1-forms and vector fields[1]

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Abstract. I show that under a mild condition on a differential 1-form ω_0 on a compact n-manifold M, the classification of global cooriented hyperplane fields (or global Pfaff equations) on M that are described by 1-forms close to ω_0 reduces to the global orbital classification of vector fields on M.

1. Introduction and main results.

Given a differential 1-form on a orientable compact n-manifold $M = M^n$ denote by (ω) one of the following: if ω vanishes at no points of M then (ω) is the (cooriented) hyperplane field in TM which is the field of kernels of ω; if ω vanishes at least at one point of M then (ω) is the Pfaff equation (1-generated module of 1-forms over the ring of functions) generated by ω. In the both cases the equivalence of (ω_1) and (ω_0) with respect to the natural action of the group of global diffeomorphisms of M is the equivalence of ω_1 and ω_0 with respect to the group of diffeomorphisms and multiplication by nonvanishing functions.

Fix ω_0 and assume that ω_1 is sufficiently close to ω_0 in the C^r-topology. If n is odd, $r \geq 1$, and ω_0 is a contact 1-form then (ω_1) is equivalent to (ω_0). This follows from the classical Gray theorem [Gra].

If n is even or if n is odd and ω_0 is not a contact 1-form then there is an obstacle for the equivalence of (ω_1) and (ω_0). This obstacle is related to a *characteristic vector field* of (ω).

A characteristic vector field X of (ω) is not unique, but any two characteristic vector fields differ by multiplication by a nonvanishing function. The construction of a single characteristic vector field requires fixing a 1-form ω generating the Pfaff equation (ω) or describing the hyperplane field (ω) (two 1-forms making this job differ by multiplication by a nonvanishing function) and fixing a volume form Ω on M^n.

If $n = 2k$ then X is a vector field on M^n defined by the relation

$$X \rfloor \Omega = \omega \wedge (d\omega)^{k-1}.$$

If $n = 2k+1$ then X is defined on the set S consisting of points at which ω is not contact:

$$S = \{H = 0\}, \quad H = (\omega \wedge (d\omega)^k)/\Omega,$$

$$X(p) \rfloor \Omega(p) = \left(\omega \wedge (d\omega)^{k-1} \wedge dH\right)\ (p), \quad p \in S.$$

It is easy to see that $X(p) \in T_pS$ for any point $p \in S$ at which S is a smooth codimension one submanifold of M^n and that the set of points at which X vanishes includes all other points of S. The set S is called the *Martinet hypersurface*.

One can easily check that for any parity of n the multiplication of ω and Ω by nonvanishing functions leads to multiplication of X by a nonvanishing function. Therefore the equivalence of (ω_1) and (ω_0) implies the equivalence of corresponding characteristic vector fields X_1 and X_0 with respect to the group of diffeomorphisms of M and multiplication by nonvanishing functions. This equivalence of vector fields is called *orbital equivalence*. The condition of orbital equivalence of X_1 and X_0 is the obstacle for the equivalence of (ω_1) and (ω_0).

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In the present paper I show that under a mild condition on the singular points of X_0 this is the only obstacle: if ω_1 is sufficiently close to ω_0 and X_1 is orbitally equivalent to X_0 via a sufficiently close to the identity diffeomorphism of M, then (ω_1) is equivalent to (ω_0).

The results of this paper are related to the question that J. Martinet asked me in 1989 and to conjectures in my book [Zh].

Convention. In what follows $M = M^n$ is a compact orientable manifold of dimension $n \geq 3$. All objects are assumed to be of the class C^∞ and all results hold in the same category.

To formulate the main results I will use the following definition.

Definition, cf. [Mou]. A vector field X on a manifold Q^d of dimension d has s-division property if for any $i = 1, \dots s$ and any differential form $(d-i)$-form θ_{d-i} on Q^d such that $X \rfloor \theta_{d-i} = 0$ there exists a differential $(d-i+1)$-form θ_{d-i+1} on Q^d such that $\theta_{d-i} = X \rfloor \theta_{d-i+1}$.

In the C^∞ category, due to the division of unity, this property is equivalent to local s-division property (the same property for germs at any fixed point of Q^d). If $X_1, \dots, X_d$ are the coefficients of the germ of X at a point $p \in Q^d$ in some coordinate system then the s-division property of the germ of X at p is the property of the ideal $I = (X_1, \dots X_d)$ generated by the functions $X_1, \dots X_d$. The simplest cases where the s-division property holds for the germ of X at $p \in Q^d$ are as follows: (a) $X(p) \neq 0$; (b) $X(p) = 0$ and the ideal $(X_1, \dots X_d)$ contains $(s+1)$ differentially independent functions. The simplest case where the s-division property does not hold is the case where X vanishes on a submanifold of Q^d of codimension s or less. The 1-division property can be easily expressed without using exterior forms: X has the 1-division property if and only if for any vector field Y on Q^d such that $Y \wedge X = 0$ there exists a function F such that $Y = FX$. Here $Y \wedge X$ is a field of exterior 2-vectors, and the condition $Y \wedge X = 0$ means that at any point $p \in Q^d$ either the vectors $X(p)$ and $Y(p)$ are proportional or one of these vectors is equal to zero.

In this paper I need the 2-division property of a characteristic vector field X_0 of (ω_0) on M^{2k} (then $Q^d = M^{2k}$) and the 1-division property of a characteristic vector field X_0 of (ω_0) on M^{2k+1}, $k \geq 2$. In the latter case X_0 is defined on the Martinet hypersurface $S \subset M^{2k+1}$, and I assume that S is a smooth codimension one submanifold of M^{2k+1}, then $d = 2k$ and $Q^d = S$. In the 3-dimensional case the division property is not required.

Theorem 1. *Let ω_0 be a 1-form on $M = M^{2k}$, X_0 - a characteristic vector field of (ω_0). Assume that the following conditions hold:*

(a) X_0 has the 2-division property;

(b) there exists an integer $r \geq 0$ such that $j^r_p X_0 \neq 0, p \in M$.

Set $q = 2r + 2$. Then (ω_0) is globally equivalent to (ω_1) provided that the 1-form ω_1 is sufficiently C^q-close to ω_0 and a characteristic vector field X_1 of (ω_1) is orbitally equivalent to X_0 via a sufficiently C^{q+1}-close to the identity diffeomorphism of M.

Note that the assumptions here do not depend on the choice of the characteristic vector fields X_0 and X_1 - I recitate that a characteristic vector field is unique up to multiplication by a nonvanishing function. It is clear that the latter is irrelevant for the orbital equivalence and for the division property.

Note also that Theorem 1 contains no assumptions on points at which ω_0 or ω_1 vanish. The assumption (b) holds for some r if and only if the vector field X_0 is flat (has zero formal series) at no points of M.

The analysis of the condition (a) is not so simple since the vector field X_0 is not generic in the space of all vector fields on M. In particular, the degeneartion $\omega_0 \wedge (d\omega_0)^{k-1}(p) = 0$ has codimension 3, see [Mar], therefore there exists an open set of 1-forms ω_0 on M such that X_0

vanishes on a codimension 3 submanifold of M. Nevertheless, conjecturally the condition (a) is violated only for infinite codimension degenerations of 1-forms. In this paper I prove the simplest (and the main) result in this direction.

