Centralizers in free Poisson algebras

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Abstract

We prove an analog of the Bergman Centralizer Theorem [1] for free Poisson algebras over an arbitrary field of characteristic 0. Some open problems are formulated.

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1 Introduction

There are many results, some of them quite deep, known about the structure of polynomial algebras, free associative algebras, and free Lie algebras. Though free Poisson algebras are very closely connected with these algebras, only few results are known about them up to now.

Free Poisson algebras are sometimes useful in research of polynomial algebras. For example free Poisson algebra of rank three and Poisson brackets were used recently in [20, 14, 15, 16] to prove that the Nagata automorphism (see [12])

$$\sigma = (x + (x^2 - yz)z,\ y + 2(x^2 - yz)x + (x^2 - yz)^2 z,\ z)$$

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of the polynomial algebra $F[x, y, z]$ over a field F of characteristic 0 is wild.

Among very few works devoted exclusively to Poisson algebras let us mention [6] were all homogeneous quadratic Poisson structures on $F[x, y, z]$ are described and [13] where algebraic quantizations of free Poisson algebras are constructed.

One of the fundamental results about free associative algebras is the Bergman Centralizer Theorem (see [1]) which says that the centralizer of any nonconstant element is a polynomial algebra on a single variable. This Theorem plays crucial role in the study of algorithmic and combinatorial questions.

An analog of this theorem for polynomial algebras was proved in [15], see also [22].

In Section 2 of this paper we prove that in the characteristic zero case the centralizer of any nonconstant element of a free Poisson algebra is a polynomial algebra on a single variable. Though the Bergman's Theorem is characteristic-free, in our setting we should assume that the characteristic of the ground field F is zero. When characteristic is finite free Poisson algebras have large centers and the centralizers are much larger than in the zero characteristic case.

Some open problems are formulated in Section 3.

2 Centralizers in free Poisson algebras

First, we recall the definition of Poisson algebras. A vector space B over a field F endowed with two bilinear operations $x \cdot y$ (a multiplication) and $[x, y]$ (a Poisson bracket) is called *a Poisson algebra* if B is a commutative associative algebra under $x \cdot y$, B is a Lie algebra under $[x, y]$, and B satisfies the following identity (the Leibniz identity):

$$[x \cdot y, z] = [x, z] \cdot y + x \cdot [y, z].$$

Of course, the Leibniz identity just says that for every $x \in B$ the map

$$ad\, x : B \longrightarrow B, \quad (y \mapsto [y, x]),$$

is a derivation of B as an associative algebra.

An important class of Poisson algebras is given by the following construction. Let L be a Lie algebra with a linear basis $l_1, l_2, \ldots, l_k, \ldots$ over F. Denote by $P(L)$ the polynomial algebra on the variables $l_1, l_2, \ldots, l_k, \ldots$. Using the Leibniz identity we can uniquely extend the Lie bracket $[x, y]$ of L to a Poisson bracket $[x, y]$ on $P(L)$, and $P(L)$ becomes a Poisson algebra. This algebra is called a Poisson-Lie algebra.

We will say that elements f and g of $P(L)$ are *polynomially dependent* if there exists a nonzero polynomial $h(x, y) \in F[x, y]$ such that $h(f, g) = 0$.

Lemma 1 *If $f, g \in P(L)$ are polynomially dependent then $[f, g] = 0$.*

Proof. The proof is quite standard. Take a polynomial dependence h of f and g with, say minimal degree relative to g possible. Then $0 = [f, h(f, g)] = [f, g]\frac{\partial h}{\partial g}(f, g)$ and since $\frac{\partial h}{\partial g}(f, g) \neq 0$ we see that $[f, g] = 0$. $\square$

Remark. Though the statement of this Lemma is correct when characteristic is finite, this proof is good only for the characteristic zero case.

Let $A = F[z_1, \ldots, z_n \ldots]$ be a polynomial algebra in any number of variables. Let $f \in A$. Denote by $C(f)$ the set of all elements of A which are polynomially dependent with f. By a theorem of A. Zaks [22] $C(f) \cong F[t]$. So in order to prove "the centralizer" theorem for a Poisson-Lie algebra P it is sufficient to check that commuting elements of P are also polynomial dependent. Of course it cannot be done without additional restrictions. As we already mentioned characteristic is essential. It is also clear that the structure of L plays role: if e. g. L is abelian then $P(L)$ is commutative, etc.. So from now on we assume that L is a free Lie algebra.

Let L be a free Lie algebra with free (Lie) generators $x_1, x_2, \ldots, x_n$. We will call $P(L)$ a free Poisson algebra on the same set of generators and denote this algebra by $P = P\langle x_1, x_2, \ldots, x_n\rangle$.

We can introduce a degree function deg on L by taking $\deg[\ldots[x_{i_1}, x_{i_2}], \ldots, x_{i_k}] = k$ if $[\ldots[x_{i_1}, x_{i_2}], \ldots, x_{i_k}] \neq 0$ and extending it on L linearly.

