A K – theoretic relative index theorem

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Abstract

We prove a relative index theorem for Dirac operators with C^*-coefficients.

Contents

1 Introduction

Let $D : C^\infty(M,E) \to C^\infty(M,E)$ be a generalized Dirac operator acting on sections of a $\mathbf{Z}_2$-graded bundle E over a complete Riemannian manifold. If 0 is not in the essential spectrum of D then the index

$$ind\, D = dim\ ker\ D^+ - dim\ ker\ D^-$$

is well defined. 0 is not in the essential spectrum if e.g. D is positive at infinity, i.e. there is a constant $c > 0$ and a compact set $K \subset M$ such that $r_{|M\setminus K} \geq c$ where $r := D^2 - \Delta$ is the endomorphism occuring in the Weizenboeck formula.

The original version of the relative index theorem due to Gromov/Lawson [8] computes $ind\, D_1 - ind\, D_2$ for two Dirac operators which are positive at infinity and which coincide outside of compact sets, i.e. D_i live on manifolds M_i, $i = 1,2$ and there are open cocompact sets $U_i \subset M_i$ with smooth boundary such that $D_{1|U_1} \cong D_{2|U_2}$ and $r_{i|U_i} \geq c > 0$. Let $M^\sharp := M_1 \setminus U_1 \cup_{\partial U} M_2 \setminus U_2$ and glue the bundles using the odd morphism given by Clifford multiplication with the unit normal vector at ∂U with grading induced from $E_{M_1\setminus U_1}$. Let $D^\sharp$ be the associated Dirac operator.

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Theorem 1.1 (Gromov/Lawson)

$$ind\ D_1 - ind\ D_2 = ind\ D^\sharp$$

Another way to look upon this theorem is as follows. Consider $M = M_1 \cup M_2$ and the opposite grading of the Clifford bundle over M_2. Let D be the Dirac operator over M. Obviously $ind\ D = ind\ D_1 - ind\ D_2$. We can now cut M at $\partial U_1 \cup \partial U_2$ and glue together again using the diffeomorphism interchanging the two boundary components obtaining $\tilde{M}$ together with a new Clifford bundle and a Dirac operator $\tilde{D}$. In fact $\tilde{M} = M^\sharp \cup (U_1 \cup_{\partial U} U_2)$ and $\tilde{D}$ is invertible over $U_1 \cup_{\partial U} U_2$ (here we assume for simplicity a product collar at ∂U_i in order to glue smoothly). Hence $ind\ \tilde{D} = ind\ D^\sharp$. The relative index theorem states that cutting and glueing as decribed above does not change the index:

$$ind\, D = ind\, \tilde{D}.$$

There are several generalizations of the relative index theorem [7],[5], [6],[1],[2],[4].

The aim of this paper is to give a K-theoretic variant of this theorem which applies also for operators acting on C^*-Hilbert-bundles over the base field k, which is $\mathbf{R}$ or $\mathbf{C}$. Such opertors have been considered first by Miščenko/Fomenko [9]. Let M be a complete Riemannian manifold and A be a $\mathbf{Z}_2$-graded C^*-algebra. A C^*-Clifford bundle S is a bundle of projective finitely generated graded A-C^*-right-Hilbert modules together with a metric connection and a Clifford multiplication satifying Leibnitz rule and compatibility with the scalar products of the fibres. We think the tangent vectors and the connection acting from the left. Let D be the associated Dirac operator. We define Sobolev spaces H^l, $l \geq 0$ using scalar products defined with D as usual (see [9]). In fact the H^l are A-C^*-right-Hilbert modules. We have $D \in B(H^1, H^0)$. Our basic assumption is

Assumption 1 *There is a $S \in K(H^0)$ such that $D + S$ is invertible and $S \in B(H^0, H^1)$, $DS \in K(H^0)$, $SD \in K(H^1)$.*

Note that K stands for compact operators between A-C^*-right-Hilbert modules (see [3], [9]). In general S fails to be odd or selfadjoint. We can now construct a Kasparov module (see [3]) representing the index of D. Let $A := D + S$ and $F := [D(AA^*)^{-1/2}]^{odd}$ where $[]^{odd}$ is the projection onto the odd part. We have $F \in B(H^0)$ and $deg\ F = 1$. Let $C_g(M)$ be the C^*-algebra generated by the bounded functions $f \in C^\infty(M)$ with vanishing gradient at infinity equipped with the supremum norm. There is a $*$-homomorphism $C_g(M) \to B(H^0)$ given by multiplication.

