Nonlocal Properties and Local Invariants for Bipartite Systems

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Abstract

The nonlocal properties of generic N-dimensional bipartite quantum systems are investigated. A complete set of invariants under local unitary transformations is presented. It is shown that two generic density matrices are locally equivalent if and only if all these invariants have equal values in these density matrices.

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As a fundamental phenomenon in quantum mechanics, nonlocality has been given a lot of attention in foundational considerations, in the discussion of Bell type inequalities [1] and hidden variable models, see e.g. [2]. Nonlocality turns out to be also very important in quantum information processing such as quantum computation [3], quantum teleportation [4, 5, 6, 7], dense coding [8] and quantum cryptographic schemes [9, 10, 11]. Nonlocal correlations in quantum systems imply a kind of entanglement among the quantum subsystems. The nonlocal properties as well as the entanglement of two parts of a quantum system remain invariant under local transformations of these parts.

The method developed in [12, 13], in principle, allows one to compute all the invariants of local unitary transformations, though it is not easy to perform it operationally. In [14], the invariants for general two-qubit systems are studied and a complete set of 18 polynomial invariants is presented. It is proven that two qubit mixed states are locally equivalent if and only if all these 18 invariants have equal values in these states. Therefore any nonlocal characteristics of entanglement is a function of these invariants. In [15] three qubits states are also discussed in detail from a similar point of view.

In the present paper we discuss the locally invariant properties of arbitrary dimensional bipartite quantum systems. We present a complete set of invariants and show that two generic density matrices are locally equivalent if and only if all these invariants have equal values in these density matrices.

We first consider the case of pure states. Let H be an N-dimensional complex Hilbert space, with e_i, $i = 1, ..., N$, as an orthonormal basis. A general pure state on $H \otimes H$ is of the form

$$|\Psi\rangle = \sum_{i,j=1}^{N} a_{ij} e_i \otimes e_j, \qquad a_{ij} \in \mathbb{C} \tag{1}$$

with the normalization $\sum_{i,j=1}^{N} a_{ij} a_{ij}^* = 1$ ($*$ denoting complex conjugation).

A quantity is called an invariant associated with the state $|\Psi\rangle$ if it is invariant under all local unitary transformations, i.e. all maps of the form $U \otimes U$ from $H \otimes H$ to itself, where U is a unitary transformation on the Hilbert space H. Let A denote the matrix given by $(A)_{ij} = a_{ij}$. The following quantities are known to be invariants associated with the state

$|\Psi\rangle$ given by (1), see [16, 17, 18, 19]:

$$I_\alpha = Tr(AA^\dagger)^\alpha, \qquad \alpha = 1, ..., N; \tag{2}$$

(with $A^\dagger$ the adjoint of the matrix A).

In terms of the Schmidt decomposition, a given $|\Psi\rangle$ can always be written in the following form, using a suitable orthonormal basis $\{e_i\}$, $i = 1, ..., N$:

$$|\Psi\rangle = \sum_{i=1}^{N} \sqrt{\Lambda_i} e_i \otimes e_i,$$

where $\sum\limits_{i=1}^{N} \Lambda_i = 1$, $\Lambda_i \geq 0$. The Λ_i, $i = 1, ..., N$, are the eigenvalues of the matrix $AA^\dagger$. As $AA^\dagger$ is self-adjoint, there always exists a unitary matrix V, $VV^\dagger = V^\dagger V = 1$, such that $VAA^\dagger V^\dagger = diag\{\Lambda_1, ..., \Lambda_N\}$. The invariants (2) can then be written in the form:

$$I_\alpha = \sum_{i=1}^{N} \Lambda_i^\alpha, \qquad \alpha = 1, ..., N.$$

[Lemma 1]. Two pure states (on $H \otimes H$) are equivalent under locally unitary transformations if and only if they have the same values of the invariants I_α, $\alpha = 1, ..., N$.

[Proof]. Let

$$|\Psi'\rangle = \sum_{i,j=1}^{N} a'_{ij} e_i \otimes e_j, \quad a'_{ij} \in \mathbb{C}, \quad \sum_{i,j=1}^{N} a'_{ij} a'^{*}_{ij} = 1.$$

Let A' denote the matrix given by $(A)'_{ij} = a'_{ij}$, Λ'_i the eigenvalues of the matrix $A'A'^\dagger$, and I'_α the corresponding invariants: $I'_\alpha = Tr(A'A'^\dagger)^\alpha = \sum_{i=1}^{N} \Lambda_i'^\alpha$, $\alpha = 1, ..., N$.

