

# **Modular Forms and Differential Operators**

**Don Zagier**

Max-Planck-Institut für Mathematik  
Gottfried-Claren-Straße 26  
53225 Bonn  
Germany



## ABSTRACT

In 1956, Rankin described which polynomials in the derivatives of modular forms are again modular forms, and in 1977, H. Cohen defined for each  $n \geq 0$  a bilinear operation which assigns to two modular forms  $f$  and  $g$  of weight  $k$  and  $l$  a modular form  $[f, g]_n$  of weight  $k + l + 2n$ . In the present paper we study these "Rankin-Cohen brackets" from two points of view. On the one hand we give various explanations of their modularity and various algebraic relations among them by relating the modular form theory to the theories of theta series, of Jacobi forms, and of pseudodifferential operators. In a different direction, we study the abstract algebraic structure ("RC algebra") consisting of a graded vector space together with a collection of bilinear operations  $[ , ]_n$  of degree  $+2n$  satisfying all of the axioms of the Rankin-Cohen brackets. Under certain hypotheses, these turn out to be equivalent to commutative graded algebras together with a derivation  $\partial$  of degree 2 and an element  $\Phi$  of degree 4, up to the equivalence relation  $(\partial, \Phi) \sim (\partial - \phi E, \Phi - \phi^2 + \partial(\phi))$  where  $\phi$  is an element of degree 2 and  $E$  is the Euler operator (= multiplication by the degree).

# MODULAR FORMS AND DIFFERENTIAL OPERATORS

DON ZAGIER

The derivative of a modular form is not a modular form. Nevertheless, there are many interesting connections between differential operators and the theory of modular forms. For instance, every modular form (by which we shall always mean a holomorphic modular form in one variable of integral weight) satisfies a non-linear third order differential equation with constant coefficients; in another direction, if such a form  $f(\tau)$  is expressed as a power series  $\varphi(t(\tau))$  in a local parameter  $t(\tau)$  which is a meromorphic modular function of  $\tau$ , then the power series  $\varphi(t)$  satisfies a linear differential equation of order  $k + 1$  with algebraic coefficients, where  $k$  is the weight of  $f$ . This latter fact, which leads to many connections between the theory of modular forms and the theory of hypergeometric and other special differential equations, played an important role in the development of both theories in the 19th century and up to the work of Fricke and Klein, but surprisingly little role in more modern investigations.

In 1956, R.A. Rankin [Ra] gave a general description of the differential operators which send modular forms to modular forms. A very interesting special case of this general setup are certain bilinear operators on the graded ring  $M_*(\Gamma)$  of modular forms on a fixed group  $\Gamma \subset PSL(2, \mathbf{R})$  which were introduced by H. Cohen [Co] and which have had many applications since then. These operators, which we call the Rankin-Cohen brackets, will be the main object of study in the present paper. On the one hand, we will be interested in understanding from various points of view "why" these operators on modular forms have to exist. These different approaches (in particular, via Jacobi forms and via pseudodifferential operators) give different explanations and even different definitions of the operators, and although these definitions differ only by constants, the constants turn out to depend in a subtle way on the parameters involved and to lead to quite complicated combinatorial problems. On the other hand, we will try to understand what kind of an additional algebraic structure these operators give to the ring  $M_*(\Gamma)$  and what other examples of the same algebraic structure can be found in (mathematical) nature. We will give a partial structure theorem showing that the algebraic structure in question is more or less equivalent to that of a graded algebra together with a derivation of degree 2 and an element of degree 4. The results will be far from definitive, our main object being to formulate certain questions and perhaps arouse some interest in them.

**§1. The Rankin-Cohen bilinear operators.** Let  $f(\tau)$  and  $g(\tau)$  denote two modular forms of weight  $k$  and  $l$  on some group  $\Gamma \subset PSL(2, \mathbf{R})$ . We denote by  $D$  the differential operator  $\frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$  (where  $q = e^{2\pi i \tau}$  as usual) and use  $f', f'', \dots, f^{(n)}$  freely instead of  $Df, D^2f, \dots, D^n f$ . The  $n$ th Rankin-Cohen bracket of  $f$  and  $g$  is defined by the formula

$$[f, g]_n(\tau) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} f^{(r)}(\tau) g^{(s)}(\tau). \quad (1)$$

(The normalization here is different from that in [Co] and has been chosen so that  $[f, g]_n$  is in  $\mathbf{Z}[[q]]$  if  $f$  and  $g$  are.) The basic fact is that this is a modular form of weight  $k + l + 2n$  on  $\Gamma$ , so that

the graded vector space  $M_*(\Gamma)$  possesses not only the well-known structure as a commutative graded ring, corresponding to the 0th bracket, but also an infinite set of further bilinear operations  $[\cdot, \cdot]_n : M_* \otimes M_* \rightarrow M_{*+*+2n}$ . We shall be interested in seeing what kind of an algebraic structure this is and where other examples of such a structure arise. Let us start by recalling why  $[f, g]_n$  is a modular form. There are at least two ways to see this.

The first is to associate to the modular form  $f(\tau)$  the formal power series

$$\tilde{f}(\tau, X) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\tau)}{n!(n+k-1)!} (2\pi i X)^n \quad (2)$$

introduced by Kuznetsov [Ku] and Cohen [Co]. Then the higher brackets of  $f$  and  $g$  are given by

$$\tilde{f}(\tau, -X) \tilde{g}(\tau, X) = \sum_{n=0}^{\infty} \frac{[f, g]_n(\tau)}{(n+k-1)!(n+l-1)!} (2\pi i X)^n \quad (f \in M_k, g \in M_l). \quad (3)$$

On the other hand,  $\tilde{f}$  satisfies the transformation law ([Ku], Theorem 1, [Co], Theorem 7.1a)

$$\tilde{f}(\gamma(\tau), \frac{X}{(c\tau+d)^2}) = (c\tau+d)^k e^{cX/(c\tau+d)} \tilde{f}(\tau, X) \quad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \gamma(\tau) := \frac{a\tau+b}{c\tau+d}). \quad (4)$$

Indeed, this identity is equivalent by comparison of coefficients to the sequence of identities

$$\frac{f^{(n)}(\gamma(\tau))}{n!(n+k-1)!} = \sum_{m=0}^n \frac{(2\pi i c)^{n-m} (c\tau+d)^{k+n+m}}{(n-m)!} \frac{f^{(m)}(\tau)}{m!(m+k-1)!} \quad (n \geq 0),$$

and these are easily proved by induction on  $n$ . [For a non-inductive proof of (4), observe that (2) is the unique power series solution of the differential equation  $(\frac{\partial}{\partial \tau} - k \frac{\partial}{\partial X} - X \frac{\partial^2}{\partial X^2}) \tilde{f} = 0$  with initial conditions  $\tilde{f}(\tau, 0) = (k-1)!^{-1} f(\tau)$ , and verify that  $(c\tau+d)^{-k} e^{-cX/(c\tau+d)} \tilde{f}(\gamma(\tau), X/(c\tau+d)^2)$  satisfies the same conditions.] Now identity (4) and the corresponding formula for  $\tilde{g}$  imply that the product occurring on the left-hand side of (3) is multiplied by  $(c\tau+d)^{k+l}$  under the transformation  $(\tau, X) \mapsto (\gamma(\tau), (c\tau+d)^{-2}X)$  (the exponential factors drop out because of the minus sign in (3)), and this says that the coefficient of  $X^n$  in this product transforms like a modular form of weight  $k+l+2n$  for all  $n$ . Since the holomorphy at the cusps is also easy to check, this proves the assertion.

