow in the commutative diagram

$$\begin{array}{ccccc} \mathcal{S}\_F(\mathfrak{L}) & \xrightarrow{\mathcal{S}\_F(\mathfrak{f})} & \mathcal{S}\_F(\mathcal{L}) & & \\ \lambda^F_{\mathfrak{f}}(\mathfrak{L})\downarrow & & \downarrow\wr & & \\ \lim\limits_{\mathfrak{e},\gamma} Ker(F(\gamma'_{\mathfrak{e}_{\mathfrak{f}}})) & \xrightarrow{\mathfrak{k}(\bar{\zeta})} & \lim\limits_{\mathfrak{e},\gamma} Ker(F(\mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_{\mathfrak{f}}})) & \xrightarrow{\bar{\zeta}} & \lim\limits_{\mathcal{M}\xrightarrow{\mathfrak{e}}\mathcal{L}} Ker(F(\mathfrak{f}'_{\mathfrak{e}})) \end{array} \qquad (5)$$

whose right vertical arrows is a canonical isomorphism. Here $\bar{\zeta} = \lim\limits_{\mathfrak{e},\gamma}(\zeta_{\mathfrak{e},\gamma})$, where the limit is taken with respect to deflations $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ and arbitrary morphisms $\mathfrak{L}_{\gamma} \xrightarrow{\gamma} \mathfrak{L}$.

3.4.2. The limit of connecting morphisms. Suppose now that $\mathfrak{L} \xrightarrow{\mathfrak{f}} \mathcal{L}$ is a deflation. In particular, for any arrow $\mathcal{L}_{\xi} \xrightarrow{\xi} \mathcal{L}$, there is a cartesian square

$$\begin{array}{ccc} \mathfrak{L}_{\xi,\mathfrak{f}} & \xrightarrow{\mathfrak{f}_{\xi}} & \mathcal{L}_{\xi} \\ \xi'_{\mathfrak{f}}\downarrow & cart & \downarrow\xi \\ \mathfrak{L} & \xrightarrow{\mathfrak{f}} & \mathcal{L} \end{array}$$

In this case, the diagram 3.4.1(3) can be replaced by the commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & & \mathfrak{L}_{\mathfrak{e}} & \xrightarrow{\mathfrak{e}} \ \mathfrak{L} & \xrightarrow{\mathfrak{f}} & \mathcal{L} \\
 & & & & & \mathfrak{f}'_{\mathfrak{f}\mathfrak{e}}\uparrow & cart \quad \mathfrak{f}'_{\mathfrak{f}}\uparrow & cart & \uparrow \mathfrak{f} \\
 & & Ker(F(\mathfrak{f}'_{\mathfrak{e}})) & \xrightarrow{\mathfrak{k}(F(\mathfrak{f}'_{\mathfrak{e}}))} & F & \mathfrak{L}_{\mathfrak{f},\mathfrak{f}\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}'_{\mathfrak{f}}}} \ \mathfrak{L}_{\mathfrak{f},\mathfrak{f}} & \xrightarrow{\mathfrak{f}_{\mathfrak{f}}} & \mathfrak{L} \\
 & & \zeta_{\mathfrak{e},\gamma}\uparrow & cart & & \gamma'_{\mathfrak{e}_{\mathfrak{f}'_{\mathfrak{f}}}}\uparrow & cart \quad \gamma'_{\mathfrak{f}_{\mathfrak{f}}}\uparrow & cart & \uparrow\gamma \\
\mathcal{S}_{-}F(\mathcal{L}) & \longrightarrow & Ker(F(\mathfrak{f}'_{\mathfrak{e}}\gamma'_{\mathfrak{e}_{\mathfrak{f}}})) & \xrightarrow{\mathfrak{k}(..)} & & \mathfrak{L}_{\mathfrak{f}\gamma,\mathfrak{f}\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\mathfrak{f}'_{\mathfrak{f}}\gamma'_{\mathfrak{f}_{\mathfrak{f}}}}} \ \mathfrak{L}_{\mathfrak{f}\gamma,\mathfrak{f}} & \xrightarrow{\mathfrak{f}_{\mathfrak{f}\gamma}} & \mathfrak{L}_{\gamma} \\
\uparrow & & \mathfrak{k}(\zeta_{\mathfrak{e},\gamma})\uparrow & & & \mathfrak{k}(..)\uparrow & cart & \uparrow \mathfrak{k}(\mathfrak{f}'_{\mathfrak{f}}\gamma'_{\mathfrak{f}_{\mathfrak{f}}}) & \\
\mathcal{S}_{-}F(\mathfrak{L}) & \longrightarrow & Ker(F(\gamma'_{\mathfrak{e}_{\mathfrak{f}}})) & \xrightarrow{id} & & Ker(F(\gamma'_{\mathfrak{e}_{\mathfrak{f}}})) & \longrightarrow & Ker(\mathfrak{f}'_{\mathfrak{f}}\gamma'_{\mathfrak{f}_{\mathfrak{f}}}) &
\end{array}
\tag{1}
$$

with $\mathfrak{L}_{\mathfrak{e}} \xrightarrow{\mathfrak{e}} \mathfrak{L}$ running through deflations of $\mathfrak{L}$ and $\mathfrak{L}_{\gamma} \xrightarrow{\gamma} \mathfrak{L}$ through all arrows to $\mathfrak{L}$.

By definition,

$$
\mathcal{S}_{-}F(\mathcal{L}) = \lim_{\mathcal{L}_{\xi} \xrightarrow{\xi} \mathcal{L}} \big(\lim_{\mathcal{M}_{\mathfrak{u}} \xrightarrow{\mathfrak{u}} \mathcal{L}} Ker(F(\xi'_{\mathfrak{u}}))\big) \xrightarrow{\sim} \lim_{\mathfrak{L}_{\gamma} \xrightarrow{\gamma} \mathfrak{L}} \big(\lim_{\mathfrak{L}_{\mathfrak{e}} \xrightarrow{\mathfrak{e}} \mathfrak{L}} Ker(F((\mathfrak{f}'_{\mathfrak{f}}\gamma'_{\mathfrak{f}_{\mathfrak{f}}})'_{\mathfrak{e}}))\big)
$$

$$
\Big\downarrow \wr
$$

$$
\lim_{\mathfrak{L}_{\mathfrak{e}} \xrightarrow{\mathfrak{e}} \mathfrak{L}} \big(\lim_{\mathfrak{L}_{\gamma} \xrightarrow{\gamma} \mathfrak{L}} Ker(F((\mathfrak{f}'_{\mathfrak{f}}\gamma'_{\mathfrak{f}_{\mathfrak{f}}})'_{\mathfrak{e}}))\big)
$$

where $\mathcal{M}_{\mathfrak{u}} \xrightarrow{\mathfrak{u}} \mathcal{L}$ runs through deflations of the object $\mathcal{L}$ and $\mathcal{L}_{\xi} \xrightarrow{\xi} \mathcal{L}$ through arbitrary morphisms, and morphisms $\mathfrak{e}$, γ in the right limits come from the diagram (1) above. Finally, $(\mathfrak{f}'_{\mathfrak{f}}\gamma'_{\mathfrak{f}_{\mathfrak{f}}})'_{\mathfrak{e}}$ is a suggestive notation for the composition $\mathfrak{f}'_{\mathfrak{e}\mathfrak{f}}\gamma'_{\mathfrak{e}_{\mathfrak{f}'_{\mathfrak{f}}}}$ of the left vertical arrows inside of the brackets in the diagram (1) – the pull-back of the composition $\mathfrak{f}'_{\mathfrak{f}}\gamma'_{\mathfrak{f}_{\mathfrak{f}}}$ along $\mathfrak{L}_{\mathfrak{e}} \xrightarrow{\mathfrak{e}} \mathfrak{L}$.

In particular, there is a canonical morphism

$$
S_{-}F(\mathcal{L}) \xrightarrow{\mathfrak{d}_{\mathfrak{f}}} \lim_{\mathfrak{L}_{\gamma} \xrightarrow{\gamma} \mathfrak{L}} (Ker(F(\mathfrak{L}_{\mathfrak{f}\gamma,\mathfrak{f}} \xrightarrow{\mathfrak{f}'_{\mathfrak{f}}\gamma'_{\mathfrak{f}_{\mathfrak{f}}}} \mathfrak{L})), \tag{2}
$$

3.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category and C_Y a virtually semi-complete category with limits of filtered diagrams. Suppose that the category C_X is quasi-filtered. Then, for any functor $C_X \xrightarrow{F} C_Y$, there exists a (unique up to isomorphism) universal ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ such that $T_0 = F$. In other words, the functor*

$$
\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \tag{3}
$$

which assigns to (morphism of) ∂^*-functors their zero components has a right adjoint, Ψ_*.

Proof. Applying the iterations of the functor S_- and the connecting morphism constructed in 3.3.4 and 3.3.4 to the functor F, we obtain a ∂^*-functor

$$S_-^\bullet(F) = (S_-^i(F), \mathfrak{d}_i^F \mid i \geq 0).$$

The claim is that this ∂^*-functor is universal. The proof of the claim is an adaptation of the argument of 3.2, which is left to the reader. ∎

3.5.1. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a virtually semi-complete category with filtered diagrams. Then the functor*

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y)$$

is a continuous localization.

Proof. By 3.2, we have a pair of adjoint functors

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \xrightarrow{\Psi_*} \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$$

and the adjunction morphism $\Psi^*\Psi_* \longrightarrow Id$ is an isomorphism. The latter means that Ψ_* is a fully faithful functor and Ψ^* is a localization functor at a left multiplicative system. ∎

3.6. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a ∂^*-functor from $(C_X, \mathcal{E}_X)$ to a virtually semi-complete category C_Y. Let C_Z be another virtually semi-complete category and $C_Y \xrightarrow{F} C_Z$ a functor which preserves virtually trivial morphisms and limits of filtered diagrams. Then*

(a) If T is a universal ∂^*-functor, then $F \circ T = (F \circ T_i, F\mathfrak{d}_i | \; i \geq 0)$ is universal.

(b) If, in addition, the functor F is fully faithful, then the ∂^*-functor $F \circ T$ is universal iff the ∂^*-functor T is universal.

Proof. (a) The fact that $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ is a universal ∂^*-functor from $(C_X, \mathcal{E}_X)$ to C_Y means precisely that $(T_{i+1}, \mathfrak{d}_i) = (\mathcal{S}_- T_i, \mathfrak{d}^{T_i})$, because, by (the argument of) 3.5, the ∂^*-functor T is isomorphic to the ∂^*-functor

$$S_-^\bullet(T_0) = (S_-^i(T_0), \mathfrak{d}_i^{T_0} \mid i \geq 0).$$

Since the functor F preserves kernels of morphisms and filtered limits, and only these types of limits appear in the construction of $S_-(G)(L)$ (cf. 3.3.1, 3.3.2), the natural morphism

$$F \circ S_-(G)(L) \longrightarrow S_-(F \circ G)(L)$$

is an isomorphism for any functor $C_X \xrightarrow{G} C_Y$ such that $S_-(G)(L)$ exists.

In particular, the canonical morphism $F \circ S_-(T_i)(L) \longrightarrow S_-(F \circ T_i)(L)$ is an isomorphism for all $i \geq 0$ and all $L \in ObC_X$.

(b) The argument is the same as in II.3.4(b). ∎

3.6.1. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category and C_Y a virtually semi-complete category. A ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y is universal iff the ∂^*-functor $\widehat{T} \stackrel{\text{def}}{=} h_Y^{\mathfrak{r}\mathfrak{c}} \circ T = (\widehat{T}_i, \widehat{\mathfrak{d}}_i \mid i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to the category $C_{Y^{\mathfrak{r}\mathfrak{c}}}$ is universal.*

Proof. The Yoneda embedding $C_Y \xrightarrow{h_Y^{\mathfrak{r}\mathfrak{c}}} C_{Y^{\mathfrak{r}\mathfrak{c}}}$ is a fully faithful functor which preserves limits and maps virtually trivial morphisms to virtually trivial morphisms. In particular, it satisfies the conditions of 3.6(b). ∎

3.6.2. Note. Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category and C_Y a virtually semi-complete category. Then , for any functor $C_X \xrightarrow{G} C_{Y^{\mathfrak{r}\mathfrak{c}}}$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i | \, i \geq 0) = \Psi_*(G)$ from $(C_X, \mathfrak{E}_X)$ to $C_{Y^{\mathfrak{r}\mathfrak{c}}}$ whose zero component coincides with G. In particular, for every functor $C_X \xrightarrow{F} C_Y$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ such that $T_0 = h_Y^{\mathfrak{r}\mathfrak{c}} \circ F = \widetilde{F}$. It follows from 3.6(b) that there exists a universal ∂^*-functor whose zero component coincides with F iff for all $L \in ObC_X$ and all $i \geq 1$, the presheaves $T_i(L)$ are representable.

3.7. Contravariant functoriality for universal ∂^*-functors.

3.7.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ be right exact categories and $(C_X, \mathfrak{E}_X) \xrightarrow{\Phi} (C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ a fully faithful 'exact' functor. Let $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$ denote the class of all arrows $\mathfrak{M} \xrightarrow{\mathfrak{t}} \mathfrak{L}$ of $\mathfrak{E}_{\mathfrak{X}}$ such that, for any morphism $\Phi(\mathcal{L}) \xrightarrow{\mathfrak{f}} \mathfrak{L}$, there exists a commutative square*

$$\begin{array}{ccc} \Phi(\mathcal{M}) & \xrightarrow{\mathfrak{f}'} & \mathfrak{M} \\ {\scriptstyle \Phi(\mathfrak{s})}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} \\ \Phi(\mathcal{L}) & \xrightarrow{\mathfrak{f}} & \mathfrak{L} \end{array}$$

where $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{L}$ is a deflation.

(a) The class $\mathfrak{E}_{\mathfrak{X}}^{\Phi}$ is a right exact structure on the category $C_{\mathfrak{X}}$.

(b) Suppose that

(i) the category $C_{\mathfrak{X}}$ is quasi-filtered,

(ii) the functor $C_X \xrightarrow{\Phi} C_{\mathfrak{X}}$ has the following property: for any $\mathcal{L}_\xi \xrightarrow{\xi} \Phi(\mathcal{L})$, there is an arrow $\Phi(\mathfrak{L}) \xrightarrow{\gamma} \mathcal{L}_\xi$ for some $\mathfrak{L} \in ObC_X$.

Then, for any universal ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from the right exact category $(C_{\mathfrak{X}}, \mathcal{E}^{\Phi}_{\mathfrak{X}})$ to a virtually semi-complete category C_Y, the composition

$$T \circ \Phi = (T_i \circ \Phi, \mathfrak{d}_i\Phi \mid i \geq 0)$$

is a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y.

Proof. (a) One can refer to (the argument of) II.4.1(a), or notice that the class $\mathfrak{E}^{\Phi}_{\mathfrak{X}}$ is the intersection of the coinduced right exact structure ${}^{\Phi}\mathfrak{E}_{\mathfrak{X}}$ (see III.1.3.3) and the right exact structure $\mathfrak{E}_{\mathfrak{X}}$. So that $\mathfrak{E}^{\Phi}_{\mathfrak{X}}$ is a right exact structure on the category $C_{\mathfrak{X}}$.

(b) We can assume (using the observation 3.1.4.1) that the categories C_X, $C_{\mathfrak{X}}$ and C_Y are connected. In particular, by assumption, the category C_Y is virtually complete.

Thanks to 3.6.1, we can and will replace universal ∂^*-functors by their composition with the "reduced" Yoneda embedding $C_Y \xrightarrow{h^{\mathfrak{r}\mathfrak{c}}_Y} C_{Y^{\mathfrak{r}\mathfrak{c}}}$. The fact that the category $C_{Y^{\mathfrak{r}\mathfrak{c}}}$ has limits of small diagrams allows to use the formula

$$\mathcal{S}_{-}F(\mathcal{L}) = \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} \lim_{\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}} Ker(F(\xi'_{\mathfrak{e}})) \tag{1}$$

for any functor $C_{\mathfrak{X}} \xrightarrow{F} C_{Y^{\mathfrak{r}\mathfrak{c}}}$. Here $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ runs through deflations of the $\mathcal{L}$ which belong to $\mathfrak{E}^{\Phi}_{\mathfrak{X}}$, $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$ through arbitrary morphisms, and $\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\xi'_{\mathfrak{e}}} \mathcal{M}$ is the pull-back of $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$ along the deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$.

There is a canonical morphism

$$\mathcal{S}_{-}(F) \circ \Phi \longrightarrow \mathcal{S}_{-}(F \circ \Phi) \tag{2}$$

which is due to the fact that Φ maps $\mathfrak{E}_X$ to $\mathfrak{E}^{\Phi}_{\mathfrak{X}}$ and preserves pull-backs of deflations.

The claim is that, under conditions (b), the morphism (2) is an isomorphism.

(b1) It follows from the definition of the right exact structure $\mathfrak{E}^{\Phi}_{\mathfrak{X}}$ that, for any object $\mathcal{L}$ of the category C_X, the images $\Phi(\mathcal{L}_{\mathfrak{s}} \xrightarrow{\mathfrak{s}} \mathcal{L})$ of deflations of $\mathcal{L}$ contain refinements of any deflation $\mathfrak{L}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \Phi(\mathcal{L})$ from $\mathfrak{E}^{\Phi}_{\mathfrak{X}}$. This implies that, for any morphism $\mathcal{L}_\xi \xrightarrow{\xi} \Phi(\mathcal{L})$, the canonical morphism

$$\lim_{(\mathfrak{L}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \Phi(\mathcal{L})) \in \mathfrak{E}^{\Phi}_{\mathfrak{X}}} Ker(F(\xi'_{\mathfrak{t}}))) \longrightarrow \lim_{(\mathcal{L}_{\mathfrak{s}} \xrightarrow{\mathfrak{s}} \mathcal{L}) \in \mathfrak{E}_X} Ker(F(\xi'_{\Phi(\mathfrak{s})})$$

is an isomorphism. Therefore, it follows from (1) above that the canonical morphism

$$\mathcal{S}_{-}F(\Phi(\mathcal{L})) \longrightarrow \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \Phi(\mathcal{L})} \lim_{(\mathcal{L}_{\mathfrak{s}} \xrightarrow{\mathfrak{s}} \mathcal{L}) \in \mathfrak{E}_X} Ker(F(\xi'_{\Phi(\mathfrak{s})}) = \lim_{(\mathcal{L}_{\mathfrak{s}} \xrightarrow{\mathfrak{s}} \mathcal{L}) \in \mathfrak{E}_X} \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \Phi(\mathcal{L})} Ker(F(\xi'_{\Phi(\mathfrak{s})})$$

is an isomorphism.

(b2) Now we fix a deflation $\mathcal{L}_{\mathfrak{s}} \stackrel{\mathfrak{s}}{\to} \mathcal{L}$ of the object $\mathcal{L}$ and consider the limit

$$\lim_{\mathcal{L}_\xi \stackrel{\xi}{\to} \Phi(\mathcal{L})} Ker(F(\xi'_{\Phi(\mathfrak{s})}).$$

The claim is that the canonical morphism

$$\lim_{\mathcal{L}_\xi \stackrel{\xi}{\to} \Phi(\mathcal{L})} Ker(F(\xi'_{\Phi(\mathfrak{s})}) \longrightarrow \lim_{\mathcal{L}_\zeta \stackrel{\zeta}{\to} \mathcal{L}} Ker(F(\Phi(\zeta'_{\mathfrak{s}})) \qquad (3)$$

is an isomorphism.

In fact, consider a cone

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\lambda_{\Phi(\zeta)}} & \\ \lambda_{\mathfrak{x}} \big\downarrow & & F \circ \Phi \left(\begin{array}{ccc} \mathcal{L}_{\zeta,\mathfrak{s}} & \xrightarrow{\mathfrak{s}_\zeta} & \mathcal{L}_\zeta \\ \zeta'_{\mathfrak{s}} \big\downarrow & cart & \big\downarrow \zeta \\ \mathcal{L}_{\mathfrak{s}} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \end{array} \right), \\ \mathfrak{x} & \longrightarrow & \end{array} \qquad (4)$$

where $\mathfrak{x}$ is an initial object of the category C_Y^* and $(\mathcal{L}_\zeta, \mathcal{L}_\zeta \stackrel{\zeta}{\to} \mathcal{L})$ runs through objects of the category $C_X/\mathcal{L}$. The claim is that the cone (4) uniquely extends to a cone

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\lambda_\xi} & \\ \lambda_{\mathfrak{x}} \big\downarrow & & F \left(\begin{array}{ccc} \mathcal{L}_{\xi,\Phi(\mathfrak{s})} & \xrightarrow{\Phi(\mathfrak{s})_\xi} & \mathcal{L}_\zeta \\ \xi'_{\Phi(\mathfrak{s})} \big\downarrow & cart & \big\downarrow \xi \\ \Phi(\mathcal{L}_{\mathfrak{s}}) & \xrightarrow{\Phi(\mathfrak{s})} & \Phi(\mathcal{L}) \end{array} \right), \\ \mathfrak{x} & \longrightarrow & \end{array} \qquad (5)$$

where $(\mathcal{L}_\xi, \mathcal{L}_\xi \stackrel{\xi}{\to} \Phi(\mathcal{L}))$ runs through objects of the category $C_{\mathfrak{X}}/\Phi(\mathcal{L})$.