Proposition 1. *If M is a manifold of dimension 4,6, or 8, then in the space of all 1-forms on M there is an open and dense set in the C^2-topology such that any 1-form ω_0 of this set satisfies the assumptions (a) and (b) of Theorem 1.*

One of the steps in the proof of Theorem 1 is the reduction to the following deformation theorem. This theorem requires *the s-division property with parameter $t \in [0,1]$*. The definition is the same except that the forms θ_{d-i} and θ_{d-i+1} must be replaced by families $\theta_{d-i,t}$ and $\theta_{d-i+1,t}$, $t \in [0,1]$. Here and in what follows all families depend C^∞ on the parameter. The simplest cases where a vector field has the s-division property with parameter $t \in [0,1]$ are the cases (a) and (b) given after the definition of the s-division property without parameter.

Theorem 1.A. *Let ω_t be a family of 1-forms on $M = M^{2k}$, $t \in [0,1]$. Assume that a single vector field X_0 is a characteristic vector field for all (ω_t). If X_0 has the 2-division property with parameter $t \in [0,1]$ then there exists a family Φ_t of global diffeomorphsims of M such that $\Phi_t^*(\omega_t) = (\omega_0)$.*

Remark. If ω_t depends on t polynomially then the 2-division property with parameter $t \in [0,1]$ can be replaced by the same division property without parameter.

The results in the odd-dimensional case, $M = M^{2k+1}$, require a bit more assumptions since they reduce the classification of hyperplane fields or Pfaff equations on M^n to the orbital classification of vector fields on a *hypersurface* of M. Within this paper I assume that the Martinet hypersurface S_0 of a Pfaff equation or hyperplane field (ω_0) is a smooth codimension one submanifold of M^{2k+1} and that the same remains true under a small deformation of ω_0. This condition (that can be called the *structural smoothness* of the Martinet hypersurface, see [JaZh1]) can be expressed as follows: $j_p^1\left(\omega_0 \wedge (d\omega_0)^k\right) \neq 0,\ p \in M$. In fact, S_0 is the zero level of a function $H_0 = \left(\omega_0 \wedge (d\omega_0)^k\right)/\Omega$, where Ω is a volume form, and the condition above means that the 1-form dH_0 does not vanish at any point of S_0.

Like in the even-dimensional case, the theorem below for the odd-dimensional case admits vanishing of ω_0 on a subset of M, but at any point of this subset a certain transversality condition is required.

Theorem 2. *Let ω_0 be a 1-form on $M = M^{2k+1}$, X_0 a characteristic vector field of (ω_0) defined on the Martinet hypersurface S_0. Assume that the following conditions hold:*

(a) $j_p^1(\omega_0 \wedge (d\omega_0)^k) \neq 0,\ p \in S_0$;

(b) if $\omega_0(p) = 0$ at a point $p \in S_0$ then the kernel of the 2-form $d\omega_0(p)$ is transversal to S_0;

(c) If $k \geq 2$ then X_0 has the 1-division property. (This assumption is not required if $k = 1$);

(d) there exists an integer $r \geq 0$ such that $j_p^r X_0 \neq 0,\ p \in S_0$.

Set $q = r+2$ if $k = 1$ and $q = max(2r+2, r+3)$ if $k \geq 2$. Then (ω_0) is globally equivalent to (ω_1) provided that the 1-form ω_1 is sufficiently C^q-close to ω_0 and a characteristic vector field X_1 of (ω_1), defined on the Martinet hypersurface S_1 of (ω_1), is orbitally equivalent to X_0 via a sufficiently C^{q+1}-close to the identity diffeomorphism of M sending S_1 to S_0.

I explain the condition (b). If $\omega_0(p) = 0$ then the space $ker\ d\omega_0(p)$ consisting of tangent vectors $a \in T_pM^{2k+1}$ such that $a \rfloor d\omega_0(p) = 0$ is invariantly related to the Pfaff equation (ω_0) (if $\omega_0(p) \neq 0$ then this is, of course, not so). The structural smoothness of S_0 at p, i.e. the condition $j_p^1(\omega_0 \wedge (d\omega_0)^k) \neq 0$, implies that if $\omega_0(p) = 0$ then $(d\omega_0)^k(p) \neq 0$, and consequently the space $ker\ d\omega_0(p)$ is a line in T_pM. The condition (b) excludes degenerations when this line is tangent to S_0.

One can expect that the condition (a) (the structural smoothness of S_0) holds for a generic 1-form ω_0, but this is so only if M is a 3 or 5-dimensional manifold. The thing is that S_0 fails to be a smooth hypersurface near points p such that $\omega_0 \wedge (d\omega_0)^{k-1}(p) = 0$. This degeneration has codimension 6, see [Mar], and therefore it is typical if $dim M \geq 7$. This explains the following proposition.

Proposition 2. *If M is a 3-dimensional or a 5-dimensional manifold then in the space of all 1-forms on M there is an open and dense set in the C^3-topology such that any 1-form ω_0 of this set satisfies the assumptions (a-d) of Theorem 2.*

The deformation theorem, corresponding to Theorem 2, is as follows.

Theorem 2.A. *Let ω_t be a family of 1-forms on $M = M^{2k+1}$ such that $j^1_p\left(\omega_t \wedge (d\omega_t)^{k-1}\right) \neq 0$, $p \in M, t \in [0,1]$. Assume that a single hypersurface S_0 and a single vector field X_0 on S_0 are the Martinet hypersurface and a characteristic vector field respectively for all (ω_t). Assume that for any point $(p,t) \in S_0 \times [0,1]$ such that $\omega_t(p) = 0$ the line $ker\ d\omega_t(p) \subset T_pM$ is transversal to S_0. If $k \geq 2$ (and only in this case) assume that X_0 has 1-division property with parameter $t \in [0,1]$. Then there exists a family Φ_t of global diffeomorphsims of M such that $\Phi_t^*(\omega_t) = (\omega_0)$.*

Remark. If ω_t depends on t polynomially then the 1-division property with $t \in [0,1]$ can be replaced by the same division property without parameter.

The proofs of Theorem 1 and 1.A are contained in sections 2 and 3. The proof of Theorems 2 and 2.A for the 3-dimensional case are contained in section 4. The proofs of Theorems 2 and 2.A for the case $k \geq 2$ are the most complicated since they combine the techniques of of all previous proofs. They are contained in sections 5 and 6. The Propositions 1 and 2 are proved in section 7.

The reduction of Theorem 1 to Theorem 1.A is based on the techniques in [MoZh, Appendix A], where Theorems 1 and 1.A are proved for the simplest case where X_0 vanishes at no points of the manifold, i.e. (ω_0) is a quasi-contact structure. Some steps in the proof of Theorems 2 and 2.A involve the techniques of the paper [JaZh2], where the significance of the division properties of a characteristic vector field for reduction theorems was realized.

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2. Proof of Theorem 1.A.

Theorems like Theorem 1.A are usually proved by the homotopy method. The required family Φ_t of diffeomorphisms will be constructed as the solution of the ordinary differential equation with a parameter $p \in M = M^{2k}$ of the form

$$\frac{d\Phi_t(p)}{dt} = Z_t\left(\Phi_t(p)\right), \quad \Phi_0(p) = p, \quad p \in M, \tag{2.1}$$

where Z_t is a family of vector fields on M. The requirement $\Phi_t^*(\omega_t) = (\omega_0)$ holds if

$$L_{Z_t}\omega_t + h_t\omega_t + \frac{d}{dt}\omega_t = 0, \tag{2.2}$$

where h_t is any family of functions on M, and L_{Z_t} is the Lie derivative along the vector field Z_t. In fact, due to the compactness of M the equation (2.1) has a solution for all t, and the basic uniquiness theorems imply that for any t the mapping $p \to \Phi^t(p)$ is a diffeomorphism of M. The function $A(t) = \Phi_t^*\omega_t$ with values in the space of 1-forms on M satisfies the equation

$$A'(t) = \Phi_t^*\left(L_{Z_t}\omega_t + \frac{d}{dt}\omega_t\right) = -h_t(\Phi_t)A(t),$$

and it follows that $A(t) = H_t A(0) = H_t \omega_0$, where H_t is a family of nonvanishing functions.