Proposition 1.2 *(H^0, F) is a Kasparov modul over the pair of C^*-algebras $(C_g(M), A)$*

Let us think of all structures over M be compressed in the symbol M. Then we let $[M] \in KK(C_g(M), A)$ be the class represented by (H^0, F) (in fact [M] does not depend on the choice of S since the difference of the F's for different S's is compact). Note that we work with KK-groups over the base field k. The equivalence relation used here is compact perturbation (see Blackadar [3] for details).

Let $N \subset M$ be a compact hypersurface cutting a normal neighbourhood $U(N)$ in two pieces $U(N)_\pm$. Assume that there is a diagram

$$\begin{array}{ccccc} \Gamma & : & S_{|U(N)_-} & \longrightarrow & S_{|U(N)_-} \\ & & \downarrow & & \downarrow \\ \gamma & : & U(N)_- & \longrightarrow & U(N)_- \end{array}$$

intertwining all structures. Then we can form a new manifold $\tilde{M}$ cutting at N and glueing together using γ and a new bundle $\tilde{S}$ using Γ with associated Dirac operator $\tilde{D}$. Suppose that D and $\tilde{D}$ satisfy Assumption 1. Then we can form $[M] \in KK(C_g(M), A)$ and $[\tilde{M}] \in KK(C_g(\tilde{M}), A)$. Restricting to constant functions we have elements $\{M\}, \{\tilde{M}\} \in KK(k, A)$. The main theorem in this paper is

Theorem 1.3 (K-theoretic relative index theorem) $\{M\} = \{\tilde{M}\}$

This theorem can be interpreted in special cases a relative index theorem for families or as equivariant relative index theorem.

One of our main motivations comes from the following situation. Let $k := \mathbf{R}$, M^n be spin, E be the real Clifford bundle with fibres isomorphic to the Clifford algebra C_n and V be a flat bundle of A-C^*-right-Hilbert modules. Set $S := E \otimes V$. Assume that there is a compact set $K \subset M$ and a constant $c > 0$ such that for the scalar curvature s we have the estimate $s_{|M \setminus K} \geq c$. Then D is invertible at infinity, i.e. there is a $f \in C_c^\infty(M)$ such that $D^2 + f$ is invertible. We want to know wether D satisfies Assumption 1. In fact

Theorem 1.4 *If D is invertible at infinity then D satisfies Assumption 1.*

As an application we construct for any discrete group π a group homomorphism

$$R_n(\pi) \to KK_n(\mathbf{R}, C_r^*(\pi))$$

where $R_n(\pi)$ is a group of n-dimensional bordisms M with prescribed positive scalar curvature metric at ∂M. (see section 5 for details).

The author thanks Stefan Stolz for the very stimulating discussion.

2 Commutator estimates

Let M be a complete Riemannian manifold and S be a Clifford-C^*-bundle with associated Dirac operator D. We form the completitions H^l, $l \geq 0$, of $C_c^\infty(M, S)$ with respect to the norms

$$\|\phi\|_l^2 = \sum_{k=0}^{l} \int_M \|D^k \phi(x)\|^2, \quad \phi \in C_c^\infty(M, S)$$

where the norm of the right hand side is the point wise norm coming from the A-C^*-Hilbert module structure of the fibres. Note that the H^l are A-C^*-right-Hilbert modules with scalar product

$$< \phi, \psi >_l = \sum_{k=0}^{l} \int_M < D^k \phi(x), D^k \psi(x) > .$$

There is an analog of Rellich's theorem

Proposition 2.1 (Miščenko/Fomenko,[9]) *For any* $f \in C_c^\infty(M)$ *the multiplication* $f : H^l \to H^k$ *is compact for* $k < l$.