By condition (b)(ii), there is a morphism $\Phi(\mathcal{L}_\gamma) \xrightarrow{\widetilde{\gamma}} \mathcal{L}_\xi$. Since the functor Φ is fully faithful, the composition $\Phi(\mathcal{L}_\gamma) \xrightarrow{\xi\circ\widetilde{\gamma}} \Phi(\mathcal{L})$ is $\Phi(\gamma)$ for a unique morphism $\mathcal{L}_\gamma \xrightarrow{\gamma} \mathcal{L}$.

Since the functor $C_X \xrightarrow{\Phi} C_{\mathfrak{X}}$ preserves pull-backs of deflations, it maps the cartesian square

$$\begin{array}{ccc} \mathcal{L}_{\gamma,\mathfrak{s}} & \xrightarrow{\mathfrak{s}_\gamma} & \mathcal{L}_\gamma \\ \gamma'_{\mathfrak{s}} \big\downarrow & cart & \big\downarrow \gamma \\ \mathcal{L}_{\mathfrak{s}} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \end{array}$$

to a cartesian square, and there is a unique cartesian square

$$\begin{array}{ccc} \Phi(\mathcal{L}_{\gamma,\mathfrak{s}}) & \xrightarrow{\Phi(\mathfrak{s}_\gamma)} & \Phi(\mathcal{L}_\gamma) \\ {\scriptstyle \widetilde{\gamma}''_{\Phi(\mathfrak{s})}}\big\downarrow & cart & \big\downarrow {\scriptstyle \widetilde{\gamma}} \\ \mathcal{L}_{\xi,\Phi(\mathfrak{s})} & \xrightarrow{\Phi(\mathfrak{s})_\xi} & \mathcal{L}_\xi \end{array}$$

in particular, a unique morphism $\Phi(\mathcal{L}_{\gamma,\mathfrak{s}}) \xrightarrow{\widetilde{\gamma}''_{\Phi(\mathfrak{s})}} \mathcal{L}_{\xi,\Phi(\mathfrak{s})}$, such that the cartesian square

$$\Phi\left(\begin{array}{ccc} \mathcal{L}_{\gamma,\mathfrak{s}} & \xrightarrow{\mathfrak{s}_\gamma} & \mathcal{L}_\gamma \\ {\scriptstyle \gamma'_{\mathfrak{s}}}\big\downarrow & cart & \big\downarrow {\scriptstyle \gamma} \\ \mathcal{L}_{\mathfrak{s}} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \end{array}\right)$$

is the composition of the cartesian squares

$$\begin{array}{ccc} \Phi(\mathcal{L}_{\gamma,\mathfrak{s}}) & \xrightarrow{\Phi(\mathfrak{s}_\gamma)} & \Phi(\mathcal{L}_\gamma) \\ {\scriptstyle \widetilde{\gamma}''_{\Phi(\mathfrak{s})}}\big\downarrow & cart & \big\downarrow {\scriptstyle \widetilde{\gamma}} \\ \mathcal{L}_{\xi,\Phi(\mathfrak{s})} & \xrightarrow{\Phi(\mathfrak{s})_\xi} & \mathcal{L}_\xi \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{L}_{\xi,\Phi(\mathfrak{s})} & \xrightarrow{\Phi(\mathfrak{s})_\xi} & \mathcal{L}_\zeta \\ {\scriptstyle \xi'_{\Phi(\mathfrak{s})}}\big\downarrow & cart & \big\downarrow {\scriptstyle \xi} \\ \Phi(\mathcal{L}_{\mathfrak{s}}) & \xrightarrow{\Phi(\mathfrak{s})} & \Phi(\mathcal{L}) \end{array}$$

We define the claimed morphism $\mathfrak{M} \xrightarrow{\lambda_\xi} F(\mathcal{L}_{\xi,\Phi(\mathfrak{s})})$ in (5) as the composition of

$$\mathfrak{M} \xrightarrow{\lambda_{\Phi(\gamma)}} F\Phi(\mathcal{L}_{\gamma,\mathfrak{s}}) \quad \text{and} \quad F(\Phi(\mathcal{L}_{\gamma,\mathfrak{s}}) \xrightarrow{\widetilde{\gamma}''_{\Phi(\mathfrak{s})}} \mathcal{L}_{\xi,\Phi(\mathfrak{s})}). \tag{6_γ}$$

(b2.1) The composition of the morphisms (6) does not depend on the choice of the morphism $\Phi(\mathcal{L}_\gamma) \xrightarrow{\widetilde{\gamma}} \mathcal{L}_\xi$.

Indeed, let $\Phi(\mathcal{L}_\beta) \xrightarrow{\widetilde{\beta}} \mathcal{L}_\xi$ be another morphism. By hypothesis, the category C_X is quasi-filtered; in particular, there is a commutative square

$$\begin{array}{ccc} \mathfrak{L}_\alpha & \xrightarrow{\alpha} & \Phi(\mathcal{L}_\gamma) \\ {\scriptstyle \alpha'}\big\downarrow & & \big\downarrow {\scriptstyle \widetilde{\gamma}} \\ \Phi(\mathcal{L}_\beta) & \xrightarrow{\widetilde{\beta}} & \mathcal{L}_\xi \end{array}$$

By condition (b)(ii), there exists a morphism $\Phi(\mathcal{L}_\rho) \xrightarrow{\widetilde{\rho}} \mathfrak{L}_\alpha$. Since the functor $C_X \xrightarrow{\Phi} C_{\mathfrak{X}}$ is fully faithful, $\alpha \circ \widetilde{\rho} = \Phi(\alpha_1)$ and $\alpha' \circ \widetilde{\rho} = \Phi(\alpha'_1)$ for uniquely determined morphisms $\mathcal{L}_\rho \xrightarrow{\alpha_1} \mathcal{L}_\gamma$ and $\mathcal{L}_\rho \xrightarrow{\alpha'_1} \mathcal{L}_\beta$; and we have a commutative square

$$\begin{array}{ccc} \Phi(\mathcal{L}_\rho) & \xrightarrow{\Phi(\alpha_1)} & \Phi(\mathcal{L}_\gamma) \\ {\scriptstyle \Phi(\alpha'_1)}\big\downarrow & & \big\downarrow {\scriptstyle \widetilde{\gamma}} \\ \Phi(\mathcal{L}_\beta) & \xrightarrow{\widetilde{\beta}} & \mathcal{L}_\xi \end{array} \tag{7}$$

Since (4) is a cone, this implies that the composition of the morphisms (6_γ) equals to the composition of the morphisms

$$\mathfrak{M} \xrightarrow{\lambda_{\Phi(\beta)}} F\Phi(\mathcal{L}_{\beta,\mathfrak{s}}) \quad \text{and} \quad F(\Phi(\mathcal{L}_{\beta,\mathfrak{s}}) \xrightarrow{\widetilde{\beta}''_{\Phi(\mathfrak{s})}} \mathcal{L}_{\xi,\Phi(\mathfrak{s})}). \tag{6_β}$$

obtained from $\Phi(\mathcal{L}_\beta) \xrightarrow{\widetilde{\beta}} \mathcal{L}_\xi$ the same way as morphisms $\Phi(\mathcal{L}_\gamma) \xrightarrow{\widetilde{\gamma}} \mathcal{L}_\xi$, because

$$F(\widetilde{\beta}''_{\Phi(\mathfrak{s})}) \circ \lambda_{\Phi(\beta)} = F(\widetilde{\beta}''_{\Phi(\mathfrak{s})}) \circ F\Phi(\alpha'_1) \circ \lambda_{\Phi(\rho)} =$$
$$F(\widetilde{\gamma}''_{\Phi(\mathfrak{s})}) \circ F\Phi(\alpha_1) \circ \lambda_{\Phi(\rho)} = F(\widetilde{\gamma}''_{\Phi(\mathfrak{s})}) \circ \lambda_{\Phi(\gamma)}$$

thanks to the commutativity of the square (7).

(b2.2) Thus, any cone of the form (4) extends uniquely up to isomorphism to a cone of the form (5). In particular, the universal cone

$$\begin{array}{ccc} \lim\limits_{\mathcal{L}_\zeta \xrightarrow{\zeta} \mathcal{L}} Ker(F(\Phi(\zeta'_{\mathfrak{s}})) & \xrightarrow{\lambda_{\Phi(\zeta)}} & \\ {\scriptstyle \lambda_{\mathfrak{x}}}\big\downarrow & & F \circ \Phi \left(\begin{array}{ccc} \mathcal{L}_{\zeta,\mathfrak{s}} & \xrightarrow{\mathfrak{s}_\zeta} & \mathcal{L}_\zeta \\ {\scriptstyle \zeta'_{\mathfrak{s}}}\big\downarrow & cart & \big\downarrow {\scriptstyle \zeta} \\ \mathcal{L}_{\mathfrak{s}} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \end{array} \right), \\ \mathfrak{x} & \longrightarrow & \end{array} \tag{8}$$

extends to a cone

$$\begin{array}{ccc} \lim\limits_{\mathcal{L}_\zeta \xrightarrow{\zeta} \mathcal{L}} Ker(F(\Phi(\zeta'_{\mathfrak{s}})) & \xrightarrow{\lambda_\xi} & \\ {\scriptstyle \lambda_{\mathfrak{x}}}\big\downarrow & & F \left(\begin{array}{ccc} \mathcal{L}_{\xi,\Phi(\mathfrak{s})} & \xrightarrow{\Phi(\mathfrak{s})_\xi} & \mathcal{L}_\zeta \\ {\scriptstyle \xi'_{\Phi(\mathfrak{s})}}\big\downarrow & cart & \big\downarrow {\scriptstyle \xi} \\ \Phi(\mathcal{L}_{\mathfrak{s}}) & \xrightarrow{\Phi(\mathfrak{s})} & \Phi(\mathcal{L}) \end{array} \right), \\ \mathfrak{x} & \longrightarrow & \end{array} \tag{9}$$

and the latter determines uniquely a morphism

$$\lim_{\mathcal{L}_\zeta \xrightarrow{\zeta} \mathcal{L}} Ker(F(\Phi(\zeta'_{\mathfrak{s}})) \longrightarrow \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \Phi(\mathcal{L})} Ker(F(\xi'_{\Phi(\mathfrak{s})}) \tag{10}$$

due to the fact that the cone

$$\begin{array}{ccc} \lim\limits_{\mathcal{L}_\xi \overset{\xi}{\to} \Phi(\mathcal{L})} Ker(F(\xi'_{\Phi(\mathfrak{s})}) & \xrightarrow{\mathfrak{d}_{\xi,\Phi(\mathfrak{s})}} & F\left(\begin{array}{ccc} \mathcal{L}_{\xi,\Phi(\mathfrak{s})} & \xrightarrow{\Phi(\mathfrak{s})_\xi} & \mathcal{L}_\zeta \\ \xi'_{\Phi(\mathfrak{s})}\big\downarrow & cart & \big\downarrow \xi \\ \Phi(\mathcal{L}_\mathfrak{s}) & \xrightarrow{\Phi(\mathfrak{s})} & \Phi(\mathcal{L}) \end{array}\right) \\ \lambda_\mathfrak{x}\big\downarrow & & \\ \mathfrak{x} & \longrightarrow & \end{array}$$

is universal. It follows from the universal properties of limits that the morphism (10) is inverse to the canonical morphism

$$\lim_{\mathcal{L}_\xi \overset{\xi}{\to} \Phi(\mathcal{L})} Ker(F(\xi'_{\Phi(\mathfrak{s})}) \longrightarrow \lim_{\mathcal{L}_\zeta \overset{\zeta}{\to} \mathcal{L}} Ker(F(\Phi(\zeta'_\mathfrak{s})) \qquad (3)$$

(b3) The fact that (3) is an isomorphism implies that

$$\lim_{(\mathcal{L}_\mathfrak{s} \overset{\mathfrak{s}}{\to} \mathcal{L}) \in \mathfrak{E}_X} \ \lim_{\mathcal{L}_\xi \overset{\xi}{\to} \Phi(\mathcal{L})} Ker(F(\xi'_{\Phi(\mathfrak{s})}) \longrightarrow \lim_{(\mathcal{L}_\mathfrak{s} \overset{\mathfrak{s}}{\to} \mathcal{L}) \in \mathfrak{E}_X} \ \lim_{\mathcal{L}_\zeta \overset{\zeta}{\to} \mathcal{L}} Ker(F(\Phi(\zeta'_\mathfrak{s}))$$

is an isomorphism. But,

$$\lim_{(\mathcal{L}_\mathfrak{s} \overset{\mathfrak{s}}{\to} \mathcal{L}) \in \mathfrak{E}_X} \ \lim_{\mathcal{L}_\zeta \overset{\zeta}{\to} \mathcal{L}} Ker(F(\Phi(\zeta'_\mathfrak{s})) = \mathcal{S}_-(F \circ \Phi)(\mathcal{L})$$

and, by (b1), the canonical morphism

$$\mathcal{S}_-F(\Phi(\mathcal{L})) \longrightarrow \lim_{(\mathcal{L}_\mathfrak{s} \overset{\mathfrak{s}}{\to} \mathcal{L}) \in \mathfrak{E}_X} \ \lim_{\mathcal{L}_\xi \overset{\xi}{\to} \Phi(\mathcal{L})} Ker(F(\xi'_{\Phi(\mathfrak{s})})$$

is an isomorphism. Since $\mathcal{L}$ in this argument is an arbitrary object of the category C_X, this shows that, under the assumptions (b), the canonical functor morphism

$$(\mathcal{S}_-F) \circ \Phi \longrightarrow \mathcal{S}_-(F \circ \Phi)$$

is an isomorphism for any functor $C_\mathfrak{X} \xrightarrow{F} C_Y$. This, in turn, proves that, for any universal ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from the right exact category $(C_\mathfrak{X}, \mathcal{E}_\mathfrak{X}^\Phi)$ to a virtually semi-complete category C_Y, the composition $T \circ \Phi = (T_i \circ \Phi, \mathfrak{d}_i \Phi \mid i \geq 0)$ is a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y. ■

3.8. Right derived functors via the category of sheaves.

3.8.1. The "reduced" category of sheaves on a subcanonical presite. Let C_X be a svelte category having the decomposition $C_X = \coprod\limits_{\mathfrak{i} \in \pi_0(X)} C_{X_\mathfrak{i}}$ into the disjoint union of its connected components; and let τ be a subcanonical pretopology on C_X. We associate with the presite (C_X, τ) the category

$$C_{X^{\mathfrak{re}}_{\tau}} \stackrel{\mathrm{def}}{=} \coprod_{\mathfrak{i} \in \pi_0(X)} C_{X^{\circledast}_{\mathfrak{i},\tau_{\mathfrak{i}}}}, \tag{1}$$

which we call the *"reduced" category of sheaves* on (C_X, τ). Here $\tau_{\mathfrak{i}}$ is the restriction of the pretopology τ to the connected component $C_{X_{\mathfrak{i}}}$ and $C_{X^{\circledast}_{\mathfrak{i},\tau_{\mathfrak{i}}}}$ denotes the "reduced" category of sheaves on the connected component $(C_{X_{\mathfrak{i}}}, \tau_{\mathfrak{i}})$, which is defined by the formula

$$C_{X^{\circledast}_{\mathfrak{i},\tau_{\mathfrak{i}}}} = \lim h_{X_{\mathfrak{i}}} \backslash (C_{X_{\mathfrak{i}}}, \tau_{\mathfrak{i}})^{\wedge}.$$

3.8.1.1. Proposition. *(a) Each of the subcategories $C_{X^{\circledast}_{\mathfrak{i},\tau_{\mathfrak{i}}}}$ is a connected component of the category $C_{X^{\mathfrak{re}}_{\tau}}$. So that (1) is the decomposition of the category $C_{X^{\mathfrak{re}}_{\tau}}$ into the disjoint union of connected components.*

(b) The sheafification functor $C^{\wedge}_X \xrightarrow{\mathfrak{q}^*_{\tau}} (C_X, \tau)^{\wedge}$ induces an exact continuous (that is having a right adjoint) localization functor

$$C_{X^{\mathfrak{re}}} \xrightarrow{\mathfrak{q}^{\mathfrak{re}}_{\tau}} C_{X^{\mathfrak{re}}_{\tau}}. \tag{2}$$

which we call the "reduced" sheafification functor.

(c) If the category C_X is virtually semi-complete, then the category $C_{X^{\mathfrak{re}}_{\tau}}$ is semi-complete and semi-cocomplete; that is each of the connected components $C_{X^{\circledast}_{\mathfrak{i},\tau_{\mathfrak{i}}}}$, $\mathfrak{i} \in \pi_0(X)$, is a complete and a cocomplete category.

Proof. The argument is left to the reader. ■

3.8.2. The "reduced" canonical embedding. Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category. The construction of 3.8.1 assigns to the right exact category $(C_X, \mathfrak{E}_X)$ the "reduced" category

$$C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}} = \coprod_{\mathfrak{i} \in \pi_0(X)} C_{X^{\circledast}_{\mathfrak{i},\mathfrak{E}_{\mathfrak{i}}}}, \tag{1}$$

of sheaves on $(C_X, \mathfrak{E}_X)$. Here $\mathfrak{E}_{\mathfrak{i}}$ denotes the restriction of the right exact structure $\mathfrak{E}_X$ to the connected component $C_{X_{\mathfrak{i}}}$.

We denote by

$$C_X \xrightarrow{\mathfrak{j}^{\mathfrak{re}}_X} C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}} \tag{2}$$

the composition of the Yoneda embedding

$$C_X \xrightarrow{h_X^{\mathfrak{re}}} C_{X^{\mathfrak{re}}} = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X_{\mathfrak{i}}^{\circledast}}$$

with the sheafification functor

$$C_{X^{\mathfrak{re}}} \xrightarrow{\mathfrak{q}_{\mathfrak{E}_X}^{\mathfrak{re}}} C_{X_{\mathfrak{E}_X}^{\mathfrak{re}}} = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X_{\mathfrak{i},\mathfrak{E}_{\mathfrak{i}}}^{\circledast}}.$$

Since right exact structures are subcanonical pretopologies, the functor (2) is fully faithful. We denote by $\mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}$ the canonical right exact structure on the category $C_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}$.

3.8.2.1. It follows from I.2.1 that the embedding (2) is a fully faithful 'exact' functor from the right exact category $(C_X, \mathfrak{E}_X)$ to the right exact category $(C_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}_X}^{\mathfrak{re}}})$.

3.8.2.2. It follows from 3.8.1.1 that if the category C_X is virtually semi-complete, then the category $C_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}$ is semi-complete.

3.8.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category and*

$$(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^{\mathfrak{re}}} (C_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}_X}^{\mathfrak{re}}})$$

the canonical embedding. Suppose that the category C_X is virtually semi-complete.

Then, for any universal ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from the right exact category $(C_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}_X}^{\mathfrak{re}}})$ to a virtually semi-complete category C_Y, the composition

$$T \circ \mathfrak{j}_X^{\mathfrak{re}} = (T_i \circ \mathfrak{j}_X^{\mathfrak{re}}, \mathfrak{d}_i \mathfrak{j}_X^{\mathfrak{re}} \mid i \geq 0)$$

is a universal ∂^*-functor from the right exact category $(C_X, \mathfrak{E}_X)$ to the category C_Y.

Proof. It follows from I.2.2.1(b) that the canonical (that is the finest) right exact structure $\mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}$ on the category $C_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}$ coincides with the right exact structure coinduced by the embedding $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^{\mathfrak{re}}} (C_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}_X}^{\mathfrak{re}}})$. The condition that the category C_X is virtually semi-complete implies (actually, is equivalent to) that the category $C_{X_{\mathfrak{E}_X}^{\mathfrak{re}}}$ is quasi-filtered. Finally, if $\mathfrak{F}$ is a presheaf of sets on C_X, then $\mathfrak{F}(\mathcal{M}) \neq \emptyset$ for some $\mathcal{M} \in ObC_X$ iff there exists morphisms from $\widehat{\mathcal{M}} = C_X(-, \mathcal{M})$ to $\mathfrak{F}$. In particular, for any object $\mathfrak{F}$ of the category $C_{X_{\mathfrak{E}}}$ there exist morphisms $\mathfrak{j}_X^{\mathfrak{re}}(\mathcal{M}) \longrightarrow \mathfrak{F}$ for some $\mathcal{M} \in ObC_X$. All together shows that the assumptions of 3.7.1 hold, hence the assertion. ∎

3.8.4. Right derived functors via the category of sheaves. Following the pattern of II.4.3, we can apply 3.8.3 to replace the computation of derived functors of any functor from a *virtually semi-complete* right exact category $(C_X, \mathfrak{E}_X)$ to a virtually semi-complete category C_Y by computation of derived functors of the associated functor from the semi-complete right exact category $(C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}}, \mathfrak{E}^{\mathfrak{s}}_{X^{\mathfrak{re}}_{\mathfrak{E}_X}})$ to the semi-complete category $C_{Y^{\mathfrak{re}}} = \coprod_{\mathfrak{i}\in\pi_0(Y)} C_{Y^{\circledast}_{\mathfrak{i}}}$ associated with the category C_Y.