For the second proof, which will also make it clear that the operator  $[\cdot, \cdot]_n$  is the only bilinear differential operator of degree  $2n$  sending modular forms to modular forms—a fact which can be seen in many other ways—is to look at the effect of this operator on theta series. Recall that if  $Q : \mathbf{Z}^m \rightarrow \mathbf{Z}$  is a positive definite quadratic form in  $m$  variables and  $P : \mathbf{Z}^m \rightarrow \mathbf{C}$  a spherical function of even degree  $d$  with respect to  $Q$  (i.e. a homogeneous polynomial of degree  $d$  in  $m$  variables which is annihilated by the Laplacian  $\Delta_Q$  associated to  $Q$ ), then the theta series  $f(\tau) = \Theta_{Q,P}(\tau) = \sum_{x \in \mathbf{Z}^m} P(x) q^{Q(x)}$  is a modular form of weight  $k = d + m/2$  on some subgroup  $\Gamma \subset PSL(2, \mathbf{Z})$  of finite index. If  $g = \Theta_{Q',P'}$  is a second such theta series of weight  $l = d' + m'/2$ , then the function

$$h(\tau) = \sum_{r+s=n} c_{r,s} f^{(r)}(\tau) g^{(s)}(\tau) = \sum_{(x,x') \in \mathbf{Z}^{m+m'}} \left( P(x)P'(x') \sum_{r+s=n} c_{r,s} Q(x)^r Q'(x')^s \right) q^{Q(x)+Q'(x')}$$

will be a modular form (of weight  $k+l+2n$ ) if and only if the homogeneous polynomial of weight  $d+d'+2n$  appearing in parentheses is spherical with respect to the combined Laplacian  $\Delta_Q + \Delta_{Q'}$ . But a short calculation, facilitated by choosing coordinates in which  $Q(x) = \sum_{i=1}^m x_i^2$ , shows that  $\Delta_Q(P(x)Q(x)^r)$  equals  $4r(r+k-1)P(x)Q(x)^{r-1}$ , so this will happen if and only if  $r(r+k-1)c_{r,s} + (s+1)(s+l)c_{r-1,s+1}$  vanishes for all  $r$  and  $s$ , i.e., if the  $c_{r,s}$  are proportional to  $(-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r}$ .

**§2. Algebraic properties of the Rankin-Cohen brackets.** The brackets introduced in §1 satisfy a number of algebraic identities. First, we have the obvious (anti-)commutativity property

$$[f, g]_n = (-1)^n [g, f]_n, \quad (5)$$

for all  $n$ . The 0th bracket, as already mentioned, is usual multiplication, so satisfies the identities

$$[[f, g]_0, h]_0 = [f, [g, h]_0]_0 \quad (6)$$

making  $(M_*, [, ]_0)$  into a commutative and associative algebra. We also have the formulas

$$[f, 1]_0 = [1, f]_0 = f, \quad [f, 1]_n = [1, f]_n = 0 \quad (n > 0) \quad (7)$$

(because the binomial coefficient  $\binom{n-1}{n}$  in (1) is zero), which say that the unit of this algebra structure has trivial higher brackets with all of  $M_*$ . The 1st bracket, given by

$$[f, g]_1 = -[g, f]_1 = kfg' - lf'g \in M_{k+l+2} \quad (f \in M_k, g \in M_l),$$

satisfies the Jacobi identity

$$[[fg]_1h]_1 + [[gh]_1f]_1 + [[hf]_1g]_1 = 0, \quad (8)$$

giving  $M_{*-2}$  the structure of a graded Lie algebra. (From now on, we often drop the comma in the notation for the brackets.) The double brackets  $[[\cdot]_0]_1$  and  $[[\cdot]_1]_0$  satisfy the identities

$$[[fg]_0h]_1 + [[gh]_0f]_1 + [[hf]_0g]_1 = 0 \quad (9)$$

and

$$m[[fg]_1h]_0 + l[[gh]_1f]_0 + k[[hf]_1g]_0 = 0 \quad (f \in M_k, g \in M_l, h \in M_m), \quad (10)$$

(the first one in which the weights play a role) as well as the mixed relations

$$[[fg]_0h]_1 = [[gh]_1f]_0 - [[hf]_1g]_0, \quad (k+m+l)[[fg]_1h]_0 = k[[hf]_0g]_1 - l[[gh]_0f]_1, \quad (11)$$

the first of which says that the Lie bracket with a fixed element of  $M_*$  acts as a derivation with respect to the associative algebra structure  $[\cdot, \cdot]_0$ . (A space having simultaneously the structures of an associative and a Lie algebra, with the latter acting via derivations on the former, is called a *Poisson algebra*.) The relations (6)–(11), which are not all independent, describe all identities relating the 0th and 1st brackets. At the next level, the relations involving the second bracket

$$[f, g]_2 = \binom{k+1}{2} fg'' - (k+1)(l+1) f'g' + \binom{l+1}{2} f''g \in M_{k+l+4} \quad (f \in M_k, g \in M_l)$$

are already quite complicated. Starting with  $f \otimes g \otimes h \in M_k \otimes M_l \otimes M_m$  we can already make nine trilinear expressions of weight  $k+l+m+4$ , namely  $[[fg]_0h]_2$ ,  $[[fg]_1h]_1$ ,  $[[fg]_2h]_0$  and their cyclic permutations. (The non-cyclic permutations give the same elements up to sign by (5).) The space they span has dimension 3, a basis being given by the first or the last group, which are mutually related by

$$(k+1)(l+1)[[fg]_0h]_2 = -m(m+1)[[fg]_2h]_0 + (k+1)(k+l+1)[[gh]_2f]_0 + (l+1)(k+l+1)[[hf]_2g]_0 \quad (12a)$$

$$(k+l+m+1)(k+l+m+2)[[fg]_2h]_0 = (k+1)(l+1)[[fg]_0h]_2 - (k+1)(k+l+1)[[gh]_0f]_2 - (l+1)(k+l+1)[[hf]_0g]_2, \quad (12b)$$

while the second group (which is linearly dependent by virtue of the Jacobi identity (8)) is expressed in terms of these by

$$[[fg]_1h]_1 = [[gh]_0f]_2 - [[hf]_0g]_2 + [[gh]_2f]_0 - [[hf]_2g]_0. \quad (13)$$

Of course we could go on in this way, giving more and more axioms for the bracket operations of various degrees. However, it is not obvious how the whole set of relations looks, or even when we have a complete defining set for a bracket of given order. For instance, although the bracket  $[\cdot]_2$  satisfies no trilinear relations like (6) or (8), a simple dimension count shows that the permutations of the  $r$ -fold 2-brackets  $[\dots[[fg]_2h]_2\dots]_2$  are linearly dependent for all sufficiently large  $r$ , but it is not clear how far we would have to go to get the first relation or how much further to ensure that all subsequent relations obtained would be consequences of ones already found. In §3 we will give an infinite collection of trilinear relations among the Rankin-Cohen brackets which possibly may generate all relations, though we do not know this.

However, even not knowing a complete (let alone minimal) collection of universal identities satisfied by the Rankin-Cohen brackets, one can investigate the class of graded vector spaces having bracket operations which satisfy these identities and try to elucidate their structure. This will be done in §5. First, however, we look at two other structures on modular forms which give new explanations of the existence of the bracket operations (1) and shed further light on their algebraic nature.

**§3. Rankin-Cohen operators and Jacobi-like forms.** We fix a subgroup  $\Gamma$  of  $PSL(2, \mathbf{R})$ . For each integer  $k > 0$  let  $J_k = J_k(\Gamma)$  be the set of all holomorphic functions  $\phi(\tau, X)$  on  $\mathcal{H} \times \mathbf{C}$  ( $\mathcal{H} =$  upper half-plane) satisfying

$$\phi(\gamma(\tau), \frac{X}{(c\tau + d)^2}) = (c\tau + d)^k e^{cX/(c\tau + d)} \phi(\tau, X) \quad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma) \quad (14)$$

(i.e., equation (4) with  $\phi$  in place of  $\tilde{f}$ ) as well as the usual holomorphy conditions at the cusps. We call the elements of  $J_k$  *Jacobi-like of weight  $k$*  because they satisfy one of the two characteristic functional equations of Jacobi forms. (The other one, which does not concern us here, involves translations of  $z$  by elements of the lattice  $\mathbf{Z}\tau + \mathbf{Z}$ , where  $X$  is proportional to  $z^2$ . See [EZ] for the theory of Jacobi forms, and in particular §3 of [EZ] for many calculations related to the ones here.)

Clearly the restriction of a Jacobi-like form to  $X = 0$  is a modular form of weight  $k$  on  $\Gamma$ , and the kernel of this map  $J_k \rightarrow M_k = M_k(\Gamma)$  is just  $X$  times  $J_{k+2}(\Gamma)$ . The Kuznetsov-Cohen functional equation (4) says that we have a canonical section  $f \rightarrow (k-1)!\tilde{f}$  of  $J_k \rightarrow M_k$ , so that the sequence

$$0 \rightarrow J_{k+2}(\Gamma) \xrightarrow{\cdot X} J_k(\Gamma) \xrightarrow{X=0} M_k(\Gamma) \rightarrow 0$$

is exact and splits canonically. This implies that there is a bijection between Jacobi-like forms of weight  $k$  and sequences of modular forms of weight  $k + 2n$  ( $n \geq 0$ ). Then the multiplication of Jacobi-like forms induces bilinear pairings  $M_* \otimes M_* \rightarrow M_{*+*+2n}$ , and these must be multiples of the Rankin-Cohen brackets. We now look at the details.