The first step in solving the equation (2.2) with respect to Z_t and h_t is to exclude h_t using the following division property of ω_t.

Lemma 2.1. *Let μ_t be any family of 1-forms on M such that $\mu_t \wedge \omega_t = 0,\ t \in [0,1]$. Then there exists a family of functions h_t such that $\mu_t = h_t \omega_t$.*

This property is not trivial since ω_t might vanish at some points $(p,t) \in M \times [0,1]$. The proof uses the division property of a characteristic vector field. Lemma 2.1 allows to reduce the equation (2.2) to the equation

$$\left(L_{Z_t}\omega_t + \frac{d}{dt}\omega_t\right) \wedge \omega_t = 0 \tag{2.3}$$

with respect to the family Z_t of vector fields. The solution of this equation will be constructed as follows.

Lemma 2.2. *There exists a family Z_t of vector fields on M such that*

$$Z_t \rfloor \left(\omega_t \wedge (d\omega_t)^{k-1}\right) = (1-k)\left(\frac{d}{dt}\omega_t\right) \wedge \omega_t \wedge (d\omega_t)^{k-2}. \tag{2.4}$$

The proof of this lemma is based on the assumption that (ω_t) have the same characteristic vector field X_0. I will show that the family of vector field Z_t satisfying (2.4) is a solution of the equation (2.3).

At first note that (2.4) implies that the form $(Z_t \rfloor \omega_t) \cdot (d\omega_t)^{k-1}$ is the external product of ω_t with a certain $(2k-3)$-form. It follows that

$$(Z_t \rfloor \omega_t) \cdot \omega_t \wedge (d\omega_t)^{k-1} = 0. \tag{2.5}$$

Notation. Denote by R the subset of M consisting of points at which the vector field X_0 does not vanish.

Since X_0 is a characteristic vector field of (ω_t) for all t then $\left(\omega_t \wedge (d\omega_t)^{k-1}\right)(p) \neq 0$ for any $p \in R$ and any t, and (2.5) implies that the function $Z_t \rfloor \omega_t$ vanishes at any point $p \in R$, for any t. The set R is everywhere dense in M - it is easy to see that otherwice the field X_0 would not have the 2-division property. Therefore $Z_t \rfloor \omega_t \equiv 0$. Then $L_{Z_t}\omega_t = Z_t \rfloor d\omega_t$ and the relation (2.4) easily implies

$$\left(L_{Z_t}\omega_t + \frac{d}{dt}\omega_t\right) \wedge \omega_t \wedge (d\omega_t)^{k-2} = 0. \tag{2.6}$$

Now the following simple algebraic statement will be used. Assume that ω is a quasi-contact form at a point $p \in M^{2k}$, i.e. $(\omega \wedge (d\omega)^{k-1})(p) \neq 0$. Let μ be a 1-form such that $\left(\mu \wedge \omega \wedge (d\omega)^{k-2}\right)(p) = 0$. Then $(\mu \wedge \omega)(p) = 0$. In fact, if $(\mu \wedge \omega)(p) \neq 0$ then $(d\omega)^{k-2}(p)$ vanishes when being restricted to the space $ker\omega(p) \cap ker\mu(p)$ which is a hyperplane in the space $ker\omega(p)$. This contradicts to the assumption $(\omega \wedge (d\omega)^{k-1})(p) \neq 0$.

Using the given algebraic statement and the relation (2.6) I conclude that (2.3) holds for all points at which ω_t is quasi-contact, i.e. for all points (p,t) such that $p \in R$. Since R is everywhere dense in M then (2.3) holds for all t and all points of M. Theorem 1.A is proved modulo Lemmas 2.1 and 2.2.

Proof of Lemma 2.2. Since X_0 is a characteristic vector field of (ω_t) then

$$X_0 \rfloor \left(\omega_t \wedge (d\omega_t)^{k-1}\right) = 0. \tag{2.7}$$

Let $p \in R$. Then $\left(\omega_t \wedge (d\omega_t)^{k-1}\right)(p) \neq 0$ and it is easy to see that near p the relation (2.7) implies that $X_0 \rfloor \omega_t = 0$ and $X_0 \rfloor d\omega_t = h_t\omega_t$ for some function h_t. It follows that near p we have, for any t:

$$X_0 \rfloor \left((\frac{d}{dt}\omega_t) \wedge \omega_t \wedge (d\omega_t)^{k-2} \right) = 0. \tag{2.8}$$

Since the set R is everywhere dense in M then (2.8) holds for all t and all points of the manifold. Using the 2-division property of X_0 with parameter $t \in [0,1]$ I conclude that (2.8) implies the existence of a family $\theta_{t,2k-1}$ of $(2k-1)$-form on M such that

$$(\frac{d}{dt}\omega_t) \wedge \omega_t \wedge (d\omega_t)^{k-2} = X_0 \rfloor \ \theta_{t,2k-1}. \tag{2.9}$$

The relation (2.7) and the 1-division property of X_0 (following from the 2-division property) imply the existence of a family Ω_t of $2k$-forms such that

$$X_0 \rfloor \Omega_t = \omega_t \wedge (d\omega_t)^{k-1}. \tag{2.10}$$

The condition that X_0 is one of the characteristic vector fields of (ω_t) implies that Ω_t vanishes at no point $(p,t) \in M \times [0,1]$. Therefore one can define a family Z_t of vector fields by the relation

$$Z_t \rfloor \Omega_t = (k-1)\theta_{t,2k-1},$$

and then (2.9) and (2.10) imply (2.4). ■

Proof of Lemma 2.1. Let $\mu_t \wedge \omega_t = 0$. Take any point $p \in R$. It was shown that near p the vector field X_0 is annihilated by ω_t and $X_0 \rfloor d\omega_t = h_t\omega_t$ for some function h_t (see the proof of Lemma 2.2). Then $X_t \rfloor \mu_t = 0$ and $X_0 \rfloor \left(\mu_t \wedge (d\omega_t)^{k-1}\right) = 0$ for all t. Since the set R is dense, these relations hold at all points $p \in M$ and for all t. The latter relation implies, together with the 1-division property of X_0, the existence of a family Ω_t' of $(2k)$-forms on M such that $\mu_t \wedge (d\omega_t)^{k-1} = X_0 \rfloor \Omega_t'$. Let Ω_t be the family of nondegenerate volume forms in (2.10), and let $\Omega_t' = h_t\Omega_t$. Then $(\mu_t - h_t\omega_t) \wedge (d\omega_t)^{k-1} = 0$. Let $\gamma_t = \mu_t - h_t\omega_t$. Note that $\gamma_t \wedge \omega_t = 0$. This relation and the relation $\gamma_t \wedge (d\omega_t)^{k-1} = 0$ easily imply that $\gamma_t(p) = 0$ for all points (p,t) such that ω_t is quasi-contact at p, i.e. for all points (p,t) such that $p \in R$. Using again the density of the set R I conclude that $\gamma_t(p) = 0$ for all p and t, i.e. $\mu_t = h_t\omega_t$. ■

Proof of remark after Theorem 1.A. It is clear that if X_0 has the 1-division property without parameter then it has the same property with parameter t provided that all families depend on t polynomially. If ω_t depends on t polynomially then all objects in the proof of Theorem 1.A depend on t polynomially, and the proof remains the same.

