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Idempotents in the Griess Algebra

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ÜBUNGEN ZUR INFINITESIMALRECHNUNG II

Sommersemester 90 Hirzebruch/Skoruppa

Blatt **6** (17. Mai 90)

Hausaufgaben

1. Sei $f \in C^2(U)$, U eine offene Teilmenge im $\mathbb{R}^2$, $a \in U$ mit $\operatorname{grad} f(a) = 0$. Es sei $\Delta = f_{x,x}(a)f_{y,y}(a) - f_{x,y}(a)^2$. Zeigen Sie:

(i) Der Punkt a ist ein lokales Maximum (Minimum) von f genau dann, wenn $\Delta > 0$ und $f_{x,x}(a) < 0$ (> 0) gilt.

(ii) Der Punkt a ist ein Sattelpunkt von f genau dann, wenn $\Delta < 0$ ist.

2. Bestimmen Sie die kritischen Punkte der Funktion $f(x,y) = \cos x + \sin y$ und das Verhalten von f in den kritischen Punkten. Skizzieren Sie den Verlauf der Niveaukurven der Funktion auf dem $\mathbf{R}^2$.

3. Finden Sie die kritischen Punkte der Funktion $f(x,y) = y(3x^2 - y^2) - (x^2 + y^2)^2$, entscheiden Sie, welche davon Maxima oder Minima sind, und skizzieren Sie den Verlauf der Niveaukurven der Funktion auf dem ganzen $\mathbb{R}^2$.

4. Sei U eine offene, den Nullpunkt enthaltene Teilmenge des $\mathbb{R}^n$, und sei $f \in C^\infty(U)$. Es gebe homogene Polynome $p_\nu(x_1, \ldots, x_n)$ vom Grad ν und ein k, sodaß

$$\lim_{x \to 0} \left[f(x_1, \ldots, x_n) - (p_0 + p_1(x_1, \ldots, x_n) + \cdots + p_k(x_1, \ldots, x_n)) \right] / |x|^k = 0$$

ist, wobei $|x|$ die euklidische Norm von $x = (x_1, \ldots, x_n)$ bezeichnet. Zeigen Sie:

$$p_\nu(x_1, \ldots, x_n) = \frac{f^{(\nu)}(0)(x)^\nu}{\nu!}$$

für $0 \le \nu \le k$.

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IDEMPOTENTS IN THE GRIESS ALGEBRA

by Werner Meyer and Wolfram Neutsch

Abstract

In an earlier publication, we have shown that in order to investigate the structure of associative subalgebras of the Griess algebra (whose automorphism group is the Monster sporadic group) it is important to have detailed information on the distribution of the idempotent elements.
We show that the variety of idempotents consists of several algebraic components of various dimensions, ranging from 0 to at least 144. On the other hand, among the connected components of the set of idempotents are some which are invariant under the action of the Monster. We feel that these constructions are of great geometric interest and may lead to a better understanding of the properties of the sporadic groups.
Furthermore we consider the Jordan elements in the Griess algebra (introduced by Conway).

Introduction

In this paper we continue the considerations of the structure of the Griess algebra $\mathfrak{G}$ begun in Meyer and Neutsch [1990]. We shall freely use the notation of this publication. In particular, the algebra product in $\mathfrak{G}$ is chosen in the same way as in Conway [1984] while our scalar product $<.,.>$ is twice as large as Conway's.

While the emphasis in the above-mentioned investigations is on the algebraic structure, we shall here discuss the set of the idempotent elements in $\mathfrak{G}$ which play a prominent rôle in both previous papers.

Definitions and Notation

We shall need some further notation. The set of all **idempotents** in $\mathbb{O}$ is

$$\mathbb{I} = \left\{ i \in \mathbb{O} \;\middle|\; i^2 = i \right\} \tag{1}$$

while the square roots of the unit element (shortly called **roots**) are contained in

$$\mathbb{K} = \left\{ x \in \mathbb{O} \;\middle|\; x^2 = 1 \right\} \tag{2}$$

Moreover, the **Jordan elements**

$$\mathbb{J} = \left\{ x \in \mathbb{O} \;\middle|\; (x^2)^2 = x \cdot x^3 \right\} \tag{3}$$

will be important. Note that

$$x^3 = x^2 \cdot x = x \cdot x^2 \tag{4}$$

is well-defined since $\mathbb{O}$ is commutative. The Jordan condition guarantees by Conway [1984] that $x \in \mathbb{J}$ associates with its square,

$$(x \cdot a) \cdot x^2 = x \cdot (a \cdot x^2) \tag{5}$$

for all $a \in \mathbb{O}$. Another (trivial) observation will be useful. The sets $\mathbb{I}$ and $\mathbb{K}$ are related by a simple affine transformation,

$$i \in \mathbb{I} \Leftrightarrow \frac{1}{2}(1 + i) \in \mathbb{K} \tag{6}$$

or, vice versa,

$$x \in \mathbb{K} \Leftrightarrow (1 - 2x) \in \mathbb{I} \tag{7}$$

Obviously, all elements in $\mathbb{I}$ and $\mathbb{K}$ are Jordan.