Let us choose a homogeneous basis

$$x_1, x_2, \ldots, x_n,\ [x_1, x_2], \ldots, [x_1, x_n], \ldots, [x_{n-1}, x_n],\ [[x_1, x_2], x_3], \ldots$$

of L. Algebra $P\langle x_1, x_2, \ldots, x_n\rangle$ coincides with the algebra of polynomials on these elements. We can extend the function deg on P by $\deg fg = \deg f + \deg g$. Evidently P is graded by this degree and for every element $f \in P$ the highest homogeneous part $\bar{f}$ can be defined in the ordinary way. Note that

$$\overline{fg} = \bar{f}\bar{g},\ \deg[f, g] \leq \deg f + \deg g.$$

For every $f \in P$ the set of elements

$$\mathcal{C}(f) = \{g \in A \mid [f, g] = 0\}$$

is called the *centralizer of* f. It follows immediately from Leibniz and Jacoby identities that $\mathcal{C}(f)$ is a subalgebra of P.

Take $f \in P \setminus F$. We would like to prove that any $g \in \mathcal{C}(f)$ is polynomially dependent with f. So let us assume that some $g \in \mathcal{C}$ is not polynomially dependent with f. It is well known (see, for example [8] or [15]) that there exists $h \in F[f, g] \subset \mathcal{C}$ such that $\bar{f}$ and $\bar{h}$ are also polynomially independent. Since $[f, h] = 0$ and consequently $[\bar{f}, \bar{h}] = 0$ we may assume that there is a pair of homogeneous independent commuting elements.

There is another natural degree function on P, just the total degree on P as a polynomial ring. Denote it by pdeg and observe that $\operatorname{pdeg}[a, b] = \operatorname{pdeg} a + \operatorname{pdeg} b - 1$ for any p-homogeneous a, $b \in P$ if $[a, b] \neq 0$. Therefore if $[f, g] = 0$ and $\tilde{a}$ denotes the highest p-homogeneous form of a then $[\tilde{f}, \tilde{g}] = 0$ as well. Of course, the result mentioned above

([8] or [15]) implies that if f and g are independent then there exists an $h \in F[f,g]$ for which $\widetilde{f}$ and $\widetilde{h}$ are independent. So if we start with a pair of independent commuting elements we can obtain a pair of independent commuting elements which are homogeneous an p-homogeneous. Let us call such elements bi-homogeneous.

Lemma 2 *Let f and g be bi-homogeneous elements of $P \setminus F$ such that $[f,g]=0$. Then there exists $a \in P$ such that $f, g \in F[a]$ i. e. f and g are polynomially dependent.*

Proof. Let us denote elements of the basis

$$x_1, x_2, \ldots, x_n,\ [x_1,x_2], \ldots, [x_1,x_n], \ldots, [x_{n-1},x_n],\ [[x_1,x_2],x_3], \ldots$$

by $e_1, e_2, \ldots, e_n, \ldots$. Put $e_i < e_j$ if $i < j$. It is clear that $[e_m, e_n]$ is a bi-homogeneous element of P and that $\deg[e_m,e_n] = \deg e_m + \deg e_n$ if $m \neq n$. So if $k \leq m < n$ then $[e_m,e_n]$ is a linear combination of $\{e_i\}$ where all $i > k$.

For every element $f \in P$ put

$$S(f) = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$$

if $f \in F[S(f)]$ and $f \notin F[S(f) \setminus \{e_{i_j}\}]$, where $1 \leq j \leq k$.

Assume now that f and g are bi-homogeneous independent elements of P such that $[f,g]=0$ and the number of elements $|S(f)|$ in $S(f)$ is minimal possible; of course then $|S(f)| \leq |S(g)|$.

Let x be the minimal element of $S(f)$. Then

$$f = f_0 + f_1 x + \ldots + f_m x^m,$$

where $f_m \neq 0$, $m > 0$, and $x \notin S(f_i)$ and

$$g = g_0 + g_1 x + \ldots + g_n x^n,$$

where $g_n \neq 0$ and $x \notin S(g_i)$. We can assume that $f_m x^m$ and $g_n x^n$ are independent. Indeed, we introduce a new degree function on P (considered as a polynomial ring) which is induced by $\deg x = 1$ and $\deg e_i = 0$ if $e_i \neq x$ and use the result we mentioned before ([8] or [15]) to find $h \in F[f,g]$ such that $\bar{f}$ and $\bar{h}$ are polynomially independent.