If we write  $\phi(\tau, X) \in J_k(\Gamma)$  as  $\sum_{n=0}^{\infty} \phi_n(\tau) (2\pi i X)^n$ , then comparing coefficients of  $X^n$  in the defining functional equation (14) gives the functional equations

$$(c\tau + d)^{-k-2n} \phi_n(\gamma(\tau)) = \sum_{m=0}^n \frac{1}{m!} \left( \frac{1}{2\pi i} \frac{c}{c\tau + d} \right)^m \phi_{n-m}(\tau) \quad (n \geq 0, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma), \quad (15)$$

and conversely any sequence of holomorphic functions  $\phi_n(\tau)$  satisfying (15) and a growth condition at cusps defines an element of  $J_k(\Gamma)$ . Equations (15) are in turn equivalent to the sequence of transformation laws

$$\phi_0 \in M_k, \quad k\phi_1 - \phi'_0 \in M_{k+2}, \quad 2(k+2)(k+1)\phi_2 - 2(k+1)\phi'_1 + \phi''_0 \in M_{k+4} \dots \quad (16)$$

and in general

$$h_n := \sum_{m=0}^n (-1)^m \frac{(2n-m+k-2)!}{m!} \phi_{n-m}^{(m)} \in M_{k+2n} \quad (n \geq 0). \quad (17)$$

This can be proved from (15) by induction on  $n$  just as (4) was proved, or alternatively deduced from (4), since a simple binomial coefficient identity lets us invert (17) to write

$$\phi_n(\tau) = \sum_{r+m=n} \frac{2m+k-1}{r!(r+2m+k-1)!} h_m^{(r)}(\tau) \quad (18)$$

or equivalently as

$$\phi(\tau, X) = \sum_{n=0}^{\infty} (2n+k-1) \tilde{h}_n(\tau, X) (2\pi i X)^n \quad (19)$$

and then the modularity of  $h_n$  follows inductively from (4) (applied to  $\tilde{h}_{n'}$ ,  $n' < n$ ) and the Jacobi-like property of  $\phi$ . Equations (17) and (18) realize the afore-mentioned bijection between  $J_k(\Gamma)$  and  $\prod_n M_{k+2n}(\Gamma)$ . Now to get the bracket operations we consider the Cohen-Kuznetsov lifts of two modular forms  $f \in M_k$ ,  $g \in M_l$ . If  $\alpha$  and  $\beta$  are two complex numbers, then the product  $\phi(\tau) = \tilde{f}(\tau, \alpha X) \tilde{g}(\tau, \beta X)$  will be Jacobi-like with respect to the variable  $(\alpha+\beta)X$ , since the exponential factors in (14) multiply. The case  $\alpha + \beta = 0$  (when we can normalize to  $\beta = -\alpha = 1$ ) was the case used to obtain the Rankin-Cohen brackets in §1. If  $\alpha + \beta$  is different from 0, we can normalize it to be equal to 1 by rescaling  $X$ . Then  $\phi$  belongs to  $J_{k+l}$  and has an expansion of the form (19). The coefficient of  $(2\pi i X)^n$  in  $\phi$  is given by

$$\phi_n(\tau) = \sum_{r+s=n} \frac{\alpha^r \beta^s}{r!s!(r+k-1)!(s+l-1)!} f^{(r)}(\tau) g^{(s)}(\tau),$$

so by Leibniz's rule the modular forms  $h_n$  defined by (17) (with  $k+l$  in place of  $k$ ) are given by

$$h_n(\tau) = \sum_{p+q+r+s=n} \frac{(2n-p-q+k+l-2)!}{p!q!r!s!(r+k-1)!(s+l-1)!} \alpha^r \beta^s f^{(p+r)}(\tau) g^{(q+s)}(\tau) \quad (n \geq 0), \quad (20)$$

This is a combination of products of derivatives of  $f$  and  $g$  which is modular of weight  $k+l+2n$  and hence must be a multiple  $\kappa_n = \kappa_n(k, l; \alpha, \beta)$  of the Rankin-Cohen bracket  $[f, g]_n$ , so as  $\alpha$  and  $\beta = 1 - \alpha$  vary we get infinitely many explanations of the existence of these brackets. Our next job is to compute the scalar  $\kappa_n$ .

We define for each  $n$  a polynomial  $H_n$  of four variables, of degree  $n$  in the first two and homogeneous of degree  $n$  in the last two, by

$$H_n(k, l; X, Y) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} X^r Y^s, \quad (21)$$

so that equation (1) can be rewritten as

$$[f, g]_n = H_n(k, l; D_{\tau_1}, D_{\tau_2})(f(\tau_1)g(\tau_2))\Big|_{\tau_1=\tau_2=\tau} \quad (f \in M_k, \quad g \in M_l). \quad (22)$$

The polynomials  $H_n$ , whose definition can also be written

$$H_n(k, l; X, Y) = \frac{1}{n!} \left( -X \frac{\partial}{\partial \eta} + Y \frac{\partial}{\partial \xi} \right)^n (\xi^{k+n-1} \eta^{l+n-1}) \Big|_{\xi=\eta=1},$$

satisfy many algebraic identities. We mention in particular

$$H_n(k, l; X, Y) = (-1)^n H_n(l, k; Y, X), \quad (23)$$

$$H_n(k, l; X, Y) = H_n(l, -k-l-2n+2; Y, -X-Y), \quad (24)$$

$$\begin{aligned} & \frac{n!(n+k+l-2)!}{(n+k-1)!(n+l-1)!} H_n(k, l; X, Y) H_n(k, l; \alpha, \beta) \\ &= \sum_{r+s+t=n} \frac{(2r+2s+t+k+l-2)!}{r!s!t!(r+k-1)!(s+l-1)!} (\alpha X)^r (\beta Y)^s (-\gamma Z)^t \quad (\alpha + \beta + \gamma = X + Y + Z = 0). \end{aligned} \quad (25)$$

The first two of these say that the 6-argument function

$$\begin{bmatrix} X & Y & Z \\ k & l & m \end{bmatrix} = H_n(k, l; X, Y) \quad (k+l+m = 2-2n, \quad X+Y+Z = 0)$$

is symmetric under even and  $(-1)^n$ -symmetric under odd permutations of its three columns, and the third (for  $k+l \notin \mathbf{Z}$ ) can then be rewritten more symmetrically as

$$\sum_{r+s+t=n} \frac{(\alpha X)^r (\beta Y)^s (\gamma Z)^t}{r!s!t!(r+k-1)!(s+l-1)!(t+m-1)!} = \frac{\begin{bmatrix} X & Y & Z \\ k & l & m \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \\ k & l & m \end{bmatrix}}{(n+k-1)!(n+l-1)!(n+m-1)!},$$

where  $x!$  denotes  $\Gamma(x+1)$  for  $x \notin \mathbf{Z}$ .

Identity (23) is trivial. To prove (24) and (25), we observe that  $H_n = H_n(k, l; X, Y)$  satisfies the differential equations

$$\left( X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} \right) H_n = 0, \quad \left( k \frac{\partial}{\partial X} + \frac{\partial^2}{\partial X^2} + l \frac{\partial}{\partial Y} + \frac{\partial^2}{\partial Y^2} \right) H_n = 0 \quad (26)$$

(the first is Euler's equation saying that  $H_n$  is homogeneous of degree  $n$  in  $X$  and  $Y$ , and the second was already used implicitly in the proof of modularity of  $[\Theta_{P,Q}, \Theta_{P',Q'}]_n$  in §1), and these characterize  $H_n$  uniquely up to a scalar factor as a function of  $X$  and  $Y$ . Thus to prove (24) we verify, using (26), that the right-hand side also satisfies (26), and then fix the normalizations by taking  $Y = 0$  and using (23). Similarly, to prove (25) we verify that the expression on the right satisfies (26) and hence is a multiple (depending on  $\alpha$  and  $\beta$ ) of  $H_n(k, l; X, Y)$ ; by the symmetry in  $(X, Y, Z)$  and  $(\alpha, \beta, \gamma)$ , this multiple must be a scalar multiple  $\lambda_n(k, l)$  of  $H_n(k, l; \alpha, \beta)$ , and the value of  $\lambda_n(k, l)$  is fixed by

specializing to  $\alpha = Y = 0$ . One can also prove both identities using generating functions; for instance, we have

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(k-n+1, l; X, Y) T^n &= \sum_{r,s \geq 0} (-1)^r \binom{k}{s} \binom{r+s+l-1}{r} X^r Y^s \\ &= \sum_{s \geq 0} \binom{k}{s} (TY)^s (1+TX)^{-l-s} = \frac{(1-TZ)^k}{(1+TX)^{k+l}} \quad (Z = -X - Y) \end{aligned}$$

and hence  $H_n(k-n+1, l; X, Y) = (-1)^n H_n(-k-l-n+1, l; Z, Y)$ , which is equivalent to (24).