The conditions for an element of $\mathbb{G}$ to be contained in either of $\mathbb{I},\mathbb{J},\mathbb{K}$ are systems of 196884 real algebraic equations each (with coefficients in $\mathbb{Q}$). This implies that all three sets are affine varieties over the field $\mathbb{R}$ (or $\mathbb{Q}$). In the sequel, we shall consider some of the properties of these varieties and their embedding in $\mathbb{G}$.

The set of idempotents in $\mathfrak{G}$

Choose an arbitrary central involution $z \in 2B$. The centraliser of z in the Monster F_1 is (Griess [1982])

$$C = C_F(z) = N_F(z) \cong 2^{1+24} \cdot Co_1 \tag{8}$$

and the algebra decomposes into C-invariant, pairwise orthogonal subspaces,

$$\mathfrak{G} = U \oplus V \oplus W \tag{9}$$

U splits into two irreducible C-invariant subspaces of dimensions 1 and 299 (the one generated by the unit element 1 and its orthogonal complement), while V and W are irreducible of dimensions 98280 and 98304, respectively. For more information concerning the action of C on these spaces, see Griess [1982] or Conway [1984].

Multiplication of U with U,V,W reproduces the other factor,

$$U \cdot U = U \tag{10}$$

$$U \cdot V = V \cdot U = V \tag{11}$$

$$U \cdot W = W \cdot U = W \tag{12}$$

therefore the rules for products involving elements of U are particularly simple. U itself is a subalgebra which can be represented by the set of all symmetric real (24,24)-matrices. The product then is given (up to a constant factor) by the Jordan product.

Unfortunately, Conway normalises the product in U such that the product of two elements u_1 and u_2 deviates from the Jordan product by a factor of 4,

$$u_1 \cdot u_2 = 2\,(u_1 u_2 + u_2 u_1) \tag{13}$$

where on the left-hand side the algebra multiplication, and on the other the matrix product are involved. Since we shall have to consider both kinds of products at the same time, we shall prefer to attach to each element $u \in U$ the matrix

$$\hat{u} = 4\,u \tag{14}$$

Then we get simply

$$u_1 \cdot u_2 = \frac{1}{2}\left(\hat{u}_1\hat{u}_2 + \hat{u}_2\hat{u}_1\right) \tag{15}$$

such that, in particular, the square of u is represented by the matrix square of $\hat{u}$.
It is now easy to determine the intersections of $\mathbb{I}$ and $\mathbb{K}$ with U:

Theorem 1:
The set $\mathbb{I} \cap U$ consists of all symmetric (24,24)-matrices $\hat{u}$ which have only eigenvalues 0 and 1, while $\mathbb{K} \cap U$ contains the symmetric (24,24)-matrices with eigenvalues -1 and 1.
All elements in U are Jordan.

Proof:
Trivial.

We observe that only discrete values of the norm of idempotents u in U occur, namely the numbers of the form $\frac{k}{8}$, where k is the trace of $\hat{u}$, i. e. the multiplicity of the eigenvalue 1. The latter can obviously attain all integer values between 0 and 24.
This motivates to split $\mathbb{I}$ and $\mathbb{K}$ into subvarieties $\mathbb{I}_k$ and $\mathbb{K}_k$:

$$\mathbb{I}_k = \left\{ x \in \mathbb{I} \;\middle|\; \langle 1, x\rangle = \frac{k}{8} \right\} \tag{16}$$

$$\mathbb{K}_k = \left\{ x \in \mathbb{K} \;\middle|\; \langle 1, x\rangle = \frac{k-12}{4} \right\} \tag{17}$$

Here, of course, non-integer values of k are possible. We take the notation

in this form because we shall mainly be concerned with the structure of the sets $\mathbb{I}_k \cap U$ and $\mathbb{K}_k \cap U$.
Another trivial fact is

Theorem 2:
The intersections of $\mathbb{I}_k$ and $\mathbb{K}_k$ with U are smooth irreducible affine varieties of dimension

$$\dim (\mathbb{I}_k \cap U) \quad = \quad \dim (\mathbb{K}_k \cap U) \quad = \quad k \cdot (24 - k) \tag{18}$$

Proof:
For $\mathbb{I}_k \cap U$, this follows immediately from the well-known structure of U, since the matrices $\hat{u}$ are the orthogonal projectors onto the k-dimensional linear subspaces of $\mathbb{R}^{24}$. Therefore, $\mathbb{I}_k \cap U$ is isomorphic to the Graßmann manifold of all k-dimensional subspaces of $\mathbb{R}^{24}$ and consequently has the dimension given in the theorem. The result for $\mathbb{K}_k \cap U$ is obtained similarly, or from equation (6) or (7).