Now,

$$[f,g] = \sum_{i \leq m+n} d_i x^i = 0,$$

where $x \notin S(d_i)$. Consequently, $d_i = 0$. So $d_{m+n} = [f_m, g_n] = 0$. Since $|S(f_m)| < |S(f)|$ we can assume that f_m and g_n are dependent and since they are homogeneous there exists $a \in P \setminus F$ such that $f_m = a^s$ and $g_p = a^t$. Therefore

$$d_{m+n-1} = [ta^{t-1} f_{m-1} - sa^{s-1} g_{n-1} + (mt - ns)a^{s+t-1} x, a] = 0$$

where $mt - ns \neq 0$ since otherwise $f_m x^m$ and $g_n x^n$ are dependent. So a and $ta^{t-1} f_{m-1} - sa^{s-1} g_{n-1} + (mt - ns) a^{s+t-1} x$ are polynomially independent. Hence $s = 0$ since if $s > 0$ then $|S(a)| < |S(f)|$ and we would have a contradiction with our choice of f. So $s = 0$ and $[f_{m-1} + mx, a] = 0$. Since $|S(f_{m-1} + mx)| \leq |S(f)|$ we can assume that $f = f_{m-1} + mx$ and $g = a$. Also f is a linear polynomial in which all variables have the same degree deg because f is bi-homogeneous.

Let y be the minimal element of $S(g)$. Then

$$f = f_0 + f_1 y$$

where $f_1 \in F$ and $y \notin S(f_0)$ and

$$g = g_0 + g_1 y + \ldots + g_n y^n,$$

where $g_n \neq 0$, $n > 0$, and $y \notin S(g_i)$. Of course $y \neq x$ since $S(g) = S(a) \not\ni x$. As above, $[f, g] = 0$ implies $[f_0 + f_1 y, g_n] = 0$ and by induction on $|S(g_n)|$ we can conclude that up to multiplication by a constant $g_n = (f_0 + f_1 y)^t$. Hence $t = 0$ since otherwise $S(g_n) \ni x$. So $g_n \in F$. Next, $[f_0, g_{n-1}] + n g_n [f_0, y] + f_1 [y, g_{n-1}] = 0$. But $[f_0, g_{n-1}] + n g_n [f_0, y] + f_1 [y, g_{n-1}] = [f_0 + f_1 y, g_{n-1} + n g_n y]$. Both $f_0 + f_1 y$ and $g_{n-1} + n g_n y$ are linear polynomials. Therefore we can treat them as elements of L. These elements are linearly independent since $x \in S(f_0 + f_1 y)$ and $x \notin S(g_{n-1} + n g_n y)$. So they cannot commute because L is a free Lie algebra. Lemma is proved. □

Lemmas 1 and 2 give us the main result.

Theorem 1 *In free Poisson algebras over a field of characteristic* 0 *the centralizer of every nonconstant element is a polynomial algebra on single variable.*

3 Problems

Let $P\langle x_1, x_2, \ldots, x_n \rangle$ be a free Poisson algebra over a field F in variables $x_1, x_2, \ldots, x_n$. For every $f \in P\langle x_1, x_2, \ldots, x_n \rangle$ we denote by (f) the two-sided ideal of $P\langle x_1, x_2, \ldots, x_n \rangle$ generated by f.

Problem 1 (Freiheitssatz) *If $f \in P\langle x_1, x_2, \ldots, x_n \rangle$ and $f \notin P\langle x_1, x_2, \ldots, x_{n-1} \rangle$, then is it true that $(f) \cap P\langle x_1, x_2, \ldots, x_{n-1} \rangle = 0$?*

Freiheitssatz was proved by A. I. Shirshov [18] for free Lie algebras and by L. Makar-Limanov [11] for free associative algebras over a field of characteristic 0.

If L is a finitely presented Lie algebra with undecidable word problem [2], then $P(L)$ is a finitely presented Poisson algebra with undecidable word problem. Thus the word problem is undecidable for Poisson algebras in general formulation.

Problem 2 *Is the word problem decidable for Poisson algebras with a single defining relation?*

The decidability of the word problem for Lie algebras with a single defining relation was also proved by A. I. Shirshov [18]. The analogs of this question for associative algebras and for semigroups are open.

It is well known [4] that elements f and g of a free associative algebra are algebraically dependent if and only if $[f, g] = 0$.

Problem 3 *If F is a field of characteristic 0, then is it true that elements f and g of $P\langle x_1, x_2, \ldots, x_n\rangle$ are algebraically dependent if and only if $[f, g] = 0$?*

There exists an algorithm which decides whether a finite set of elements in a free Lie algebra is algebraically dependent (see, for example [17]) and it is known that for free associative algebras algebraic dependence of a finite set of elements is algorithmically unrecognizable [19].

Problem 4 *Is algebraic dependence of a finite set of elements of a free Poisson algebra algorithmically recognizable?*

In 1968 P.Cohn [3] proved that the automorphisms of a free Lie algebra with a finite set of generators are tame. It is well known [5, 7, 9, 10] that automorphisms of polynomial algebras and free associative algebras in two variables are also tame.

Problem 5 *Is it true that automorphisms of two generated free Poisson algebras are tame?*

Note that Nagata automorphism is a wild automorphism of free Poisson algebras in three variables.

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