Now returning to (20), where  $\alpha + \beta = 1$ , we see from (25) and (21) that

$$h_n(\tau) = \frac{n!(n+k+l-2)!}{(n+k-1)!(n+l-1)!} H_n(k, l; \alpha, \beta) [f, g]_n(\tau).$$

(This actually gives another proof of (25), since we already knew that  $h_n$  had to be a multiple of  $[f, g]_n$ , so the right-hand side of (25) *must* be a multiple of  $H_n(k, l; X, Y)$ .) Changing to the inhomogeneous notation, we can summarize what we have proved as

**Proposition.** For  $f \in M_k(\Gamma)$ ,  $g \in M_l(\Gamma)$  ( $k, l > 0$ ) and  $\alpha, \beta \in \mathbf{C}$  we have the identity

$$\tilde{f}(\tau, \alpha X) \tilde{g}(\tau, \beta X) = \sum_{n=0}^{\infty} c_n(k, l; \alpha, \beta) \widetilde{[f, g]_n}(\tau, (\alpha + \beta)X) (2\pi i X)^n \quad (27)$$

with

$$c_n = (2n+k-1) \frac{n!(n+k+l-2)!}{(n+k-1)!(n+l-1)!} H_n(k, l; \alpha, \beta). \quad (28)$$

Applying this proposition twice, we find that, if  $h \in M_m(\Gamma)$  is a third modular form on  $\Gamma$ , then

$$\tilde{f}(\tau, \alpha X) \tilde{g}(\tau, \beta X) \tilde{h}(\tau, \gamma X) = \sum_{n,p \geq 0} c_n(k, l; \alpha, \beta) c_p(k+l+2n, m; \alpha+\beta, \gamma) \tilde{F}_{n,p}(\tau, (\alpha+\beta+\gamma)X) (2\pi i X)^{n+p}$$

with  $F_{n,p}(\tau) = [[f, g]_n h]_p$ . Since the expression on the left is symmetric in its arguments, we get:

**Corollary.** For  $f \in M_k(\Gamma)$ ,  $g \in M_l(\Gamma)$  and  $h \in M_m(\Gamma)$  and  $\alpha, \beta, \gamma \in \mathbf{C}$ , the expression

$$\sum_{n=0}^r c_n(k, l; \alpha, \beta) c_{r-n}(k+l+2n, m; \alpha+\beta, \gamma) [[f, g]_n, h]_{r-n} \in M_{k+l+m+2r}(\Gamma) \quad (r \in \mathbf{Z}_{\geq 0}),$$

with  $c_n$  given by (28), is symmetric under all permutations of  $(f, k, \alpha)$ ,  $(g, l, \beta)$ ,  $(h, m, \gamma)$ .

Varying  $r$  and comparing coefficients of the various monomials in  $\alpha$ ,  $\beta$  and  $\gamma$ , we systematically obtain in this way universal identities satisfied by the Rankin-Cohen brackets of the sort studied in §2. For instance, the triple brackets  $[[fg]_* h]_*$  can always be expressed (in general, in many ways) as linear combinations of the triple brackets  $[[fh]_* g]_*$ .

Finally, we mention that combinatorial identities similar to (24) and (25) occur, in a somewhat related context, in the paper [IZ].

**§4. The Rankin-Cohen operators and pseudodifferential operators.** This connection was suggested to the author by Yu. Manin and will be treated in detail in the joint paper [MZ], so we give only a few indications.

Let  $D$  as before be the differential operator  $(2\pi i)^{-1} d/d\tau$ . (The factor  $2\pi i$ , introduced earlier for convenience, is more of a nuisance now, but we will let it be.) Then by a formal *pseudodifferential operator* we mean a formal power series  $\sum_{n=0}^{\infty} g_n(\tau) D^{-n}$  where the  $g_n$  are holomorphic functions in the upper half-plane. We can multiply two such series by Leibniz's rule

$$\left( \sum_{m=0}^{\infty} f_m(\tau) D^{-m} \right) \left( \sum_{n=0}^{\infty} g_n(\tau) D^{-n} \right) = \sum_{m,r,n \geq 0} \binom{-m}{r} f_m(\tau) g_n^{(r)}(\tau) D^{-m-r-n},$$

and the pseudodifferential operators in this way form an associative, but of course not commutative, ring.

Now if we consider some modular group  $\Gamma$  acting on the upper half-plane, then  $\Gamma$  also acts on  $D$  via  $D \mapsto (c\tau + d)^2 D$ , so it makes sense to speak of a pseudodifferential operator  $\sum_{n=0}^{\infty} g_n(\tau) D^{-n}$  being  $\Gamma$ -invariant. If  $n_0 > 0$  is the smallest index with  $g_{n_0} \neq 0$  for such an operator, then it is easily seen that  $g_{n_0}$  is a modular form of weight  $k = 2n_0$ ,  $g_{n_0+1} + \frac{1}{2}(n_0 + 1)g'_{n_0}$  is a modular form of weight  $k + 2$ , etc. This is reminiscent of the equations (15), and indeed, a calculation shows that the power series

$$\sum_{n=n_0}^{\infty} \frac{g_n(\tau)}{n!(n-1)!} (-2\pi i X)^{n-n_0}$$

belongs to  $J_k(\Gamma)$ , setting up a 1:1 correspondence between invariant pseudodifferential operators of the form  $\sum_{n \geq n_0} g_n D^{-n}$  and Jacobi-like forms of weight  $k$ . Combining this with the Kuznetsov-Cohen lifting (4), we find that there is a canonical lifting

$$f(\tau) \mapsto \mathcal{D}[f] = \sum_{r=0}^{\infty} (-1)^r \frac{(r+k/2)!(r+k/2-1)!}{r!(r+k-1)!} f^{(r)}(\tau) D^{-r-k/2} \quad (f \in M_k, \quad k > 0 \text{ even})$$

from modular forms to pseudodifferential operators, and that conversely any  $\Gamma$ -invariant pseudodifferential operator can be expanded as a sum of such lifts. In particular, since the product of two  $\Gamma$ -invariant pseudodifferential operators is another one, we can associate to two modular forms  $f \in M_k$ ,  $g \in M_l$  a sequence of modular forms  $\{h_n\}_{n \geq 0}$  via

$$\mathcal{D}[f] \cdot \mathcal{D}[g] = \sum_{n=0}^{\infty} \mathcal{D}[h_n] \quad (h_n \in M_{k+l+2n}).$$

Then, just as in §3, the uniqueness of the Rankin-Cohen brackets implies that  $h_n$  must be some universal factor  $t_n = t_n(k, l)$  of  $[f, g]_n$ . Since, unlike the situation in §3 where the definition of the modular forms  $h_n$  depended on an arbitrary parameter  $\alpha$ , the present operation is completely canonical, one would expect the scalar factor occurring to be very simple. Surprisingly, it is not: the combinatorial calculations needed here are far worse than the already complicated ones in §3. The formula for  $t_n(k, l)$ , as well as other aspects of the connection between pseudodifferential operators and modular forms (including a connection with super-pseudodifferential operators in the case of modular forms of odd weight), will be discussed in [MZ].