Namely, we associate with a functor $C_X \xrightarrow{F} C_Y$ the composition $C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}} \xrightarrow{F^{\mathfrak{re}}_{\mathfrak{E}_X}} C_{Y^{\mathfrak{re}}}$ of the functor $C^{\mathfrak{re}}_X \xrightarrow{F^{\mathfrak{re}}} C_{Y^{\mathfrak{re}}}$ with the inclusion functor $C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}} \longrightarrow C^{\mathfrak{re}}_X$. The composition of $F^{\mathfrak{re}}_{\mathfrak{E}_X}$ with the canonical embedding $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}^{\mathfrak{re}}_X} (C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}}, \mathfrak{E}^{\mathfrak{s}}_{X^{\mathfrak{re}}_{\mathfrak{E}_X}})$ is isomorphic to the composition of $C_X \xrightarrow{F} C_Y$ with the Yoneda embedding $C_Y \xrightarrow{h^{\mathfrak{re}}_Y} C_{Y^{\mathfrak{re}}}$.

By 3.8.3, the universal ∂^*-functor $\mathcal{S}^{\bullet}_{-}(h^{\mathfrak{re}}_Y \circ F)$, whose zero component is $h^{\mathfrak{re}}_Y \circ F$, is isomorphic to the composition $\mathcal{S}^{\bullet}_{-}(F^{\mathfrak{re}}_{\mathfrak{E}_X})\mathfrak{j}^{\mathfrak{re}}_X$ of the universal ∂^*-functor, whose zero component is the functor $F^{\mathfrak{re}}_{\mathfrak{E}_X}$, with the canonical embedding

$$(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}^{\mathfrak{re}}_X} (C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}}, \mathfrak{E}^{\mathfrak{s}}_{X^{\mathfrak{re}}_{\mathfrak{E}_X}}).$$

It follows that the universal ∂^*-functor $(C_X, \mathfrak{E}_X) \xrightarrow{\mathcal{S}^{\bullet}_{-}F} C_Y$ exists iff the functors $\mathcal{S}^n_{-}(F^{\mathfrak{re}}_{\mathfrak{E}_X})\mathfrak{j}^{\mathfrak{re}}_X$ factor through the Yoneda embedding $C_Y \xrightarrow{h^{\mathfrak{re}}_Y} C_{Y^{\mathfrak{re}}}$ for every $n \geq 1$.

In this case, $h^{\mathfrak{re}}_X \circ \mathcal{S}^{\bullet}_{-}F \simeq \mathcal{S}^{\bullet}_{-}(F^{\mathfrak{re}}_{\mathfrak{E}_X})\mathfrak{j}^{\mathfrak{re}}_X$.

3.8.4.1. Remark. One of the immediate advantages of having the isomorphism

$$\mathcal{S}^{\bullet}_{-}(F^{\mathfrak{re}}_{\mathfrak{E}_X})\mathfrak{j}^{\mathfrak{re}}_X \xrightarrow{\sim} \mathcal{S}^{\bullet}_{-}(h^{\mathfrak{re}}_Y \circ F)$$

is that it reduces the computation of derived functors to the case already studied in the the previous chapters: when both the source of the derived functor – a right exact category, and its target are categories with initial objects.

In fact, the computation of derived functors of functors from a right exact category $(C_X, \mathfrak{E}_X)$ to a category C_Y is reduced to the case when both C_X and C_Y are connected. Since, by hypothesis, the categories C_X and C_Y are virtually semi-complete, their connectedness means that they are virtually complete; that is the categories $C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}}$ and $C_{Y^{\mathfrak{re}}}$ have initial objects.

3.9. The dual picture: ∂-functors and universal ∂-functors.

Let $(C_X, \mathfrak{I}_X)$ be a left exact category. A ∂-functor on $(C_X, \mathfrak{I}_X)$ is the data which becomes a ∂^*-functor in the dual right exact category. A ∂-functor on $(C_X, \mathfrak{I}_X)$ is *universal* if its dualization is a universal ∂^*-functor. A universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ will be also called the *left derived* functor of its zero component T_0.

We leave to the reader the reformulation in the context of ∂-functors of all notions and facts about ∂^*-functors.

3.10. Remark. If $(C_X, \mathfrak{E}_X)$ is a category with initial objects, the definition of ∂^*-functors given in 3.1 differs from the one we used prior to Chapter VII (see II.2.0). But, it follows from the formula for derived functors that universal ∂^*-functors are the same. By duality, same holds for universal ∂-functors from left exact categories with final objects.

4. Universal problems for universal ∂^*- and ∂-functors.

We extend the setting of Section II.8 to arbitrary right and left exact categories.

4.1. The categories of universal ∂^*-functors. Fix a svelte right exact category $(C_X, \mathfrak{E}_X)$. Let $\partial^*\widetilde{\mathfrak{Un}}(X, \mathfrak{E}_X)$ denote the category whose objects are universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to virtually semi-complete categories (see 3.1.0).

Let T be a universal ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to a category C_Y and $\widetilde{T}$ a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to a category C_Z. A morphism from T to T' is a pair (F, ϕ), where F is a functor from C_Y to C_Z which preserves filtered limits and maps virtually trivial morphisms to virtually trivial morphisms, and ϕ is a ∂^*-functor isomorphism $F \circ T \xrightarrow{\sim} T'$.

If (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by

$$(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi).$$

4.1.0. We denote by $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ the full subcategory of the category $\partial^*\widetilde{\mathfrak{Un}}(X, \mathfrak{E}_X)$ generated by those universal ∂^*-functors whose zero component maps virtually trivial morphisms to virtually trivial morphisms.

4.1.1. The category $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$. We denote by $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ the subcategory of $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ whose objects are ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to *semi-complete* categories C_Y and morphisms are pairs (F, ϕ) such that the functor F preserves limits.

4.2. The categories of universal ∂-functors. Dually, for a left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, we denote by $\partial\widetilde{\mathfrak{Un}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal ∂-functors from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to virtually semi-cocomplete categories. Given two universal ∂-functors T and T' from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to respectively C_Y and C_Z, a morphism from T to T' is a pair (F, ψ), where F is a functor from C_Y to C_Z preserving filtered colimits and ψ is a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by

$$(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi').$$

4.2.0. We denote by $\partial\mathfrak{Un}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ the full subcategory of the category $\widetilde{\partial\mathfrak{Un}}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ generated by the universal ∂-functors from $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$ to virtually semi-cocomplete categories whose zero component maps virtually cotrivial morphisms to virtually cotrivial morphisms.

4.2.1. The category $\partial\mathfrak{Un}^{\mathfrak{c}}(X,\mathfrak{I}_X)$. We denote by $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial\mathfrak{Un}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ whose objects are ∂-functors with values in semi-cocomplete categories and morphisms are pairs (F,ψ) such that the functor F preserves colimits and maps virtually cotrivial morphisms to virtually cotrivial morphisms.

4.3. Proposition. *Let $(C_X,\mathfrak{E}_X)$ be a svelte right exact category and $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$ a svelte left exact category. Suppose that the category C_X is virtually semi-complete and the category $C_{\mathfrak{X}}$ is virtually semi-cocomplete. Then the categories $\partial^*\mathfrak{Un}(X,\mathfrak{E}_X)$, $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X,\mathfrak{E}_X)$, $\partial\mathfrak{Un}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$, and $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. It is convenient to start with the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$. Consider the "reduced" Yoneda embedding

$$C_{\mathfrak{X}} \xrightarrow{h^{\mathfrak{r}\mathfrak{e}}_{\mathfrak{X}}} C^{\mathfrak{r}\mathfrak{e}}_{X} = \coprod_{\mathfrak{i}\in\pi_0(\mathfrak{X})} C_{\mathfrak{X}^{\circledast}_{\mathfrak{i}}}, \quad \mathcal{M}\longmapsto \widehat{\mathcal{M}} \stackrel{\mathrm{def}}{=} C_{\mathfrak{X}}(-,M). \tag{1}$$

Let $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ be the universal ∂-functor from the left exact category $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$ to the semi-bicomplete category $C^{\mathfrak{r}\mathfrak{e}}_{\mathfrak{X}}$, whose zero component coincides with the Yoneda embedding (1), that is $Ext^0_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}} = h^{\mathfrak{r}\mathfrak{e}}_{\mathfrak{X}}$. The claim is that

The universal ∂-functor $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is an initial object of the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$.

In fact, let C_Y be a semi-cocomplete category. The category $\mathcal{H}om^{\mathfrak{r}\mathfrak{e}}(C_{\mathfrak{X}},C_Y)$ of functors from $C_{\mathfrak{X}}$ and C_Y which map virtually cotrivial morphisms to cotrivial morphisms, is naturally equivalent to the category $\mathcal{H}om_{\mathfrak{c}}(C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}},C_Y)$ of continuous (i.e. having a right adjoint, or, what is the same, preserving colimits) functors from the category $C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}}$ of presheaves of sets on $C_{\mathfrak{X}}$ to the category C_Y. Let $F^{\mathfrak{r}\mathfrak{e}}$ denote the determined uniquely up to isomorphism continuous functor corresponding to F, i.e. $F = F^{\mathfrak{r}\mathfrak{e}}\circ h^{\mathfrak{r}\mathfrak{e}}_{\mathfrak{X}}$.

(a) Each of the connected components $C_{\mathfrak{X}^{\circledast}_{\mathfrak{i}}}$ of the category $C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}} = \coprod_{\mathfrak{i}\in\pi_0(\mathfrak{X})} C_{\mathfrak{X}^{\circledast}_{\mathfrak{i}}}$ has a final object – the constant presheaf taking values in the one-element set. Therefore, for every object $\mathcal{L}$ of the category $C_{\mathfrak{X}}$, the presheaf of sets $Ext^1_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}(\mathcal{L}) = Ext_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}(\mathcal{L})$ is the colimit of the diagram

$$\begin{array}{ccccc} \widehat{\mathcal{L}} & \xrightarrow{\ \widehat{\mathfrak{j}}\ } & \widehat{\mathcal{M}} & & \\ {\scriptstyle\widehat{\xi}}\downarrow & & \downarrow{\scriptstyle\widehat{\xi}_{\mathfrak{j}}} & & \\ \widehat{\mathcal{L}}_{\xi} & \xrightarrow{\ \widehat{\mathfrak{j}}_{\xi}\ } & \widehat{\mathcal{M}}_{\mathfrak{j},\xi} & \xrightarrow{\ \mathfrak{c}(\widehat{\mathfrak{j}}_{\xi})\ } & Cok(\widehat{\mathfrak{j}}_{\xi}) \end{array} \tag{2}$$

where

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\ \mathfrak{j}\ } & \mathcal{M} \\ {\scriptstyle \xi}\big\downarrow & cocart & \big\downarrow{\scriptstyle \xi_{\mathfrak{j}}} \\ \mathcal{L}_\xi & \xrightarrow{\ \widehat{\mathfrak{j}}_\xi\ } & \mathcal{M}_{\mathfrak{j},\xi} \end{array}$$

runs through the (objects of the) category of push-forwards of inflations of the object $\mathcal{L}$.

(b) Since the functor $F^{\mathfrak{re}}$ preserves colimits, the formula for $S_+F(N)$ can be rewritten as follows:

$$\begin{aligned} S_+F(\mathcal{L}) &= \operatorname{colim}(Cok(F(\mathcal{M} \xrightarrow{\xi_{\mathfrak{j}}} \mathcal{M}_{\mathfrak{j},\xi}))) = \operatorname{colim}(Cok(F^{\mathfrak{re}}(\widehat{\mathcal{M}} \xrightarrow{\widehat{\xi}_{\mathfrak{j}}} \widehat{\mathcal{M}}_{\mathfrak{j},\xi}))) = \\ &= F^{\mathfrak{re}}(\operatorname{colim}(Cok(\widehat{\mathcal{M}} \xrightarrow{\widehat{\xi}_{\mathfrak{j}}} \widehat{\mathcal{M}}_{\mathfrak{j},\xi}))) = F^{\mathfrak{re}} S_+ h^{\mathfrak{re}}_{\mathfrak{X}}(\mathcal{L}) = F^{\mathfrak{re}} Ext^1_{\mathfrak{X}}(\mathcal{L}), \end{aligned} \tag{3}$$

where colimit is taken by the diagram of all push-forwards of inflations of the object $\mathcal{L}$.

The proof of the remaining assertions follows (with obvious adjustments) the arguments of the corresponding parts of the proof of II.8.1. ■

5. The structure of universal ∂-functors to semi-cocomplete categories.

5.1. Observations. Let $(C_X, \mathfrak{I}_X)$ be a svelte left exact category and C_Y a virtually cocomplete category with colimits of cofiltered diagrams. Then, by (the dual version of) 3.3.4, we have an endofunctor S_+ of the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y and, for every push-forward

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\ \mathfrak{j}\ } & \mathcal{M} \\ {\scriptstyle \xi}\big\downarrow & cocart & \big\downarrow{\scriptstyle \xi_{\mathfrak{j}}} \\ \mathcal{L}_\xi & \xrightarrow{\ \mathfrak{j}_\xi\ } & \mathcal{M}_{\mathfrak{j},\xi} \end{array} \tag{1}$$

of an inflation $\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M}$, the *connecting morphism*

$$F(\mathcal{M}_{\mathfrak{j},\xi}) \xrightarrow{\mathfrak{d}_0^F(\xi,\mathfrak{j})} \mathcal{S}_+F(\mathcal{L}) \tag{2}$$

which depends functorially on the pair of arrows $\mathcal{L}_\xi \xleftarrow{\xi} \mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M}$ and factors through the cokernel of the morphism $F(\mathcal{M}) \xrightarrow{F(\mathfrak{j}_\xi)} F(\mathcal{M}_{\mathfrak{j},\xi})$. Explicitly,

$$S_+F(\mathcal{L}) = \operatorname{colim}(Cok(F(\mathcal{M} \xrightarrow{\xi_{\mathfrak{j}}} \mathcal{M}_{\mathfrak{j},\xi}))), \tag{3}$$

where the colimit is taken by the diagram of all push-forwards (1) of inflations of the object $\mathcal{L}$ (see the dual version of 3.3.4).

5.1.1. Triangles. If an object $\mathcal{L}$ of the category $C_{\mathfrak{X}}$ belongs to a connected component $C_{\mathfrak{X}_{\mathfrak{i}}}$, $\mathfrak{i} \in \pi_0(\mathfrak{X})$, then there is a morphism of functors $\mathfrak{y}^F_{\mathfrak{i}} \xrightarrow{\lambda^F_{\mathfrak{i}}} S_+F_{\mathfrak{i}}$, where $F_{\mathfrak{i}}$ is the functor $C_{\mathfrak{X}_{\mathfrak{i}}} \longrightarrow C_{Y_{\pi_0(F)(\mathfrak{i})}}$ induced by the functor $C_{\mathfrak{X}} \xrightarrow{F} C_Y$ and $\mathfrak{y}^F_{\mathfrak{i}}$ is the constant functor with the values in a final object of the category $C_{Y_{\pi_0(F)(\mathfrak{i})}}$ such that the diagram

$$\begin{array}{ccccc} F(\mathcal{L}) & \xrightarrow{F(\mathfrak{j})} & F(\mathcal{M}) & \longrightarrow & \mathfrak{y}^F_{\mathfrak{i}} \\ {\scriptstyle F(\xi_{\mathfrak{j}})}\big\downarrow & & \big\downarrow{\scriptstyle F(\mathfrak{j}_\xi)} & & \big\downarrow{\scriptstyle \lambda^F_{\mathfrak{i}}(\mathcal{L})} \\ F(\mathcal{L}_\xi) & \xrightarrow{F(\xi_{\mathfrak{j}})} & F(\mathcal{M}_{\mathfrak{j},\xi}) & \xrightarrow{\mathfrak{d}^F_0(\xi,\mathfrak{j})} & \mathcal{S}_+F(\mathcal{L}) \end{array} \qquad (4)$$

commutes.

5.2. A structure of a $\mathbb{Z}_+$-category on $C_{\mathfrak{X}^{\mathfrak{re}}}$. For any presheaf of sets $\mathcal{G}$ on $C_{\mathfrak{X}}$, we set

$$\widehat{\Theta}_{\mathfrak{X}*}(\mathcal{G})(-) = C_{\mathfrak{X}^{\mathfrak{re}}}(Ext^1_{\mathfrak{X}}(-), \mathcal{G}). \qquad (1)$$

The map $\mathcal{G} \longmapsto \widehat{\Theta}_{\mathfrak{X}*}(\mathcal{G})$ extends to an endofunctor $C_{\mathfrak{X}^{\mathfrak{re}}} \xrightarrow{\widehat{\Theta}_{\mathfrak{X}*}} C_{\mathfrak{X}^{\mathfrak{re}}}$. It follows from the definition of $\widehat{\Theta}_{\mathfrak{X}*}$ (and the Yoneda's formula) that

$$C_{\mathfrak{X}^{\mathfrak{re}}}(Ext^1_{\mathfrak{X}}(-), \mathcal{G}) = \widehat{\Theta}_{\mathfrak{X}*}(\mathcal{G})(-) \simeq C_{\mathfrak{X}^{\mathfrak{re}}}(-, \widehat{\Theta}_{\mathfrak{X}*}(\mathcal{G})). \qquad (2)$$

Let $\widehat{\Theta}^*_{\mathfrak{X}}$ denote the continuous functor $C_{\mathfrak{X}^{\mathfrak{re}}} \longrightarrow C_{\mathfrak{X}^{\mathfrak{re}}}$ corresponding to $Ext^1_{\mathfrak{X}}$. It follows from the definition of the functor $\widehat{\Theta}_{\mathfrak{X}*}$ (see (2)) that

$$C_{\mathfrak{X}^{\mathfrak{re}}}(\widehat{\Theta}^*_{\mathfrak{X}}(-), \mathcal{G}) \simeq C_{\mathfrak{X}^{\mathfrak{re}}}(-, \widehat{\Theta}_{\mathfrak{X}*}(\mathcal{G})),$$

that is the functor $\widehat{\Theta}_{\mathfrak{X}*}$ is a right adjoint to $\widehat{\Theta}^*_{\mathfrak{X}}$.

5.3. Standard "triangles". Applying 5.1.1 to the Yoneda functor $h^{\mathfrak{re}}_{\mathfrak{X}}$, we obtain from the diagram 5.1.1(4) the diagram

$$\begin{array}{ccccc} \widehat{\mathcal{L}} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{\mathcal{M}} & \longrightarrow & y_{\mathfrak{i}} \\ {\scriptstyle \widehat{\xi}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\mathfrak{j}}_\xi} & & \big\downarrow{\scriptstyle \lambda(\mathcal{L})} \\ \widehat{\mathcal{L}}_\xi & \xrightarrow{\widehat{\xi}_{\mathfrak{j}}} & \widehat{\mathcal{M}}_{\mathfrak{j},\xi} & \xrightarrow{\mathfrak{d}_0(\xi,\mathfrak{j})} & \widehat{\Theta}^*_{\mathfrak{X}}(\widehat{\mathcal{L}}) \end{array} \qquad (1)$$

Here $y_{\mathfrak{i}}$ is an final object of the category $C_{\mathfrak{X}^{\circledast}_{\mathfrak{i}}}$ corresponding to the connected component $C_{\mathfrak{X}_{\mathfrak{i}}}$ of the category $C_{\mathfrak{X}}$ containing the object $\mathcal{L}$; the left square is the image of a push-forward

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\ \mathfrak{j}\ } & \mathcal{M} \\ \xi \downarrow & cocart & \downarrow \xi_{\mathfrak{j}} \\ \mathcal{L}_{\xi} & \xrightarrow{\ \mathfrak{j}_{\xi}\ } & \mathcal{M}_{\mathfrak{j},\xi} \end{array}$$

of an inflation $\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M}$. We call the diagram (1) a *standard "triangle"*.