A more interesting and important question is: What do the sets $\mathbb{I}_k \cap U$ and $\mathbb{K}_k \cap U$ look like ?
The fact that $\mathbb{I} \cap U$ splits into components on which the norm is constant, can be generalised:

Theorem 3:
The norm is constant on each connected component of $\mathbb{I}$.

Proof:
Let a and b be two points on the same component $\mathfrak{C}$ of $\mathbb{I}$. Then there is a curve

$$\gamma : \quad [0,1] \longrightarrow \mathfrak{C} \tag{19}$$

with

$$\gamma(0) = a \tag{20}$$

and

$$\gamma(1) = b \tag{21}$$

because each connected component of $\mathbb{I}$ is also path-connected.
Furthermore, we may assume γ to be piecewise differentiable. Consider the function $F \circ \gamma$, where

$$F(x) = \frac{\langle x^2, x^2 \rangle}{\langle x, x \rangle^2} \tag{22}$$

has been defined in Meyer and Neutsch [1990]. There (Lemma 2) it was shown that the gradient of F at $x \in \mathfrak{G}$ is zero if and only if the third power of x depends linearly on x.
This holds for all idempotents. Thus

$$\frac{d}{dt} F \circ \gamma(t) = \nabla F(\gamma(t)) \frac{d}{dt} \gamma(t) = 0 \tag{23}$$

and F is constant on $\mathfrak{C}$. This shows that

$$F(a) = \langle a, a \rangle^{-1} = F(b) = \langle b, b \rangle^{-1} \tag{24}$$

The assertion is immediate.

By Whitney [1957], a real algebraic variety in a finite-dimensional vector space over $\mathbb{R}$ cannot have infinitely many topological components. Together with Theorem 3, this leads to

<u>Theorem 4:</u>

(a) Each of the sets $\mathbb{I}$, $\mathbb{J}$, and $\mathbb{K}$ consists of a finite number of connected components;

(b) The norm attains only finitely many values on the idempotents in $\mathfrak{G}$.

We now aim at investigating the structure of $\mathbb{I}$ in greater detail.
By Meyer and Neutsch [1990], each transposition idempotent forms a 0-dimensional component on its own.
More interesting are the algebraic components of $\mathbb{I}$ which intersect U (or one of its conjugates):

Theorem 5:
For all $k \neq 8, 16$, the set $\mathbb{I}_k \cap U$ is an irreducible component of $\mathbb{I}$.

Proof:
By assertion (b) of Theorem 4 (or, simpler, by the boundedness of F), the zero element is an isolated idempotent, and the same holds for 1, since for each $i \in \mathbb{I}$, $1-i$ is also contained in $\mathbb{I}$.
This proves the theorem in the special cases $k \in \{0, 24\}$. We may thus assume from now on that $8 \nmid k$. Furthermore, we shall prove the equivalent fact for the set $\mathbb{K}$ of roots instead of $\mathbb{I}$, since this is slightly simpler.
It suffices to find some x in $\mathbb{K}_k \cap U$ such that the tangent cone $\mathbb{T}\mathbb{K}_x$ of $\mathbb{K}$ at x fulfills

$$\mathbb{T}\mathbb{K}_x \subseteq TU_x = U \qquad (25)$$

This holds because under this condition, since $\mathbb{K}_k \cap U$ is a submanifold of U, the tangent spaces in x at $\mathbb{K} \cap U$ and $\mathbb{K}$ coincide.
Linearising the root condition

$$x^2 = 1 \qquad (26)$$

we get easily that each vector ε in $\mathbb{T}\mathbb{K}_x$ obeys

$$x \cdot \varepsilon = 0 \qquad (27).$$

In other words, we have to show that an $x \in \mathbb{K}_k \cap U$ exists with

$$\left\{ \varepsilon \;\middle|\; x\cdot\varepsilon = 0 \right\} \subseteq U \tag{28}$$