D extends to an operator $D \in B(H^l, H^{l-1})$, $\forall l \in \mathbf{N}$. Suppose that D satisfies Assumption 1 and form $A := D+S$. Then we have $(AA^*)^{-1/2}, (A^*A)^{-1/2} \in B(H^0, H^1)$. Note the integral representation

$$(A^*A)^{-1/2} = \frac{2}{\pi}\int_0^\infty (A^*A+\lambda^2)^{-1}d\lambda$$

where the integral converges in $B(H^0)$. For a bounded function f we have

$$\begin{aligned} f(A^*A)A^* &= A^*f(AA^*) \\ Af(A^*A) &= f(AA^*)A. \end{aligned}$$

Since we want to commute A and $(A^*A)^{-1/2}$ we need

Lemma 2.2 $(A^*A)^{-1/2} - (AA^*)^{-1/2} \in K(H^0, H^2)$

Proof: We have

$$\begin{aligned} & (A^*A)^{-1/2} - (AA^*)^{-1/2} \\ =\ & \frac{2}{\pi}\int_0^\infty ((A^*A+\lambda^2)^{-1} - (AA^*+\lambda^2)^{-1})d\lambda \\ =\ & \frac{2}{\pi}\int_0^\infty (A^*A+\lambda^2)^{-1}(AA^* - A^*A)(AA^*+\lambda^2)^{-1}d\lambda \\ =\ & \frac{2}{\pi}\int_0^\infty (A^*A+\lambda^2)^{-1}(SD+DS^*+SS^*-DS-S^*D-S^*S)(AA^*+\lambda^2)^{-1}d\lambda. \end{aligned}$$

By the following decomposition we see that every term is bounded in $B(H^0, H^2)$ by $C(1+\lambda^2)^{-1}$ and compact:

$$\begin{aligned} &H^0 \overset{(AA^*+\lambda^2)^{-1}}{\longrightarrow} H^0 \overset{SS^*+SS^*+DS^*+DS}{\longrightarrow} H^0 \overset{(A^*A+\lambda^2)^{-1}}{\longrightarrow} H^2 \in K(H^0,H^2) \\ &H^0 \overset{(AA^*+\lambda^2)^{-1}}{\longrightarrow} H^1 \overset{SD+S^*D}{\longrightarrow} H^1 \overset{(A^*A+\lambda^2)^{-1}}{\longrightarrow} H^2 \in K(H^0,H^2) \end{aligned}$$

□

Also we need a commutator estimate for $(AA^*)^{-1/2}$ with functions in $C_g(M)$.

Lemma 2.3 *For* $f \in C_g(M)$ *we have*

$$[f, (AA^*)^{-1/2}] \in K(H^0, H^1)$$

Proof: W.l.o.g we can assume that $f \in C^\infty(M)$ is bounded with *grad* $f \in C_0(M, TM)$. Using the integral representation for $(AA^*)^{-1/2}$ we have

$$\begin{aligned} [f, (AA^*)^{-1/2}] &= \frac{2}{\pi}\int_0^\infty [f, (AA^*+\lambda^2)^{-1}]d\lambda \\ &= \frac{2}{\pi}\int_0^\infty (AA^*+\lambda^2)^{-1}[f, D^2+SD+DS^*+SS^*](AA^*+\lambda^2)^{-1}d\lambda \end{aligned}$$

Note that $[f, D^2] = -Dgrad\, f - grad\, fD$ is of first order. By the decomposition

$$H^0 \overset{(AA^*+\lambda^2)^{-1}}{\longrightarrow} H^1 \overset{[f,AA^*]}{\longrightarrow} H^0 \overset{(AA^*+\lambda^2)^{-1}}{\longrightarrow} H^1 \in K(H^0,H^1)$$

we see that the integrand is bounded by $C(1+\lambda^2)^{-1}$. Compactness follows from Proposition 2.1. □

3 The relative index theorem

Let M be a complete Riemannian manifold and S be a $\mathbf{Z}_2$-graded Clifford-C^*-bundle with associated Dirac operator D. Suppose Assumption 1. Set $F := [D(AA^*)^{-1/2}]^{odd}$.