From the equation

$$(-1)^N det(AA^\dagger - \lambda) = \lambda^N + c_1 \lambda^{N-1} + ... + c_N,$$

where

$$c_1 = I_1, \quad c_k = -[I_k + c_1 I_{k-1} + ... + c_{k-1} I_1], \quad k \geq 2,$$

we have that if $I'_\alpha = I_\alpha$, $\forall \alpha = 1, ..., N$, then $A'A'^\dagger$ and $AA^\dagger$ have the same eigenvalues, $\Lambda'_i = \Lambda_i$, $i = 1, ..., N$. Therefore $|\Psi'\rangle$ and $|\Psi\rangle$ are equivalent under local unitary transformations: $|\Psi'\rangle = u \otimes w |\Psi\rangle$ for some $u, w \in \mathcal{U}$, $\mathcal{U}$ denoting the family of all unitary operations on H. Two pure states on $H \otimes H$ are equivalent under local unitary transformations if and only if they have the same set of invariants (2). $\blacksquare$

From the proof of Lemma 1, one can deduce the following conclusion:

[Lemma 2]. Two Hermitian $m \times m$ matrices A and B are unitary equivalent (*i.e.*, there exists a unitary matrix u on an m-dimensional complex vector space satisfying $A = uBu^\dagger$) if and only if

$$Tr(A^\alpha) = Tr(B^\alpha), \quad \text{for} \;\; \alpha = 1, ..., m.$$

We consider now mixed states on $H \otimes H$. Let ρ be a density matrix defined on $H \otimes H$ with $rank(\rho) = n \leq N^2$. ρ can be decomposed according to its eigenvalues and eigenvectors:

$$\rho = \sum_{i=1}^{n} \lambda_i |\nu_i\rangle\langle\nu_i|,$$

where λ_i and $|\nu_i\rangle$, $i = 1, ..., n$, are the nonzero eigenvalues and eigenvectors respectively of the density matrix ρ. $|\nu_i\rangle$ has the form

$$|\nu_i\rangle = \sum_{k,l=1}^{N} a^i_{kl} e_k \otimes e_l, \quad a^i_{kl} \in \mathbb{C}, \quad \sum_{k,l=1}^{N} a^i_{kl} a^{i*}_{kl} = 1, \quad i = 1, ..., n.$$

Let A_i denote the matrix given by $(A_i)_{kl} = a^i_{kl}$. We introduce $\{\rho_i\}$, $\{\theta_i\}$,

$$\rho_i = Tr_1|\nu_i\rangle\langle\nu_i| = A_i A_i^\dagger, \quad \theta_i = Tr_2|\nu_i\rangle\langle\nu_i| = A_i^\dagger A_i, \quad i,j = 1, ..., n, \tag{3}$$

Tr_1 and Tr_2 stand for the traces over the first and second Hilbert spaces respectively, and therefore, ρ_i and θ_i can be regarded as reduced density matrices. Let $\Omega(\rho)$ and $\Theta(\rho)$ be two "metric tensor" matrices, with entries given by

$$\Omega(\rho)_{ij} = Tr(\rho_i \rho_j), \quad \Theta(\rho)_{ij} = Tr(\theta_i \theta_j), \quad \text{for } i,j = 1, ..., n, \tag{4}$$

and

$$\Omega(\rho)_{ij} = \Theta(\rho)_{ij} = 0, \quad \text{for } N^2 \geq i,j > n.$$

A mixed state ρ is *generic* if the corresponding "metric tensor" matrices Ω, Θ satisfy $det(\Omega(\rho)) \neq 0$ and $det(\Theta(\rho)) \neq 0$. Obviously, a generic state implies $n = N^2$ or $det(\rho) \neq 0$.

Similarly we also introduce $X(\rho)$ and $Y(\rho)$ as

$$X(\rho)_{ijk} = Tr(\rho_i \rho_j \rho_k), \quad Y(\rho)_{ijk} = Tr(\theta_i \theta_j \theta_k), \quad i,j,k = 1, ..., n. \tag{5}$$

[Theorem]. Two generic density matrices are equivalent under local unitary transformations if and only if the following invariants have the same values for both density matrices:

$$\begin{aligned} &J^s(\rho) = Tr_2(Tr_1 \rho^s), \quad s = 1, ..., N^2; \\ &\Omega(\rho), \quad \Theta(\rho), \quad X(\rho), \quad Y(\rho). \end{aligned} \tag{6}$$