5.4. "Triangles" in the category of presheaves of sets. A *"triangle"* is any diagram in $C_{\mathfrak{X}^{\mathfrak{re}}}$ of the form

$$\begin{array}{ccccc} \mathfrak{L} & \xrightarrow{\ \mathfrak{j}\ } & \mathfrak{M} & \longrightarrow & y_{\mathfrak{i}} \\ \xi \downarrow & & \downarrow \mathfrak{j}_{\xi} & & \downarrow \lambda(\mathfrak{L}) \\ \mathfrak{L}_{\xi} & \xrightarrow{\ \xi_{\mathfrak{j}}\ } & \mathfrak{M}_{\mathfrak{j},\xi} & \xrightarrow{\ \mathfrak{d}\ } & \widehat{\Theta}^*_{\mathfrak{X}}(\mathfrak{L}) \end{array} \qquad (2)$$

which is isomorphic to a standard "triangle". Here $\mathfrak{L}$ belongs to a connected component $C_{\mathfrak{X}^{\circledast}_{\mathfrak{i}}}$ of the category $C_{\mathfrak{X}^{\mathfrak{re}}}$ and $y_{\mathfrak{i}}$ is a final object of the component $C_{\mathfrak{X}^{\circledast}_{\mathfrak{i}}}$. "Triangles" form a category $\mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{re}}}$.

5.5. Prestable category of presheaves. Thus, the left exact structure $\mathfrak{I}_{\mathfrak{X}}$ on the category $C_{\mathfrak{X}}$ produces the data $(C_{\mathfrak{X}^{\mathfrak{re}}}, \mathfrak{I}_{\mathfrak{X}^{\mathfrak{re}}}, \widehat{\Theta}^*_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{re}}})$, where $\mathfrak{I}_{\mathfrak{X}^{\mathfrak{re}}}$ is the coarsest left exact structure on $C_{\mathfrak{X}^{\mathfrak{re}}}$ which is closed under filtered colimits and makes the Yoneda embedding $C_{\mathfrak{X}} \xrightarrow{h^{\mathfrak{re}}_{\mathfrak{X}}} C_{\mathfrak{X}^{\mathfrak{re}}}$ an 'exact' functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C_{\mathfrak{X}^{\mathfrak{re}}}, \mathfrak{I}_{\mathfrak{X}^{\mathfrak{re}}})$ (see the argument of II.9.1), $\widehat{\Theta}^*_{\mathfrak{X}}$ a continuous endofunctor of $C_{\mathfrak{X}^{\mathfrak{re}}}$ corresponding to $Ext^1_{\mathfrak{X}}$, $\mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{re}}}$ the category of "triangles" on the category of presheaves. We call the data $(C_{\mathfrak{X}^{\mathfrak{re}}}, \mathfrak{I}_{\mathfrak{X}^{\mathfrak{re}}}, \widehat{\Theta}^*_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{re}}})$ the *prestable category of presheaves* on the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$.

5.5.1. Note. The prestable category $(C_{\mathfrak{X}^{\mathfrak{re}}}, \mathfrak{I}_{\mathfrak{X}^{\mathfrak{re}}}, \widehat{\Theta}^*_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{re}}})$ contains all the information about the universal ∂-functor $Ext^{\bullet}_{\mathfrak{X}} = (Ext^i_{\mathfrak{X}}, \mathfrak{d}_i \mid i \geq 0)$, and, therefore, due to the universality of $Ext^{\bullet}_{\mathfrak{X}}$, all the information about all universal ∂-functors from the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to cocomplete categories. In fact, the universal ∂-functor $Ext^{\bullet}_{\mathfrak{X}}$ is of the form $(\widehat{\Theta}^{*n}_{\mathfrak{X}} \circ h^{\mathfrak{re}}_{\mathfrak{X}}, \widehat{\Theta}^{*n}_{\mathfrak{X}}(\mathfrak{d}_0) \mid n \geq 0)$; and for any functor F from $C_{\mathfrak{X}}$ to a category C_Y with colimits and final objects, the universal ∂-functor $(T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to C_Y such that $T_0 = F$ is isomorphic to

$$F^{\mathfrak{re}} \circ Ext^{\bullet}_{\mathfrak{X}} = (F^{\mathfrak{re}} \widehat{\Theta}^{*n}_{\mathfrak{X}}, F^{\mathfrak{re}} \widehat{\Theta}^{*n}_{\mathfrak{X}}(\mathfrak{d}_0) \mid n \geq 0) \circ h^{\mathfrak{re}}_{\mathfrak{X}}. \qquad (1)$$

Here $\mathfrak{d}_0 = (\mathfrak{d}_0(\xi, \mathfrak{j}))$, where $\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M}$ is an inflation, $\mathcal{L} \xrightarrow{\xi} \mathcal{L}_{\xi}$ an arbitrary morphism, and $\mathfrak{d}_0(\xi, \mathfrak{j})$ is the connecting morphism in the diagram 5.4(2).

5.6. The stable category of presheaves of sets on a left exact category. Given a left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, we denote by $C_{\mathfrak{X}_{\mathfrak{s}}^{\mathfrak{r}\mathfrak{e}}}$ the quotient category $\Sigma_{\widehat{\Theta}_{\mathfrak{X}}^*}^{-1} C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}}$, where $\Sigma_{\widehat{\Theta}_{\mathfrak{X}}^*}$ denotes the class of all arrows $\mathfrak{t}$ of $C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}}$ such that $\widehat{\Theta}_{\mathfrak{X}}^*(\mathfrak{t})$ is an isomorphism. The endofunctor $\widehat{\Theta}_{\mathfrak{X}}^*$ induces a conservative endofunctor $\Theta_{\mathfrak{X}_{\mathfrak{s}}^{\mathfrak{r}\mathfrak{e}}}$ of the category $C_{\mathfrak{X}_{\mathfrak{s}}^{\mathfrak{r}\mathfrak{e}}}$.

We denote by $\mathfrak{Tr}_{\mathfrak{X}_{\mathfrak{s}}^{\mathfrak{r}\mathfrak{e}}}$ the category of all diagrams of the form

$$\begin{array}{ccccc} \mathfrak{L} & \xrightarrow{\mathfrak{j}} & \mathfrak{M} & \longrightarrow & y \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}_\xi} & & \downarrow{\scriptstyle \lambda(\mathfrak{L})} \\ \mathfrak{L}_\xi & \xrightarrow{\xi_{\mathfrak{j}}} & \mathfrak{M}_{\mathfrak{j},\xi} & \xrightarrow{\mathfrak{d}} & \Theta_{\mathfrak{X}_{\mathfrak{s}}}^*(\mathfrak{L}) \end{array} \qquad (1)$$

in the category $C_{\mathfrak{X}_{\mathfrak{s}}^{\mathfrak{r}\mathfrak{e}}}$, which are isomorphic to the images of (standard) triangles. The objects of the category $\mathfrak{Tr}_{\mathfrak{X}_{\mathfrak{s}}^{\mathfrak{r}\mathfrak{e}}}$ will be also called *triangles*.

We call the triple $(C_{\mathfrak{X}_{\mathfrak{s}}^{\mathfrak{r}\mathfrak{e}}}, \Theta_{\mathfrak{X}_{\mathfrak{s}}^{\mathfrak{r}\mathfrak{e}}}, \mathfrak{Tr}_{\mathfrak{X}_{\mathfrak{s}}^{\mathfrak{r}\mathfrak{e}}})$ the *stable category of presheaves of sets* on the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$.

6. Prestable and stable category of a left exact category.

6.1. The category $C_{\mathfrak{X}^{\mathfrak{p}}}$. Let $C_{\mathfrak{X}^{\mathfrak{p}}}$ be the smallest strictly full subcategory of the category $C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}}$ which contains all representable presheaves and the trivial presheaf – the final object of $C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}}$, and is $\widehat{\Theta}_{\mathfrak{X}}^*$-stable. We denote by $\theta_{\mathfrak{X}^{\mathfrak{p}}}$ the endofunctor $C_{\mathfrak{X}^{\mathfrak{p}}} \longrightarrow C_{\mathfrak{X}^{\mathfrak{p}}}$ induced by the endofunctor $C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}} \xrightarrow{\widehat{\Theta}_{\mathfrak{X}}^*} C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}}$.

6.2. Triangles. *Triangles* are the same as in 5.4. That is a *triangle* is a diagram of the form

$$\begin{array}{ccccc} \mathfrak{L} & \xrightarrow{\mathfrak{j}} & \mathfrak{M} & \longrightarrow & y \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}_\xi} & & \downarrow{\scriptstyle \lambda(\mathfrak{L})} \\ \mathfrak{L}_\xi & \xrightarrow{\xi_{\mathfrak{j}}} & \mathfrak{M}_{\mathfrak{j},\xi} & \xrightarrow{\mathfrak{d}} & \widehat{\Theta}_{\mathfrak{X}}^*(\mathfrak{L}) \end{array} \qquad (1)$$

which is isomorphic to a standard "triangle" defined in 5.3. Here, as in 5.4 and 5.3, y is the constant presheaf with values in one-element set – the standard final object of the category of presheaves of sets $C_{\mathfrak{X}^{\mathfrak{r}\mathfrak{e}}}$. We denote $\mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{p}}}$ the category of triangles.

6.3. The prestable category of a left exact category. Given a left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, we call the data $(C_{\mathfrak{X}^{\mathfrak{p}}}, \theta_{\mathfrak{X}^{\mathfrak{p}}}, \mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{p}}})$ the *prestable* category of $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$.

6.4. The stable category of a left exact category. Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ be an arbitrary left exact category and $(C_{\mathfrak{X}^{\mathfrak{p}}}, \theta_{\mathfrak{X}^{\mathfrak{p}}}, \mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{p}}})$ the associated with $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ presuspended category. Let $\Sigma = \Sigma_{\theta_{\mathfrak{X}^{\mathfrak{p}}}}$ be the class of all arrows $\mathfrak{t}$ of $C_{\mathfrak{X}^{\mathfrak{p}}}$ such that $\theta_{\mathfrak{X}^{\mathfrak{p}}}(\mathfrak{t})$ is an isomorphism.

Consider the quotient category $C_{\mathfrak{X}_{\mathfrak{s}}} = \Sigma^{-1} C_{\mathfrak{X}^{\mathfrak{p}}}$. The endofunctor $\theta_{\mathfrak{X}^{\mathfrak{p}}}$ of the category $C_{\mathfrak{X}^{\mathfrak{p}}}$ determines a conservative endofunctor $\theta_{\mathfrak{X}_{\mathfrak{s}}}$ of the category $C_{\mathfrak{X}_{\mathfrak{s}}}$.

We denote by $\mathfrak{Tr}_{\mathfrak{X}_\mathfrak{s}}$ the *essential* image of the category $\mathfrak{Tr}_{\mathfrak{X}^\mathfrak{p}}$ of triangles in $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ and continue to call objects of the category $\mathfrak{Tr}_{\mathfrak{X}_\mathfrak{s}}$ *"triangles"*.

We call the triple $(C_{\mathfrak{X}_\mathfrak{s}}, \theta_{\mathfrak{X}_\mathfrak{s}}, \mathfrak{Tr}_{\mathfrak{X}_\mathfrak{s}})$ (and, sometimes, loosely, the category $C_{\mathfrak{X}_\mathfrak{s}}$) the *stable* category of the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$.

7. 'Exactness' properties.

7.0. Right 'semi-exact' functors. Let $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories. We say that a functor $C_{\mathfrak{X}} \xrightarrow{F} C_Y$ is a *right 'semi-exact'* functor from $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ to $(C_Y, \mathfrak{E}_Y)$, if, for any deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ and any morphism $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$, the canonical morphism from $F(M_{\xi,\mathfrak{e}})$ to the pull-back of $F(\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L})$ along $F(\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L})$ is a deflation.

7.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories. Suppose that*
– the category C_X is quasi-filtered,
– the category C_Y is virtually semi-complete and has limits of filtered diagrams,
– the right exact category $(C_Y, \mathfrak{E}_Y)$ satisfies $(CE5^*)$; that is limits of filtered diagrams of deflations are deflations.

Then, for any right 'semi-exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{F} (C_Y, \mathfrak{E}_Y)$ and any deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ in $(C_X, \mathfrak{E}_X)$,

(i) The canonical morphism

$$S\_F(\mathcal{L}) \xrightarrow{\mathfrak{d}_\mathfrak{e}} \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} (Ker(F(\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\xi'_\mathfrak{e}} \mathcal{M})), \tag{1}$$

is a deflation. Here $\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\xi'_\mathfrak{e}} \mathcal{M}$ is the pull-back of $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$ along the deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$.

(ii) The pair of arrows

$$S\_F(\mathcal{M}) \xrightarrow{S\_F(\mathfrak{e})} S\_F(L) \xrightarrow{\mathfrak{d}_\mathfrak{e}} \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} (Ker(F(\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\xi'_\mathfrak{e}} \mathcal{M})), \tag{1.1}$$

is 'exact'.

(b) The functor S_F is a right 'semi-exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.

Proof. (a1) Let $\mathcal{M}_\mathfrak{t} \xrightarrow{\mathfrak{t}} \mathcal{M}$ be a deflation. The diagram

$$F\begin{pmatrix} \mathcal{M}_{\xi,\mathfrak{et}} & \xrightarrow{\mathfrak{t}_{\xi'_\mathfrak{e}}} & \mathcal{M}_{\xi,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\xi} & \mathcal{L}_\xi \\ \xi'_{\mathfrak{et}}\big\downarrow & cart & \xi'_\mathfrak{e}\big\downarrow & cart & \big\downarrow \xi \\ \mathcal{M}_\mathfrak{t} & \xrightarrow{\mathfrak{t}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{pmatrix} \tag{2}$$

decomposes into a commutative diagram

$$\begin{array}{ccccc} F(\mathcal{M}_{\xi,\mathfrak{e}\mathfrak{t}}) & \xrightarrow{\gamma} & \mathfrak{M} & \longrightarrow & \\ F(\xi_{\mathfrak{e}\mathfrak{t}}) \downarrow & & \zeta_{\mathfrak{e},\gamma} \downarrow & cart & F \left(\begin{array}{ccc} \mathcal{M}_{\xi,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_\xi} & \mathcal{L}_\xi \\ \xi'_{\mathfrak{e}} \downarrow & cart & \downarrow \xi \\ \mathcal{M}_{\mathfrak{e}} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array} \right) \\ F(\mathcal{M}_{\mathfrak{t}}) & \xrightarrow{id} & F(\mathcal{M}_{\mathfrak{t}}) & \xrightarrow{F(\mathfrak{t})} & \end{array} \qquad (3)$$

whose middle square is cartesian (as well as the square inside of the brackets). By 1.3, there is a natural isomorphism $Ker(\zeta_{\mathfrak{e},\gamma}) \xrightarrow{\sim} Ker(F(\xi'_{\mathfrak{e}}))$. It follows from 1.4 and the right square of the diagram (3) that the pull-back of $Ker(\zeta) \xrightarrow{\mathfrak{k}(\zeta)} \mathfrak{M}$ along $F(\mathcal{M}_{\xi,\mathfrak{e}\mathfrak{t}}) \xrightarrow{\gamma} \mathfrak{M}$ is (isomorphic to) the canonical morphism $Ker(F(\xi'_{\mathfrak{e}\mathfrak{t}})) \longrightarrow F(\mathcal{M}_{\xi,\mathfrak{e}\mathfrak{t}})$.

Since the functor F is right 'semi-exact', the arrow $F(\mathcal{M}_{\xi,\mathfrak{e}\mathfrak{t}}) \xrightarrow{\gamma} \mathfrak{M}$ is a deflation. Therefore, its pull-back $Ker(F(\xi'_{\mathfrak{e}\mathfrak{t}}))\longrightarrow Ker(\zeta)$ is a deflation. So that

$$Ker(F(\xi'_{\mathfrak{e}\mathfrak{t}})) \xrightarrow{\gamma_{\mathfrak{e}\mathfrak{t},\xi}} Ker(F(\xi'_{\mathfrak{e}})) \qquad (4_{\mathfrak{e}\mathfrak{t},\xi})$$

is a deflation for any deflation $\mathcal{M}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \mathcal{M}$ of the object $\mathcal{M}$.

(a2) By hypothesis, the right exact category $(C_Y, \mathfrak{E}_Y)$ has the property $(CE5^*)$: the limit of filtered diagram of deflations is a deflation. Since deflations to a given object form a filtered diagram, it follows from the fact that $(4_{\mathfrak{t},\xi})$ is a deflation for any deflation $\mathcal{M}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \mathcal{M}$ of the object $\mathcal{M}$, which depends functorially on $\mathfrak{t}$, the limit

$$\lim_{\mathcal{M}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \mathcal{M}} Ker(F(\xi'_{\mathfrak{e}\mathfrak{t}})) \xrightarrow{\gamma_{\mathfrak{e},\xi}} Ker(F(\xi'_{\mathfrak{e}})) \qquad (5_{\mathfrak{e},\xi})$$

of the diagrams $(4_{\mathfrak{t},\xi})$ exists and is a deflation.

(a3) Notice that deflations $(5_{\mathfrak{e},\xi})$ depend functorially on $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$. More precisely, they define a functor from the category $C_X/\mathcal{L}$ to the category of deflations of $(C_Y, \mathfrak{E}_Y)$. By hypothesis, the category $C_X/\mathcal{L}$ is filtered for any object $\mathcal{L}$ of C_X. Therefore, by the property $(CE5^*)$, the limit

$$\begin{gathered} \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} \Big(\lim_{\mathcal{M}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \mathcal{M}} Ker(F(\xi'_{\mathfrak{e}\mathfrak{t}})) \xrightarrow{\gamma_{\mathfrak{e},\xi}} Ker(F(\xi'_{\mathfrak{e}}))\Big) = \\ \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} \Big(\lim_{\mathcal{M}_{\mathfrak{t}} \xrightarrow{\mathfrak{t}} \mathcal{M}} Ker(F(\xi'_{\mathfrak{e}\mathfrak{t}}))\Big) \xrightarrow{\gamma_{\mathfrak{e}}} \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} \big(Ker(F(\xi'_{\mathfrak{e}}))\big) \end{gathered} \qquad (5)$$

exists and is a deflation.

(a4) It remains to observe that, by definition,

$$\mathcal{S}_{-}F(\mathcal{L}) = \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} \Big(\lim_{\mathcal{M}_{\mathfrak{u}} \xrightarrow{\mathfrak{u}} \mathcal{L}} Ker(F(\xi'_{\mathfrak{u}}))\Big)$$

and the (following from this formula) canonical morphism

$$\mathcal{S}_{-}F(\mathcal{L}) \longrightarrow \lim_{\mathcal{L}_\xi \stackrel{\xi}{\to} \mathcal{L}} \big(\lim_{\mathcal{M}_\mathfrak{t} \stackrel{\mathfrak{t}}{\to} \mathcal{M}} Ker(F(\xi'_{\mathfrak{e}\mathfrak{t}}))\big)$$

is an isomorphism. The latter follows from the fact that the compositions deflations $\mathcal{M}_\mathfrak{t} \xrightarrow{\mathfrak{e}\mathfrak{t}} \mathcal{L}$ form a final subdiagram in the category of all deflations of the object $\mathcal{L}$.

(b) For any right 'semi-exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{F} (C_Y, \mathfrak{E}_Y)$, the derived functor $\mathcal{S}_{-}F$ is right 'semi-exact'. ■

7.2. Weakly right 'semi-exact' functors. Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories. We say that a functor $C_X \xrightarrow{F} C_Y$ is a *weakly right 'semi-exact'* functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$, if, for any deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ and any morphism $\mathcal{L}_\gamma \xrightarrow{\gamma} \mathcal{L}$, there exists a morphism $\mathcal{L}_\xi \xrightarrow{\lambda} \mathcal{L}_\gamma$ such that the canonical morphism from $F(M_{\xi,\mathfrak{e}})$ to the pull-back of $F(\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L})$ along $F(\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L})$, where $\xi = \gamma \circ \lambda$, is a deflation.

One can replace *right 'semi-exact'* in Proposition 7.1 by *weakly right 'semi-exact'*:

7.2.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories satisfying the conditions of 7.1; and let $C_X \xrightarrow{F} C_Y$ be a weakly right 'semi-exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. Then,*

(a) For any deflation $M \xrightarrow{\mathfrak{e}} L$ in $(C_X, \mathfrak{E}_X)$, the canonical morphism

$$S_{-}F(L) \xrightarrow{\mathfrak{d}_\mathfrak{e}} \lim(Ker(F(M_{\xi,\mathfrak{e}} \xrightarrow{\mathfrak{e}_\xi} M)), \tag{1}$$

where the limit is taken by all morphisms $L_\xi \xrightarrow{\xi} L$, is a deflation.

(b) The functor $S_{-}F$ is a weakly right 'semi-exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.

Proof. The argument is an easy adoptation of the proof of 7.1. ■

7.3. 'Exact' ∂^*-functors.