Lemma 3.1 *The even part of $D(AA^*)^{-1/2}$ is compact.*

Proof: Let ϵ be the $\mathbf{Z}_2$-grading of H^0 and $\sim$ denote equality modulo $K(H^0)$.

$$
\begin{aligned}
2[D(AA^*)^{-1/2}]^{ev} &= \epsilon D(AA^*)^{-1/2}\epsilon + D(AA^*)^{-1/2} \\
&= D\epsilon[\epsilon,(AA^*)^{-1/2}] \\
&= D\epsilon\frac{2}{\pi}\int_0^\infty [\epsilon,(AA^*+\lambda^2)^{-1}]d\lambda \\
&= D\epsilon\frac{2}{\pi}\int_0^\infty (AA^*+\lambda^2)^{-1}[\epsilon,AA^*](AA^*+\lambda^2)^{-1}d\lambda \\
&= D\epsilon\frac{2}{\pi}\int_0^\infty (AA^*+\lambda^2)^{-1}[\epsilon,SS^*+DS^*+SD](AA^*+\lambda^2)^{-1}d\lambda \\
&\sim 0
\end{aligned}
$$

□

Proposition 3.2 *(H^0,F) is a Kasparov module over the pair of C^*-algebras $(C_g(M),A)$.*

Proof: We have to verify

$$
\begin{aligned}
F-F^* &\in K(H^0) \\
F^2-1 &\in K(H^0) \\
[f,F] &\in K(H^0) \quad \forall f\in C_g(M)
\end{aligned}
$$

Then

$$
\begin{aligned}
F^*-F &\overset{Lemma\ 3.1}{\sim} (AA^*)^{-1/2}D - D(AA^*)^{-1/2} \\
&\sim (AA^*)^{-1/2}A - D(AA^*)^{-1/2} \\
&= A(A^*A)^{-1/2} - D(AA^*)^{-1/2} \\
&\overset{Lemma\ 2.2}{\sim} A(AA^*)^{-1/2} - D(AA^*)^{-1/2} \\
&\sim D(AA^*)^{-1/2} - D(AA^*)^{-1/2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
F^2-1 &\overset{Lemma\ 3.1}{\sim} D(AA^*)^{-1/2}D(AA^*)^{-1/2}-1 \\
&\sim A^*(AA^*)^{-1/2}A(AA^*)^{-1/2}-1 \\
&= A^*A(A^*A)^{-1/2}(AA^*)^{-1/2}-1 \\
&\overset{Lemma\ 2.2}{\sim} A^*A(A^*A)^{-1/2}(A^*A)^{-1/2}-1 \\
&= 0
\end{aligned}
$$

W.l.o.g. we can assume f to be smooth and $grad\, f \in C_0(M, TM)$.

$$\begin{aligned} [f,F] &\overset{Lemma\ 3.1}{\sim} [f, D(AA^*)^{-1/2}] \\ &= [f,D](AA^*)^{-1/2} + D[f,(AA^*)^{-1/2}] \\ &\overset{Lemma\ 2.3}{\sim} -grad\, f(AA^*)^{-1/2} \\ &\overset{Prop.\ 2.1}{\sim} 0. \end{aligned}$$

□

Let $[M] \in KK(C_g(M), A)$ denote the class represented by (H^0, F) (as above we compress all structures in the symbol M) and $\{M\} \in KK(k, A)$ be the class obtained from $[M]$ restricting to the constant functions in $C_g(M)$. Clearly $\{M\}$ is represented by (H^0, F) too.

Let $N \subset M$ be a compact hypersurface cutting a normal neighbourhood $U(N)$ in two pieces $U(N)_\pm$. Assume that there is a diagram

$$\begin{array}{ccccc} \Gamma & : & S_{|U(N)_-} & \longrightarrow & S_{|U(N)_-} \\ & & \downarrow & & \downarrow \\ \gamma & : & U(N)_- & \longrightarrow & U(N)_- \end{array}$$

intertwining all structures. We form a new manifold $\tilde{M}$ cutting at N and glueing together using γ and a new bundle $\tilde{S}$ using Γ with associated Dirac operator $\tilde{D}$. Suppose that $\tilde{D}$ also satisfies Assumption 1. Let $\{\tilde{M}\} \in KK(k, A)$ be the class given by $\tilde{D}$.