**§5. Definition and examples of Rankin-Cohen algebras.** We define a *Rankin-Cohen algebra* (or RC algebra for short) over a field  $K$  as a graded  $K$ -vector space  $M_* = \bigoplus_{k \geq 0} M_k$  (with  $M_0 = K \cdot 1$  and  $\dim_K M_k$  finite for all  $k$ ) together with bilinear operations  $[ , ]_n : M_k \otimes M_l \rightarrow M_{k+l+2n}$  ( $k, l, n \geq 0$ ) which satisfy (5)–(13) and all the other algebraic identities satisfied by the Rankin-Cohen brackets. In view of the remarks at the end of §2, this may seem like a strange definition, since we do not know how to give a complete set of axioms. Nevertheless, we will be able to construct examples and, to a large extent, to clarify the structure of these objects. The situation should be thought of as analogous to building up the theory of Lie algebras starting with the observation that the operation  $[X, Y] = XY - YX$  in an associative algebra seems to have interesting properties. One could then define Lie algebras as algebras with a bracket satisfying all algebraic identities universally satisfied by this standard bracket in any associative algebra, and a good many results could be proved without knowing a complete generating set for these identities. One would initially be forced to look at subspaces of associative algebras closed under the standard bracket, but would eventually prove that all Lie algebras arise this way (existence of the universal enveloping algebra) and also that all universal identities satisfied by the bracket are in fact consequences of anticommutativity and the Jacobi identity. In the same way, we will start by considering RC algebras which are subspaces closed under all bracket operations of some standard examples, and then show that, under some general hypotheses, all RC algebras in fact arise in this way.

We will suppose the ground field  $K$  to be of characteristic 0 (in our examples it is usually  $\mathbf{Q}$  or  $\mathbf{C}$ ) although it is clear that the theory makes sense in any characteristic or, for that matter, even if we work over  $\mathbf{Z}$  rather than a field.

**Example 1.** Since modular forms and their derivatives do not satisfy any universal relation, the only identities satisfied by the Rankin-Cohen brackets on  $M_*(\Gamma)$  are those following from the formula (1) and Leibniz's rule. The basic example of an RC algebra is therefore given by

*Definition* Let  $R_*$  be a commutative graded algebra with unit over  $K$  together with a derivation  $D : R_* \rightarrow R_*$  of degree 2 (i.e.  $D(R_k) \subseteq R_{k+2}$  for all  $k$  and  $(fg)' = f'g + fg'$ , where as before  $f', f'', \dots, f^{(r)}$  denote  $Df, D^2f, \dots, D^r f$ ), and define  $[ , ]_{D,n}$  by

$$[f, g]_{D,n} = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} f^{(r)} g^{(s)} \in R_{k+l+2n} \quad (f \in R_k, g \in R_l). \quad (29)$$

Then  $(R_*, [ , ]_{D,n})$  is an RC algebra which we will call the *standard RC algebra on  $(R_*, D)$* .

Since a subspace of an RC algebra which contains 1 and is closed under all the bracket operations is obviously again an RC algebra, this gives us a large further class of examples, the sub-RC algebras of the standard ones. A basic question (the analogue of the question of the existence of universal enveloping algebras in the Lie algebra case) is whether every RC algebra can in fact be realized in this way. We will give an affirmative answer under a weak additional hypothesis below.

**Example 2.** Our original example of an RC algebra,  $M_*(\Gamma)$  with the brackets defined by (1), is not a standard algebra, since  $M_*(\Gamma)$  is not closed under  $D = (2\pi i)^{-1} d/d\tau$ . Of course it is a subalgebra of a standard RC-algebra in a variety of ways, since we can take  $R_*$  to be any algebra of functions on the upper half-plane which contains  $M_*(\Gamma)$  and is closed under differentiation (e.g. the space of all  $C^\infty$  or of all holomorphic functions). However, we would like an  $R_*$  which is not too big. Let us look in more detail at the case  $\Gamma = PSL(2, \mathbf{Z})$ . Here  $M_*(\Gamma) = \mathbf{C}[Q, R]$ , where  $Q = 1 + 240q + \dots$  and  $R = 1 - 504q - \dots$  are the normalized Eisenstein series of weights 4 and 6 (in Ramanujan's notation). As is well-known, their derivatives are given by  $Q' = \frac{1}{3}(PQ - R)$  and  $R' = \frac{1}{2}(PR - Q^2)$ , where  $P = 1 - 24q - \dots$  is the normalized Eisenstein series of weight 2, and since we also have

$P' = \frac{1}{12}(P^2 - Q)$  this says that  $M_*(\Gamma)$  is contained in the standard RC algebra on  $(\mathbb{C}[P, Q, R], D)$ . Now, forgetting modular forms and the interpretation of  $P, Q$  and  $R$  as functions, we can express this example purely algebraically: let  $K$  be a field of characteristic 0 and define a derivation on the polynomial algebra over  $K$  on three graded generators  $P, Q, R$  of degrees 2, 4 and 6 by

$$D = \frac{P^2 - Q}{12} \frac{\partial}{\partial P} + \frac{PQ - R}{3} \frac{\partial}{\partial Q} + \frac{PR - Q^2}{2} \frac{\partial}{\partial R} : K[P, Q, R]_* \rightarrow K[P, Q, R]_{*+2}; \quad (30)$$

then the subalgebra generated by  $Q$  and  $R$  is closed under the bracket operations  $[ \ ]_n = [ \ ]_{D,n}$  defined by (29) for all  $n \geq 0$ . From an algebraic point of view this is not at all obvious (except for  $n = 0$ ), although one can easily check a few examples, e.g. (with  $1728\Delta = Q^3 - R^2$ )

$$\begin{aligned} [Q, R]_1 &= -3456 \Delta, & [Q, \Delta]_1 &= 4 R \Delta, & [R, \Delta]_1 &= 6 Q^2 \Delta, \\ [Q, Q]_2 &= 4800 \Delta, & [Q, R]_2 &= 0, & [R, R]_2 &= -21168 Q \Delta, & [\Delta, \Delta]_2 &= -13 Q \Delta^2. \end{aligned} \quad (31)$$

**Example 3.** We try to understand the last example by observing that we also have a derivation  $\partial$  of degree 2 on the subalgebra  $M_* = K[Q, R]$  of  $R_* = K[P, Q, R]$ , defined in terms of  $D$  by

$$\partial f = Df - \frac{k}{12} P f \in M_{k+2} \quad (f \in M_k) \quad (32)$$

or directly by

$$\partial = -\frac{R}{3} \frac{\partial}{\partial Q} - \frac{Q^2}{2} \frac{\partial}{\partial R} : M_* \rightarrow M_{*+2} \quad (33)$$

(this is a well-known fact about derivatives of modular forms, but is also clear algebraically from (30)). Of course the standard RC algebra structure on  $M_*$  associated to  $\partial$  is completely different from the one inherited from  $(R_*, D)$ . But we now see that we can reconstruct  $(R_*, D)$  from  $(M_*, \partial)$  by using (32) to define  $Df$  for  $f \in M_k$  and defining  $D(P)$  as  $\frac{1}{12}(P^2 - Q)$ . We generalize this example in the following result.

**Proposition 1.** *Let  $M_*$  be a commutative and associative graded  $K$ -algebra with  $M_0 = K \cdot 1$  together with a derivation  $\partial : M_* \rightarrow M_{*+2}$  of degree 2, and let  $\Phi \in M_4$ . Define brackets  $[ \ ]_{\partial, \Phi, n}$  ( $n \geq 0$ ) on  $M_*$  by*

$$[f, g]_{\partial, \Phi, n} = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} f_r g_s \in M_{k+l+2n} \quad (f \in M_k, g \in M_l) \quad (34)$$

where  $f_r \in M_{k+2r}$ ,  $g_s \in M_{l+2s}$  ( $r, s \geq 0$ ) are defined recursively by

$$f_{r+1} = \partial f_r + r(r+k-1) \Phi f_{r-1}, \quad g_{s+1} = \partial g_s + s(s+l-1) \Phi g_{s-1} \quad (r, s \geq 0) \quad (35)$$

with initial conditions  $f_0 = f$ ,  $g_0 = g$  (so  $f_1 = \partial f$ ,  $f_2 = \partial^2 f + k\Phi f$  and similarly for  $g_s$ ). Then  $(M_*, [ \ ]_{\partial, \Phi, *})$  is an RC algebra.

**Definition.** An RC algebra will be called *canonical* if its brackets are given as in Proposition 1 for some derivation  $\partial : M_* \rightarrow M_{*+2}$  of degree +2 and some element  $\Phi \in M_4$ .