7.3.1. Definition. Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories. A ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y is called *'exact'*, if all functors T_i are right 'semi-exact' and, for every deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ in $(C_X, \mathfrak{E}_X)$, and every $i \geq 0$, the canonical morphism

$$T_{i+1}(L) \xrightarrow{\bar{\mathfrak{d}}_i(\mathfrak{e})} \lim_{\mathcal{L}_\xi \stackrel{\xi}{\to} \mathcal{L}} (Ker(T_i(\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\mathfrak{e}_\xi} \mathcal{M})) \tag{1}$$

is a deflation, and the pair of morphisms

$$T_{i+1}(M) \xrightarrow{T_{i+1}(\mathfrak{e})} T_{i+1}(L) \xrightarrow{\bar{\partial}_i(\mathfrak{e})} \lim_{\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}} (Ker(T_i(\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\mathfrak{e}_\xi} \mathcal{M})) \tag{1.1}$$

is 'exact. Here $\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\xi'_{\mathfrak{e}}} \mathcal{M}$ is the pull-back of a morphism $L_\xi \xrightarrow{\xi} L$ along the deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$.

7.3.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte, quasi-filtered right exact category and $(C_Y, \mathfrak{E}_Y)$ a virtually semi-complete right exact categories with limits of filtered diagrams and such that limits of filtered diagrams of deflations are deflations. Let $T = (T_i|\ i \geq 0)$ be a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. If the functor T_0 is weakly right 'exact', then the universal ∂^*-functor T is 'exact'.*

Proof. The assertion follows from 7.2.1. ■

7.4. Coeffaceable functors. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a virtually semi-complete category. A functor $C_X \xrightarrow{F} C_Y$ is called *coeffaceable*, if, for every $\mathcal{L} \in ObC_X$, there exists a deflation $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{L}$ which the functor F maps to a *virtually trivial* morphism of the category C_Y.

7.4.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a virtually semi-complete category.*

(a) Any coeffaceable functor from $(C_X, \mathfrak{E}_X)$ to C_Y maps all projective objects of $(C_X, \mathfrak{E}_X)$ to initial objects of connected components of the category C_Y.

(b) If $(C_X, \mathfrak{E}_X)$ has enough projective objects, then a functor $C_X \xrightarrow{F} C_Y$ is coeffaceable iff it maps all projective objects of the right exact category $(C_X, \mathfrak{E}_X)$ to trivial objects of the category C_Y.

Proof. (a) Let $C_X \xrightarrow{F} C_Y$ be a coeaffaceable functor and $\mathcal{P}$ a projective object of the right exact category $(C_X, \mathfrak{E}_X)$. The functor F being coeffaceable, there exists a deflation $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{P}$ such that $F(\mathfrak{t})$ is a trivial morphism. Since the object $\mathcal{P}$ is projective, there is a morphism $\mathcal{P} \xrightarrow{\gamma} \mathcal{M}$ such that $\mathfrak{t} \circ \gamma = id_{\mathcal{P}}$. Since the composition of a virtually trivial morphism with any morphism is a virtually trivial morphism, $id_{F(\mathcal{P})} = F(\mathfrak{t}) \circ F(\gamma)$ is a virtually trivial, hence trivial, morphism. So that $F(\mathcal{P})$ is a trivial object of the category C_Y. This means, precisely, that $F(\mathcal{P})$ is an initial object of the connected component of the object $F(\mathcal{P})$.

(b) It follows from (a) that any deflation $\mathcal{P} \xrightarrow{\mathfrak{e}} \mathcal{L}$ with $\mathcal{P}$ a projective object is mapped, by any coeffaceable functor $C_X \xrightarrow{F} C_Y$, to a morphism from an initial object of a connected component of the category C_Y. All such morphisms are trivial; in particular, they are virtually trivial. ■

7.4.2. Corollary. *Let* $(C_X, \mathfrak{E}_X)$ *be a right exact category and* C_Y *a virtually complete category. Suppose that the right exact category has projective objects and there exists a coeffaceable functor from* C_X *to* C_Y*. Then the category* C_Y *has initial objects.*

Proof. The assertion follows from 7.4.1(a). ∎

7.5. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *be a quasi-filtered right exact category and* $(C_Y, \mathfrak{E}_Y)$ *a virtually semi-complete right exact category. Let* $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ *be an 'exact'* ∂^**-functor from* $(C_X, \mathcal{E}_X)$ *to* $(C_Y, \mathfrak{E}_Y)$*. Suppose that* $\mathfrak{E}_Y^{\circledast} = Iso(C_Y)$ *and the functors* T_i *are* $\mathfrak{E}_X$*-coeffaceable for* $i \geq 1$*. Then* T *is a universal* ∂^**-functor.*

Proof. Considering restriction of the ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ (or, what is the same, its zero component T_0) to connected components of the right exact category $(C_X, \mathfrak{E}_X)$, we reduce to the case when the category C_Y is virtually complete. Taking the composition with the canonical 'exact' embedding $(C_Y, \mathfrak{E}_Y) \xrightarrow{\mathfrak{j}_Y^{\circledast}} (C_{Y_{\mathfrak{E}_Y^{\circledast}}}, \mathfrak{E}^{\mathfrak{s}}_{Y_{\mathfrak{E}_Y^{\circledast}}})$, we reduce the assertion to the case of complete and cocomplete right exact category.

The rest of the argument is an adaptation of the argument of III.2.6. Details are left to the reader. ∎

7.6. Projective objects and coeffaceability.

7.6.1. Pointed objects and pointed projective objects. We call an object M of a category C_X *pointed*, if there exists a cone $M \longrightarrow Id_{C_{X_\mathfrak{i}}}$, where $C_{X_\mathfrak{i}}$ is the connected component of M.

7.6.1.1. Proposition. *(a) Let* $\mathcal{M}$ *be an object of a category* C_X*. The following conditions are equivalent:*

(a1) the object $\mathcal{M}$ *is pointed;*

(a2) the connected component $C_{X_\mathfrak{i}}$ *of* $\mathcal{M}$ *is virtually complete and there is a morphism from* $\widehat{\mathcal{M}}$ *to an initial object of* $C_{X_\mathfrak{i}^{\circledast}}$*.*

(b) The following conditions on a category C_X *are equivalent:*

(i) For every object $\mathcal{L}$*, there exists a pointed object* $\mathcal{M}$ *and a morphism* $\mathcal{M} \longrightarrow \mathcal{L}$*.*

(ii) The category C_X *is virtually semi-complete.*

(c) If the category C_X *has an initial object* x*, then pointed objects are precisely those objects, which have morphisms to* x*.*

Proof. The assertion follows from definitions. ∎

7.6.2. Pointed objects and pointed sheaves. Let C_X be a svelte category having the decomposition $C_X = \coprod_{\mathfrak{i} \in \pi_0(X)} C_{X_\mathfrak{i}}$ into the disjoint union of its connected components; and let τ be a subcanonical pretopology on C_X. Recall that $C_{X_\tau^{\mathfrak{r}\mathfrak{e}}}$ denotes the *associated* category with colimits:

$$C_{X^{\mathfrak{re}}_{\tau}} = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X^{\circledast}_{\mathfrak{i},\tau_{\mathfrak{i}}}}, \qquad (1)$$

where $\tau_{\mathfrak{i}}$ is the restriction of the pretopology τ to the connected component $C_{X_{\mathfrak{i}}}$.

An object $\mathcal{M}$ is pointed iff $\widehat{\mathcal{M}} = C_X(-,\mathcal{M})$ is a pointed object of the category $C_{X^{\mathfrak{re}}_{\tau}}$.

In fact, if $\widehat{\mathcal{M}}$ is a pointed object of the category of sheaves, that is there is a cone $\widehat{\mathcal{M}} \longrightarrow Id_{(C_{X_{\mathfrak{i}}},\tau)^{\mathfrak{re}}}$, then, since the pretopology τ is subcanonical, the corestriction of this cone to representable sheaves defines a cone $\mathcal{M} \longrightarrow Id_{C_{X_{\mathfrak{i}}}}$.

Conversely, let M be a pointed object of C_X; i.e. there is a cone $\mathcal{M} \longrightarrow Id_{C_{X_{\mathfrak{i}}}}$, where $C_{X_{\mathfrak{i}}}$ is the connected component of M. Since every presheaf of sets is a colimit of a canonically dependent on it diagram of representable presheaves, the cone $\mathcal{M} \longrightarrow Id_{C_{X_{\mathfrak{i}}}}$ determines a cone $\widehat{\mathcal{M}} \longrightarrow Id_{C^{\circledast}_{X_{\mathfrak{i}}}}$ in the "reduced" category of presheaves of sets on $C_{X_{\mathfrak{i}}}$. The sheafification functor maps this cone to a cone $\widehat{\mathcal{M}} \longrightarrow Id_{C^{\circledast}_{X_{\mathfrak{i},\tau_{\mathfrak{i}}}}}$.

7.6.3. Pointed objects in a right exact category. Let $(C_X,\mathfrak{E}_X)$ be a right exact category. We are interested in the case when $(C_X,\mathfrak{E}_X)$ has *enough* pointed objects; that is every object $\mathcal{L}$ of the category C_X has a deflation $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{L}$ with $\mathcal{M}$ a pointed object.

7.6.3.1. Proposition. *Suppose that a right exact category $(C_X,\mathfrak{E}_X)$ has enough pointed objects. Then all projective objects of $(C_X,\mathfrak{E}_X)$ are pointed.*

Proof. Any deflation to a projective object splits. In particular, every projective object has a morphism to a pointed object; and any object having a morphism to a pointed object is pointed. ■

7.6.4. Proposition. *Let $T = (T_i,\mathfrak{d}_i|\ i \geq 0)$ be a universal ∂^*-functor from a right exact category $(C_X,\mathcal{E}_X)$ to a virtually semi-complete category C_Y. Then each of the functors T_i, $i \geq 1$, maps all pointed projective objects to trivial objects.*

In particular, if the right exact category $(C_X,\mathcal{E}_X)$ has enough pointed projective objects, then the functors T_i are coeffaceable for $i \geq 1$.

Proof. Being a universal ∂^*-functor, $T = (T_i,\mathfrak{d}_i|\ i \geq 0)$ is isomorphic to $\mathcal{S}^{\bullet}_{-}(F)$, where $F = T_0$. Taking the composition with the Yoneda embedding $C_Y \xrightarrow{h^{\mathfrak{re}}_Y} C_{Y^{\mathfrak{re}}}$, we can use the isomorphism $h^{\mathfrak{re}}_X \circ \mathcal{S}^{\bullet}_{-}F \simeq \mathcal{S}^{\bullet}_{-}(F^{\mathfrak{re}}_{\mathfrak{E}_X})\mathfrak{j}^*_X$ of 3.7.1.2, which allows to reduce the assertion to the universal ∂^*-functors from the right exact category $(C_{X_{\mathfrak{E}}},\mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}})$ of sheaves of sets on $(C_X,\mathcal{E}_X)$ endowed with the canonical right exact structure. Given a pointed projective $\mathcal{P}$ of the right exact category $(C_X,\mathcal{E}_X)$, we replace $(C_X,\mathcal{E}_X)$ by the connected component $(C_{X_{\mathfrak{i}}},\mathcal{E}_{X_{\mathfrak{i}}})$ of the object $\mathcal{P}$ and exclude the trivial (that is taking value in $\emptyset$) sheaf from the category of sheaves of sets on $(C_{X_{\mathfrak{i}}},\mathcal{E}_{X_{\mathfrak{i}}})$. By III.1.3.4 and 7.6.2, the canonical embedding $(C_{X_{\mathfrak{i}}},\mathfrak{E}_{X_{\mathfrak{i}}}) \xrightarrow{\mathfrak{j}^*_{X_{\mathfrak{i}}}} (C_{X_{\mathfrak{E}}},\mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}})$ maps pointed projective objects to pointed projective objects and the category of non-trivial sheaves on $(C_{X_\mathfrak{i}}, \mathcal{E}_{X_\mathfrak{i}})$ has initial objects. The assertion follows now from III.2.2.1. ∎

8. Homology of 'spaces' with coefficients in a right exact category. Let C_X be a svelte category and $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ a svelte right exact category with colimits of functors $C_X \longrightarrow C_{\mathcal{Z}}$. We define the zero *homology object* of a 'space' X (represented by a category C_X) with coefficients in $C_X \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$ by setting $H_0(X, \mathcal{F}) = \mathrm{colim}\mathcal{F}$.

8.1. Higher homology. Suppose that the category $C_{\mathcal{Z}}$ is virtually semi-complete. The higher homology groups, $H_n(X, \mathcal{F})$, $n \geq 1$, are values at $\mathcal{F}$ of satellites of the functor

$$C_{\mathfrak{H}(\mathcal{Z},X)} \xrightarrow{H_0(X,-)} C_{\mathcal{Z}}$$

with respect to the (object-wise) right exact structure $\mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)}$ induced by $\mathfrak{E}_{\mathcal{Z}}$ (cf. 6.0.1).

If the category $C_{\mathcal{Z}}$ has limits of filtered diagrams, then, since the category $(C_{\mathfrak{H}(\mathcal{Z},X)}$ of functors from C_X to $C_{\mathcal{Z}}$ inherits this property, there exists a universal ∂^*-functor

$$H_\bullet(X, -) = (H_n(X, -), \ \mathfrak{d}_n \mid n \geq 0)$$

from the right exact category *of coefficients* $(C_{\mathfrak{H}(\mathcal{Z},X)}, \mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)})$ to $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$.

8.2. Proposition. *Suppose that the right exact category* $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ *is virtually semi-complete and satisfies* $(CE5^*)$*: the limit of filtered diagram of deflations is a deflation.*

Then the universal ∂^**-functor* $H_\bullet(X, -)$ *is 'exact'.*

Proof. The argument is similar to the proof of III.7.1.1. Namely, the condition that the right exact category $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ satisfies $(CE5^*)$, implies that

$$\mathcal{F} \longmapsto H_0(X, \mathcal{F}) = \mathrm{colim}\mathcal{F}$$

defines a right 'exact' functor from the right exact category of functors $(C_{\mathfrak{H}(\mathcal{Z},X)}, \mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)})$ to the right exact category $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$. Therefore, by 7.3.1, the universal ∂^*-functor

$$H_\bullet(X, -) = (H_n(X, -), \ \mathfrak{d}_n \mid n \geq 0)$$

is 'exact'. ∎

9. Towards the "absolute" higher K-theory of right exact 'spaces'.

9.1. The basic higher K-theory. We take the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces' endowed with the left exact structure $\mathfrak{I}_\mathfrak{c}^{\rightarrow}$.

9.2. Proposition. *Let* $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ *be a left exact category and* $\mathfrak{F}$ *a weakly 'exact' functor* $(C_\mathfrak{S}, \mathfrak{I}_\mathfrak{B}) \longrightarrow (\mathfrak{Esp}_\mathfrak{r}, \mathfrak{I}_\mathfrak{c}^{\rightarrow})$, *Let* C_Z *be a virtually semi-complete category with limits of filtered diagrams and* $\mathcal{G}$ *be a functor from* $(\mathfrak{Esp}_\mathfrak{r})^{op}$ *to* C_Z*. Then* *(a) There exists a universal* ∂^**-functor* $\mathcal{G}_\bullet^{\mathfrak{S},\mathfrak{F}} = (\mathcal{G}_i^{\mathfrak{S},\mathfrak{F}}, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$ *from the right exact category* $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})^{op}$ *to the category* C_Z *whose zero component,* $\mathcal{G}_0^{\mathfrak{S},\mathfrak{F}}$*, is the composition of the functor*

$$\mathcal{C}_{\mathfrak{S}}^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{re}})^{op}$$

and the functor $\mathcal{G}$*.*

(b) If $(C_Z, \mathfrak{E}_Z)$ *is a right exact virtually semi-complete category satisfying* $(CE5^*)$ *and the functor* $\mathcal{G}$ *is left 'exact', then the* ∂^**-functor* $\mathcal{G}_\bullet^{\mathfrak{S},\mathfrak{F}}$ *is 'exact'. In particular, the* ∂^**-functor* $\mathcal{G}_\bullet = (\mathcal{G}_i, \mathfrak{d}_i \mid i \geq 0)$ *from* $(\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{re}}, \mathfrak{I}_{\mathfrak{c}}^{\rightarrow})$ *to* $(C_Z, \mathfrak{E}_Z)$ *is 'exact'.*

Proof. The assertion is a special case of II.3.4. ∎

9.2.1. Corollary. *Let* $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ *be a left exact category, and*

$$(C_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}}) \xrightarrow{\mathfrak{F}} (\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{I}_{\mathfrak{c}}^{\rightarrow})$$

a weakly 'exact' functor. Then there exists a universal ∂^**-functor*

$$K_\bullet^{\mathfrak{S},\mathfrak{F}} = (K_i^{\mathfrak{S},\mathfrak{F}}, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$$

from $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})^{op}$ *to* $\mathbb{Z} - mod$*, whose zero component,* $K_0^{\mathfrak{S},\mathfrak{F}}$*, is the composition of the functor*

$$\mathcal{C}_{\mathfrak{S}}^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{re}})^{op}$$

and the functor K_0*. The* ∂^**-functor* $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ *is 'exact'.*

In particular, the ∂^**-functor* $K_\bullet = (K_i, \mathfrak{d}_i \mid i \geq 0)$ *from the right exact category* $(\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{re}}, \mathfrak{I}_{\mathfrak{cre}}^{\rightarrow})^{op}$ *to the abelian category* $\mathbb{Z} - mod$ *is 'exact'.*

Appendix K

Exact Categories and (Co)suspended Categories.

The main purpose of this Appendix is to sketch definition and general properties of suspended categories, following the approach of B. Keller and D. Vossieck [KeV], [KV1], [Ke2], starting with their definition of an exact category.

K1. Exact categories.

For the convenience of applications, we consider mostly k-linear categories and k-linear functors, where k is a commutative associative unital ring.

K1.1. Definition. Let C_X be a k-linear category and $\mathcal{E}_X$ a class of pairs of morphisms $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ of C_X such that the sequence $0 \longrightarrow L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N \longrightarrow 0$ is exact (i.e. $\mathfrak{j}$ is a kernel of $\mathfrak{e}$ and $\mathfrak{e}$ a cokernel of $\mathfrak{j}$). The elements of $\mathcal{E}_X$ are called *conflations.* The morphism $\mathfrak{e}$ (resp. $\mathfrak{j}$) of a conflation $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ is called a *deflation* (resp. *inflation*). The pair $(C_X, \mathcal{E}_X)$ is called an *exact* category if $\mathcal{E}_X$ is closed under isomorphisms and the following conditions hold.

(Ex0) id_0 is a deflation.

(Ex1) The composition of two deflations is a deflation.

(Ex2) For every diagram $M' \xrightarrow{f} M \xleftarrow{\mathfrak{e}} L$, where $\mathfrak{e}$ is a deflation, there is a cartesian square

$$\begin{array}{ccc} L' & \xrightarrow{\mathfrak{e}'} & M' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ L & \xrightarrow{\mathfrak{e}} & M \end{array}$$

where $\mathfrak{e}'$ is a deflation.

(Ex2^{op}) For every diagram $M' \xleftarrow{f} M \xrightarrow{\mathfrak{j}} L$, where $\mathfrak{j}$ is an inflation, there is a cocartesian square

$$\begin{array}{ccc} L' & \xleftarrow{\mathfrak{j}'} & M' \\ {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle f} \\ L & \xleftarrow{\mathfrak{j}} & M \end{array}$$

where $\mathfrak{j}'$ is an inflation.

For an exact category $(C_X, \mathcal{E}_X)$, we denote by $\mathfrak{E}_X$ the class of all deflations and by $\mathfrak{M}_X$ the class of all inflations of $(C_X, \mathcal{E}_X)$.

K1.2. Remarks.

K1.2.1. Applying (Ex2) to (Ex0), we obtain that id_M is a deflation for every $M \in ObC_X$. Thus, axioms (Ex0), (Ex1), (Ex2) mean simply that the class $\mathfrak{E}_X$ of deflations forms a right multiplicative system, or, what is the same, a pretopology on C_X. The invariance of $\mathcal{E}_X$ under isomorphisms implies that all isomorphisms of C_X are deflations.

The fact that all arrows of $\mathfrak{E}_X$ are cokernels of their kernels means precisely that the pretopology $\mathfrak{E}_X$ on C_X is *subcanonical*, i.e. every representable presheaf of sets on C_X is a sheaf on $(C_X, \mathfrak{E}_X)$. Thus, one can start from a class $\mathfrak{E}_X$ of arrows of C_X which forms a subcanonical pretopology (equivalently, it is a right multiplicative system formed by strict epimorphisms) and define $\mathfrak{M}_X$ as kernels of arrows of $\mathfrak{E}_X$. The only remaining requirement is the axiom (Ex^{op}) – the invariance of the class $\mathfrak{M}_X$ of inflations under a cobase change.

This shows, in particular, that the first three axioms make sense in any category and the last axiom, (Ex2^{op}), makes sense in any pointed category.