Theorem 3.3 (K-theoretic relative index theorem) $\{\tilde{M}\} = \{M\}$

Proof: Note that $H^0 = \tilde{H}^0$ in a canonical way. Thus it is enough to show that

$$\Delta := F - \tilde{F} \in K(H^0).$$

Recall that we use the compact perturbation as equivalence relation in the KK-groups. Let $\psi, \phi \in C^\infty(M)$, $\phi \equiv 1$ outside of some small neighbourhood of N and $\psi, \phi \equiv 0$ inside a smaller one such that $\psi\phi = \phi$. Set $\chi := (1 - \phi)$ and let $\rho \in C_c^\infty(U(N))$ such that $\rho\chi = \chi$. Let $\tilde{\Delta} := \psi\Delta\phi + \rho\Delta\chi$. Then

$$\begin{aligned} \tilde{\Delta} - \Delta &= \psi\Delta\phi + \rho\Delta\chi - \Delta \\ &= (1-\psi)\Delta\chi + (1-\rho)\Delta\chi \\ &\overset{Lemma\ 2.3}{\sim} (1-\psi)\phi\Delta + (1-\rho)\chi\Delta \\ &= 0 \end{aligned}$$

Thus it is enough to show the compactness of $\tilde{\Delta}$. Let us consider e.g. $\psi\Delta\phi$.

$$\begin{aligned} & \psi\Delta\phi \\ \overset{Lemma\ 3.1}{\sim}\ & \frac{2}{\pi}\int_0^\infty \psi[D(AA^* + \lambda^2)^{-1} - \tilde{D}(\tilde{A}\tilde{A}^* + \lambda^2)^{-1}]\phi d\lambda \\ \overset{Prop.\ 2.1}{\sim}\ & \frac{2}{\pi}\int_0^\infty D\psi[(AA^* + \lambda^2)^{-1} - (\tilde{A}\tilde{A}^* + \lambda^2)^{-1}]\phi d\lambda \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\pi}\int_0^\infty D(AA^*+\lambda^2)^{-1}(AA^*+\lambda^2)\psi[(AA^*+\lambda^2)^{-1}-(\tilde{A}\tilde{A}^*+\lambda^2)^{-1}]\phi d\lambda \\
&\sim \frac{2}{\pi}\int_0^\infty D(AA^*+\lambda^2)^{-1} \\
&\qquad [(AA^*+\lambda^2)\psi(AA^*+\lambda^2)^{-1}\phi-(\tilde{A}\tilde{A}^*+\lambda^2)\psi(\tilde{A}\tilde{A}^*+\lambda^2)^{-1}\phi]d\lambda \\
&\overset{Prop.\,2.1}{\sim} \frac{2}{\pi}\int_0^\infty D(AA^*+\lambda^2)^{-1}[\psi\phi-\psi\phi]d\lambda \\
&= 0
\end{aligned}$$

Analogously we handle $\rho\Delta\chi$. Thus $\tilde{\Delta}\in K(H^0)$ and also $\Delta\in K(H^0)$. □

4 Invertibility at infinity

Let M be a complete Riemannian manifold and S be a $\mathbf{Z}_2$-graded Clifford-C^*-bundle with associated Dirac operator D. We say that D is invertible at infinity if there is some $f\in C_c^\infty(M)$ such that D^2+f is invertible as operator in $B(H^1,H^0)$.

Proposition 4.1 *If D is invertible at infinity then $D\in B(H^1,H^0)$ is Fredholm.*

Proof: We construct a parametrix $R\in B(H^0,H^1)$ such that $DR\sim 1$ and $RD\sim 1$. Let $\psi,\phi\in C_c^\infty(M)$ such that $\phi\equiv 1$ on *supp* f and such that $\psi\phi=\phi$. Moreover let $\chi\in C^\infty(M)$ such that $\chi\equiv 0$ on *supp* f and $\chi(1-\phi)=1-\phi$. Let $R_U:=D(D^2+f)^{-1}$ and R_K be a parametrix of D with support on some compact set containing *supp* ψ. R_K can be constructed using pseudodifferential calculus as in [9]. Set $R=\chi R_U(1-\phi)+\psi R_K\phi$. Then we have $R\in B(H^0,H^1)$. Apply now D.