**Proof.** As already observed, our only way to verify that something is an RC algebra is to embed it into a standard RC algebra  $(R_*, [\cdot, \cdot]_{D,*})$  for some larger graded ring  $R_*$  with derivation  $D$ . We take  $R_* = M[\phi]_* := M_* \otimes_K K[\phi]$ , where  $\phi$  has degree 2, and define  $D$  by

$$D(f) = \partial(f) + k\phi f \in R_{k+2} \quad (f \in M_k), \quad D(\phi) = \Phi + \phi^2 \in R_4. \quad (36)$$

(This defines  $D$  on generators of  $R_*$ , and we extend  $D$  uniquely as a derivation.) If we show that  $[f, g]_{D,n} = [f, g]_{\partial, \Phi, n}$  for  $f$  and  $g$  in  $M_*$  then we are done, since  $M_*$  is obviously closed under the brackets  $[\cdot]_{\partial, \Phi, n}$ . To this end, we observe that the brackets  $[\cdot]_{D,n}$ , just as in §1, can be described by the generating function

$$\sum_{n=0}^{\infty} \frac{[f, g]_{D,n}}{(n+k-1)!(n+l-1)!} X^n = \tilde{f}(-X) \tilde{g}(X) \in R_*[[X]] \quad (f \in R_k, g \in R_l)$$

where

$$\tilde{f}(X) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!(n+k-1)!} X^n, \quad \tilde{g}(X) = \sum_{n=0}^{\infty} \frac{g^{(n)}}{n!(n+l-1)!} X^n.$$

(These make sense only for  $k$  and  $l$  strictly positive, but since  $M_0 = K.1$  and the brackets (34) clearly satisfy (7), there is no harm in assuming this.) We claim that

$$e^{-\phi X} \tilde{f}(X) = \sum_{r=0}^{\infty} \frac{f_r}{r!(r+k-1)!} X^r \quad (37)$$

with  $f_r$  defined by (35), and similarly of course for  $g$ ; the assertion follows immediately since the exponential terms  $e^{\pm\phi X}$  drop out in the product  $\tilde{f}(-X)\tilde{g}(X)$ .

To prove (37), we *define*  $f_r$  by the generating function (37) and prove the recursion (35) by induction (the initial condition  $f_0 = f$  is obvious). Clearly (37) is equivalent to the closed formula

$$f_r = \sum_{n=0}^r \frac{(-1)^{r-n} r!(r+k-1)!}{n!(n+k-1)!(r-n)!} \phi^{r-n} f^{(n)} \in R_{k+2r}.$$

Assume inductively that we have proved that  $f_r \in M_{k+2r}$  for some  $r$ . Then

$$\begin{aligned} \partial f_r &= f'_r - (k+2r)\phi f_r \\ &= \sum_{n=0}^r \frac{(-1)^{r-n} r!(r+k-1)!}{n!(n+k-1)!(r-n)!} [\phi^{r-n} f^{(n+1)} + (r-n)\phi^{r-n-1}(\phi^2 + \Phi)f^{(n)} - (k+2r)\phi^{r-n+1}f^{(n)}] \\ &= \sum_{n=0}^{r+1} \frac{(-1)^{r+1-n} r!(r+k-1)!}{n!(n+k-1)!(r+1-n)!} [n(n+k-1) - (r-n)(r+1-n) \\ &\quad + (k+2r)(r+1-n)] \phi^{r-n+1} f^{(n)} + \Phi \sum_{n=0}^{r-1} \frac{(-1)^{r-n} r!(r+k-1)!}{n!(n+k-1)!(r-n-1)!} \phi^{r-n-1} f^{(n)} \\ &= f_{r+1} - r(r+k-1)\Phi f_{r-1} \in M_{k+2r+2}, \end{aligned}$$

and this simultaneously proves the recursion (35) and the inductively used assumption  $f_r \in M_{k+2r}$ .  $\square$

We observe that the definition (36) is motivated by (32) in the special case  $M_* = K[Q, R]$ ,  $\phi = \frac{1}{12}P$ ,  $\Phi = \frac{-1}{144}Q$ , and that the proof just given is merely the algebraic abstraction of the proposition on page 94 of [VZ] in that case (compare also iv) and v) on page 95 of [VZ] for the case when  $M_*$  is the ring of modular forms on some other group  $\Gamma$  other than  $PSL(2, \mathbf{Z})$ .

**§6. A structure theorem for Rankin-Cohen algebras.** A priori one would not expect that a subring of a ring  $R_*$  with derivative would ever be closed under all the infinitely many bracket operations  $[\cdot]_{D,\phi,n}$  unless it were already closed under  $D$ . The only non-trivial example which we had where this happened, the rings of modular forms on subgroups of  $PSL(2, \mathbf{R})$ , has just been explained by the construction given in the Proposition above. It is then natural to expect that this construction may suffice to yield all examples of RC algebras. In this section we will show that this is “almost” true, and write down conditions under which it is exactly true.

We therefore assume given an RC algebra  $M_*$  over a field  $K$ , and want to realize its brackets as the brackets  $[\cdot]_{\partial,\Phi,n}$  for some derivation  $\partial$  of degree 2 and some element  $\Phi$  of degree 4. Since the 0th bracket makes  $M_*$  into an ordinary commutative algebra (by virtue of equations (5)–(7)), we already have a ring structure, which we will denote from now on in the usual way by juxtaposition (i.e.  $fg$  instead of  $[f, g]_0$ ). Let us assume that this ring is an integral domain, or at least that there is one homogeneous element  $F$  of some positive degree  $N$  which is not a zero-divisor, and let  $\widehat{M}_*$  be the quotient field of  $M_*$  or the ring  $M[1/F]_*$ , respectively. (It has elements of positive and negative grading and hence is not quite the kind of object considered up to now.) The compatibility of all the brackets in the case of a standard RC algebra now implies that we can canonically extend the bracket operations to  $\widehat{M}_*$ . For instance, the first equation in (11), which says that the Lie bracket  $[\cdot, h]_1$  with a fixed element  $h$  acts as a derivation with respect to the ring structure, forces us to define  $[f/F, h]_1$  as  $[f, h]_1/F - f[F, h]_1/F^2$ . We now define a derivation  $\partial: \widehat{M}_* \rightarrow \widehat{M}_{*+2}$  and an element  $\Phi \in \widehat{M}_4$  by

$$\partial(f) = \frac{[F, f]_1}{NF} \quad (f \in \widehat{M}_*), \quad \Phi = \frac{[F, F]_2}{N^2(N+1)F^2} \quad (38)$$

We claim that the brackets  $[\cdot]_{\partial,\Phi,n}$  associated to  $\partial$  and  $\Phi$  agree with the given brackets on  $\widehat{M}_*$ . Indeed, since all formal identities among brackets which are satisfied by standard RC algebras are by definition satisfied in all RC algebras, it is enough to check this for  $(M_*, [\cdot]_*)$  a subalgebra of a standard RC algebra  $(R_*, [\cdot]_{D,*})$ . The bracket  $[\cdot]_{D,*}$  extends to  $\widehat{R}_* = R_* \otimes_{M_*} \widehat{M}_*$  for the same reason as before. Define  $\phi \in \widehat{R}_2$  by

$$\phi = \frac{F'}{NF} \quad (F' = D(F) \in R_{N+2}).$$

Then for  $f \in R_k$  we have

$$D(f) - k\phi f - \partial(f) = \frac{1}{NF} (NF f' - k f F' - [F, f]_1) = 0$$

by (29) with  $n = 1$  and

$$D(\phi) - \phi^2 - \Phi = \left(\frac{F'}{NF}\right)' - \left(\frac{F'}{NF}\right)^2 - \frac{N(N+1)FF'' - (N+1)^2 F'^2}{N^2(N+1)F^2} = 0$$

by (29) with  $n = 2$ , so  $D$  and  $\partial$  are indeed related by (36) and consequently  $[\cdot]_* = [\cdot]_{D,*}|_{M_*} = [\cdot]_{\partial,\Phi,*}$  by the calculation already given in the proof of Proposition 1. This shows that *any RC algebra  $M_*$  which contains at least one homogeneous element  $F$  of positive degree which is not a zero-divisor is a subalgebra of a canonical RC algebra (namely,  $(\widehat{M}_*, [\cdot]_{\partial,\Phi,*})$  with  $\widehat{M}_* = M_*[1/F]$  and  $\partial, \Phi$  given by (38)) and hence also a sub RC algebra of a standard algebra (namely  $(\widehat{M}_*[\phi], [\cdot]_{D,*})$  with  $\phi$  of degree 2 and  $D: \widehat{M}[\phi]_* \rightarrow \widehat{M}[\phi]_{*+2}$  defined by (36))*. Note that if  $M_*$  is already embedded as a sub RC algebra of a standard RC algebra  $(R_*, [\cdot]_{D,*})$ , then this embedding extends to an embedding of  $\widehat{M}_*[\phi]$  into  $\widehat{R}_* = R[1/F]_*$  by  $\phi \mapsto D(F)/NF$  and this extension is compatible with the differentials by the calculations just done. We state the special case when  $M_*$  is closed under  $\partial$  and contains  $\Phi$  as:

**Proposition 2.** *Let  $M_\star$  be an RC algebra and suppose that  $M_\star$  contains a homogeneous element  $F$  of some degree  $N > 0$  such that*

- (i)  $F$  is not a zero-divisor;
- (ii)  $[F, M_\star]_1 \subseteq (F) = M_\star \cdot F$ ;
- (iii)  $[F, F]_2 \in (F^2)$ .