The fact that all identical morphisms are deflations implies that arrows $0 \longrightarrow M$ are inflations for all objects M of C_X. Applying the axiom (Ex2^{op}) to arbitrary pair of inflations $L \longleftarrow 0 \longrightarrow M$, we obtain the existence of coproducts of any two objects; i.e. the category C_X is additive.

K1.2.2. Quillen's original definition of an exact category contains some additional axioms. B. Keller showed that they follow from the axioms (Ex0) – (Ex2) and (Ex2^{op}) (cf. [Ke1, Appendix A]). Moreover, he observes (in [Ke1, A.2]) that the axiom (Ex2) follows from (Ex2^{op}) and a weaker version of (Ex2):

(Ex2') For every diagram $M' \xrightarrow{f} M \xleftarrow{\mathfrak{e}} L$, where $\mathfrak{e}$ is a deflation, there is a commutative square

$$\begin{array}{ccc} L' & \xrightarrow{\mathfrak{e}'} & M' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ L & \xrightarrow{\mathfrak{e}} & M \end{array}$$

where $\mathfrak{e}'$ is a deflation.

Quillen's description of exact categories is self-dual which implies self-duality of Keller's axioms: if $(C_X, \mathcal{E}_X)$ is an exact category, then $(C_X^{op}, \mathcal{E}_X^{op})$ is an exact category too.

K1.2.3. In the axioms (Ex2) and (Ex2^{op}), the conditions "there exists a cartesian (resp. cocartesian) square" can be replaced by "for any cartesian (resp. cocartesian) square". This implies that for any family $\{\mathcal{E}_i \mid i \in J\}$ of exact category structures on an additive category C_X, the intersection $\mathcal{E}_J = \bigcap_{i \in J} \mathcal{E}_i$ is a structure of an exact category.

K2. Suspended and cosuspended categories.

Suspended categories were introduced in [KeV]. In a sequel, we shall mostly use their dual version – *cosuspended* categories. They are defined as follows.

K2.1. Definitions. A *cosuspended k-linear category* is a triple $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$, where $C_{\mathfrak{X}}$ is an additive k-linear category, $\theta_{\mathfrak{X}}$ a k-linear functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}}$, and $Tr^-_{\mathfrak{X}}$ is a class of sequences of the form

$$\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L \qquad (1)$$

called *triangles* and satisfying the following axioms:

(SP0) Every sequence of the form (1) isomorphic to a triangle is a triangle.

(SP1) For every $M \in ObC_{\mathfrak{X}}$, the sequence $0 \longrightarrow M \xrightarrow{id_M} M \longrightarrow 0$ is a triangle.

(SP2) If $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$ is a triangle, then

$$\theta_{\mathfrak{X}}(M) \xrightarrow{-\theta_{\mathfrak{X}}(u)} \theta_{\mathfrak{X}}(L) \xrightarrow{\ w\ } N \xrightarrow{\ v\ } M$$

is a triangle.

(SP3) Given triangles $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$ and $\theta_{\mathfrak{X}}(L') \xrightarrow{w'} N' \xrightarrow{v'} M' \xrightarrow{u'} L'$ and morphisms $L \xrightarrow{\alpha} L'$ and $M \xrightarrow{\beta} M'$ such that the square

$$\begin{array}{ccc}
L & \xleftarrow{\ u\ } & M \\
\alpha \downarrow & & \downarrow \beta \\
L' & \xleftarrow{\ u'\ } & M'
\end{array}$$

commutes, there exists a morphism $N \xrightarrow{\gamma} N'$ such that the diagram

$$\begin{array}{ccccccc}
L & \xleftarrow{\ u\ } & M & \xleftarrow{\ v\ } & N & \xleftarrow{\ w\ } & \theta_{\mathfrak{X}}(L) \\
\alpha \downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow \theta_{\mathfrak{X}}(\alpha) \\
L' & \xleftarrow{\ u'\ } & M' & \xleftarrow{\ v'\ } & N' & \xleftarrow{\ w'\ } & \theta_{\mathfrak{X}}(L')
\end{array}$$

commutes.

(SP4) For every pair of morphisms $M \xrightarrow{u} L$ and $M' \xrightarrow{x} M$, there exists a commutative diagram

$$\begin{array}{ccccccc}
L & \xleftarrow{\ u\ } & M & \xleftarrow{\ v\ } & N & \xleftarrow{\ w\ } & \theta_{\mathfrak{X}}(L) \\
id \uparrow & & \uparrow x & & \uparrow y & & \uparrow id \\
L & \xleftarrow{\ u'\ } & M' & \xleftarrow{\ v'\ } & N' & \xleftarrow{\ w'\ } & \theta_{\mathfrak{X}}(L) \\
 & & s \uparrow & & \uparrow t & & \uparrow \theta_{\mathfrak{X}}(u) \\
 & & \widetilde{M} & \xleftarrow{\ id\ } & \widetilde{M} & \xleftarrow{\ r\ } & \theta_{\mathfrak{X}}(M) \\
 & & r \uparrow & & \uparrow & & \\
 & & \theta_{\mathfrak{X}}(M) & \xleftarrow{\ \theta_{\mathfrak{X}}(v)\ } & \theta_{\mathfrak{X}}(N) & &
\end{array}$$

whose two upper rows and two central columns are triangles.

K2.2. Suspended categories. A *suspended k-linear category* is defined dually; i.e. it is a triple $\mathfrak{T}_+ C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^+_{\mathfrak{X}})$, where $C_{\mathfrak{X}}$ is an additive k-linear category, $\theta_{\mathfrak{X}}$ a k-linear functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}}$, and $Tr^+_{\mathfrak{X}}$ is a class of sequences of the form

$$L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{w} \theta_{\mathfrak{X}}(L) \tag{2}$$

such that the dual data is a cosuspended category.

K2.3. Triangulated categories and (co)suspended categories. A suspended category $\mathfrak{T}_+ C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^+_{\mathfrak{X}})$ (resp. a cosuspended category $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$) is a triangulated category iff the translation functor $\theta_{\mathfrak{X}}$ is an auto-equivalence.

K2.4. Properties of cosuspended and suspended categories. The following properties of a cosuspended category $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$ follow directly from the axioms:

(a) Every morphism $M \xrightarrow{u} L$ of $C_{\mathfrak{X}}$ can be embedded into a triangle

$$\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L.$$

(b) For every triangle $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$, the sequence of representable functors

$$\ldots \longrightarrow C_{\mathfrak{X}}(-, \theta_{\mathfrak{X}}(L)) \xrightarrow{C_{\mathfrak{X}}(-,w)} C_{\mathfrak{X}}(-, N) \xrightarrow{C_{\mathfrak{X}}(-,v)} C_{\mathfrak{X}}(-, M) \xrightarrow{C_{\mathfrak{X}}(-,u)} C_{\mathfrak{X}}(-, L) \tag{3}$$

is exact. In particular, the compositions $u \circ v$, $v \circ w$, $w \circ \theta_{\mathfrak{X}}(u)$ are zero morphisms.

(c) If the rows of the commutative diagram

$$\begin{array}{ccccccc} L & \xleftarrow{u} & M & \xleftarrow{v} & N & \xleftarrow{w} & \theta_{\mathfrak{X}}(L) \\ \alpha \downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow \theta_{\mathfrak{X}}(\alpha) \\ L' & \xleftarrow{u'} & M' & \xleftarrow{v'} & N' & \xleftarrow{w'} & \theta_{\mathfrak{X}}(L') \end{array}$$

are triangles and the two left vertical arrows, α and β, are isomorphisms, then γ is an isomorphism too (see the axiom K2.1 (SP3)).

(d) Direct sum of triangles is a triangle.

(e) If $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$, is a triangle, then the sequence

$$\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \longrightarrow 0$$

is split exact iff $u = 0$.

(f) For an arbitrary choice of triangles starting with u, x and xu in the diagram K2.1 (SP4), there are morphisms y and t such that the second central column is a triangle and the diagram commutes.

If $\mathfrak{T}\_C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}}))$ is a triangulated category, i.e. the translation functor $\theta_{\mathfrak{X}}$ is an auto-equivalence, then, in addition, we have the following properties:

(g) A diagram $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$, is a triangle if (by (SP2), iff)

$$\theta_{\mathfrak{X}}(M) \xrightarrow{-\theta_{\mathfrak{X}}(w)} \theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M$$

is a triangle.

(h) Given triangles $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$ and $\theta_{\mathfrak{X}}(L') \xrightarrow{w'} N' \xrightarrow{v'} M' \xrightarrow{u'} L'$ and morphisms $M \xrightarrow{\beta} M'$ and $N \xrightarrow{\gamma} N'$ such that the square

$$\begin{array}{ccc} N & \xrightarrow{v} & M \\ \gamma \big\downarrow & & \big\downarrow \beta \\ N' & \xrightarrow{v'} & M' \end{array}$$

commutes, there exists a morphism $L \xrightarrow{\alpha} L'$ such that the diagram

$$\begin{array}{ccccccc} L & \xleftarrow{u} & M & \xleftarrow{v} & N & \xleftarrow{w} & \theta_{\mathfrak{X}}(L) \\ \alpha \big\downarrow & & \big\downarrow \beta & & \big\downarrow \gamma & & \big\downarrow \theta_{\mathfrak{X}}(\alpha) \\ L' & \xleftarrow{u'} & M' & \xleftarrow{v'} & N' & \xleftarrow{w'} & \theta_{\mathfrak{X}}(L') \end{array}$$

commutes.

(i) For every triangle $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$, the sequence of corepresentable functors

$$\ldots \longleftarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(L), -) \xleftarrow{C_{\mathfrak{X}}(w,-)} C_{\mathfrak{X}}(N, -) \xleftarrow{C_{\mathfrak{X}}(v,-)} C_{\mathfrak{X}}(M, -) \xleftarrow{C_{\mathfrak{X}}(u,-)} C_{\mathfrak{X}}(L, -) \qquad (3^o)$$

is exact.

K2.5. Triangle functors. Let $\mathfrak{T}\_C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$ and $\mathfrak{T}\_C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, Tr^-_{\mathfrak{Y}})$ be cosuspended k-linear categories. A *triangle* k-linear functor from $\mathfrak{T}\_C_{\mathfrak{X}}$ to $\mathfrak{T}\_C_{\mathfrak{Y}}$ is a pair (F, ϕ), where F is a k-linear functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{Y}}$ and ϕ is a functor morphism $\theta_{\mathfrak{Y}} \circ F \longrightarrow F \circ \theta_{\mathfrak{X}}$ such that for every triangle $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$ of $\mathfrak{T}\_C_{\mathfrak{X}}$, the sequence

$$\theta_{\mathfrak{Y}}(F(L)) \xrightarrow{F(w)\phi(L)} F(N) \xrightarrow{F(v)} F(M) \xrightarrow{F(u)} F(L)$$

is a triangle of $\mathfrak{T}_-C_{\mathfrak{Y}}$. It follows from this condition and the property K2.4(b) (applied to the case $M = 0$) that ϕ is invertible. The composition of triangle functors is defined naturally: $(G, \psi) \circ (F, \phi) = (G \circ F, G\phi \circ \psi F)$.

If (F, ϕ) and (F', ϕ') are triangle functors from $\mathfrak{T}_-C_{\mathfrak{X}}$ to $\mathfrak{T}_-C_{\mathfrak{Y}}$. A morphism from (F, ϕ) to (F', ϕ') is given by a functor morphism $F \xrightarrow{\lambda} F'$ such that the diagram

$$\begin{array}{ccc} \theta_{\mathfrak{Y}} \circ F & \xrightarrow{\phi} & F \circ \theta_{\mathfrak{X}} \\ {\scriptstyle \theta_{\mathfrak{Y}}\lambda}\downarrow & & \downarrow{\scriptstyle \lambda\theta_{\mathfrak{X}}} \\ \theta_{\mathfrak{Y}} \circ F' & \xrightarrow{\phi'} & F' \circ \theta_{\mathfrak{X}} \end{array}$$

commutes. The composition is the composition of the functor morphisms.

Altogether gives the definition of a large bicategory $\widetilde{\mathfrak{Tr}}_k^-$ formed by cosuspended k-linear categories, triangle k-linear functors as 1-morphisms and morphisms between them as 2-morphisms. Restricting to svelte cosuspended categories, we obtain the bicategory $\mathfrak{Tr}_k^-$.

We denote by $\widetilde{\mathfrak{Tr}}_k$ (resp. by $\mathfrak{Tr}_k$) the full subbicategory of $\widetilde{\mathfrak{Tr}}_k^-$ whose objects are triangulated (resp. svelte triangulated) categories.

Finally, dualizing (i.e. inverting all arrows in the constructions above), we obtain the large bicategory $\widetilde{\mathfrak{Tr}}_k^+$ of suspended categories and triangular functors and its subbicategory $\mathfrak{Tr}_k^+$ whose objects are svelte suspended categories. Thus, we have a diagram of natural full embeddings

$$\begin{array}{ccccc} \widetilde{\mathfrak{Tr}}_k^+ & \longleftarrow & \widetilde{\mathfrak{Tr}}_k & \longrightarrow & \widetilde{\mathfrak{Tr}}_k^- \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{Tr}_k^+ & \longleftarrow & \mathfrak{Tr}_k & \longrightarrow & \mathfrak{Tr}_k^- \end{array}$$

K2.6. Triangle equivalences. A triangle k-linear functor $\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{(F,\phi)} \mathfrak{T}_-C_{\mathfrak{Y}}$ is called a *triangle equivalence* if there exists a triangle functor $\mathfrak{T}_-C_{\mathfrak{Y}} \xrightarrow{(G.\psi)} \mathfrak{T}_-C_{\mathfrak{X}}$ such that the compositions $(F, \phi) \circ (G, \psi)$ and $(G, \psi) \circ (F, \phi)$ are isomorphic to respective identical triangle functors.

It follows from K7.1.1 that the quasi-inverse triangle functor (G, ψ) is k-linear.

K2.6.1. Lemma [Ke1]. *A triangle k-linear functor (F, ϕ) is a triangle equivalence iff F is an equivalence of the underlying categories.*

K3. Stable and costable categories of an exact category.

Let C_X be a k-linear category and $\mathcal{B}$ its full subcategory. The class $\mathcal{J}_{\mathcal{B}}$ of all arrows of C_X which factor through some objects of $\mathcal{B}$ is an ideal in $HomC_X$. We denote by $\mathcal{B}\backslash C_X$, or by $C_{\mathcal{B}\backslash X}$ the category having same objects as C_X; its morphisms are classes of morphisms of C_X modulo the ideal $\mathcal{J}_{\mathcal{B}}$, that is two morphisms with the same source and target are equivalent if their difference belongs to the ideal $\mathcal{J}_{\mathcal{B}}$.

We are particularly interested in this construction when $(C_X, \mathcal{E}_X)$ is an exact k-linear category and $\mathcal{B}$ is the fully exact subcategory of C_X generated by $\mathcal{E}_X$-projective or $\mathcal{E}_X$-injective objects of $(C_X, \mathcal{E}_X)$. In the first case, we denote the category $\mathcal{B}\backslash C_X$ by $C_{\mathfrak{S}_-X}$ and will call it the *costable* category of $(C_X, \mathcal{E}_X)$. In the second case, the notation is $C_{\mathfrak{S}_+X}$ and the name of this category is the *stable* category of $(C_X, \mathcal{E}_X)$.

K3.1. Example. Let C_X be an additive k-linear category endowed with the smallest exact structure $\mathcal{E}_X^{spl}$ (cf. K2.1). Then the correponding costable category is trivial: all its objects are isomorphic to zero.

K3.2. Exact categories with enough projective objects and their costable categories. Let $(C_X, \mathcal{E}_X)$ be an exact k-linear category with *enough projective objects*; i.e. for each object M of C_X, there exists a deflation $P \longrightarrow M$, where P is a projective object. Then the costable category $C_{\mathfrak{S}_-X}$ of $(C_X, \mathcal{E}_X)$ has a natural structure of a cosuspended k-linear category defined as follows. The endofunctor $\theta_{\mathfrak{S}_-X}$ assigns to an object M the (image in $C_{\mathfrak{S}_-X}$ of) the kernel of a deflation $P \longrightarrow M$, where P is a projective object. For any morphism $L \xrightarrow{f} M$, the morphism $\theta_{\mathfrak{S}_-X}(f)$ is the image of the morphism h in the commutative diagram

$$\begin{array}{ccccc}
\theta_{\mathfrak{S}_-X}(L) & \xrightarrow{\mathfrak{j}} & P_L & \xrightarrow{\mathfrak{e}} & L \\
h \downarrow & & \downarrow g & & \downarrow f \\
\theta_{\mathfrak{S}_-X}(M) & \xrightarrow{\mathfrak{j}'} & P_M & \xrightarrow{\mathfrak{e}'} & M
\end{array}$$

A standard argument shows that objects $\theta_{\mathfrak{S}_-X}(L)$ are determined uniquely up to isomorphism and the morphism $\theta_{\mathfrak{S}_-X}(f)$ is uniquely determined by the choice of the objects $\theta_{\mathfrak{S}_-X}(L)$ and $\theta_{\mathfrak{S}_-X}(M)$.

With each conflation $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$ of $(C_X, \mathcal{E}_X)$, it is associated a sequence

$$\theta_{\mathfrak{S}_-X}(L) \xrightarrow{\partial} N \xrightarrow{\bar{\mathfrak{j}}} M \xrightarrow{\bar{\mathfrak{e}}} L$$

called a *standard triangle* and determined by a commutative diagram

$$\begin{array}{ccccc}
\theta_{\mathfrak{S}_-X}(L) & \xrightarrow{\widetilde{\mathfrak{j}}} & P_L & \xrightarrow{\widetilde{\mathfrak{e}}} & L \\
\widetilde{\partial} \downarrow & & \downarrow g & & \downarrow id_L \\
N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L
\end{array}$$

The morphism g here exists thanks to the projectivity of P_L. The *connecting morphism* $\theta_{\mathfrak{S}_-X}(L) \xrightarrow{\partial} N$ is, by definition, the image of $\widetilde{\partial}$.

Triangles are defined as sequences of the form $\theta_{\mathfrak{S}_-X}(L') \xrightarrow{\partial'} N' \xrightarrow{\mathfrak{j}'} M' \xrightarrow{\mathfrak{e}'} L'$ which are isomorphic to a standard triangle.

K3.2.1. Proposition ([KeV]). *For any exact k-linear category $(C_X, \mathcal{E}_X)$ with enough projective objects, the triple $\mathfrak{T}_-C_{\mathfrak{S}_-X} = (C_{\mathfrak{S}_-X}, \theta_{\mathfrak{S}_-X}, \mathfrak{Tr}_{\mathfrak{S}_-X})$ constructed above is a cosuspended k-linear category.*

If $(C_X, \mathcal{E}_X)$ is an exact category with enough injective objects, then the dual construction provides a structure of a suspended category on the stable category $C_{\mathfrak{S}_+X}$ of $(C_X, \mathcal{E}_X)$.

K3.2.2. The case of Frobenius categories. Recall that an exact category $(C_X, \mathcal{E}_X)$ is called a Frobenius category, if it has enough injective and projective objects and projective objects coincide with injective objects.

K3.2.1. Proposition. *If $(C_X, \mathcal{E}_X)$ is a Frobenius category, then its costable cosuspended category $\mathfrak{T}_-C_{\mathfrak{S}_-X}$ and (therefore) the stable suspended category $\mathfrak{T}_+C_{\mathfrak{S}_+X}$ are triangulated, and are triangular equivalent one to another.*

Proof. It is easy to check that if $(C_X, \mathcal{E}_X)$ is a Frobenius category, then the translation functor $\theta_{\mathfrak{S}_-X}$ is an auto-equivalence of the category $C_{\mathfrak{S}_-X}$. The rest follows from this fact. Details are left to the reader. ■

K3.3. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories with enough projective objects. Every 'exact' k-linear functor $(C_X, \mathcal{E}_X) \xrightarrow{f^*} (C_Y, \mathcal{E}_Y)$ which maps projective objects to projective objects induces a triangle k-linear functor $\mathfrak{T}_-C_{\mathfrak{S}_-X} \xrightarrow{\mathfrak{T}_-f^*} \mathfrak{T}_-C_{\mathfrak{S}_-Y}$ between the corresponding costable cosuspended categories.*

Proof. The argument is left to the reader. ■

K3.3.1. Corollary. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories with enough projective objects and*

$$(C_X, \mathcal{E}_X) \xrightarrow{f^*} (C_Y, \mathcal{E}_Y) \xrightarrow{f_*} (C_X, \mathcal{E}_X)$$

a pair of 'exact' functors such that f^* is k-linear and a left adjoint of f_*. Then the functor f^* induces a triangle k-linear functor $\mathfrak{T}_-C_{\mathfrak{S}_-X} \xrightarrow{\mathfrak{T}_-f^*} \mathfrak{T}_-C_{\mathfrak{S}_-Y}$ between the corresponding costable cosuspended categories.