[Proof]. We first show that the quantities given in (6) are invariant under local unitary transformations. Let u and w be unitary transformations, $uu^\dagger = u^\dagger u = ww^\dagger = w^\dagger w = 1$. Under the local unitary transformation $u \otimes w$, we have $\rho \to \rho' = u \otimes w \; \rho \; u^\dagger \otimes w^\dagger$. Correspondingly, we have $|\nu_i\rangle \to |\nu_i'\rangle = u \otimes w|\nu_i\rangle$, or equivalently A_i is mapped to $A_i' = u^t A_i w$, where u^t is the transpose of u. Therefore

$$\rho_i' = A_i' A_i'^\dagger = u^t A_i A_i^\dagger u^* = u^t \rho_i u^*, \quad \theta_i' = A_i'^\dagger A_i' = w^\dagger A_i^\dagger A_i w = w^\dagger \theta_i w. \tag{7}$$

By using (7), it is straightforward to check the following relations:

$$\begin{aligned}
&J^s(\rho) \to J^s(\rho') = Tr_2[\sum_{i=1}^n \lambda_i^s Tr_1(|\nu_i'\rangle\langle\nu_i'|)] = Tr_2[\sum_{i=1}^n \lambda_i^s A_i' A_i'^\dagger] = J^s(\rho),\\
&\Omega(\rho)_{ij} \to \Omega(\rho')_{ij} = Tr(\rho_i'\rho_j') = Tr(u^t \rho_i \rho_j u^*) = \Omega(\rho)_{ij},\\
&\Theta(\rho)_{ij} \to \Theta(\rho')_{ij} = Tr(\theta_i'\theta_j') = Tr(w^\dagger \theta_i \theta_j w) = \Theta(\rho)_{ij}\\
&X(\rho)_{ijk} \to X(\rho')_{ijk} = Tr(\rho_i'\rho_j'\rho_k') = Tr(u^t \rho_i \rho_j \rho_k u^*) = X(\rho)_{ijk},\\
&Y(\rho)_{ijk} \to Y(\rho')_{ijk} = Tr(\theta_i'\theta_j'\theta_k') = Tr(w^\dagger \theta_i \theta_j \theta_k w) = Y(\rho)_{ijk},
\end{aligned}$$

where $i, j, k = 1, ..., n$. Hence the quantities in (6) are invariants of local unitary transformations. If two density matrices are equivalent under local unitary transformations, then their corresponding invariants in (6) have the same values.

Now suppose conversely that the states $\rho = \sum_{i=1}^n \lambda_i |\nu_i\rangle\langle\nu_i|$ and $\rho' = \sum_{i=1}^n \lambda_i' |\nu_i'\rangle\langle\nu_i'|$ give the same values to the invariants in (6). We are going to prove that ρ and ρ' are equivalent under local unitary transformations.

a) As

$$J^s(\rho) = Tr_2(\sum_{i=1}^n \lambda_i^s Tr_1(|\nu_i\rangle\langle\nu_i|)) = Tr_2(\sum_{i=1}^n \lambda_i^s A_i A_i^\dagger) = \sum_{i=1}^n \lambda_i^s,$$

from $J^s(\rho') = J^s(\rho)$ we have

$$\sum_{i=1}^n \lambda_i'^s = \sum_{i=1}^n \lambda_i^s, \qquad \forall\; s = 1, ..., N^2.$$

From Lemma 2, we have that ρ' and ρ have the same nonzero eigenvalues, i.e., $\lambda_i' = \lambda_i$, $i = 1, ..., n$.

b) From (4), the generic condition $det(\Omega(\rho)) \neq 0$, $det(\Theta(\rho)) \neq 0$, implies that ρ_i (resp. θ_i), $i = 1, ..., n(= N^2)$, span the space of $N \times N$ matrices and form an N-dimensional representation of the associative algebra $gl(N, \mathbb{C})$ (in the sense of universal enveloping algebras),

$$\rho_i \rho_j = \sum_{k=1}^{n} C_{ij}^k \rho_k, \qquad \theta_i \theta_j = \sum_{k=1}^{n} D_{ij}^k \theta_k, \qquad C_{ij}^k\ ,\ D_{ij}^k \in \mathbb{C} \tag{8}$$

Taking trace of (8) and using the condition $Tr\rho_i = Tr\theta_i = 1$ one gets

$$\Omega_{ij} = \sum_{k=1}^{n} C_{ij}^k, \qquad \Theta_{ij} = \sum_{k=1}^{n} D_{ij}^k. \tag{9}$$

From (9) and (5) we obtain

$$X_{ijk} = \sum_{l=1}^{n} C_{ij}^l \Omega_{lk}, \qquad Y_{ijk} = \sum_{l=1}^{n} D_{ij}^l \Theta_{lk}.$$

Therefore

$$C_{ij}^l = \sum_{k=1}^{n} X_{ijk} \Omega^{lk}, \qquad D_{ij}^l = \sum_{k=1}^{n} Y_{ijk} \Theta^{lk}, \tag{10}$$

where the matrices Ω^{ij} and Θ^{ij} are the corresponding inverses of the matrices Ω_{ij} and Θ_{ij} (which exist by the genericity assumption). (10) implies that the structure coefficients C_{ij}^l and D_{ij}^l of the algebras (8) are given by $\{\Omega_{ij}, X_{ijk}\}$ and $\{\Theta_{ij}, Y_{ijk}\}$ respectively.