3. Proof of Theorem 1.

The proof is based on the following lemma.

Lemma 3.1. *Let ω_0 be a 1-form on $M = M^{2k}$ satisfying the conditions (a) and (b) of Theorem 1. Let X_0 be a characteristic vector field of (ω_0). Let ω_1 be a 1-form on M that is sufficiently C^{r+1}-close to ω_0 and such that*

$$\omega_1 \wedge (d\omega_1)^{k-1} = \omega_0 \wedge (d\omega_0)^{k-1}. \tag{3.1}$$

Then X_0 is a characteristic vector field of (ω_t), where $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$.

At first I will show that Theorem 1 follows from this lemma and Theorem 1.A. Let ω_0 and ω_1 be 1-forms satisfying the conditions of Theorem 1. Let Φ be a diffeomorphism realizing the orbital equivalence of a characteristic vector field X_1 of (ω_1) and the characteristic vector field X_0 of (ω_0). Set $\omega_2 = \Phi^*\omega_1$. Then

$$\omega_2 \wedge (d\omega_2)^{k-1} = F\omega_0 \wedge (d\omega_0)^{k-1}$$

for some function F on M. Since Φ is C^{q+1}-close to the identity and ω_1 is C^q-close to ω_0 then ω_2 is C^q-close to ω_0 and consequently the form $\omega_2 \wedge (d\omega_2)^{k-1}$ is C^{q-1}-close to $\omega_0 \wedge (d\omega_0)^{k-1}$. By the condition (b) of Theorem 1 the r-jet of the form $\omega_0 \wedge (d\omega_0)^{k-1}$ vanishes at no points of M, therefore the function F is C^{q-r-1}-close to 1. Note that $q-r-1 = r+1$. Set $\omega_3 = F^{1/k}\omega_2$. Then ω_3 is C^{r+1}-close to ω_0, and ω_3 satisfies the relation

$$\omega_3 \wedge (d\omega_3)^{k-1} = \omega_0 \wedge (d\omega_0)^{k-1}.$$

By Lemma 3.1 the vector field X_0 is a characteristic vector field of $(\hat{\omega}_t)$, where $\hat{\omega}_t = \omega_0 + t(\omega_3 - \omega_0)$. By Theorem 1.A (ω_t) is equivalent to (ω_0). In particular, (ω_3) is equivalent to (ω_0) and consequently (ω_1) is equivalent to (ω_0).

Proof of Lemma 3.1. I use again the set $R \subset M$ consisting of points at which the field X_0 does not vanish. The fact that X_0 is a characteristic vector field of (ω_0) and of (ω_1) implies that near any point $p \in R$ the following holds:

$$X_0 \rfloor \omega_0 = X_0 \rfloor \omega_1 = 0, \tag{3.2}$$

$$X_0 \rfloor d\omega_0 = h_0\omega_0, \quad X_0 \rfloor d\omega_1 = h_1\omega_1 \tag{3.3}$$

for some functions h_0 and h_1 defined near p. I will show that the relation (3.1) implies that

$$h_0 = h_1. \tag{3.4}$$

In order to prove this I take the Lie derivative of the relation (3.1) along the vector field X_0. Using (3.2) and (3.3) one obtains the following:

$$L_{X_0}\left(\omega_0 \wedge (d\omega_0)^{k-1}\right) = (k-1)h_0\omega_0 \wedge (d\omega_0)^{k-1},$$

$$L_{X_0}\left(\omega_1 \wedge (d\omega_1)^{k-1}\right) = (k-1)h_1\omega_1 \wedge (d\omega_1)^{k-1}.$$

These relations and (3.1) imply (3.4). The relation (3.4) easily implies

$$X_0 \rfloor d\omega_t = h_0\omega_t.$$

Using this and (3.2) one obtains that near $p \in R$ the field X_0 is annihilated by the form $\omega_t \wedge (d\omega_t)^{k-1}$:

$$X_0 \rfloor \left(\omega_t \wedge (d\omega_t)^{k-1}\right) = 0.$$

The set R is dense, therefore the latter relation holds for all points $p \in M$ and for all t. This relation and the 2-division property of X_0 allow to conclude that

$$\omega_t \wedge (d\omega_t)^{k-1} = Q_t \cdot \omega_0 \wedge (d\omega_0)^{k-1}, \tag{3.5}$$

where Q_t is a family of functions on M. To show that X_0 is a characteristic vector field of (ω_t) one has to prove that the function Q_t vanishes at no points of M. This follows from the C^{r+1}-closeness of ω_1 to ω_0. In fact, the latter implies that the form $\omega_t \wedge (d\omega_t)^{k-1}$ is C^r-close to $\omega_0 \wedge (d\omega_0)^{k-1}$ (the fact that t varies on the compact segment $[0,1]$ is used here).

By the condition (b) of Theorem 1 the r-jet of the form $\omega_0 \wedge (d\omega_0)^{k-1}$ vanishes at no points of M, therefore the function Q_t is C^0-close to 1. ■

4. Proof of Theorems 2 and 2.A, $k = 1$.

At first I prove Theorem 2.A. Fix the function

$$H_0 = (\omega_0 \wedge d\omega_0)/\Omega,$$

where Ω is a fixed volume form on $M = M^3$. The assumption that (ω_t) have the same characteristic vector field X_0 on the Martinet surface $S_0 = \{H_0 = 0\}$ implies

$$(\omega_t \wedge dH_0)(p) = (Q_t\omega_0 \wedge dH_0)(p), \ p \in S_0,$$

where Q_t is a family of functions vanishing at no points of M. There is no loss of generality to assume that $Q_t \equiv 1$ since the family ω_t can be replaced by $\hat{\omega}_t = Q_t\omega_t$. Then the following relation holds

$$\left((\omega_t - \omega_0) \wedge dH_0\right)(p) = 0, \ p \in S_0.$$

Since the 1-jet of the form $\omega_0 \wedge d\omega_0$ does not vanish at points of S_0 then $dH_0(p) \neq 0, p \in S_0$. It follows that the latter display relation implies the existence of a family F_t of functions on M and a family α_t of 1-forms on M such that

$$\omega_t = \omega_0 + F_t dH_0 + H_0\alpha_t. \tag{4.1}$$

Like in the even-dimensional case, the homotopy method is used: I will show that (4.1) allows to solve the equation

$$L_{Z_t}\omega_t + h_t\omega_t + \frac{d}{dt}\omega_t = 0 \tag{4.2}$$

with respect to the unknown family Z_t of vector fields on M and the unknown family h_t of functions on M. Here L_{Z_t} is the Lie derivative along the vector field Z_t.

The equation (4.2) will be solved in two steps. At first I will show that the equation

$$L_{Y_t}\omega_t + \frac{d}{dt}\omega_t = 0 \ mod \ (H_0) \tag{4.3}$$

is solvable with respect to the unknown family Y_t of vector fields on M. Here and below $\nu_t = 0 \ mod \ (H_0)$ for an s-form ν_t means that $\nu_t = H_0\mu_t$ for some s-form μ_t (s is any integer). The solvability of the equation (4.3) reduces the equation (4.2) to the equation

$$L_{Z_t}\omega_t + h_t\omega_t = H_0\gamma_t, \tag{4.4}$$

where γ_t is a certain family of 1-forms on M. The second step is to prove that (4.4) is solvable for any family γ_t.

So, to prove Theorem 2.A in the 3-dimensional case, it suffices to prove the solvability of the equations (4.3) and (4.4).