$$\begin{aligned}
DR &= D\chi R_U(1-\phi)+D\psi R_K\phi \\
&= grad\,\chi R_U(1-\phi)+grad\,\psi R_K\phi+\chi DR_U(1-\phi)+\psi DR_K\phi \\
&\overset{Prop.\,2.1}{\sim} \chi D^2(D^2+f)^{-1}(1-\phi)+\psi\phi \\
&= \chi(D^2+f)(D^2+f)^{-1}(1-\phi)+\psi\phi \\
&= \chi(1-\phi)+\psi\phi \\
&= 1
\end{aligned}$$

$$\begin{aligned}
RD &= \chi R_U(1-\phi)D+\psi R_K\phi D \\
&\overset{Prop.\,2.1}{\sim} \chi R_U D(1-\phi)+\psi R_K D\phi \\
&\sim \chi D(D^2+f)^{-1}D(1-\phi)+\psi\phi \\
&= \chi(D^2+f)^{-1}D^2(1-\phi)-\chi(D^2+f)^{-1}grad\,f\,(D^2+f)^{-1}(1-\phi)+\psi\phi \\
&\sim \chi(1-\phi)+\psi\phi \\
&= 1
\end{aligned}$$

□

Note that $RD, DR\in B(H^k)$ for any $k\geq 1$ and $DR-1, RD-1\in K(H^k)$ by the same proof. Assume now that the fibre V of S is a free A-C^*-Hilbert module.

Theorem 4.2 *Let $D \in B(H^1, H^0)$ be invertible at infinity. Then there is an operator S such that $D + S$ is invertible and $S \in K(H^0, H^k)$ for any given $k \in \mathbf{N}$.*

Proof: We construct first isomorphisms $H^l \cong l^2 \otimes V$. Let $M = \cup_\alpha K_\alpha$ be a countable triangulation such that $S_{|K_\alpha} \cong K \times V$. For every α fix an orthonormal basis $\{\psi_\alpha^i\}_{i \in \mathbf{N}}$ in $L^2(K_\alpha)$ where $\psi_\alpha^i \in C_c^\infty(int(K_\alpha))$. With respect to this basis we have

$$L^2(K_\alpha, S_{|K_\alpha}) \cong l^2 \otimes V.$$

Fix an enumeration of the ψ_α^i. Then we get also

$$H^0 \cong \oplus_\alpha L^2(K_\alpha, S_{|K_\alpha}) = \oplus_\alpha l^2 \otimes V = l^2 \otimes V.$$

For $v \in V$ let $v_i = (0, \ldots, v, 0, \ldots)$ with v at the i'th entry and $L_n \subset H^0$ be the subspace generated by the v_i with $i \leq n$. By construction we have in fact for any $n, k \in \mathbf{N}$ that $L_n \subset H^k$ compactly embedded. For $l \geq 0$ we use the identification

$$l^2 \otimes V \cong H^0 \overset{(1+D^2)^{-l/2}}{\longrightarrow} H^l$$

in order to construct the desired isomorphism. Define the subspaces $L_n \subset H^l$ as above. Again $L_n \in H^k$ for any k, n compactly embedded (do not confuse the L_n in different H^l).

We construct now decompositions $H^1 = U_1 \oplus W_1$, $H^0 = U_2 \oplus W_2$ such that

$$D = \begin{pmatrix} D^1 & 0 \\ 0 & D^2 \end{pmatrix}$$

and D^1 is invertible, $W_1, W_2 \subset H^k$ compactly for any given $k \in \mathbf{N}$ (this construction is essentially due to Miščenko/Fomenko [9]). Let $DR = 1 + K_1$ where R is the parametrix obtained above. We construct decompositions $H^0 = M_i \oplus N_i$, $i = 1, 2$ such that $N_i \subset H^k$ compactly for any k and

$$1 + K_1 = \begin{pmatrix} 1 + K^1 & 0 \\ 0 & * \end{pmatrix}.$$

Since K_1 is compact we can find by definition (see [9]) a n_0 such that for all $n \geq n_0$ we have $\|K_{1|L_n^\perp}\| < 1$. Let