The generic density matrix ρ' also gives rise to N-dimensional representation of the associative algebra $gl(N, \mathbb{C})$,

$$\rho'_i \rho'_j = \sum_{k=1}^{n} C_{ij}^{\prime k} \rho'_k, \qquad \theta'_i \theta'_j = \sum_{k=1}^{n} D_{ij}^{\prime k} \theta'_k, \qquad C_{ij}^{\prime k}\ ,\ D_{ij}^{\prime k} \in \mathbb{C}. \tag{11}$$

If $\Omega(\rho') = \Omega(\rho)$, $\Theta(\rho') = \Theta(\rho)$, $X(\rho') = X(\rho)$ and $Y(\rho') = Y(\rho)$, we have $C_{ij}^{\prime l} = C_{ij}^l$ and $D_{ij}^{\prime l} = D_{ij}^l$. Due to the fact that all the Casimir operators of the algebra $gl(N, \mathbb{C})$ can be written as polynomials in the algebraic generators, we conclude that the values of all the Casimir operators given by the two N-dimensional representations $\{\rho_i\}$ and $\{\rho'_i\}$ (resp. $\{\theta_i\}$ and $\{\theta'_i\}$) are equal. Hence, these two sets of representations (primed and unprimed) of the algebra $gl(N, \mathbb{C})$ are equivalent, i.e.,

$$\rho'_i = u^t \rho_i u^*, \qquad \theta'_i = w^\dagger \theta_i w \tag{12}$$

for some $u, w \in \mathcal{U}$. Therefore $|\nu'_i\rangle = u \otimes w |\nu_i\rangle$, $i = 1, ..., n$, and $\rho' = u \otimes w\ \rho\ u^\dagger \otimes w^\dagger$. Hence ρ' and ρ are equivalent under local unitary transformations. ■

Remark. For a degenerate state ρ, $det(\Omega(\rho)) = 0$ or $det(\Theta(\rho)) = 0$, the above invariants (6) are not complete in the sense that two degenerate density matrices can not be equivalent under local unitary transformations even if they give the same values to the invariants in (6). This is due to the fact that there exist null vectors for the degenerate state. For example in the case $det(\Omega(\rho))=0$, there exists at least one Hermitian matrix B which satisfies $Tr(B\rho_i)=0$ for $i=1,...,n$. Hence $\Omega(\rho')_{ij}=\Omega(\rho)_{ij}$ and $X(\rho')_{ijk}=X(\rho)_{ijk}$ are not enough to get the first equivalence relation of (12). In this case some new invariants have to be introduced to get a complete set of invariants. From the algebraic relations (8) and formula (10), other generalized invariants like $Tr((\rho_i)^{m_i}(\rho_j)^{m_j}...(\rho_k)^{m_k})$ and $Tr((\theta_i)^{m_i}(\theta_j)^{m_j}...(\theta_k)^{m_k})$, $i,j,...,k=1,...,n$; $m_i,m_j,...,m_k\in I\!N$, can be represented in terms of $\{\Omega_{ij},X_{ijk}\}$ and $\{\Theta_{ij},Y_{ijk}\}$, for example,

$$Tr\left(\rho_{i_1}\rho_{i_2}\cdots\rho_{i_m}\right)=\sum_{\{\alpha_1,\cdots,\alpha_{m-2}\}}C_{i_1\ i_2}^{\alpha_1}C_{\alpha_1\ i_3}^{\alpha_2}\cdots C_{\alpha_{m-3}\ i_{m-1}}^{\alpha_{m-2}}\Omega_{\alpha_{m-2}\ i_m}.$$

Hence by doing so we do not get new invariants.

To summarize, we have discussed here the local invariants for arbitrary dimensional bipartite quantum systems and have presented a set of invariants of local unitary transformations. The set of invariants is not necessarily independent but it is complete in the sense that two generic density matrices are equivalent under local unitary transformations if and only if all these invariants have equal values for these density matrices.

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