A solution Y_t of the equation (4.3) will be found in the form $Y_t = H_0X_t$, where X_t is a family of vector fields on M. Then $L_{Y_t}\omega_t = (X_t \rfloor \omega_t) \cdot dH_0$. Since $\frac{d}{dt}\omega_t = G_t dH_0$ with $G_t = \frac{d}{dt}F_t$ (by the relation (4.1)), then to solve (4.3) it suffices to solve the equation

$$X_t \rfloor \omega_t + G_t = 0. \tag{4.5}$$

The solvability of (4.5) is not trivial since ω_t might vanish at some points. Differentiating the relation $\omega_t \wedge d\omega_t = 0 \ mod \ (H_0)$ one obtains $\frac{d}{dt}\omega_t \wedge d\omega_t = \delta_t \wedge \omega_t \ mod \ (H_0)$, where δ_t is a certain family of 2-forms. Then (4.1) implies

$$G_t dH_0 \wedge d\omega_t = \delta_t \wedge \omega_t \ mod \ (H_0). \tag{4.6}$$

Let (p_0, t_0) be a fixed point of $S_0 \times [0,1]$ such that $\omega_t(p) = 0$. The transversality condition in Theorem 2.A means that $dH_0 \wedge d\omega_t \neq 0$ near (p_0, t_0). Therefore the relation (4.6) implies that the equation (4.5) is solvable locall near (p_0, t_0). Consequently, (4.5) is solvable near any point $(p,t) \in M \times [0,1]$. By the division of unity (4.5) has a global solution.

It remains to solve the equation (4.4). Since S_0 is the Martinet surface of (ω_t) for all t then

$$\omega_t \wedge d\omega_t = H_0 \Omega_t \tag{4.7}$$

for some family Ω_t of volume forms on M. By the condition $j^1_p(\omega_t \wedge d\omega_t) \neq 0, p \in S_0$ the volume forms Ω_t are not degenerated. Define families Z_t and V_t of vector fields fields on M by the relations

$$Z_t \rfloor \Omega_t = \gamma_t \wedge \omega_t, \tag{4.8}$$

$$V_t \rfloor \Omega_t = d\omega_t \tag{4.9}$$

and a family h_t of functions by the relation

$$h_t = V_t \rfloor \gamma_t.$$

I will show that (Z_t, h_t) is a solution of (4.4). In fact, (4.8) implies that $Z_t \rfloor \omega_t = 0$ and then by (4.8) and (4.9)

$$L_{Z_t}\omega_t = Z_t \rfloor d\omega_t = Z_t \rfloor (V_t \rfloor \Omega_t) =$$

$$= -V_t \rfloor (Z_t \rfloor \Omega_t) = -V_t \rfloor (\gamma_t \wedge \omega_t) = -h_t\omega_t + (V_t \rfloor \omega_t) \cdot \gamma_t. \tag{4.10}$$

Note now that (4.9) imples that $V_t \rfloor d\omega_t = 0$. Using this, (4.7) and (4.9) one obtains that $(V_t \rfloor \omega_t) \cdot d\omega_t = H_0 d\omega_t$. Then the function $V_t \rfloor \omega_t - H_0$ vanishes at points at which the 2-form $d\omega_t$ does not vanish. In particular, this function vanishes at points away from S_0 and consequently $V_t \rfloor \omega_t \equiv H_0$. This and (4.10) imply the required relation (4.4). Theorem 2.A is proved.

Theorem 2 can be reduced to Theorem 2.A by the folowing arguments. Let ω_0 and ω_1 be 1-forms satisfying the conditions of Theorem 2. Arguing like in the proof of Theorem 1, it is easy to see that there is no loss of generality to assume that ω_1 is C^{r+2}-close to ω_0, that these forms have the same Martinet surface $S_0 = \{H_0 = 0\}$, where $H_0 = (\omega_0 \wedge d\omega_0)/\Omega$, Ω is a fixed volume form, and that (ω_0) and (ω_1) have the same characteristic vector field X_0. The latter means that

$$\omega_1 \wedge dH_0 = Q\omega_0 \wedge dH_0 \ mod \ (H_0),$$

where Q is a nonvanishing function on M. The condition that $j^r_p X_0 \neq 0, p \in S_0$, implies that $j^r_p(\omega_0 \wedge dH_0) \neq 0, p \in S_0$. Therefore the C^{r+2}-closeness of ω_1 to ω_0 implies the C^2-closeness of the function Q to 1. Set $\omega_2 = Q\omega_1$. Then the form ω_2 is C^2-close to ω_0 and these forms satisfy the relation

$$\omega_2 \wedge dH_0 = \omega_0 \wedge dH_0 \ mod \ (H_0). \tag{4.11}$$

It follows that $\omega_2 - \omega_0 = FdH_0 \ mod \ (H_0)$ for some function F (the condition $dH_0(p) \neq 0, p \in S_0$ is used). Consider the family $\omega_t = \omega_0 + t(\omega_2 - \omega_0), t \in [0,1]$. Note that at any point $p \in S_0$ the 3-form $\omega_t \wedge d\omega_t$ depends on t linearly: $(\omega_t \wedge d\omega_t)(p) = (\omega_0 \wedge d\omega_0)(p) + t\Omega_1$, where Ω_1 is a certain 3-form. Since S_0 is the Martinet surface of ω_0 and of ω_2 then the forms $\omega_0 \wedge d\omega_0$ and Ω_1 vanish on S_0 and therefore $\omega_t \wedge d\omega_t$ vanishes on S_0 for all t.

The C^2-closemess of ω_2 to ω_0 implies the C^1-closeness of $\omega_t \wedge d\omega_t$ to $\omega_0 \wedge d\omega_0$ (the compactness of the segment $[0,1]$ is used). Since dH_0 does not vanish on S_0, it follows that the 1-jet of $\omega_t \wedge d\omega_t$ does not vanish on S_0 and that S_0 is the Martinet syrface of (ω_t) for all t. Note also that the 3-form $d\omega_t \wedge dH_0$ does not vanish at points of S_0 at which ω_t vanishes; this follows from the assumption (b) of Theorem 2 and the C^2-closeness of ω_2 to ω_0. Therefore the family ω_t satisfies the assimptions of Theorem 2.A and by this theorem

(ω_t) is equaivalent to (ω_0). In particular, (ω_2) is equivalent to (ω_0) and consequently (ω_1) is equivalent to (ω_0).

5. Proof of Theorem 2.A, $k \geq 2$.

Like in previous sections the homotopy method is used, and to prove Theorem 2.A it suffices to solve the equation

$$L_{Z_t}\omega_t + h_t\omega_t + \frac{d}{dt}\omega_t = 0.$$

The solvability of this equation with respect to the unknown family Z_t of vector fields and the unknown family h_t of functions on $M = M^{2k+1}$ follows from the Propositions 5.1 and 5.2 below. The proof of these propositions uses the techniques involved in the both even-dimensional and 3-dimensional cases.

Fix a volume form Ω on M and a function

$$H_0 = \omega_0 \wedge (d\omega_0)^k/\Omega.$$

The Martinet hypersurface S_0 of (ω_0) is the zero level of H_0. I will use the following notations. If A_t and B_t are families of s-forms on M then $A_t = B_t \; mod \; (H_0)$ means that $A_t - B_t = H_0 C_t$ for some family C_t of 1-forms on M. Here s is any integer.