$$K = \begin{pmatrix} K^1 & K^2 \\ K^3 & K^4 \end{pmatrix}$$

with respect to $H^0 = L_n^\perp \oplus L_n$ for some $n \geq n_0$. Then $1 + K^1$ is invertible. Set

$$X_2 := \begin{pmatrix} 1 & 0 \\ -K^3(1 + K^1)^{-1} & 1 \end{pmatrix} \quad X_1 := \begin{pmatrix} 1 & -(1 + K^1)^{-1}K^2 \\ 0 & 1 \end{pmatrix}.$$

Then

$$X_2(1 + K_1)X_1 = \begin{pmatrix} 1 + K^1 & 0 \\ 0 & 1 + K^4 - K^3(1 + K^1)^{-1}K^2 \end{pmatrix}.$$

Set $M_1 \oplus N_1 := X_1(L_n^\perp \oplus L_n)$ and $M_2 \oplus N_2 := X_2^{-1}(L_n^\perp \oplus L_n)$. Note that $K_i \in K(H^k)$ for any $k \geq 0$. Thus choosing n large enough we have $(1 + K^1)^{-1} \in B(H^k)$. Then $N_i \subset H^k$, $i = 1,2$. Let $P : H^0 \to N_2$ be the projection onto N_2 along M_2 and set $D_1 := (1-P)D$. Then $D_1 R = 1 + \tilde{K}_1$ and $RD_1 = 1 + \tilde{K}_2$ with $\tilde{K}_1 = (1-P)K_1 - P$ and $\tilde{K}_2 = 1 + K_2 - RPD$. We construct decompositions $H^1 = \bar{M}_i \oplus \bar{N}_i$, $i = 1,2$ such that

$$1 + \tilde{K}_2 = \begin{pmatrix} 1 + \tilde{K}^2 & 0 \\ 0 & * \end{pmatrix}$$

and $\bar{N}_i \subset H^k$ compactly for k as above. Consider the composition

$$H^1 = \bar{M}_1 \oplus \bar{N}_1 \xrightarrow{D_1} M_2 \oplus N_2 \xrightarrow{R} \bar{M}_2 \oplus \bar{N}_2 = H^1.$$

$RD_{|\bar{M}_1} : \bar{M}_1 \to \bar{M}_2$ is an isomorphism. Hence $D_1(\bar{M}_1) \subset M_2$ is a closed subspace. Since $D_1(H^1) = M_2$ we have the factorization

$$\bar{M}_1 \oplus \bar{N}_1 \to D_1(\bar{M}_1) \oplus [D_1(\bar{N}_1) \oplus N_2] \cong M_2 \oplus N_2 = H^0.$$

Let $\Pi : H^1 \to \bar{N}_1$ be the projection onto $\bar{N}_1$ anlong $\bar{M}_1$ and $Q : H^0 \to D_1(\bar{N}_1) \oplus N_2$ be the projection along $D_1(\bar{M}_1)$. Then $(1-Q)D(1-\Pi) : \bar{M}_1 \to D_1(\bar{M}_1)$ is invertible. Let $U_1 := (1-Q)H^1$, $W_1 := QH^1$, $U_2 := (1-\Pi)H^0$, $W_2 := \Pi H^0$. Then we have

$$D = \begin{pmatrix} (1-Q)D(1-\Pi) & 0 \\ 0 & * \end{pmatrix}.$$

Note that $D_1(\bar{N}_1) \oplus N_2 \subset D_1 H^k + H^k \subset H^{k-1}$. Thus $W_i \subset H^{k-1}$, $i = 1,2$ compactly.