*Then  $[\cdot]_\star = [\cdot]_{\partial, \Phi, n}$  for  $\partial : M_\star \rightarrow M_{\star+2}$  and  $\Phi \in M_4$  as in (38), so  $M_\star$  is a canonical RC algebra.*

Examples of RC algebras which satisfy the conditions of Proposition 2 are the rings of modular forms  $M_\star(\Gamma)$  with  $\Gamma \subset PSL(2, \mathbf{R})$  commensurable with  $\Gamma_1 = PSL(2, \mathbf{Z})$  and the RC bracket defined by (1). Indeed, on such a group we can define a modular form  $F(\tau) = \prod \Delta|_{12}\gamma(\tau)$ , where  $\gamma$  runs over the set of left cosets  $(\Gamma \cap \Gamma_1) \backslash \Gamma_1$  and  $f|_k\gamma(\tau) = (c\tau + d)^k f(\gamma\tau)$  for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  as usual. This is a modular form of weight  $N = 12[\Gamma_1 : \Gamma_1 \cap \Gamma]$  which has no zeros in the upper half-plane. Thus  $[F, f]_1/F$  ( $f \in M_k(\Gamma)$ ) and  $[F, F]_2/F^2$  are certainly holomorphic in the upper half-plane, and of course they transform with respect to  $\Gamma_1$  like modular forms (of weights  $k+2$  and  $4$ , respectively). To see that they actually belong to  $M_\star(\Gamma)$ , we must check that they are holomorphic at the cusps, but this is clear because it is obvious from (1) that  $\text{ord}_\infty([f, g]_n) \geq \text{ord}_\infty(f) + \text{ord}_\infty(g)$  for any  $f, g \in M_\star(\Gamma_1)$  and the identity  $[f|_k\gamma, g|_l\gamma]_n = [f, g]_n|_{k+l+2n}\gamma$  shows that the same inequality is true at any cusp.

I do not know whether it is true that any RC algebra which is finitely generated (over the ground field  $K$ ) and an integral domain satisfies the conditions of Proposition 2 for some  $F$ . (The stated hypotheses are easily seen to be necessary.) But even apart from this, Proposition 2 is not really a satisfactory characterization, since there is no obvious way to pick  $F$ , which a priori could have arbitrarily high degree. The following sharpening of Proposition 2 gives a criterion for an RC algebra to be canonical which can be checked in a finite amount of time.

**Theorem.** *Let  $(M_\star, [\cdot]_\star)$  be an RC algebra which is finitely generated over a field of characteristic 0. Then the following are equivalent:*

- (a)  $(M_\star, [\cdot]_\star)$  is canonical.
- (b) for every homogeneous element  $F \in M_\star$  there is an element  $G \in M_{\star+2}$  such that
  - (i)  $[F, f]_1 \equiv k f G \pmod{F}$  for all  $k \geq 0$  and all  $f \in M_k$ .
  - (ii)  $[F, F]_2 \equiv (N+1)G^2 - (N+1)[F, G]_1 \pmod{F^2}$ .
- (c) Property (b) holds for some homogeneous  $F \in M_\star$  which is not a divisor of zero.

*Specifically, if  $(F, G)$  are a pair of elements satisfying (i) and (ii), and with  $F \in M_N$  not a divisor of zero, then the bracket on  $M_\star$  agrees with the canonical bracket associated to*

$$\partial_{F,G}(f) := \frac{[F, f]_1 - k f G}{NF} \quad (f \in M_k), \quad \Phi_{F,G} := \frac{[F, F]_2 + (N+1)([F, G]_1 - G^2)}{N^2(N+1)F^2}. \quad (39)$$

**Remarks.** The special case when  $F$  can be chosen in (c) with  $G = 0$  is Proposition 2, but because of our freedom to pick *any* element (homogeneous and not a divisor of 0) to verify (c), we now get an effective criterion to check whether a given RC algebra is canonical. Indeed, pick any  $F$ , say of weight  $N$ , and check whether the elements  $[F, f_i]$  are proportional to  $k_i f_i$  modulo the ideal  $(F)$ , where  $f_i$  ( $i = 1, \dots, I$ ) are homogeneous generators of  $M_\star$  of weight  $k_i$ . If this is not the case, then  $M_\star$  is not canonical by the implication (a)  $\Rightarrow$  (b). If it is, then pick an element  $G$  of  $M_{n+2}$  satisfying (i) and verify whether (ii) is true. If it is, then  $M_\star$  is canonical by the implication (c)  $\Rightarrow$  (a) of the theorem. If it is not, then  $M_\star$  is not canonical, because of the implication (a)  $\Rightarrow$  (b) and the fact that the truth of (ii) is independent of the choice of  $G$ . (Any two choices differ by a multiple of  $F$ , and if  $G_1 = G + F\phi$

with  $\phi \in M_2$  then  $[F, G_1]_1 - G_1^2 - [F, G]_1 + G^2 = F([F, \phi]_1 - 2G\phi) - \phi^2 F^2$  belongs to  $(F^2)$  by the defining property of  $G$ .)

**Example.** Let  $M_\star = M_\star(PSL(2, \mathbf{Z})) = \mathbf{C}[Q, R]$  with the original Rankin-Cohen bracket (1). Of course we already know that this satisfies the conditions of Proposition 2 with  $F = \Delta = \frac{Q^3 - R^2}{1728}$ , giving  $M_\star$  the canonical structure associated to the derivation (33) and the element  $\Phi = -Q/144$ . But suppose that we had not noticed this nice element  $\Delta$  and instead wanted to check the canonicalness of  $M_\star$  starting with  $F = Q$ , the homogeneous element of lowest positive weight in  $M_\star$ . According to the theorem, we must find an element  $G \in M_6$  satisfying (i) and (ii). Since  $M_\star$  has only two generators  $Q$  and  $R$ , and by the derivation property of  $[\cdot]_1$ , it is enough to check (i) for  $f = Q$  and  $f = R$ . Using (31) we find

$$[F, Q]_1 = 0 \equiv 4Q \cdot \frac{R}{3} \pmod{Q}, \quad [F, R]_1 = -2Q^3 + 2R^2 \equiv 6R \cdot \frac{R}{3} \pmod{Q}$$

and hence (i) holds with  $G = R/3$ . Then, using (31) again, we find

$$[F, F]_2 - (N+1)[F, G]_1 - (N+1)G^2 = \frac{25}{9}(Q^3 - R^2) - \frac{10}{3}(Q^3 - R^2) - \frac{5}{9}R^2 = -\frac{5}{9}Q^3 \equiv 0 \pmod{Q^2}$$

and hence (ii) also holds, proving that  $M_\star$  is canonical with respect to the derivation  $\partial$  and element  $\Phi$  given by  $\partial(Q) = 0$ ,  $\partial(R) = -Q^2/2$ ,  $\Phi = -Q/36$ .

**Proof of the theorem.** The statement of the theorem indicates the proof. Assume first that  $M_\star$  is canonical with respect to some  $\partial : M_\star \rightarrow M_{\star+2}$  and  $\Phi \in M_4$ , and choose any homogeneous element  $F \in M_N$ ,  $N > 0$ . Then properties (i) and (ii) in (b) hold with  $G = -\partial(F)$  because of the identities

$$\begin{aligned} [F, f]_1 - k f G &= [F, f]_{\partial, \Phi, 1} + k f \partial(F) = N \partial(f) F \quad (f \in M_k, k \geq 0), \\ [F, F]_2 + (N+1)[F, G]_1 - (N+1)G^2 &= (N(N+1)F\partial^2(F) - (N+1)^2\partial(F)^2 + N^2(N+1)\Phi F^2) \\ &\quad - (N+1)(N\partial(F)F - (N+1)F\partial^2(F)) - (N+1)(\partial(F))^2 = N^2(N+1)\Phi F^2. \end{aligned}$$

Conversely, suppose that  $M_\star$  contains elements  $F \in M_N$ ,  $G \in M_{N+2}$  for some  $N > 0$  satisfying (i) and (ii) (and with  $F$  not a zero-divisor), and define  $\partial$  and  $\Phi$  by (39). Then we claim that the brackets  $[\cdot]_{\partial, \Phi, \star}$  induced by  $\partial$  and  $\Phi$  agree with the given bracket. As in earlier proofs, we can assume here that  $(M_\star, [\cdot]_\star)$  is a sub RC algebra of a standard RC algebra  $(R_\star, [\cdot]_{D, \star})$ , since the assertion to be proved is equivalent to a collection of universal identities for the brackets of RC algebras and such identities are true by definition if they are true for standard algebras. Now the larger algebra  $(R_\star, [\cdot]_{D, \star})$  is canonical, with derivation  $D$  and weight 4 element 0, so we have to show that in a ring with more than one choice of  $(F, G)$  as in (b) of the theorem, the induced bracket operations agree.