Proposition 5.1. *There exists a family Y_t of vector fields on M such that*

$$\left(L_{Y_t}\omega_t + \frac{d}{dt}\omega_t\right) \wedge \omega_t \wedge (d\omega_t)^{k-1} = 0 \; mod \; (H_0). \tag{5.1}$$

Proposition 5.2. *Assume that γ_t is a family of 1-forms on M such that*

$$\gamma_t \wedge \omega_t \wedge (d\omega_t)^{k-1} = 0 \; mod \; (H_0). \tag{5.2}$$

Then there exist a family Z_t of vector fields and a family h_t of functions such that

$$L_{Z_t}\omega_t + h_t\omega_t = \gamma_t. \tag{5.3}$$

To prove Proposition 5.1 fix a characteristic vector field X_0 of (ω_0). Since X_0 is a characteristic vector field of (ω_t) for all t then

$$X_0 \rfloor \left(\omega_t \wedge (d\omega_t)^{k-1} \wedge dH_0\right) = 0 \; mod \; (H_0). \tag{5.4}$$

Lemma 5.1. *The relation (5.4) implies that there exists a family F_t of functions on M such that*

$$(\frac{d}{dt}\omega_t) \wedge \omega_t \wedge (d\omega_t)^{k-1} = F_t \cdot \omega_t \wedge (d\omega_t)^{k-1} \wedge dH_0 \; mod \; (H_0). \tag{5.5}$$

Lemma 5.1 allows to find a solution Y_t of the equation (5.1) in the form

$$Y_t = H_0 X_t,$$

where X_t is a family of vector fields on M. Then

$$L_{Y_t}\omega_t = (X_t \rfloor \omega_t) \cdot dH_0 \; mod \; (H_0)$$

and by Lemma 5.1 to solve (5.1) it suffices to solve the equation

$$X_t \rfloor \omega_t - F_t = 0 \ mod \ (H_0). \tag{5.6}$$

Note that the form ω_t might vanish at some points, therefore the solvability of (5.6) is not trivial. By the division of unity, it suffices to solve (5.6) locally.

Lemma 5.2. *The equation (5.6) has a local solution, near any point $(p,t) \in M \times [0,1]$.*

Now I will prove Proposition 5.2. A solution (Z_t, h_t) of the equation (5.3) will be constructed explicitly.

Since S_0 is the Martinet hypersurface of (ω_t) then

$$\omega_t \wedge (d\omega_t)^k = H_0 \Omega_t, \tag{5.7}$$

where Ω_t is a family of nondegenerated volume forms on M. Define a family V_t of vector fields on M by the relation

$$V_t \rfloor \Omega_t = (d\omega_t)^k. \tag{5.8}$$

The relations (5.7), (5.8) and the fact that the form $(d\omega_t)^k$ does not vanish at points of a dense set in M (at points away from S_0) imply

$$V_t \rfloor d\omega_t = 0, \quad V_t \rfloor \omega_t = H_0. \tag{5.9}$$

Using (5.2) and (5.9) one obtains

$$(V_t \rfloor \gamma_t) \cdot \omega_t \wedge (d\omega_t)^{k-1} = 0 \ mod \ (H_0).$$

Notation. Denote by R the set of points of S_0 at which the field X_0 does not vanish.

Since X_0 is a characteristic vector field of (ω_t) then the form $\omega_t \wedge (d\omega_t)^{k-1}$ does not vanish at points of the set R. The set R is dense in S_0 (it is easy to see that otherwice X_0 cannot have the 1-division property). Therefore the latter relation implies that the function $V_t \rfloor \gamma_t$ vanishes at any point of S_0. Since $dH_0(p) \neq\neq 0, p \in S_0$ then there exists a family h_t of functions on M such that

$$V_t \rfloor \gamma_t = h_t H_0. \tag{5.10}$$

I will show that the pair (h_t, Z_t), where h_t is defined by (5.10) and Z_t by the relation

$$Z_t \rfloor \Omega_t = k \frac{\gamma_t \wedge \omega_t \wedge (d\omega_t)^{k-1}}{H_0} \tag{5.11}$$

is a solution of the equation (5.3).

Note that (5.11) implies that $Z_t \rfloor \omega_t = 0$ and therefore $L_{Z_t}\omega_t = Z_t \rfloor d\omega_t$. Using this and (5.8) evaluate

$$(L_{Z_t}\omega_t) \wedge (d\omega_t)^{k-1} = \frac{Z_t \rfloor (d\omega_t)^k}{k} = \frac{Z_t \rfloor (V_t \rfloor \Omega_t)}{k} =$$
$$= -\frac{V_t \rfloor (Z_t \rfloor \Omega_t)}{k} = -V_t \rfloor \left(\frac{\gamma_t \wedge \omega_t \wedge (d\omega_t)^{k-1}}{H_0} \right).$$

Using again (5.9) and (5.10), I obtain

$$(L_{Z_t}\omega_t + h_t\omega_t - \gamma_t) \wedge (d\omega_t)^{k-1} = 0.$$

This relation implies the required relation (5.3) at points $(p,t) \in M \times [0,1]$ such that $(d\omega_t)^{k-1}(p) \neq 0$. Therefore (5.3) holds at points (p,t) such that $p \notin S_0$. Consequently (5.3) holds for all points $(p,t) \in M \times [0,1]$. Theorem 2.A is proved modulo Lemmas 5.1 and 5.2.

Proof of Lemma 5.1. I use the set $R \subset S_0$ defined above. The relation (5.4) easily implies:

$$X_0(p) \rfloor \omega_t(p) = 0, \quad X_0(p) \rfloor dH_0(p) = 0,$$

$$X_0(p) \rfloor d\omega_t(p) = a(t,p)\omega_t(p) + b(t,p)dH_0(p), \quad p \in R,$$

where $a(t,p)$ and $b(t,p)$ are some numbers depending on t and p. Since S_0 is the Martinet hypersurface of (ω_t) for all t then $\left(\omega_t \wedge (d\omega_t)^k\right)(p) = 0, \quad p \in S_0$, and it follows that $b(t,p) = 0, \ p \in R, t \in [0,1]$. This and the density of the set R in S_0 allow to conclude that

$$X_0(p) \rfloor \left((\frac{d}{dt}\omega_t) \wedge \omega_t \wedge (d\omega_t)^{k-1} \right)(p) = 0, \ p \in R. \tag{5.12}$$

Define a family A_t of vector fields on M by the relation

$$A_t(p) \rfloor \Omega(p) = \left((\frac{d}{dt}\omega_t) \wedge \omega_t \wedge (d\omega_t)^{k-1} \right)(p), \ p \in S_0.$$

Then by (5.12) $X_0(p) \wedge A_t(p) = 0, \ p \in S_0$. It follows that the field A_t is tangent to S_0 at any point of the set R and consequently at any point of S_0. By the 1-division property of X_0 there exists a family F_t of functions on S_0 such that $A_t = F_t X_0$. This relation and the condition $dH_0(p) \neq 0, \ p \in S_0$ imply (5.5). ∎

Proof of Lemma 5.2. I will use the following notations. The relation

$$A_t = B_t \ mod \ (H_0, \omega_t),$$

where A_t and B_t are s-forms, means that $A_t - B_t = \omega_t \wedge \nu_t + H_0\mu_t$ for some $(s-1)$-form ν_t and s-form μ_t. Write the relation (5.5) in the form

$$\frac{d}{dt}\omega_t \wedge \omega_t \wedge (d\omega_t)^{k-1} = F_t \cdot \omega_t \wedge (d\omega_t)^{k-1} \wedge dH_0 + H_0\alpha_t, \tag{5.13}$$

where α_t is a family of $(2k)$-forms. Taking the differential of this relation gives

$$(\frac{d}{dt}\omega_t) \wedge (d\omega_t)^k = F_t \cdot (d\omega_t)^k \wedge dH_0 + dH_0 \wedge \alpha_t \ mod \ (H_0, \omega_t). \tag{5.14}$$

To prove the solvability of the equation (5.6) I will show that the following relations hold:

$$(\frac{d}{dt}\omega_t) \wedge (d\omega_t)^k = 0 \ mod \ (H_0, \omega_t), \tag{5.15}$$

$$\alpha_t = 0 \ mod \ (H_0, \omega_t). \tag{5.16}$$

These relations and (5.14) imply

$$F_t \cdot (d\omega_t)^k \wedge dH_0 = 0 \ mod \ (H_0, \omega_t). \tag{5.17}$$

The assumption of Theorem 2.A that the kernel of $d\omega_t$ is transversal to S_0 at points at which ω_t vanishes implies that at such points the $(2k+1)$-form $(d\omega_t)^k \wedge dH_0$ does not vanish. Therefore (5.17) implies that locally (near any fixed point (p,t)) the equation (5.6) can be solved locally.