Proof. By K7.1, the functor f^* maps projective objects of $(C_X, \mathcal{E}_X)$ to projective objects of $(C_Y, \mathcal{E}_Y)$. The assertion follows now from K3.3. ■

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Glossary of notations

Chapter I

$(C_X, \mathfrak{E}_X)$ a right exact category, 1.1
$\mathfrak{E}_X^{\mathfrak{s}}$ the class of universally strict epimorphisms of a category C_X, 1.1.1
$\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}}$ the category of *right exact 'spaces'* and weakly 'exact' morphisms, 1.6
$\mathfrak{Esp}_{\mathfrak{r}}$ the category of *right exact 'spaces'* and 'exact' morphisms, 1.6
$\bar{\emptyset}_X$ the *trivial* presheaf of sets on C_X: it maps C_X to the empty set, 2.0.1.1
C_X^* the category of non-trivial presheaves of sets, 2.0.1.1, 4.2.6
$(C_X, \tau)^\wedge$ the category of all sheaves of sets on the (pre)site (C_X, τ), 2.0.1.2
$C_{X_\tau} = (C_X, \tau)^\wedge \cap C_X^*$ the category of non-trivial sheaves of sets on (C_X, τ), 2.0.1.3
$C_{X_{\mathfrak{E}}} \stackrel{\mathrm{def}}{=} C_{X_{\mathfrak{E}_X}}$ the category of *non-trivial* sheaves of sets on $(C_X, \mathfrak{E}_X)$, 2.0.1.3
$C_X^{\circledast} \stackrel{\mathrm{def}}{=} \widehat{\mathfrak{x}}\backslash C_X^* = \widehat{\mathfrak{x}}\backslash C_X^\wedge$ 2.0.2(b)
$C_X \xrightarrow{h_X^{\circledast}} C_X^{\circledast}$ the canonical fully faithful functor induced by Yoneda embedding, 2.0.2(b)
$\mathcal{H}om^{\circledast}(C_X, C_Y)$ the full subcategory of $\mathcal{H}om(C_X, C_Y)$ generated by functors mapping initial objects to initial objects, 2.0.2(b2)
$\mathcal{H}om(C_X, C_Y)^{\circledast}$ the full subcategory of the category $\mathcal{H}om(C_X, C_Y)$ generated by functors mapping final objects to final objects, 2.0.2(c2)
$\mathcal{H}om_{\mathfrak{c}}(C_X^\wedge, C_Y)^{\circledast} \stackrel{\mathrm{def}}{=} \mathcal{H}om(C_X^\wedge, C_Y)^{\circledast} \bigcap \mathcal{H}om_{\mathfrak{c}}(C_X^\wedge, C_Y)$ 2.0.2(c2)
$C_X^* \xrightarrow{F^\diamond} C_Y$, assigns to every non-trivial presheaf of sets $\mathcal{G}$ on C_X the colimit of the composition of the forgerful functor $h_X^*/\mathcal{G} \longrightarrow C_X$ and $C_X \xrightarrow{F} C_Y$, argument 2.0.2(a)
$C_{X_\tau^{\circledast}} \stackrel{\mathrm{def}}{=} \widehat{\mathfrak{x}}\backslash C_{X_\tau} = \widehat{\mathfrak{x}}\backslash (C_X, \tau)^\wedge$, where $\mathfrak{x}$ is an initial object of C_X, 2.0.5
$C_X \xrightarrow{\mathfrak{j}_{X_\tau}^{\circledast}} \widehat{\mathfrak{x}}\backslash C_{X_\tau} = C_{X_\tau^{\circledast}}$ the composition of $C_X \xrightarrow{\mathfrak{h}_X^{\circledast}} \mathfrak{x}\backslash C_X^\wedge = \mathfrak{x}\backslash C_X^*$ and the functor $\widehat{\mathfrak{x}}\backslash C_X^* = \widehat{\mathfrak{x}}\backslash C_X^\wedge \xrightarrow{\mathfrak{q}_\tau^*} \widehat{\mathfrak{x}}\backslash (C_X, \tau)^\wedge = \widehat{\mathfrak{x}}\backslash C_{X_\tau}$ induced by the sheafification functor $C_X^* \xrightarrow{\mathfrak{q}_\tau^*} C_{X_\tau}$, 2.0.5
$\mathcal{H}om_k(C_X, C_Y)$ the category of k-linear functors from C_X to C_Y, 2.0.6
$\mathcal{M}_k(X) \stackrel{\mathrm{def}}{=} \mathcal{H}om_k(C_X^{op}, k-mod)$ the category of k-linear presheaves of k-modules on a k-linear category C_X, 2.0.6.1, 2.3
$C_X \xrightarrow{\mathfrak{h}_X} \mathcal{M}_k(X)$, $\mathcal{L} \longmapsto \widehat{\mathcal{L}} \stackrel{\mathrm{def}}{=} C_X(-, \mathcal{L})$, the *k-linear Yoneda embedding*, 2.0.6.1
$Sh_k(X, \tau)$ the category of k-linear sheaves of k-modules on (C_X, τ), 2.0.7
$C_{X_{\mathfrak{E}}^{\circledast}} \stackrel{\mathrm{def}}{=} \widehat{\mathfrak{x}}\backslash (C_X, \mathfrak{E}_X)^\wedge$ the category of sheaves of sets on $(C_X, \mathfrak{E}_X)$ over $\widehat{\mathfrak{x}}$ endowed with the canonical (– the finest) right exact structure $\mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}}$, 2.1.1
$(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^{\circledast}} (C_{X_{\mathfrak{E}}}, \mathfrak{E}^{\mathfrak{s}}_{X_{\mathfrak{E}}^{\circledast}})$ the canonical fully faithful 'exact' functor, 2.1.1

$C_{X^{\circledast}_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^{\circledast}} C_{Y^{\circledast}_{\mathfrak{E}}}$ the functor induced by $(C_X, \mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathfrak{E}_Y)$, 2.1.1
$Sh_k(X, \mathfrak{E}_X)$ the category of k-linear sheaves of k-modules on $(C_X, \mathfrak{E}_X)$, 2.3
C_{X_K} the Karoubian envelope of a category C_X, 3.3
$(C_{X_K}, \mathfrak{E}_{X_K})$ the Karoubian envelope of a right exact category $(C_X, \mathfrak{E}_X)$, 3.4.3
$Ker(f)$ the kernel of a morphism f, 4.1
$Cok(f)$ the cokernel of a morphism f, 4.1
Alg_k the category of associative unital k-algebras, 4.2.1
$\mathbf{Aff}_k \stackrel{\mathrm{def}}{=} Alg_k^{op}$ the category of non-commutative affine k-schemes, 4.2.3.3
$Coim(f)$ the coimage of f, 4.5
$\mathfrak{E}_X^{\circledast}$ the class of *deflations* whose pull-backs contain isomorphisms, 5.5.1, 5.5.2
$ExCat_k$ the category of exact k-linear categories and 'exact' k-linear functors, 7.1
$\mathfrak{E}_X^{spl}$ the class of split epimorphisms of a category C_X, 7.2.1
Add_k the category of additive k-linear categories and k-linear functors, 7.2.1
$\mathbf{C}(\mathcal{A})$ the category of complexes of an additive k-linear category $\mathcal{A}$, 7.2.2

Chapter II

$\mathcal{K}(C_X)$ the category of complexes, 1.5
$\mathcal{K}^b(C_X)$ the category of bounded complexes, 1.5
$T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ a ∂^*-functor, 2
$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$ the category of ∂^*-functors from $(C_X, \mathcal{E}_X)$ to C_Y, 2.1
$\mathcal{E}x_*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), (C_X, \mathfrak{E}_X))$ the category of preserving conflations functors, 2.2
$\partial^*\mathfrak{Un}((C_X, \mathfrak{E}_X), C_Y)$ the category of universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to C_Y, 3.0.1
$S_-(F)(L) = \lim\big(Ker(F(\mathfrak{k}(\mathfrak{e}))) \mid (M \stackrel{\mathfrak{e}}{\to} L) \in \mathfrak{E}_X\big)$ the satellite of F, 3.2
$\mathfrak{E}_{\mathfrak{X}}^{\Phi}$ a subclass of deflations coinduced by a functor $C_X \xrightarrow{\Phi} C_{\mathfrak{X}}$, 3.6.1
$\mathfrak{E}_X^{\circledast}$ the class of *deflations* (– arrows of $\mathfrak{E}_X$) with trivial kernel, 3.7
$C_{X^{\circledast}_{\mathfrak{E}}} \xrightarrow{F^{\circledast}_{\mathfrak{E}_X}} C_{Y^{\circledast}}$ composition of the inclusion $C_{X^{\circledast}_{\mathfrak{E}}} \longrightarrow C_{X^{\circledast}}$ and $C_{X^{\circledast}} \xrightarrow{F^{\circledast}} C_{Y^{\circledast}}$, 4.3
$h_X^{\circledast} \circ \mathcal{S}_-^{\bullet}F \simeq \mathcal{S}_-^{\bullet}(F^{\circledast}_{\mathfrak{E}_X}) \circ \mathfrak{j}_X^{\circledast}$, 4.3
by $\mathcal{H}om_k^*((C_X, \mathfrak{E}_X), C_Y)$ the category of k-linear ∂^*-functors, 4.6.2.2
$\partial_k^*\mathfrak{Un}((C_X, \mathfrak{E}_X), C_Y)$ the full subcategory of $\mathcal{H}om_k^*(C_X, \mathfrak{E}_X), C_Y)$ generated by universal ∂^*-functors, 4.6.2.2
$\partial_k^*\mathfrak{Un}(Sh_k(X, \mathfrak{E}_X), C_Y) \longrightarrow \partial_k^*\mathfrak{Un}((C_X, \mathfrak{E}_X), C_Y)$, $T = (T_i, \mathfrak{d}_i \mid i \geq 0) \longmapsto T \circ \mathfrak{j}_X^* = (T_i \circ \mathfrak{j}_X^*, \mathfrak{d}_i\mathfrak{j}_X^* \mid i \geq 0)$ localization functor, 4.6.6
$Ext^{\bullet}_{X, \mathfrak{E}_X}(-, L)$ 5.1
$\mathcal{E}xt^{\bullet}_{X, \mathfrak{E}_X}(-, \mathcal{L})$ 5.2
$(CE5)$, $(CE5^*)$ properties of right exact categories, 6.1
$\partial^*\widetilde{\mathfrak{Un}}(X, \mathfrak{E}_X)$ the category of universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$, 8.1.0
$\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ the category of universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$ whose zero components map initial objects to initial objects, 8.1.1

$\partial^*\mathfrak{Un}_\mathfrak{c}(X,\mathfrak{E}_X)$ the category of universal ∂^*-functors from $(C_X,\mathfrak{E}_X)$ to categories with limits and morphisms (F,ϕ) with F preserving limits, 8.1.2

$\partial\widetilde{\mathfrak{Un}}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of universal ∂-functors from a left exact category $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$, 8.1.3

$\partial\mathfrak{Un}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of universal ∂^*-functors from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ whose zero components map final objects to final objects, 8.1.3

$\partial^*\mathfrak{Un}_\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of universal ∂-functors from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to categories with limits and morphisms (F,ϕ) with F preserving limits, 8.1.4

$Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ the universal ∂-functor from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to $C^*_\mathfrak{X}$ such that $Ext^0_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}} = h_\mathfrak{X}$, 8.2

$\partial^*_k\mathfrak{Un}(X,\mathfrak{E}_X)$ the category of universal k-linear ∂^*-functors from $(C_X,\mathfrak{E}_X)$ to k-linear additive categories, 8.3

$\partial_k\mathfrak{Un}^\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of k-linear ∂-functors with values in cocomplete categories and morphisms (F,ψ) such that the functor F preserves limits, 8.3

$\partial_k\mathfrak{Un}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of universal k-linear ∂-functors from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to k-linear additive categories, 8.3

$\partial_k\mathfrak{Un}^\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of universal k-linear ∂-functors to cocomplete categories and morphisms (F,ψ) such that the functor F preserves colimits, 8.3

$\partial^*\mathfrak{UEx}(X,\mathfrak{E}_X)$ the category of the universal 'exact' ∂^*-functors, 9.0

$\partial^*\mathfrak{UEx}_\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of universal ∂^*-functors to complete right exact categories satisfying $(CE5^*)$ and morphisms (F,ψ) with F preserving limits, 9.0

$\partial\mathfrak{UEx}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of universal 'exact' ∂-functors from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to left exact categories satisfying $(CE5)$, 9.0.1

$\partial\mathfrak{UEx}^\mathfrak{c}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of universal ∂-functors to cocomplete left exact categories satisfying $(CE5)$ and morphisms (F,ψ) with F preserving colimits, 9.0.1

$\partial^*_k\mathfrak{UEx}(X,\mathfrak{E}_X)$ the category of universal 'exact' k-linear ∂^*-functors from $(C_X,\mathfrak{E}_X)$ to right exact k-linear categories satisfying $(CE5^*)$ whose zero component maps deflations to deflations, 9.2

$\partial_k\mathfrak{UEx}(\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ the category of universal 'exact' k-linear ∂-functors from $(C_\mathfrak{X},\mathfrak{I}_\mathfrak{X})$ to k-linear left exact categories satisfying $(CE5)$ whose zero component maps inflations to inflations, 9.2

Chapter III

${}^\Phi\mathfrak{E}_Y$ the right exact structure coinduced via $C_X \xrightarrow{\Phi} C_Y$, 1.3.3

$\mathcal{F}-mod$ the category of modules over a monad $\mathcal{F}$, 1.5

$\mathfrak{R}ex((C_X,\mathfrak{E}_X),(C_Y,\mathfrak{E}_Y))$ the category of right 'exact' functors, 2.8

$\mathcal{E}ff^o((C_X,\mathfrak{E}_X),C_Y)$ the category of coeffaceable functors from $(C_X,\mathfrak{E}_X)$ to C_Y, 2.8

$\mathfrak{Ex}_k((C_X,\mathfrak{E}_X),k-mod)$ the category of 'exact' k-linear functors from $(C_X,\mathfrak{E}_X)$ to $k-mod$, 2.8.3

$\mathcal{H}_X(F)(N) = \mathrm{colim}(Ker(F(M) \rightrightarrows F(M\times_N M)))$, where $M \longrightarrow N$ runs through deflations of N, Heller functor, 2.9

$(C_X^{\wedge}, \mathfrak{I}_X^{\wedge}, \widehat{\Theta}_X^*, \mathfrak{Tr}_{X^{\wedge}})$ the *prestable category of presheaves* on $(C_X, \mathfrak{I}_X)$, 3.5
$\mathfrak{Im}(f)$ the image of a morphism f, 4.2.1
$(C_{X_\mathfrak{p}}, (\theta_{X_\mathfrak{p}}, \lambda), \mathfrak{Tr}_{X_\mathfrak{p}})$ the *prestable* category of the left exact category $(C_X, \mathfrak{I}_X)$, 5.2
$(C_\mathfrak{X}, \widetilde{\theta}_\mathfrak{X}, \mathfrak{Tr}_\mathfrak{X})$ *presuspended* category, 6.1
$H_\bullet(X, \mathcal{F}) = (H_n(X, \mathcal{F}),\ \mathfrak{d}_n \mid n \geq 0)$ homology with coefficients in $\mathcal{F}$, 7.1
$\mathfrak{Pa}(X)$ the *'space' of paths* of the 'space' X, 7.2
$\pi_n(X, \mathfrak{O}_X)$ the *n-th homotopy group* of the pointed 'space' $(X, \mathfrak{O}_X)$, 6.4

Chapter IV

$\mathfrak{I}^{\mathfrak{st}}_{\mathfrak{Sp}}$ the class of strict monomorphisms of 'spaces', 2.1
$\mathfrak{I}^{\mathfrak{s}}$ the canonical left exact structure on $|Cat|^o$, 2.3
$\mathfrak{I}^{\mathfrak{s}}_{\mathfrak{c}}$ the class of all conservative morphisms from $\mathfrak{I}^{\mathfrak{s}}$, 2.4.4
$|KCat|^o$ the category of Karoubian 'spaces', 2.5
$\mathfrak{I}^{\mathfrak{Ka}}$ the canonical left exact structure on $\mathfrak{I}^{\mathfrak{Ka}}$, 2.5.2
$\mathfrak{L}$, $\mathfrak{L}_\ell$, $\mathfrak{L}_\mathfrak{r}$, $\mathfrak{L}_\mathfrak{e}$, $\mathfrak{L}^\mathfrak{c}$, $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}$ structures of a left exact category on the category of 'spaces', 3.1
$\mathfrak{I}^{\mathfrak{s}}_\ell$ and $\mathfrak{I}^{\mathfrak{s}}_\mathfrak{r}$ left exact structures on the category of 'spaces', 3.3
$|Cat_k|^o$ the category of k-'spaces', 4.1
$\mathfrak{I}^{\mathfrak{st}}_k$ the finest left exact structure on $|Cat_k|^o$, 4.4.1
$\mathfrak{I}^{\mathfrak{s}}_k$ the k-linear version of the left exact structure $\mathfrak{I}^{\mathfrak{s}}$, 4.4.2
$\mathfrak{L}(k)$, $\mathfrak{L}_\ell(k)$, $\mathfrak{L}_\mathfrak{r}(k)$, $\mathfrak{L}_\mathfrak{e}(k)$, $\mathfrak{L}^\mathfrak{c}(k)$, $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}(k)$ left exact structures on $|Cat_k|^o$, 4.4.6
$\mathfrak{I}^{\mathfrak{es}}$ a left exact structure on the category of right exact 'spaces', 5.3
$\mathfrak{K}\mathfrak{Esp}_\mathfrak{r}$ the category of right exact Karoubian 'spaces', 5.4
$\mathfrak{I}^{\mathfrak{Ke}}$ a canonical left exact structure on $\mathfrak{K}\mathfrak{Esp}_\mathfrak{r}$, 5.4.2
$\mathfrak{L}_{\mathfrak{es}}$ a left exact structure on the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces', 5.5
$\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ a left exact structure on the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces', 5.5.3
$\mathfrak{Esp}^\mathfrak{r}_k$ the category of right exact k-spaces, 6
$\mathfrak{I}^{\mathfrak{es}}_k$ the preimage of the left exact structure $\mathfrak{I}^{\mathfrak{es}}$, 6
$\mathfrak{K}\mathfrak{Esp}^\mathfrak{r}_k$ Karoubian right exact k-'spaces', 6.4
$\mathfrak{I}^{\mathfrak{Ke}}_k$ a canonical left exact structure on $\mathfrak{K}\mathfrak{Esp}^\mathfrak{r}_k$, 6.4
$\mathfrak{L}_{\mathfrak{es}}(k)$, $\mathfrak{L}^{\mathfrak{es}}_\ell(k)$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}(k)$, $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{es}}(k)$, $\mathfrak{L}^{\mathfrak{c},\mathfrak{c}}_{\mathfrak{es}}(k)$ left exact structures on $\mathfrak{Esp}^\mathfrak{r}_k$, 6.5
$\mathfrak{I}^{\mathfrak{es}}_\ell(k)$ and $\mathfrak{I}^{\mathfrak{es}}_\mathfrak{r}(k))$ canonical left exact structures on $\mathfrak{Esp}^\mathfrak{r}_k$, 6.6
$(\mathfrak{Esp}^\mathfrak{e}_k, \mathfrak{I}^\mathfrak{e}_k)$ left exact category of exact k-'spaces', 7.2
$C_{X^\mathfrak{D}} = C_X^\mathfrak{D}$ the category of diagrams $\mathfrak{D} \longrightarrow C_X$, 8
$\mathcal{P}^\mathfrak{D}$ the endofunctor $X \longmapsto X^\mathfrak{D}$ of the category $|Cat|^o$, 8
$\mathfrak{I}^\mathfrak{D} = \mathcal{P}^{\mathfrak{D}^{-1}}(\mathfrak{I}^\mathfrak{s})$ a left exact structure on $|Cat|^o$, 8.2
$\mathfrak{E}^\mathfrak{c}_{X^\mathfrak{D}}$ a right exact structure on the category $C_X^\mathfrak{D} = C_{X^\mathfrak{D}}$, 8.3.3
$\Sigma_{G,\mathfrak{E}_X}$ the class of arrows whose pull-backs along all deflations belong to Σ_G, 8.3.5.1
$\mathfrak{I}^\mathfrak{D}_\mathfrak{c}$ a left exact structure on the category $\mathfrak{Esp}_r$ of right exact 'spaces', 8.5.2
$\mathfrak{Pa}(X, \mathfrak{E}_X)$ the path 'space' of a right exact 'space' $(X, \mathfrak{E}_X)$, 10.1.1