The general solution of (27) can be found explicitly. To achieve this, we split ε into its parts in U, V, W, respectively, like

$$\varepsilon = u + v + w \tag{29}$$

By the invariance of U, V, W under multiplication with elements of U, see equations (10), (11), (12), we find that the required condition is tantamount to the three relations

$$x\cdot u = 0 \tag{30}$$

$$x\cdot v = 0 \tag{31}$$

$$x\cdot w = 0 \tag{32}$$

Let us begin with the last one. We may write w in the form

$$w = \sum_{i=1}^{4096} q_i \otimes \lambda_i \tag{33}$$

where

$$\left\{ q_i \;\middle|\; i = 1,\ldots,4096 \right\} \tag{34}$$

is a basis of the 4096-dimensional space occurring in the C-invariant decomposition

$$W = \underline{4096} \otimes \underline{24} \tag{35}$$

(Conway [1984]) and the λ_i are vectors in $\mathbb{R}^{24}$. By the formulas given in the just cited reference, we deduce

$$0 = x \cdot w = \sum_{i=1}^{4096} x \cdot \left\{ q_i \otimes \lambda_i \right\} = \sum_{i=1}^{4096} q_i \otimes \left\{ \frac{1}{4} \lambda_i \hat{x} + \sigma \lambda_i \right\} \tag{36}$$

if we put

$$\sigma = \frac{1}{8} \operatorname{trace}(x) = \frac{1}{32} \operatorname{trace}(\hat{x}) = \frac{k-12}{16} \tag{37}$$

Since the q_i are linearly independent, we get for all $i \in \{1, \ldots, 4096\}$:

$$\frac{1}{4} \lambda_i \hat{x} + \sigma \lambda_i = 0 \tag{38}$$

or

$$\lambda_i \hat{x} = -4 \sigma \lambda_i \tag{39}$$

But x is a root; hence

$$\lambda_i = \lambda_i \hat{x}^2 = 16 \sigma^2 \lambda_i \tag{40}$$

which is obviously possible for nonzero λ_i only if

$$\sigma = \pm \frac{1}{4} \tag{41}$$

This means that k must have one of the excluded values 8 or 16.
Under the assumption $8 \nmid k$, equation (32) thus has no nontrivial solutions, and

$$\mathbb{T}K_x \subseteq U \oplus V \tag{42}$$

We next consider relation (31). Let

$$v = \sum_{r \in \Lambda_2} v_r X_r \tag{43}$$

be the canonical decomposition with coefficients $v_r \in \mathbb{R}$ obeying

$$v_{-r} = - v_r \tag{44}$$

for all minimal vectors $r \in \Lambda_2 \subseteq \mathbb{R}^{24}$ in the Leech-lattice, Λ. Conway's choice of the normalisation is such that

$$(r,r) = 32 \tag{45}$$

After these preparations, the formula in question leads to

$$0 = x \cdot v = \sum_{r \in \Lambda_2} v_r \, x \cdot X_r = \sum_{r \in \Lambda_2} \frac{1}{4} \left[r \, \hat{x} \, {}^t r \right] v_r \, X_r \tag{46}$$

Therefore, v_r vanishes except possibly if

$$r \, \hat{x} \, {}^t r = 0 \tag{47}$$

It is clear that we can find some $x \in \mathbb{K}_k \cap U$ such that (47) is not fulfilled for any of the 196560 vectors $r \in \Lambda_2$. For all x with this property, we get

$$T\mathbb{K}_x \subseteq U \tag{48}$$

This concludes the proof of the theorem.

Remark:
We have not been able to calculate the tangent space in the remaining cases $k = 8$ and $k = 16$. It is conceivable that for these k, the dimension of the irreducible component of $\mathbb{K}_k$ (or $\mathbb{I}_k$) which intersects U may be much larger than the value predicted by the above formula (18), namely $k \cdot (24-k) = 128$. If so, all points in $\mathbb{I}_k \cap U$ would be singular, and the configuration clearly of high interest.

On the other hand, the connected components of $\mathbb{I}$ are much larger:

Theorem 6:

For all $k \in \{0,\dots,24\}$, the topological component $\mathbb{C}_k$ of $\mathbb{I}$ which contains $\mathbb{I}_k \cap U$ is invariant under the action of the Monster F_1.

Proof:

The maximal F_1-subgroup C leaves U and $\mathbb{I}_k$ invariant, thus the stabiliser of $\mathbb{C}_k$ is either C or F_1 itself. If the latter case holds, we are done. If otherwise, each two F_1-conjugates of $\mathbb{I}_k \cap U$ would be disjoint.
This is not true, however, since the triality element θ which cyclically permutes the three "languages" of Conway's construction of $\mathbb{G}$, leaves all elements $u \in U$ invariant whose corresponding (24,24)-matrices $\hat{u}$ are diagonal. Since all $\mathbb{I}_k \cap U$ contain diagonal matrices, namely those with exactly k 1's in the diagonal, all other entries being 0, the proposition immediately follows.

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