It remains to prove the relations (5.15) and (5.16). The relation (5.15) can be obtained by differiantiating by t of the relation $\omega_t \wedge (d\omega_t)^k = 0 \ mod \ (H_0)$. The proof of (5.16) is as follows. The relation (5.13) can be written in the form

$$H_0 \alpha_t = \omega_t \wedge \eta_t \tag{5.17}$$

for some $(2k-1)$-form η_t. I work near a point $(p,t) \in S_0 \times [0,1]$ such that $\omega_t(p) = 0$, otherwice (5.16) is trivial. The transversality of $ker d\omega_t$ to S_0 allows to fix a vector field B_t which is transversal to S_0 and such that $B_t \rfloor d\omega_t = 0$. Note that $(B_t \rfloor \omega_t)(p) = 0, p \in S_0$ since the form $\omega_t \wedge (d\omega_t)^k$ vanishes on S_0. Then by (5.17) the function $B_t \rfloor \nu_t = 0$ vanishes at points of S_0 at which ω_t does not vanish and consequently it vanishes at any point of S_0. Take the Lie derivative of the relation (5.17) along B_t at a point $p \in S_0$. Using the relations

$$(B_t \rfloor \omega_t)(p) = (B_t \rfloor d\omega_t)(p) = (B_t \rfloor \nu_t)(p) = 0, \ p \in S_0$$

one obtains

$$(B_t \rfloor dH_0)(p) \cdot \alpha_t(p) = \omega_t(p) \wedge \beta_t(p), \ p \in S_0,$$

where β_t is a family of 1-forms on S_0. The transversality of B_t to S_0 means that $(B_t \rfloor dH_0)(p) \neq 0$, therefore the latter relation implies (5.16). ■

6. Proof of Theorem 2, $k \geq 2$

Fix a volume form Ω on $M = M^{2k+1}$, the function $H_0 = \omega_0 \wedge (d\omega_0)^k / \Omega$ and a characteristic vector field X_0 of (ω_0) defined by the relation

$$X_0(p) \rfloor \Omega_0(p) = \left(\omega_0 \wedge (d\omega_0)^{k-1} \wedge dH_0\right)(p), \quad p \in S_0 = \{H_0 = 0\}.$$

There is no loss of generality to assume that ω_1 is C^q-close to ω_0 (recitate that $q = max(2r+2, r+3)$), that (ω_1) and (ω_0) have the same Martinet hypersurface $S_0 = \{H_0 = 0\}$ and that X_0 is a characteristic vector field of (ω_1). Then

$$\left(\omega_1 \wedge (d\omega_1)^{k-1} \wedge dH_0\right)(p) = \left(Q\omega_0 \wedge (d\omega_0)^{k-1} \wedge dH_0\right)(p), \ p \in S_0$$

for some function Q on S_0. The $(2k)$-forms in this relation are C^{q-1}-close. Since the r-jet of X_0 does not vanish on S_0 then the r-jet of the form $\omega_0 \wedge (d\omega_0)^{k-1} \wedge dH_0$ does not vanish on S_0 and one can conclude that the function Q is C^{q-r-1}-close to 1. Define $\omega_2 = Q^{1/k}\omega_1$. Then ω_2 is C^{q-r-1}-close to ω_0 and these forms satisfy the relation

$$\left(\omega_2 \wedge (d\omega_2)^{k-1} \wedge dH_0\right)(p) = \left(\omega_0 \wedge (d\omega_0)^{k-1} \wedge dH_0\right)(p), \ p \in S_0. \tag{6.1}$$

Let R be the set of points of S_0 at which the characteristic vector field X_0 does not vanish. In section 5 it was shown that $X_0(p) \rfloor \omega_0(p) = 0, \quad X_0(p) \rfloor d\omega_0(p) = h_0(p)\omega_0(p), \ p \in R$ for some function h_0 defined in a neighbourhood of p of S_0. Using this relations and taking the Lie derivative of the form $\omega_0 \wedge (d\omega_0)^{k-1} \wedge dH_0$ along X_0, one obtains

$$\left(L_{X_0}\left(\omega_0 \wedge (d\omega_0)^{k-1} \wedge dH_0\right)\right)(p) = k\left(h_0\omega_0 \wedge (d\omega_0)^{k-1} \wedge dH_0\right)(p), \ p \in R. \tag{6.2}$$

Since X_0 is also a characteristic vector field of (ω_2), one obtains:

$$X_0(p) \rfloor \omega_2(p) = 0, \quad X_0(p) \rfloor d\omega_2(p) = h_2(p)\omega_0(p), \ p \in R,$$

$$\left(L_{X_0}\left(\omega_2 \wedge (d\omega_2)^{k-1} \wedge dH_0\right)\right)(p) = k\left(h_2\omega_2 \wedge (d\omega_2)^{k-1} \wedge dH_0\right)(p), \ p \in R \tag{6.3}$$

for some function h_2 on S_0. The relations (6.1)-(6.3) imply

$$h_0(p) = h_2(p).$$

This allows to conclude that

$$X_0(p) \rfloor \omega_t(p) = 0, \quad X_0(p) \rfloor d\omega_t(p) = h_0(p)\omega_t(p), \quad p \in R,$$

where $\omega_t = \omega_0 + t(\omega_2 - \omega_0)$ is the segment joining ω_0 to ω_2. It follows

$$\left(\omega_t \wedge (d\omega_t)^k\right)(p) = 0, \ p \in R, \tag{6.4}$$

$$X_0(p) \rfloor \left(\omega_t \wedge (d\omega_t)^{k-1} \wedge dH_0\right)(p) = 0, \ p \in R. \tag{6.5}$$

Since the set R is dense in S_0 (otherwice the condition (d) of Theorem 2 is violated) then the relations (6.4) and (6.5) hold at all points $p \in S_0$ (not only at points of R).

Using these relations and the C^{q-r-1}-closeness of ω_t to ω_0 (following from the C^{q-r-1}-closeness of ω_2 to ω_0) I will show that S_0 is the Martinet hypersurface of (ω_t) for all t and X_0 is a characteristic vector field of (ω_t) for all t. If this is so then the family (ω_t) satisfies the assumptions of Theorem 2.A. This follows from the C^{q-r-1}-closeness of ω_t to ω_0, the observation that $q - r - 1 \geq 2$ and the condition (b) of Theorem 2. By Theorem 2.A (ω_t) is equivalent to (ω_0). In particular (ω_2) is equivalent to ω_0 and consequently (ω_1) is equivalent to (ω_0).