In fact, a little reflection shows that the key thing to check is that the property (b) in the theorem in a given RC algebra is independent of the choice of  $F$ , corresponding to the equivalence of (b) with the apparently much weaker (c). So now suppose that  $(F, G)$  satisfy (i) and (ii) and let  $\tilde{F} \in M_{\tilde{N}}$  be an arbitrary homogeneous element of  $M_\star$ . We must show that there is an element  $\tilde{G} \in M_{\tilde{N}+2}$  so that  $(\tilde{F}, \tilde{G})$  also satisfy (ii). We may start by choosing any  $\tilde{G}$  which satisfies (ii), since we have already seen (in the ‘‘Remarks’’ above) that the truth or falsity of property (ii) is independent of the choice of  $\tilde{G}$  given  $\tilde{F}$ . We set

$$\tilde{G} = \frac{\tilde{N} G \tilde{F} - [F, \tilde{F}]_1}{N F}, \quad (40)$$

which belongs to  $M_{\tilde{N}+2}$  by property (i) of  $(F, G)$ . Then for  $f \in M_k$  we find

$$\partial_{F,G}(f) - \partial_{\tilde{F},\tilde{G}}(f) = \frac{\tilde{N}\tilde{F}[f, F]_1 + NF[\tilde{F}, f]_1 + k[\tilde{F}, F]_1}{N\tilde{N}F\tilde{F}} = 0$$

by the identity (10) of §2, and similarly  $\Phi_{F,G} - \Phi_{\tilde{F},\tilde{G}} = 0$  by virtue of the more complicated identity

$$N^2(N+1)F^2[\tilde{F}, \tilde{F}]_2 = \tilde{N}^2(\tilde{N}+1)\tilde{F}^2[F, F]_2 - (N+1)(\tilde{N}+1)[F, \tilde{F}]_1^2 + \tilde{N}(\tilde{N}+1)\tilde{F}[[F, \tilde{F}]_1, F]_1 \quad (41)$$

which could have been (but was not) included in the list of universal identities in RC algebras given in §2. Thus the brackets constructed with  $\partial_{F,G}$  and  $\Phi_{F,G}$  are the same as those constructed from  $\tilde{F}$  and  $\tilde{G}$  chosen as in (40), and therefore the same as those constructed from any pair  $(\tilde{F}, \tilde{G})$  satisfying (i)–(ii) at all. (Changing  $G$  to  $G + \phi F$  changes  $\partial(f)$  ( $f \in M_k$ ) to  $\partial(f) + k\phi f$  and  $\Phi$  to  $\Phi + \phi^2 - \partial(\phi)$  but does not change the associated brackets, by the proof of Proposition 1.)  $\square$

We remark that the reason for the truth of the theorem is that we have the identities (10) and (41). The former says that, once we have fixed the multiplication (0th bracket) on an RC algebra, the first bracket for any two elements  $f, g \in M_*$  is determined once we have given the first brackets of  $f$  and  $g$  with a fixed homogeneous element  $F$  of  $M_*$  which is not a zero divisor. Similarly, the identity (41) tells us how to compute the second bracket  $[\tilde{F}, \tilde{F}]_2$  for any homogeneous  $\tilde{F} \in M_*$  (and hence also how to compute the second bracket  $[g, h]_2$  for any elements  $g, h \in M_*$ , by the usual polarization procedure for recovering a bilinear form from its associated quadratic form) knowing only the 0th and 1st brackets and the second bracket of  $F$  with itself. In other words, to specify the brackets in an RC algebra (assumed to contain one homogeneous non-zero divisor  $F$ ), we need to know only

1. The 0th bracket  $[f, g]_0$  for arbitrary  $f$  and  $g$ , which is arbitrary subject only to the conditions of bilinearity, associativity, and commutativity,
2. The 1st bracket of arbitrary elements  $f$  with the fixed element  $F$ , i.e. the derivation  $f \mapsto [f, F]_1$ , and
3. The 2nd bracket of  $F$  with itself, i.e. a single further element of  $M_*$ .

**§7. Other occurrences of Rankin-Cohen algebras.** We end by raising the question where else RC algebras arise naturally in mathematics. One possible candidate, pointed out to me by T. Springer, is in invariant theory, where the algebras of invariants have natural bilinear operations called the “transvectant” or “Überschiebung” (cf. [Sp], p. 66). These operations are indexed by integers  $n \geq 0$  and satisfy some universal identities of the general form of those occurring in §2, but they decrease rather than increasing the total weight (i.e., they send  $M_k \otimes M_l$  to  $M_{k+l-2n}$  rather than  $M_{k+l+2n}$ ). The relationship between the two types of algebraic structures remains to be determined. Another possibility are the so-called Moyal brackets in quantum theory, which are related to symplectic structures and seem to have similar algebraic properties to the brackets considered in this paper. Finally, the most natural source of interesting algebras with an infinite number of bilinear operations seems to be conformal field theories and more specifically vertex operator algebras. The axioms for vertex operator algebras as given in [Bo] or the appendix of [Ge] are different from ours, but discussions with Yu. Manin and W. Eholzer suggest that there may be a reformulation of the axioms of vertex operator algebras which is much closer to the RC algebras studied here. We hope to discuss this in a future publication.

**Note added in proof.** Following a remark of W. Eholzer, it transpired that there is a further universal identity satisfied by the brackets in RC-algebras which is particularly simple and appealing: the multiplication on  $\bigoplus_k M_k$  defined by  $f * g = \sum_{n \geq 0} [f, g]_n$  is associative. That implies in turn infinitely many identities of the sort considered in §2 (possibly including all identities whose coefficients are independent of the weights, like (6), (8), (9), (11a) and (13)). Moreover, this multiplication turns out to be one of a whole 1-parameter family of associative multiplications, all the rest of which do explicitly involve the weights of the forms involved, and one of which is the one arising from the correspondence with pseudodifferential operators discussed in §4. Details will be included in the paper [MZ].

#### REFERENCES

- [Bo] R. Borcherds, *Vertex operator algebras, Kac-Moody algebras and the monster*, Proc. Nat. Acad. Sci. USA **83** (1986) 3068–3071.
- [Co] H. Cohen, *Sums involving the values at negative integers of L functions of quadratic characters*, Math. Ann. **217** (1977), 81–94.
- [EZ] M. Eichler and D. Zagier, “The Theory of Jacobi Forms,” Progress in Math. **55**, Birkhäuser, Boston–Basel–Stuttgart (1985).
- [Ge] E. Getzler, *Manin triples and  $N = 2$  superconformal field theory*, preprint, MIT 1993.
- [IZ] T. Ibukiyama and D. Zagier, *Higher spherical polynomials*, in preparation.
- [Ku] N.V. Kuznetsov, *A new class of identities for the Fourier coefficients of modular forms* (in Russian), Acta Arith. **27** (1975), 505–519.
- [MZ] Yu.I. Manin and D. Zagier, *Automorphic pseudodifferential operators*, in preparation.
- [Ra] R.A. Rankin, *The construction of automorphic forms from the derivatives of a given form*, J. Indian Math. Soc. **20** (1956), 103–116.
- [Sp] T.A. Springer, “Invariant Theory,” Lecture Notes **585**, Springer, Berlin–Heidelberg–New York (1977).
- [VZ] F. Rodriguez Villegas and D. Zagier, *Square roots of central values of Hecke L-series*, in “Advances in Number Theory, Proceedings of the Third Conference of the Canadian Number Theory Association” (F. Gouvea and N. Yui, eds.), Clarendon Press, Oxford (1993), 81–99.