The formal difference of projective finitely generated A-modules

$$[W_1] - [W_2] \in K_0(A)$$

is the index of D. Since D is selfadjoint we have $[W_1] = [W_2]$ in $K_0(A)$. Thus there is a number $r \geq 0$ such that $W_1 \oplus A^r \cong W_2 \oplus A^r$. Choosing our n large enough we can assume that $W_1 = W_2$. It is here where the assumption on the fibre of S enters. Choose an isomorphism $I : W_1 \to W_2$ and set

$$\bar{D} := (1-Q)D(1-\Pi) + QI\Pi.$$

$\bar{D}$ is invertible and $S := \bar{D} - D$ is in $K(H^0, H^l)$ for any given $l \geq 0$. This proves the theorem. □

If the fibre of S is not free we can circumvent the stabilization problem as follows. We consider instead of H^l the spaces $\tilde{H}^l := H^l \oplus A^r$ for some large r and extend the action of D and $C_g(M)$ by zero. Then Theorem 4.2 holds on these spaces. The resulting classes $[M] \in KK(C_g(M), A)$ represented by $(\tilde{H}^0, \tilde{F})$ do not depend on r. There is also a corresponding modification of the relative index theorem 3.3.

5 An application

Fix a finitely generated group π. Any spin manifold N with $\pi_1(N) = \pi$ gives rise to a $B := BSpin \times B\pi$-manifold (see [11]). The B structure

$$f : N \to B$$

is given by the product of the classifying maps of the spin structure and of the universal cover of N. Consider the set $S_n(\pi)$ of tuples (M^n, N, F, h) where (M, N, F) is a n-dimensional B-bordism, $N = \partial M$ and h is a positive scalar curvature metric on N. S is a semigroup under disjoint union. Let $\sim$ be the equivalence relation given by B-bordism. A B-bordism of (M, N, F, h) and (M_1, N_1, F_1, h_1) consists of a B-bordism (W, N, N_1, Φ) between $(N, F_{|N})$ and $(N_1, F_{1|N_1})$, a positive scalar curvature metric g on W which is product near ∂W and restricts to h, h_1 at N, N_1 and a zero-B-bordism (V, Ψ) of $(M \cup_N W \cup_{N_1} M_1, (F, \Phi, F_1))$. Note that $R_n(\pi) := S_n(\pi)/\sim$ is a goup. A similar group has been considered by B.Hajduk. It is a special case of a construction due to S.Stolz [10].

Theorem 5.1 *There is a canonical homomorphism $R_n(\pi) \to KK_n(\mathbf{R}, C_r^*(\pi))$.*

Proof: Let $(M, N, F, h) \in S_n(\pi)$. Choose a metric on M such that it is product near N and restricts to h. Glue a metric cylinder $[0, \infty) \times N$ at the boundary of M obtaining the complete manifold $\bar{M}$ and extend F constantly. $F^*E\pi$ is a π-principal fibre bundle. Associate $C_r^*(\pi)$ and obtain a flat bundle with fibre $C_r^*(\pi)$ using the canonical action of π on $C_r^*(\pi)$ from the left. Let E be the real Clifford bundle with fibre C_n associated to the spin structure and form $S := E \otimes V$. S is a C^*-Clifford bundle over $C_n \otimes C_r^*(\pi)$. Let D be the associated Dirac operator. Since the scalar curvature is positive at infinity, D is invertible at infinity and we can form $\{\bar{M}\} \in KK(\mathbf{R}, C_n \otimes C_r^*(\pi))$. Clearly the map associating to $(M, N, F, h) \in S_n(\pi)$ the class $\{\bar{M}\}$ is additive. We must show that it factors through $R_n(\pi)$. Let (W, N, Φ) be a zero-B-bordism of (N, F_N), g be a positive scalar curvature metric on W which is product near ∂W and restricts to h on N and (V, Ψ) be a zero-B-bordism of $(M \cup_N W, (F, \Phi))$. Let $L := \bar{M} \cup \bar{W}$ and $\tilde{L} := W \cup_N M \cup \mathbf{R} \times N$. Then $\{L\} = \{\bar{M}\}$ and $\{\tilde{L}\} = \{W \cup_N M\}$ since on the remaining components there are positive scalar curvature metrics and the Dirac operator is invertible there. By the relative index theorem $\{L\} = \{\tilde{L}\}$. But $\{\tilde{L}\} = 0$ since the Dirac operator is zero-bordant. Hence $\{\bar{M}\} = 0$. This proves the theorem. □

The idea of this construction is due to Stefan Stolz.

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