The proof that S_0 is the Martinet hypersurface of (ω_t) for all t is as follows. The relation (6.4) (which is valid for all $p \in S_0$) and the condition $dH_0(p) \neq 0, p \in S_0$ imply

$$\omega_t \wedge (d\omega_t)^k = E_t H_0 \Omega = E_t \omega_0 \wedge (d\omega_0)^k \tag{6.6}$$

for some family E_t of functions on M. Since the form $\omega_t \wedge (d\omega_t)^k$ is C^{q-r-2}-close to $\omega_0 \wedge (d\omega_0)^k$ and the 1-jet of the function H_0 does not vanish then E_t is C^{q-r-3}-close to 1. Since $q \geq r + 3$ then E_t vanishes at no points of M for any $t \in [0, 1]$. Therefore S_0 is the Martinet hypersurface of (ω_t) for all t.

It remains to show that X_0 is a characteristic vector field of (ω_t) for all t. Define a family C_t of vector fields on S_0 by the relation $C_t(p) \rfloor \Omega(p) = \left(\omega_t \wedge (d\omega_t)^{k-1} \wedge dH_0\right)(p), \ p \in S_0$. It is easy to see that C_t is tangent to S_0. The relation (6.5) implies $C_t \wedge X_0 = 0$. Using the 1-division property of X_0 gives $C_t = K_t X_0$ for some family K_t of functions on S_0. The field C_t is a characteristic vector field of (ω_t), and to prove that X_0 is also a characteristic vector field of (ω_t) one has show that the function K_t vanishes at no points of S_0. The C^{q-r-1}-closeness of ω_t to ω_0 implies the C^{q-r-2}-closeness of C_t to X_0. The r-jet of X_0 vanishes at no points of S_0, therefore the relation $C_t = K_t X_0$ implies that the function K_t is C^{q-2r-2}-close to 1. Since $q \geq 2r + 2$ then K_t vanishes at no points of M. Theorem 2 is proved.

7. Proof of Propositions 1 and 2.

The proof is based on the analysis of singularities of 1-forms in [Mar] and [Zh] and the following simple facts:

(a) The global s-division property reduces, in the C^∞ category, to the local s-division property (due to the division of unity);

(b) If X is a germ at 0 of a vector field vanishing at $0 \in \mathbb{R}^n$ then the simplest sufficient condition guarantying the s-division property of X is the existence of at least $(s+1)$ differentially independent germs in the ideal I_X generated by the function germs $f_1, \ldots, f_n$ that are the coefficients of X in a coordinate system near 0 (the ideal does not depend on the choice of a coordinate system).

Lemma 7.1, [Mar]. *Let $k \geq 2$, $0 \leq p \leq k-2$. The set of 1-jets $0 \in \mathbb{R}^{2k}$ of 1-forms ω on $\mathbb{R}^{2k}$ such that $\omega(0) \neq 0$ and*

$$\left(\omega \wedge (d\omega)^{k-1-p}\right)(0) = 0 \tag{7.1}$$

is a stratified submanifold of codimension $(p+1)(2p+3)$ of the manifold consisting of all 1-jets at 0 of 1-forms on $\mathbb{R}^{2k}$.

By this lemma the set of 1-jets at 0 of 1-forms such that

$$\left(\omega \wedge (d\omega)^{k-1}\right)(0) = 0 \tag{7.1}$$

is a stratified submanifold of codimension 3 in the space of all 1-jets at 0. A characteristic vector field of (ω) vanishes at 0 if and only if (7.1) holds. This explains the following statement.

Lemma 7.2, [Zh, sec.25,13]. *In the space of all 2-jets at 0 of 1-forms on $\mathbb{R}^{2k}$, $k \geq 2$, there is a stratified submanifold E of codimension $(2k+1)$ such that any 1-form ω whose 2-jet at 0 is away from E satisfies the following property. Let I be the ideal generated by the coefficients of the germ at 0 of a characteristic vector field X of (ω). If $X(0) = 0$ and $\omega(0) \neq 0$ then I is generated by 3 differentially independent at 0 functions provided that*

$$\left(\omega \wedge (d\omega)^{k-2}\right)(0) \neq 0. \tag{7.2}$$

If $\omega(0) = 0$ (and consequently $X(0) = 0$) then I is generated by $2k$ differentially independent at 0 functions.

By Lemma 7.1 the degeneration (7.2) has codimension 10. Therefore Proposition 1 follows from Lemmas 7.1, 7.2, the transversality theorem and the mentioned above sufficient condition for the s-division property, $s = 2$.

In the odd-dimensional case, the main lemma used in the proof of Proposition 2 is as follows.

Lemma 7.3, [Mar]. *Let $0 \leq p \leq k-1$. The set J_p consisting of 1-jets at a nonfixed point $x \in M^{2k+1}$ of 1-forms ω on M^{2k+1} such that $\omega(x) \neq 0$ and*

$$\left(\omega \wedge (d\omega)^{k-p}\right)(x) = 0 \tag{7.4}$$

is a stratified submanifold of codimension $(p+1)(2p+1)$ in the space J of all 1-jets (at a nonfixed point of the manifold). The set J_0 is not a smooth hypersurface near a point $j^1_x\omega$ if and only if

$$\left(\omega \wedge (d\omega)^{k-1}\right)(x) = 0 \tag{7.4}$$

Consequently, the set of points at which J_0 is not smooth is a stratified codimension 6 submanifold of J.

The condition (a) of Theorem 2 (the structural smoothness of the Martinet hypersurface) means that the set J_0 is a smooth hypersurface near the point $j^1_p\omega_0$ and the mapping $x \to j^1_x\omega_0$ is transversal to J_0. Therefore by the transversality theorem and Lemma 7.3 the condition (a) of Theorem 2 holds for an open and dense subset of the space of all nonvanishing 1-forms on $M = M^{2k+1}$ in the C^2-topology provided that $dim M \leq 5$.

The genericity of the conditions (c) and (d) of Theorem 2 for the case where $dim M \leq 5$ and ω_0 vanishes at no points of the manifold follows from the genericity of the condition (a) and the following lemma.

Lemma 7.4, [Zh, sec.17]. *In the space of all 3-jets at 0 of 1-forms on $\mathbb{R}^{2k+1}$ there is a set E of codimension $2k+2$ such that any 1-form ω whose 3-jet at 0 is away from E satisfies the following property. If $\omega(0) \neq 0$ and the Martinet hypersurface S is structurally smooth at 0 (i.e. $j^1_0\left(\omega \wedge (d\omega)^k\right) \neq 0$) then the ideal generated by the coefficients of the germ at 0 of a characteristic vector field X of (ω) is generated by 2 differentially independent at 0 function germs.*

Lemmas 7.3 and 7.4 prove Proposition 2 for the case where (ω_0) is a hyperplane field, i.e. ω_0 vanishes at no points of the manifold. To complete the proof I note that the conditions (a)-(d) of Theorem 2 can be easily checked for the example where the germ of at p of ω_0 has the normal form

$$(x_1dy_1 - y_1dx_1) + \cdots + (x_kdy_k - y_kdx_k) + d\phi(x,y) \pm zdz$$

in some coordinate system centered at p, where $\phi(x,y)$ is a generic quadratic function, and use a result in [Zh, sec. 13] stating that this normal form serves for all germs at 0 of 1-forms ω on $\mathbb{R}^{2k+1}$ such that $\omega(0) = 0$ and $j^2_0\omega$ does not belong to a certain (fixed) stratified submanifold of codimension $(2k+2)$ of the space of all 2-jets at 0.

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