Chapter V

$|C_X|$ the set of isomorphism classes of objects of C_X, 1.1
$\mathbb{Z}|C_X|$ the free abelian group generated by $|C_X|$, 1.1
$\mathbb{Z}_0(C_X)$ the subgroup of $\mathbb{Z}|C_X|$ generated by $\{[M]-[N] \mid (M \to N) \in HomC_X\}$, 1.1
$K_0(X, \mathfrak{E}_X)$ the group K_0 of a right exact 'space' $(X, \mathfrak{E}_X)$, 1.4
$\mathfrak{Esp}_{\mathfrak{r}}^w$ the category of right exact 'spaces' and morphisms preserving conflations, 1.7
$\mathfrak{Esp}_{\mathfrak{r}}^*$ the subcategory of $\mathfrak{Esp}_{\mathfrak{r}}^w$ formed by 'exact' morphisms, 1.7
$|\mathfrak{Gr}|^o$ the category of 'spaces' represented by groupoids, 2.1
$\mathfrak{Gr}_*$ the embedding $|\mathfrak{Gr}|^o \longrightarrow |Cat|^o$, 2.1
$|\mathfrak{Ord}|^o$ the subcategory of 'spaces' represented by groupoids, 2.2
$\mathfrak{Sk}_*$ the full embedding $|\mathfrak{Ord}|^o \longrightarrow |Cat|^o$, 2.2
$K_0^{\mathcal{Y}}(\mathcal{X}, \xi) = K_0^{\mathcal{Y}}(\mathcal{X}, \mathcal{X} \xrightarrow{\xi} \mathcal{Y}) \stackrel{\text{def}}{=} Cok(K_0(\mathcal{Y}) \xrightarrow{K_0(\xi)} K_0(\mathcal{X}))$, relative K_0
$K_i^{\mathcal{Y},\mathfrak{I}}(\mathcal{X}, \xi)$ *universal* K-groups of the right exact 'space' $(\mathcal{X}, \xi)$ over $\mathcal{Y}$, 2
$K_\bullet^{\mathfrak{S},\mathfrak{F}} = (K_i^{\mathfrak{S},\mathfrak{F}}, \mathfrak{d}_i \mid i \geq 0)$ universal K-functor from $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})^{op}$ to $\mathbb{Z}-mod$, 3.4
$|Cat_*|^o \stackrel{\text{def}}{=} |Cat|^o/x$ the category of 'spaces' over the trivial 'space', 4
$\mathfrak{Esp}_{\mathfrak{r}}^* \stackrel{\text{def}}{=} \mathfrak{Esp}_{\mathfrak{r}}^*/x$ the category of right exact 'spaces' over the trivial 'space', 4
$\mathfrak{I}_{\mathfrak{c}*}^{\to}$ the canonical left exact structure on $\mathfrak{Esp}_{\mathfrak{r}}^*$, 4.1
$\mathfrak{Esp}_k^{\mathfrak{a}}$ the category of k-linear abelian categories, 5
$\mathfrak{I}_k^{\mathfrak{a}}$ a canonical left exact structure on $\mathfrak{Esp}_k^{\mathfrak{a}}$, 5.2.2
$K_\bullet^{\mathfrak{a}} = (K_i^{\mathfrak{a}}, \mathfrak{d}_i^{\mathfrak{a}} \mid i \geq 0)$ the universal Grothendieck K-functor, 5.4
$K_\bullet^{\mathfrak{Q}} = (K_i^{\mathfrak{Q}}, \mathfrak{d}_i^{\mathfrak{Q}} \mid i \geq 0)$ the Quillen's K-functor, 5.5

Chapter VI

$\mathbb{S} \bullet \mathbb{T}$ the Gabriel product of subcategories of a right exact category, 1
$\mathcal{B}^{(n+1)}$ the *upper* n^{th} *infinitesimal neighborhood* of $\mathcal{B}$, 2
$\mathcal{B}_{(n+1)}$ the *lower* n^{th} *infinitesimal neighborhood* of $\mathcal{B}$, 2
$\mathfrak{REsp}_{\mathfrak{r}}$ the category of right exact relative 'spaces', 6.2
$(\mathfrak{REsp}_{\mathfrak{r}}, \mathfrak{I}_{\mathfrak{r}}^{\to})$ the left exact category of relative 'spaces', 6.2

Complementary Facts

$\mathfrak{Mon}^+(C_X)$ the category of augmented monads on C_X, C1.9.3
$\mathfrak{Pr}(X, \mathfrak{E}_X)$ the full subcategory of C_X whose objects are projective objects, C5.1
$\Omega_{\mathcal{F}}$ the functor of *Kähler differentials*, C5.3

Chapter VII

$C_X/\mathcal{L}^+$ the disjoint union of the category $C_X/\mathcal{L}$ and the “point” – the category with one morphism, 1.0

$\mathfrak{D}_f$ the functor $C_X/\mathcal{L}^+ \longrightarrow C_X/\mathcal{L}$, which is identical on $C_X/\mathcal{L}$ and maps the "point" to the object $(\mathcal{M}, \mathcal{M} \xrightarrow{f} \mathcal{L})$, 1.0

$(Ker(f), Ker(f) \xrightarrow{\mathfrak{k}(f)} \mathcal{M})$, the kernel of the morphism $\mathcal{M} \xrightarrow{f} \mathcal{L}$, 1.0

$\mathcal{M}_{\xi,f} \xrightarrow{\xi'_f} \mathcal{M}$ the pull-back of the morphism $\mathcal{L}_\xi \xrightarrow{\xi} \mathcal{L}$ along $\mathcal{M} \xrightarrow{f} \mathcal{L}$, 1.1

$C_X/\mathcal{L} \xrightarrow{\mathfrak{P}_f} C_X/\mathcal{M}$ the functor of pull-back along $\mathcal{M} \xrightarrow{f} \mathcal{L}$, 1.1

$(Ker(f), Ker(f) \xrightarrow{\mathfrak{k}(f)} \mathcal{M}) = \lim(\mathfrak{P}_f)$, 1.1

$(Ker(f) \xrightarrow{0_f} M$ the canonical morphism, 1.5

$Coim(f) = Cok(Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M)$ 1.5

$M \xrightarrow{p_f} Coim(f) \xrightarrow{j_f} N$ a canonical decomposition associated with $M \xrightarrow{f} N$, 1.5

$(Ker_{\mathfrak{v}}(\mathfrak{f}), Ker_{\mathfrak{v}}(\mathfrak{f}) \xrightarrow{\mathfrak{k}_{\mathfrak{v}}(\mathfrak{f})} \widehat{\mathcal{M}})$ the virtual kernel of a morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$, 2.1

$$\begin{array}{ccc} Ker_{\mathfrak{v}}(\mathfrak{f}) & \xrightarrow{\mathfrak{f}_\xi} & \lim h_{X/\mathcal{L}} \\ \mathfrak{k}_{\mathfrak{v}}(\mathfrak{f}) \downarrow & cart & \downarrow \mathfrak{j}_{\mathcal{L}} \\ \widehat{\mathcal{M}} & \xrightarrow{\widehat{\mathfrak{f}}} & \widehat{\mathcal{L}} \end{array}$$

cartesian square defining the virtual kernel of a morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{L}$, 2.2

$\lim(C_X \xrightarrow{h_X} C_X^\wedge$ virtually initial object of C_X, 2.3.0

$C_{X^\circledast} \stackrel{\mathrm{def}}{=} \lim h_X \backslash C_X^\wedge$ the "reduced" category of presheaves of sets on C_X, 2.3.0.2

$C_X \xrightarrow{h_X^*} C_X^*$ the corestriction to C_X^* of the Yoneda embedding, 2.3.5.2

$C_X = \coprod_{\mathfrak{i} \in \pi_0(X)} C_{X_{\mathfrak{i}}}$ be the decomposition of a category C_X into the disjoint union of its connected components, 2.4.4

$C_{X^{\mathfrak{rc}}} = \coprod_{\mathfrak{i} \in \pi_0(X)} C_{X_{\mathfrak{i}}^\circledast}$ "reduced" category associated with a category C_X, 2.4.0

$C_X \xrightarrow{h_X^{\mathfrak{rc}}} C_{X^{\mathfrak{rc}}} = \coprod_{\mathfrak{i} \in \pi_0(X)} C_{X_{\mathfrak{i}}^\circledast}$ the "reduced" Yoneda embedding, 2.4.0

$Cat_{\mathfrak{sc}}$ the category of semi-complete categories, 2.10.2

$Cat_{\mathfrak{sc}}^{\mathfrak{v}}$ the category of virtually semi-complete categories, 2.10.3

$\mathfrak{E}'_{\mathcal{B},X}$ the class of all deflations $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ such that $\mathcal{M}$, $\mathcal{L}$ and $Ker(\mathfrak{e})$ are objects of a subcategory $\mathcal{B}$, 2.11.3

$\mathfrak{E}_{\mathcal{B},X} = \mathfrak{E}'_{\mathcal{B},X} \bigcup Iso(\mathcal{B})$, 2.11.3

$T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ ∂^*-functor, 3.1

$T_{i+1}(\mathcal{L}) \xrightarrow{\mathfrak{d}_i(\mathfrak{e},\xi)} T_i(\mathcal{M}_{\xi,\mathfrak{e}})$ connecting morphism for $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xleftarrow{\xi} \mathcal{L}_\xi$, 3.1

$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$ the category of ∂^*-functors from $(C_X, \mathcal{E}_X)$ to C_Y, 3.1.2

$\mathcal{H}om^*((C_X,\mathfrak{E}_X),C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X,C_Y)$ assigns to (morphisms of) ∂^*-functors their zero components, 3.2.1

$\mathfrak{D}\mathcal{S}_-F(\mathcal{L})$ the diagram $\begin{array}{ccc} \mathcal{V}_{\alpha,F(\xi_{\mathfrak{e}})} & \xrightarrow{\mathfrak{d}^{\alpha}_{\xi,\mathfrak{e}}} & \\ \downarrow & cart & \\ \mathcal{V}_{\alpha} & \xrightarrow{\alpha} & \end{array} F\left(\begin{array}{ccc} \mathcal{M}_{\xi,\mathfrak{e}} & \xrightarrow{\mathfrak{e}_{\xi}} & \mathcal{L}_{\xi} \\ \xi_{\mathfrak{e}}\downarrow & cart & \downarrow\xi \\ \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array}\right)$, 3.3.1

$\mathcal{S}_-F(\mathcal{L}) \stackrel{\mathrm{def}}{=} \mathfrak{D}\mathcal{S}_-F(\mathcal{L})$ the first right derived functor of a functor F, 3.3.2

$\begin{array}{ccccc} \mathcal{S}_-F(\mathcal{L}) & \xrightarrow{\mathfrak{d}_{\xi,\mathfrak{e}}} & F(\mathcal{M}_{\xi,\mathfrak{e}}) & \xrightarrow{F(\mathfrak{e}_{\xi})} & F(\mathcal{L}_{\xi}) \\ & & F(\xi_{\mathfrak{e}})\downarrow & & \downarrow F(\xi) \\ & & F(\mathcal{M}) & \xrightarrow{F(\mathfrak{e})} & F(\mathcal{L}) \end{array}$ the canonical diagram, 3.3.3

$\mathcal{S}_-F(\mathcal{L}) \xrightarrow{\mathfrak{d}_{\xi,\mathfrak{e}}} F(\mathcal{M}_{\xi,\mathfrak{e}})$ the connecting morphism, 3.3.3
$S^{\bullet}_-(F) = (S^i_-(F),\mathfrak{d}^F_i \mid i \geq 0)$ the derived ∂^*-functor of F, 3.3.5
$\mathfrak{E}^{\Phi}_{\mathfrak{X}}$ 3.7.1
$C_{X^{\mathfrak{re}}_{\tau}} = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X^{\circledast}_{\mathfrak{i},\tau_{\mathfrak{i}}}}$, the reduced category of sheaves of sets on a (pre)site (C_X,τ), 3.8.1
$C_{X^{\mathfrak{re}}} \xrightarrow{\mathfrak{q}^{\mathfrak{re}}_{\tau}} C_{X^{\mathfrak{re}}_{\tau}}$ the "reduced" sheafification functor 3.8.1.1
$C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}} = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X^{\circledast}_{\mathfrak{i},\mathfrak{E}_{\mathfrak{i}}}}$, the "reduced" category of sheaves on a right exact category $(C_X,\mathfrak{E}_X) = \coprod_{\mathfrak{i}\in\pi_0(X)} (C_{X_{\mathfrak{i}}},\mathfrak{E}_{\mathfrak{i}})$, 3.8.2
$C_X \xrightarrow{\mathfrak{j}^{\mathfrak{re}}_X} C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}}$ the composition of the embedding $C_X \xrightarrow{h^{\mathfrak{re}}_X} C_{X^{\mathfrak{re}}} = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X^{\circledast}_{\mathfrak{i}}}$ with the "reduced" sheafification functor $C_{X^{\mathfrak{re}}} \xrightarrow{\mathfrak{q}^{\mathfrak{re}}_{\mathfrak{E}_X}} C_{X^{\mathfrak{re}}_{\mathfrak{E}_X}} = \coprod_{\mathfrak{i}\in\pi_0(X)} C_{X^{\circledast}_{\mathfrak{i},\mathfrak{E}_{\mathfrak{i}}}}$, 3.8.2
$\mathcal{S}^{\bullet}_-(F^{\mathfrak{re}}_{\mathfrak{E}_X})\circ\mathfrak{j}^{\mathfrak{re}}_X \xrightarrow{\sim} \mathcal{S}^{\bullet}_-(h^{\mathfrak{re}}_Y\circ F)$, 3.8.4, 3.8.4.1
$h^{\mathfrak{re}}_X\circ\mathcal{S}^{\bullet}_-F \simeq \mathcal{S}^{\bullet}_-(F^{\mathfrak{re}}_{\mathfrak{E}_X})\circ\mathfrak{j}^{\mathfrak{re}}_X$ when $\mathcal{S}^{\bullet}_-F$ exists, 3.8.4
$\partial^*\mathfrak{Un}(X,\mathfrak{E}_X)$ the category of universal ∂^*-functors from $(C_X,\mathfrak{E}_X)$ to virtually semi-complete categories, 4.1
$\partial^*\mathfrak{Un}_{\mathfrak{c}}(X,\mathfrak{E}_X)$ the subcategory of universal ∂^*-functors from $(C_X,\mathfrak{E}_X)$ to semi-complete categories and morphisms preserving limits, 4.1.1
$\partial\mathfrak{Un}(X,\mathfrak{I}_X)$ the category of universal ∂-functors from a left exact category $(C_X,\mathfrak{I}_X)$ to virtually semi-cocomplete categories, 4.2
$\partial\mathfrak{Un}^{\mathfrak{c}}(X,\mathfrak{I}_X)$ the subcategory of ∂-functors from a left exact category $(C_X,\mathfrak{I}_X)$ to semi-cocomplete categories and morphisms preserving colimits, 4.2.1
$Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}} \stackrel{\mathrm{def}}{=} \mathcal{S}^{\bullet}_+(h^{\mathfrak{re}}_{\mathfrak{X}})$ the left derived functor of the Yoneda embedding $C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} C^{\wedge}_{\mathfrak{X}}$, 4.3

$S_+F(\mathcal{L}) = \mathrm{colim}(Cok(F(\mathcal{M} \xrightarrow{\xi_{\mathfrak{j}}} \mathcal{M}_{\mathfrak{j},\xi})))$, where $\xi_{\mathfrak{j}}$ runs through push-forwards of morphisms $\mathcal{L} \xrightarrow{\xi} \mathcal{L}$ along inflations $\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M}$, 5.1

$S_+F(\mathcal{L}) = F^*S_+h_X(\mathcal{L}) = F^*Ext^1_X(\mathcal{L})$, 5.1.2

$C_{\mathfrak{X}^{\mathfrak{re}}} \xrightarrow{\widehat{\Theta}_{\mathcal{X}*}} C_{\mathfrak{X}^{\mathfrak{re}}}$ the structure of $\mathbb{Z}_+$-category on $C_{\mathfrak{X}^{\mathfrak{re}}}$ determined by Ext^1_X 5.2

$\widehat{\Theta}_{X*}(\mathcal{G})(-) = C_{\mathfrak{X}^{\mathfrak{re}}}(Ext^1_X(-), \mathcal{G})$ the right adjoint to $\widehat{\Theta}^*_X$, 5.2

$$\begin{array}{ccccc} \widehat{\mathcal{L}} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{\mathcal{M}} & \longrightarrow & y_{\mathfrak{i}} \\ \widehat{\xi}\downarrow & & \downarrow \widehat{\mathfrak{j}}_{\xi} & & \downarrow \lambda(\mathcal{L}) \\ \widehat{\mathcal{L}}_{\xi} & \xrightarrow{\widehat{\xi}_{\mathfrak{j}}} & \widehat{\mathcal{M}}_{\mathfrak{j},\xi} & \xrightarrow{\mathfrak{d}_0(\xi,\mathfrak{j})} & \widehat{\Theta}^*_X(\widehat{\mathcal{L}}) \end{array}$$ a standard "triangle" in $C_{\mathfrak{X}^{\mathfrak{re}}}$, 5.3

$\mathfrak{I}_{\mathfrak{X}^{\mathfrak{re}}}$ the coarsest left exact structure on $C_{\mathfrak{X}^{\mathfrak{re}}}$ closed under filtered colimits and making the embedding $C_{\mathfrak{X}} \xrightarrow{h^{\mathfrak{re}}_{\mathfrak{X}}} C_{\mathfrak{X}^{\mathfrak{re}}}$ an 'exact' functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C_{\mathfrak{X}^{\mathfrak{re}}}, \mathfrak{I}_{\mathfrak{X}^{\mathfrak{re}}})$, 5.5

$(C_{\mathfrak{X}^{\mathfrak{re}}}, \mathfrak{I}_{\mathfrak{X}^{\mathfrak{re}}}, \widehat{\Theta}^*_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{re}}})$ the *prestable category of presheaves of sets* on the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, 5.5

$C_{\mathfrak{X}^{\mathfrak{re}}_{\mathfrak{s}}}$ the quotient category $\Sigma^{-1}_{\widehat{\Theta}^*_{\mathfrak{X}}} C_{\mathfrak{X}^{\mathfrak{re}}}$, where

$\Sigma_{\widehat{\Theta}^*_{\mathfrak{X}}} = \{\mathfrak{t} \in HomC_{\mathfrak{X}^{\mathfrak{re}}} \mid \widehat{\Theta}^*_{\mathfrak{X}}(\mathfrak{t})$ is an isomorphism$\}$, 5.6

$(C_{\mathfrak{X}^{\mathfrak{re}}_{\mathfrak{s}}}, \Theta_{\mathfrak{X}^{\mathfrak{re}}_{\mathfrak{s}}}, \mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{re}}_{\mathfrak{s}}})$ the *stable category of presheaves of sets* on $(C_X, \mathfrak{I}_X)$, 5.6

$C_{\mathfrak{X}^{\mathfrak{p}}} \xrightarrow{\theta_{\mathfrak{X}^{\mathfrak{p}}}} C_{\mathfrak{X}^{\mathfrak{p}}}$ the $\mathbb{Z}_+$-structure induced by $C_{\mathfrak{X}^{\mathfrak{re}}} \xrightarrow{\widehat{\Theta}^*_{\mathfrak{X}}} C_{\mathfrak{X}^{\mathfrak{re}}}$, 6.1

$(C_{\mathfrak{X}^{\mathfrak{p}}}, \theta_{\mathfrak{X}^{\mathfrak{p}}}, \mathfrak{Tr}_{\mathfrak{X}^{\mathfrak{p}}})$ the *prestable* category of $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, 6.3

$(C_{\mathfrak{X}_{\mathfrak{s}}}, \theta_{\mathfrak{X}_{\mathfrak{s}}}, \mathfrak{Tr}_{\mathfrak{X}_{\mathfrak{s}}})$ the *stable* category of the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, 6.4

$S_-F(\mathcal{L}) \xrightarrow{\mathfrak{d}_{\mathfrak{e}}} \lim_{\mathcal{L}_{\xi} \xrightarrow{\xi} \mathcal{L}} (Ker(F(\mathcal{M}_{\xi,\mathfrak{e}} \xrightarrow{\xi'_{\mathfrak{e}}} \mathcal{M}))$ the canonical morphism, 7.1

$H_0(X, \mathcal{F}) = \mathrm{colim}\mathcal{F}$ the zero homology of X with coefficients in $C_X \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$, 8

$H_\bullet(X, \mathcal{F}) = (H_n(X, \mathcal{F}), \mathfrak{d}_n \mid n \geq 0)$ homology of X with coefficients in $C_X \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$, 8

$K^{\mathfrak{S},\mathfrak{F}}_\bullet = (K^{\mathfrak{S},\mathfrak{F}}_i, \mathfrak{d}_i \mid i \geq 0)$ universal K-functor from $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})^{op}$ to $\mathbb{Z} - mod$, 9.2.1

Index